Algebraic and geometric convergence
of discrete representations into PSL₂ℂ

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Anderson and Canary have shown that if the algebraic limit of a sequence of discrete, faithful representations of a finitely generated group into $\mathrm{PSL}_2\mathbb{C}$ does not contain parabolics, then it is also the sequence's geometric limit. We construct examples that demonstrate the failure of this theorem for certain sequences of unfaithful representations, and offer a suitable replacement.

1   Introduction

When Γ is a finitely generated group, let $\mathcal{D}(\Gamma)$ be the set of all representations $\rho\colon \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ with discrete, torsion free and nonelementary image. Here, a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ is called elementary if it is virtually abelian. The set $\mathcal{D}(\Gamma)$ sits naturally in the variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ and inherits the topology given by pointwise convergence; this is called the *algebraic topology* on $\mathcal{D}(\Gamma)$ and a pointwise convergent sequence of representations is usually called *algebraically convergent*. The goal of this note is to investigate the relationship between the algebraic convergence of a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ and the *geometric convergence* of the subgroups $\rho_i(\Gamma) \subset \mathrm{PSL}_2\,\mathbb{C}$. Recall that a sequence of closed subgroups (G_i) of $\mathrm{PSL}_2\,\mathbb{C}$ converges to a subgroup $G \subset \mathrm{PSL}_2\,\mathbb{C}$ geometrically if it does in the Chabauty topology.

Assume from now on that (ρ_i) is a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation $\rho \in \mathcal{D}(\Gamma)$, and that the groups $\rho_i(\Gamma)$ converge geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. For convenience, we will often say that ρ_i converges geometrically to G. While it is clear that $\rho(\Gamma) \subset G$, it was Jørgensen [22] who first realized that the geometric limit may be larger than the algebraic limit. For example, Thurston [38] constructed an algebraically convergent sequence in $\mathcal{D}(\pi_1(\Sigma_g))$ that converges geometrically to a subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ that is not even finitely generated. Other examples of this phenomenon, each one dramatic in its own way, were constructed by Kerckhoff and Thurston [25], Anderson and Canary [2] and Brock [11].

All these examples are related to the appearance of new parabolic elements in the algebraic limit; in fact, the following holds:

Theorem 1.1 (Anderson–Canary [3]) *Let Γ be a finitely generated group and assume that (ρ_i) is a sequence of faithful representations in $\mathcal{D}(\Gamma)$ converging algebraically to some $\rho \in \mathcal{D}(\Gamma)$. If $\rho(\Gamma)$ does not contain parabolic elements, then the groups $\rho_i(\Gamma)$ converge geometrically to $\rho(\Gamma)$.*

In [4], Anderson and Canary extended this result to the case where ρ and ρ_i map the same elements to parabolics for all i. Evans proved in [20] that the same conclusion holds under the weaker assumption that if an element of Γ is sent to a parabolic by ρ then it is also parabolic in ρ_i for all i. All these results were obtained in the presence of certain technical assumptions rendered unnecessary by work of Brock and Souto [14], and a fortiori by the resolution of the tameness conjecture by Agol [1] and Calegari and Gabai [15].

Motivated by questions of a different nature, related to the attempt to understand the structure of closed hyperbolic 3–manifolds whose fundamental group can be generated by, say, 10 elements, we revisited Theorem 1.1 convinced that it would remain true after dropping the assumption that the representations ρ_i are faithful. To our surprise, we found the following examples showing that Theorem 1.1 fails dramatically in this more general setting.

Example 1 Let Γ be the fundamental group of a closed surface of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a subgroup $G \subset \mathrm{PSL}_2\,\mathbb{C}$, such that

- G does not contain any parabolic elements.
- $\rho(\Gamma)$ has index 2 in G.

Example 2 Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a group G, such that

- G does not contain any parabolic elements.
- $\rho(\Gamma)$ has infinite index in G.

Example 3 Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a faithful representation ρ and geometrically to a group G, such that

- $\rho(\Gamma)$ does not contain any parabolic elements.
- G is not finitely generated.

Apart from discussing the examples above, our main goal is to understand the failure of Theorem 1.1 and see what is still true without the assumption that the representations are faithful. We prove:

Theorem 1.2 *Let* Γ *be a finitely generated group and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$*. Assume that* (ρ_i) *is algebraically convergent and converges geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$*. If* G *does not contain parabolic elements, then* G *is finitely generated.*

Before going further, observe that Example 3 shows that for the conclusion of Theorem 1.2 to hold it does not suffice to assume that the algebraic limit has no parabolics. Similarly, Example 2 shows that under the assumptions of Theorem 1.2 the algebraic limit $\rho(\Gamma)$ can have infinite index in the geometric limit G.

The reader may find it surprising that we mention Example 1 at all; the other two seem to be much more dramatic. However, Example 1 is *the mother of all examples*. It also shows that the following result, due to Anderson, fails in the nonfaithful setting.

Theorem 1.3 (Anderson) *Assume that* Γ *is a finitely generated group, and that* (ρ_i) *is a sequence of* faithful *representations in* $\mathcal{D}(\Gamma)$ *converging algebraically to some representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$*. Then there is no* $\gamma \in G \setminus \rho(\Gamma)$ *with* $\gamma^k \in \rho(\Gamma)$ *for some* $k \geq 2$*. In particular, if the image* $\rho(\Gamma)$ *of the algebraic limit has finite index in the geometric limit* G*, then* $\rho(\Gamma) = G$*.*

We mention Theorem 1.3 because its failure is the heart of the failure of Theorem 1.1 for sequences which are not necessarily faithful:

Theorem 1.4 *Let* Γ *be a finitely generated group and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$*. Assume that* (ρ_i) *converges algebraically to a representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$*. If*

- $\rho(\Gamma)$ *does not contain parabolic elements, and*
- *there is no* $\gamma \in G \setminus \rho(\Gamma)$ *with* $\gamma^k \in \rho(\Gamma)$ *for some* $k \geq 2$,

then $G = \rho(\Gamma)$*.*

The second condition here is used only to ensure that any degenerate ends of $\mathbb{H}^3/\rho(\Gamma)$ inject into $\mathbb{H}^3/G$. (See Section 2 for definitions.) So, we will actually prove the following stronger version of Theorem 1.4:

Theorem 1.5 *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) converges algebraically to a representation ρ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If*

- *$\rho(\Gamma)$ does not contain parabolic elements, and*

- *every degenerate end of $\mathbb{H}^3/\rho(\Gamma)$ has a neighborhood which embeds under the covering $\mathbb{H}^3/\rho(\Gamma) \to \mathbb{H}^3/G$,*

then $G = \rho(\Gamma)$. □

The proofs of Theorems 1.4 and 1.5, like the proofs of all results in this note, are completely independent of Anderson and Canary's Theorem 1.1. In fact, combining Theorem 1.4 with Anderson's Theorem 1.3 we obtain a new simpler proof of Theorem 1.1, which we will use in constructing Examples 1–3. It should be observed that while Anderson and Canary [3] and Evans [20] used fairly involved arguments to bypass the question of tameness, we use here the resolution of the tameness conjecture by Agol [1] and Calegari and Gabai [15] in a crucial way.

To conclude this fairly long introduction, we describe the paper section by section. After some preliminaries in Section 2 we construct the examples mentioned above in Section 3. In Section 4 we prove Theorem 1.4, obtaining a new proof of Theorem 1.1. In Section 5 we obtain a technical result, Proposition 5.1, which asserts that under some conditions if $M \to N$ is a cover then there is a tower $M \to M' \to N$ where $|\chi(M')| < |\chi(M)|$. Proposition 5.1 and Theorem 1.5 are the key ingredients in the proof of Theorem 1.2 in Section 6: the idea is to show that if $H \subset G$ is finitely generated, contains $\rho(\Gamma)$ and is such that $|\chi(\mathbb{H}^3/H)|$ is minimal then $G = H$ and hence G is finitely generated. In Section 7 we discuss extensions of Theorem 1.2 and Theorem 1.4 to the case that the algebraic limit has cusps; for instance, this permits us to recover Evans' general version of Theorem 1.1. Once this is done, we discuss briefly in Appendix A to which extent other well-known theorems about faithful representations remain true if the condition of faithfulness is dropped. See Section 3.

2 Preliminaries

In this section we recall some well-known facts and definitions about hyperbolic 3–manifolds.

2.1 Hyperbolic manifolds

By a hyperbolic 3–manifold we will mean any Riemannian 3–manifold isometric to $\mathbb{H}^3/\Gamma$, where Γ is a discrete, torsion-free group of isometries of hyperbolic 3–space. We will usually assume that the elements of Γ are orientation preserving, or equivalently that $\mathbb{H}^3/\Gamma$ is orientable. The full group of orientation preserving isometries of hyperbolic 3–space is written $\mathrm{Isom}_+(\mathbb{H}^3)$, and is often identified with $\mathrm{PSL}_2\,\mathbb{C}$ through its action on the boundary of $\mathbb{H}^3$.

Conjugate subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ give isometric quotients of $\mathbb{H}^3$; in order to remove this indeterminacy we consider *pointed* hyperbolic 3–manifolds, ie pairs (M,ω) where ω is an orthonormal frame of some tangent space of M. Choosing once and forever a fixed frame $\omega_{\mathbb{H}^3}$ of some tangent space of $\mathbb{H}^3$, every quotient manifold $\mathbb{H}^3/\Gamma$ has an induced framing $\omega_{\mathbb{H}^3/\Gamma}$ given by the projection of $\omega_{\mathbb{H}^3}$. Now, if (M,ω) is a pointed hyperbolic 3–manifold then there is a unique $\Gamma \subset \mathrm{PSL}_2\,\mathbb{C}$ such that the manifolds (M,ω) and $(\mathbb{H}^3/\Gamma, \omega_{\mathbb{H}^3/\Gamma})$ are isometric as pointed manifolds.

Remark It would be more natural to speak of *framed* hyperbolic 3–manifolds instead of pointed; however, it is customary to use the given terminology.

2.2 Tameness

We will be mainly interested in hyperbolic 3–manifolds M with finitely generated fundamental group. Any such manifold is *tame* by the work of Agol [1] and Calegari and Gabai [15]:

Tameness Theorem (Agol, Calegari–Gabai) *Let M be a hyperbolic 3–manifold with finitely generated fundamental group. Then M is homeomorphic to the interior of a compact 3–manifold.*

A *compact core* of a hyperbolic 3–manifold M with finitely generated fundamental group is a compact submanifold $C \subset M$ with $M \setminus C$ homeomorphic to $\partial C \times \mathbb{R}$. Observe that M is homeomorphic to the interior of every such compact core. It follows from the Tameness Theorem that every such M admits an exhaustion by nested compact cores. If $C \subset M$ is a standard compact core, then the ends of M correspond

naturally to components of $M \setminus C$. The component $U_{\mathcal{E}}$ of $M \setminus C$ corresponding to an end $\mathcal{E}$ is said to be a *standard neighborhood* of $\mathcal{E}$ and the component of ∂C contained in the closure of $U_{\mathcal{E}}$ is said to *face $\mathcal{E}$*. We will often denote the component of ∂C facing $\mathcal{E}$ by $\partial \mathcal{E}$. Observe that $U_{\mathcal{E}}$ is homeomorphic to $\partial \mathcal{E} \times \mathbb{R}$.

2.3 Geometry of ends in the absence of cusps

Let M be a hyperbolic 3–manifold with finitely generated fundamental group and without cusps. An end $\mathcal{E}$ of M is *convex cocompact* if it has a neighborhood in M disjoint from the convex-core $\mathrm{CC}(M)$ of M. Recall that the convex core $\mathrm{CC}(M)$ is the smallest convex submanifold of M whose inclusion is a homotopy equivalence. A manifold with compact $\mathrm{CC}(M)$ is said to be *convex cocompact* ; equivalently, all ends of M are convex cocompact.

For every $d > 0$, the set of points in M within distance d of $\mathrm{CC}(M)$ is homeomorphic to M and has *strictly convex C^1–boundary*: that is, $\langle \nabla_X \nu, X \rangle > 0$, where ∇ is the Levi-Civita connection, X is a vector tangent to ∂K_i and ν is the outer normal field along ∂K_i.

