The local Donaldson–Thomas theory of curves

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The local Donaldson–Thomas theory of curves is solved by localization and degeneration methods. The results complete a triangle of equivalences relating Gromov–Witten theory, Donaldson–Thomas theory, and the quantum cohomology of the Hilbert scheme of points of the plane.

Dedicated to the memory of Raoul Bott

1 Introduction

1.1 Overview

Let X be a nonsingular projective variety of dimension 3 over $\mathbb{C}$. Gromov–Witten theory is defined by integration over the moduli space of stable maps to X, and Donaldson–Thomas theory is defined by integration over the moduli space of ideal sheaves of X (see Donaldson–Thomas [5], Maulik et al [24; 25] and Thomas [34]).

If X is quasi-projective, the Gromov–Witten and Donaldson–Thomas theories may not be well-defined. However, if X is the total space of a rank 2 bundle over a nonsingular projective curve,

$$N \to C,$$

local Gromov–Witten and Donaldson–Thomas theories are defined via equivariant residues (see Bryan–Pandharipande [4] and Maulik et al [24]).

The Gromov–Witten and Donaldson–Thomas theories of X *relative* to a nonsingular surface

$$S \subset X$$

are defined via moduli spaces of maps and sheaves with boundary conditions along S. See Eliashberg–Givental–Hofer [6], Ionel–Parker [12], Li–Ruan [16], Li [17; 18] and Maulik et al [25] for various treatments of the subject.

If X is the total space of a rank 2 bundle, a natural set of surfaces is determined by the fibers of N over points of C. The data

(1) $(N \to C, \ p_1, \ldots, p_r)$

determine relative local Gromov–Witten and Donaldson–Thomas theories of N over C.

The *local theory of curves* refers to all relative local Gromov–Witten and Donaldson–Thomas theories[1] specified by data of type (1). The main result of the paper is a proof of the Gromov–Witten/Donaldson–Thomas correspondence for the local theory of curves.

The local Gromov–Witten theory of curves is determined in Bryan–Pandharipande [4]. Our focus here is on the local Donaldson–Thomas theory of curves. The paper has at least three motivations:

  (i)   local curves provide a rich class of non-toric examples of the GW/DT correspondence,

 (ii)   the proof yields a computation of the 1–legged equivariant vertex,

(iii)   the correspondence for local curves will likely play a basic role in the proof of the GW/DT correspondence for all 3–folds.

The 1–legged equivariant vertex (ii) contains the Calabi–Yau 1–legged vertex governed by the Gopakumar–Mariño–Vafa formula (see Liu–Liu–Zhou [21], Mariño–Vafa [23] and Okounkov–Pandharipande [29]) as a special case. The computation of the 1–legged equivariant vertex may be viewed as a Hodge integral result on the Gromov–Witten side or a vertex measure result on the Donaldson–Thomas side.

The main foundational reference for Donaldson–Thomas theory in algebraic geometry is Thomas [34], see also Maulik et al [24]. The required foundational development of relative Donaldson–Thomas theory has yet to be written.[2] We will assume two basic properties of relative Donaldson–Thomas theory: existence and degeneration (see Maulik et al [25]).

1.2 Definitions

1.2.1 Ideal sheaves Let C be a nonsingular projective curve, and let N be the total space of a rank 2 bundle,

$$N \to C.$$

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[1] We study here theories with *no* internal insertions.
[2] See Li–Wu [19] for recent developments.

Let $I_n(N, d)$ denote the moduli space of of ideal sheaves

$$0 \longrightarrow I_Z \longrightarrow \mathcal{O}_N \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

of proper subschemes $Z \subset N$ of degree d and Euler characteristic

$$n = \chi(\mathcal{O}_Z).$$

The *degree* of Z is simply the length of the intersection

$$Z \cap N_p,$$

where $p \in C$ is a generic point.

1.2.2 Partition functions If N decomposes as a direct sum of line bundles,

$$(2) \qquad\qquad N = L_1 \oplus L_2,$$

the splitting determines a scaling action of a 2–dimensional torus T on N. The *level* of the splitting is the pair of integers (k_1, k_2) where,

$$k_i = \deg(L_i).$$

Of course, the scaling action and the level depend upon the splitting (2).

Since every T–fixed proper subscheme Z must be supported on the zero section of N, the T–fixed point set

$$I_n(N, d)^T \subset I_n(N, d)$$

is proper.

The moduli space $I_n(N, d)$ carries a T–equivariant perfect obstruction theory obtained from (traceless) $\mathrm{Ext}_0(I, I)$, see Thomas [34]. Though N is quasi-projective, $\mathrm{Ext}_0(I, I)$ is well-defined since the associated quotient scheme $Z \subset N$ is proper. Alternatively, for any T–equivariant compactification,

$$N \subset \bar{N},$$

the obstruction theory on

$$I_n(N, d) \subset I_n(\bar{N}, d)$$

is obtained by restriction.

The definition of the local Donaldson–Thomas invariants of N follows the strategy of Bryan–Pandharipande [4] and Maulik et al [24]. We define $\mathsf{Z}(N)_d$ formally by:

$$(3) \qquad\qquad \mathsf{Z}(N)_d \text{ `` } = \text{ '' } \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(N, d)]^{\mathrm{vir}}} 1.$$

The variable q indexes the Euler characteristic n. The integral on the right of (3) is well-defined by the virtual localization formula as an equivariant residue.

Definition 1 *The partition function for the degree d local Donaldson–Thomas invariants of N is defined by:*

$$(4) \qquad \mathsf{Z}(N)_d = \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(N,d)^T]^{\mathrm{vir}}} \frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})}.$$

The T–fixed part of the perfect obstruction theory for $I_n(N, d)$ induces a perfect obstruction theory for $I_n(N, d)^T$ and hence a virtual class (see Graber–Pandharipande [9] and Maulik et al [24]). The equivariant virtual normal bundle of the embedding,

$$I_n(N, d)^T \subset I_n(N, d),$$

is $\mathrm{Norm}^{\mathrm{vir}}$ with equivariant Euler class $e(\mathrm{Norm}^{\mathrm{vir}})$. The integral in (4) denotes equivariant push-forward.

Following the notation of Maulik et al [24; 25], $\mathsf{Z}(N)_d$ is *unprimed* since the degree 0 contributions have not yet been removed. Since a geometrical method of removing the degree 0 contribution from Donaldson–Thomas theory is not available, a formal method is followed.

Definition 2 *The reduced partition function $\mathsf{Z}'_{DT}(N)_d$ for the degree d local Donaldson–Thomas invariants of N is defined by:*

$$\mathsf{Z}'(N)_d = \frac{\mathsf{Z}(N)_d}{\mathsf{Z}(N)_0}.$$

The residues defined by the localization formula take values in the localized T–equivariant cohomology of a point,

$$\int_{[I_n(N,d)^T]^{\mathrm{vir}}} \frac{1}{e(\mathrm{Norm}^{\mathrm{vir}})} \in \mathbb{Q}(t_1, t_2).$$

Here, t_1, t_2 are the weights of the standard representations of the factors of T.

If N is an *indecomposable* rank 2 vector bundle, the total space of N admits a scaling action of a 1–dimensional algebraic torus. The local Donaldson–Thomas theory of N can be defined as above with respect to the 1–dimensional scaling torus. However, since every indecomposable bundle N is deformation equivalent to a split bundle over C, the indecomposable case is recovered from the split case via restriction to the diagonal torus.

In our study of the local Donaldson–Thomas theory of N, we will *always* assume a splitting (2) of N and an action of a 2–dimensional scaling torus T.

1.2.3 Relative geometry The fiber of N over a point $p \in C$ determines a T–equivariant divisor

$$N_p \subset N$$

isomorphic to $\mathbb{C}^2$ with the standard T–action. We will consider the local theories of N relative to the divisor

$$S = \bigcup_{i=1}^{r} N_{p_i} \subset N$$

determined by the fibers over $p_1, \ldots, p_r \in C$.

Let $I_n(N/S, d)$ denote the relative moduli space of ideal sheaves, see Li [18] and Maulik et al [25] for a discussion. The construction of $I_n(N/S, d)$, and the existence of a canonical T–equivariant $\mathrm{Ext}_0(I, I)$ obstruction theory will be assumed here.

For each p_i, let η^i be a partition of d weighted by the equivariant Chow ring,

$$A_T^*(N_{p_i}, \mathbb{Q}) \cong \mathbb{Q}[t_1, t_2],$$

of the fiber N_{p_i}. By Nakajima's construction, a weighted partition η^i determines a T–equivariant class

$$C_{\eta^i} \in A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$$

in the Chow ring of the Hilbert scheme of points. In Donaldson–Thomas theory, the weighted partition η^i specifies relative conditions via the boundary map

$$\epsilon_i \colon I_n(N/S, d) \to \mathrm{Hilb}(N_{p_i}, d).$$

An element $\eta \in \mathcal{P}(d)$ of the set of partitions of d may be viewed as a weighted partition with all weights set to the identity class $1 \in H_T^*(N_{p_i}, \mathbb{Q})$. The Nakajima basis of $A_T^*(\mathrm{Hilb}(N_{p_i}, d), \mathbb{Q})$ consists of identity weighted partitions indexed by $\mathcal{P}(d)$. The T–equivariant intersection pairing in the Nakajima basis is:

$$\int_{\mathrm{Hilb}(N_{p_i}, d)} C_\mu \cup C_\nu = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{(-1)^{d-\ell(\mu)}}{\mathfrak{z}(\mu)} \delta_{\mu,\nu},$$

where

$$\mathfrak{z}(\mu) = \prod_{i=1}^{\ell(\mu)} \mu_i \cdot |\mathrm{Aut}(\mu)|.$$

The notation $\eta([0])$ will be used to set all weights to $[0] \in A_T^*(N_{p_i}, \mathbb{Q})$. Since

$$[0] = t_1 t_2 \in A_T^*(N_{p_i}, \mathbb{Q}),$$

the weight choice has only a mild effect.

Following the notation of Bryan–Pandharipande [4] and Maulik et al [25], the relative local Donaldson–Thomas partition function,

$$Z(N/S)_{d,\eta^1,\ldots,\eta^r} = \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(N/S,d)^T]^{\mathrm{vir}}} \frac{\prod_{i=1}^r \epsilon_i^*(C_{\eta^i})}{e(\mathrm{Norm}^{\mathrm{vir}})},$$

is well-defined. Let

$$Z'(N/S)_{d,\eta^1,\ldots,\eta^r} = \frac{Z(N/S)_{d,\eta^1,\ldots,\eta^r}}{Z(N/S)_0}.$$

denote the reduced relative partition function.

1.3 Degeneration

Simpler notation for the partition functions will often be used. If N is split with level (k_1, k_2) over a genus g base curve C, then let

$$Z(g|k_1, k_2)_{d,\eta^1,\ldots,\eta^r} = Z(N/S)_{d,\eta^1,\ldots,\eta^r},$$

$$Z'(g|k_1, k_2)_{d,\eta^1,\ldots,\eta^r} = Z'(N/S)_{d,\eta^1,\ldots,\eta^r}.$$

For many formulas, the q–shifted function,

$$\mathsf{DT}(g|k_1, k_2)_{d,\eta^1,\ldots,\eta^r} = q^{-d(1-g)} Z'(g|k_1, k_2)_{d,\eta^1,\ldots,\eta^r},$$

will be more convenient. In the relative cases, the redundant degree subscript d will often be dropped.

Let $\bigwedge_d$ be the inverse of the T–equivariant intersection form of the Nakajima basis of $\mathrm{Hilb}(\mathbb{C}^2, d)$,

(5) $$\bigwedge_d (\mu, \nu) = (-1)^{d-\ell(\mu)} (t_1 t_2)^{\ell(\mu)} \mathfrak{z}(\mu) \, \delta_{\mu,\nu}.$$

The indices of the DT partition function are raised by $\bigwedge_d$:

$$\mathsf{DT}(g|k_1, k_2)_{\mu^1,\ldots,\mu^s}^{\nu^1,\ldots,\nu^t} = \mathsf{DT}(g|k_1, k_2)_{\mu^1,\ldots,\mu^s,\nu^1,\ldots,\nu^t} \prod_{i=1}^t \bigwedge_d (\nu^i, \nu^i).$$

All the partition functions discussed here lie in the ring $\mathbb{Q}(t_1, t_2)((q))$ of Laurent series in q.

The degeneration formulas for the local Donaldson–Thomas theory of curves are conveniently expressed in terms of the DT partition functions:

$$\mathsf{DT}(g|k_1, k_2)_{\mu^1,\ldots,\mu^s}^{\nu^1,\ldots,\nu^t} = \sum_{\gamma} \mathsf{DT}(g'|k_1', k_2')_{\mu^1,\ldots,\mu^s}^{\gamma} \mathsf{DT}(g''|k_1'', k_2'')_{\gamma}^{\nu^1,\ldots,\nu^t},$$

where $g = g' + g''$, and $k_i = k'_i + k''_i$, and

$$\text{DT}(g|k_1, k_2)_{\mu^1,\ldots,\mu^s} = \sum_\gamma \text{DT}(g-1|k_1, k_2)^\gamma_{\mu^1,\ldots,\mu^s,\gamma},$$

see Maulik et al [25] for a discussion. The above degeneration formulas will be assumed here.

1.4 Localization

Our localization formulas trace their origins to Bott's remarkable paper [1] with many stops along the way.

The virtual localization formula of Graber–Pandharipande [9], proven in the abstract setting of perfect obstruction theories, applies to the Donaldson–Thomas virtual class. Applications in absolute Donaldson–Thomas theory (without relative conditions) are treated foundationally in Maulik et al [24]. Applications in the relative setting follow from the existence of a T–equivariant $\text{Ext}_0(I, I)$ obstruction theory.

1.5 GW/DT correspondence

Let N be the a rank 2 bundle over a genus g curve C with splitting of level (k_1, k_2). The GW/DT correspondence for the local theory of curves consists of three results.

First, the relative local degree 0 series of N is determined in terms of the McMahon function,

$$M(q) = \prod_{n\geq 1} \frac{1}{(1-q^n)^n},$$

the generating series for 3–dimensional partitions.

Theorem 1 *The degree 0 Donaldson–Thomas partition function is determined by:*

$$Z(N/S)_0 = M(-q)^{\int_N c_3(T_N[-S]\otimes K_N[S])}.$$

Here, $T_N[-S]$ is the sheaf of tangent fields on N with logarithmic zeros, $K_N[S]$ is the logarithmic canonical bundle, and the integral in the exponent is defined via localization on N,

$$\int_N c_3(T_N[-S] \otimes K_N[S]) = \int_C \frac{c_3(T_N[-S]\otimes K_N[S])}{e(N)}$$

$$= (2g - 2 + r)\frac{(t_1 + t_2)^2}{t_1 t_2} - (k_1 + k_2),$$

where r is the number of relative points on C.

Second, the reduced Donaldson–Thomas series are proven to satisfy a basic rationality condition.

Theorem 2 *The reduced series* $Z'(N/S)_{d,\eta^1,\dots,\eta^r}$ *is a rational function in the variables* t_1, t_2, *and* q.

Finally, the local Gromov–Witten theory and the local Donaldson–Thomas theories of curves are proven to exactly match.

Theorem 3 *After the change of variables* $e^{iu} = -q$,

$$(-iu)^{d(2-2g+k_1+k_2)-\delta} Z'_{GW}(N/S)_{d,\eta^1,\dots,\eta^r} =$$
$$(-q)^{-\frac{d}{2}(2-2g+k_1+k_2)} Z'_{DT}(N/S)_{d,\eta^1,\dots,\eta^r},$$

For the Gromov–Witten side, we follow the definitions and notations of Bryan–Pandharipande [4]. In particular,

$$\delta = \sum_{i=1}^{r} d - \ell(\eta^i).$$

1.6 Method

Theorems 1–3 are proven by solving the local Donaldson–Thomas theory of curves. The GW/DT correspondence is obtained by matching the Gromov–Witten results of [4] with the Donaldson–Thomas results here.

The solution of the local Donaldson–Thomas theory of curves follows the TQFT strategy of [4]. The first step is the determination of the level $(0,0)$ theory of $\mathbf{P}^1$. In the Gromov–Witten case, integral evaluations over the moduli space of curves are required [4]. Parallel Donaldson–Thomas integrals are evaluated here via connections to the quantum cohomology of the Hilbert scheme points of the plane. Next, the level $(-1,0)$ theory of $\mathbf{P}^1$ is determined by a direct calculation. Together, the results solve the local Donaldson–Thomas theory of curves and prove the GW/DT correspondence.

The GW/DT correspondence for the local theory of curves has been studied in the absolute case for the anti-diagonal action by Y Song. The correspondence for $\mathbf{P}^2$–bundles over curves is closely related (see Gholampour [7] and Gholampour–Song [8]).

1.7 Quantum cohomology of $\text{Hilb}(\mathbb{C}^2, d)$

For $\lambda, \mu, \nu \in \mathcal{P}(d)$, define the series $\langle \lambda, \mu, \nu \rangle^{\text{Hilb}_d}$ of genus 0, 3–pointed, T–equivariant Gromov–Witten invariants of $\text{Hilb}(\mathbb{C}^2, d)$ by a sum over curve degrees

$$\langle \lambda, \mu, \nu \rangle^{\text{Hilb}_d} = \sum_{n \geq 0} q^n \langle \lambda, \mu, \nu \rangle^{\text{Hilb}_d}_{0,3,n}.$$

The insertions on the right stand for Nakajima basis elements. See Okounkov–Pandharipande [31] for a complete discussion of the Gromov–Witten invariants of the Hilbert scheme $\text{Hilb}(\mathbb{C}^2, d)$.

The results of [31] together with our calculation of the local Donaldson–Thomas theory of curves yields a *Donaldson–Thomas/Hilbert correspondence*.

Theorem 4 $\text{DT}(0|0,0)_{d,\lambda,\mu,\nu} = \langle \lambda, \mu, \nu \rangle^{\text{Hilb}_d}$.

Our results complete the triangle of equivalences studied in the earlier papers:

Quantum cohomology of $\text{Hilb}_d(\mathbb{C}^2)$ [31]
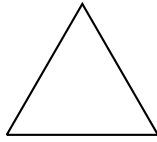
Gromov–Witten theory of $\mathbf{P}^1 \times \mathbb{C}^2$ [4] Donaldson–Thomas theory of $\mathbf{P}^1 \times \mathbb{C}^2$

A fourth vertex of equivalence is obtained from the orbifold quantum cohomology of the stack symmetric product $(\mathbb{C}^2)^d / \Sigma_d$, see Bryan–Graber [2].

1.8 Acknowledgments

2 Degree 0

2.1 Vertex measures

The degree 0 calculation of the local Donaldson–Thomas theory of curves is easily obtained from the results of Maulik et al [24; 25]. Let

(6) $$\mathsf{W}(\varnothing, \varnothing, \varnothing) = M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}$$

denote the generating series of the equivariant vertex measures of finite 3–dimensional partitions at a 3–fold fixed point with tangent weights s_1, s_2, s_3. Let

$$\text{(7)} \qquad \mathsf{W}(\varnothing, \varnothing, \varnothing)_S = M(-q)^{-\frac{s_2+s_3}{s_1}}$$

denote series of the finite vertex measures at a 3–fold fixed point on a relative divisor S with normal weight s_1 and tangent weights s_2, s_3.

The evaluations (6) and (7) are Theorem 1 and Corollary 1 of [25] with an appropriate adjustment of signs to match the conventions here.

2.2 Localization

Let S denote the 1–dimension torus acting on $\mathbf{P}^1$ with respective tangent weights s and $-s$ at the fixed points $0, \infty \in \mathbf{P}^1$.

