A local curvature bound in Ricci flow

PENG LU

In this note we give a proof of a result which is closely related to Perelman's theorem in Section 10.3 of the paper *The entropy formula for the Ricci flow and its geometric applications* [4].

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1 Introduction

In [4, Section 10.3], G Perelman gives the following theorem.

Theorem 1.1 (Perelman) *There exist ϵ, $\delta > 0$ with the following property. Suppose $g_{ij}(t)$ is a smooth solution to the Ricci flow on $[0, (\epsilon r_0)^2]$, and assume that at $t = 0$ we have $|\mathrm{Rm}|(x) \le r_0^{-2}$ in $B(x_0, r_0)$, and $\mathrm{Vol}\, B(x_0, r_0) \ge (1-\delta)\omega_n r_0^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then the estimate $|\mathrm{Rm}|(x,t) \le (\epsilon r_0)^{-2}$ holds whenever $0 \le t \le (\epsilon r_0)^2$, $\mathrm{dist}_t(x, x_0) < \epsilon r_0$.*

He continues: "The proof is a slight modification of the proof of theorem 10.1, and is left to the reader. A natural question is whether the assumption on the volume of the ball is superfluous."

In this note by using the idea in the proof of Perelman's pseudolocality theorem [4, Theorem 10.1] (see Theorem 2.1 below for the statement), we will show:

Theorem 1.2 *Given $n \ge 2$ and $v_0 > 0$, there exists $\epsilon_0 > 0$, depending only on n and v_0, which has the following property. For any $r_0 > 0$ and $\epsilon \in (0, \epsilon_0]$ suppose that $(M^n, g(t))$, $t \in [0, (\epsilon r_0)^2]$, is a complete smooth solution to the Ricci flow with bounded sectional curvature, and suppose that at $t = 0$ for some $x_0 \in M$ we have curvature bound $|\mathrm{Rm}|(x, 0) \le r_0^{-2}$ for $x \in B_{g(0)}(x_0, r_0)$ and volume lower bound $\mathrm{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \ge v_0 r_0^n$. Then $|\mathrm{Rm}|(x,t) \le (\epsilon_0 r_0)^{-2}$ for $t \in [0, (\epsilon r_0)^2]$ and $x \in B_{g(t)}(x_0, \epsilon_0 r_0)$.*

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In Section 2 we will give a proof of Theorem 1.2 using two technical lemmas which will be proved in Section 3. In Section 4 we will give two examples and a remark. The first example shows that the curvature bound in Theorem 1.2 is false without the assumption $\mathrm{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^n$. The second example shows that the curvature bound in Theorem 1.2 is false without the assumption that the Ricci flow is complete. The remark says that Theorem 1.2 follows from Theorem 1.1 and the proof of Lemma 3.1.

2 Proof of Theorem 1.2

First we give a proof of Theorem 1.2 assuming Proposition 2.1 below. Then we will prove the proposition.

Proposition 2.1 *Given $n \geq 2$ and $v_0 > 0$, there exists $\epsilon_0 > 0$, depending only on n and v_0, which has the following property. For any $r_0 > 0$ and $\epsilon \in (0, \epsilon_0]$ suppose that $(M^n, g(t))$, $t \in [0, (\epsilon r_0)^2]$, is a complete smooth solution to the Ricci flow with bounded sectional curvature, and suppose that at $t = 0$ for some $x_0 \in M$ we have curvature bound $|\mathrm{Rm}|(x, 0) \leq r_0^{-2}$ for $x \in B_{g(0)}(x_0, r_0)$ and volume lower bound $\mathrm{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^n$. Then $|\mathrm{Rm}|(x, t) \leq (\epsilon_0 r_0)^{-2}$ for $t \in [0, (\epsilon r_0)^2]$ and $x \in B_{g(0)}(x_0, e^{n-1}\epsilon_0 r_0)$.*

Proof of Theorem 1.2 It suffices to prove the following statement. For the solution $g(t)$ in Proposition 2.1 we have

$$B_{g(t)}(x_0, \epsilon_0 r_0) \subset B_{g(0)}(x_0, e^{n-1}\epsilon_0 r_0) \quad \text{for any } t \in [0, (\epsilon r_0)^2]. \tag{1}$$

We will prove (1) by contradiction.

If (1) is not true, there is a point $x \in B_{g(t)}(x_0, \epsilon_0 r_0) \setminus B_{g(0)}(x_0, e^{n-1}\epsilon_0 r_0)$. Let $\gamma(s)$, $0 \leq s \leq s_0$, be a unit-speed minimal geodesic with respect to metric $g(t)$ such that $\gamma(0) = x_0$ and $\gamma(s_0) = x$. Then $s_0 < \epsilon_0 r_0$, and there is a $s_1 \in (0, s_0]$ such that $\gamma(s_1) \in \partial(B_{g(0)}(x_0, e^{n-1}\epsilon_0 r_0))$ and $\gamma([0, s_1]) \subset B_{g(0)}(x_0, e^{n-1}\epsilon_0 r_0)$. In particular, the length satisfies

$$L_{g(0)}(\gamma|_{[0,s_1]}) \geq e^{n-1}\epsilon_0 r_0. \tag{2}$$

From the curvature bound $|\mathrm{Rm}|(x, t) \leq (\epsilon_0 r_0)^{-2}$ in Proposition 2.1 and the Ricci flow equation, we have

$$|\gamma'(s)|_{g(0)} \leq e^{(n-1)}|\gamma'(s)|_{g(t)} \quad \text{for } t \in [0, (\epsilon r_0)^2] \text{ and } s \in [0, s_1].$$

Hence $\displaystyle L_{g(0)}(\gamma|_{[0,s_1]}) \leq \int_0^{s_1} e^{n-1}|\gamma'(s)|_{g(t)}\, ds \leq e^{n-1} \cdot s_0 < e^{n-1}\epsilon_0 r_0.$

This contradicts (2). Hence (1) is proved, and Theorem 1.2 is proved assuming Proposition 2.1. $\square$

Proof of Proposition 2.1 Let $\widetilde{g}(t) \doteqdot (r_0)^{-2} g((r_0)^2 t)$ be the parabolically scaling of $g(t)$. The proposition holds for $g(t)$ and r_0 if and only if the proposition holds for $\widetilde{g}(t)$ and $r_0 = 1$. Hence it suffices to prove the proposition for $r_0 = 1$ which we assume from now on. We will prove the proposition for $r_0 = 1$ by contradiction argument.