Smoothing the boundary, we obtain the following well-known fact:

Lemma 2.1 *Let M be a hyperbolic 3–manifold with finitely generated fundamental group. There is an exhaustion of M by a nested sequence of submanifolds K_i such that*

(1) *the boundary ∂K_i is smooth and strictly convex.*

(2) *every convex cocompact end of M has a neighborhood disjoint of K_i.*

(3) *the inclusion of K_i into M is a homotopy equivalence.*   □

Continuing with the same notation as in Lemma 2.1, convexity implies that there is a well-defined map $\kappa_{K_i} \colon M \to K_i$ that takes a point in M to the point in K_i closest to it. Strict convexity implies that the preimage of a point $x \in \partial K_i$ under this projection is a geodesic ray. It follows that the map

$$M \setminus K_i \to \partial K_i \times (0, \infty), \quad x \mapsto (\kappa_{K_i}(x), d(x, \kappa_{K_i}(x)))$$

is a diffeomorphism; in fact, its inverse is the radial coordinate map

$$\partial K_i \times (0, \infty) \to M \setminus K_i, \quad (x, t) \mapsto \exp_x(t\nu(x)),$$

where ν is the outer unit normal vector-field along ∂K_i.

An end $\mathcal{E}$ which is not convex cocompact is said to be *degenerate*. It follows from the Tameness Theorem and earlier work of Bonahon [8] and Canary [17] that degenerate ends have very well-behaved geometry. For instance, every degenerate end $\mathcal{E}$ has a neighborhood which is completely contained in the convex core $\mathrm{CC}(M)$. From our point of view, the most important fact about degenerate ends is the Thurston–Canary [18] Covering Theorem, of which we state the following weaker version:

Theorem 2.2 (Thurston, Canary [18]) *Let* M *and* N *be noncompact hyperbolic* 3*–manifolds, assume that* M *has finitely generated fundamental group and no cusps, let* $\pi\colon M \to N$ *be a Riemannian cover and let* $\mathcal{E}$ *be a degenerate end of* M*. Then* $\mathcal{E}$ *has a standard neighborhood* $U_{\mathcal{E}}$ *such that the restriction*

$$\pi|_{U_{\mathcal{E}}}\colon U_{\mathcal{E}} \to \pi(U_{\mathcal{E}})$$

of the covering π *to* $U_{\mathcal{E}}$ *is a covering map onto a standard neighborhood of a degenerate end* $\mathcal{E}'$ *of* N*. More precisely, there is a finite covering* $\sigma\colon \partial\mathcal{E} \to \partial\mathcal{E}'$ *and homeomorphisms*

$$\phi\colon \partial\mathcal{E} \times \mathbb{R} \to U_{\mathcal{E}}, \quad \psi\colon \partial\mathcal{E}' \times \mathbb{R} \to \pi(U_{\mathcal{E}})$$

with $(\psi^{-1} \circ \pi|_{U_{\mathcal{E}}} \circ \phi)(x,t) = (\sigma(x),t)$*. In particular, the covering* $\pi|_{U_{\mathcal{E}}}$ *has finite degree.*

Combining the Tameness Theorem, Lemma 2.1 and the Covering Theorem (Theorem 2.2) we obtain:

Proposition 2.3 *Let* M *and* N *be hyperbolic* 3*–manifolds with infinite volume, assume that* M *has finitely generated fundamental group and no cusps, and let* $\pi\colon M \to N$ *be a Riemannian cover. Then* M *admits an exhaustion by nested standard compact cores* $C_i \subset C_{i+1}$ *such that the following holds:*

(1) *If a component* S *of* ∂C_i *faces a convex cocompact end of* M *then* S *is smooth and strictly convex.*

(2) *If a component* S *of* ∂C_i *faces a degenerate end of* M *then the restriction* $\pi|_S\colon S \to \pi(S)$ *is a finite covering onto an embedded surface in* N*.*   □

2.4 Conformal boundaries and Ahlfors–Bers theory

As in the previous section, assume that we have a hyperbolic 3–manifold $M = \mathbb{H}^3/\Gamma$ with finitely generated fundamental group and no cusps.

The *limit set* of Γ, written $\Lambda(\Gamma)$, is the closure of the set of fixed points in $\mathbb{S}^2_\infty = \partial\mathbb{H}^3$ of hyperbolic elements of Γ. The complement of $\Lambda(\Gamma)$ is the *domain of discontinuity*

$\Omega(\Gamma) = \mathbb{S}^2_\infty \setminus \Lambda(\Gamma)$, which is the largest open subset of $\mathbb{S}^2_\infty$ on which Γ acts properly discontinuously. The quotient $\partial_c M = \Omega(\Gamma)/\Gamma$ is called the *conformal boundary* of M. In fact, Γ acts properly discontinuously on $\mathbb{H}^3 \cup \Omega(\Gamma)$, and the quotient $M \cup \partial_c M$ is a manifold with boundary having interior M.

The action of Γ on $\mathbb{S}^2_\infty$ is by Mobius transformations, so $\partial_c M$ inherits a natural conformal structure. This structure is closely tied with the geometry of M: for instance, the unique hyperbolic metric on $\partial_c M$ compatible with this conformal structure, called the *Poincaré metric*, is similar to the intrinsic metric on $\partial\mathrm{CC}(M)$. Specifically, the closest point projection $\kappa\colon M \to \mathrm{CC}(M)$ extends continuously to a map $\overline{\kappa}\colon \partial_c M \to \partial\mathrm{CC}(M)$, and we have the following theorem of Canary:

Proposition 2.4 (Canary [16]) *For every $\epsilon > 0$ there exists $K > 0$ so that the following holds. Let M be a hyperbolic 3–manifold with finitely generated fundamental group, such that every component of $\partial_c M$ has injectivity radius at least ϵ in the Poincaré metric. Then the closest point projection $\kappa\colon \partial_c M \to \partial\mathrm{CC}(M)$ is K–lipschitz, where $\partial_c M$ has the Poincaré metric and $\partial\mathrm{CC}(M)$ is considered with the Riemannian metric induced by its inclusion into M.*

The conformal boundary plays an important role in the deformation theory of hyperbolic 3–manifolds; in particular, a convex-cocompact hyperbolic 3–manifold is determined up to isometry by its topology and conformal boundary. A more precise statement of this is as follows.

Let M be the interior of a compact hyperbolizable 3–manifold $\overline{M}$ in which each boundary component has negative Euler characteristic. Define $\mathcal{CH}(M)$ to be the set of all convex-cocompact hyperbolic metrics on M, where two metrics are identified if they differ by an isometry isotopic to the identity map. It follows from Thurston's hyperbolization theorem [39] that $\mathcal{CH}(M)$ is nonempty, and it inherits a natural complex structure through its relation to the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2\,\mathbb{C})$ (see Matsuzaki and Taniguchi [28, Section 4.3]). Then we have:

Theorem 2.5 (Ahlfors–Bers Parameterization, see [28]) *The map $\mathcal{CH}(M) \to \mathcal{T}(\partial\overline{M})$, induced from the map taking a convex-cocompact uniformization of M to its conformal boundary, is a biholomorphic equivalence. Therefore, $\mathcal{CH}(M)$ is a complex manifold of dimension*

$$\frac{3}{2}|\chi(\partial\overline{M})| = 3|\chi(M)|.$$

Here, $\mathcal{T}(\partial\overline{M})$ is the *Teichmüller space* of conformal structures on $\partial\overline{M}$, where two conformal structures are identified if there is a conformal map between them that is

isotopic to the identity. The conformal boundary of a convex-cocompact uniformization of M is only identified with $\partial\overline{M}$ up to isotopy, so a point of $\mathcal{CH}(M)$ gives a point in $\mathcal{T}(\partial\overline{M})$ rather than a specific conformal structure on $\partial\overline{M}$.

The space $\mathcal{CH}(M)$ is natural with respect to certain coverings. If M' is a cover of M with $\pi_1(M')$ finitely generated, then the Tameness Theorem and Canary's Covering Theorem (Theorem 2.2) imply that convex-cocompact metrics on M lift to convex-cocompact metrics on M'. In fact:

Lemma 2.6 *If M' is a 3–manifold with finitely generated fundamental group and $\tau: M' \to M$ is a covering map, there is a holomorphic map*

$$\tau^*\colon \mathcal{CH}(M) \to \mathcal{CH}(M')$$

induced by the map taking a hyperbolic structure on M to its pullback under τ.

2.5 Geometric convergence

Recall that a sequence (G_i) of closed subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ converges *geometrically* to a subgroup G if it does in the Chabauty topology. More concretely, (G_i) converges geometrically to G if G is the subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ consisting precisely of those elements $g \in \mathrm{PSL}_2\,\mathbb{C}$ such that there are $g_i \in G_i$ with $g_i \to g$ in $\mathrm{PSL}_2\,\mathbb{C}$. In other words, G is the accumulation set of the groups G_i.

Most of our arguments are based on an interpretation of geometric convergence in terms of the quotient manifolds $\mathbb{H}^3/G_i$.

Definition A sequence (M_i, ω_i) of pointed hyperbolic 3–manifolds converges *geometrically* to a pointed manifold $(M_\infty, \omega_\infty)$ if for every compact $K \subset M_\infty$ that contains the origin of ω_∞, there is a sequence $\phi_i\colon K \to M_i$ of smooth maps with $\phi_i(\omega_\infty) = \omega_i$ converging in the C^k–topology to an isometric embedding for all $k \in \mathbb{N}$. We will refer to the maps ϕ_i as the *almost isometric embeddings provided by geometric convergence*.

Remark Note that although the phrase "converging in the C^k–topology to an isometric embedding" is suggestive and pleasing to the ear, it has no meaning. One way to formalize this would be to say that for each point $x \in K$, there is an $\epsilon > 0$ and a sequence of isometric embeddings

$$\beta_i\colon B(\phi_i(x), \epsilon) \to \mathbb{H}^3$$

from ϵ–balls around $\phi_i(x) \in M_i$ so that $\beta_i \circ \phi_i$ converges to an isometric embedding of some neighborhood of $x \in M_\infty$ into $\mathbb{H}^3$.

Recall that by our convention above, a pointed hyperbolic manifold is a manifold together with a base frame and that choosing a base frame $\omega_{\mathbb{H}^3}$ of hyperbolic space we obtain a bijection between the sets of discrete torsion free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ and of pointed hyperbolic 3–manifolds. Under this identification, the notions of geometric convergence of groups and manifolds are equivalent (see for instance Benedetti and Petronio [6] or Kapovich [24]):

Proposition 2.7 *Suppose $G_1, G_2, \ldots, G_\infty$ are discrete and torsion-free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$ and consider for all $i = 1, \ldots, \infty$ the pointed hyperbolic 3–manifold (M_i, ω_i) where $M_i = \mathbb{H}^3/G_i$ and ω_i is the projection of the frame $\omega_{\mathbb{H}^3}$ of $\mathbb{H}^3$. The groups G_i converge geometrically to G_∞ if and only if the pointed manifolds (M_i, ω_i) converge geometrically to $(M_\infty, \omega_\infty)$.*

2.6 Algebraic convergence

Let Γ be a finitely generated group. Recall that a sequence (ρ_i) of representations $\rho_i\colon \Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ converges *algebraically* to a representation ρ if for every $\gamma \in \Gamma$ we have $\rho_i(\gamma) \to \rho(\gamma)$ in $\mathrm{PSL}_2\,\mathbb{C}$. Jørgensen proved in [23] that if each image $\rho_i(\Gamma)$ is discrete and nonelementary then the same is true of $\rho(\Gamma)$. His argument also shows that torsion cannot suddenly appear in the limit, so we have the following theorem:

Proposition 2.8 *Let Γ be a finitely generated group. Then the subset $\mathcal{D}(\Gamma) \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ consisting of (not necessarily faithful) representations with discrete, torsion free and nonelementary image is closed with respect to the algebraic topology.*

When a sequence (ρ_i) converges both algebraically to a representation ρ and geometrically to some group G, it is easy to see that $\rho(\Gamma) \subset G$. In other words, the manifold $\mathbb{H}^3/\rho(\Gamma)$ covers the manifold $\mathbb{H}^3/G$. In particular, given a compact subset $C \subset \mathbb{H}^3/\rho(\Gamma)$ we can project it down to $\mathbb{H}^3/G$ and then map the image to M_i under the almost isometric embeddings given by geometric convergence. This produces maps $C \to M_i = \mathbb{H}^3/\rho_i(\Gamma)$ which look more and more like the restriction of a covering to C.

More generally, assume that H is a finitely generated subgroup of G containing $\rho(\Gamma)$. By the Tameness Theorem, the manifold $\mathbb{H}^3/H$ contains a standard compact core C_H. Composing the restriction to C_H of the covering $\mathbb{H}^3/H \to \mathbb{H}^3/G$ with the almost isometric embeddings given by geometric convergence, we obtain for sufficiently large i maps $C_H \to M_i$ similar to those described above. Using the induced homomorphisms $H \to \pi_1(M_i, \omega_i)$ one can then construct a sequence of representations $\sigma_i\colon H \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to the inclusion of $H \hookrightarrow G \hookrightarrow \mathrm{PSL}_2\,\mathbb{C}$.