Let $N = L_1 \oplus L_2$ be a splitting of level (k_1, k_2) over $\mathbf{P}^1$. The S–action on $\mathbf{P}^1$ can be lifted to N with fiber weights $(k_1 s, k_2 s)$ over $0 \in \mathbf{P}^1$ and fiber weights $(0, 0)$ over $\infty \in \mathbf{P}^1$. Since the scaling T–action on N commutes with S, the 3–dimensional torus,

$$\mathbf{T} = S \times T,$$

acts on N.

We may calculate the degree 0 Donaldson–Thomas series $\mathsf{Z}(0|k_1, k_2)_0$ via virtual localization with respect to $\mathbf{T}$, see Graber–Pandharipande [9] and Maulik et al [24; 25]. By the evaluation of the finite vertex measure (6),

$$\mathsf{Z}(0|k_1, k_2)_0 = \left[\mathsf{W}(\varnothing, \varnothing, \varnothing)\big|_{s,\, t_1+k_1 s,\, t_2+k_2 s} \cdot \mathsf{W}(\varnothing, \varnothing, \varnothing)\big|_{-s,t_1,t_2} \right]_{s=0}$$

$$= M(-q)^{-2\frac{(t_1+t_2)^2}{t_1 t_2}-(k_1+k_2)}.$$

The series $\mathsf{Z}(0|k_1, k_2)_{0,\varnothing}$ relative to $\infty \in \mathbf{P}^1$ is also determined by virtual localization. Here, measure (6) arises at the $\mathbf{T}$–fixed point over 0, and measure (7) arises at the $\mathbf{T}$–fixed point over ∞ of the relative divisor:

$$\mathsf{Z}(0|k_1, k_2)_{0,\varnothing} = \left[\mathsf{W}(\varnothing, \varnothing, \varnothing)\big|_{s,\, t_1+k_1 s,\, t_2+k_2 s} \cdot \mathsf{W}(\varnothing, \varnothing, \varnothing)_{N_\infty}\big|_{-s,t_1,t_2} \right]_{s=0}$$

$$= M(-q)^{-\frac{(t_1+t_2)^2}{t_1 t_2}-(k_1+k_2)}.$$

By the above evaluations, the proof of Theorem 1 is complete for the cases $\mathsf{Z}(0|k_1, k_2)_0$ and $\mathsf{Z}(0|k_1, k_2)_{0,\varnothing}$.

2.3 Degeneration

The degeneration formulas for the degree 0 theory take the following two forms:

$$Z(g|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r}} = Z(g'|k_1', k_2')_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r'}, \varnothing} \cdot Z(g''|k_1'', k_2'')_{0, \varnothing, \underbrace{\varnothing, \ldots, \varnothing}_{r''}},$$

where $g = g' + g''$, $k_i = k_i' + k_i''$, $r = r' + r''$, and

$$Z(g|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r}} = Z(g - 1|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r}, \varnothing, \varnothing}.$$

The first degeneration formula yields a recursive equation for $r \geq 2$:

$$Z(0|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r}} = Z(0|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r-1}} / Z(0|0, 0)_{0, \varnothing}.$$

From the $r = 0, 1$ cases, we deduce

$$Z(0|k_1, k_2)_{0, \underbrace{\varnothing, \ldots, \varnothing}_{r}} = M(-q)^{(-2+r)\frac{(t_1+t_2)^2}{t_1 t_2} - (k_1+k_2)}.$$

Finally, Theorem 1 is obtained for $g > 0$ by applications of the second degeneration formula. $\qquad \square$

3 TQFT

3.1 Overview

The degeneration structure of local Donaldson–Thomas theory of curves is most concisely formulated as a functor of tensor categories,

$$\mathbf{DT}(-) \colon 2\mathbf{Cob}^{L_1, L_2} \to R\mathbf{mod}.$$

Our treatment here exactly follows the TQFT construction in Bryan–Pandharipande [4] for the local Gromov–Witten theory of curves. A more detailed discussion can be found there.

3.2 $2\mathbf{Cob}^{L_1, L_2}$

The objects of the category $2\mathbf{Cob}^{L_1, L_2}$ are compact oriented 1–manifolds. A morphism in $2\mathbf{Cob}^{L_1, L_2}$,

$$Y_1 \to Y_2,$$

is an equivalence class of triples (W, L_1, L_2) where W is an oriented cobordism from Y_1 to Y_2 and L_1, L_2 are complex line bundles on W, trivialized on ∂W. The triples (W, L_1, L_2) and (W', L_1', L_2') are equivalent if there exists a boundary preserving oriented diffeomorphism,

$$f \colon W \to W',$$

and bundle isomorphisms

$$L_i \cong f^* L_i'.$$

Composition is given by concatenation of the cobordisms and gluing of the bundles along the concatenation using the trivializations.

The isomorphism class of L_i is determined by the Euler class

$$e(L_i) \in H^2(W, \partial W),$$

which assigns an integer to each component of W. For a connected cobordism W, we refer to the pair of integers (k_1, k_2), determined by the Euler classes of L_1 and L_2, as the *level*. Under concatenation, the levels simply add. For example:
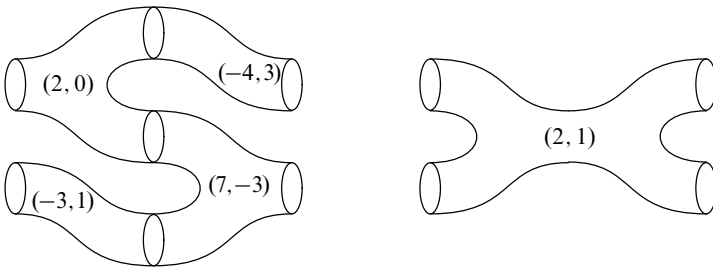
The empty manifold is a distinguished object in $2\mathbf{Cob}^{L_1, L_2}$. A morphism in $2\mathbf{Cob}^{L_1, L_2}$ from the empty manifold to itself is given by a compact, oriented, closed 2–manifold X together with a pair of complex line bundles $L_1 \oplus L_2 \to X$.

The category $2\mathbf{Cob}^{L_1, L_2}$ is generated by the following finite set of morphisms [4]:
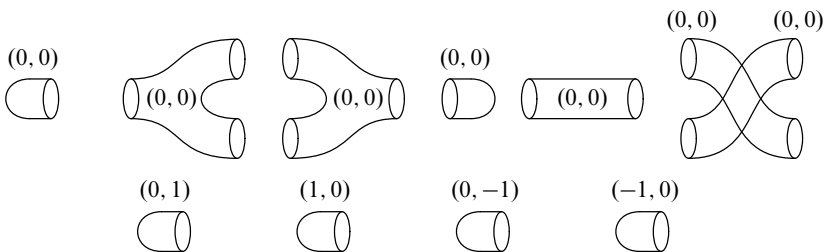
3.3 The functor DT(−)

Let R be the ring of Laurent series in q with coefficients given by rational functions in t_1 and t_2,

$$R = \mathbb{Q}(t_1, t_2)((q)).$$

The collection of partition functions $\mathsf{DT}(g|k_1, k_2)_{\lambda^1,\dots,\lambda^r}$ of degree d gives rise to a functor

$$\mathbf{DT}(-)\colon 2\mathbf{Cob}^{L_1, L_2} \to R\mathbf{mod}$$

as follows. Define

$$\mathbf{DT}(S^1) = H = \bigoplus_{\lambda \vdash d} Re_\lambda$$

to be the free R–module with basis $\{e_\lambda\}_{\lambda \vdash d}$ labelled by partitions of d, and let

$$\mathbf{DT}\left(S^1 \coprod \cdots \coprod S^1\right) = H \otimes \cdots \otimes H.$$

Let $W_s^t(g|k_1, k_2)$ be the connected genus g cobordism from a disjoint union of s circles to a disjoint union of t circles, equipped with lines bundles L_1 and L_2 of level (k_1, k_2). We define the R–module homomorphism

$$\mathbf{DT}\left(W_s^t(g|k_1, k_2)\right)\colon H^{\otimes s} \to H^{\otimes t}$$

by

$$e_{\eta^1} \otimes \cdots \otimes e_{\eta^s} \mapsto \sum_{\mu^1 \dots \mu^t \vdash d} \mathsf{DT}(g|k_1, k_2)^{\mu^1,\dots,\mu^t}_{\eta^1,\dots,\eta^s} \, e_{\mu^1} \otimes \cdots \otimes e_{\mu^t}.$$

We extend the definition of $\mathbf{DT}(-)$ to disconnected cobordisms by tensor product:

$$\mathbf{DT}\left(W[1] \coprod \cdots \coprod W[n]\right) = \mathbf{DT}(W[1]) \otimes \cdots \otimes \mathbf{DT}(W[n]).$$

Proposition 3 $\mathbf{DT}(-)\colon 2\mathbf{Cob}^{L_1, L_2} \to R\mathbf{mod}$ *is a well-defined functor.*

Proof The degeneration formula of Donaldson–Thomas theory (see Maulik et al [25]) implies the following compatibility:

$$\mathbf{DT}\big((W, L_1, L_2) \circ (W', L_1', L_2')\big) = \mathbf{DT}(W, L_1, L_2) \circ \mathbf{DT}(W', L_1', L_2').$$

We must also prove $\mathbf{DT}(-)$ takes identity morphisms to identity morphisms. Since the tube $W_1^1(0|0, 0)$ is the identity morphism from S^1 to S^1 in $2\mathbf{Cob}^{L_1, L_2}$, we require

$$(8) \qquad\qquad \mathsf{DT}(0|0, 0)^\nu_\mu = \delta^\nu_\mu.$$

Equation (8) is proven in Lemma 4 below. $\qquad\square$

Lemma 4 $DT(0|0,0)_\mu^\nu = \delta_\mu^\nu$.

Proof   Let $N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ be the trivial bundle with level $(0,0)$ splitting on $\mathbf{P}^1$. The moduli space $I_d(N/N_0 \cup N_\infty, d)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2, d)$. The q–constant terms of $DT(0|0,0)_{\mu,\nu}$ are therefore determined by the intersection form in the Nakajima basis:

$$DT(0|0,0)_{\mu,\nu} = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{(-1)^{d-\ell(\mu)}}{\mathfrak{z}(\mu)} \delta_{\mu,\nu}.$$

Hence, the Lemma is proven for q–constant terms.

The degeneration formula in Donaldson–Thomas theory yields the following factorization

$$\sum_\nu DT(0|0,0)_\mu^\nu DT(0|0,0)_\nu^\rho = DT(0|0,0)_\mu^\rho.$$

Since the matrix $DT(0|0,0)_\mu^\nu$ is invertible by the q–constant analysis, $DT(0|0,0)_\mu^\nu$ must be the identity matrix (with no q–dependence).   □

4   Vanishing

4.1   Summary

Consider the local Donaldson–Thomas theory of level $(0,0)$ on a nonsingular curve C. Let

$$T = \mathbb{C}^* \times \mathbb{C}^*$$

be the 2–dimensional scaling torus. Let $T^\pm$ denote the 1–dimensional anti-diagonal subtorus,

$$T^\pm = \{(\xi, \xi^{-1}) \mid \xi \in \mathbb{C}^*\} \subset T.$$

The $T^\pm$–equivariant Donaldson–Thomas invariants are obtained from the T–equivariant invariants by the substitution

$$t_1 = t, \quad t_2 = -t,$$

where t is the weight of the standard representation of $T^\pm$. We prove vanishing results for the $T^\pm$–equivariant Donaldson–Thomas invariants of level $(0,0)$.

4.2 Descendent insertions

Let N be a split rank 2 bundle on a nonsingular curve C with a scaling T–action. We will consider the local Donaldson–Thomas theory of N with descendent insertions. We review the definitions of Maulik et al [25].

The moduli space $I_n(N, d)$ is canonically isomorphic to the Hilbert scheme of curves of N. Let π_1 and π_2 denote the projections to the respective factors of $I_n(N, d) \times N$. Consider the universal ideal sheaf $\mathfrak{I}$,

$$\mathfrak{I} \to I_n(N, d) \times N.$$

Since $\mathfrak{I}$ is π_1–flat and N is nonsingular, a finite resolution of $\mathfrak{I}$ by locally free sheaves on $I_n(N, d) \times N$ exists. Hence, the Chern classes of $\mathfrak{I}$ are well-defined.

For $\gamma \in A_T^l(N, \mathbb{Q})$, let $\mathrm{ch}_{k+2}(\gamma)$ denote the following operation on the Chow homology of $I_n(N, d)$:

$$\mathrm{ch}_{k+2}(\gamma)\colon A_*^T(I_n(N, d), \mathbb{Q}) \to A_{*-k+1-l}^T(I_n(N, d), \mathbb{Q}),$$
$$\mathrm{ch}_{k+2}(\gamma)(\xi) = \pi_{1*}\big(\mathrm{ch}_{k+2}(\mathfrak{I}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)\big).$$

Though π_1 is *not* proper, the T–equivariant push-forward π_{1*} is well-defined by localization.

Descendent fields in Donaldson–Thomas theory, denoted by $\sigma_k(\gamma)$, correspond to the operations $\mathrm{ch}_{k+2}(\gamma)$. The T–equivariant descendent invariants are defined by

$$(9) \qquad \big\langle \sigma_{k_1}(\gamma_{l_1}) \cdots \sigma_{k_r}(\gamma_{l_r})\big\rangle_{n,d}^N = \int_{[I_n(N,d)^T]^{\mathrm{vir}}} \frac{\prod_{i=1}^r \mathrm{ch}_{k_i+2}(\gamma_{l_i})}{e(\mathrm{Norm}^{\mathrm{vir}})},$$

where the latter integral is the push-forward to a point of the class

$$\mathrm{ch}_{k_1+2}(\gamma_{l_1}) \circ \cdots \circ \mathrm{ch}_{k_r+2}(\gamma_{l_r})\left(\frac{[I_n(N, d)^T]^{\mathrm{vir}}}{e(\mathrm{Norm}^{\mathrm{vir}})}\right).$$

The descendent invariants of N may be viewed equivalently as equivariant residues:

$$\big\langle \sigma_{k_1}(\gamma_{l_1}) \cdots \sigma_{k_r}(\gamma_{l_r})\big\rangle_{n,d}^N = \mathrm{Res}_{I_n(N,d)^T}\left[\int_{[I_n(N,d)]^{\mathrm{vir}}} \prod_{i=1}^r \mathrm{ch}_{k_i+2}(\gamma_{l_i})\right].$$

The definition of T–equivariant descendent invariants in relative Donaldson–Thomas theory of N is identical. The boundary condition over a relative point $p_i \in C$ is determined by a partition η^i weighted by $H_T^*(N_{p_i}, \mathbb{Q})$.

Brackets with relative conditions on the right will often be used. For example,

$$(10) \qquad \big\langle \sigma_{k_1}(\gamma_{l_1}) \cdots \sigma_{k_r}(\gamma_{l_r}) \,\big|\, \nu^1, \ldots, \nu^s\big\rangle_{n,d}^N$$

denotes a descendent invariant relative to s points of C.

4.3 Brackets

Efficient bracket notation for Donaldson–Thomas invariants will be used throughout the paper.

For absolute brackets (without relative conditions), the degree subscript d is always required. If the Euler characteristic subscript n is omitted, a sum is signified,

$$\left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \right\rangle_d^N = \sum_n q^n \left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \right\rangle_{n,d}^N.$$

If a relative condition occurs in a bracket, the degree subscript is redundant and therefore may be omitted,

$$\left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle_n^N = \left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle_{n,|v|}^N.$$

If all subscripts in a relative bracket are omitted, a sum is signified,

$$\left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle^N = \sum_n q^n \left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle_{n,|v|}^N,$$

as in the absolute case.

Most of our Donaldson–Thomas computations will concern the local theory of $\mathbf{P}^1$. If the superscript N is replaced by a level (k_1, k_2), the theory of $\mathbf{P}^1$ is signified,

$$\left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle^{(m_1, m_2)} = \left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle^{\mathcal{O}_{\mathbf{P}^1}(m_1) \oplus \mathcal{O}_{\mathbf{P}^1}(m_2)}.$$

If the superscript is omitted altogether, then the level $(0,0)$ theory of $\mathbf{P}^1$ is signified,

$$\left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle = \left\langle \prod_i \sigma_{k_i}(\gamma_{l_i}) \,\Big|\, v \right\rangle^{(0,0)}.$$

Of course, redundant labels may be kept in various formulas for emphasis.

4.4 Restriction to $T^{\pm}$

Consider the level $(0,0)$ theory on a nonsingular curve C. Since the $T^{\pm}$–fixed locus of $I_n(\mathcal{O}_C \oplus \mathcal{O}_C, d)$ is proper, the $T^{\pm}$–equivariant descendent invariants are well-defined by residues. The restriction of the T–equivariant descendent invariants to $T^{\pm}$ yield the $T^{\pm}$–equivariant descendent invariants. The restriction to $T^{\pm}$ can also be seen to be well-defined by the following more precise result.

Lemma 5 *The relative T–equivariant descendent invariants of level $(0,0)$ on C take values in the subring*

$$\mathbb{Q}[t_1, t_2]_{(t_1 t_2)} \subset \mathbb{Q}(t_1, t_2).$$

Proof Let $N = \mathcal{O}_C \oplus \mathcal{O}_C$. As before, let

$$\mathfrak{J} \to \mathcal{N}$$

denote the universal ideal sheaf over the universal total space

$$\mathcal{N} \to I_n(N/S, d).$$

Since $N = C \times \mathbb{C}^2$, there is a proper morphism

$$\rho: \mathcal{N} \to I_n(N/S, d) \times \mathbb{C}^2.$$

Moreover, $\rho_*(\mathcal{O}_{\mathcal{N}}/\mathfrak{J})$ is flat family over $I_n(N/S, d)$ of torsion sheaves of $\mathbb{C}^2$ of length n if the genus of C is 0. There is an associated morphism of Hilbert–Chow type,

$$\iota: I_n(N/S, d) \to \text{Sym}^n(\mathbb{C}^2).$$

If the genus $g_C > 0$, a parallel construction yields

$$\iota: I_n(N/S, d) \to \text{Sym}^{n+dg_C}(\mathbb{C}^2).$$

A T–equivariant, proper morphism,

$$\iota': \text{Sym}^{n+dg_C}(\mathbb{C}^2) \to \oplus_1^{n+dg_C}(\mathbb{C}^2),$$

is obtained via the higher moments,

$$\iota'\left(\{(x_i, y_i)\}\right) = \left(\sum_i x_i, \sum_i y_i\right) \oplus \left(\sum_i x_i^2, \sum_i y_i^2\right) \oplus \cdots \oplus \left(\sum_i x_i^{n+dg_C}, \sum_i y_i^{n+dg_C}\right).$$

Let $j = \iota' \circ \iota$.