Suppose the proposition is not true. Then there are $n \geq 2$, $v_0 > 0$, a sequence of $\epsilon_{0i} \to 0^+$, a sequence of $\epsilon_i \in (0, \epsilon_{0i}]$, a sequence of complete smooth solutions to the Ricci flow $\left(M_i^n, g_i(t)\right)$, $t \in [0, \epsilon_i^2]$, with bounded sectional curvature, and a sequence of points $x_{0i} \in M_i$, such that the following is true for each i:

(i) $|\mathrm{Rm}_{g_i}|(x, 0) \leq 1$ for $x \in B_{g_i(0)}(x_{0i}, 1)$.

(ii) $\mathrm{Vol}_{g_i(0)}(B_{g_i(0)}(x_{0i}, 1)) \geq v_0$.

(iii) There are $t_i \in (0, \epsilon_i^2]$ and $x_i \in B_{g_i(0)}(x_{0i}, e^{n-1}\epsilon_{0i})$ such that $|\mathrm{Rm}_{g_i}|(x_i, t_i) > \epsilon_{0i}^{-2}$.

(iv) $\epsilon_{0i} \leq 1/(8e^{n-1})$.

To get a contradiction from the existence of sequence $\{(M_i, g_i(t))\}$, we need the following point-picking statement whose proof is simpler than the proof of the point-picking claim used by Perelman in [4, Section 10.1]. Let $A_i \doteqdot 1/(100n\epsilon_{0i})$.

Claim A *Fix any i, there is a point $(\overline{x}_i, \overline{t}_i) \in B_{g_i(0)}(x_{0i}, (2A_i + e^{n-1})\epsilon_{0i}) \times (0, \epsilon_i^2]$ with $\overline{Q}_i \doteqdot |\mathrm{Rm}_{g_i}|(\overline{x}_i, \overline{t}_i) > \epsilon_{0i}^{-2}$ such that*

$$|\mathrm{Rm}_{g_i}|(x, t) \leq 4\overline{Q}_i \quad \text{for } (x, t) \in B_{g_i(0)}\big(\overline{x}_i, A_i \overline{Q}_i^{-1/2}\big) \times (0, \overline{t}_i].$$

Proof of Claim A Let $Q_i^0 \doteqdot |\mathrm{Rm}_{g_i}|(x_i, t_i)$. If (x_i, t_i) from (iii) satisfies the curvature bound of the claim, ie,

$$|\mathrm{Rm}_{g_i}|(x, t) \leq 4Q_i^0 \quad \text{for } (x, t) \in B_{g_i(0)}\big(x_i, A_i (Q_i^0)^{-1/2}\big) \times (0, t_i],$$

we choose $(\overline{x}_i, \overline{t}_i) = (x_i, t_i)$ and the claim is proved.

If (x_i, t_i) does not satisfy the curvature bound of the claim, then there is a point

$$(x_i^1, t_i^1) \in B_{g_i(0)}\big(x_i, A_i (Q_i^0)^{-1/2}\big) \times (0, t_i]$$

such that $|\mathrm{Rm}_{g_i}|(x_i^1, t_i^1) > 4Q_i^0$. We compute using $Q_i^0 > \epsilon_{0i}^{-2}$

$$\begin{aligned} d_{g_i(0)}(x_i^1, x_{0i}) &\leq d_{g_i(0)}(x_i, x_{0i}) + A_i (Q_i^0)^{-1/2} \\ &\leq e^{n-1}\epsilon_{0i} + A_i \epsilon_{0i} \\ &< (2A_i + e^{n-1})\epsilon_{0i}. \end{aligned}$$

If (x_i^1, t_i^1) satisfies the curvature bound of the claim, we choose $(\overline{x}_i, \overline{t}_i) = (x_i^1, t_i^1)$ and the claim is proved.

If (x_i^1, t_i^1) does not satisfy the claim, let $Q_i^1 \doteqdot |\mathrm{Rm}_{g_i}|(x_i^1, t_i^1)$, then there is a point

$$(x_i^2, t_i^2) \in B_{g_i(0)}\big(x_i^1, A_i(Q_i^1)^{-1/2}\big) \times (0, t_i^1]$$

such that $|\mathrm{Rm}_{g_i}|(x_i^2, t_i^2) > 4Q_i^1$. We compute using $Q_i^1 > 4Q_i^0$

$$\begin{aligned} d_{g_i(0)}(x_i^2, x_{0i}) &\le d_{g_i(0)}(x_i^1, x_{0i}) + A_i(Q_i^1)^{-1/2} \\ &\le (e^{n-1} + A_i)\epsilon_{0i} + A_i \cdot \tfrac{1}{2}\epsilon_{0i} \\ &< (2A_i + e^{n-1})\epsilon_{0i}. \end{aligned}$$

If (x_i^2, t_i^2) satisfies the curvature bound of the claim, we choose $(\overline{x}_i, \overline{t}_i) = (x_i^2, t_i^2)$ and the claim is proved.

If (x_i^2, t_i^2) does not satisfy the claim, then there will be a point (x_i^3, t_i^3) and we can continue the above process of arguments. Hence for each i either we get a finite sequence of points $\{(x_i^k, t_i^k)\}_{k=0}^{k_i}$ where $(x_i^0, t_i^0) \doteqdot (x_i, t_i)$ such that the claim holds by taking $(\overline{x}_i, \overline{t}_i) = (x_i^{k_i}, t_i^{k_i})$, or there is an infinite sequence of points $\{(x_i^k, t_i^k)\}_{k=0}^{\infty}$ which satisfies the following. Let $Q_i^k \doteqdot |\mathrm{Rm}_{g_i}|(x_i^k, t_i^k)$. Then for each integer $k \ge 0$

$$(x_i^{k+1}, t_i^{k+1}) \in B_{g_i(0)}\big(x_i^k, A_i(Q_i^k)^{-1/2}\big) \times (0, t_i^k]$$

such that $|\mathrm{Rm}_{g_i}|(x_i^{k+1}, t_i^{k+1}) > 4Q_i^k$.

