

On the classification of gradient Ricci solitons

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We show that the only shrinking gradient solitons with vanishing Weyl tensor and Ricci tensor satisfying a weak integral condition are quotients of the standard ones S^n , $S^{n-1} \times \mathbb{R}$ and \mathbb{R}^n . This gives a new proof of the Hamilton–Ivey–Perelman classification of 3–dimensional shrinking gradient solitons. We also show that gradient solitons with constant scalar curvature and suitably decaying Weyl tensor when noncompact are quotients of \mathbb{H}^n , $\mathbb{H}^{n-1} \times \mathbb{R}$, \mathbb{R}^n , $S^{n-1} \times \mathbb{R}$ or S^n .

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1 Introduction

A Ricci soliton is a Riemannian metric together with a vector field (M, g, X) that satisfies

$$\text{Ric} + \frac{1}{2}L_X g = \lambda g.$$

It is called shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. In the case $X = \nabla f$ the equation can also be written as

$$\text{Ric} + \text{Hess } f = \lambda g$$

and the metric is called a gradient Ricci soliton.

In dimension 2 Hamilton proved that the shrinking gradient Ricci solitons with bounded curvature are S^2 , $\mathbb{R}P^2$ and \mathbb{R}^2 with constant curvature [20]. Ivey proved the first classification result in dimension 3 showing that compact shrinking gradient solitons have constant positive curvature [21]. In the noncompact case Perelman has shown that the 3–dimensional shrinking gradient Ricci solitons with bounded nonnegative sectional curvature are S^3 , $S^2 \times \mathbb{R}$ and \mathbb{R}^3 or quotients thereof [33]. (In Perelman’s paper he also includes the assumption that the manifold is κ –noncollapsed but this assumption is not necessary to the argument; see for example, Chow, Lu and N [15].) The Hamilton–Ivey estimate shows that all 3–dimensional shrinking Ricci solitons with bounded curvature have nonnegative sectional curvature [15, Theorem 6.44]. So

the work of Perelman, Hamilton and Ivey together give the following classification in dimension 3.¹

Theorem 1.1 *The only three dimensional shrinking gradient Ricci solitons with bounded curvature are the finite quotients of \mathbb{R}^3 , $S^2 \times \mathbb{R}$ and S^3 .*

Recently Ni and Wallach [31] have given an alternative approach to proving the classification of 3-dimensional shrinkers which extends to higher dimensional manifolds with zero Weyl tensor. (Every 3-manifold has zero Weyl tensor.) Their argument also requires nonnegative Ricci curvature. Also see Naber's paper [29] for a different argument in the 3-dimensional case. By using a different set of formulas we remove the nonnegative curvature assumption.

Theorem 1.2 *Let (M^n, g, f) be a complete shrinking gradient Ricci soliton of dimension $n \geq 3$ such that $\int_M |\text{Ric}|^2 e^{-f} d\text{vol}_g < \infty$ and $W = 0$ then M is a finite quotient of \mathbb{R}^n , $S^{n-1} \times \mathbb{R}$ or S^n .*

Recall that a result of Morgan [27] implies $e^{-f} d\text{vol}_g$ is a finite measure so as a corollary we obtain a new direct proof of Theorem 1.1 that does not require the Hamilton–Ivey estimate. When M is compact Theorem 1.2 was established using similar techniques by Eminenti, LaNave and Mantegazza in [16].

We note that a shrinking soliton has finite fundamental group by work of the second author [37] and that Naber has shown that it can be made into a gradient soliton by adding an appropriate Killing field to X [29]. In the compact case this was proven by Perelman [32].

If we relax the Weyl curvature condition and instead assume that the scalar curvature is constant we also get a nice general classification.

Theorem 1.3 *Let (M^n, g, f) be a complete gradient Ricci soliton with $n \geq 3$, constant scalar curvature, and $W(\nabla f, \cdot, \cdot, \nabla f) = o(|\nabla f|^2)$, then M is a flat bundle of rank 0, 1 or n over an Einstein manifold.*

Note that the Weyl curvature condition is vacuous when $n = 3$ or M is compact. Moreover, when M is compact or the soliton is steady it is already known that it has to be rigid when the scalar curvature is constant by work of the authors [35]. It is also worth pointing out that the theorem is in a sense optimal. Namely, rigid solitons with zero, one or n -dimensional Euclidean factors have $W(\nabla f, \cdot, \cdot, \nabla f) = 0$. When $n = 3$ the theorem yields the following new result.

¹We thank Ben Chow for pointing this out to us.

Corollary 1 *The only 3–dimensional expanding gradient Ricci solitons with constant scalar curvature are quotients of \mathbb{R}^3 , $\mathbb{H}^2 \times \mathbb{R}$ and \mathbb{H}^3 .*

There are now many examples of nontrivial gradient solitons, but we do not know of any with constant scalar curvature. Moreover, we have shown that any gradient soliton which is homogeneous or has constant scalar curvature and is radially flat (ie $\sec(\nabla f, E) = 0$) is a product of Einstein and Euclidean manifolds [34; 35]. There are a number nontrivial homogeneous expanding Ricci solitons, even in dimension 3 (see Baird and Danielo [1], Lauret [23] and Lott [25]). Unlike the shrinking case, these metrics do not support a *gradient* soliton structure.

Our results follow from considering elliptic equations for various curvature quantities on solitons. While there are well-known Ricci flow versions of a number of these formulas, the elliptic proofs are surprisingly straightforward and give some interesting extra rigidity. For example, by considering the equation for the curvature operator we show that if the *second* eigenvalue of the curvature operator of a shrinking gradient Ricci soliton is nonnegative then the metric has nonnegative curvature operator. This then extends a number of rigidity theorems for nonnegative (or 2–nonnegative) curvature operator (see Böhm and Wilking [4], Cao [8], Naber [29] and Petersen and Wylie [34]).

The paper is organized as follows. We start by deriving the formulas for f –Laplacians of various functions and tensors related to curvature. Next we give our constant scalar curvature characterization. The proof of this result is almost entirely algebraic. By contrast the proof of the theorems for shrinking solitons relies far more heavily on analytic techniques. In the appendix we review the proof of the classification of 2–dimensional solitons giving a proof that follows from an Obata-type characterization of warped product manifolds originally due to Brinkmann [5].

Remark 1.4 In the period between the completion of this paper and its publication there have been a number of related results which we summarize for the reader here. In [9] X Cao, Wang and Zhang give a proof of the classification of locally conformally flat shrinking gradient Ricci solitons assuming a different integral curvature condition than the one considered in this paper. In [39] Z H Zhang shows that locally conformally flat shrinking gradient Ricci soliton must have positive curvature. This, combined with Ni and Wallach’s result, proves the classification of locally conformally flat shrinking gradient Ricci solitons with no additional assumptions. On the other hand, Munteanu and Sesum show in [28] that both the Cao–Wang–Zhu condition and the condition $\int_M |\text{Ric}|^2 e^{-f} d\text{vol}_g < \infty$ are always satisfied for shrinking gradient Ricci solitons. Thus, there are a number of independent proofs of the classification of locally conformally flat shrinking gradient Ricci solitons. More recently, H Cao and Chen [7]

and Catino and Mantegazza [10] independently give the complete classification of locally conformally flat steady and expanding gradient solitons. In particular, these results, combined with the classification of rotationally symmetric shrinking gradient Ricci solitons by Kotschwar [22], gives yet another route to the classification of shrinking gradient Ricci solitons on locally conformally flat manifolds.