Since j is a T–equivariant, proper morphism, there is T–equivariant push-forward

$$j_*: A_*^T(I_n(N/S, d), \mathbb{Q}) \to A_*^T(\oplus_1^{n+dg_C}(\mathbb{C}^2), \mathbb{Q}).$$

Descendent invariants are defined via the T–equivariant residue of

$$\left(\prod_i \text{ch}_{k_i}(\gamma_{l_i}) \cup \epsilon_{\text{rel}}\right) \cap [I_n(N/S, d)]^{\text{vir}} \in A_*^T(I_n(N/S, d), \mathbb{Q}),$$

where ϵ_{rel} denotes the relative conditions. We may instead calculate the T–equivariant residue of

$$j_* \left(\left(\prod_i \text{ch}_{k_i}(\gamma_{l_i}) \cup \epsilon_{\text{rel}} \right) \cap [I_n(N/S, d)]^{\text{vir}} \right) \in A_*^T(\oplus_1^{n+dg_C}(\mathbb{C}^2), \mathbb{Q}).$$

Since the space $\oplus_1^{n+dg_C}(\mathbb{C}^2)$ has a unique T–fixed point with tangent weights,

$$t_1, t_2, 2t_1, 2t_2, \ldots, (n + dg_C)t_1, (n + dg_C)t_2,$$

we conclude the descendent invariant has only monomial poles in the variables t_1 and t_2. $\qquad \square$

We denote the restriction of the T–equivariant descendent invariants to the anti-diagonal subtorus by an additional superscript $\pm$. For example, the restriction of (10) is denoted by

$$\left\langle \sigma_{k_1}(\gamma_{l_1}) \cdots \sigma_{k_r}(\gamma_{l_r}) \mid v^1, \ldots, v^s \right\rangle_{n,d}^{N\pm}.$$

4.5 Absolute $\mathbf{P}^1$

The bundle $N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ admits a natural action of the 3–dimensional torus,

$$\mathbf{T} = S \times T,$$

via the canonical lifting of S, see Section 2.2. The T–equivariant descendent invariants of N can be calculated by localization on $I_n(N, d)$ with respect to the $\mathbf{T}$–action.

The localization of the virtual class $[I_n(N, d)]^{\text{vir}}$ to the $\mathbf{T}$–fixed points of $I_n(N, d)$ is determined by the formulas of Maulik et al [24; 25] in terms of vertex and edge terms.

Consider first the vertex terms. Let $\Pi(\lambda, \varnothing, \varnothing)$ be the set of 3–dimensional partitions with outgoing 2–dimensional partitions λ, $\varnothing$, and $\varnothing$. The partitions $\pi \in \Pi(\lambda, \varnothing, \varnothing)$ are finite in two of the three outgoing directions. The generating series $\mathsf{W}(\lambda, \varnothing, \varnothing)$ is defined by

$$\mathsf{W}(\lambda, \varnothing, \varnothing) = \sum_{\pi \in \Pi(\lambda, \varnothing, \varnothing)} \mathsf{w}(\pi) q^{|\pi|}.$$

Here, $\mathsf{w}(\pi)$ is the equivariant vertex measure, and $|\pi|$ is the number of boxes of π which remain after removing the infinite outgoing cylinder [24; 25].

Lemma 6 *For $\pi \in \Pi(\lambda, \varnothing, \varnothing)$ satisfying $|\pi| > 0$, the measure $\mathsf{w}(\pi)|_{s,t_1,t_2}$ is divisible by $t_1 + t_2$.*

Proof The proof exactly follows the derivation of [25, Lemma 4]. We determine here the precise positive power of $t_1 + t_2$ dividing the measure $w(\pi)|_{s,t_1,t_2}$.

Let λ and μ be two partition diagrams satisfying $\lambda \supset \mu$. The difference λ/μ between λ and μ is a *skew diagram*. The content $c(\square)$ of a square of a partition diagram with coordinates (i, j) is defined by
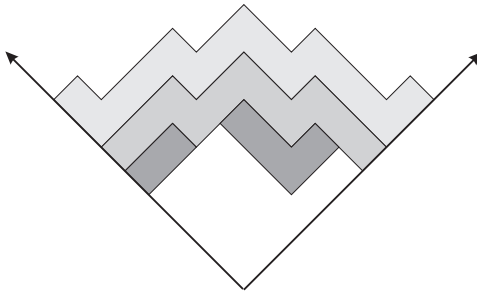
$$c(\square) = j - i .$$

A *rim hook* is a connected skew diagram which does not contain two squares of equal content.

For any skew diagram λ/μ there is a minimal integer r for which

$$\mu = \nu_0 \subset \nu_1 \subset \cdots \subset \nu_r = \lambda$$

and each ν_{k+1}/ν_k is a rim hook. The minimal r is the *rank* of λ/μ. The rank can be determined by repeatedly peeling off maximal rim hooks from λ. The process can be seen in a rank 4 example:

Let a_k be the number of squares in λ/μ of content k. The rank is determined in terms of a_k by

$$\text{(11)} \qquad \text{rank}\,\lambda/\mu = \frac{1}{2} \sum_k |a_k - a_{k+1}|$$

since the right side of (11) receives contributions from the beginning and end of every rim hook. Since each summand in (11) is either 0 or 1, each term can be squared

$$\text{(12)} \qquad \text{rank}\,\lambda/\mu = \frac{1}{2} \sum_k (a_k - a_{k+1})^2 .$$

A 3–dimensional partition with one leg of profile λ can be viewed as a sequence of slices by planes perpendicular to the direction of the leg:

$$\lambda^0 \supset \lambda^{(1)} \supset \lambda^{(2)} \supset \lambda^{(M)} = \cdots = \lambda^{(\infty)} = \lambda , \quad M \gg 0 .$$

The order of divisibility of $w(\pi)|_{s,t_1,t_2}$ by $t_1 + t_2$ is

$$(13) \qquad \operatorname{ord}_{(t_1+t_2)} w(\pi)|_{s,t_1,t_2} = \sum_{k=0}^{\infty} \operatorname{rank} \lambda^{(k)}/\lambda^{(k+1)},$$

where, in fact, only finitely many terms are nonzero. Formula (13) is a direct consequence of (12) and the proof of [25, Lemma 4]. $\qquad\square$

The $\mathbf{T}$–fixed points of $I_n(N, d)$ are isolated and correspond bijectively to triples (π, λ, π') where $\pi, \pi' \in \Pi(\lambda, \varnothing, \varnothing)$ and

$$|\pi| + |\pi'| = n - |\lambda|.$$

The vertex partitions π, π' determine the nonreduced structure of the $\mathbf{T}$–fixed ideal over $0, \infty \in \mathbf{P}^1$. The edge partition λ determines the nonreduced structure over $\mathbf{P}^1 \setminus \{0, \infty\}$.

The $\mathbf{T}$–equivariant localization of the virtual class $[I_n(N, d)]^{\mathrm{vir}}$ to the fixed point (π, λ, π') is

$$(14) \qquad w(\pi)|_{s,t_1,t_2} \cdot E(\lambda) \cdot w(\pi')|_{-s,t_1,t_2},$$

where the edge terms $E(\lambda)$ are the (inverse) tangent T–weights to $\mathrm{Hilb}(\mathbb{C}^2, |\lambda|)$ at the T–fixed point indexed by λ. The edge terms are easily seen to be prime to $t_1 + t_2$.

By Lemma 6, the localization (14) vanishes when restricted to the anti-diagonal torus $T^{\pm}$ if either $|\pi|$ or $|\pi'|$ are positive.

Lemma 7 For $n > d$,

$$\left\langle \sigma_{k_1}(\gamma_{l_1}) \cdots \sigma_{k_r}(\gamma_{l_r}) \right\rangle_{n,d}^{\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \pm} = 0.$$

Proof The result is a consequence of the vanishing (14) of the virtual class when localized to the $\mathbf{T}$–fixed points (and restricted to $T^{\pm}$). The descendent integrand plays no role. $\qquad\square$

Lemma 7 is the first vanishing result for the $T^{\pm}$–equivariant Donaldson–Thomas invariants of level $(0, 0)$.

4.6 The matrix M_d

Let $N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ be the trivial bundle with level $(0, 0)$ splitting over $\mathbf{P}^1$. Let N_0 denote the fiber of N over $0 \in \mathbf{P}^1$. Let

$$[N_0] \in A_T^1(N, \mathbb{Q})$$

be the associated class.

We define a matrix M_d of descendent invariants of N relative to N_∞ indexed by the set $\mathcal{P}(d)$ of partitions of d. For partitions $\mu, \nu \in \mathcal{P}(d)$, let

$$M_d(\mu, \nu) = q^{-d} \left\langle \prod_i \sigma_{\mu_i - 1}([N_0]) \,\middle|\, \nu([0]) \right\rangle^N,$$

following the bracket conventions of Section 4.3. The partition μ specifies a descendent insertion, and the partition ν specifies a relative condition along N_∞. Since, the minimal Euler characteristic of a degree d subscheme of N is d, the elements of M_d lie in the ring $\mathbb{Q}[t_1, t_2]_{(t_1 t_2)}[\![q]\!]$.

Define the *length* partial order on $\mathcal{P}(d)$ by the following rule: $\mu \geq \mu'$ if $\ell(\mu) > \ell(\mu')$ or if $\mu = \mu'$.

Lemma 8 *M_d is upper triangular with respect to the length partial ordering.*

Proof Let $N \subset \bar{N}$ denote the T–equivariant compactification over $\mathbf{P}^1$ defined by

$$\bar{N} = \mathbf{P}(N \oplus \mathcal{O}_{\mathbf{P}^1}).$$

Here, T acts trivially on the additional $\mathcal{O}_{\mathbf{P}^1}$. We will only consider the curve classes on $\bar{N}$ obtained from N.

Let $\bar{N}_\infty$ denote the fiber of the compactification over $\infty \in \mathbf{P}^1$. Let

$$0_\uparrow, 0_\rightarrow \in \bar{N}_\infty$$

be the two *new* T–fixed points with normal T–weights

$$t_1 - t_2, -t_2, \text{ and } t_2 - t_1, -t_1$$

respectively.

By the residue definition, the T–equivariant Donaldson–Thomas descendent invariant

$$(15) \qquad \left\langle \prod_i \sigma_{\mu_i - 1}([N_0]) \,\middle|\, \nu([0]) \right\rangle^N_n$$

occurs as a summand in the localization computation of the T–equivariant descendent

$$(16) \qquad \left\langle \prod_i \sigma_{\mu_i - 1}([\bar{N}_0]) \,\middle|\, \nu([0]) \right\rangle^{\bar{N}}_n$$

for *any* partition μ — not necessarily a partition of d.

The virtual dimension of the moduli space $I_n(\bar{N}/\bar{N}_\infty, d)$ is $2d$. The integrand and relative constraints of (16) impose

$$|\mu| - \ell(\mu) + d + \ell(\nu)$$

conditions. Therefore, since $\bar{N}$ is proper, the integral (16) vanishes if

$$|\mu| - \ell(\mu) + \ell(\nu) < d.$$

The T–localization formula expresses (16) as the following sum of triple products of relative local Donaldson–Thomas invariants of N:

$$(17) \quad \sum_{n_0+n_\uparrow+n_\rightarrow=n} \sum_{A_0\cup A_\uparrow\cup A_\rightarrow=\{1,\dots,\ell(\mu)\}} \left\langle \prod_{i\in A_0} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \nu([0]) \right\rangle_{n_0}^N$$

$$\cdot \left\langle \prod_{i\in A_\uparrow} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \varnothing \right\rangle_{n_\uparrow}^N \Bigg|_{t_1-t_2,-t_2} \cdot \left\langle \prod_{i\in A_\rightarrow} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \varnothing \right\rangle_{n_\rightarrow}^N \Bigg|_{t_2-t_1,-t_1}.$$

The relative localization formula is applied with a factorization rule — the T–fixed loci do *not* geometrically factor.[3]

By induction on n and $\ell(\mu)$, we conclude

$$(18) \quad \left\langle \prod_i \sigma_{\mu_i-1}([N_0]) \,\middle|\, \nu([0]) \right\rangle_n^N = 0$$

if $|\mu| - \ell(\mu) + \ell(\nu) < d$. In particular,

$$M_d(\mu, \nu) = 0$$

if $\ell(\mu) > \ell(\nu)$.

If $|\mu| - \ell(\mu) + \ell(\nu) = d$, the constraints of (16) impose exactly $2d$ conditions. The nonequivariant integral

$$(19) \quad \left\langle \prod_i \sigma_{\mu_i-1}([\bar{N}_0]) \,\middle|\, \nu_1([\gamma]), \nu_2([0]), \dots, \nu_{\ell(\nu)}([0]) \right\rangle_n^{\bar{N}}$$

is independent of $\gamma \in \bar{N}_\infty$ — the parts of the relative condition over ∞ are written explicitly here. After specializing γ to a T–fixed point of $\bar{N}_\infty$, the invariant (19) can be computed by localization with respect to T.

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[3] In the absolute case, the T–fixed loci of $I_n(\bar{N}, d)$ factor as a triple product. The T–fixed loci relative to a fiber, $I_n(\bar{N}/\bar{N}_\infty, d)$ do *not* factor. However, a factorization rule holds. Factorization can be deduced from the relative localization formula applied to disjoint unions following the discussion of connected/disconnected issues in relative Gromov–Witten theory in Maulik–Pandharipande [26, Section 1.8].

If γ is specialized to $0 \in \bar{N}_\infty$, T–equivariant localization expresses (19), as before, as a sum of triple product (17). The vanishing (18) removes most terms.

If γ is specialized to $0_\uparrow$, the T–equivariant localization formula for (19) takes a different form:

(20)
$$\sum_{n_0+n_\uparrow+n_\rightarrow=n} \sum_{A_0\cup A_\uparrow\cup A_\rightarrow=\{1,\dots,\ell(\mu)\}} \left\langle \prod_{i\in A_0} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \nu_2([0]),\dots,\nu_{\ell(\nu)}([0]) \right\rangle_{n_0}^N$$
$$\cdot \left\langle \prod_{i\in A_\uparrow} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \nu_1([0]) \right\rangle_{n_\uparrow}^N \bigg|_{t_1-t_2,-t_2} \cdot \left\langle \prod_{i\in A_\rightarrow} \sigma_{\mu_i-1}([N_0]) \,\middle|\, \varnothing \right\rangle_{n_\rightarrow}^N \bigg|_{t_2-t_1,-t_1}.$$

By repeated use of the comparison of the two evaluations of (19), we find

$$\left\langle \prod_i \sigma_{\mu_i-1}([N_0]) \,\middle|\, \nu([0]) \right\rangle_n^N = 0$$

unless there are disjoint subpartitions $\mu[i] \subset \mu$ such that

(21)
$$|\mu[i]| - \ell(\mu[i]) + 1 = \nu_i.$$

If $\mu \in \mathcal{P}(d)$ and $\ell(\mu) = \ell(\mu)$, condition (21) implies $\mu = \nu$. Hence,

$$M_d(\mu, \nu) = 0$$

if $\ell(\mu) = \ell(\nu)$ unless $\mu = \nu$. $\qquad\square$

Lemma 9 M_d *is invertible in the ring of matrices with* $\mathbb{Q}[t_1, t_2]_{(t_1 t_2)}[\![q]\!]$ *coefficients.*

Proof The minimal Euler characteristic of a degree d subscheme of N is d. Since

$$I_d(N, d) \cong \mathrm{Hilb}(N_\infty, d),$$

the matrix of q–constant terms of M_d is determined by the classical (equivariant) intersection theory of the Hilbert scheme of points of the plane and is well-known to be invertible. $\qquad\square$

Let $M_d^\pm$ denote the restriction of M_d to the anti-diagonal torus. The following vanishing result holds.

Lemma 10 $M_d^\pm$ *has no q dependence.*

Proof Let C_d be a matrix indexed by partitions $\mathcal{P}(d)$ with the coefficients
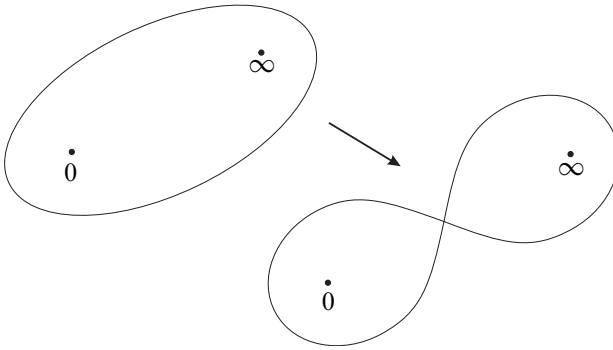
$$C_d(\mu, \nu) = q^{-d} \left\langle \prod_i \sigma_{\mu_i - 1}([N_0]) \cdot \prod_j \sigma_{\nu_j - 1}([N_\infty]) \right\rangle_d^N.$$

The degeneration formula in Donaldson–Thomas theory, yields the factorization

$$M_d \bigwedge_d M_d^t = C_d,$$ (22)

where $\bigwedge_d$ is defined by (5). The matrix $\bigwedge_d$ has *no q* dependence.

Equation (22) is obtained by a degeneration of the base to a reducible nodal curve:

The bundle N specializes to a sum of trivial bundles on the reducible curve. The degeneration is equivariant for the scaling torus T.

The restriction $C_d^\pm$ has no q dependence by Lemma 7. The $T^\pm$–restriction of (22) is a Gauss decomposition of $C_d^\pm$. Uniqueness of the Gauss decomposition implies the Lemma. $\square$

Lemma 10 can be restated as an explicit vanishing of Donaldson–Thomas invariants. For $n > d$,

$$\left\langle \prod_i \sigma_{\mu_i - 1}([N_0]) \,\middle|\, \nu([0]) \right\rangle_{n,d}^{\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \pm} = 0,$$ (23)

where $\mu, \nu \in \mathcal{P}(d)$.

4.7 Degeneration

Let $N = \mathcal{O}_C \oplus \mathcal{O}_C$ be the trivial bundle with level $(0,0)$ splitting over a nonsingular, genus g curve C. The class

$$[N_z] \in A_T^1(N, \mathbb{Q})$$

is independent of $z \in C$. We will denote the fiber class $[N_z]$ by F.

Lemma 11 *For $n > d(1-g)$,*

$$\tag{24} \left\langle \prod_i \sigma_{k_i}(F) \,\Big|\, v^1, \dots, v^s \right\rangle_{n,d}^{\mathcal{O}_C \oplus \mathcal{O}_C \pm} = 0.$$

Proof Since the matrix $M_d^{\pm}$ is invertible and has no q dependence, the relative conditions in the Donaldson–Thomas integral (24) can be systematically traded for descendent insertions. The Lemma is then equivalent to the vanishing for $n > d(1-g)$ of *all* absolute invariants

$$\tag{25} \left\langle \prod_{i'} \sigma_{k'_{i'}}(F) \right\rangle_{n,d}^{\mathcal{O}_C \oplus \mathcal{O}_C \pm} = 0.$$

After degenerating C to a nodal rational curve (and again trading relative conditions for descendent insertions), we need only prove the vanishing (25) in case C is $\mathbf{P}^1$. The latter vanishing is a consequence of Lemma 7. $\qquad\square$

4.8 Cotangent lines, rubber and topological recursion

Let $N = \mathcal{O}_C \oplus \mathcal{O}_C$ be the trivial bundle with level $(0,0)$ splitting over a nonsingular curve C. Consider the moduli space of ideal sheaves $I_n(N/N_p, d)$ relative to the fiber over $p \in C$. The cotangent line bundle,

$$\mathbb{L}_p \to I_n(N/N_p, d),$$

is defined by the cotangent space at the relative point p of the target curve. The torus T acts trivially on $\mathbb{L}_p$. Let

$$\psi_p \in A_T^1(I_n(N/N_p, d), \mathbb{Q})$$

denote the first Chern class of $\mathbb{L}_p$.