The assumption that $\rho(\Gamma) \subset H$ implies then that $\sigma_i(H) = \rho_i(\Gamma)$ for all i. In particular, (compare with McMullen [31, Lemma 4.4]):

Proposition 2.9 *Let Γ be a finitely generated group, (ρ_i) a sequence in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ_∞ and geometrically to a group $G \subset \mathrm{PSL}_2\mathbb{C}$. If $H \subset G$ is a finitely generated subgroup of G containing $\rho(\Gamma)$ then for large i there are representations*

$$\sigma_i \colon H \to \mathrm{PSL}_2\mathbb{C} \quad \text{with } \sigma_i(H) = \rho_i(\Gamma)$$

converging to the inclusion $H \hookrightarrow \mathrm{PSL}_2\mathbb{C}$. In particular, the groups $\sigma_i(H)$ converge geometrically to G. □

2.7 Roots

Recall that the nontrivial elements in $\mathrm{PSL}_2\mathbb{C}$ are either hyperbolic, parabolic or elliptic depending on their dynamical behaviour. Every hyperbolic element $\gamma \in \mathrm{PSL}_2\mathbb{C}$ stabilizes a geodesic A in $\mathbb{H}^3$, and if $\alpha \in \mathrm{PSL}_2\mathbb{C}$ is a k–th root of γ, ie $\gamma = \alpha^k$, then $\alpha A = A$. It follows easily that the set of k–th roots of γ is finite. A similar argument applies in the parabolic and elliptic case, so we obtain the following well-known, and in this paper surprisingly important, fact:

Lemma 2.10 *For every $k \in \mathbb{Z}$ and nontrivial element $\gamma \in \mathrm{PSL}_2\mathbb{C}$, the set $\{\alpha \in \mathrm{PSL}_2\mathbb{C} \mid \alpha^k = \gamma\}$ is finite.* □

Lemma 2.10 has the following immediate consequence:

Corollary 2.11 *Let $\Gamma \subset \Gamma'$ be groups and assume that Γ' contains a torsion-free subgroup H such that*

(1) Γ *and H generate Γ', and*

(2) $\Gamma \cap H$ *has finite index in H.*

Then for every faithful representation $\rho \colon \Gamma \to \mathrm{PSL}_2\mathbb{C}$, the set of representations $\{\rho' \colon \Gamma' \to \mathrm{PSL}_2\mathbb{C} \mid \rho'|_\Gamma = \rho\}$ is finite. □

Lemma 2.10 also has the following consequence, which was stated in the introduction.

Theorem 1.3 (Anderson) *Assume that Γ is a finitely generated group and (ρ_i) is a sequence of faithful representations in $\mathcal{D}(\Gamma)$ that converges algebraically to a representation ρ and geometrically to a group G. Then there is no $\gamma \in G \setminus \rho(\Gamma)$ with $\gamma^k \in \rho(\Gamma)$ for some $k \geq 2$. In particular, if the image $\rho(\Gamma)$ of the algebraic limit has finite index in the geometric limit G, then $\rho(\Gamma) = G$.*

Proof Assume there is some $g \in G \setminus \rho(\Gamma)$ with $g^k = \rho(\eta)$ for some $\eta \in \Gamma$ and $k \geq 2$. Since $g \in G$ there are $\gamma_i \in \Gamma$ with $\lim_{i \to \infty} \rho_i(\gamma_i) = g$. Taking powers we then have

$$\lim_{i \to \infty} \rho_i(\gamma_i^k) = g^k = \rho(\eta) = \lim_{i \to \infty} \rho_i(\eta).$$

It follows from the Margulis lemma that $\rho_i(\gamma_i^k) = \rho_i(\eta)$ for sufficiently large i. *Since the representations ρ_i are faithful* , this implies that $\gamma_i^k = \eta$ for large i. But the group Γ embeds in $\mathrm{PSL}_2 \, \mathbb{C}$, by say ρ_7, so Lemma 2.10 shows that each of its elements has only finitely many k–th roots. This implies that up to passing to a subsequence we may assume that $\gamma_i = \gamma_j$ for all pairs (i, j), and hence g belongs to the algebraic limit. This is a contradiction, so the claim follows. $\qquad\square$

We included this proof because its failure for nonfaithful sequences is at the heart of the examples presented in Section 3.

2.8 Maximal cusps

Finally, we describe a class of hyperbolic manifolds that we will use as building blocks in constructing our examples in Section 3. A good reference for this section is Canary, Culler, Hersonsky and Shalen [5].

Assume that M is a compact, orientable, irreducible and atoroidal 3–manifold with interior N and no torus boundary components. If N has a geometrically finite hyperbolic metric, then there is a collection C of disjoint simple closed curves in ∂M such that

$$N \cup \partial_c N \cong M \setminus \bigcup_{\gamma \in C} \gamma,$$

by a homeomorphism whose restriction to N is isotopic to the inclusion $N \subset M$. Here, $\partial_c N$ is the *conformal boundary* defined in Section 2.4.

The curves in C are determined up to isotopy, and correspond to the rank one cusps in N. We will say that the collection C has been *pinched*. Each curve in C is homotopically nontrivial in M and no two curves in C are freely homotopic in M. It follows from Thurston's Hyperbolization Theorem [24; 36] that any collection of curves on ∂M satisfying these two topological constraints can be obtained as above from a hyperbolic structure on N; it is therefore said that such a collection is *pinchable*.

One says that a component $S \subset \partial M$ is *maximally cusped* with respect to some geometrically finite structure on N if the associated collection C contains a pants decomposition for S. In this case, any component of $\partial \mathrm{CC}(N)$ that faces S is a totally geodesic hyperbolic thrice-punctured sphere. Given two hyperbolic 3–manifolds with

maximally cusped ends, we can topologically glue their convex cores together along these thrice-punctured spheres. Since every homeomorphism between hyperbolic thrice-punctured spheres can be isotoped to an isometry, after altering the identifications we can ensure that our gluing produces a hyperbolic 3–manifold. Here is a precise description of the result of this process.

Lemma 2.12 (Gluing maximal cusps) *Let M_1 and M_2 be compact, orientable 3–manifolds with interiors N_i that have geometrically finite hyperbolic metrics. Assume that S_i are maximally cusped components of ∂M_i with pinched pants decompositions $P_i \subset S_i$. Then if $h\colon S_1 \to S_2$ is a homeomorphism with $h(P_1) = P_2$, there is a hyperbolic 3–manifold N with the following properties:*

- *$N \cong (M_1 \setminus P_1) \sqcup_h (M_2 \setminus P_2)$, so N has a rank 2 cusp corresponding to each element of P_1 (or P_2).*
- *N is the union of two subspaces with disjoint interiors, which are isometric to $N_1 \setminus E_1$ and $N_2 \setminus E_2$, where E_i are the components of $N_i \setminus \mathrm{CC}(N_i)$ adjacent to S_i.*

3 Examples

In this section we construct the examples mentioned in the introduction. All our constructions follow the same general strategy. We will describe this here before starting with the examples in detail.

We first construct a finitely generated group $\widehat{\Gamma}$ and a sequence of (unfaithful) discrete representations (σ_i) in $\mathcal{D}(\widehat{\Gamma})$ that converges *strongly* to a faithful representation σ_∞ whose image $\sigma_\infty(\widehat{\Gamma})$ has no parabolics. Recall that $\sigma_i \to \sigma_\infty$ *strongly* if it does so algebraically and $\sigma_i(\widehat{\Gamma}) \to \sigma_\infty(\widehat{\Gamma})$ geometrically. Next, we find a proper subgroup $\Gamma \subset \widehat{\Gamma}$ such that

$$\sigma_i(\Gamma) = \sigma_i(\widehat{\Gamma}) \text{ for all } i \in \mathbb{N}$$

and let $\rho_i = \sigma_i|_\Gamma$. By construction, $\rho_i \to \rho_\infty = \sigma_\infty|_\Gamma$ algebraically and $\rho_i(\Gamma) \to \sigma_\infty(\widehat{\Gamma})$ geometrically. Since Γ is a proper subgroup of $\widehat{\Gamma}$ and σ_∞ is faithful, we obtain that the algebraic limit $\rho_\infty(\Gamma) = \sigma_\infty(\Gamma)$ is a proper subgroup of the geometric limit $\sigma_\infty(\widehat{\Gamma})$.

Example 1 *Let Γ be the fundamental group of a closed orientable surface of genus 3. There is a sequence of representations (ρ_i) in $\mathcal{D}(\Gamma)$ that converges algebraically to a faithful representation ρ and geometrically to a group G such that*

- *G has no parabolics.*
- *$\rho(\Gamma)$ has index 2 in G.*

In the following, we let S be a closed orientable surface of genus g. The group $\widehat{\Gamma}$ from the strategy above will be $\pi_1 S$, while Γ will be the fundamental group of a degree 2 cover of S. The concise statement of the example above is the special case $g = 2$, but our argument below works for any closed orientable surface.

To begin with, let H be a handlebody with boundary S. Recall that the *Masur domain* $O_H \subset \mathcal{PML}(S)$ is the set of all projective measured laminations on S that intersect positively with every element of $\mathcal{PML}(S)$ that is a limit of meridians. One calls a pseudo-Anosov map $f: S \to S$ *generic* if its attracting lamination λ lies in O_H. To see that such maps exist, note that O_H is open and nonempty and that the attracting laminations of pseudo-Anosov maps are dense in $\mathcal{PML}(S)$. The density follows from the density of simple closed curves, for conjugating any given pseudo-Anosov map by powers of a Dehn twist produces a sequence of pseudo-Anosov maps whose attracting laminations limit to the twisting curve in $\mathcal{PML}(S)$.

A sequence of convex-cocompact handlebodies Fix a generic pseudo-Anosov map $f: S \to S$ with attracting lamination λ and repelling lamination $\bar{\lambda}$. Theorem 2.5 implies that the deformation space of convex-cocompact hyperbolic metrics on a hyperbolizable 3–manifold is parameterized by the Teichmuller space of its boundary. So, we can produce a sequence of convex-cocompact metrics on H corresponding to an orbit of f on $\mathcal{T}(S)$. In other words, after fixing a conformal structure X on S, we have a sequence of convex-cocompact hyperbolic manifolds (N_i) and a sequence of homeomorphisms

$$h_i: (H, S) \to (N_i, \partial_c N_i)$$

such that $h_i \circ f^i: S \to \partial_c N_i$ is conformal with respect to X. Here, $\partial_c N_i$ is the *conformal boundary* of N_i discussed in Section 2.4.

Good markings By Proposition 2.4 and the fact that the Poincaré metric on $\partial_c N_i$ is constant in moduli space, the nearest point projection $\eta_i: \partial_c N_i \to \partial \mathrm{CC}(N_i)$ is K–lipschitz for some constant K independent of i. Considering S with its Poincaré metric, the map

$$\sigma_i := \eta_i \circ h_i \circ f^i: S \to N_i$$

is K–lipschitz as well. Since it is π_1–surjective, we may use σ_i to mark the fundamental group of N_i with $\pi_1(S)$. Specifically, after taking a base point $p \in S$, the surjections $(\sigma_i)_*: \pi_1(S, p) \to \pi_1(N_i, \sigma_i(p))$ determine up to conjugacy a sequence of representations

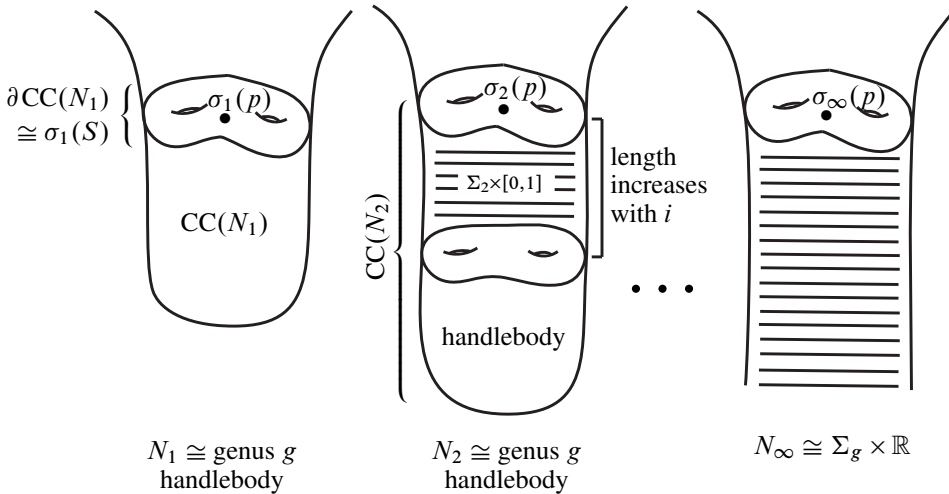$$\rho_i: \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C}), \quad \mathbb{H}^3/\rho_i(\pi_1(S, p)) = N_i.$$

Figure 1: A schematic of the convergence $N_i \to N_\infty$

Strong convergence and the limit The goal here is to establish the following proposition.