2 The f -Laplacian of curvature

In this section we are interested in deriving elliptic equations for the curvature of a gradient Ricci soliton. Let V be a tensor bundle on a gradient Ricci soliton, ∇ be the Riemannian connection on V , and T a self-adjoint operator on V . The X -Laplacian and f -Laplacian of T is the operator

$$\begin{aligned} (\Delta_X T) &= (\Delta T) - (\nabla_X T), \\ (\Delta_f T) &= (\Delta T) - (\nabla_{\nabla f} T), \end{aligned}$$

where Δ is the connection Laplacian induced by ∇ . We are interested in the cases where $V = \wedge^2 M$ and T is the curvature operator \mathcal{R} and where $V = TM$ and T is the $(1, 1)$ Ricci tensor. The formulas in these cases are the following.

Lemma 2.1 *For a gradient Ricci soliton:*

$$\begin{aligned} \Delta_f \mathcal{R} &= 2\lambda \mathcal{R} - \mathcal{R}^2 - \mathcal{R}^\# \\ \Delta_f \text{Ric} &= 2\lambda \text{Ric} - 2 \sum_{i=1}^n R(\cdot, E_i)(\text{Ric}(E_i)) \\ \Delta_f \text{scal} &= 2\lambda \text{scal} - 2|\text{Ric}|^2 \end{aligned}$$

Remark 2.2 The last equation is well known; see [35] for a proof. A similar equation for the Ricci tensor appears in [16]. Some other interesting formulas for the curvature operator of gradient solitons appear in [8].

Remark 2.3 Since Ricci solitons are special solutions to the Ricci flow the above equations can be derived from the parabolic formulas derived by Hamilton [18] for the Ricci flow:

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{R} &= \Delta \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^\# \\ \frac{\partial}{\partial t} \text{Ric} &= \Delta_L \text{Ric} \\ \frac{\partial}{\partial t} \text{scal} &= \Delta \text{scal} + 2|\text{Ric}|^2 \end{aligned}$$

However, we will simply perform the elliptic calculation which is more straightforward (for example no ‘‘Uhlenbeck trick’’ is necessary). We also expect similar calculations will give formulas for elliptic equations which do not come directly from a Ricci flow.

$\mathcal{R}^\#$ is the Lie-Algebra square of \mathcal{R} introduced by Hamilton in [19]. Recall that if we change $\mathcal{R}^\#$ into a (0,4)-tensor its formula is

$$g(\mathcal{R}^\#(X \wedge Y), W \wedge Z) = R^\#(X, Y, Z, W) = 2(B(X, W, Y, Z) - B(X, Z, Y, W)),$$

where
$$B(X, Y, W, Z) = - \sum_{i=1}^n g(R(X, E_i)Y, R(W, E_i)Z)$$

and $\{E_i\}$ is an orthonormal basis of T_pM . It is also convenient to identify $\wedge^2 T_pM$ with $\mathfrak{so}(n)$. Then $\wedge^2 T_pM$ becomes a Lie Algebra and the formula for $\mathcal{R}^\#$ becomes

$$g(\mathcal{R}^\#(U), V) = \sum_{\alpha, \beta} g([\mathcal{R}(\phi_\alpha), \mathcal{R}(\phi_\beta)], U) g([\phi_\alpha, \phi_\beta], V)$$

for any two bivectors U and V , where $\{\phi_\alpha\}$ is an orthonormal basis of $\wedge^2 T_pM$. (See page 186 of Chow and Knopf [14] for the derivation of the equivalence of these two formulas. See Böhm and Wilking [4] and Chen [12] for more about $\mathcal{R}^\#$.)

Before the main calculation we recall some curvature identities for gradient Ricci solitons.

Proposition 1 *For a gradient Ricci soliton:*

(2-1)
$$\nabla \text{scal} = 2\text{div}(\text{Ric}) = 2\text{Ric}(\nabla f)$$

(2-2)
$$(\nabla_X \text{Ric})(Y) - (\nabla_Y \text{Ric})(X) = -R(X, Y)\nabla f$$

(2-3)
$$\nabla_{\nabla f} \text{Ric} + \text{Ric} \circ (\lambda I - \text{Ric}) = R(\cdot, \nabla f)\nabla f + \frac{1}{2}\nabla \cdot \nabla \text{scal}$$

(2-4)
$$\sum_{i=1}^n (\nabla_{E_i} R)(E_i, X, Y) = R(\nabla f, X)Y$$

Proof The proofs of the first three identities can be found in [35] (also see [8]). For the fourth formula consider

$$\begin{aligned} \sum_{i=1}^n (\nabla_{E_i} R)(E_i, X, Y, Z) &= - \sum_{i=1}^n (\nabla_{E_i} R)(Y, Z, X, E_i) \\ &= -(\operatorname{div} R)(Y, Z, X) \\ &= -(\nabla_Y \operatorname{Ric})(Z, X) + (\nabla_Z \operatorname{Ric})(Y, X) \\ &= g(R(Y, Z)\nabla f, X) \\ &= g(R(\nabla f, X)Y, Z) \end{aligned}$$

where in the third line we have used the (contracted) 2nd Bianchi identity and in the fourth line we have used (2-2). □

We are now ready to derive the formula for the f -Laplacian of curvature.

Proof of Lemma 2.1 We will begin the calculation by considering the (0,4)-curvature tensor R . Fix a point p , let X, Y, Z, W be vector fields with $\nabla X = \nabla Y = \nabla Z = \nabla W = 0$ at p and let E_i be normal coordinates at p . Then

$$\begin{aligned} (\Delta R)(X, Y, Z, W) &= \sum_{i=1}^n (\nabla_{E_i, E_i}^2 R)(X, Y, Z, W) \\ &= \sum_{i=1}^n (\nabla_{E_i, X}^2 R)(E_i, Y, Z, W) - (\nabla_{E_i, Y}^2 R)(E_i, X, Z, W) \\ &= \sum_{i=1}^n (\nabla_{X, E_i}^2 R)(E_i, Y, Z, W) - (\nabla_{Y, E_i}^2 R)(E_i, X, Z, W) \\ &\quad + (R_{E_i, X} R)(E_i, Y, Z, W) - (R_{E_i, Y} R)(E_i, X, Z, W) \\ &= \nabla_X (R(\nabla f, Y, Z, W)) - \nabla_Y (R(\nabla f, X, Z, W)) \\ &\quad + \sum_{i=1}^n (R_{E_i, X} R)(E_i, Y, Z, W) - (R_{E_i, Y} R)(E_i, X, Z, W) \\ &= (\nabla_X R)(\nabla f, Y, Z, W) + R(\nabla_X \nabla f, Y, Z, W) \\ &\quad - (\nabla_Y R)(\nabla f, X, Z, W) - R(\nabla_Y \nabla f, X, Z, W) \\ &\quad + \sum_{i=1}^n (R_{E_i, X} R)(E_i, Y, Z, W) - (R_{E_i, Y} R)(E_i, X, Z, W) \end{aligned}$$

where in the fourth line we have applied (2-4). The second Bianchi identity implies

$$(\nabla_{\nabla f} R)(X, Y, Z, W) = (\nabla_X R)(\nabla f, Y, Z) - (\nabla_Y R)(\nabla f, X, Z),$$

and the gradient soliton equation gives

$$R(\nabla_X \nabla f, Y, Z, W) = \lambda R(X, Y, Z, W) - R(\text{Ric}(X), Y, Z, W).$$