The Donaldson–Thomas theory of *rubber* naturally arises at the boundary of the moduli space $I_n(N/N_p, d)$. Let R be a rank 2 bundle of level $(0,0)$ over $\mathbf{P}^1$. Let

$$R_0, R_\infty \subset R$$

denote the fibers over $0, \infty \in \mathbf{P}^1$. The 1–dimensional torus S acts on R via the symmetries of $\mathbf{P}^1$. Let $I_n(R/R_0 \cup R_\infty, d)$ be the relative moduli space of ideal sheaves, and let

$$I_n(R/R_0 \cup R_\infty, d)^\circ \subset I_n(R/R_0 \cup R_\infty, d)$$

denote the open set with finite stabilizers for the S–action and *no* destabilization over $\infty \in \mathbf{P}^1$. The rubber moduli space,

$$I_n(R/R_0 \cup R_\infty, d)^\sim = I_n(R/R_0 \cup R_\infty, d)^\circ/S,$$

denoted by a superscripted tilde, is determined by the (stack) quotient. The moduli space is empty unless $n > d$. The rubber theory of R is defined by integration against the rubber virtual class,

$$[I_n(R/R_0 \cup R_\infty, d)^\sim]^{\mathrm{vir}}.$$

All of the above rubber constructions are T–equivariant.

The rubber moduli space $I_n(R/R_0 \cup R_\infty, 0)^\sim$ carries cotangent lines at the dynamical points 0 and ∞ of $\mathbf{P}^1$. Let

$$\psi_0, \psi_\infty \in A^1_T(I_n(R/R_0 \cup R_\infty, d)^\sim, \mathbb{Q})$$

denote the associated cotangent line classes. Rubber integrals with relative conditions μ over 0 and ν over ∞ are denoted by the bracket

(26) $$\left\langle \mu \mid \psi_0^a \psi_\infty^b \mid \nu \right\rangle^\sim_{n,d}.$$

Cotangent line classes in Donaldson–Thomas theory of N can be removed via topological recursion relations. For the relative theory of $N/\cup_{i=1}^s N_{p_i}$, the topological recursion relation is:

$$\left\langle \prod_i \sigma_{k_i}(F) \prod_{j=1}^s \psi_{p_j}^{a_j} \mid \nu^1, \ldots, \nu^s \right\rangle^N_{n,d} =$$

$$\sum_{|\eta|=d} \sum_{n_1+n_2=n+d} \left\langle \prod_i \sigma_{k_i}(F) \prod_{j \neq 1} \psi_{p_j}^{a_j} \mid \eta, \nu^2, \ldots \nu^s \right\rangle^N_{n_1,d} \cdot \bigwedge_d (\eta, \eta) \langle \eta \mid \psi_\infty^{a_1-1} \mid \nu^1 \rangle^\sim_{n_2,d}.$$

The proof follows from the degeneration formula of Donaldson–Thomas theory applied to the boundary expression for ψ_{p_1} on the Artin stack of target destabilizations. The relative conditions away from p_1 and the descendent insertions are bystanders in the topological recursion relation.

4.9 Rubber calculus

The rubber integrals (26) are determined via a Donaldson–Thomas rubber calculus. The technique, following Gromov–Witten theory (see Maulik–Pandharipande [26] and Okounkov–Pandharipande [30]), involves rigidification and topological recursion.

The universal target over the rubber moduli space is no longer a product. Let

$$\pi \colon \mathcal{R} \to I_n(R/R_0 \cup R_\infty, d)^\sim$$

denote the universal target. The space $\mathcal{R}$ can be viewed as a moduli space of rubber ideal sheaves *together* with a point r of the target rubber. The point r is *not* permitted to lie on the relative divisors R_0 and R_∞. The stability condition is given by finiteness of the associated automorphism group. The virtual class of $\mathcal{R}$ is obtained via π–flat pull-back,

$$[\mathcal{R}]^{\mathrm{vir}} = \pi^* \Big([I_n(R/R_0 \cup R_\infty, d)^\sim]^{\mathrm{vir}} \Big).$$

As before, let

$$\mathfrak{J} \to \mathcal{R}$$

denote the universal ideal sheaf on $\mathcal{R}$.

The target point r together with R_0 and R_∞ specifies 3 distinct points of the destabilized $\mathbf{P}^1$ over which the rubber is fibered. By viewing the target point as $1 \in \mathbf{P}^1$, we obtain a rigidification map,

$$\phi \colon \mathcal{R} \to I_n(N/N_0 \cup N_\infty, d),$$

where $N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ is the trivial bundle with level $(0, 0)$ splitting over $\mathbf{P}^1$. By a comparison of deformation theories,

$$(27) \qquad [\mathcal{R}]^{\mathrm{vir}} = \phi^* \Big([I_n(N/N_0 \cup N_\infty, d)]^{\mathrm{vir}} \Big).$$

Rubber calculus transfers rubber integrals (26) to descendent integrals on rigid (non-rubber) targets via the maps π and ϕ. To start,

$$(d - n)[I_n(R/R_0 \cup R_\infty, d)]^{\mathrm{vir}} = \pi_* \Big(\mathrm{ch}_3(\mathfrak{J}) \cap [\mathcal{R}]^{\mathrm{vir}} \Big)$$

by a π–fiberwise calculation. Since $n > d$, for nonempty rubber moduli spaces, $d - n$ is negative. By the push-pull formula,

$$(28) \qquad (d - n) \big\langle \mu \mid \psi_0^a \psi_\infty^b \mid v \big\rangle_{n,d}^\sim = \big\langle \mu \mid \mathrm{ch}_3(\mathfrak{J})\, \pi^*(\psi_0^a \psi_\infty^b) \mid v \big\rangle_{n,d}^{\mathcal{R}\sim}.$$

Next, we compare the cotangent lines $\pi^*(\psi_0)$ and $\phi^*(\psi_0)$ on $\mathcal{R}$. A standard argument yields:

$$\pi^*(\psi_0) = \phi^*(\psi_0) - \phi^*(D_0),$$

where

$$D_0 \subset I_n(N/N_0 \cup N_\infty, d)$$

is the virtual boundary divisor for which the rubber over ∞ carries Euler characteristic n. Similarly,

$$\pi^*(\psi_\infty) = \phi^*(\psi_\infty) - \phi^*(D_\infty).$$

We will apply the cotangent line comparisons to the right side of (28).

Consider the Hilbert scheme of points $\mathrm{Hilb}(R_0, d)$ of the relative divisor. The boundary condition μ corresponds to a Nakajima basis element of $A_T^*(\mathrm{Hilb}(R_0, d), \mathbb{Q})$. Let $\mathfrak{J}_0$ be the universal ideal sheaf on

$$\mathrm{Hilb}(R_0, d) \times R_0,$$

and let

$$\sigma_1 = \pi_*\big(\mathrm{ch}_3(\mathfrak{J}_0)\big) \in A_T^1(\mathrm{Hilb}(R_0, d), \mathbb{Q}).$$

The class σ_1 in $A_T^1(\mathrm{Hilb}(R_\infty, d)$ is defined in the same way.

The cotangent line comparisons and equation (28) together yield the following result:

$$(29) \quad (d-n)\langle \mu \mid \psi_0^a \psi_\infty^b \mid \nu \rangle_{n,d}^{\sim} =$$
$$\langle \mu \mid \sigma_1(F)\, \psi_0^a \psi_\infty^b \mid \nu \rangle_{n,d}^N - \langle \sigma_1 \cdot \mu \mid \psi_0^{a-1} \psi_\infty^b \mid \nu \rangle_{n,d}^{\sim} - \langle \mu \mid \psi_0^a \psi_\infty^{b-1} \mid \sigma_1 \cdot \nu \rangle_{n,d}^{\sim}.$$

The rubber integrals (26) are expressed in terms of the relative Donaldson–Thomas theory of N with descendent insertions $\sigma_k([F])$ by repeated applications of equation (29) and the topological recursion relations.

The following two vanishing statements are a consequence of the rubber calculus and Lemma 11.

Lemma 12 *For $n > d(1-g)$,*

$$\left\langle \prod_i \sigma_{k_i}(F) \prod_{j=1}^s \psi_{p_j}^{a_j} \mid \nu^1, \ldots, \nu^s \right\rangle_{n,d}^{\mathcal{O}_C \oplus \mathcal{O}_C \pm} = 0.$$

Lemma 13 *For $n > d$,*

$$\langle \mu \mid \psi_0^a \psi_\infty^b \mid \nu \rangle_{n,d}^{\sim \pm} = 0.$$

4.10 Parallels

The anti-diagonal vanishing in level $(0,0)$ holds for all vertices in the triangle of equivalences of Section 1.7.

The Gromov–Witten vanishing follows easily from Mumford's Hodge bundle relation

$$c(\mathbb{E}) \cdot c(\mathbb{E}^{\vee}) = 1$$

on the moduli space of curves $\overline{M}_g$, see Bryan–Pandharipande [4]. The Hilbert scheme vanishing is obtained from the existence of a modified virtual class in the hyperkähler setting. The proof requires a restriction of the obstruction theory of maps to $\mathrm{Hilb}(\mathbb{C}^2, d)$, see Okounkov–Pandharipande [31]. The Donaldson–Thomas vanishing proven here could also be pursued via a construction of a modified virtual class.[4] Instead, our derivation proceeds formally from the equivariant vertex calculations of Maulik et al [24; 25] using localization and degeneration.

5 Additivity

5.1 Summary

The level $(0,0)$ Donaldson–Thomas theory of $\mathbf{P}^1$ relative to $0, \infty \in \mathbf{P}^1$ will play a crucial role in the study of local curves. The main results of the Section are vanishing and additivity properties for the invariants

$$(30) \qquad \left\langle \mu \mid \sigma_1(F) \mid \nu \right\rangle.$$

We follow here the bracket conventions of Section 4.3.

The vanishing of the invariants (30) in most cases is established by the following three results.

Lemma 14 *If $|\ell(\mu) - \ell(\nu)| > 1$, then $\left\langle \mu \mid \sigma_1(F) \mid \nu \right\rangle$ vanishes.*

Lemma 15 *If $|\ell(\mu) - \ell(\nu)| = 1$, then $\left\langle \mu \mid \sigma_1(F) \mid \nu \right\rangle_{n,d}$ vanishes for $n > d$.*

Lemma 16 *If $\ell(\mu) = \ell(\nu)$, then $\left\langle \mu \mid \sigma_1(F) \mid \nu \right\rangle$ vanishes unless $\mu = \nu$.*

---

[4]Recently, such vanishing arguments have appeared: see Kiem–Li [13] and Maulik–Pandharipande–Thomas [27].

In the diagonal case, we will obtain the form

$$
(31) \qquad \langle \mu \mid \sigma_1(F) \mid \mu \rangle_n = \gamma_{\mu,n} (t_1 t_2)^{-\ell(\mu)} (t_1 + t_2)
$$

for $\gamma_{\mu,n} \in \mathbb{Q}$ and prove a crucial additivity property parallel to Equation (25) of Okounkov–Pandharipande [31].

Proposition 17 *An additivity relation holds:*

$$
\frac{\langle \mu \mid \sigma_1(F) \mid \mu \rangle}{\langle \mu \mid \mu \rangle_{|\mu|,|\mu|}} = \sum_i q^{|\mu| - \mu_i} \frac{\langle \mu_i \mid \sigma_1(F) \mid \mu_i \rangle}{\langle \mu_i \mid \mu_i \rangle_{\mu_i,\mu_i}} - (\ell(\mu) - 1)(t_1 + t_2) q^{|\mu|} \Phi(q).
$$

The bracket $\langle \mu \mid \mu \rangle_{|\mu|,|\mu|}$ is the intersection form

$$
\langle \mu \mid \mu \rangle_{|\mu|,|\mu|} = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{(-1)^{d - \ell(\mu)}}{\mathfrak{z}(\mu)}
$$

in the Nakajima basis. The function $\Phi(q)$ is defined by

$$
\Phi(q) = q \frac{d}{dq} \log M(-q),
$$

where $M(q)$ is the MacMahon series.

5.2 Proofs of Lemmas 14 and 15

Let $N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$ be the trivial bundle with splitting of level $(0,0)$. Let $\bar{N}$ denote the compactification of N defined in Section 4.6. The proofs are obtained from dimensional analysis for integrals in the Donaldson–Thomas theory of $\bar{N}$.

By linearity, we can trade the invariants with unweighted relative conditions for weighted relative conditions:

$$
\langle \mu \mid \sigma_1(F) \mid \nu \rangle = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \langle \mu([0]) \mid \sigma_1(F) \mid \nu \rangle
$$

$$
= \frac{1}{(t_1 t_2)^{\ell(\nu)}} \langle \mu \mid \sigma_1(F) \mid \nu([0]) \rangle
$$

By T–localization on $\bar{N}$ and the vanishing of Lemma 4, we find

$$
\langle \mu([0]) \mid \sigma_1(F) \mid \nu \rangle_n^{\bar{N}} = \langle \mu([0]) \mid \sigma_1(F) \mid \nu \rangle_n
$$
$$
+ q^d \langle \mu([0]) \mid \nu \rangle_{d,d} \cdot \langle \varnothing \mid \sigma_1(F) \mid \varnothing \rangle_{n-d} \big|_{t_1 - t_2, -t_2}
$$
$$
+ q^d \langle \mu([0]) \mid \nu \rangle_{d,d} \cdot \langle \varnothing \mid \sigma_1(F) \mid \varnothing \rangle_{n-d} \big|_{t_2 - t_1, -t_1}
$$

If $1 + \ell(\mu) - \ell(\nu) < 0$, then

$$\big\langle \mu([0]) \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle_n^{\overline{N}} = 0$$

by dimension considerations since $\overline{N}$ is proper. If $\ell(\mu) - \ell(\nu) < 0$, then

$$\big\langle \mu([0]) \,\big|\, \nu \big\rangle_{d,d} = 0.$$

Therefore, $1 + \ell(\mu) - \ell(\nu) < 0$ implies

$$\big\langle \mu([0]) \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle_n = 0.$$

Similarly, if $1 - \ell(\mu) + \ell(\nu) < 0$, then

$$\big\langle \mu \,\big|\, \sigma_1(F) \,\big|\, \nu([0]) \big\rangle_n = 0.$$

We conclude $\big\langle \mu \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle$ vanishes unless $|\ell(\mu) - \ell(\nu)| \leq 1$. Lemma 14 is proven.
□

If the equality $|\ell(\mu) - \ell(\nu)| = 1$ holds, then the argument yields a finer result. Either

$$\big\langle \mu([0]) \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle_{n,d} = \big\langle \mu([0]) \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle_{n,d}^{\overline{N}} \in \mathbb{Q}$$

or

$$\big\langle \mu \,\big|\, \sigma_1(F) \,\big|\, \nu([0]) \big\rangle_{n,d} = \big\langle \mu \,\big|\, \sigma_1(F) \,\big|\, \nu([0]) \big\rangle_{n,d}^{\overline{N}} \in \mathbb{Q}.$$

If $n > d$, the factor $(t_1 + t_2)$ must divide the invariant

$$(32) \qquad \big\langle \mu \,\big|\, \sigma_1(F) \,\big|\, \nu \big\rangle_{n,d}$$

by Lemma 11. Thus, if $|\ell(\mu) - \ell(\nu)| = 1$ and $n > d$, the integral (32) must vanish. Lemma 15 is proven. □

5.3 Degree 0 calculation

We calculate the integral $\big\langle \varnothing \,\big|\, \sigma_1(F) \,\big|\, \varnothing \big\rangle$. By the degeneration formula and Theorem 1,

$$\big\langle \varnothing \,\big|\, \sigma_1(F) \,\big|\, \varnothing \big\rangle = \frac{\big\langle \sigma_1(F) \big\rangle_0}{\big\langle \varnothing \,\big|\, \big\rangle \cdot \big\langle \,\big|\, \varnothing \big\rangle}$$

$$= M(-q)^{2 \frac{(t_1 + t_2)^2}{t_1 t_2}} \big\langle \sigma_1(F) \big\rangle_0.$$

To complete the calculation, we determine the series $\big\langle \sigma_1(F) \big\rangle_0$ via localization with respect to the **T**–action defined in Section 2.2. Let the **T**–equivariant lift of F be

specified by $[N_0]$. Then,

$$\langle \sigma_1(F) \rangle_0 = \left[W_{\sigma_1(s)}(\varnothing, \varnothing, \varnothing)|_{s,t_1,t_2} \cdot W(\varnothing, \varnothing, \varnothing)|_{-s,t_1,t_2} \right]_{s=0}.$$

Lemma 18 *The vertex measure* $W_{\sigma_1(s)}(\lambda, \varnothing, \varnothing)$ *with a descendent insertion is determined by*

$$W_{\sigma_1(s)}(\lambda, \varnothing, \varnothing)|_{s,t_1,t_2} = s\left(-q\frac{d}{dq} + c(\lambda; t_1, t_2) + \frac{|\lambda|}{2}(t_1 + t_2) \right) W(\lambda, \varnothing, \varnothing)|_{s,t_1,t_2}.$$

Here, $c(\lambda; t_1, t_2)$ is the sum of the (t_1, t_2)–contents of all squares in λ

$$c(\lambda; t_1, t_2) = \sum_{(i,j)\in\lambda} \left(i\,t_1 + j\,t_2 \right).$$

Viewing λ as a Young diagram, the sum is over the *interior* corners of the squares — the corners closest to the origin. For $|\lambda| \leq 1$, the total content $c(\lambda; t_1, t_2)$ vanishes.

Proof Let $\mathbf{T}$ act of $\mathbb{C}^3$ with tangent weights s, t_1, t_2 at the origin. Let π be a 3–dimensional partition with a single infinite leg of cross-section λ in the direction of the tangent weight s. Let I_π denote the corresponding $\mathbf{T}$–fixed ideal. We first compute the restriction

$$\mathrm{ch}_3(\mathfrak{J})|_{[I_\pi]\times 0},$$

where $\mathfrak{J}$ is the universal ideal sheaf.

Let s_1, s_2, s_3 be the associated weights of the $\mathbf{T}$–action on the coordinates x_1, x_2, x_3 of $\mathbb{C}^3$,

$$s_1 = -s, \quad s_2 = -t_1, \quad s_3 = -t_2.$$

Consider a graded free resolution of I_π of length 3,

$$0 \to \bigoplus_k x^{c_k} A \to \bigoplus_j x^{b_j} A \to \bigoplus_i x^{a_i} A \to I_\pi \to 0,$$

where $A = \mathbb{C}[x_1, x_2, x_3]$ is the coordinate ring and

$$a_i, b_j, c_k \in \mathbb{Z}^3$$

denote the degrees of the generators in each step.

Computing the Chern character via the resolution, we find

$$\mathrm{ch}_3(\mathfrak{J})|_{I_\pi \times 0} = \frac{1}{3!}\left(\sum_i (s, a_i)^3 - \sum_j (s, b_j)^3 + \sum_k (s, c_k)^3 \right).$$

Here, $s = (s_1, s_2, s_3)$, and (s, v) denotes the standard inner product of s and v in $\mathbb{Z}^3$.

By calculating the trace of the **T**–representation defined by A/I_π, we obtain a second relation:

$$(33) \quad 1 - \sum_i e^{(s,a_i)} + \sum_j e^{(s,b_j)} - \sum_k e^{(s,c_k)} =$$
$$(1 - e^{s_2})(1 - e^{s_3}) \sum_{(i,j)\in\lambda} e^{is_2+js_3} + (1 - e^{s_1})(1 - e^{s_2})(1 - e^{s_3}) \sum_{p\in\pi'} e^{(s,p)}$$

where π' denotes π minus the infinite leg. In particular, the renormalized volume $|\pi|$ is the number of squares in π'.