Proposition 3.1 *The sequence* (ρ_i) *converges strongly to a representation* $\rho_\infty\colon \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C})$ *that is faithful and purely loxodromic. Its quotient* $N_\infty = \mathbb{H}^3/\rho_\infty(\pi_1(S, p))$ *is homeomorphic to* $S \times \mathbb{R}$ *and has no cusps. One of the ends of* N_∞ *is convex-cocompact with conformal boundary* X *and the other is degenerate with ending lamination* $\overline{\lambda}$.

Here, X is the base conformal structure on S fixed above and $\overline{\lambda}$ is the repelling lamination of f. Recall that an algebraically convergent sequence $\rho_i \to \rho_\infty$ converges *strongly* if in addition the images of (ρ_i) converge to the image of ρ_∞ geometrically.

The resolution of Thurston's Ending Lamination Conjecture by Minsky [32] and Brock, Canary and Minsky [13] implies that N_∞ is uniquely characterized by the description given in Proposition 3.1. So to prove Proposition 3.1, it suffices to show that every subsequence of (ρ_i) has a subsequence that converges strongly to some ρ_∞ as described above.

Start with some subsequence (ρ_{i_j}) of (ρ_i). Passing to a further subsequence, we may assume that (ρ_{i_j}) converges algebraically to some

$$\rho_\infty\colon \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C}), \quad \text{where } N_\infty := \mathbb{H}^3/\rho_\infty(\pi_1(S, p)).$$

We may also assume that the images converge geometrically:

$$\rho_{i_j}(\pi_1(S, p)) \to G \subset \mathrm{PSL}(2, \mathbb{C}), \quad \text{where } N_G := \mathbb{H}^3/G.$$

If base-frames are chosen for N_{i_j} on $\partial\mathrm{CC}(N_{i_j})$, the resulting pointed manifolds converge in the Gromov–Hausdorff topology to N_G. By the Arzela–Ascoli Theorem, we can pass to a final subsequence such that the K–lipschitz maps $\sigma_{i_j} \colon S \to N_{i_j}$ converge uniformly to some

$$\sigma_G \colon S \longrightarrow N_G.$$

There is a natural covering map $\pi \colon N_\infty \to N_G$ that corresponds to the inclusion $\rho_\infty(\pi_1(S, p)) \subset G$, and σ_G lifts to a map $\sigma_\rho \colon S \to N_\infty$ inducing the marking given by ρ_∞.

Claim σ_∞ *is an embedding, and its image bounds a neighborhood E' of a convex-cocompact end of N_∞. Furthermore, $E' \cong \Sigma_g \times \mathbb{R}$ and $\pi|_{E'} \colon E' \to N_G$ is an embedding.*

Proof We will show that the image of σ_G bounds a convex-cocompact end in N_G homeomorphic to $\Sigma_g \times \mathbb{R}$. It will follow immediately that this end lifts to an end E' as desired.

Let $K \subset N_G$ be a Gromov–Hausdorff limit of the sequence of convex cores $\mathrm{CC}(N_{i_j})$, passing to another subsequence as necessary. Note that K is convex and contains $\mathrm{CC}(N_G)$. Its boundary ∂K is the limit of the boundaries $\partial\mathrm{CC}(N_{i_j})$, and is therefore the image of σ_G. Using convexity, it is then not hard to see that σ_G is an embedding. Setting $E = N_G \setminus K$, we obtain a convex-cocompact end of N_G. As $\partial E \cong \Sigma_g$, the nearest point retraction gives a homeomorphism $N_G \setminus K \cong \Sigma_g \times (0, \infty)$. □

The following claim is the reason our pseudo-Anosov map $f \colon S \to S$ was chosen to have attracting lamination λ in the Masur domain O_H.

Claim ρ_∞ *is faithful.*

Proof An equivalent statement is that the embedding $\sigma_\infty \colon S \to N_\infty$ is π_1–injective. So by the Loop Theorem, it suffices to check that ρ_∞ is injective on elements of $\pi_1(S, p)$ representable by simple loops on S.

Let $\gamma \subset S$ be a simple closed curve. Then $f^{i_j}(\gamma) \to \lambda$ in $\mathcal{PML}$; since $\lambda \in O_H$, which is an open subset of $\mathcal{PML}$, for large i_j we have $f^{i_j}(\gamma) \in O_H$ as well. In particular, $f^{i_j}(\gamma)$ is not compressible in H. It follows that

$$\sigma_{i_j}(\gamma) = \eta_{i_j} \circ h_{i_j} \circ f^{i_j}(\gamma)$$

is incompressible in N_{i_j}. Therefore $\rho_{i_j}(\gamma) \neq \mathrm{Id}$ for sufficiently large i_j. A standard application of the Margulis Lemma shows that $\rho_\infty(\gamma) \neq \mathrm{Id}$ as well (see, eg [28, Theorem 7.1]). $\square$

Geometrically, the reason that ρ_∞ is faithful is that for large i all compressible curves on $\partial\mathrm{CC}(N_i)$ are very long. Since a fixed generating set for $\pi_1(S, p)$ maps under our markings to a set of loops on $\partial\mathrm{CC}(N_i)$ with uniformly bounded length, this implies that elements of $\pi_1(S, p)$ representing compressible curves in N_i must for large i be expressible only with very large words in the generators. So in the limit, there are no compressible curves.

As $\rho_\infty\colon \pi_1(S, p) \to \mathrm{PSL}(2, \mathbb{C})$ is faithful, the Tameness Theorem implies that $N_\infty \cong S \times \mathbb{R}$. From above, we know that one of the two topological ends of N_∞ is convex-cocompact with associated conformal structure X. To analyze the geometry of the other end, we must first prove the following:

Claim *The lamination $\bar\lambda$ is not realized in N_∞. In particular, N_∞ has a degenerate end with ending lamination λ.*

Proof Fix a meridian m. By definition, the curve $f^{-i_j}(m)$ is in the kernel of the representation ρ_{i_j} for all i_j. On the other hand, the sequence $(f^{-i_j}(m))$ converges to $\bar\lambda$ in $\mathcal{PML}(S)$. This implies that $\bar\lambda$ is not realized; see for instance Hossein and Souto [34, Section 4]. $\square$

Since N_∞ is homeomorphic to $S \times \mathbb{R}$ and has a convex-cocompact end and a degenerate end, we deduce that ρ_∞ is purely loxodromic.

Claim ρ_i *converges to* ρ_∞ *strongly.*

Proof We need to show that the covering $\pi\colon N_\infty \to N_G$ is trivial. There is a neighborhood of the degenerate end of N_∞ that is completely contained in the convex core $\mathrm{CC}(N_\infty)$. So, $N_\infty \setminus E'$ is the union of $\mathrm{CC}(N_\infty)$ and some compact set. A result of Thurston [25] and Bonahon [8] implies that the injectivity radius of N_∞ is bounded above inside of its convex core, so by extension there is an upper bound K for the injectivity radius of N_∞ outside of E'. Pick a point $x \in N_G$ deep enough inside its convex-cocompact end so that $\mathrm{inj}(x, N_G) > K$. Then no element of $N_\infty \setminus E'$ can project to x. Since π is injective on E', $|\pi^{-1}(x)| = 1$. Therefore π is trivial. $\square$

This finishes the proof of Proposition 3.1.

Restricting the markings Finally, we show that we can restrict our representations (ρ_i) to a subgroup of $\pi_1(S, p)$ to create a sequence with the properties desired for our example.

Claim *After passing to a subsequence, there exists an index 2 subgroup $\Gamma \subset \pi_1(S, p)$ such that for each $i > 0$,*

$$(\sigma_i)_*|_\Gamma : \Gamma \to \pi_1(N_i, \sigma_i(p))$$

is a surjection.

Proof The proof consists of two simple observations. First, if M is a handlebody then there is an index 2 subgroup $\Gamma \subset \pi_1(\partial M)$ that surjects onto $\pi_1(M)$. One can take, for example, the kernel of the map taking an element of $\pi_1(\partial M)$ to its mod–2 intersection number with any longitude in a standard meridian-longitude basis for $H_1(\partial M)$. Therefore, we can construct for each i an index 2 subgroup Γ_i with $(\sigma_i)_*: \Gamma_i \to \pi_1(N_i, \sigma_i(p))$ a surjection. Since there are only finitely many index 2 subgroups of $\pi_1(S, p)$, we can pass to a subsequence so that a single $\Gamma \subset \pi_1(S, p)$ works for all i. $\qquad\square$

Consider now the sequence of representations $\rho_i|_\Gamma: \Gamma \to \mathrm{PSL}(2, \mathbb{C})$. The claim implies that $\rho_i(\Gamma) = \rho_i(\pi_1(S, p))$, so

$$\rho_i(\Gamma) \to \rho_\infty(\pi_1(S, p))$$

geometrically. Now $\rho_i \to \rho_\infty$ algebraically, so $\rho_i|_\Gamma \to \rho_\infty|_\Gamma$. Therefore, since ρ_∞ is faithful, $\rho_\infty(\Gamma)$ has index 2 in $\rho_\infty(\pi_1(S, p))$. We have therefore provided a sequence of representations converging algebraically to a faithful representation whose image is an index 2 subgroup of the geometric limit, and so that the geometric limit has no parabolics. This concludes our example.

An alternate method The difficult part of the example above is constructing a sequence of hyperbolic 3–manifolds (N_i), each homeomorphic to the interior of a genus g handlebody, that converges geometrically to a hyperbolic 3–manifold N_∞ homeomorphic to $\Sigma_g \times \mathbb{R}$. Such a sequence can be assembled differently by working backwards from the limit.

Fix a hyperbolic 3–manifold $N_\infty \cong \Sigma_g \times \mathbb{R}$ with no cusps, one degenerate end and one convex-cocompact end. Such manifolds exist, for instance, by [30, Theorem 3.7]. If we use that construction then a theorem of McMullen [29] shows that N_∞ is an algebraic limit of one-sided maximal cusps. (In fact this is true no matter how N_∞ is constructed, but that relies on the recent resolution of the Density Conjecture for surface groups [13]).
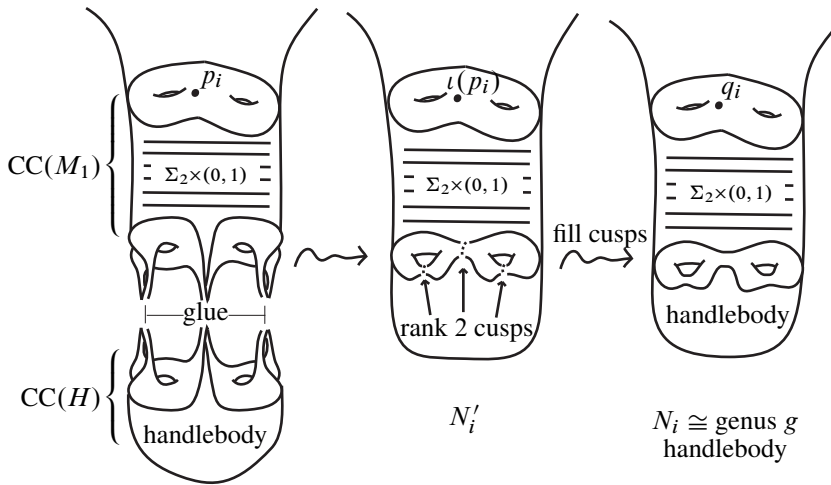
Figure 2: Producing long handlebodies by gluing maximal cusps

This means that there is a sequence of marked hyperbolic 3–manifolds $M_i \cong \Sigma_g \times \mathbb{R}$ converging algebraically to N_∞, where each M_i has one convex cocompact end and one maximally cusped end.

As N_∞ has no cusps, Theorem 1.1 implies that there are base points $p_i \in M_i$ so that $(M_i, p_i) \to (N_\infty, p_\infty)$ geometrically. Furthermore, p_i can be chosen to lie on the component of $\partial \mathrm{CC}(M_i)$ facing the convex-cocompact end of M_i. Note that since the thrice-punctured sphere components of $\partial \mathrm{CC}(M_i)$ disappear in the geometric limit, their distances to p_i grow without bound.