So we have

$$\begin{aligned} (\Delta_f R)(X, Y, Z, W) &= 2\lambda R(X, Y, Z, W) - R(\text{Ric}(X), Y, Z, W) + R(\text{Ric}(Y), X, Z, W) \\ &\quad + \sum_{i=1}^n (R_{E_i, X} R)(E_i, Y, Z, W) - (R_{E_i, Y} R)(E_i, X, Z, W). \end{aligned}$$

We now must unravel the terms remaining inside the sum. By definition

$$\begin{aligned} (R_{E_i, X} R)(E_i, Y, Z, W) &= R(E_i, X, R(E_i, Y)Z, W) - R(R(E_i, X)E_i, Y, Z, W) \\ &\quad - R(E_i, R(E_i, X)Y, Z, W) - R(E_i, Y, R(E_i, X)Z, W). \end{aligned}$$

A straightforward calculation involving the Bianchi identity then gives

$$\begin{aligned} \sum_{i=1}^n (R_{E_i, X} R)(E_i, Y, Z, W) - (R_{E_i, Y} R)(E_i, X, Z, W) &= R(\text{Ric}(X), Y, Z, W) - R(\text{Ric}(Y), X, Z, W) \\ &\quad + \sum_{i=1}^n [-2R(X, E_i, R(E_i, Y)Z, W) + 2R(Y, E_i, R(E_i, X)Z, W)] \\ &\quad + \sum_{i=1}^n R(E_i, R(X, Y)E_i, Z, W). \end{aligned}$$

We now have

$$\begin{aligned} (\Delta_f R)(X, Y, Z, W) &= 2\lambda R(X, Y, Z, W) \\ &\quad + 2 \sum_{i=1}^n [-R(X, E_i, R(E_i, Y)Z, W) + R(Y, E_i, R(E_i, X)Z, W)] \\ &\quad + \sum_{i=1}^n R(E_i, R(X, Y)E_i, Z, W). \end{aligned}$$

However,

$$\begin{aligned} \sum_{i=1}^n R(E_i, R(X, Y)E_i, Z, W) &= - \sum_{i=1}^n R(W, Z, E_i, R(X, Y)E_i) \\ &= - \sum_{i=1}^n g(R(W, Z)E_i, R(X, Y)E_i) \\ &= -R^2(X, Y, Z, W) \end{aligned}$$

and

$$\begin{aligned} &2 \sum_{i=1}^n -R(X, E_i, R(E_i, Y)Z, W) + R(Y, E_i, R(E_i, X)Z, W) \\ &= 2 \sum_{i=1}^n -g(R(X, E_i)W, R(Y, E_i)Z) + g(R(Y, E_i)W, R(X, E_i)Z) \\ &= -R^\#(X, Y, Z, W). \end{aligned}$$

So we have obtained the desired formula for the curvature operator.

To compute the formula for the Ricci tensor we could trace the formula for the curvature tensor, or we can give the following direct proof.

Again fix a point p , extend $Y(p)$ to a vector field in a neighborhood of p such that $\nabla Y = 0$, and let E_i be normal coordinates at p . Then

$$\begin{aligned} (\Delta \text{Ric})(Y) &= \sum_{i=1}^n (\nabla_{E_i, E_i}^2 \text{Ric})(Y) \\ &= \sum_{i=1}^n ((\nabla_{E_i, Y}^2 \text{Ric})(E_i) - \nabla_{E_i}(R(E_i, Y)\nabla f)) \\ &= \sum_{i=1}^n ((\nabla_{Y, E_i}^2 \text{Ric})(E_i) - (R_{Y, E_i} \text{Ric})(E_i) \\ &\quad - (\nabla_{E_i} R)(E_i, Y, \nabla f) - R(E_i, Y)(\nabla_{E_i} \nabla f)) \\ &= \nabla_Y(\text{div}(\text{Ric})) + \text{Ric}(\text{Ric}(Y)) + R(Y, \nabla f)\nabla f + \lambda \text{Ric}(Y) \\ &\quad - 2 \sum_{i=1}^n R(Y, E_i)(\text{Ric}(E_i)) \\ &= (\nabla_{\nabla f} \text{Ric})(Y) + 2\lambda \text{Ric}(Y) - 2 \sum_{i=1}^n R(Y, E_i)(\text{Ric}(E_i)) \end{aligned}$$

where in going from the first to second lines we have applied (2-2), in going from the third to fourth lines we apply (2-4) and in obtaining the last line we apply (2-3). \square

From the Ricci equation we can also derive the following formula:

Lemma 2.4

$$\begin{aligned} \Delta_f(\text{Ric}(\nabla f, \nabla f)) &= 4\lambda \text{Ric}(\nabla f, \nabla f) - 2D_{\nabla f}|\text{Ric}|^2 \\ &\quad + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 2 \sum_{i=1}^n R(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{1}{2} \Delta_f(D_{\nabla f} \text{scal}) &= D_{\nabla f} \Delta_f \text{scal} + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 2 \sum_{i=1}^n R(\nabla f, E_i, \text{Ric}(E_i), \nabla f). \end{aligned}$$

Proof From the above equation we get:

$$\begin{aligned} \Delta_f(\text{Ric}(\nabla f, \nabla f)) &= (\Delta_f \text{Ric})(\nabla f, \nabla f) + 2\text{Ric}(\Delta_f \nabla f, \nabla f) + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) \\ &\quad + 4(\nabla_{E_i} \text{Ric})(\nabla_{E_i} \nabla f, \nabla f) \\ &= (\Delta_f \text{Ric})(\nabla f, \nabla f) - 2\lambda \text{Ric}(\nabla f, \nabla f) + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) \\ &\quad + 4\lambda(\nabla_{E_i} \text{Ric})(E_i, \nabla f) - 4(\nabla_{E_i} \text{Ric})(\text{Ric}(E_i), \nabla f) \\ &= -2R(\nabla f, E_i, \text{Ric}(E_i), \nabla f) + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 4\lambda \text{Ric}(\nabla f, \nabla f) \\ &\quad - 4(\nabla_{\nabla f} \text{Ric})(\text{Ric}(E_i), E_i) + 4R(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \\ &= 4\lambda \text{Ric}(\nabla f, \nabla f) - 2D_{\nabla f}|\text{Ric}|^2 \\ &\quad + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 2R(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \end{aligned}$$

The second formula follows from:

$$\begin{aligned} \Delta_f \text{scal} &= 2\lambda \text{scal} - 2|\text{Ric}|^2 \\ 2\text{Ric}(\nabla f, \nabla f) &= D_{\nabla f} \text{scal} \quad \square \end{aligned}$$

We are now going to see how the Weyl decomposition affects the formula for the Ricci tensor.