Now we show that for any i there can not be an infinite sequence $\{(x_i^k, t_i^k)\}_{k=0}^{\infty}$ from which the claim follows. We compute

$$\begin{aligned} &d_{g_i(0)}(x_i^{k+1}, x_{0i}) \\ &\qquad \le d_{g_i(0)}(x_{0i}, x_i^0) + d_{g_i(0)}(x_i^0, x_i^1) + d_{g_i(0)}(x_i^1, x_i^2) + \cdots + d_{g_i(0)}(x_i^k, x_i^{k+1}) \\ &\qquad \le e^{n-1}\epsilon_{0i} + A_i(Q_i^0)^{-1/2} + A_i(Q_i^1)^{-1/2} + \cdots + A_i(Q_i^k)^{-1/2} \\ &\qquad \le e^{n-1}\epsilon_{0i} + A_i\epsilon_{0i} + A_i\tfrac{1}{2}\epsilon_{0i} + \cdots + A_i\tfrac{1}{2^k}\epsilon_{0i} \\ &\qquad < (2A_i + e^{n-1})\epsilon_{0i}, \end{aligned}$$

where we have used $Q_i^{k+1} > 4Q_i^k > 4^{k+1}Q_i^0 > 4^{k+1}\epsilon_{0i}^{-2}$. For any fixed i, from $A_i = 1/(100n\epsilon_{0i})$ and $\epsilon_{0i} \le 1/(8e^{n-1})$, we conclude that (x_i^k, t_i^k) is in the compact set $\overline{B_{g_i(0)}(x_{0i}, 1)} \times [0, \epsilon_i^2]$ for all k. On the other hand we have

$$\lim_{k\to\infty} |\mathrm{Rm}_{g_i}|(x_i^k, t_i^k) \ge \lim_{k\to\infty} 4^k\epsilon_{0i}^{-2} = \infty,$$

which is impossible. Now Claim A is proved. □

Let $(\overline{x}_i, \overline{t}_i)$ be the point given by Claim A. We divide the rest of the proof of Proposition 2.1 into three cases according to the value of

$$\overline{\lim}_{i\to\infty}\overline{t}_i \cdot |\mathrm{Rm}_{g_i}|(\overline{x}_i, \overline{t}_i) \doteqdot \widetilde{\alpha} \tag{3}$$

equals to infinite, positive finite number, or zero. We will derive contradictions in each of the three cases.

Case 1 $\widetilde{\alpha} = +\infty$. From Claim A and the choice of $A_i = 1/(100n\epsilon_{0i})$, by passing to a subsequence (still indexed by i) we have

(1i) $\overline{t}_i \le \epsilon_i^2$,

(1ii) $\lim_{i\to\infty}\overline{t}_i \cdot |\mathrm{Rm}_{g_i}|(\overline{x}_i, \overline{t}_i) = \infty$,

(1iii) $d_{g_i(0)}(\overline{x}_i, x_{0i}) < 1/4$. In particular, $B_{g_i(0)}(\overline{x}_i, 3/4) \subset B_{g_i(0)}(x_{0i}, 1)$.

From the assumptions (i) and (ii) given at the beginning of the proof of Proposition 2.1 and the Bishop–Gromov volume comparison theorem there is a constant $v_1 > 0$, depending only on n and v_0, such that

$$\mathrm{Vol}_{g_i(0)}\left(B_{g_i(0)}(x_{0i}, 1/4)\right) \ge v_1.$$

Since the ball $B_{g_i(0)}(\overline{x}_i, 1/2)$ contains the ball $B_{g_i(0)}(x_{0i}, 1/4)$ we have

$$\mathrm{Vol}_{g_i(0)}\left(B_{g_i(0)}(\overline{x}_i, 1/2)\right) \ge v_1. \tag{4}$$

We define a *regular domain* in a smooth manifold to be a bounded domain with a C^1–boundary. Recall Perelman's pseudolocality theorem [4, Theorem 10.1] says the following (for an expository account, see, for example, Chow et al [3, Chapter 21]).

Theorem 2.1 (Perelman) *For every $\alpha > 0$ and $n \ge 2$ there exist $\delta > 0$ and $\epsilon_0 > 0$ depending only on α and n with the following property. Let $(M^n, g(t))$, $t \in [0, (\epsilon r_0)^2]$, where $\epsilon \in (0, \epsilon_0]$ and $r_0 \in (0, \infty)$, be a complete solution of the Ricci flow with bounded curvature and let $x_0 \in M$ be a point such that*

$$R(x, 0) \ge -r_0^{-2} \quad \text{for } x \in B_{g(0)}(x_0, r_0)$$

and

$$\left(\mathrm{Area}_{g(0)}(\partial\Omega)\right)^n \ge (1-\delta)c_n\left(\mathrm{Vol}_{g(0)}(\Omega)\right)^{n-1}$$

for any regular domain $\Omega \subset B_{g(0)}(x_0, r_0)$, where $c_n = n^n\omega_n$ is the Euclidean isoperimetric constant. Then we have the curvature estimate

$$|\mathrm{Rm}(x,t)| \le \frac{\alpha}{t} + \frac{1}{(\epsilon_0 r_0)^2}$$

for $x \in B_{g(t)}(x_0, \epsilon_0 r_0)$ and $t \in (0, (\epsilon r_0)^2]$.

Let $\delta \doteqdot \delta_0 > 0$ be the constant in Theorem 2.1 corresponding to $\alpha = 1$. Applying Lemma 3.1 below to metric $4g_i(0)$ and ball $B_{4g_i(0)}(\overline{x}_i, 1) = B_{g_i(0)}(\overline{x}_i, 1/2)$ we conclude that there is a $r_1 < 1/2$, depending only on n, δ_0 and v_1 but not depending on i, such that

$$(5)\qquad \big(\mathrm{Area}_{g_i(0)}(\partial\Omega)\big)^n \geq (1-\delta_0)c_n\big(\mathrm{Vol}_{g_i(0)}(\Omega)\big)^{n-1}$$

for any regular domain $\Omega \subset B_{g_i(0)}(\overline{x}_i, r_1)$.