Passing to a subsequence, we may assume that the pinched pants decompositions on the maximally cusped ends of M_i all have the same topological type. In other words, we may choose a pants decomposition $P \subset \Sigma_g$ and homeomorphisms

$$M_i \cup \partial_c M_i \cong (\Sigma_g \times [0, 1]) \setminus (P \times \{0\})$$

for all i. Pick an identification of Σ_g with the boundary of a genus g handlebody H so that P is a pinchable collection of curves on ∂H; the latter condition can be ensured, for instance, by composing any fixed identification with a high power of a pseudo-Anosov homeomorphism of ∂H whose attracting lamination lies in the Masur domain. This allows us to endow H with a geometrically finite hyperbolic metric in which P has been pinched. Then both $\partial \mathrm{CC}(H)$ and the bottom boundary components of $\mathrm{CC}(M_i)$ are identified with $\Sigma_g \setminus P$, so we may use Lemma 2.12 to glue them together. This produces a sequence of hyperbolic 3–manifolds N_i' equipped with isometric embeddings of $\mathrm{CC}(H)$ and the subset $K_i \subset M_i$ that is the union of the

convex core of M_i and its convex-cocompact end; we will denote the inclusion of the latter by $\iota_i \colon K_i \to N_i'$. Since the frontier of K_i in M_i consists of the thrice-punctured sphere components of $\partial \mathrm{CC}(M_i)$, its distance to the base point $p_i \in M_i$ tends to infinity with i. The same holds for the distances from $\iota_i(p_i)$ to the frontier of $\iota_i(K_i) \subset N_i'$. Therefore, the sequence of based manifolds $(N_i', \iota_i(p_i))$ converges geometrically to the same limit, (N_∞, p_∞), as our original sequence.

Observe that N_i' is homeomorphic to the manifold obtained from H by pushing $P \subset \partial H$ into the interior of H and then drilling it out. The curves in P correspond to rank 2 cusps of N_i', so we may choose (x, y)–coordinates for the Dehn filling space of each cusp so that $(1, 0)$ corresponds to filling a curve that is contractible in H and $(0, 1)$ represents filling a curve homotopic into P. Then the manifold $N_{i,n}'$ obtained from $(1, n)$–Dehn filling each cusp of N_i' is homeomorphic to H (compare with [25, Section 3]). If n is large, an extension of Thurston's Dehn filling theorem due to Bonahon and Otal [9] and Comar [19] implies that $N_{i,n}'$ admits a unique hyperbolic structure with the same conformal boundary as N_i'. Furthermore, there are base points $p_{i,n} \in N_{i,n}'$ such that $(N_{i,n}', p_{i,n}) \to (N_i', \iota_i(p_i))$ geometrically. An appropriate sequence (n_i) can then be chosen so that (N_{i,n_i}', p_{i,n_i}) converges geometrically to (N_∞, p_∞). Setting $N_i = N_{i,n_i}'$ and $q_i = p_{i,n_i}$ finishes our work. $\qquad\square$

Example 2 *Let $\Gamma \cong \pi_1(\Sigma_3) \star \mathbb{Z}$ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to ρ and geometrically to G such that*

- *G does not contain any parabolic elements.*
- *$\rho(\Gamma)$ has infinite index in G.*

In the following, $\widehat{\Gamma} \cong \pi_1(\Sigma_2) \star \mathbb{Z}$ will be the fundamental group of a compression body with genus 3 exterior boundary and genus 2 interior boundary. The group Γ will be the subgroup generated by the fundamental group of a double cover of the interior boundary and a loop going around the remaining handle.

The representations here are constructed from those in the previous example by using Klein–Maskit combination:

The Combination Theorem (See [28].) *Let G_1 and G_2 be two discrete and torsion free subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. Suppose that there exist fundamental domains $D_i \subset \Omega(G_i)$ for G_i, each containing the exterior of the other. Then $G = \langle G_1, G_2 \rangle$ is discrete, torsion free and is isomorphic to $G_1 \star G_2$. Moreover, if the groups G_i do not contain parabolics then the same is true for G.*
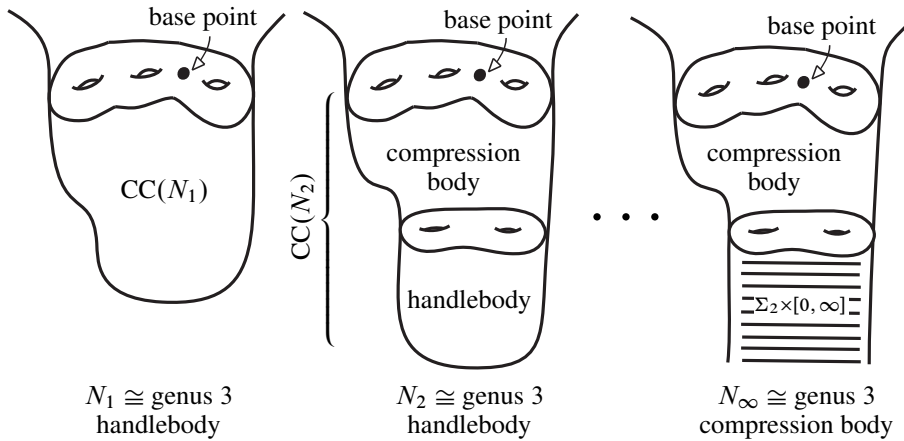
Figure 3: The geometric convergence $N_i \to N_\infty$

Let us recall the players from Example 1. For clarity, we will append apostrophes to all letters that represent objects from the first example. So, (ρ'_i) is a sequence of representations in $\mathcal{D}(\pi_1(\Sigma_2))$ converging strongly to a faithful representation ρ'_∞ without parabolics, and $\Gamma' \subset \pi_1(\Sigma_2)$ is an index 2 subgroup with $\rho_i(\Gamma') = \rho_i(\pi_1(\Sigma_2))$ for all i. For convenience, set $G'_i = \rho_i(\pi_1(\Sigma_2))$ and $G'_\infty = \rho_\infty(\pi_1(\Sigma_2))$.

Our previous analysis implies that the convex cores of $\mathbb{H}^3/G'_i$ converge to the convex core of $\mathbb{H}^3/G'_\infty$ in the Gromov–Hausdorff topology, so the limit sets $\Lambda(G'_i)$ converge to $\Lambda(G'_\infty)$ in the Hausdorff topology. Since the domains of discontinuity $\Omega(G'_i)$ and $\Omega(G'_\infty)$ are all connected and nonempty, it follows that there are fundamental domains D_i for $G'_i \curvearrowright \Omega(G'_i)$ converging in the Hausdorff topology to a fundamental domain D_∞ for $G'_\infty \curvearrowright \Omega(G'_\infty)$.

Pick some loxodromic element $\alpha \in \mathrm{PSL}(2, \mathbb{C})$ with fixed points contained in the interior of D_∞ (which is connected). Moreover, assume that its translation distance is large enough so that there is a fundamental set for $\Omega(\langle\alpha\rangle)$ whose complement is entirely contained within D_∞. After discarding a finite number of terms, its complement will also be contained in D_i for all i. We now construct new representations

$$\rho_i\colon \widehat{\Gamma} \longrightarrow \mathrm{PSL}(2, \mathbb{C}), \quad \text{where } \widehat{\Gamma} := \pi_1(\Sigma_2) \star \mathbb{Z}$$

from ρ'_i by sending $1 \in \mathbb{Z}$ to α..If $G_i \subset \mathrm{PSL}(2, \mathbb{C})$ is the image of ρ_i, the Klein–Maskit combination theorem implies that $G_i = G'_i \star \langle\alpha\rangle$ and is discrete, torsion free and has no parabolics. The same statements apply when $i = \infty$, showing that ρ_∞ is faithful with image

$$G_\infty = G'_\infty \star \langle\alpha\rangle .$$

The manifolds $N_i = \mathbb{H}^3/G_i$, $i = 1, 2, \ldots$ are all convex-cocompact and homeomorphic to the interior of a genus 3 handlebody. The manifold $N_\infty = \mathbb{H}^3/G_\infty$ is homeomorphic to a compression body with genus 3 exterior boundary and connected, genus 2 interior boundary.

It follows, for instance, from the argument in [3, Prop 10.2] that $\rho_i \to \rho_\infty$ strongly. Consider the subgroup

$$\Gamma := \Gamma' \star \mathbb{Z} \subset \pi_1(\Sigma_2) \star \mathbb{Z} =: \widehat{\Gamma}.$$

Then $\rho_i|_\Gamma$ converges algebraically to $\rho_\infty|_\Gamma$. Since $\rho_i'(\Gamma') = \rho_i'(\pi_1(\Sigma_2))$ for all $i \in \mathbb{N}$, we also have that

$$\rho_i(\Gamma) = \rho_i(\widehat{\Gamma}) \text{ for all } i \in \mathbb{N}.$$

So $\rho_i(\Gamma)$ converges geometrically to $\rho_\infty(\widehat{\Gamma}) = G_\infty$. As ρ_∞ is faithful and $[\widehat{\Gamma} : \Gamma] = \infty$, it follows that $\rho_\infty(\Gamma)$ has infinite index in $\rho_\infty(\widehat{\Gamma}) = G_\infty$. Therefore $\rho_i \colon \Gamma \to PSL(2, \mathbb{C})$ is a sequence of representations for which the algebraic limit has infinite index in the geometric limit. As mentioned above, the geometric limit G_∞ has no parabolics. Since $\mathbb{H}^3/\rho_\infty(\Gamma)$ is homeomorphic to a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3, we have provided the desired example.

Example 3 *Let Γ be the fundamental group of a compression body with exterior boundary of genus 4 and connected interior boundary of genus 3. There is a sequence (ρ_i) in $\mathcal{D}(\Gamma)$ converging algebraically to a representation ρ and geometrically to a group G such that*

- *$\rho(\Gamma)$ does not contain any parabolic elements.*
- *G is infinitely generated.*

In [38], Thurston exhibited a sequence of representations of a closed surface group into $PSL(2, \mathbb{C})$ converging geometrically to a group that is not finitely generated. However, the algebraic limit of these representations contains parabolic elements. The idea here is to attach pieces of the handlebodies in Example 2 to the manifolds in his sequence so that the parabolics are hidden outside the algebraic limit.

To facilitate such a combination, we must build a variant of Example 2 in which each of the handlebodies in the sequence is maximally cusped. Let M be a compression body with genus 3 exterior boundary S_E and connected, genus 2 interior boundary S_I. Assume that $P_E \subset S_E$ is a pants decomposition consisting of curves in the Masur domain of M.

Claim *There is a sequence of maximally cusped pointed hyperbolic 3–manifolds* (N_i), *each homeomorphic to the interior of a genus 3 handlebody, that converges geometrically to a hyperbolic 3–manifold* N_∞ *homeomorphic to the interior of* M *in which* P_E *has been pinched. Moreover, the subsets* $\mathrm{CC}(N_i) \subset N_i$ *converge to* $\mathrm{CC}(N_\infty)$ *geometrically.*

Proof The sequence (N_i) will be constructed in two steps. First, we will produce a sequence of hyperbolic 3–manifolds M_i homeomorphic to the interior of M in which both ends are maximally cusped. The sequence will converge geometrically to the manifold N_∞ referenced in the statement of the claim. We will then use the same gluing trick exploited in Example 1 to cap off the interior ends of each M_i without changing the sequence's geometric limit, thus producing the desired sequence of handlebodies (N_i).

Fix pants decompositions P_E and P_I for the boundary components of M, and assume that every curve in P_E lies in the Masur domain of M. If we choose a pseudo-Anosov diffeomorphism $f\colon S_I \to S_I$, then for each i we have a new pants decompositions $f^i(P_I)$ for S_I. It is not hard to check that for each i, $P_E \cup f^i(P_I)$ is a pinchable collection of curves on ∂M (see Section 2.8). So, there is a sequence of marked hyperbolic 3–manifolds $M_i \in AH(M)$ in which $P_E \cup f^i(P_I)$ have been pinched. Note that in fact M_i lies in the deformation space $AH(M, P_E)$ of hyperbolic structures on the interior of M in which the curves of P_E represent parabolics. Since P_E consists of curves lying in the Masur domain, the pared manifold (M, P_E) has incompressible and acylindrical boundary. It follows from a theorem of Thurston [39, Theorem 7.1] that $AH(M, P_E)$ is compact. So after passing to a subsequence, we may assume that (M_i) converges algebraically to some $N_\infty \in AH(M, P_E)$.

We claim that the only parabolic loops in N_∞ are those that are freely homotopic into P_E. The end of N_∞ facing S_E is maximally cusped by P_E, so there is no room for additional cusps there. It therefore suffices to show that the end facing S_I has no cusps. Work of Thurston, Bonahon and Brock, implies that there is a continuous map

$$\mathrm{length}\colon AH(M) \times \mathcal{ML}(S_I) \to \mathbb{R}$$

that extends the function that assigns to an element $N \in AH(M)$ and a simple closed curve $\gamma \in S_I$ the shortest length of a curve in N homotopic to γ (see [10]). Since $\mathrm{length}_{M_i}(f^i(P_I)) = 0$ for all i, in the limit we have $\mathrm{length}_{M_\infty}(\lambda) = 0$, where λ is the attracting lamination of f. This implies that λ cannot be geodesically realized by a pleated surface in N_∞ homotopic to S_I. Let $\widehat{N}_\infty$ be the cover of N_∞ corresponding to $\pi_1(S_I)$. The end of N_∞ facing S_I lifts homeomorphically to an end $\mathcal{E}$ of $\widehat{N}_\infty$. The other end of $\widehat{N}_\infty$ has no cusps and is therefore convex-cocompact by the Tameness

Theorem and Thurston's Covering Theorem (Theorem 2.2). Since λ is filling and unrealizable in $\hat{N}_\infty$, the argument in [28, Theorem 6.34] then shows that $\mathcal{E}$ is degenerate with ending lamination λ. In particular, $\mathcal{E}$ has no cusps. Projecting down, the same is true for the end of N_∞ facing S_I.