Lemma 2.5 *The following relationships hold:*

$$\begin{aligned} \Delta_f \text{Ric} &= 2\lambda \text{Ric} - \frac{2n \text{scal}}{(n-1)(n-2)} \text{Ric} + \frac{4}{n-2} \text{Ric}^2 \\ &\quad - \frac{2}{(n-2)} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) I + W(\cdot, E_i, \text{Ric}(E_i)) \\ \frac{1}{2} \Delta_f (D_{\nabla f} \text{scal}) &= D_{\nabla f} \Delta_f \text{scal} + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) \\ &\quad + \frac{2n \text{scal}}{(n-1)(n-2)} \text{Ric}(\nabla f, \nabla f) - \frac{4}{n-2} \text{Ric}(\text{Ric}(\nabla f), \nabla f) \\ &\quad + \frac{2}{(n-2)} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) |\nabla f|^2 + W(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \end{aligned}$$

Proof The Weyl decomposition looks like

$$\begin{aligned} R &= W + \frac{1}{n-2} \text{Ric} \circ g - \frac{\text{scal}}{2(n-1)(n-2)} g \circ g, \\ h \circ g(x, y, y, x) &= h(x, x)g(y, y) + h(y, y)g(x, x) - 2h(x, y)g(x, y) \end{aligned}$$

where W is absent when $n = 3$. More specifically we need

$$\begin{aligned} R(x, y, y, x) &= \frac{1}{n-2} (\text{Ric}(x, x)g(y, y) + \text{Ric}(y, y)g(x, x) - 2\text{Ric}(x, y)g(x, y)) \\ &\quad - \frac{\text{scal}}{(n-1)(n-2)} (|x|^2|y|^2 - (g(x, y))^2) + W(x, y, y, x). \end{aligned}$$

If we assume that E_i is an orthonormal frame that diagonalizes the Ricci tensor $\text{Ric}(E_i) = \rho_i E_i$, then

$$\begin{aligned} \text{Ric}(Y) &= g(Y, E_i)\rho_i E_i \\ \text{Ric}(Y, Y) &= \rho_i (g(Y, E_i))^2 \\ \text{Ric}(Y, \text{Ric}(Y)) &= \text{Ric}(Y, g(Y, E_i)\rho_i E_i). \end{aligned}$$

Using this the Weyl free part of the formula for

$$R(Y, E_i, \text{Ric}(E_i), Y) = \rho_i R(Y, E_i, E_i, Y)$$

becomes

$$\begin{aligned} & \frac{1}{n-2}(\text{scal} \cdot \text{Ric}(Y, Y) + \rho_i^2 |Y|^2 - 2\text{Ric}(Y, \rho_i E_i)g(Y, E_i)) \\ & \qquad - \frac{\text{scal}}{(n-1)(n-2)}(|Y|^2 \text{scal} - \rho_i(g(Y, E_i))^2) \\ & = \frac{1}{n-2}(\text{scal} \cdot \text{Ric}(Y, Y) + |\text{Ric}|^2 |Y|^2 - 2\text{Ric}(Y, \text{Ric}(Y))) \\ & \qquad + \frac{\text{scal}}{(n-1)(n-2)}(\text{Ric}(Y, Y) - |Y|^2 \text{scal}) \\ & = \frac{1}{n-2} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) |Y|^2 - \frac{2}{n-2} \text{Ric}(Y, \text{Ric}(Y)) \\ & \qquad + \left(\frac{\text{scal}}{(n-1)(n-2)} + \frac{\text{scal}}{n-2} \right) \text{Ric}(Y, Y) \\ & = \frac{1}{n-2} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) |Y|^2 - \frac{2}{n-2} \text{Ric}(Y, \text{Ric}(Y)) + \frac{n\text{scal}}{(n-1)(n-2)} \text{Ric}(Y, Y). \end{aligned}$$

This establishes the first formula and the second by using $Y = \nabla f$. □

3 Constant scalar curvature

We now turn our attention to the case where scal is constant in dimensions $n \geq 3$.

Recall the following results from [35, Propositions 5 and 7].

Proposition 2 *Assume that we have a shrinking (resp. expanding) gradient soliton*

$$\text{Ric} + \text{Hess } f = \lambda g$$

with constant scalar curvature. Then $0 \leq \text{scal} \leq n\lambda$ (resp. $n\lambda \leq \text{scal} \leq 0$.) Moreover, the metric is flat when $\text{scal} = 0$ and Einstein when $\text{scal} = n\lambda$. In addition f is unbounded when M is noncompact and $\text{scal} \neq n\lambda$.

This in conjunction with the above formulas allow us to prove:

Theorem 3.1 *Any gradient soliton which has constant scalar curvature, $\lambda \neq 0$ and $W(\nabla f, \cdot, \cdot, \nabla f) = o(|\nabla f|^2)$ is rigid.*

Proof We can assume that M is noncompact and that f is unbounded. The fact that the scalar curvature is constant in addition shows that

$$0 = \Delta_f \text{scal} = \lambda \text{scal} - |\text{Ric}|^2$$

and from the formula for $\Delta_f \text{Ric}(\nabla f, \nabla f)$ we get

$$\begin{aligned} 0 &= \frac{1}{2} \Delta_f (D_{\nabla f} \text{scal}) \\ &= D_{\nabla f} \Delta_f \text{scal} + 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 2W(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \\ &\quad + \frac{2n\text{scal}}{(n-1)(n-2)} \text{Ric}(\nabla f, \nabla f) - \frac{4}{n-2} \text{Ric}(\text{Ric}(\nabla f), \nabla f) \\ &\quad + \frac{2}{(n-2)} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) |\nabla f|^2 \\ &= 2\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) + 2W(\nabla f, E_i, \text{Ric}(E_i), \nabla f) \\ &\quad + \frac{2}{(n-2)} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) |\nabla f|^2 \end{aligned}$$

Since $|\text{Ric}|^2 = \lambda \text{scal}$ is constant we see that both Ric and Hess f are bounded. Thus $\text{Ric}(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f)$ is bounded and $W(\nabla f, E_i, \text{Ric}(E_i), \nabla f) = o(|\nabla f|^2)$. Recall that

$$\text{scal} + |\nabla f|^2 - 2\lambda f = \text{const}$$

so if the scalar curvature is constant and f is unbounded we see that $|\nabla f|^2$ is unbounded. This implies that

$$|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} = 0$$

as it is constant. We know in addition that Ric has one zero eigenvalue when $\nabla f \neq 0$, so in that case the Cauchy–Schwarz inequality shows that

$$|\text{Ric}|^2 \geq \frac{\text{scal}^2}{n-1}$$

with equality holding only if all the other eigenvalues are the same.

If ∇f vanishes on an open set, then the metric is Einstein on that set, in particular $\text{scal} = n\lambda$ everywhere and so the entire metric is Einstein. This means that we can assume $\nabla f \neq 0$ on an open dense set. Thus Ric has a zero eigenvalue everywhere and the other eigenvalues are given by the constant

$$\rho = \frac{\text{scal}}{n-1}.$$

But by [Corollary 2](#) which we will prove below this implies that $\widetilde{M} = N^{n-1} \times \mathbb{R}$ where N is Einstein if $n > 3$. When $n = 3$, N is a surface and so must also have constant curvature if M does. □

We now prove that a gradient Ricci soliton whose Ricci curvature has one nonzero eigenvalue of multiplicity $n - 1$ at every point must split. This will follow from the following more general lemma.