Let $r_2 \doteqdot \min\{r_1, 1/\sqrt{n(n-1)}\}$, and let $\widehat{g}_i(t) = (r_2)^{-2} g_i((r_2)^2 t)$, $0 \leq t \leq (r_2)^{-2}\epsilon_i^2$. It follows from assumption (i) that the scalar curvature $R_{\widehat{g}_i}(\,\cdot\,, 0) \geq -1$ on $B_{\widehat{g}_i(0)}(\overline{x}_i, 1)$. From (5) we have

$$\big(\mathrm{Area}_{\widehat{g}_i(0)}(\partial\Omega)\big)^n \geq (1-\delta_0)c_n\big(\mathrm{Vol}_{\widehat{g}_i(0)}(\Omega)\big)^{n-1}$$

for any regular domain $\Omega \subset B_{\widehat{g}_i(0)}(\overline{x}_i, 1)$.

For i large enough we can apply Theorem 2.1 (using $\alpha = 1$) to $(B_{\widehat{g}_i(0)}(\overline{x}_i, 1), \widehat{g}_i(t))$, $0 \leq t \leq (r_2)^{-2}\epsilon_i^2$, and conclude

$$|\mathrm{Rm}_{\widehat{g}_i}|(x,t) \leq \frac{1}{t} + \frac{1}{(r_2)^{-2}\epsilon_i^2}$$

for $t \in (0, (r_2)^{-2}\epsilon_i^2]$ and $x \in B_{\widehat{g}_i(t)}(\overline{x}_i, (r_2)^{-1}\epsilon_i)$. Equivalently we have

$$|\mathrm{Rm}_{g_i}|(x,t) \leq \frac{1}{t} + \frac{1}{\epsilon_i^2}$$

for $t \in (0, \epsilon_i^2]$ and $x \in B_{g_i(t)}(\overline{x}_i, \epsilon_i)$. In particular

$$|\mathrm{Rm}_{g_i}|(\overline{x}_i, \overline{t}_i) \leq \frac{1}{\overline{t}_i} + \frac{1}{\epsilon_i^2} \leq \frac{2}{\overline{t}_i}$$

for i large enough. This contradicts with the assumption of Case 1 that $\widetilde{\alpha}$ in (3) is infinity.

Case 2 $\widetilde{\alpha} \in (0, \infty)$. Let $\widehat{t}_i \doteqdot \overline{Q}_i \overline{t}_i$. Let $\widehat{g}_i(t) \doteqdot \overline{Q}_i g_i\big((\overline{Q}_i)^{-1} t\big)$, $t \in [0, \widehat{t}_i]$. Let b_0 be a constant bigger than $(11/3)(n-1)(\widetilde{\alpha}+1)+1$ to be chosen later (see (10) below). By passing to a subsequence we have

(2i) $|\mathrm{Rm}_{\widehat{g}_i}|(x,t) \leq 4$ for $x \in B_{\widehat{g}_i(0)}(\overline{x}_i, A_i)$ and $t \in [0, \widehat{t}_i]$,

(2ii) $|\mathrm{Rm}_{\widehat{g}_i}|(\overline{x}_i, \widehat{t}_i) = 1$,

(2iii) $|\mathrm{Rm}_{\widehat{g}_i}|(x, 0) \leq \overline{Q}_i^{-1}$ for $x \in B_{\widehat{g}_i(0)}(\overline{x}_i, A_i)$,

(2iv) $\widehat{t}_i \leq \widetilde{\alpha} + 1$, $\widehat{t}_i \to \widetilde{\alpha}$, $A_i > 2e^{4(n-1)(\widetilde{\alpha}+1)} b_0$, and $A_i \to \infty$.

Applying Lemma 3.2 to $\widehat{g}_i(t)$ with $b=b_0$ we get a function $h_i\colon M_i\times[0,\widehat{t}_i]\to[0,1]$ such that the support

$$\operatorname{supp} h_i(\,\cdot\,,t)\subset \overline{B}_{\widehat{g}_i(t)}\left(\overline{x}_i, 2b_0-\frac{11}{3}(n-1)t\right)\subset B_{\widehat{g}_i(0)}(\overline{x}_i,A_i)$$

and

$$\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i(t)}\right)h_i\le \frac{10}{b_0^2}h_i.$$

Recall the curvature $\mathrm{Rm}_{\widehat{g}_i}$ of Ricci flow $\widehat{g}_i(t)$ satisfies

$$\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)|\mathrm{Rm}_{\widehat{g}_i}|^2\le -2|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|^2+16|\mathrm{Rm}_{\widehat{g}_i}|^3.$$

Now we compute the evolution equation of $h_i|\mathrm{Rm}_{\widehat{g}_i}|^2$.

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)(h_i|\mathrm{Rm}_{\widehat{g}_i}|^2)\\
&=\left(\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)h_i\right)|\mathrm{Rm}_{\widehat{g}_i}|^2+h_i\left(\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)|\mathrm{Rm}_{\widehat{g}_i}|^2\right)-2\nabla_{\widehat{g}_i}h_i\cdot\nabla_{\widehat{g}_i}|\mathrm{Rm}_{\widehat{g}_i}|^2\\
&\le \frac{10}{b_0^2}\,h_i|\mathrm{Rm}_{\widehat{g}_i}|^2+h_i\big(-2|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|^2+16|\mathrm{Rm}_{\widehat{g}_i}|^3\big)+\frac{4\sqrt{10}}{b_0}|\mathrm{Rm}_{\widehat{g}_i}|\cdot h_i^{1/2}|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|\\
&\le\left(\frac{10}{b_0^2}+64\right)h_i|\mathrm{Rm}_{\widehat{g}_i}|^2-2h_i|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|^2+\frac{16\sqrt{10}}{b_0}\cdot h_i^{1/2}|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|\\
&\le\left(\frac{10}{b_0^2}+64\right)(h_i|\mathrm{Rm}_{\widehat{g}_i}|^2)+\frac{320}{b_0^2},
\end{aligned}$$

where we have used

$$|\nabla_{\widehat{g}_i}h_i|=\frac{|\phi'(w)|}{b_0}|\nabla_{\widehat{g}_i}d_{\widehat{g}_i(t)}(x,\overline{x}_i)|_{\widehat{g}_i}\le\frac{\sqrt{10}}{b_0}h_i^{1/2}$$

and $|\mathrm{Rm}_{\widehat{g}_i}|\le 4$ on $\operatorname{supp}h_i(\,\cdot\,,t)$. Here ϕ is the function defined in the proof of Lemma 3.2.