This shows that the convergence $M_i \to N_\infty$ is type preserving, so Theorem 1.1 implies that the convergence is strong. So, base frames for M_i can be chosen so that the sequence converges geometrically to N_∞. The rest of the argument follows that given at the end of Example 1. Fix a genus 2 handlebody H and choose a hyperbolic metric on its interior in which a pants decomposition $P \subset \partial H$ with the same topological type as $P_I \subset S_I$ has been pinched. We can then create for each i a hyperbolic 3–manifold N_i' by removing from M_i the component of $M_i \setminus CC(M_i)$ facing S_I and gluing $CC(H)$ in its place. As in Example 1, the sequence (N_i') converges geometrically to N_∞. Performing an appropriate Dehn filling on each N_i' yields a sequence of hyperbolic 3–manifolds (N_i), each homeomorphic to the interior of a genus 3 handlebody, that also converges geometrically to N_∞.

We must show that $CC(N_i) \to CC(N_\infty)$ geometrically. First, every component of $\partial CC(N_\infty)$ is contained in a geometric limit of some sequence of components of $\partial CC(N_i)$. These are all thrice-punctured spheres, however, so in fact we have that for large i there are components of $\partial CC(N_i)$ that closely approximate each component of $\partial CC(N_\infty)$. However, $\partial CC(N_i)$ and $\partial CC(N_\infty)$ both have 4 components, so for large i they must almost coincide. From this, it is easy to check that $CC(N_i) \to CC(N_\infty)$ geometrically. $\qquad\square$

Fix a geometrically finite hyperbolic structure on $\Sigma_3 \times \mathbb{R}$ in which both ends are maximally cusped, and let C be its convex core. Then there are pants decompositions $P_{+,-} \subset \Sigma_3$ so that

$$C \cong (\Sigma_3 \times [-1, +1]) \setminus (P_- \times \{-1\} \cup P_+ \times \{+1\}),$$

and we label the components of ∂C as positive and negative accordingly, so that $\partial C = \partial_+ C \cup \partial_- C$. Let E be the union of all components of the complement of C that face its positive boundary components. Assume that the pairs $(\Sigma_3, P_{+,-})$ and (S_E, P_E) have the same topological type, and that every curve in P_+ intersects some curve in P_-.

We now glue these pieces to the manifolds N_i and N_∞ from the previous Lemma. To begin with, let

$$N_\infty' = CC(N_\infty) \sqcup_h C \sqcup_g C \sqcup_g \cdots .$$

Here, the gluing maps $h \colon \partial CC(N_\infty) \to \partial_- C$ and $g \colon \partial_- C \to \partial_+ C$ can be any isometries that extend to maps $(S_E, P_E) \to (\Sigma_3, P_-)$ and $(\Sigma_3, P_+) \to (\Sigma_3, P_-)$. This

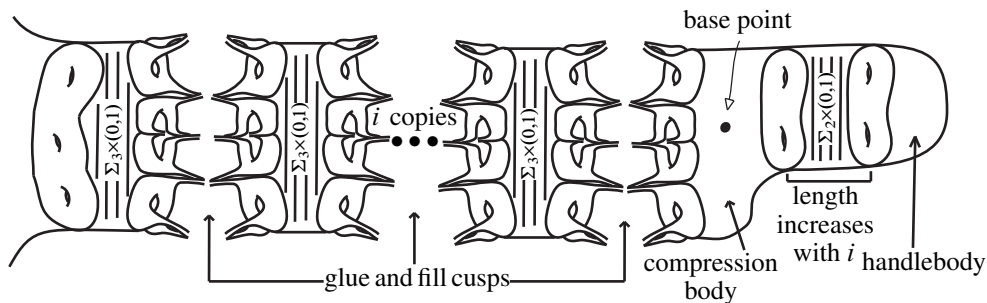Figure 4: The construction of the manifold N_i^*

ensures that N_∞' is constructed as in Lemma 2.12. Note that the inclusion map $\mathrm{CC}(N_\infty) \to N_\infty'$ is π_1–injective.

Next, for large i geometric convergence and the gluing map h determine an identification $h_i \colon \partial \mathrm{CC}(N_i) \to \partial_- C$. Define

$$N_i' = \mathrm{CC}(N_i) \sqcup_{h_i} \underbrace{C \sqcup_g \cdots \sqcup_g C}_{i \text{ times}} \sqcup_{\mathrm{id}} E.$$

Recall that $\mathrm{CC}(N_i) \to \mathrm{CC}(N_\infty)$ geometrically. From this it follows that if N_i' is given the base frame of N_i produced in the previous Lemma, then (N_i') converges geometrically to N_∞'.

As in Example 1, performing $(1, n)$–Dehn filling on each of the cusps in N_i' produces, for large n, a new hyperbolic manifold $N_{i,n}'$ homeomorphic to the interior of a genus 3 handlebody. Also, an appropriate diagonal sequence $N_i^* = N_{i,n_i}'$ can be chosen to converge geometrically to N_∞'. In summary, we have proven the following claim.

Claim *There is a sequence of convex-cocompact pointed hyperbolic 3–manifolds N_i^*, each homeomorphic to the interior of a genus 3 handlebody, that converges geometrically to a hyperbolic 3–manifold N_∞' with infinitely generated fundamental group.*

We now obtain the sequence of representations advertised in the statement of this example by marking the manifolds N_i^* appropriately. Recall that the fundamental group of the compression body M splits as a free product

$$\pi_1(M) = \pi_1(S_I) \star \langle \alpha \rangle \cong \pi_1(\Sigma_2) \star \mathbb{Z},$$

for some element $\alpha \in \pi_1(M)$. The inclusion map $M \cong \mathrm{CC}(N_\infty) \to N_\infty'$ is π_1–injective, so it determines an embedding

$$\rho_\infty \colon \pi_1(\Sigma_2) \star \mathbb{Z} \longrightarrow \pi_1(N_\infty').$$

Then ρ_∞ identifies a finite generating set for $\pi_1(\Sigma_2) \star \mathbb{Z}$ with a finite set of loops in N'_∞. For large i, geometric convergence provides an almost isometric embedding of these loops into N_i^*; therefore, there are induced homomorphisms

$$\rho_i \colon \pi_1(\Sigma_2) \star \mathbb{Z} \longrightarrow \pi_1(N_i^*).$$

In fact, ρ_i is surjective, and therefore is a marking of $\pi_1(N_i^*)$.

The sequence of marked hyperbolic manifolds (N_i^*, ρ_i) converges algebraically to the cover of N'_∞ corresponding to the image of ρ_∞, and converges geometrically to N'_∞ (as noted above). Note that $\pi_1(N'_\infty)$ is not finitely generated. We are not quite done, however, because the algebraic limit here has cusps. To hide the cusps, we will use the finite index trick exploited in Example 1, but for this to work the pants decomposition P_E used above must be chosen more carefully.

Claim   *There is a pants decomposition $P_E \subset S_E$ consisting of curves $\{\gamma_1, \ldots, \gamma_n\}$ in the Masur domain with the property that no power of any γ_i is conjugate into any subgroup of $\pi_1(M)$ of the form $\Gamma' \star \langle \alpha \rangle$, where $\Gamma' < \pi_1(S_I, p)$ has index 2.*

Deferring the proof for a moment, pick as in Example 1 and Example 2 an index 2 subgroup $\Gamma' \subset \pi_1(\Sigma_2)$ so that the restriction of

$$\rho_i \colon \pi_1(\Sigma_2) \star \mathbb{Z} \longrightarrow \mathrm{PSL}(2, \mathbb{C})$$

to the subgroup $\Gamma = \Gamma' \star \mathbb{Z}$ surjects onto each $\pi_1(N_i^*)$. If in constructing (N_i^*), the pants decomposition P_E is chosen as indicated in the above claim, there will be no parabolics in the algebraic limit of $(\rho_i|_\Gamma)$. However, since this is still a sequence of markings for N_i^*, the geometric limit will be N'_∞, which has infinitely generated fundamental group. This finishes the example.

Proof of Claim   Although the claim is purely topological, the proof we give uses 3–dimensional hyperbolic geometry. It would be nice to give a more straightforward proof; also, it is possible that the second part of the conclusion is satisfied by any pants decomposition of curves in the Masur domain.

To begin, construct by some means a hyperbolic manifold N homeomorphic to the interior of M that has no cusps and in which both ends are degenerate. One way to produce N is as follows. First, find a hyperbolic manifold homeomorphic to $\Sigma_3 \times \mathbb{R}$ that has one degenerate end and one maximally cusped end. This can be done using an argument similar to the construction of N_∞ above. We can then glue its convex core to the convex core of N_∞ as in Lemma 2.12 and fill the resulting cusps to create a totally degenerate hyperbolic manifold N homeomorphic to the interior of M. See Bonahon and Otal [9] for information on Dehn filling geometrically infinite manifolds.

Fix an index 2 subgroup $\Gamma' \subset \pi_1(S_I, p)$, and let $N_{\Gamma'}$ be the cover of N corresponding to $\Gamma' \star \langle \alpha \rangle$. Then $N_{\Gamma'}$ has a degenerate end homeomorphic to $\Sigma_3 \times [0, \infty)$ that double covers the genus 2 end of N. Adjoining a loop in $N_{\Gamma'}$ representing α to a level surface of this end and thickening produces a compact core for $N_{\Gamma'}$ homeomorphic to the interior of a compression body with genus 4 exterior boundary and connected, genus 3 interior boundary. The Tameness Theorem of Agol [1] and Calegari and Gabai [15] implies that $N_{\Gamma'}$ is itself homeomorphic to the interior of such a compression body, and a theorem of Canary [17] implies that its genus 4 end, $\widehat{\mathcal{E}}$, is either degenerate or convex-cocompact. It cannot be that both ends of $N_{\Gamma'}$ are degenerate, for then Canary's Covering Theorem (Theorem 2.2) would imply that $\Gamma' \star \langle \alpha \rangle$ is finite index in $\pi_1(N)$. Therefore, $\widehat{\mathcal{E}}$ is convex-cocompact.

Since the genus 3 end, $\mathcal{E}$, of N is degenerate, it has an ending lamination $\lambda \subset S_E$. Canary has shown that λ lies in the Masur domain of M (see [17] for a proof of this and the uniqueness of λ). Choose a pants decomposition $P_E \subset S_E$ consisting of curves that lie close to λ in $\mathcal{PML}(S_E)$. The Masur domain of M is an open subset of $\mathcal{PML}(S_E)$, so we may assume that each curve in P_E lies inside of it. Furthermore, their geodesic representatives in N lie very deep inside of $\mathcal{E}$. As $\widehat{\mathcal{E}}$ is convex-cocompact, the convex core of $N_{\Gamma'}$ covers a subset of N that has bounded intersection with $\mathcal{E}$. So, we may assume that the geodesic representatives of curves in P_E do not intersect its image. If some power of a curve in P_E were conjugate into $\Gamma' \star \langle \alpha \rangle$, then its geodesic representative in N would lift to a closed geodesic in $N_{\Gamma'}$. Every closed geodesic in $N_{\Gamma'}$ is contained in its convex core, so this is impossible.

This last paragraph shows that as long as the curves in P_E are chosen within a small neighborhood around the ending lamination λ, then no powers of them are conjugate into subgroup of $\pi_1(M)$ of the form $\Gamma' \star \langle \alpha \rangle$, where Γ' is a *fixed* index 2 subgroup of $\pi_1(S_I)$. However, there are only finitely many index 2 subgroups of $\pi_1(S_I)$, so it follows that if we choose P_E from the intersection of all such neighborhoods we can ensure that no power of any of its curves is conjugate into *any* such $\Gamma' \star \langle \alpha \rangle$. $\qquad\square$

4 Proofs of Theorems 1.4 and 1.5

Before beginning the bulk of the proof, we will present a technical lemma whose proof requires a bit of differential geometry. Afterwards, Theorems 1.4 and 1.5 will follow from purely topological arguments.

Recall from Proposition 2.3 that a hyperbolic 3–manifold M with finitely generated fundamental group and no cusps contains a compact core $C \subset M$ for which each component of ∂C facing a convex-cocompact end of M is smooth and strictly convex.