Lemma 3.2 *Let T be a constant rank, symmetric, nonnegative tensor on some (tensor) bundle. If $g((\Delta_X T)(s), s) \leq 0$ for $s \in \ker T$, then the kernel is a parallel subbundle.*

Proof We are assuming that $\ker T$ is a subbundle. Select an orthonormal frame E_1, \dots, E_n and let s be section of $\ker T$. First note that

$$(\Delta_X T)(s) = \Delta_X(T(s)) - 2 \sum_{i=1}^n ((\nabla_{E_i} T)(\nabla_{E_i} s)) + T(\Delta_X s)$$

so from the hypothesis we have

$$\begin{aligned} 0 \geq g((\Delta_X T)(s), s) &= -2 \sum_{i=1}^n g((\nabla_{E_i} T)(\nabla_{E_i} s), s) + g(T(\Delta_X s), s) \\ &= -2 \sum_{i=1}^n g(\nabla_{E_i} s, (\nabla_{E_i} T)(s)) + g(\Delta_X s, T(s)) \\ &= -2 \sum_{i=1}^n g(\nabla_{E_i} s, (\nabla_{E_i} T)(s)) \\ &= 2 \sum_{i=1}^n g(\nabla_{E_i} s, T(\nabla_{E_i} s)). \end{aligned}$$

The nonnegativity of T then gives that $\nabla s \in \ker T$. □

Corollary 2 *Let (M, g, f) be a gradient Ricci soliton such that, at each point, the Ricci tensor has one nonzero eigenvalue of multiplicity $n - 1$, then $\tilde{M} = N^{n-1} \times \mathbb{R}$. Moreover, if $n > 3$ then N is Einstein.*

Proof Let E_1, \dots, E_n be an orthonormal frame such that $\text{Ric}(E_1) = 0$ and $\text{Ric}(E_i) = \rho E_i$ for $i > 1$. Then

$$\begin{aligned} (\Delta_f \text{Ric})(Y) &= 2\lambda \text{Ric}(Y) - 2 \sum_{i=1}^n R(Y, E_i) \text{Ric}(E_i) \\ &= 2\lambda \text{Ric}(Y) - 2\rho \sum_{i=2}^n R(Y, E_i) E_i \\ &= 2(\lambda - \rho) \text{Ric}(Y) + 2\rho R(Y, E_1) E_1. \end{aligned}$$

Since this vanishes on E_1 we see that the previous lemma can be applied. □

4 Shrinkers

For gradient shrinking solitons we use an approach due to Naber [29, Section 7]. There is a natural measure $e^{-f} d\text{vol}_g$ which makes the f -Laplacian self-adjoint. From the perspective of comparison geometry the tensor $\text{Ric} + \text{Hess} f$ is the Ricci tensor for this measure and Laplacian. (see eg Lichnerowicz [24], Morgan [27] and Wei and Wylie [36]). In particular for a shrinking soliton the measure must be bounded above by a Gaussian measure, note that no assumption on the boundedness of Ricci curvature is necessary.

Lemma 4.1 [27; 36] *On a shrinking gradient Ricci soliton the measure $e^{-f} d\text{vol}_g$ is finite and if $u = O(e^{\alpha d^2(\cdot, p)})$ for some $\alpha < \lambda/2$ and fixed point p then $u \in L^2(e^{-f} d\text{vol}_g)$.*

In [29] Naber combines a similar volume comparison with a refinement of a Liouville theorem of Yau [38, Theorem 3]. We will apply the Liouville theorem to nonsmooth functions such as the smallest eigenvalue of the Ricci tensor so we need to refine these arguments further.

Theorem 4.2 (Yau–Naber Liouville Theorem) *Let (M, g, f) be a manifold with finite f -volume: $\int e^{-f} d\text{vol} < \infty$. If u is a locally Lipschitz function in $L^2(e^{-f} d\text{vol}_g)$ which is bounded below such that*

$$\Delta_f(u) \geq 0$$

in the sense of barriers, then u is constant.

Proof Note that since the measure is finite and u is bounded from below we can assume u is positive by adding a suitable constant to u .

To prove the theorem we must modify slightly the techniques of Yau and Naber. First we apply a heat kernel smoothing procedure of Greene and Wu [17, Section 3].

Let K be a smooth compact subset of M and let $U(x, t)$ be the solution to the equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_f\right)U &= 0 \\ U(x, 0) &= \tilde{u}(x) \end{aligned}$$

on the double of a smooth open set that contains K where \tilde{u} is a continuous extension of u to the larger open set. Then, by the standard theory, U_t is a smooth function that

converges in $W^{1,2}(K)$ to u as $t \rightarrow 0$. Moreover, Green and Wu show that given $\varepsilon > 0$ there is t_0 such that for all $t < t_0$

$$\Delta_f(U(\cdot, t)) \geq -\varepsilon,$$

on K .

Now to the proof of the theorem. Let $x \in M$ and $r_k \rightarrow \infty$. Using the procedure described above we construct smooth functions u_k such that

$$\begin{aligned} |u_k - u|_{(W^{1,2}(B(x, r_k+1)))} &< \frac{1}{k}, \\ \Delta_f(u_k) &\geq -\frac{1}{k}. \end{aligned}$$

Let ϕ_k be a cut-off function which is 1 on $B(x, 1)$, 0 outside of $B(x, r_k + 1)$, and has $|\nabla\phi_k| \leq 2/r_k$. First we integrate by parts:

$$\begin{aligned} \int_M \Delta_f(u_k)\phi_k^2 u_k (e^{-f} d\text{vol}_g) \\ = - \int_M 2\phi_k u_k g(\nabla u_k, \nabla\phi_k)(e^{-f} d\text{vol}_g) - \int_M \phi_k^2 |\nabla u_k|^2 (e^{-f} d\text{vol}_g) \end{aligned}$$

Then we complete the square

$$\left| \sqrt{\frac{1}{2}}\phi_k \nabla u_k + \sqrt{2}u_k \nabla\phi_k \right|^2 = 2\phi_k u_k g(\nabla u_k, \nabla\phi_k) + \frac{1}{2}\phi_k^2 |\nabla u_k|^2 + 2u_k^2 |\nabla\phi_k|^2$$

to obtain

$$\begin{aligned} \int_M \Delta_f(u_k)\phi_k^2 u_k (e^{-f} d\text{vol}_g) \\ \leq -\frac{1}{2} \int_{B(x, r_k+1)} \phi_k^2 |\nabla u_k|^2 (e^{-f} d\text{vol}_g) + 2 \int_M u_k^2 |\nabla\phi_k|^2 (e^{-f} d\text{vol}_g). \end{aligned}$$

On the other hand

$$\begin{aligned} \int_M \Delta_f(u_k)\phi_k^2 u_k (e^{-f} d\text{vol}_g) &\geq -\frac{1}{k} \int_M \phi_k^2 u_k (e^{-f} d\text{vol}_g) \\ &\geq -\frac{2}{k} \int_{B(x, r_k+1)} u (e^{-f} d\text{vol}_g). \end{aligned}$$