Extracting the cubic term in (33), we find

$$\mathrm{ch}_3(\mathfrak{I})|_{I_\pi \times 0} = -s_2 s_3 \left(c(\lambda; s_2, s_3) + \frac{|\lambda|}{2}(s_2 + s_3) \right) + s_1 s_2 s_3 |\pi|$$
$$= t_1 t_2 \left(-s|\pi| + c(\lambda; t_1, t_2) + \frac{|\lambda|}{2}(t_1 + t_2) \right).$$

When applied to the computation of $\mathsf{W}_{\sigma_1(s)}(\lambda, \varnothing, \varnothing)|_{s,t_1,t_2}$, the argument of the descendent $\sigma_1(s)$ and the equivariant push-forward together remove the prefactor $t_1 t_2$. $\quad\square$

By evaluation (6) of $\mathsf{W}(\varnothing, \varnothing, \varnothing)$ and Lemma 18,

$$\mathsf{W}_{\sigma_1(s)}(\varnothing, \varnothing, \varnothing) = \frac{(t_1 + t_2)(t_1 + s)(t_2 + s)}{t_1 t_2} \Phi(q) \, \mathsf{W}(\varnothing, \varnothing, \varnothing)|_{s,t_1,t_2}.$$

After multiplying all the factors,

$$(34) \quad \left\langle \varnothing \mid \sigma_1(F) \mid \varnothing \right\rangle = (t_1 + t_2) \, \Phi(q).$$

5.4 Proofs of Lemma 16 and Proposition 17

The proof is via T–localization on $\bar{N}$. We follow the notation of Section 4.6 for the T–fixed points of $\bar{N}_\infty$. Let

$$0_\uparrow, 0_\rightarrow \in \bar{N}_\infty$$

be the two new T–fixed points with normal T–weights

$$t_1 - t_2, -t_2, \text{ and } t_2 - t_1, -t_1$$

respectively.

For $\ell(\mu) = \ell(\nu)$, consider the T–equivariant localization formula for the integral

$$
\begin{aligned}
\langle \mu \mid \sigma_1(F) \mid \nu([0]) \rangle^{\bar{N}} =\ & \langle \mu \mid \sigma_1(F) \mid \nu([0]) \rangle \\
& + q^d \langle \mu \mid \nu([0]) \rangle_{d,d} \cdot \langle \varnothing \mid \sigma_1(F) \mid \varnothing \rangle \big|_{t_1 - t_2, -t_2} \\
& + q^d \langle \mu \mid \nu([0]) \rangle_{d,d} \cdot \langle \varnothing \mid \sigma_1(F) \mid \varnothing \rangle \big|_{t_2 - t_1, -t_1}.
\end{aligned}
$$

After evaluation and rearrangement,

$$
\text{(35)} \quad \langle \mu \mid \sigma_1(F) \mid \nu([0]) \rangle = \langle \mu \mid \sigma_1(F) \mid \nu[(0)] \rangle^{\bar{N}}
$$
$$
+ \frac{(-1)^{d-\ell(\mu)} \delta_{\mu,\nu}}{3(\mu)} (t_1 + t_2) q^d \, \Phi(q)
$$

We find $\langle \mu \mid \sigma_1(F) \mid \nu[(0)] \rangle$ is a linear function of t_1 and t_2. By symmetry, the integral must be a function of $t_1 + t_2$.

We draw two conclusions. First, in the diagonal case $\mu = \nu$, the form (31) is proven. Second, $\langle \mu \mid \sigma_1(F) \mid \nu[(0)] \rangle$ is *determined* by restriction to $t_2 = 0$.

If $\ell(\nu) = 1$, the Lemma and Proposition are empty. Let $\ell(\nu) \geq 2$. We proceed by induction on $\ell(\nu)$.

The strategy of the proof is to compare two T–equivariant integrals:

$$
m(\nu_1) \langle \mu \mid \sigma_1(F) \mid \nu[(0)] \rangle^{\bar{N}}_n
$$

and

$$
\langle \mu \mid \sigma_1(F) \mid \nu_1([0_{\uparrow}]), \nu_2[(0)], \dots, \nu_{\ell(\nu)}([0]) \rangle^{\bar{N}}_n,
$$

where $m(\nu_1)$ is the multiplicity of the part ν_1 in ν. Both integrals evaluate to linear functions of t_1 and t_2. The answers are *not* equal — the integrands are different T–equivariant classes. Consider the 1–dimensional subtorus

$$
T_1 \subset T
$$

determined by the first factor. The points

$$
0, 0_{\uparrow} \in \bar{N}_{\infty}
$$

define *equivalent* T_1–equivariant classes on $\bar{N}$. Hence, the two integrals above are equal after the restriction $t_2 = 0$.

Consider the T–equivariant localization formula for the second integral. The formula immediately yields 0 unless there is a part μ_1 of μ equal to ν_1. Let μ^* and ν^* denote

the subpartitions obtained by removing the first parts of μ and ν. Then,

$$\langle \mu \mid \sigma_1(F) \mid \nu_1([0_\uparrow]), \nu^*([0]) \rangle^{\overline{N}} = q^{\mu_1} \frac{(-1)^{\mu_1-1}}{\mu_1} \langle \mu^* \mid \sigma_1(F) \mid \nu^*([0]) \rangle$$

$$+ q^{|\mu^*|} \frac{(-1)^{|\mu^*|-\ell(\mu^*)} \delta_{\mu^*,\nu^*}}{\mathfrak{z}(\mu^*)} \langle \mu_1 \mid \sigma_1(F) \mid \nu_1([0]) \rangle \big|_{t_1-t_2,-t_2}$$

$$+ q^d \frac{(-1)^{\mu_1-1}}{\mu_1} \frac{(-1)^{|\mu^*|-\ell(\mu^*)} \delta_{\mu^*,\nu^*}}{\mathfrak{z}(\mu^*)} (-2t_1 + t_2) \Phi(q)$$

Comparing the $t_2 = 0$ restriction with (35) and using induction, we find

$$\langle \mu \mid \sigma_1(F) \mid \nu[(0)] \rangle = 0$$

unless $\mu = \nu$ proving Lemma 16.

We now assume $\mu = \nu$. Combining all the equations yields the following additivity relation:

$$\frac{\langle \mu \mid \sigma_1(F) \mid \mu \rangle}{\langle \mu \mid \mu \rangle_{|\mu|,|\mu|}} = q^{\mu_1} \frac{\langle \mu^* \mid \sigma_1(F) \mid \mu^* \rangle}{\langle \mu \mid \mu \rangle_{|\mu^*|,|\mu^*|}} + q^{|\mu^*|} \frac{\langle \mu_1 \mid \sigma_1(F) \mid \mu_1 \rangle}{\langle \mu_1 \mid \mu_1 \rangle_{\mu_1,\mu_1}} - (t_1 + t_2) q^d \Phi(q).$$

The induction step is complete and Proposition 17 is proven. □

6 Degree 1

6.1 Vertex measure

The following formula for the degree 1 vertex measure was stated (without proof) in Maulik et al [25].

Proposition 19 *At a 3–fold fixed point with tangent weights s_1, s_2, and s_3, the degree 1 vertex measure is*

$$\mathsf{W}(1, \varnothing, \varnothing) = (1+q)^{\frac{s_2+s_3}{s_1}} M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}},$$

where the unique degree 1 leg extends in the s_1 direction.

As an easy application of Proposition 19, we calculate the series

$$\langle \, \rangle_1 = \sum_{n \geq 1} q^n \langle \, \rangle_{n,1}$$

via **T**–localization:

$$\langle \, \rangle_1 = \left[W(1, \varnothing, \varnothing)|_{s,t_1,t_2} \cdot \frac{q}{t_1 t_2} \cdot W(1, \varnothing, \varnothing)|_{-s,t_1,t_2} \right]_{s=0}$$

$$= \frac{q}{t_1 t_2} M(-q)^{-2\frac{(t_1+t_2)^2}{t_1 t_2}}.$$

Empty Donaldson–Thomas brackets $\langle \, \rangle$ denote the integrand 1,

$$\langle \, \rangle_1 = \langle 1 \rangle_1.$$

6.2 Proof of Proposition 19

The Proposition is proven by using two geometric constraints for the vertex measure $W(1,0,0)$.

The effective curve classes of $\mathbf{P}^1 \times \mathbf{P}^2$ are generated by β_1 and β_2 where

$$\beta_1 = [\mathbf{P}^1 \times 0], \qquad \beta_2 = [0 \times \mathbf{P}^1].$$

We will calculate Donaldson–Thomas series associated to these two classes.

Consider first β_1. The virtual dimension of $I_n(\mathbf{P}^1 \times \mathbf{P}^2, \beta_1)$ is 2. There is a Hilbert–Chow morphism

$$\epsilon \colon I_n(\mathbf{P}^1 \times \mathbf{P}^2, \beta_1) \to \mathbf{P}^2$$

which specifies the location of the line over the second factor of $\mathbf{P}^1 \times \mathbf{P}^2$. We will compute the Donaldson–Thomas series

$$(36) \qquad \frac{\langle \epsilon^*(P) \rangle_{\beta_1}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\langle \, \rangle_0^{\mathbf{P}^1 \times \mathbf{P}^2}} = \frac{\sum_{n \geq 1} q^n \langle \epsilon^*(P) \rangle_{n,\beta_1}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\sum_{n \geq 0} q^n \langle \, \rangle_{n,0}^{\mathbf{P}^1 \times \mathbf{P}^2}},$$

where P is the class of a point in $\mathbf{P}^2$. The integrand in the numerator of (36) has dimension 2, so the integrals are well-defined.

We will calculate (36) by equivariant localization. Let the 1–dimensional torus S act on $\mathbf{P}^1$ with fixed points $0, \infty \in \mathbf{P}^1$ and tangent weights $s, -s$. Let the 2–dimensional torus T act on $\mathbf{P}^2$ with fixed points p, p', p''. Let t_1, t_2 be the tangent T–weights at p. Localization of (36) by the action of the 3–dimensional torus,

$$\mathbf{T} = S \times T,$$

yields

$$\frac{\langle \epsilon^*(P) \rangle_{\beta_1}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\langle \, \rangle_0^{\mathbf{P}^1 \times \mathbf{P}^2}} = t_1 t_2 \frac{W(1, \varnothing, \varnothing)}{W(\varnothing, \varnothing, \varnothing)} \bigg|_{s,t_1,t_2} \cdot \frac{q}{t_1 t_2} \cdot \frac{W(1, \varnothing, \varnothing)}{W(\varnothing, \varnothing, \varnothing)} \bigg|_{-s,t_1,t_2}.$$

Here, the $\mathbf{T}$–equivariant lift of P is specified by $P = [p]$. We conclude

$$(37) \qquad W'(1, \varnothing, \varnothing)|_{s,t_1,t_2} W'(1, \varnothing, \varnothing)|_{-s,t_1,t_2} \in \mathbb{Q}[\![q]\!],$$

where

$$W'(1, \varnothing, \varnothing) = \frac{W(1, \varnothing, \varnothing)}{W(\varnothing, \varnothing, \varnothing)}.$$

Next, consider the class β_2. The virtual dimension of $I_n(\mathbf{P}^2 \times \mathbf{P}^1, \beta_1)$ is 3. There is a Hilbert–Chow morphism

$$\epsilon \colon I_n(\mathbf{P}^1 \times \mathbf{P}^2, \beta_2) \to \mathbf{P}^1 \times (\mathbf{P}^2)^\vee$$

which specifies the line component. We will compute the Donaldson–Thomas series

$$(38) \qquad \frac{\langle \epsilon^*(Q) \rangle_{\beta_2}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\langle \ \rangle_0^{\mathbf{P}^1 \times \mathbf{P}^2}} = \frac{\sum_{n \geq 1} q^n \langle \epsilon^*(Q) \rangle_{n,\beta_2}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\sum_{n \geq 0} q^n \langle \ \rangle_{n,0}^{\mathbf{P}^1 \times \mathbf{P}^2}},$$

where Q is the class of a point in $\mathbf{P}^1 \times (\mathbf{P}^2)^\vee$. The integrand in the numerator of (36) has dimension 3, so the integrals are well-defined.

The $\mathbf{T}$–equivariant localization formula yields the following evaluation of the series (38):

$$\frac{\langle \epsilon^*(Q) \rangle_{\beta_2}^{\mathbf{P}^1 \times \mathbf{P}^2}}{\langle \ \rangle_0^{\mathbf{P}^1 \times \mathbf{P}^2}} = st_2(t_2 - t_1) \frac{W(1, \varnothing, \varnothing)}{W(\varnothing, \varnothing, \varnothing)} \bigg|_{t_1,s,t_2} \cdot \frac{q}{st_2(t_2 - t_1)} \cdot \frac{W(1, \varnothing, \varnothing)}{W(\varnothing, \varnothing, \varnothing)} \bigg|_{-t_1,s,t_2 - t_1}.$$

The $\mathbf{T}$–equivariant lift of Q is specified by the line over $0 \in \mathbf{P}^1$ connecting T–fixed points $p, p' \in \mathbf{P}^2$ where the tangent T–weights at p' are $-t_1, t_2 - t_1$. We conclude

$$W'(1, \varnothing, \varnothing)|_{t_1,s,t_2} W'(1, \varnothing, \varnothing)|_{-t_1,s,t_2-t_1} \in \mathbb{Q}[\![q]\!].$$

After renaming the variables, we obtain

$$(39) \qquad W'(1, \varnothing, \varnothing)|_{s,t_1,t_2} W'(1, \varnothing, \varnothing)|_{-s,t_1,t_2-s} \in \mathbb{Q}[\![q]\!],$$

and, by symmetry,

$$(40) \qquad W'(1, \varnothing, \varnothing)|_{s,t_1,t_2} W'(1, \varnothing, \varnothing)|_{-s,t_1-s,t_2} \in \mathbb{Q}[\![q]\!],$$

By definition of the equivariant vertex measure,

$$W(1, \varnothing, \varnothing)|_{s,t_1,t_2} \in \mathbb{Q}(s, t_1, t_2)[\![q]\!].$$

The q^0 coefficient is 1. By Lemma 6, the coefficient of q^n is divisible by $t_1 + t_2$ for $n > 0$.

By repeated applications of the logarithms of equations (37), (39), and (40), we find

$$\log\left(W'(1,\varnothing,\varnothing)\right)\big|_{s,t_1,t_2} + \log\left(W'(1,\varnothing,\varnothing)\right)\big|_{-s,t_1-is,t_2-js} \in \mathbb{Q}[\![q]\!]$$

for all non-negative integers i and j. The coefficients

$$\log\left(W'(1,\varnothing,\varnothing)\right)\big|_{s,t_1,t_2} = \sum_{n\geq 1} f_n(s,t_1,t_2)q^n$$

must therefore satisfy

$$f_n(s,t_1,t_2) + f_n(-s,t_1-xs,t_2-ys) = g_n(x,y)$$

for variables x and y. Differentiation with respect to x yields

$$-s\frac{\partial f_n}{\partial t_1}(-s,t_1,t_2) = \frac{\partial g_n}{\partial x}(0,0).$$

Similarly

$$-s\frac{\partial f_n}{\partial t_2}(-s,t_1,t_2) = \frac{\partial g_n}{\partial y}(0,0).$$

Hence, by integration and symmetry,

$$f_n(s,t_1,t_2) = \gamma_n\frac{t_1+t_2}{s},$$

where $\gamma_n \in \mathbb{Q}$. Since f_n *must* be divisible by t_1+t_2, the constant of integration vanishes.

After specializing to Calabi–Yau weights $s+t_1+t_2 = 0$, we find

$$f_n = -\gamma_n.$$

However, after Calabi–Yau specialization, the equivariant vertex measure takes the simple form

$$\mathsf{w}(\pi) = (-1)^{|\pi|}$$

for 3–dimensional partitions π, see Maulik et al [24].

Lemma 20 $\log\left(W'(1,\varnothing,\varnothing)\right)\big|_{s+t_1+t_2=0} = -\log(1+q).$

The Lemma is a special case of the vertex evaluation required for the calculation of the level $(-1,0)$ cap (see Okounkov–Reshetikhin [32]), see Section 10.7 for a detailed discussion.

Lemma 20 determines the constants γ_n and completes the proof of Proposition 19.
□

6.3 Descendent calculation

We calculate the series $\langle (1) \mid \sigma_1(F) \mid (1) \rangle$. By the degeneration formula,

$$\langle (1) \mid \sigma_1(F) \mid (1) \rangle = \frac{q^2}{(t_1 t_2)^2} \frac{\langle \sigma_1(F) \rangle_1}{\langle (1) \mid \rangle \cdot \langle \mid (1) \rangle}$$

$$= \frac{q}{t_1 t_2} \frac{\langle \sigma_1(F) \rangle_1}{\langle \, \rangle_1}$$

$$= M(-q)^{2 \frac{(t_1 + t_2)^2}{t_1 t_2}} \langle \sigma_1(F) \rangle_1.$$

To complete the calculation, we determine the series $\langle \sigma_1(F) \rangle_1$ via localization with respect to the **T**–action. Let the **T**–equivariant lift of F be specified by $[N_0]$. Then,

$$\langle \sigma_1(F) \rangle_1 = \left[\mathsf{W}_{\sigma_1(s)}(1, \varnothing, \varnothing)|_{s, t_1, t_2} \cdot \frac{q}{t_1 t_2} \cdot \mathsf{W}(1, \varnothing, \varnothing)|_{-s, t_1, t_2} \right]_{s=0}.$$

By [Proposition 19](#) and [Lemma 18](#),

$$\mathsf{W}_{\sigma_1(s)}(1, \varnothing, \varnothing) = (t_1 + t_2) \left(\frac{-q}{1+q} + \frac{1}{2} \right) \mathsf{W}(1, \varnothing, \varnothing)_{s, t_1, t_2}$$

$$+ \frac{(t_1 + t_2)(t_1 + s)(t_2 + s)}{t_1 t_2} \Phi(q) \mathsf{W}(1, \varnothing, \varnothing)|_{s, t_1, t_2}.$$

After multiplying all the factors,

(41)
$$\frac{\langle (1) \mid \sigma_1(F) \mid (1) \rangle}{\langle (1) \mid (1) \rangle_{1,1}} = \frac{(t_1 + t_2)}{2} q \frac{1-q}{1+q} + (t_1 + t_2) q \Phi(q).$$

6.4 Cap

We calculate the partition function $\mathsf{DT}(0|0, 0)_\lambda$ corresponding to the cap in the TQFT formalism.

Lemma 21 *The invariants of the level $(0, 0)$ cap are given by*

$$\mathsf{DT}(0|0, 0)_\lambda = \begin{cases} \frac{1}{d!(t_1 t_2)^d} & \text{if } \lambda = (1^d) \\ 0 & \text{if } \lambda \neq (1^d). \end{cases}$$

Proof If $\lambda \neq (1^d)$, then $\ell(\lambda) < d$. By equation [(18)](#),

$$\langle \mid \lambda([0]) \rangle = 0$$

if $\ell(\lambda) < d$. Since

$$\mathsf{DT}(0|0,0)_\lambda = q^{-d} \frac{\langle\,|\,\lambda\rangle}{\langle\,|\,\varnothing\rangle} = \frac{q^{-d}}{(t_1 t_2)^{\ell(\lambda)}} \frac{\langle\,|\,\lambda([0])\rangle}{\langle\,|\,\varnothing\rangle},$$

the cap $\mathsf{DT}(0|0,0)_\lambda$ vanishes if $\lambda \neq (1^d)$.