Let $u_i\doteqdot h_i|\mathrm{Rm}_{\widehat{g}_i}|^2$. We have proved

$$\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)u_i\le\left(\frac{10}{b_0^2}+64\right)u_i+\frac{320}{b_0^2}$$

on $M_i\times[0,\widehat{t}_i]$.

Let $H_i > 0$ be the backward heat kernel to the conjugate heat equation on $(M_i, \widehat{g}_i(t))$, $t \in [0, \widehat{t}_i]$, centered at $\overline{x}_i$, ie,

$$\left(\frac{\partial}{\partial t} + \Delta_{\widehat{g}_i} - R_{\widehat{g}_i}\right) H_i = 0,$$
$$\lim_{t \to \widehat{t}_i} H_i(x,t) = \delta_{\overline{x}_i}.$$

Note that $\int_{M_i} H_i(\cdot\,, t)\, d\mu_{\widehat{g}_i(t)} = 1$.

Now we compute

$$\begin{aligned}
\frac{d}{dt}&\int_{M_i} u_i H_i \, d\mu_{\widehat{g}_i} \\
&= \int_{M_i}\left(\left(\frac{\partial}{\partial t} - \Delta_{\widehat{g}_i}\right)u_i\right) H_i \, d\mu_{\widehat{g}_i} + \int_{M_i} u_i \left(\left(\frac{\partial}{\partial t} + \Delta_{\widehat{g}_i} - R_{\widehat{g}_i}\right) H_i\right) d\mu_{\widehat{g}_i} \\
&\le \int_{M_i}\left(\left(\frac{10}{b_0^2} + 64\right)u_i + \frac{320}{b_0^2}\right) H_i \, d\mu_{\widehat{g}_i} \\
&= \left(\frac{10}{b_0^2} + 64\right)\int_{M_i} u_i H_i \, d\mu_{\widehat{g}_i} + \frac{320}{b_0^2}.
\end{aligned}$$

Hence it follows from a simple integration that $U_i(t) \doteqdot \int_{M_i} u_i H_i \, d\mu_{\widehat{g}_i}$ satisfies

$$U_i(t) \le e^{(10/b_0^2+64)t}\, U_i(0) + \frac{320}{(10/b_0^2+64)b_0^2}\left(e^{(10/b_0^2+64)t} - 1\right) \tag{6}$$

for $t \in [0, \widehat{t}_i]$.

By the definition of h_i we have at $t = \widehat{t}_i$

$$U_i(\widehat{t}_i) = u_i(\overline{x}_i, \widehat{t}_i) = \phi\left(\frac{(11/3)(n-1)\widehat{t}_i}{b_0}\right) |\mathrm{Rm}_{\widehat{g}_i}|^2(\overline{x}_i, \widehat{t}_i) = 1. \tag{7}$$

On the other hand we have

$$\begin{aligned}
U_i(0) &= \int_{M_i} h_i(x,0) |\mathrm{Rm}_{\widehat{g}_i}|^2(x,0) H_i(x,0)\, d\mu_{\widehat{g}_i(0)} \\
&\le \int_{B_{\widehat{g}_i(0)}(\overline{x}_i, 2b_0)} |\mathrm{Rm}_{\widehat{g}_i}|^2(x,0) H_i(x,0)\, d\mu_{\widehat{g}_i(0)} \\
&\le \bar{Q}_i^{-2} \int_{B_{\widehat{g}_i(0)}(\widehat{x}_i, 2b_0)} H_i(x,0)\, d\mu_{\widehat{g}_i(0)} \\
&\le \bar{Q}_i^{-2} \int_{M_i} H_i(x,0)\, d\mu_{\widehat{g}_i(0)}
\end{aligned}$$

where we have used support $\operatorname{supp} h_i(\,\cdot\,,0) \subset B_{\hat g_i(0)}(\bar x_i, 2b_0)$ in the first inequality and (2iii) in the second inequality. Hence we have proved

$$U_i(0) \le \bar Q_i^{-2}. \tag{8}$$

By combining (6), (7) and (8) we get

$$1 \le e^{(10/b_0^2+64)\hat t_i}\,\bar Q_i^{-2} + \frac{320}{(10/b_0^2+64)b_0^2}\big(e^{(10/b_0^2+64)\hat t_i} - 1\big).$$

Hence

$$1 \le e^{(10/b_0^2+64)(\tilde\alpha+1)}\,\bar Q_i^{-2} + \frac{320}{(10/b_0^2+64)b_0^2}\,e^{(10/b_0^2+64)(\tilde\alpha+1)}. \tag{9}$$

Let

$$b_0 \doteqdot \max\left\{\frac{11}{3}(n-1)(\tilde\alpha+1)+1,\, 3e^{33(\tilde\alpha+1)}\right\}. \tag{10}$$

For such choice of b_0 we have

$$\frac{320}{(10/b_0^2+64)b_0^2}\,e^{(10/b_0^2+64)(\tilde\alpha+1)} < \frac{5}{9}.$$

Equation (9) is impossible since $\bar Q_i \to \infty$. We get the required contradiction for Case 2.

Case 3 $\tilde\alpha = 0$. The proof of this case is similar to the proof of Case 2. Let $\hat t_i \doteqdot \bar Q_i \bar t_i$. Let $\hat g_i(t) \doteqdot \bar Q_i g_i\big((\bar Q_i)^{-1}t\big)$, $t \in [0,\hat t_i]$. By passing to a subsequence we have

- (3i) $|\mathrm{Rm}_{\hat g_i}|(x,t) \le 4$ for $x \in B_{\hat g_i(0)}(\bar x_i, A_i)$ and $t \in [0,\hat t_i]$,
- (3ii) $|\mathrm{Rm}_{\hat g_i}|(\bar x_i, \hat t_i) = 1$,
- (3iii) $|\mathrm{Rm}_{\hat g_i}|(x,0) \le \bar Q_i^{-1}$ for $x \in B_{\hat g_i(0)}(\bar x_i, A_i)$,
- (3iv) $\hat t_i \le 1/(6(n-1))$, $\hat t_i \to 0$, $A_i > 4e^2$ and $A_i \to \infty$.