So we have

$$\begin{aligned} \frac{1}{2} \int_{B(x,1)} |\nabla u_k|^2 (e^{-f} d\text{vol}_g) \\ \leq \frac{8}{r_k^2} \int_{B(x,r_k+1)} u^2 (e^{-f} d\text{vol}_g) + \frac{2}{k} \int_{B(x,r_k+1)} u (e^{-f} d\text{vol}_g). \end{aligned}$$

Note that, since the volume is finite, $u \in L^2(e^{-f} d\text{vol}_g)$ implies $u \in L^1(e^{-f} d\text{vol}_g)$ so the right hand side will go to zero as $k \rightarrow \infty$. Taking the limit and using that u_k converge to u in $W^{1,2}$ we obtain

$$\int_{B(x,1)} |\nabla u|^2 (e^{-f} d\text{vol}_g) = 0.$$

Which implies u is constant since it is continuous. □

Remark 4.3 One consequence of this theorem is that if a gradient shrinking soliton has $\text{scal} \in L^2(e^{-f} d\text{vol}_g)$ then either $\text{scal} > 0$ or the metric is flat (see [35]).

We can also apply the Yau–Naber Liouville theorem to obtain a strong minimum principle for tensors. The strong minimum principle for tensors in the parabolic setting were developed for the study of Ricci flow by Hamilton [19].

Theorem 4.4 (Tensor Minimum Principle) *Let (M, g, f) be a manifold with finite f -volume: $\int e^{-f} d\text{vol} < \infty$, and T a symmetric tensor on some (tensor) bundle such that $|T| \in L^2(e^{-f} d\text{vol}_g)$ and*

$$\Delta_f T = \lambda T + \Phi(T) \quad \text{where } g(\Phi(T)(s), s) \leq 0 \text{ and } \lambda > 0,$$

then T is nonnegative and $\ker(T)$ is parallel.

Note that if T and $\Phi(T)$ are nonnegative then the Bochner formula shows that T is parallel.

Proof As long as T is nonnegative and has constant rank Lemma 3.2 shows that $\ker(T)$ is parallel.

Denote the eigenvalues of T by $\lambda_1 \leq \lambda_2 \leq \dots$. Let s be a unit field such that $T(s) = \lambda_1 s$ at p otherwise extended by parallel translation along geodesics emanating from p . We can then calculate at $p \in M$

$$\begin{aligned} \Delta_f \lambda_1 &\leq \Delta_f g(T(s), s) \\ &= g((\Delta_f T)(s), s) \\ &= \lambda g(T(s), s) + g(\Phi(T)(s), s) \\ &\leq \lambda \lambda_1 \end{aligned}$$

where the first inequality is in the barrier sense of Calabi [6]. Thus the first eigenvalue satisfies the differential inequality

$$\Delta_f \lambda_1 \leq \lambda \lambda_1$$

everywhere in the barrier sense. A similar analysis where we minimize over k dimensional subspaces at a point shows that

$$\Delta_f (\lambda_1 + \dots + \lambda_k) \leq \lambda (\lambda_1 + \dots + \lambda_k)$$

in the barrier sense.

To see that T is nonnegative let $u = \min \{\lambda_1, 0\}$, then $u \leq 0$,

$$\Delta_f u \leq 0$$

in the sense of barriers, and, since $|T| \in L^2(e^{-f} d\text{vol}_g)$, so is u . The Yau–Naber Liouville Theorem then implies u is constant. In other words, either $\lambda_1 \geq 0$ or λ_1 is constant and less than 0. However, this last case is impossible since if λ_1 is constant

$$0 = \Delta_f \lambda_1 \leq \lambda \lambda_1.$$

Knowing that $\lambda_1 + \dots + \lambda_k \geq 0$, now allows us to apply the strong minimum principle to show that, if $\lambda_1 + \dots + \lambda_k$ vanishes at some point, then it vanishes everywhere (see McOwen [26, page 244]). Since $\dim(\ker(T))$ is the largest k such that $\lambda_1 + \dots + \lambda_k$ vanishes this shows that the kernel is a distribution. \square

We now apply the minimum principle to our formulas for the f -Laplacian of curvature. When $T = \mathcal{R}$ we have $\Phi(\mathcal{R}) = -\mathcal{R}^2 - \mathcal{R}^\#$. Since \mathcal{R}^2 is always nonnegative we see from the minimum principle that the curvature operator of a gradient shrinking soliton is nonnegative if $\mathcal{R}^\#$ is nonnegative. In fact, by examining the proof of the minimum principle we can also obtain the result alluded to in the introduction. As notation, let

$$\lambda_1 \leq \lambda_2 \leq \dots$$

be the ordering of the eigenvalues of the curvature operator.

Corollary 3 *Let (M, g, f) be a shrinking gradient Ricci soliton with $\lambda_2 \geq 0$ and $|\mathcal{R}| \in L^2(e^{-f} d\text{vol}_g)$ then $\mathcal{R} \geq 0$, $\ker \mathcal{R}$ is parallel, and the holonomy algebra*

$$\mathfrak{hol}_p = \text{im}(\mathcal{R}: \wedge^2 T_p M \rightarrow \wedge^2 T_p M).$$

Proof Fix a point p and let ϕ_1 be a parallel bivector such that

$$g(\mathcal{R}(\phi_1), \phi_1) = \lambda_1$$

at p . Then, from the same argument as in the proof of the Tensor Minimum principle, we obtain

$$\Delta_f \lambda_1 \leq \lambda \lambda_1 - g(\mathcal{R}^\#(\phi_1), \phi_1)$$

Let $\{\phi_\alpha\}$ be a basis of orthonormal eigenvectors for \mathcal{R} . The structure constants of the Lie algebra are $C_{\alpha\beta\gamma} = g([\phi_\alpha, \phi_\beta], \phi_\gamma)$, which are fully antisymmetric in α, β and γ . Then if $\lambda_2 \geq 0$,

$$\begin{aligned} g(\mathcal{R}^\#(\phi_1), \phi_1) &= \sum_{\alpha, \beta} (C_{1\alpha\beta})^2 \lambda_\alpha \lambda_\beta \\ &= \sum_{\alpha, \beta \geq 2} (C_{1\alpha\beta})^2 \lambda_\alpha \lambda_\beta \\ &\geq 0. \end{aligned}$$

Thus we see that $\lambda_1 \geq 0$. Next the tensor minimum principle can be applied to see that $\ker \mathcal{R}$ is parallel. This shows in turn that the orthogonal complement $\text{im} \mathcal{R}$ is parallel. The Ambrose–Singer theorem on holonomy then implies the last claim (see Besse [2].) □

Remark 4.5 In dimension 3 this implies that a gradient shrinking soliton with nonnegative Ricci curvature has nonnegative curvature operator.

Remark 4.6 There are simple examples of manifolds with $\lambda_2 \geq 0$ that do not admit 2–nonnegative curvature operator metrics or even nonnegative Ricci curvature metrics. Consider the product $N \times M$ where N is a negatively curved surface and M a possibly one dimensional manifold with nonnegative curvature operator. Then $\lambda_1 < 0$ and $\lambda_2 = 0$. If $\text{scal}_M > |\text{scal}_N|$, then the metric will also have positive scalar curvature.