If $\lambda = (1^d)$, then the Donaldson–Thomas invariant for the compactified geometry

$$\overline{\langle\,|\,\lambda([0])\rangle}^{(0,0)}$$

is a constant independent of the equivariant parameters t_1 and t_2. By specializing the partition weights to different T–fixed points and localizing, we obtain

$$\frac{\langle\,|\,\lambda([0])\rangle}{\langle\,|\,\varnothing\rangle} = \frac{1}{d!} \left(\frac{\langle\,|\,1([0])\rangle}{\langle\,|\,\varnothing\rangle} \right)^d.$$

The degree 1 calculation,

$$\frac{\langle\,|\,1([0])\rangle}{\langle\,|\,\varnothing\rangle} = q,$$

completes the proof of the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

7 The operator M_σ

7.1 Fock space formalism

Let the 2–dimensional torus T act on $\mathbb{C}^2$ by standard diagonal scaling. We review the Fock space description of the T–equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$, see Grojnowski [10] and Nakajima [28].

By definition, the Fock space $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\varnothing$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\varnothing = 0, \quad k > 0,$$

and satisfy the commutation relations

$$[\alpha_k, \alpha_l] = k\, \delta_{k+l}.$$

A natural basis of $\mathcal{F}$ is given by the vectors

(42) $$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod \alpha_{-\mu_i}\, v_\varnothing.$$

indexed by partitions μ. Here,

$$\mathfrak{z}(\mu) = |\operatorname{Aut}(\mu)| \prod \mu_i$$

is the usual normalization factor.

The Nakajima basis defines a canonical isomorphism,

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \cong \bigoplus_{d \geq 0} A_T^*(\operatorname{Hilb}(\mathbb{C}^2, d), \mathbb{Q}).$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\Pi_i \mu_i} [V_\mu]$$

where $[V_\mu]$ is (the cohomological dual of) the class of the subvariety of $\operatorname{Hilb}(\mathbb{C}^2, |\mu|)$ with generic element given by a union of schemes of lengths

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of $\mathbb{C}^2$. The vacuum vector $v_\varnothing$ corresponds to the unit in $A_T^*(\operatorname{Hilb}_0, \mathbb{Q})$.

The standard inner product on the T–cohomology induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$(43) \qquad \langle \mu \mid \nu \rangle_{\mathcal{F}} = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{(-1)^{|\mu| - \ell(\mu)}}{\mathfrak{z}(\mu)} \delta_{\mu\nu}.$$

With respect to the inner product,

$$(44) \qquad (\alpha_k)^* = (-1)^{k-1} (t_1 t_2)^{\operatorname{sgn}(k)} \alpha_{-k}.$$

If there is no ambiguity, the subscript $\mathcal{F}$ will be omitted from the bracket (43).

7.2 The class D

Let $\mathcal{O}/\mathcal{I}$ be the rank d tautological bundle on $\operatorname{Hilb}(\mathbb{C}^2, d)$, and let

$$D = c_1(\mathcal{O}/\mathcal{I}).$$

A straightforward calculation shows

$$D = -\big|2, 1^{d-2}\big\rangle.$$

The classical multiplication of D on the Fock space $\mathcal{F}$ is given by the following operator:

$$(45) \quad -(t_1 + t_2) \sum_{k>0} \frac{k-1}{2} \alpha_{-k}\, \alpha_k + \frac{1}{2} \sum_{k,l>0} \left[t_1 t_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \right],$$

see Lehn [14], Lehn–Sorger [15], Li et al [20] and Okounkov–Pandharipande [31].

The first summand of (45) contains a term proportional to the energy operator,

$$|\cdot| = \sum_{k>0} \alpha_{-k}\, \alpha_k\,.$$

The energy operator acts diagonally on Fock space with eigenvalue $|\mu|$ on the vector $|\mu\rangle$.

7.3 Operators

The following operator on Fock space plays a central role in the paper:

$$(46) \quad \mathsf{M}(q, t_1, t_2) = (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1} \alpha_{-k}\, \alpha_k +$$

$$\frac{1}{2} \sum_{k,l>0} \left[t_1 t_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \right].$$

The q–dependence of M is only in the first sum in (46) which acts diagonally in the basis (42). The two terms in the second sum in (46) are known respectively as the splitting and joining terms. The operator M is self-adjoint

$$(47) \qquad\qquad\qquad \mathsf{M}^* = \mathsf{M}$$

with respect to (44).

Let the operator M_σ on Fock space be defined by matrix elements

$$\langle \mu \mid \mathsf{M}_\sigma \mid \nu \rangle_{\mathcal{F}} = q^{-d} \langle \mu \mid -\sigma_1(F) \mid \nu \rangle$$
$$= \sum_{n \geq d} q^{n-d} \langle \mu \mid -\sigma_1(F) \mid \nu \rangle_{n,d}$$

for partitions satisfying

$$|\mu| = |\nu| = d.$$

The matrix elements of M_σ are understood to vanish unless $|\mu| = |\nu|$. Define the operator

$$\widetilde{\mathsf{M}}_\sigma = \mathsf{M} - (t_1 + t_2)\Phi(q) \cdot \mathrm{Id}\,.$$

Proposition 22 $\mathsf{M}_\sigma = \widetilde{\mathsf{M}}_\sigma$

We check here three initial compatibilities required for Proposition 22. Proposition 22 will be proven in Section 8 by relating Donaldson–Thomas integrals to Gromov–Witten invariants of the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2, d)$.

First, Proposition 22 requires the negative of the q–shifted operator bracket,

$$-q^d \big\langle \mu \mid \widetilde{\mathsf{M}}_\sigma \mid \nu \big\rangle_{\mathcal{F}},$$

to satisfy the additivity property of Proposition 17. The required additivity is easily checked.

Second, Proposition 22 is valid for $q = 0$. Let N be the trivial bundle with level $(0,0)$ splitting on $\mathbf{P}^1$. There is an isomorphism of moduli spaces

$$(48) \qquad\qquad I_d(N/N_0 \cup N_\infty, d) \cong \mathrm{Hilb}(N_z, d)$$

for any $z \in \mathbf{P}^1$. Under the isomorphism (48), the Donaldson–Thomas descendent class $-\sigma_1([N_z])$ determines an element of $A_T^1(\mathrm{Hilb}(N_z, d), \mathbb{Q})$. By a Riemann–Roch calculation,

$$-\sigma_1(N_z) = D - \frac{t_1 + t_2}{2} d.$$

Proposition 22 correctly equates the $q = 0$ evaluation $\widetilde{\mathsf{M}}_\sigma(0)$ of $\widetilde{\mathsf{M}}_\sigma$ with the classical multiplication (45) of D shifted by $-\frac{t_1+t_2}{2}d$.

Third, Proposition 22 is valid in degrees 0 and 1 by the descendent evaluation (34),

$$\big\langle \varnothing \mid \mathsf{M}_\sigma \mid \varnothing \big\rangle_{\mathcal{F}} = \big\langle \varnothing \mid -\sigma_1(F) \mid \varnothing \big\rangle = -(t_1 + t_2)\Phi(q),$$

and the descendent evaluation (41),

$$\frac{\big\langle (1) \mid \mathsf{M}_\sigma \mid (1) \big\rangle_{\mathcal{F}}}{\big\langle (1) \mid (1) \big\rangle_{\mathcal{F}}} = q^{-1} \frac{\big\langle (1) \mid -\sigma_1(F) \mid (1) \big\rangle}{\big\langle (1) \mid (1) \big\rangle_{1,1}}$$

$$= -\frac{(t_1 + t_2)}{2} \frac{1-q}{1+q} - (t_1 + t_2)\Phi(q).$$

8 Proof of Proposition 22

8.1 Induction strategy

Our proof of Proposition 22 closely follows the proof of Theorem 1 in Okounkov–Pandharipande [31] though several differences occur. Since the results and the geometry differ, the closeness of the proof is somewhat surprising.

We proceed by induction on the degree d. If $d = 0$ or 1, Proposition 22 has already been proven by descendent calculations. Let $d > 1$.

Next, we induct on the Euler number n. In the minimal case $n = d$, Proposition 22 recovers classical multiplication by D on the Hilbert scheme of points. Let $n > d$.

The induction step relies upon the addition formula of Proposition 17. We will compute an invariant

$$\big\langle \gamma_1 \,\big|\, -\sigma_1(F) \,\big|\, \gamma_2 \big\rangle_n$$

for which the expansions of the classes

$$\gamma_1, \gamma_2 \in A_T^{2d}(\mathrm{Hilb}(\mathbb{C}^2, d), \mathbb{Q}),$$

in the Nakajima basis contain nontrivial multiples *not divisible by* $(t_1 + t_2)$ of the class $|(d)\rangle$. By the addition rules, if

$$(49) \qquad \big\langle \gamma_1 \,\big|\, -\sigma_1(F) \,\big|\, \gamma_2 \big\rangle_n = \big\langle \gamma_1 \,\big|\, \widetilde{\mathsf{M}}_\sigma \,\big|\, \gamma_2 \big\rangle_{\mathcal{F}, n-d} \,,$$

then Proposition 22 is proven for Euler number n.

Both sides of (49) are constant multiples of $t_1^{2d}(t_1 + t_2)$ modulo $(t_1 + t_2)^2$. By (31),

$$\big\langle (d) \,\big|\, -\sigma_1(F) \,\big|\, (d) \big\rangle_n = \frac{\gamma_{d,n}}{t_1^2}(t_1 + t_2) \quad \mathrm{mod}\ (t_1 + t_2)^2.$$

Hence, we need only verify the equality (49) modulo $(t_1 + t_2)^2$.

8.2 Induction step: I

Let $d > 1$ and let $n > d$. For the induction step, we will compute the invariant

$$(50) \qquad \big\langle \big[\mathcal{I}_{(d)}\big] \,\big|\, -\sigma_1(F) \,\big|\, \big[\mathcal{I}_{(d-1,1)}\big]\big\rangle_n \,.$$

Here, $\mathcal{I}_\lambda$ denotes the monomial ideal corresponding to the partition λ, and $[\mathcal{I}_\lambda]$ denotes the T–equivariant class of the associated fixed point in $\mathrm{Hilb}(\mathbb{C}^2, |\lambda|)$.

The T–fixed point $[\mathcal{I}_\lambda]$ corresponds to the Jack polynomial

$$J^\lambda \in \mathcal{F} \otimes \mathbb{Q}[t_1, t_2].$$

For $-t_2/t_1 = 1$, the Jack polynomials specialize to the Schur functions. Hence,

$$J^\lambda \equiv \frac{(-1)^{|\lambda|}\,|\lambda|!}{\dim \lambda} \sum_\mu \chi_\mu^\lambda\, t_1^{|\lambda| + \ell(\mu)} \,|\mu\rangle \quad \mathrm{mod}\ t_1 + t_2 \,,$$

where $\dim \lambda$ is the dimension of the representation λ of the symmetric group and χ_μ^λ is the evaluation of the corresponding character on the conjugacy class μ, see

Macdonald [22]. In particular, the coefficient of $\left|d\right\rangle$ in the expansion of both $\mathsf{J}^{(d)}$ and $\mathsf{J}^{(d-1,1)}$ is nonzero.

Lemma 23 *We have*

$$(51) \quad \left\langle \mathsf{J}^{(d)} \mid \widetilde{\mathsf{M}}_\sigma - \widetilde{\mathsf{M}}_\sigma(0) \mid \mathsf{J}^{(d-1,1)} \right\rangle_{\mathcal{F}} \equiv$$

$$(-1)^d (t_1 + t_2) \frac{t_1^{2d} (d!)^2}{d-1} \left(\frac{q}{1+q} + d \frac{(-q)^d}{1-(-q)^d} \right) \mod (t_1 + t_2)^2 .$$

Here, the operator $\widetilde{\mathsf{M}}_\sigma - \widetilde{\mathsf{M}}_\sigma(0)$ is formed by the terms of positive degree in q in the operator $\widetilde{\mathsf{M}}_\sigma$.

The proof of Lemma 23 follows exactly the derivation of equation (31) of Okounkov–Pandharipande [31]. The differences between our operator $\widetilde{\mathsf{M}}_\sigma$ and the operator M_D of [31] yield constant functions orthogonal to the non-trivial character $\chi^{(d-1,1)}$.

8.3 Localization

8.3.1 Overview Our goal now is to reproduce the answer (51) by calculating the Donaldson–Thomas invariant

$$\left\langle \left[\mathcal{I}_{(d)} \right] \mid -\sigma_1(F) \mid \left[\mathcal{I}_{(d-1,1)} \right] \right\rangle_n .$$

By the rubber calculus (29),

$$(52) \quad \left\langle \left[\mathcal{I}_{(d)} \right] \mid -\sigma_1(F) \mid \left[\mathcal{I}_{(d-1,1)} \right] \right\rangle_n = (n-d) \left\langle \left[\mathcal{I}_{(d)} \right] \mid \left[\mathcal{I}_{(d-1,1)} \right] \right\rangle_n^\sim .$$

We calculate the right side of (52) by T–equivariant localization on the rubber moduli space

$$R_{n,d} = I_n(R/R_0 \cup R_\infty)^\sim ,$$

see Section 4.8.

Since the T–fixed locus of the moduli space $R_{n,d}$ is proper, the virtual localization formula of Graber–Pandharipande [9] may be applied. However, since $R_{n,d}$ contains positive dimensional families of T–invariant ideal sheaves, a straightforward application is difficult. Our strategy for computing (52) uses comparisons to integrals in the quantum cohomology of the Hilbert scheme of points $\text{Hilb}(\mathbb{C}^2, d)$.

8.3.2 Skewers and twistors Consider the rubber moduli space $R_{n,d}$ for $n > d$. Let

$$[I] \in R_{n,d}^T$$

be a T–fixed ideal sheaf. The ideal sheaf I is defined on a rubber target fibered by $\mathbb{C}^2$ over a chain C of rational curves. The diagram below gives an example of a subscheme associated to a T–fixed ideal sheaf on a reducible rubber target.
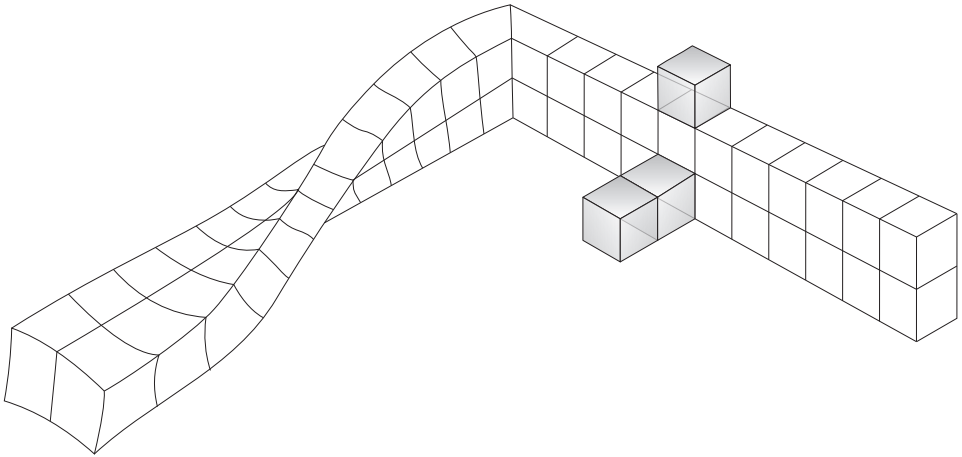
Let $P \subset C$ be a rational component. The restriction I_P of I to the component $\mathbb{C}^2 \times P$ of the target rubber is either a *skewer* or a *twistor*:

(i) A skewer is determined by an element

$$[I_{sk}] \in I_*\left(\mathbb{C}^2 \times \mathbf{P}^1, d\right)^T$$

where the factor $\mathbf{P}^1$ is rigid. Certainly the skewer I_{sk} is T–fixed on the rubber $\mathbb{C}^2 \times P$. The component on the right of the above diagram is a skewer.

(ii) A twistor is obtained from a T–fixed element

$$[f_{tw}] \in M_{0,\{0,\infty\}}\left(\mathrm{Hilb}\left((\mathbb{C}^2, d), *\right)\right)^T.$$

By pulling-back the universal ideal sheaf over the Hilbert scheme, f_{tw} determines a T–fixed ideal sheaf I_{tw} on the rubber $\mathbb{C}^2 \times P$. The component on the left of the diagram is a twistor.

The degree of the map f_{tw} to the Hilbert scheme and the Euler number n_{tw} of I_{tw} are related by

$$\deg(f_{tw}) = n_{tw} - d.$$

Elementary considerations show these disjoint constructions exhaust all T–fixed ideal sheaves on the rubber $\mathbb{C}^2 \times P$.

8.3.3 Comparison We will calculate (52) via a comparison result.

Proposition 24 *We have*

$$\big\langle \big[\mathcal{I}_{(d)}\big] \,\big|\, \big[\mathcal{I}_{(d-1,1)}\big]\big\rangle_n^{\sim} = \big\langle \big[\mathcal{I}_{(d)}\big] \,\big|\, \big[\mathcal{I}_{(d-1,1)}\big]\big\rangle_{n-d}^{Hilb(\mathbb{C}^2,d)} \quad \mod (t_1 + t_2)^2.$$

Proposition 24 is proven in Section 8.3.5. As a first step, a simpler comparison result is obtained here.

Consider the open set of stable maps,

$$U_{n,d} \subset \bar{M}_{0,\{0,\infty\}}\big(\mathrm{Hilb}(\mathbb{C}^2, d), n-d\big)$$

for which the domain is a *chain* of rational curves. The open set $U_{n,d}$ carries a T–equivariant Gromov–Witten obstruction theory via restriction. By pulling-back the universal ideal sheaf over the Hilbert scheme (as in the twistor construction), we obtain an open immersion

$$\iota \colon U_{n,d} \subset R_{n,d}.$$

Hence, $U_{n,d}$ also carries a T–equivariant Donaldson–Thomas theory.

Lemma 25 *The T–equivariant Gromov–Witten and Donaldson–Thomas obstruction theories of $U_{n,d}$ are canonically isomorphic.*

Proof Let

$$\big[f \colon C \to \mathrm{Hilb}(\mathbb{C}^2, d)\big] \in U_{n,d}$$

be a stable map, and let

$$\iota([f]) = [I] \in R_{n,d}$$

be the associated ideal sheaf. Since both obstruction theories can be defined relative to the degenerations of the domain, we need only check the Gromov–Witten complex associated to the cohomology $H^*(C, f^* T_{\mathrm{Hilb}})$ matches the shift of the complex associated to $\mathrm{Ext}_0(I, I)$. For a point $p \in C$, the tangent to the Hilbert scheme may be viewed as $\mathrm{Ext}_0^1(I_p, I_p)$. Moreover,

$$\mathrm{Ext}_0^0\big(I_p, I_p\big) = \mathrm{Ext}_0^2\big(I_p, I_p\big) = 0.$$

The required matching is then a straightforward application of the Leray spectral sequence. $\qquad\square$

8.3.4 Tangent representations Let $[I] \in R^T_{n,d}$ be a T–fixed ideal sheaf on a rubber target fibered by $\mathbb{C}^2$ over a chain C of rational curves.

The *special* points of C consist of $0, \infty \in C$ and all the nodes. Over the special points of $s \in C$, the ideal I must correspond to T–fixed points $[I_s] \in \mathrm{Hilb}(\mathbb{C}^2, d)$.

Let $P \subset C$ be a component containing the special points s, s' of C. Fractional T–weight $w_{P,s}$ and $w_{P,s'}$ are defined by the T–representation of the tangent space to P at s and s':

(i) If $P \subset C$ corresponds to a skewer, then

$$[I_s] = [I_{s'}]$$

and $w_{P,s} = w_{P,s'} = 0$.

(ii) If $P \subset C$ corresponds to a twistor, then

$$[I_s] \neq [I_{s'}]$$

and $w_{P,s}, w_{P,s'} \neq 0$ modulo $(t_1 + t_2)$, see Okounkov–Pandharipande [31].