Applying Lemma 3.2 to $\hat g_i(t)$ with $b = 2$ we get a function $h_i\colon M_i \times [0,\hat t_i] \to [0,1]$ such that the support

$$\operatorname{supp} h_i(\,\cdot\,,t) \subset \bar B_{\hat g_i(t)}\left(\bar x_i, 4 - \frac{11}{3}(n-1)t\right) \subset B_{\hat g_i(0)}(\bar x_i, A_i)$$

and
$$\left(\frac{\partial}{\partial t} - \Delta_{\hat g_i(t)}\right)h_i \le \frac{5}{2}h_i.$$

We compute

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)(h_i|\mathrm{Rm}_{\widehat{g}_i}|^2)\\
&=\left(\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)h_i\right)|\mathrm{Rm}_{\widehat{g}_i}|^2+h_i\left(\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)|\mathrm{Rm}_{\widehat{g}_i}|^2\right)-2\nabla_{\widehat{g}_i}h_i\cdot\nabla_{\widehat{g}_i}|\mathrm{Rm}_{\widehat{g}_i}|^2\\
&\le\frac{5}{2}\,h_i|\mathrm{Rm}_{\widehat{g}_i}|^2+h_i\big(-2|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|^2+16|\mathrm{Rm}_{\widehat{g}_i}|^3\big)+2\sqrt{10}|\mathrm{Rm}_{\widehat{g}_i}|\cdot h_i^{1/2}|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|\\
&\le\frac{133}{2}\,h_i|\mathrm{Rm}_{\widehat{g}_i}|^2-2h_i|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|^2+8\sqrt{10}\,h_i^{1/2}|\nabla_{\widehat{g}_i}\mathrm{Rm}_{\widehat{g}_i}|\\
&\le\frac{133}{2}\,(h_i|\mathrm{Rm}_{\widehat{g}_i}|^2)+80,
\end{aligned}$$

where we have used

$$|\nabla_{\widehat{g}_i}h_i|=\frac{|\phi'(w)|}{2}|\nabla_{\widehat{g}_i}d_{\widehat{g}_i(t)}(x,\overline{x}_i)|_{\widehat{g}_i}\le\frac{\sqrt{10}}{2}h_i^{1/2}$$

and $|\mathrm{Rm}_{\widehat{g}_i}|\le 4$ on $\operatorname{supp}h_i(\,\cdot\,,t)$. Here ϕ is the function defined in the proof of Lemma 3.2.

Let $u_i \doteqdot h_i|\mathrm{Rm}_{\widehat{g}_i}|^2$. We have proved

$$\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)u_i\le 67u_i+80$$

on $M_i\times[0,\widehat{t}_i]$.

Let $H_i>0$ be the backward heat kernel to the conjugate heat equation on $(M_i,\widehat{g}_i(t))$, $t\in[0,\widehat{t}_i]$, centered at $\overline{x}_i$. Note that $\int_{M_i}H_i(\,\cdot\,,t)\,d\mu_{\widehat{g}_i(t)}=1$. We compute

$$\begin{aligned}
\frac{d}{dt}\int_{M_i}u_iH_i\,d\mu_{\widehat{g}_i}&=\int_{M_i}\left(\left(\frac{\partial}{\partial t}-\Delta_{\widehat{g}_i}\right)u_i\right)H_i\,d\mu_{\widehat{g}_i}\\
&\le\int_{M_i}(67u_i+80)\,H_i\,d\mu_{\widehat{g}_i}\\
&=67\int_{M_i}u_iH_i\,d\mu_{\widehat{g}_i}+80.
\end{aligned}$$

Hence it follows from a simple integration that $U_i(t)\doteqdot\int_{M_i}u_iH_i\,d\mu_{\widehat{g}_i}$ satisfies

$$U_i(t)\le e^{67t}U_i(0)+\frac{80}{67}(e^{67t}-1) \tag{11}$$

for $t\in[0,\widehat{t}_i]$.

At $t = \widehat{t_i}$ we have

$$U_i(\widehat{t_i}) = u_i(\overline{x}_i, \widehat{t_i}) = \phi\left(\frac{(11/3)(n-1)\widehat{t_i}}{2}\right)|\mathrm{Rm}_{\widehat{g}_i}|^2(\overline{x}_i, \widehat{t_i}) = 1. \tag{12}$$

On the other hand by an argument similar to the proof of (8) we have

$$U_i(0) \le \overline{Q}_i^{-2}. \tag{13}$$

By combining (11), (12) and (13) we get

$$1 \le e^{67\widehat{t_i}}\,\overline{Q}_i^{-2} + \frac{80}{67}(e^{67\widehat{t_i}} - 1).$$

This is impossible since $\widehat{t_i} \to 0$ and $\overline{Q}_i \to \infty$. We get the required contradiction for Case 3.

Now we have finished the proof of Proposition 2.1 modulo the proofs of Lemmas 3.1 and 3.2. □

3 Proof of two technical lemmas

In the proof of Proposition 2.1 we have used the following two lemmas. Intuitively the first lemma says that if a ball of radius 1 has bounded sectional curvature and is volume noncollapsing, then the isoperimetric constant on small certain size ball is close to the Euclidean one. Note that the next lemma and essential the same proof are also given by Wang [5].

Lemma 3.1 *Given $n \ge 2$, $v_0 > 0$ and $\delta_0 > 0$, there is $r > 0$, depending only on n, v_0, and δ_0, which has the following property. Let $B(x_0, 1)$ be a ball in a Riemannian manifold (M^n, g) which satisfies the following:*

(I) *The closed ball $\overline{B(x_0, 1)}$ is compact in M.*

(II) *The Riemann curvature $|\mathrm{Rm}| \le 1$ on $B(x_0, 1)$.*

(III) *The volume $\mathrm{Vol}(B(x_0, 1)) \ge v_0 > 0$.*

Then we have

$$\big(\mathrm{Area}(\partial\Omega)\big)^n \ge (1-\delta_0)c_n\big(\mathrm{Vol}(\Omega)\big)^{n-1} \tag{14}$$

for any regular domain $\Omega \subset B(x_0, r)$. Here $c_n = n^n\omega_n$ is the isoperimetric constant for Euclidean space.

Proof **Step 1** (Injectivity radius bound) Under the assumption of Lemma 3.1, by a theorem of Cheeger–Gromov–Taylor [2, Theorem A.7] there is a $\iota_0 > 0$ depending only on n and v_0 such that the injectivity radius $\mathrm{inj}_{x_0} \geq \iota_0$.