For convenience, let S_{cc} be the union of those components of ∂C facing convex cocompact ends and E_{cc} the union of the adjacent components of $M \setminus C$. Then E_{cc} is homeomorphic to $S_{cc} \times (0, \infty)$ via "radial coordinates":

$$R \colon S_{cc} \times (0, \infty) \to E_{cc}, \quad R(x, t) = \exp_x(t\nu(x)),$$

where ν is the outer unit-normal vector field along S_{cc}.

If $f \colon C \to N$ is a smooth immersion into some complete hyperbolic 3–manifold N, it has a natural *radial extension* $\bar{f} \colon C \cup E_{cc} \to N$. Namely, there is a radial coordinate map along the image of S_{cc}:

$$R_f \colon S_{cc} \times (0, \infty) \to N, \quad R_f(x, t) = \exp_{f(x)}(t\nu_f(x)),$$

where $\nu_f(x)$ is the unit vector in $TN_{f(x)}$ orthogonal to $df_x(TS_x)$ that points away from $f(C)$, and one can then define

$$\bar{f}(p) = \begin{cases} R_f \circ R^{-1}(p) & p \in E_{cc}, \\ f(p) & p \in C. \end{cases}$$

Observe that $\bar{f}$ is continuous, and differentiable everywhere but on S_{cc}.

In the situations where we will find radial extensions useful, the map f will be very close in the C^2 topology to a *Riemannian* immersion. In particular, the (strict) convexity of the surface S_{cc} will persist in the image. This implies a convenient regularity in the radial extension:

Lemma 4.1 *If $f \colon C \to N$ is a smooth immersion with $f(S_{cc})$ convex, there exists some $L > 0$ so that for all $p \in E_{cc}$ and $v \in TM_p$,*

$$\frac{1}{L} \leq \frac{\|d\bar{f}_p(v)\|}{\|v\|} \leq L.$$

The following global statement comes from applying Lemma 4.1 and a compactness argument on C.

Corollary 4.2 (Radial extensions are locally bilipschitz) *If $f \colon C \to N$ is a smooth immersion with $f(S_{cc})$ convex, there exists some $L > 0$ so that every $p \in C \cup E_{cc}$ has a neighborhood on which $\bar{f}$ is L–bilipschitz.*

Proof of Lemma 4.1 Consider a component $E \subset E_{cc}$, and let $S \subset S_{cc}$ be the adjacent component of ∂C. Here, $\bar{f}$ is the composition $R_f \circ (R)^{-1}$ of radial coordinate maps,

so to prove the Lemma it suffices to find a constant L so that for all $(x, t) \in S \times (0, \infty)$ and $v \in TS_x \times \mathbb{R}$,

$$
(4\text{-}1) \qquad \frac{1}{L} \leq \frac{\|(dR_f)_{(x,t)}(v)\|}{\|dR_{(x,t)}(v)\|} \leq L.
$$

Since the ratio is one when v is contained in the $\mathbb{R}$ factor, from now on we will assume $v \in TS_x$.

We first estimate $\|dR_{(x,t)}(v)\|$. Given $x \in S$ and $v \in TS_x$, let $g(s)$ be a curve in S with $g(0) = x$ and $g'(0) = v$, and consider the geodesic variation

$$
\gamma_s(t) := R(g(s), t) = \exp_{g(s)}(tv(g(s))).
$$

The corresponding Jacobi field $J_{x,v}(t) = \frac{d}{ds} R(g(s), t)|_{s=0}$ along the geodesic $\gamma_0(t)$ then satisfies $J_{x,v}(t) = dR_{(x,t)}(v)$.

The Jacobi-field $J_{x,v}(t)$ is determined by its initial conditions

$$
J_{x,v}(0) = dR_{(x,0)}(v),
$$

and

$$
\begin{aligned}
\frac{\nabla}{dt} J_{x,v}(t)|_{t=0} &= \frac{\nabla}{dt} \frac{d}{ds} \exp_{g(s)}\left(tv(g(s))\right)\big|_{t=0} \\
&= \frac{\nabla}{ds} \frac{d}{dt} \exp_{g(s)}\left(tv(g(s))\right)\big|_{t=0} \\
&= \frac{\nabla}{ds} v(g(s)) = \nabla_{dR_{(x,0)}(v)} v
\end{aligned}
$$

Therefore, we have

$$
J_{x,v}(t) = \cosh(t) E_1(t) + \sinh(t) E_2(t),
$$

where $E_1(t)$ and $E_2(t)$ are the parallel vector fields along γ_0 with $E_1(0) = dR_{(x,0)}(v)$ and $E_2(0) = \nabla_{dR_{(x,0)}(v)} v$. That the right-hand side satisfies the Jacobi equation follows quickly from the fact that $E_1(t)$ and $E_2(t)$ are both orthogonal to $\gamma_0'(t)$.

The triangle inequality, together with the fact that the vector fields E_1 and E_2 have constant length, shows that

$$
\|J_{x,v}(t)\| \leq \cosh(t)(\|dR_{(x,0)}(v)\| + \|\nabla_{dR_{(x,0)}(v)} v\|).
$$

On the other hand we have

$$
\begin{aligned}
\|J_{x,v}(t)\|^2 = \cosh(t)^2 \|dR_{(x,0)}(v)\|^2 &+ \sinh(t)^2 \|\nabla_{dR_{(x,0)}(v)} v\|^2 \\
&+ 2\cosh(t)\sinh(t)\langle dR_{(x,0)}v, \nabla_{dR_{(x,0)}(v)} v\rangle.
\end{aligned}
$$

By convexity of S_{cc}, the last term in the sum is positive. A little bit of algebra and the fact that $J_{x,v}(t) = dR_{(x,t)}(v)$ yields

$$(4\text{-}2) \qquad \frac{\sinh(t)}{2} \leq \frac{\|dR_{(x,t)}(v)\|}{\|dR_{(x,0)}(v)\| + \|\nabla_{dR_{(x,0)}(v)}v\|} \leq \cosh(t).$$

Since $f(S_{\mathrm{cc}})$ is convex, a similar computation shows

$$(4\text{-}3) \qquad \frac{\sinh(t)}{2} \leq \frac{\|d(R_f)_{(f(x),t)}(v)\|}{\|d(R_f)_{(f(x),t)}(v)\| + \|\nabla_{d(R_f)_{(f(x),t)}v}v_f\|} \leq \cosh(t).$$

By compactness the ratio between the denominators in (4-2) and (4-3) is uniformly bounded from above and below. In other words, there is some positive constant c with

$$(4\text{-}4) \qquad \frac{\sinh(t)}{2c\cosh(t)} \leq \frac{\|d(R_f)_{(f(x),t)}(v)\|}{\|dR_{(x,t)}(v)\|} \leq \frac{2c\cosh(t)}{\sinh(t)}$$

for all $(x,t) \in S \times (0,\infty)$ and $v \in T_x S$. If t is constrained away from zero then the lower and upper bounds in (4-4) are bounded by positive numbers from below and above respectively. When $t = 0$ both $dR_{(x,t)}$ and $d(R_f)_{(x,t)}$ have maximal rank, so constant positive bounds arise from a compactness argument. This yields (4-1) and concludes the proof of the Lemma. □

We are now ready to prove the main result of this section.

Theorem 1.5 *Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) converges algebraically to a representation ρ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If*

- *$\rho(\Gamma)$ does not contain parabolic elements, and*
- *every degenerate end of $\mathbb{H}^3/\rho(\Gamma)$ has a neighborhood which embeds under the covering $\mathbb{H}^3/\rho(\Gamma) \to \mathbb{H}^3/G$,*

then $G = \rho(\Gamma)$.

Proof Before going further, set $M_A = \mathbb{H}^3/\rho(\Gamma)$, $M_G = \mathbb{H}^3/G$ and $M_i = \mathbb{H}^3/\rho_i(\Gamma)$ for $i = 1, 2, \ldots$, choose a base frame $\omega_{\mathbb{H}^3}$ for hyperbolic space $\mathbb{H}^3$ and let ω_i, ω_A and ω_G be the corresponding base frames of M_i, M_A and M_G respectively. By Proposition 2.7, the pointed manifolds (M_i, ω_i) converge geometrically to (M_G, ω_G). We may assume that M_G is noncompact, for otherwise Mostow's Rigidity Theorem implies that the sequence (M_i) must be eventually stable, so certainly $G = \rho(\Gamma)$.

We claim M_G has infinite volume. If not, it has finite volume and (M_i) is obtained by performing Dehn filling on M_G with larger and larger coefficients [6, Theorem E.2.4].

Then there must be parabolics in the algebraic limit $\rho(\Gamma)$. For otherwise, combining a result of Canary [17] with the Tameness Theorem of Agol [1] and Calegari and Gabai [15] implies that $M_A = \mathbb{H}^3/\rho(\Gamma)$ is either convex-cocompact, or has a degenerate end. The first case is impossible by Marden's Stability Theorem [27] and the second violates Canary's Covering Theorem (Theorem 2.2).

There is a covering map $\pi\colon M_A \to M_G$ induced by the inclusion $\rho(\Gamma) \subset G$, and our goal is to show that this is a homeomorphism. Recall that Proposition 2.3 provides an exhaustion of M_A by compact cores $C \subset M_A$ such that

(1) if a component S of ∂C faces a convex cocompact end of M_A then S is smooth and strictly convex.

(2) if a component S of ∂C faces a degenerate end of M_A then the restriction $\pi|_S\colon S \to \pi(S)$ is an embedding. (Proposition 2.3 only gives that $\pi|_S$ is finite-to-one, but the second assumption in the statement of the theorem implies that we may take $\pi|_S$ to be an embedding.)

Fixing a compact core $C_0 \subset M_A$, we may also assume that all C are large enough to contain C_0 and satisfy the following property:

(3) $\pi(C_0) \cap \pi(S) = \varnothing$ for any component $S \subset \partial C$ facing a degenerate end of M_A.

Then to prove that π is a homeomorphism, it clearly suffices to show that $\pi|_C$ is injective for all such compact cores $C \subset M_A$.

Fix a compact core $C \subset M_A$ as described above. For sufficiently large i, the geometric convergence $(M_i, \omega_i) \to (M_G, \omega_G)$ supplies an almost isometric embedding $\phi_i\colon \pi(C) \hookrightarrow M_i$, so for large i, we have a map

$$f_i\colon C \to M_i, \quad f_i = \phi_i \circ (\pi|_C)$$

that behaves much like the restriction of a nearly Riemannian covering map. In fact, f_i is π_1–surjective. For if $S \subset \Gamma$ is a finite generating set then C contains loops based at ω_A representing the elements of $\rho(S)$; f_i then maps these loops to loops in M_i representing $\rho_i(S)$, which generate $\pi_1(M_i) \cong \rho_i(\Gamma)$. We aim to show that f_i is actually an embedding, as the same will then be true for $\pi|_C$.

We first consider the case where M_A is convex-cocompact, as the proof is particularly simple. In this case, every component of ∂C is strictly convex and faces a convex-cocompact end of M_A, so f_i radially extends (as in the beginning of the section) to a globally defined map
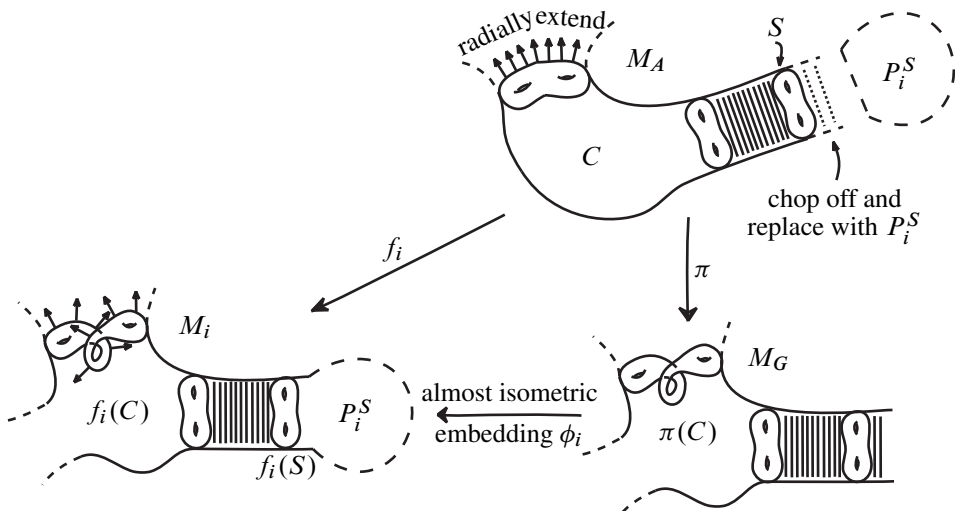
$$\bar{f}_i\colon M_A \to M_i.$$

Figure 5: Extending $f_i = \phi_i \circ \pi$ to a cover of M_i

We claim that this is a covering map for $i \gg 0$. To see this, note that when i is large, f_i is C^2–close to a local isometry, so the strict convexity of ∂C persists after applying f_i. Therefore Corollary 4.2 applies to show that $\overline{f}_i$ is (uniformly) locally bilipschitz. It is well-known that any locally isometric map between complete Riemannian manifolds is a covering map, and in fact the same proof applies to locally bilipschitz maps. So, $\overline{f}_i \colon M_A \to M_i$ is a covering map. However, f_i is π_1–surjective, so its extension $\overline{f}_i$ is a π_1–surjective covering map, and therefore a homeomorphism. This shows that f_i is injective, and in particular $\pi|_C$ is as well.