In fact, in the twistor case, if

$$[I_s] = [\mathcal{I}_\mu], \quad [I_{s'}] = [\mathcal{I}_\nu],$$

then

$$w_{P,s} = w^{n_P - d}_{\mu,\nu} = -w^{n_P - d}_{\nu,\mu} = -w_{P,s'},$$

where n_P is the Euler characteristic associated to I_P and $w^n_{\mu,\nu}$ is the universal function defined in [31, Lemma 4] viewed here as a T–weight.

8.3.5 Proof of Proposition 24 We index the T–equivariant localization contributions to the invariants

$$\big\langle [\mathcal{I}_\mu] \,\big|\, [\mathcal{I}_\nu] \big\rangle^{\sim}_n$$

by graphs following [31]. An *oriented chain* of Euler number n is a graph $\Gamma_{\mu,\nu} = (V, v_1, v_2, \rho, E, S, \delta)$,

(i) V is a finite vertex set with distinguished elements v_1 and v_2,

(ii) $\rho \colon V \to \mathcal{P}(d)$,

(iii) E is a finite edge set,

(iv) $S \subset V$,

(v) $\delta \colon E \cup S \to \mathbb{Z}_{>0}$ is an assignment,

satisfying the following conditions

(a) Γ is a connected chain with initial vertex v_1 and final vertex v_2,

(b) $\rho(v_1) = \mathcal{I}_\mu$, $\rho(v_2) = \mathcal{I}_\nu$,

(c)   if $v', v'' \in V$ are connected by an edge, then $\rho(v') \neq \rho(v'')$,

(d)   if $v \in V \setminus S$ has edge valence 2 with neighbors v', v'', then

$$w_{\rho(v),\rho(v')}^{\delta(e(v,v'))} + w_{\rho(v),\rho(v'')}^{\delta(e(v,v''))} \neq 0 \mod t_1 + t_2,$$

(e) $\sum_{e \in E} \delta(e) + \sum_{s \in S} \delta(s) = n + d(|E| + |S| - 1)$.

Let $[I] \in R^T_{n,d}$ be T–fixed ideal sheaf on a rubber target fibered by $\mathbb{C}^2$ over a chain C of rational curves. We associate an oriented chain,

$$\Gamma_I = (V, v_1, v_2, \rho, E, S, \delta),$$

of Euler number n to I by the following construction. The vertex set

$$V = V_1 \cup V_2 \cup V_3,$$

is a union of three disjoint subsets:

(1) V_1 is the set of maximal connected subcurves of skewer components of the base C,

(2) V_2 is the set of nodes of $s \in C$ for which the incident components $P, P' \subset C$ are twistors and satisfy the *breaking condition*

$$w_{P,s} + w_{P',s} \neq 0 \mod t_1 + t_2,$$

(3) V_3 is subset of the marking $0, \infty \in C$ which lie on twistors.

The markings $0, \infty \in C$ are associated to elements of the union $V_1 \cup V_3$ — the markings determine v_1 and v_2. The function ρ is obtained from I. Chains of unbroken twistors of C link the vertices of V. The edge set E is determined by such chains. The set S equals V_1. The degree assignment δ is obtained from the Euler number of the restriction of I to the associated components.

The oriented chain Γ_I is easily seen to satisfy conditions (a)–(e). Condition (c) is a consequence of [31, Lemma 5]. The chain Γ_I is invariant as $[I]$ varies in a connected component of the T–fixed locus of $R^T_{n,d}$.

Let $G_n(\mu, \nu)$ denote the finite set of oriented chains trees $\Gamma_{\mu,\nu}$ of Euler number n. Let

$$R_\Gamma \subset R^T_{n,d}$$

denote the substack of T–fixed maps corresponding to the tree $\Gamma \in G_n(\mu, \nu)$. Let

$$\left\langle [\mathcal{I}_\mu] \mid [\mathcal{I}_\nu] \right\rangle_d^{\Gamma \sim}$$

denote the localization contribution of R_Γ. By the virtual localization formula (see Graber–Pandharipande [9]),

$$(53) \qquad \left\langle [\mathcal{I}_\mu] \mid [\mathcal{I}_\nu] \right\rangle_d^{\Gamma \sim} = \int_{[R_\Gamma]^{\mathrm{vir}}} \frac{\epsilon_0^*([\mathcal{I}_\mu]) \cup \epsilon_\infty^*([\mathcal{I}_\nu])}{e(N^{\mathrm{vir}})}.$$

By decomposing the Donaldson–Thomas obstruction theory, we can express the integral (53) in terms of descendent skewer integrals corresponding to the vertices S and descendent twistor integrals corresponding to the edges E.

The edge integrals are exactly equal to associated integrals in the Gromov–Witten theory of the Hilbert scheme $\mathrm{Hilb}(\mathbb{C}^2, d)$ by Lemma 25. Moreover, the Hilbert scheme integrals which arise for each edge compute a Gromov–Witten residue *for the $T^\pm$–action*. Hence, the edge term contributions to (53) are divisible by $(t_1 + t_2)$.

We prove the skewer integrals are also divisible by $(t_1 + t_2)$ by localizing a rubber integral:

$$\left\langle [\mathcal{I}_\gamma] \mid \psi_0^a \psi_\infty^b \mid [\mathcal{I}_\gamma] \right\rangle_n^\sim = \left\langle [\mathcal{I}_\gamma] \mid \psi_0^a \psi_\infty^b \mid [\mathcal{I}_\gamma] \right\rangle^{\Gamma_{\gamma,\gamma}^0 \sim} + \sum_{\Gamma \in G_n(\gamma,\gamma) \setminus \{\Gamma^0\}} \left\langle [\mathcal{I}_\gamma] \mid \psi_0^a \psi_\infty^b \mid [\mathcal{I}_\gamma] \right\rangle^{\Gamma \sim}.$$

Here, $\Gamma_{\gamma,\gamma}^0$ is the unique chain with a single skewer vertex. By Lemma 13, the rubber integral on the left is divisible by $(t_1 + t_2)$. The Γ^0 contribution is the skewer integral of interest. The second summand on the right is expressed in terms of skewer integrals of *lower* Euler number and edge integrals. The former are divisible by $(t_1 + t_2)$ by induction. We conclude the skewer integrals are divisible by $(t_1 + t_2)$.

We now specialize to the localization analysis of the rubber integral,

$$\left\langle [\mathcal{I}_{(d)}] \mid [\mathcal{I}_{(d-1,d)}] \right\rangle_n^\sim = \sum_{\Gamma \in G_n(\mu,\nu)} \left\langle [\mathcal{I}_{(d)}] \mid [\mathcal{I}_{(d-1,1)}] \right\rangle^{\Gamma \sim}.$$

Assume $\mu \neq \nu$. Since both skewer vertices and edges contribute factors of $(t_1 + t_2)$,

$$\left\langle [\mathcal{I}_{(d)}] \mid [\mathcal{I}_{(d-1,1)}] \right\rangle_n^\sim = \left\langle [\mathcal{I}_{(d)}] \mid [\mathcal{I}_{(d-1,1)}] \right\rangle^{\Gamma_{\mu,\nu}^0 \sim} \quad \mod t_1 + t_2,$$

where $\Gamma_{\mu,\nu}^0$ is the unique single edged chain with $S = \varnothing$. The Proposition 24 is then a consequence of Lemma 25. □

8.4 Induction step: II

By Lemma 23, Proposition 24, and the Hilbert scheme calculation of Okounkov–Pandharipande [31], we obtain

$$\big\langle \big[\mathcal{I}_{(d)}\big] \,\big|\, -\sigma_1(F) \,\big|\, \big[\mathcal{I}_{(d-1,1)}\big]\big\rangle_n = \big\langle \mathsf{J}^{(d)} \,\big|\, \widetilde{\mathsf{M}}_\sigma - \widetilde{\mathsf{M}}_\sigma(0) \,\big|\, \mathsf{J}^{(d-1,1)}\big\rangle_{\mathcal{F},n-d}$$

completing the proof of Proposition 22. □

9 The level $(0,0)$ theory

9.1 The operator M_D

Let D denote the insertion of the relative condition $-(2,1^{d-2})$ in the local Donaldson–Thomas theory of curves. Let the operator M_D on Fock space be defined by matrix elements

$$\big\langle \mu \,\big|\, \mathsf{M}_D \,\big|\, v\big\rangle_{\mathcal{F}} = \mathrm{DT}(0|0,0)_{\mu,D,v}$$
$$= -\mathrm{DT}(0|0,0)_{\mu,(2,1^{d-2}),v}.$$

By definition, the insertion D vanishes in degrees $d = 0, 1$.

Proposition 26 $\quad \mathsf{M}_D = \mathsf{M} - \frac{t_1+t_2}{2}\frac{(-q)+1}{(-q)-1}\,|\,\cdot\,|\,.$

Proof By applying the degeneration formula to the definition of M_σ, we obtain

$$\big\langle \mu \,\big|\mathsf{M}_\sigma\big|v\big\rangle_{\mathcal{F}} = q^{-d}\big\langle \mu \,\big|\, -\sigma_1(F) \,\big|\, v\big\rangle$$
$$= \sum_\gamma q^{-d}\mathsf{Z}(0|0,0)_{\mu,\gamma,v}\bigwedge_d(\gamma,\gamma)q^{-d}\big\langle \gamma \,\big|\, -\sigma_1(F)\big\rangle$$
$$= \sum_\gamma \mathrm{DT}(0|0,0)_{\mu,\gamma,v}\bigwedge_d(\gamma,\gamma)q^{-d}\frac{\big\langle \gamma \,\big|\, -\sigma_1(F)\big\rangle}{\big\langle \varnothing \,\big|\, \big\rangle}$$

By equation (18),

$$\big\langle \gamma \,\big|\, -\sigma_1(F)\big\rangle = 0$$

if $\ell(\gamma) < d - 1$. Hence, there are only two nonvanishing terms in the sum over the partition γ:

$$\big\langle \mu \,\big|\mathsf{M}_\sigma\big|v\big\rangle_{\mathcal{F}} = \ \mathrm{DT}(0|0,0)_{\mu,(1^d),v}\frac{d!(t_1t_2)^d}{q^d}\frac{\big\langle (1^d)|-\sigma_1(F)\big\rangle}{\big\langle \varnothing \,\big|\, \big\rangle}$$
$$-\mathrm{DT}(0|0,0)_{\mu,(2,1^{d-2}),v}\frac{2(d-2)!(t_1t_2)^{d-1}}{q^d}\frac{\big\langle (2,1^{d-2})|-\sigma_1(F)\big\rangle}{\big\langle \varnothing \,\big|\, \big\rangle}$$

By [Lemma 21] for the cap $\mathsf{DT}(0|0,0)_\lambda$, the insertion (1^d) can be freely added or removed in the local Donaldson–Thomas theory of curves. Hence,

$$\mathsf{DT}(0|0,0)_{\mu,(1^d),\nu} = \mathsf{DT}(0|0,0)_{\mu,\nu}$$
$$= \langle \mu \mid \nu \rangle_{\mathcal{F}},$$

Similarly,

$$q^{-d} \frac{\langle (1^d) \mid -\sigma_1(F) \rangle}{\langle \varnothing \mid \rangle} = q^{-d} \frac{\langle (1^d) \mid -\sigma_1(F) \mid (1^d) \rangle}{\langle \varnothing \mid \varnothing \rangle}$$
$$= \langle (1^d) \mid \mathsf{M}_\sigma \mid (1^d) \rangle$$
$$= \frac{(t_1+t_2)}{d!(t_1 t_2)^d} \left(\frac{d}{2} \frac{(-q)+1}{(-q)-1} - \Phi(q) \right),$$

where the last equality is obtain from [Proposition 22]. Finally,

$$q^{-d} \frac{\langle (2,1^{d-2}) \mid -\sigma_1(F) \rangle}{\langle \varnothing \mid \rangle} = q^{-d} \frac{\langle (2,1^{d-2}) \mid -\sigma_1(F) \mid (1^d) \rangle}{\langle \varnothing \mid \varnothing \rangle}$$
$$= \langle (2,1^{d-2}) \mid \mathsf{M}_\sigma \mid (1^d) \rangle_{\mathcal{F}}$$
$$= -\langle (2,1^{d-2}) \mid (2,1^{d-2}) \rangle_{\mathcal{F}}.$$

We conclude

$$\langle \mu \mid \mathsf{M}_D \mid \nu \rangle_{\mathcal{F}} = -\mathsf{DT}(0|0,0)_{\mu,(2,1^{d-2}),\nu}$$
$$= \langle \mu \mid \mathsf{M}_\sigma \mid \nu \rangle_{\mathcal{F}} - (t_1+t_2)\langle \mu \mid \nu \rangle_{\mathcal{F}} \left(\frac{d}{2} \frac{(-q)+1}{(-q)-1} - \Phi(q) \right)$$
$$= \left\langle \mu \mid \mathsf{M} - \frac{t_1+t_2}{2} \frac{(-q)+1}{(-q)-1} \mid \cdot \mid \nu \right\rangle_{\mathcal{F}},$$

where the last equality is obtained from [Proposition 22].   □

9.2 Proof of [Theorem 3] in level $(0,0)$

9.2.1 Starred series The terminology of Bryan–Pandharipande [4] is convenient for discussing the GW/DT correspondence. On the Gromov–Witten side, let

$$\mathsf{GW}^*(g|k_1,k_2)_{\lambda^1 \ldots \lambda^r} = (-iu)^{d(2-2g+k_1+k_2)-\delta} \, \mathsf{Z}'(N)_{\lambda^1 \ldots \lambda^r},$$

where N is rank 2 bundle of level (k_1,k_2) on a genus g curve and

$$\delta = \sum_{i=1}^r (d - \ell(\lambda^i)).$$

On the Donaldson–Thomas side, let

$$\mathsf{DT}^*(g|k_1,k_2)_{\lambda^1\ldots\lambda^r} = (-q)^{-\frac{d}{2}(2-2g+k_1+k_2)}\, Z'_{DT}(N)_{\lambda^1\ldots\lambda^r}$$
$$= (-1)^{-d(1-g)}(-q)^{-\frac{d}{2}(k_1+k_2)}\mathsf{DT}(g|k_1,k_2).$$

[Theorem 3](#) of the Gromov–Witten/Donaldson–Thomas correspondence for local curves can be restated as the equality

(54) $$\qquad\qquad \mathsf{GW}^*(g|k_1,k_2)_{\lambda^1\ldots\lambda^r} = \mathsf{DT}^*(g|k_1,k_2)_{\lambda^1\ldots\lambda^r},$$

after the variable change $e^{iu} = -q$,

9.2.2 TQFT
Slightly altered metrics are defined for raising the indices of the starred series:

$$\mathsf{GW}^*(g|k_1,k_2)^{\nu^1\ldots\nu^t}_{\mu^1\ldots\mu^s} = \left(\prod_{i=1}^t \mathfrak{z}(\nu^i)(-t_1t_2)^{l(\nu^i)}\right)\mathsf{GW}^*(g|k_1,k_2)_{\mu^1\ldots\mu^s\nu^1\ldots\nu^t},$$

$$\mathsf{DT}^*(g|k_1,k_2)^{\nu^1\ldots\nu^t}_{\mu^1\ldots\mu^s} = \left(\prod_{i=1}^t \mathfrak{z}(\nu^i)(-t_1t_2)^{l(\nu^i)}\right)\mathsf{DT}^*(g|k_1,k_2)_{\mu^1\ldots\mu^s\nu^1\ldots\nu^t}.$$

With respect to the above metrics, the starred partition function satisfy the same degeneration rules as their unstarred counterparts:

$$\mathsf{GW}^*(g|k_1,k_2)^{\nu^1,\ldots,\nu^t}_{\mu^1,\ldots,\mu^s} = \sum_\gamma \mathsf{GW}^*(g'|k_1',k_2')^\gamma_{\mu^1,\ldots,\mu^s}\mathsf{GW}^*(g''|k_1'',k_2'')^{\nu^1,\ldots,\nu^t}_\gamma,$$

$$\mathsf{DT}^*(g|k_1,k_2)^{\nu^1,\ldots,\nu^t}_{\mu^1,\ldots,\mu^s} = \sum_\gamma \mathsf{DT}^*(g'|k_1',k_2')^\gamma_{\mu^1,\ldots,\mu^s}\mathsf{DT}^*(g''|k_1'',k_2'')^{\nu^1,\ldots,\nu^t}_\gamma,$$

where $g = g' + g''$, and $k_i = k_i' + k_i''$, and

$$\mathsf{GW}^*(g|k_1,k_2)_{\mu^1,\ldots,\mu^s} = \sum_\gamma \mathsf{GW}^*(g-1|k_1,k_2)^\gamma_{\mu^1,\ldots,\mu^s,\gamma},$$

$$\mathsf{DT}^*(g|k_1,k_2)_{\mu^1,\ldots,\mu^s} = \sum_\gamma \mathsf{DT}^*(g-1|k_1,k_2)^\gamma_{\mu^1,\ldots,\mu^s,\gamma}.$$

Hence, tensor functors

$$\mathbf{GW}^*, \mathbf{DT}^*\colon 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}.$$

can be defined just as before.

9.2.3 Matching in level $(0, 0)$ Using the TQFT structure, to prove Theorem 3 in level $(0, 0)$, we must establish the following three equalities:

$$\text{GW}^*(0|0, 0)_\lambda = \text{DT}^*(0|0, 0)_\lambda,$$
$$\text{GW}^*(0|0, 0)_{\lambda\mu} = \text{DT}^*(0|0, 0)_{\lambda\mu},$$
$$\text{GW}^*(0|0, 0)_{\lambda\mu\nu} = \text{DT}^*(0|0, 0)_{\lambda\mu\nu},$$

corresponding respectively to the cap, the tube, and the pair of pants.

The matching of the level $(0, 0)$ cap is a consequence of [4, Lemma 6.2] for the Gromov–Witten side and Lemma 21 of Section 6 for the Donaldson–Thomas side. Similarly, the level $(0, 0)$ tube matching is a consequence of [4, Lemma 6.1] and Lemma 4 of Section 3.

The pair of pants matching in level $(0, 0)$ is more subtle. The main result of the Appendix of [4] is the unique determination of the level $(0, 0)$ TQFT for the local Gromov–Witten theory of curves by the cap, the tube, and the set of series

$$\text{GW}^*(0|0, 0)_{\mu,(2,1^{d-2}),\nu}$$

for all μ and ν. Since the cap and the tube have been shown to match, the equality

(55) $$\text{GW}^*(0|0, 0)_{\mu,(2,1^{d-2}),\nu} = \text{DT}^*(0|0, 0)_{\mu,(2,1^{d-2}),\nu}$$

suffices to complete the matching in level $(0, 0)$. Equality (55) is a consequence of [4, A.3] and Proposition 26 above.   □

10 The cap of level $(-1, 0)$

10.1 TQFT

By [4, Theorem 4.1], the proof of Theorem 3 for all levels now requires only the equality

(56) $$\text{GW}^*(0|-1, 0)_\lambda = \text{DT}^*(0|-1, 0)_\lambda.$$

The Gromov–Witten side was calculated in Bryan–Pandharipande [3; 4]. By [4, Lemma 6.3 and Section 6.4.1],

$$\text{GW}^*(0|-1, 0)_\lambda = (-q)^{-d/2} t_2^{-\ell(\lambda)} \frac{(-1)^{d-\ell(\lambda)}}{\mathfrak{z}(\lambda)} t_2^{-\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-(-q)^{-\lambda_i}}$$

In order to prove (56), we must find the following evaluation:

$$\mathrm{DT}(0|-1,0)_\lambda = (-1)^d (-q)^{-d/2} \mathrm{DT}^*(0|-1,0)_\lambda$$

$$= t_2^{-\ell(\lambda)} \frac{(-1)^{d-\ell(\lambda)}}{\mathfrak{z}(\lambda)} q^{-d} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-(-q)^{-\lambda_i}}$$

$$= \frac{t_2^{-\ell(\lambda)}}{\mathfrak{z}(\lambda)} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-(-q)^{\lambda_i}}.$$

10.2 T–action

Let T be the standard 2–dimensional torus action on the bundle

$$N = \mathcal{O}(-1) \oplus \mathcal{O}$$

over $\mathbf{P}^1$ with scaling weights t_1 and t_2 on the factors $\mathcal{O}(-1)$ and $\mathcal{O}$ respectively. Let the 1–dimensional torus S act with weights $s, -s$ at the fixed points

$$0, \infty \in \mathbf{P}^1$$

and weights $(-s, 0)$ on the fiber of N_∞. We will consider the full

$$\mathbf{T} = S \times T$$

action on N.