Step 2 (Metric tensor on ball $B(x_0, 1)$) Let $x = (x^i)$ be the normal coordinates at x_0. It follows from a result of Hamilton (see Cao et al [1, Theorem 4.10, page 308]) that for any $\varepsilon > 0$ there is $\lambda_0 = \lambda_0(n, \varepsilon)$ such that metric tensor

$$(1-\varepsilon)(\delta_{ij}) \leq (g_{ij}) \leq (1+\varepsilon)(\delta_{ij}) \tag{15}$$

for $|x| \leq \lambda_0$. Note that (δ_{ij}) is the Euclidean metric in the coordinates (x^i).

Step 3 (Approximation argument) Let $r \doteqdot \min\{\iota_0, \lambda_0\}$ and let $\exp_{x_0}\colon B(r) \to B(x_0, r)$ be the exponential map. $\exp_{x_0}$ is a diffeomorphism. Now we consider a regular domain $\Omega \subset B\,(x_0, r)$. We compute

$$\begin{aligned}
\mathrm{Vol}_g(\Omega) &= \int_\Omega \sqrt{\det(g_{ij})} \cdot dx^1 \cdots dx^n \\
&\leq \int_{(\exp_{x_0})^{-1}\Omega} \sqrt{(1+\varepsilon)^n \det(\delta_{ij})} \cdot dx^1 \cdots dx^n \\
&= (1+\varepsilon)^{n/2}\, \mathrm{Vol}_{\mathrm{Euc}}((\exp_{x_0})^{-1}\Omega).
\end{aligned}$$

Let $\{\theta_a\}_{a=1}^{n-1}$ be an orthonormal frame of $(\partial\Omega, (\delta_{ij})|_{\partial\Omega})$ at some point x and let $\{\theta_a^*\}$ be the dual frame. The area form $d\sigma_{(\partial\Omega,(\delta_{ij})|_{\partial\Omega})}$ at x is given by $\theta_1^* \wedge \cdots \wedge \theta_{n-1}^*$. The area form $d\sigma_{(\partial\Omega, g|_{\partial\Omega})}$ at x is given by

$$\sqrt{\det(g(\theta_a, \theta_b))_{(n-1)\times(n-1)}} \cdot \theta_1^* \wedge \cdots \wedge \theta_{n-1}^*.$$

We can estimate

$$\sqrt{\det(g(\theta_a, \theta_b))_{(n-1)\times(n-1)}} \geq \sqrt{(1-\epsilon)^{n-1} \det((\delta_{ij})(\theta_a, \theta_b))} = (1-\epsilon)^{(n-1)/2},$$

hence

$$\begin{aligned}
\mathrm{Area}_{g|_{\partial\Omega}}(\partial\Omega) &= \int_{\partial\Omega} d\sigma_{(\partial\Omega, g|_{\partial\Omega})} \\
&\geq \int_{\partial((\exp_{x_0})^{-1}\Omega)} (1-\varepsilon)^{(n-1)/2} d\sigma_{(\partial((\exp_{x_0})^{-1}\Omega),(\delta_{ij})|_{\partial((\exp_{x_0})^{-1}\Omega)})} \\
&= (1-\varepsilon)^{(n-1)/2}\, \mathrm{Area}_{\mathrm{Euc}}(\partial((\exp_{x_0})^{-1}\Omega)).
\end{aligned}$$

Now we compute

$$\begin{aligned}\frac{\big(\mathrm{Area}_{g|_{\partial\Omega}}(\partial\Omega)\big)^n}{\big(\mathrm{Vol}_g(\Omega)\big)^{n-1}} &\geq \frac{\Big((1-\varepsilon)^{(n-1)/2}\,\mathrm{Area}_{\mathrm{Euc}}\,\big(\partial((\exp_{x_0})^{-1}\Omega)\big)\Big)^n}{\Big((1+\varepsilon)^{n/2}\,\mathrm{Vol}_{\mathrm{Euc}}((\exp_{x_0})^{-1}\Omega)\Big)^{n-1}}\\ &= \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{(n-1)n/2}\cdot\frac{\big(\mathrm{Area}_{\mathrm{Euc}}\,\big(\partial((\exp_{x_0})^{-1}\Omega)\big)\big)^n}{\big(\mathrm{Vol}_{\mathrm{Euc}}((\exp_{x_0})^{-1}\Omega)\big)^{n-1}}\\ &\geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{(n-1)n/2} c_n.\end{aligned}$$

Given δ_0 we choose ε such that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{(n-1)n/2} = 1-\delta_0,$$

which in turn requires us to choose the corresponding $\lambda_0(n,\varepsilon)$ to ensure (15). Then Lemma 3.1 holds for $r=\min\{\iota_0,\lambda_0\}$. □

The second lemma is about the existence of an auxiliary function.

Lemma 3.2 *Let $(M^n, g(t))$, $t\in[0,\hat t]$, be a solution of the Ricci flow. Let b be a constant bigger than $(11/3)(n-1)\hat t+1$ and let A be a constant bigger or equal to $2e^{4(n-1)\hat t}b$. We assume that closed ball $\overline{B_{g(0)}(\bar x, A)}\subset M$ is a compact subset and that $|\mathrm{Rm}|(x,t)\leq 4$ for $(x,t)\in B_{g(0)}(\bar x,A)\times[0,\hat t]$. Then there is a function $h\colon M\times[0,\hat t]\to[0,1]$ such that for each $t\in[0,\hat t]$ the support*

$$\operatorname{supp} h(\cdot,t)\subset \bar B_{g(t)}(\bar x, 2b-(11/3)(n-1)t)\subset B_{g(0)}(\bar x,A)$$

and

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)h\leq\frac{10}{b^2}h$$

on $M\times[0,\hat t]$.

Proof Let $\phi\colon\mathbb{R}\to[0,1]$ be a smooth function which is strictly decreasing on the interval $[1,2]$ and which satisfies

$$\phi(s)=\begin{cases}1 & \text{if } s\in(-\infty,1],\\ 0 & \text{if } s\in[2,\infty),\end{cases}\tag{16}$$

and

$$(\phi'(s))^2\leq 10\phi(s),\tag{17a}$$

$$\phi''(s)\geq -10\phi(s)\tag{17b}$$

for $s \in \mathbb{R}$. We define for any $t \in [0, T]$

$$h(x,t) = \phi\left(\frac{d_{g(t)}(x,\overline{x}) + at}{b}\right)$$

where a and b are two positive constants to be chosen. Note that $\operatorname{supp} h(\cdot, t) \subset B_{g(t)}(\overline{x}, 2b - at)$.