In the formula for f –Laplacian of the Ricci tensor we have $\Phi(\text{Ric})$ is $-2\mathcal{K}$ where

$$\mathcal{K} = \sum_{i=1}^n g(R(\cdot, E_i)(\text{Ric}(E_i))).$$

If we let $\{E_i\}$ be a basis of eigenvectors for Ric with eigenvalues ρ_i . Then

$$g(\mathcal{K}(Y), Y) = \sum_{i=1}^n \rho_i \sec(Y, E_i)$$

so that $\mathcal{K} \geq 0$ if M has nonnegative (or nonpositive) sectional curvature, or if M is Einstein. The minimum principle gives the following splitting theorem for shrinking solitons with $\mathcal{K} \geq 0$. This is the soliton version of a result of Böhm and Wilking [3] which states that any compact manifold with nonnegative sectional curvature and finite

fundamental group flows in a short time under the Ricci flow to a metric with positive Ricci curvature.

Corollary 4 *Let (M, g, f) be a shrinking gradient Ricci soliton with $\mathcal{K} \geq 0$ and $|\text{Ric}| \in L^2(e^{-f} d\text{vol}_g)$ then $\tilde{M} = N \times \mathbb{R}^k$ where N has positive Ricci curvature. In particular, a compact shrinking soliton with $\mathcal{K} \geq 0$ has positive Ricci curvature.*

Proof The minimum principle and the de Rham splitting theorem show that $\tilde{M} = N \times F$, where N has positive Ricci curvature and F is Ricci flat. From [34] we get that both N and F are gradient solitons. Finally Ricci flat solitons are Gaussians, thus proving the corollary. The last bit about compact manifolds follows from the fact that shrinking solitons have finite volume and hence finite fundamental group. \square

We now turn our attention to the proof of Theorem 1.2. We assume that M is a nonflat, gradient shrinking soliton with $W = 0$ and $|\text{Ric}| \in L^2(e^{-f} d\text{vol}_g)$. Recall that when $W = 0$

$$\Delta_f \text{Ric} = 2\lambda \text{Ric} - \frac{2n \text{scal}}{(n-1)(n-2)} \text{Ric} + \frac{4}{n-2} \text{Ric}^2 - \frac{2}{(n-2)} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) I.$$

Let $\rho_1 \leq \dots \leq \rho_n$ be the eigenvalues of Ric and E a unit field such that $\text{Ric}(E) = \rho_1 E$ at $p \in M$ and extend it to be parallel along geodesics emanating from p . Clearly $\rho_1 \leq \text{Ric}(E, E)$ with equality at p . Calculating at p we have

$$\begin{aligned} \Delta_f(\rho_1) &\leq \Delta_f \text{Ric}(E, E) \\ &= (\Delta_f \text{Ric})(E, E) \\ &= 2\lambda \rho_1 - \frac{2n \text{scal}}{(n-1)(n-2)} \rho_1 + \frac{4}{n-2} \rho_1^2 - \frac{2}{n-2} \left(|\text{Ric}|^2 - \frac{\text{scal}^2}{n-1} \right) \end{aligned}$$

where $\Delta_f(\rho_1)$ is interpreted as being in the upper barrier sense of [6].

The ratio ρ_1/scal then satisfies

$$\begin{aligned} \Delta_h \left(\frac{\rho_1}{\text{scal}} \right) &\leq 2\phi, \\ h &= f - \log(\text{scal}^2), \end{aligned}$$

where

$$\phi = \frac{\rho_1^2(n\rho_1 - \text{scal})}{(n-1)\text{scal}^2} + \frac{((n-2)\rho_1 - \text{scal})((n-1) \sum_{j=2}^n (\rho_j)^2 - (\sum_{j=2}^n \rho_j)^2)}{(n-1)(n-2)\text{scal}^2}$$

which is clearly nonpositive. (Recall that $\text{scal} > 0$ by Remark 4.3.)

We now have $\rho_1/\text{scal} \leq 1$ and $\Delta_h(\rho_1/\text{scal}) \leq 0$, so to apply the Yau–Naber Liouville Theorem, we must show the measure is finite and the function is in $L^2(e^{-h} d\text{vol}_g)$. This is clear from [Lemma 4.1](#) because

$$\int_M e^{-h} d\text{vol}_g = \int_M \text{scal}^2 e^{-f} d\text{vol}_g < \infty,$$

$$\int_M \left(\frac{\rho_1}{\text{scal}}\right)^2 e^{-h} d\text{vol}_g = \int_M (\rho_1)^2 e^{-f} d\text{vol}_g < \infty.$$

Thus ρ_1/scal is constant. In particular, ϕ must vanish and

$$\rho_1^2(n\rho_1 - \text{scal}) = 0,$$

$$((n-2)\rho_1 - \text{scal}) \left((n-1) \sum_{j=2}^n (\rho_j)^2 - \left(\sum_{j=2}^n \rho_j \right)^2 \right) = 0.$$

The first equation tells us that either $\rho_1 = 0$ or M is Einstein. When $\rho_1 = 0$ the second equation and the Cauchy–Schwarz inequality tells us that

$$\rho_2 = \rho_3 = \dots = \rho_n = \frac{\text{scal}}{n-1} > 0.$$

Then by [Corollary 2](#) the universal cover of M splits $\tilde{M} = N \times \mathbb{R}$ where N is again a shrinking gradient soliton with a Ricci tensor that has only one eigenvalue. When $n = 3$ Hamilton’s classification of surface solitons (see [Appendix A](#)) then shows that N is the standard sphere, while if $n > 3$ Schur’s lemma shows that N is Einstein.

Appendix A Surface gradient solitons

In the literature the classification of shrinking surface solitons is usually stated for metrics with bounded curvature. However, we have used this classification under the weaker condition $\text{scal} \in L^2(e^{-f} d\text{vol}_g)$. In this appendix we verify that the classification still holds in this case.

We consider warped product metrics, a slightly larger class of metrics than rotationally symmetric ones.

Definition A.1 A Riemannian metric (M, g) on either \mathbb{R}^n , S^n , or $N \times \mathbb{R}$ is a warped product if it can be written as

$$g = dr^2 + h^2(r)g_0.$$

When $M = \mathbb{R}^n$, we assume that $h(0) = 0$ and g_0 is the standard metric on the sphere. When $M = S^n$, we require $h(0) = h(r_0) = 0$ and g_0 is the standard metric on the sphere.

There is a very simple Obata-type characterization of warped product metrics originally due to Brinkmann [5]. (Also see 9.117 of Besse [2] and Cheeger and Colding [11].)

Theorem A.2 (Brinkmann) *A Riemannian manifold (M, g) is a warped product if and only if there is a nontrivial function f such that*

$$\text{Hess } f = \mu g$$

for some function $\mu: M \rightarrow \mathbb{R}$.

Remark A.3 This theorem indicates that any flow that preserves conformal classes has the property that the corresponding gradient solitons must be rotationally symmetric. This then makes it possible to classify all complete gradient solitons for such flows.