10.3 Independence

Let $L \subset N$ denote the **T**–equivariant divisor determined by the summand $\mathcal{O}(-1)$, and let

$$[L] \in A^1_{\mathbf{T}}(N, \mathbb{Q})$$

be the associated class. Let

$$\lambda([L]) = \{\lambda_1([L]), \dots, \lambda_{\ell(\lambda)}([L])\}$$

be a weighted partition of d. The reduced **T**–equivariant Donaldson–Thomas residue integral relative to N_∞,

$$\langle\,|\,\lambda([L])\rangle'_{(-1,0)} = \frac{\langle\,|\,\lambda([L])\rangle_d^{(-1,0)}}{\langle\,|\,\varnothing\rangle_0^{(-1,0)}},$$

is of degree 0 in the equivariant parameters s, t_1, and t_2.

Lemma 27 $\langle\,|\,\lambda([L])\rangle'_{(-1,0)}$ *is independent of* s, t_1, *and* t_2.

Proof Let $N \subset \bar{N}$ be a $\mathbf{T}$–equivariant compactification, and let

$$[P] \in H_2(\bar{N}, \mathbb{Z})$$

be the push-forward to $\bar{N}$ of the class of the zero section $P \subset N$. Consider the restricted moduli space of ideal sheaves

$$R = \epsilon_\infty^{-1}(C_{\lambda[L]}) \subset I_n(\bar{N}/\bar{N}_\infty, d[P]),$$

following the notation of Section 1.2.3. Since the line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^1$ has no nontrivial multisections, the elements of

$$[I_Z] \in R_n$$

for which the entire 1–dimension support of Z lies on P determine an open and closed $\mathbf{T}$–equivariant substack $R_n^0 \subset R_n$

Dimension 0 integrals over R_n^0 are certainly independent of the equivariant parameters s, t_1, and t_2. By localization,

$$\langle\,|\,\lambda([L])\rangle'_{(-1,0)} = \frac{\sum_n q^n \int_{[R_n^0]^{\mathrm{vir}}} 1}{\sum_n q^n \int_{[I_n(\bar{N}/\bar{N}_\infty, 0)]^{\mathrm{vir}}} 1}.$$

Since the denominator on the right is also independent of the equivariant parameters, the Lemma is proven. ☐

10.4 Localization

The $\mathbf{T}$–equivariant virtual localization formula for the series

$$(57) \qquad\qquad \langle\,|\,\lambda([L])\rangle'_{(-1,0)}$$

involves an edge summation over all of the T–fixed points $\mathcal{I}_\mu$ of the Hilbert scheme $\mathrm{Hilb}(N_\infty, d)$. See Maulik et al [24; 25] for a discussion of localization in relative Donaldson–Thomas theory.

We orient the partition μ so that the rows of the associated Young diagram extend in the $\mathcal{O}$ direction. Define $n(\mu)$ by a summation over rows:

$$(58) \qquad\qquad n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i.$$

With our orientation conventions, $d + n(\mu)$ is the Euler characteristic of a pure edge with profile μ.

By an application of the virtual localization formula, we find the series (57) equals

$$
(59) \qquad \sum_{|\mu|=d} \frac{\mathsf{W}(\mu,\varnothing,\varnothing)|_{s,t_1-s,t_2}}{\mathsf{W}(\varnothing,\varnothing,\varnothing)|_{s,t_1-s,t_2}} \cdot q^{n(\mu)} \mathsf{E}^{(-1,0)}(\mu) \cdot \frac{\langle [\mathcal{I}_\mu] \, | \frac{1}{-s-\psi_\infty} \, |\lambda\rangle^\sim}{\langle \varnothing | \frac{1}{-s-\psi_\infty} \, |\varnothing\rangle^\sim} \, t_2^{\ell(\lambda)} .
$$

The terms W and $\mathsf{E}^{(-1,0)}$ are respectively the equivariant vertex and the equivariant edge weight [24; 25]. The rubber integral series in the numerator is

$$
\langle [\mathcal{I}_\mu] \, | \frac{1}{-s-\psi_\infty} \, |\lambda\rangle^\sim = q^d \langle [\mathcal{I}_\mu] \, |\lambda\rangle_{d,d} + \sum_{n>d} q^n \langle [\mathcal{I}_\mu] \, | \frac{1}{-s-\psi_\infty} \, |\lambda\rangle^\sim_{n,d} .
$$

The denominator series has a parallel definition.

By Lemma 27, the series (57) is independent s, t_1, and t_2. Hence the localization formula can be evaluated after specialization of the equivariant parameters.

10.5 Rubber

We evaluate the localization formula (57) after the specialization

$$
(60) \qquad\qquad\qquad\qquad t_1 + t_2 = 0.
$$

By the vanishing of Lemma 13, the rubber integrals on the right side simplify to

$$
\frac{\langle [\mathcal{I}_\mu] | \frac{1}{-s-\psi_\infty} \, |\lambda\rangle^\sim}{\langle \varnothing | \frac{1}{-s-\psi_\infty} \, |\varnothing\rangle^\sim} = q^d \langle [\mathcal{I}_\mu] \, | \lambda\rangle_{d,d} .
$$

The matrix element on the right is the equivariant intersection form of the classical cohomology of $\mathrm{Hilb}(N_\infty, d)$.

For opposite weights (60), the Hilbert scheme intersections reduce to characters of the symmetric group, see for example Vasserot [35]. We find

$$
(61) \qquad\qquad \langle [\mathcal{I}_\mu] \, | \lambda\rangle_{d,d} = \frac{t_2^{d-\ell(\lambda)}}{\mathfrak{z}(\lambda)} \, \chi_\lambda^\mu \prod_{\square \in \mu} h(\square) .
$$

Here, $h(\square)$ denotes the hook length. A similar calculation can be found in Okounkov–Pandharipande [31].

10.6 Edge

The edge term of the localization formula (59) also simplifies after the specialization (57).

We recall the formula for the edge term adapted to the level $(-1, 0)$ geometry (see Maulik et al [24]). Given a partition μ, form the following polynomials

$$Q_\mu(x_1, x_2) = \sum_{(i,j)\in\mu} x_1^i x_2^j,$$

$$F_\mu(x_1, x_2) = -Q_\mu - \frac{\bar{Q}_\mu}{x_1 x_2} + Q_\mu \bar{Q}_\mu \frac{(1-x_1)(1-x_2)}{x_1 x_2}.$$

The sum in the first definition is over the interior corners of the squares of the Young diagram of μ — the corners closest to the origin. Also,

$$\bar{Q}_\mu(x_1, x_2) = Q_\mu(x_1^{-1}, x_2^{-1}).$$

The rational function

$$E_\mu = \frac{F_\mu(x_1, x_2)}{x_0 - 1} + \frac{F_\mu(x_1 x_0, x_2)}{x_0^{-1} - 1}$$

is readily seen to be a Laurent polynomial in the variables x_i. The edge weight $\mathsf{E}(\mu)$ is obtained from the following transformation:

$$(62) \qquad E_\mu = \sum_k a_k x_0^{k_0} x_1^{k_1} x_2^{k_2} \mapsto \mathsf{E}(\mu) = \prod_k (k_0 s + k_1(t_1 - s) + k_2 t_2)^{-a_k}.$$

Setting $t_1 + t_2 = 0$ is equivalent to substituting

$$x_0 = (x_1 x_2)^{-1}$$

in the above formulas.

Lemma 28 *We have*

$$F_\mu(x_1, x_2) = -\sum_{\square\in\mu} \left(x_1^{l(\square)} x_2^{-a(\square)-1} + x_1^{-l(\square)-1} x_2^{a(\square)} \right),$$

where $a(\square)$ and $l(\square)$ denote the arm-lengths and leg-length of a square in a diagram (number of squares to the right and below $\square$, respectively).

Proof The polynomial F_μ is, up to sign, the character of the scaling torus action on the tangent space at $[\mathcal{I}_\mu]$ to the Hilbert scheme of points in $\mathbb{C}^2$. The exponents in the formula are well known to be the weights of tangent action, see for example Haiman [11].   □

Define an auxiliary function

$$G(x_1, x_2) = -\sum_{\square\in\mu} x_1^{l(\square)} x_2^{-a(\square)-1}$$

for which $F = G + \bar{G}/(x_1 x_2)$. We compute

$$
(63) \qquad E_\mu|_{x_3=(x_1x_2)^{-1}} = -G|_{x_1=x_2^{-1}} + \frac{G - G|_{x_1=x_2^{-1}}}{(x_1x_2)^{-1} - 1} - \frac{\bar{G} - \bar{G}|_{x_2=x_1^{-1}}}{(x_1x_2) - 1}.
$$

All three terms of (63) are Laurent polynomials. The third term is minus bar of the second one. By the transformation (62), the factors corresponding to the second and third terms of (63) cancel up to a sign. This sign is the parity of the overall number of monomials in the second term in (63) equal to

$$
\sum_{\square \in \mu} l(\mu) = n(\mu).
$$

We have proven the following result:

$$
E(\mu)^{(-1,0)}|_{s, t_1-s, t_2} = (-1)^{n(\mu)} t_2^{-d} \prod_{\square \in \mu} h(\square)^{-1}.
$$

10.7 Vertex

The **T**–equivariant tangent weights of N at the fixed point over $0 \in \mathbf{P}^1$ satisfy the Calabi–Yau condition after the specialization (60):

$$
s + (t_1 - s) + t_2 = 0.
$$

The vertex $\mathsf{W}(\mu, \varnothing, \varnothing)$ has a rather simple evaluation in the Calabi–Yau case (see Maulik et al [24] and Okounkov–Reshetikhin–Vafa [33]):

$$
\mathsf{W}(\mu, \varnothing, \varnothing)|_{s, t_1-s, t_2} = \sum_\pi (-q)^{|\pi|}.
$$

The sum is over all 3–dimensional partitions π with a single infinite leg in the s direction asymptotic to μ. Here, $|\pi|$ is the renormalized volume — the number of boxes remaining after the infinite leg is removed. In particular,

$$
\mathsf{W}(\varnothing, \varnothing, \varnothing)|_{s, t_1-s, t_2} = M(-q),
$$

a specialization of (6).

Evaluation of the Calabi–Yau vertex is reduced to the enumeration of 3–dimensional partition. The enumeration for the 1–legged vertex is solved in Okounkov–Reshetikhin [32],

$$
(64) \qquad \frac{\mathsf{W}(\mu, \varnothing, \varnothing)|_{s, t_1-s, t_2}}{\mathsf{W}(\varnothing, \varnothing, \varnothing)|_{s, t_1-s, t_2}} = \prod_{\square \in \mu} \frac{1}{1 - (-q)^{h(\square)}},
$$

where the product is over all squares in the Young diagram of μ.

The origin of hook lengths in (64) is the following classical formula (see Macdonald [22]) for the value of the Schur function s_μ at the point $(1, q, q^2, \dots)$,

$$(65) \qquad s_\mu(1, q, q^2, \dots) = q^{n(\mu)} \prod_{\square \in \mu} \frac{1}{1 - q^{h(\square)}}.$$

10.8 Evaluation

Putting all pieces of the localization formula (59) together, we find:

$$\big\langle \lambda([L]) \big| \big\rangle'_{(0,-1)} = \frac{q^d}{\mathfrak{z}(\lambda)} \sum_\mu \chi^\mu_\lambda \, s_\mu(1, -q, (-q)^2, (-q)^3, \dots).$$

By a classical formula in the theory of symmetric functions, $\sum_\mu \chi^\mu_\lambda \, s_\mu$ equals the power sum symmetric function p_λ, see [22]. Since

$$p_k(1, -q, (-q)^2, (-q)^3, \dots) = \frac{1}{1 - (-q)^k},$$

we obtain the following result.

Proposition 29

$$(66) \qquad \big\langle \lambda([L]) \big| \big\rangle'_{(0,-1)} = \frac{q^d}{\mathfrak{z}(\mu)} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1 - (-q)^{\lambda_i}}$$

10.9 Proof of Theorem 3

By definition, we find

$$\big\langle \mu([L]) \big| \big\rangle'_{(0,-1)} = t_2^{\ell(\lambda)} q^d \, \mathsf{DT}(0| - 1, 0).$$

Hence, by Proposition 29,

$$\mathsf{DT}(0| - 1, 0) = \frac{t_2^{-\ell(\lambda)}}{\mathfrak{z}(\mu)} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1 - (-q)^{\lambda_i}}.$$

The matching (56) of the $(-1, 0)$ cap is established, and the proof of Theorem 3 is complete. $\square$

10.10 Proof of Theorem 2

By Theorem 3, the rationality of the Donaldson–Thomas series for local curves is a direct consequence of Theorem 6.4 of Bryan–Pandharipande [4]. The proof of Theorem 2 is complete. □

11 The 1–legged vertex

11.1 Overview

The localization formula for the level $(-1, 0)$ cap together with a differential equation for rubber integrals provides an effective determination of the 1–legged equivariant vertex.

11.2 Differential equation

Consider the following Donaldson–Thomas rubber descendent series:

$$\left\langle \mu \left| \frac{1}{1 - \psi_\infty} \right| v \right\rangle^{\sim} = q^d \langle \mu \mid v \rangle_{d,d} + \sum_{n>d} q^n \left\langle \mu \left| \frac{1}{1 - \psi_\infty} \right| v \right\rangle^{\sim}_{n,d}.$$

Define a operator S on Fock space by the matrix elements

(67) $$\left\langle \mu \left| \mathsf{S} \right| v \right\rangle_{\mathcal{F}} = q^{-d} M(-q)^{t_1+t_2} \left\langle \mu \left| \frac{1}{1 - \psi_\infty} \right| v \right\rangle^{\sim}.$$

By the rubber calculus relation (29),

$$q \frac{d}{dq} q^{-d} \left\langle \mu \left| \frac{1}{1 - \psi_\infty} \right| v \right\rangle^{\sim} = q^{-d} \left\langle \mu \left| \frac{-\sigma_1(F)}{1 - \psi_\infty} \right| v \right\rangle - q^{-d} \left\langle \mu \left| \frac{1}{1 - \psi_\infty} \right| D \cdot v \right\rangle^{\sim}.$$

By the topological recursion relation of Section 4.8,

$$q^{-d} \left\langle \mu \left| \frac{-\sigma_1(F)}{1 - \psi_\infty} \right| v \right\rangle = q^{-2d} \sum_\eta \langle \mu \mid -\sigma_1(F) \mid \eta \rangle \bigwedge_d (\eta, \eta) \left\langle \eta \left| \frac{1}{1 - \psi_\infty} \right| v \right\rangle^{\sim}.$$

Together with Proposition 22, we conclude

(68) $$q \frac{d}{dq} \mathsf{S} = \mathsf{MS} - \mathsf{SM}(0),$$

where $\mathsf{M}(0)$ denotes the q–constant terms of M. The series $\Phi(d)$ drops out of right side of (68).

The differential equation (68) for rubber descendents is almost identical to the quantum differential equation for the Hilbert scheme of points of the plane (see Okounkov–Pandharipande [31]).

Lemma 30 $\langle \varnothing \,|\, \frac{1}{1-\psi_\infty} \,|\, \varnothing \rangle^\sim = M(-q)^{-(t_1+t_2)}$.

Proof The differential equation (68) takes a simple form in degree 0:

$$q\frac{d}{dq}\langle \varnothing \,|\, \mathsf{S} \,|\, \varnothing \rangle_\mathcal{F} = 0.$$

The solution is a constant. The Lemma follows from definition (67). $\square$

By Lemma 30, we may express the matrix elements of S as ratios of Donaldson–Thomas rubber series:

$$\tag{69} \langle \mu \,|\, \mathsf{S} \,|\, \nu \rangle_\mathcal{F} = q^{-d} \frac{\langle \mu \,|\, \frac{1}{1-\psi_\infty} \,|\, \nu \rangle^\sim}{\langle \varnothing \,|\, \frac{1}{1-\psi_\infty} \,|\, \varnothing \rangle^\sim}.$$

11.3 Computation of the 1–legged vertex

Let $\mathsf{W}'(\mu,\varnothing,\varnothing)$ denote the reduced 1–legged vertex,

$$\mathsf{W}'(\mu,\varnothing,\varnothing) = \frac{\mathsf{W}(\mu,\varnothing,\varnothing)}{\mathsf{W}(\mu,\varnothing,\varnothing)}.$$

The localization formula (59),

$$\langle \,|\, \lambda([L])\rangle'_{(-1,0)} =$$

$$\sum_{|\mu|=d} \mathsf{W}'(\mu,\varnothing,\varnothing)|_{s,t_1-s,t_2} \cdot q^{n(\mu)} \mathsf{E}^{(-1,0)}(\mu) \cdot \frac{\langle [\mathcal{I}_\mu] \,|\, \frac{1}{-s-\psi_\infty} \,|\, \lambda \rangle^\sim}{\langle \varnothing \,|\, \frac{1}{-s-\psi_\infty} \,|\, \varnothing \rangle^\sim} t_2^{\ell(\lambda)},$$

has a fixed evaluation given by Proposition 29 independent of the equivariant parameters. If the matrix

$$\tag{70} \mathsf{E}^{(-1,0)}(\mu) \cdot \frac{\langle [\mathcal{I}_\mu] \,|\, \frac{1}{-s-\psi_\infty} \,|\, \lambda \rangle^\sim}{\langle \varnothing \,|\, \frac{1}{-s-\psi_\infty} \,|\, \varnothing \rangle^\sim}$$

can be calculated, the above localization formula may be viewed as a square system of linear equations for the unknown vector $\mathsf{W}'(\mu,\varnothing,\varnothing)$, where μ ranges over all partitions of a fixed size.

The matrix (70) is a product of two factors. The first is an invertible diagonal matrix of edge weights explicitly determined by (62). The second is the operator S written on

the left in the fixed point basis. The operator S is completely determined by the linear differential equation (68). The inverse of S satisfies

$$q\frac{d}{dq}\mathsf{S}^{-1} = \mathsf{M}(0)\,\mathsf{S}^{-1} - \mathsf{S}^{-1}\,\mathsf{M}\,.$$

In particular, S is invertible for q not equal to a root of unity.

An identical argument can be used in the local Gromov–Witten theory of curves to determine 1–partition Hodge integrals from the results of Bryan–Pandharipande [4]. For Calabi–Yau Hodge integrals, the results specialize to the Gopakumar–Mariño–Vafa formula proven in Liu–Liu–Zhou [21] and Okounkov–Pandharipande [29].

Alternatively, both the 1–legged vertex and the 1–partition Hodge integrals can be recovered from an parallel localization formula for the level $(0, 0)$ cap. We leave the details to the reader.

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