By the curvature assumption we have $B_{g(t)}(\overline{x}, e^{-4(n-1)\hat{t}} A) \subset B_{g(0)}(\overline{x}, A)$ for $t \in [0, \hat{t}]$. We choose $2b \le e^{-4(n-1)\hat{t}} A$ so that $\operatorname{supp} h\,(\cdot, t) \subset B_{g(0)}\,(\overline{x}, A)$.

Let $w(x,t) \doteqdot (d_{g(t)}(x,\overline{x}) + at)/b$. We compute

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \\
&\quad = \frac{\phi'(w)}{b}\left(\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x,\overline{x}) + a\right) - \frac{\phi''(w)}{b^2}\left|\nabla_{g(t)} d_{g(t)}(x,\overline{x})\right|^2_{g(t)} \\
&\quad \le \frac{\phi'(w)}{b}\left(\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x,\overline{x}) + a\right) + \frac{10}{b^2}\, h.
\end{aligned}$$

Choosing a such that $a\hat{t} < b - 1$, then for $x \in B_{g(t)}(\overline{x}, 1)$ or $x \notin \operatorname{supp} h(\cdot, t)$ we have $\phi'(w)(x,t) = 0$, hence for such x we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \le \frac{10}{b^2}\, h.$$

For $x \notin B_{g(t)}\,(\overline{x}, 1)$ and $x \in \operatorname{supp} h(\cdot, t)$, we use [4, Lemma 8.3(a)] with $r_0 = 1$ and $K = 4$ and get

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) d_{g(t)}(x,\overline{x})\bigg|_{t=t_0} \ge -(n-1)\left(\frac{2}{3} K r_0 + \frac{1}{r_0}\right) = -\frac{11}{3}(n-1).$$

By choosing $a \doteqdot \frac{11}{3}(n-1)$ and using $\phi'(w) \le 0$ we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) h \le \frac{10}{b^2}\, h.$$

The lemma is proved. □

4 Two examples

In this section we give two example showing that neither the volume lower bound assumption nor the completeness assumption in Theorem 1.2 can be dropped.

Let r be an arbitrary positive constant in $(0,1]$. Let g_r^0 be a Riemannian metric on a topological sphere Σ^2 which contains a round cylinder $S^1(r)\times[-1,1]$ of radius r and length 2. We have $\mathrm{Vol}_{g_r^0}(\Sigma)\geq 4\pi r$. We assume volume $\mathrm{Vol}_{g_r^0}(\Sigma)\leq 20r$. Let $(\Sigma^2, g_r(t))$, $t\in[0,T_r)$, be the maximal solution of the Ricci flow with $g_r(0)=g_r^0$. Then the blowup time

$$T_r=\frac{1}{8\pi}\,\mathrm{Vol}_{g_r^0}(\Sigma)\in\left(\frac{1}{2}r,\frac{5}{2\pi}r\right].$$

Let $p\in S^1(r)$. Then $x_0 \doteqdot (p,0)$ is a point in Σ. For any ϵ_0 we can choose r small enough so that $T_r<\epsilon_0$. Clearly we have $|\mathrm{Rm}_{g_r}|(x,0)=0$ for $x\in B_{g_r(0)}(x_0,1)$ and $\mathrm{Vol}_{g_r(0)}(B_{g_r(0)}(x_0,1))\leq 4\pi r$. For any $\epsilon\in((1/2)r,T_r)$, should the conclusion of Theorem 1.2 hold for $g_r(t)$ when r is small enough, we would have $|\mathrm{Rm}_{g_r}|(x_0,\epsilon)\leq \epsilon_0^{-2}$. Since ϵ is arbitrary, we have $\lim_{t\to T_r}|\mathrm{Rm}_{g_r}|(x_0,t)<\epsilon_0^{-2}$. However it is well-known that the limit should be infinity. Hence Theorem 1.2 does not hold for $g_r(t)$. By taking the product of $(\Sigma^2,g_r(t))$ with flat torus we get high dimensional examples.

The second example is a simple modification of the previous example, the idea of construction is due to Peter Topping (unpublished work). Let

$$\Phi\colon \mathbb{R}\times(-1,1)\to S^1(r)\times(-1,1)\subset\Sigma$$

be the standard universal cover map. Then $(\mathbb{R}\times(-1,1),\Phi^*g_r(t))$ is a incomplete solution of the Ricci flow. Clearly we have $\left|\mathrm{Rm}_{\Phi^*g_r}\right|(x,0)=0$ for $x\in B_{\Phi^*g_r(0)}((0,0),1)$ and $\mathrm{Vol}_{\Phi^*g_r(0)}(B_{\Phi^*g_r(0)}((0,0),1))=\pi$. Arguing as in the previous example we conclude that Theorem 1.2 does not hold for $\Phi^*g_r(t)$ with the ball center being $(0,0)$ when r is small enough.

Finally we make a remark. It follows from the proof of Lemma 3.1 that under the same assumption as the lemma there is a $\tilde{r}\in(0,1]$, depending only on n, v_0, and δ_0, such that

$$\mathrm{Vol}\,(B(x_0,\tilde{r}))\geq(1-\delta_0)\omega_n\tilde{r}^n.$$

We need to switch the notation below. Denote the r_0 in Theorem 1.1 by r_1 and denote the r_0 in Theorem 1.2 still by r_0. Let δ_0 to be the δ in Theorem 1.1. Let $g(t)$ be a solution of the Ricci flow satisfying the assumption of Theorem 1.2. Then the assumption of Theorem 1.1 holds for $g(t)$ with $r_1=r_0\tilde{r}$, hence by Theorem 1.1 we get a curvature bound which is essentially equivalent to the curvature bound given by Theorem 1.2. The reason why we do not use Theorem 1.1 and the proof of Lemma 3.1 to give a more direct proof of Theorem 1.2 is that at the time of writing this note the author is not aware of a detailed proof of Theorem 1.1 in the literature.

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Department of Mathematics, University of Oregon
Eugene, OR 97405

penglu@uoregon.edu

Proposed: Tobias Colding
Seconded: Gang Tian, David Gabai

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