Corollary 5 *Any surface gradient Ricci soliton is a warped product.*

Proof Simply use that $\text{Ric} = (\text{scal}/2)g$. □

So the problem of finding surface gradient solitons is reduced to determining which functions $h(r)$ give a soliton. For example, Hamilton's cigar is obtained by taking $h(r) = \tanh(r)$ and is the unique (up to scaling) noncompact steady gradient soliton surface with positive curvature. For a nontrivial example of an expanding surface gradient soliton see Chow, Lu and Ni [15, pages 164–167].

Now suppose we have a nonflat shrinking soliton on a surface with $\text{scal} \in L^2(e^{-f} d\text{vol}_g)$. As we have seen this implies $\text{scal} > 0$. Moreover, since we are on a surface, the Ricci curvature is positive so Proposition 1.1 in [30] implies the scalar curvature is bounded away from zero. Thus M is compact. Then, since it is also a warped product, M must be a rotationally symmetric metric on the sphere. Chen, Lu and Tian show that this implies M is a round sphere [13].

References

- [1] **P Baird, L Daniello**, *Three-dimensional Ricci solitons which project to surfaces*, J. Reine Angew. Math. 608 (2007) 65–91 [MR2339469](#)
- [2] **A L Besse**, *Einstein manifolds*, Ergebnisse der Math. und ihrer Grenzgebiete (3) 10, Springer, Berlin (1987) [MR867684](#)

- [3] **C Böhm, B Wilking**, *Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature*, *Geom. Funct. Anal.* 17 (2007) 665–681 [MR2346271](#)
- [4] **C Böhm, B Wilking**, *Manifolds with positive curvature operators are space forms*, *Ann. of Math. (2)* 167 (2008) 1079–1097 [MR2415394](#)
- [5] **H W Brinkmann**, *Einstein spaces which are mapped conformally on each other*, *Math. Ann.* 94 (1925) 119–145 [MR1512246](#)
- [6] **E Calabi**, *An extension of E Hopf's maximum principle with an application to Riemannian geometry*, *Duke Math. J.* 25 (1958) 45–56 [MR0092069](#)
- [7] **H-D Cao**, *Recent progress on Ricci solitons*, from: “Recent advances in geometric analysis”, (Y-I Lee, C-S Lin, M-P Tsui, editors), *Adv. Lect. Math.* 11, Int. Press, Somerville, MA (2010) 1–38 [MR2648937](#)
- [8] **X Cao**, *Compact gradient shrinking Ricci solitons with positive curvature operator*, *J. Geom. Anal.* 17 (2007) 425–433 [MR2358764](#)
- [9] **X Cao, B Wang, Z Zhang**, *On locally conformally flat gradient shrinking Ricci solitons* [arXiv:0807.0588v3](#)
- [10] **G Catino, C Mantegazza**, *Evolution of the Weyl tensor under the Ricci flow* [arXiv:0910.4761v4](#)
- [11] **J Cheeger, T H Colding**, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, *Ann. of Math. (2)* 144 (1996) 189–237 [MR1405949](#)
- [12] **H Chen**, *Pointwise $\frac{1}{4}$ -pinched 4-manifolds*, *Ann. Global Anal. Geom.* 9 (1991) 161–176 [MR1136125](#)
- [13] **X Chen, P Lu, G Tian**, *A note on uniformization of Riemann surfaces by Ricci flow*, *Proc. Amer. Math. Soc.* 134 (2006) 3391–3393 [MR2231924](#)
- [14] **B Chow, D Knopf**, *The Ricci flow: an introduction*, *Math. Surveys and Monogr.* 110, Amer. Math. Soc. (2004) [MR2061425](#)
- [15] **B Chow, P Lu, L Ni**, *Hamilton's Ricci flow*, *Graduate Studies in Math.* 77, Amer. Math. Soc. (2006) [MR2274812](#)
- [16] **M Eminenti, G La Nave, C Mantegazza**, *Ricci solitons: the equation point of view*, *Manuscripta Math.* 127 (2008) 345–367 [MR2448435](#)
- [17] **RE Greene, H Wu**, *C^∞ approximations of convex, subharmonic, and plurisubharmonic functions*, *Ann. Sci. École Norm. Sup. (4)* 12 (1979) 47–84 [MR532376](#)
- [18] **RS Hamilton**, *Three-manifolds with positive Ricci curvature*, *J. Differential Geom.* 17 (1982) 255–306 [MR664497](#)
- [19] **RS Hamilton**, *Four-manifolds with positive curvature operator*, *J. Differential Geom.* 24 (1986) 153–179 [MR862046](#)

- [20] **RS Hamilton**, *The Ricci flow on surfaces*, from: “Mathematics and general relativity (Santa Cruz, CA, 1986)”, (J A Isenberg, editor), Contemp. Math. 71, Amer. Math. Soc. (1988) 237–262 [MR954419](#)
- [21] **T Ivey**, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. 3 (1993) 301–307 [MR1249376](#)
- [22] **B Kotschwar**, *On rotationally invariant shrinking Ricci solitons*, Pacific J. Math. 236 (2008) 73–88 [MR2398988](#)
- [23] **J Lauret**, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. 319 (2001) 715–733 [MR1825405](#)
- [24] **A Lichnerowicz**, *Variétés kählériennes à première classe de Chern non négative et variétés riemanniennes à courbure de Ricci généralisée non négative*, J. Differential Geometry 6 (1971) 47–94 [MR0300228](#)
- [25] **J Lott**, *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. 339 (2007) 627–666 [MR2336062](#)
- [26] **R McOwen**, *Partial differential equations. Methods and applications*, Prentice Hall, Upper Saddle River, NJ (1996)
- [27] **F Morgan**, *Manifolds with density*, Notices Amer. Math. Soc. 52 (2005) 853–858 [MR2161354](#)
- [28] **O Munteanu**, **N Sesum**, *On gradient Ricci solitons* [arXiv:0910.1105](#)
- [29] **A Naber**, *Noncompact shrinking 4-solitons with nonnegative curvature* [arXiv:0710.5579](#)
- [30] **L Ni**, *Ancient solutions to Kähler–Ricci flow*, Math. Res. Lett. 12 (2005) 633–653 [MR2189227](#)
- [31] **L Ni**, **N Wallach**, *On a classification of gradient shrinking solitons*, Math. Res. Lett. 15 (2008) 941–955 [MR2443993](#)
- [32] **G Perelman**, *The entropy formula for the Ricci flow and its geometric applications* [arXiv:math.DG/0211159](#)
- [33] **G Perelman**, *Ricci flow with surgery on three manifolds* [arXiv:math.DG/0303109](#)
- [34] **P Petersen**, **W Wylie**, *On gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. 137 (2009) 2085–2092 [MR2480290](#)
- [35] **P Petersen**, **W Wylie**, *Rigidity of gradient Ricci solitons*, Pacific J. Math. 241 (2009) 329–345 [MR2507581](#)
- [36] **G Wei**, **W Wylie**, *Comparison geometry for the Bakry–Emery Ricci tensor*, J. Differential Geom. 83 (2009) 377–405 [MR2577473](#)
- [37] **W Wylie**, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. 136 (2008) 1803–1806 [MR2373611](#)

- [38] **ST Yau**, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math. J. 25 (1976) 659–670 [MR0417452](#)
- [39] **Z-H Zhang**, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. 242 (2009) 189–200 [MR2525510](#)

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