In the general case, the argument needs modification because one cannot radially extend f_i into the degenerate ends of M_A. To deal with this, we will alter the problematic parts of M_A so that an extension of f_i is obvious.

Claim *If $S \subset \partial C$ faces a degenerate end of M_A, then the restriction of f_i to S is an embedding with image a separating surface in M_i.*

Proof The map $f_i|_S = \phi_i \circ \pi|_S$ is a composition of embeddings, so it is an embedding itself. Property (3) above implies that every component $S \subset f_i(S)$ is disjoint from $f_i(C_0)$. However, the argument given to show that $f_i|_C$ is π_1–surjective also applies to $f_i|_{C_0}$, so every loop in M_i is homotopic into $f_i(C_0)$ and therefore has trivial algebraic intersection with S. Therefore S is separating.   □

For each such S, let P_i^S be the closure of the component of $M_i \setminus f_i(S)$ that does not contain $f_i(C_0)$. Then if E_{cc} is the union of the components of $M_A \setminus C$ that are

neighborhoods of convex cocompact ends, one can construct a new 3–manifold

$$M'_A = b(C \cup E_{\mathrm{cc}}b) \sqcup_{f_i} \left(\bigcup_S P_i^S \right)$$

by gluing each P_i^S to $C \cup E_{\mathrm{cc}}$ along S. The map f_i extends naturally to a continuous map $\bar{f}_i \colon M'_A \to M_i$; the extension into E_{cc} is radial and on P_i^S we use the natural inclusion into M_i. It is easy to see that $\bar{f}_i$ is a π_1–surjective covering map, so the proof ends the same way it did in the previous case and $M_A = M'_A$. $\qquad\square$

As in the introduction, Theorem 1.5 has the following more aesthetic consequence.

Theorem 1.4 Let Γ be a finitely generated group and (ρ_i) a sequence in $\mathcal{D}(\Gamma)$. Assume that (ρ_i) converges algebraically to a representation $\rho \in \mathcal{D}(\Gamma)$ and geometrically to a subgroup G of $\mathrm{PSL}_2\,\mathbb{C}$. If

- $\rho(\Gamma)$ does not contain parabolic elements, and
- there is no $\gamma \in G \setminus \rho(\Gamma)$ with $\gamma^k \in \rho(\Gamma)$ for some $k \geq 2$,

then $G = \rho(\Gamma)$.

Proof As in the previous proof, it suffices to assume that $M_A = \mathbb{H}^3/\rho(\Gamma)$ and $M_G = \mathbb{H}^3/G$ both have infinite volume. Canary's Covering Theorem (Theorem 2.2) then implies that every degenerate end $\mathcal{E}$ of M_A has a neighborhood that finitely covers a neighborhood of a degenerate end $\mathcal{E}'$ of M_G. More precisely, there are neighborhoods $U_{\mathcal{E}}$ and $U_{\mathcal{E}'}$ of $\mathcal{E}$ and $\mathcal{E}'$, a finite covering $\sigma \colon \partial\mathcal{E} \to \partial\mathcal{E}'$, and homeomorphisms

$$\phi \colon \partial\mathcal{E} \times \mathbb{R} \to U_{\mathcal{E}}, \quad \psi \colon \partial\mathcal{E}' \times \mathbb{R} \to \pi(U_{\mathcal{E}})$$

with $(\psi^{-1} \circ \pi|_{U_{\mathcal{E}}} \circ \phi)(x,t) = (\sigma(x),t)$.

We claim that since there is no $\gamma \in G \setminus \rho(\Gamma)$ with $\gamma^k \in \rho(\Gamma)$, each of these coverings must be injective. For otherwise, one could find a path $\gamma \colon [0,1] \to U_{\mathcal{E}}$ with $\gamma(0) \neq \gamma(1)$ but $\pi(\gamma(0)) = \pi(\gamma(1))$. Then $\pi \circ \gamma$ represents an element $\gamma \in G \setminus \rho(\Gamma)$. Moreover, since the covering $\pi|_{U_{\mathcal{E}}}$ is finite-to-one, we must have $\gamma^k \in \rho(\Gamma)$ for some $k \geq 2$.

Therefore, every degenerate end of M_A has a neighborhood that embeds in M_G. So, Theorem 1.5 applies. $\qquad\square$

Before concluding this section, observe that Theorem 1.4 together with Theorem 1.3 imply Theorem 1.1 of Anderson and Canary mentioned in the introduction.

5   Attaching roots

In this section we prove:

Proposition 5.1 *Let M and N be hyperbolic 3–manifolds with infinite volume and let $\tau\colon M \to N$ be a covering. Assume that N has no cusps, that $\pi_1(M)$ is finitely generated and that M has a degenerate end which does not embed under the covering τ. Then there is a hyperbolic 3–manifold M' with finitely generated fundamental group, with*

$$|\chi(M')| < |\chi(M)|$$

and coverings $\tau'\colon M \to M'$ and $\tau''\colon M' \to N$ with $\tau = \tau'' \circ \tau'$.

Note that since M and M' are not closed, the ratio of $\chi(M)$ and $\chi(M')$ is not necessarily the degree of the covering $M \to M'$.

Continuing with the notation above, by Proposition 2.3 the manifold M has a standard compact core C with the property that if a component S of ∂C faces a degenerate end of M then the restriction of τ to S is a covering onto an embedded surface in N. The assumption that M has a degenerate end which does not embed under the covering τ implies that there is actually a component S_0 of ∂C such that

$$\tau|_{S_0}\colon S_0 \to \tau(S_0)$$

is a nontrivial covering. Observe that by the Covering Theorem (Theorem 2.2) the embedded surface $\tau(S_0) \subset N$ faces a degenerate end of N.

Choosing a base point $* \in S_0$, we set

$$\Gamma = \pi_1(M, *) \quad \text{and} \quad H = \pi_1(\tau(S_0), \tau(*)).$$

The desired manifold M' will be the cover of N corresponding to the subgroup

$$\Gamma' = \langle \tau_*(\Gamma), H \rangle \subset \pi_1(N, \tau(*)).$$

By construction, $\pi_1(M') \cong \Gamma'$ is finitely generated and there are covering maps $\tau'\colon M \to M'$ and $\tau''\colon M' \to N$ with $\tau = \tau'' \circ \tau'$, so it remains only to prove that $|\chi(M')| < |\chi(M)|$.

For our purposes, the most useful way to interpret the Euler characteristic will be through its relation to the dimension of the deformation spaces $\mathcal{CH}(M)$ and $\mathcal{CH}(M')$ of convex-cocompact hyperbolic structures on M and M' (see Section 2.4). Observe that since M and M' have finitely generated fundamental group and no cusps, they are homeomorphic by the Tameness Theorem to the interiors of compact hyperbolizable 3–manifolds $\overline{M}$ and $\overline{M}'$ whose boundary components have negative Euler characteristic.

It follows from Section 2.4 that $\mathcal{CH}(M)$ and $\mathcal{CH}(M')$ are complex manifolds of $\mathbb{C}-$dimensions $3|\chi(M)|$ and $3|\chi(M)|$, respectively, and that there is a holomorphic map

$$(\tau')^* \colon \mathcal{CH}(M') \to \mathcal{CH}(M)$$

defined by lifting hyperbolic structures using $\tau' \colon M \to M'$. We will prove:

Claim $(\tau')^*$ *has discrete fibers and is not open.*

Since any holomorphic map with discrete fibers is open unless the dimension of the domain is smaller than the dimension of the image, we deduce from the claim that

$$3|\chi(M')| = \dim_{\mathbb{C}} \mathcal{CH}(M') < \dim_{\mathbb{C}} \mathcal{CH}(M) = 3|\chi(M)|.$$

So $|\chi(M')| < |\chi(M)|$ and hence Proposition 5.1 will follow once we prove the claim.

The first part of the claim is almost immediate. If $\tau'_* \colon \Gamma \to \Gamma'$ is the inclusion induced by the covering $\tau' \colon M \to M'$ and $H < \Gamma'$ is as above, then by construction

- H is torsion-free.
- Γ' is generated by $\tau'_*(\Gamma)$ and H.
- $\tau'_*(\Gamma) \cap H$ has finite index in H.

It follows from Corollary 2.11 that a faithful representation $\Gamma \to \mathrm{PSL}_2\,\mathbb{C}$ has only finitely many extensions to Γ'. Therefore, hyperbolic structures on M' that map under $(\tau')^*$ to the same element of $\mathcal{CH}(M)$ have only finitely many options for holonomy representations, up to conjugacy. However, the elements of $\mathcal{CH}(M')$ with holonomy in any fixed conjugacy class form a discrete subset of $\mathcal{CH}(M')$ [28, page 154], so $(\tau')^*$ must have discrete fibers.

To show that it is not open, we use the Ahlfors–Bers parameterization to produce from $(\tau')^*$ a holomorphic map

$$\beta \colon \mathcal{T}(\partial \overline{M}') \cong \mathcal{CH}(M') \xrightarrow{(\tau')^*} \mathcal{CH}(M) \cong \mathcal{T}(\partial \overline{M}).$$

The Teichmuller spaces of $\partial \overline{M}'$ and $\partial \overline{M}$ split as products of the Teichmuller spaces of their connected components; let $S \subset \partial \overline{M}$ be the component adjacent to the degenerate end that the surface S_0 faces. The Covering Theorem (Theorem 2.2) implies that τ' extends to a nontrivial cover $\overline{\tau} \colon S \to S'$ onto some connected component $S' \subset \partial \overline{M}'$. With respect to the decompositions

$$\mathcal{T}(\partial \overline{M}) = \mathcal{T}(S) \times \mathcal{T}(\partial \overline{M} \setminus S), \quad \mathcal{T}(\partial \overline{M}') = \mathcal{T}(S') \times \mathcal{T}(\partial \overline{M}' \setminus S')$$

the map β can be written as

$$\beta(\sigma_1, \sigma_2) = (\overline{\tau}^* \sigma_1, \hat{\beta}(\sigma_1, \sigma_2))$$

where $\overline{\tau}^*\colon \mathcal{T}(S') \to \mathcal{T}(S)$ is the map induced by the covering $\overline{\tau}\colon S \to S'$. This covering is nontrivial, so the image of $\overline{\tau}^*$ has positive codimension and hence the same holds for the image of β. Therefore β is not open, implying the same for $(\tau')^*$. $\qquad\square$

Remark   A purely homological computation yields that under the same assumptions as in Proposition 5.1 we have $b_1(M') < b_1(M)$ where $b_1(\cdot)$ is the first Betti number with $\mathbb{R}$–coefficients. However, this homological argument does not seem to work in the relative case that we will discuss in Section 7. This is why we choose to work with Euler characteristics and deformation spaces instead.

6   Proof of Theorem 1.2

Recall the statement of Theorem 1.2:

Theorem 1.2   *Let* Γ *be a finitely generated group and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$. *Assume that* (ρ_i) *is algebraically convergent and converges geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$. *If* G *does not contain parabolic elements, then* G *is finitely generated.*

If the hyperbolic 3–manifold $\mathbb{H}^3/G$ is compact then G is obviously finitely generated. Assume from now on that this is not the case; since G does not contain parabolic elements, this assumption implies that $\mathbb{H}^3/G$ has infinite volume.

Among all finitely generated subgroups of G which contain $\rho(\Gamma)$, choose H such that the Euler characteristic of the associated hyperbolic 3–manifold $\mathbb{H}^3/H$ has minimal absolute value. Since $H \subset G$ we have a covering $\mathbb{H}^3/H \to \mathbb{H}^3/G$. By minimality, we obtain from Proposition 5.1 that every degenerate end of $\mathbb{H}^3/H$ embeds under this cover.

On the other hand, the assumption that H contains $\rho(\Gamma)$ implies, by Proposition 2.9, that there is a sequence of representations $\sigma_i\colon H \to \mathrm{PSL}_2\,\mathbb{C}$ converging algebraically to the inclusion of H in G such that the groups $\sigma_i(H)$ converge geometrically to G. Theorem 1.5 implies now that $H = G$. In particular, G is finitely generated. This concludes the proof of Theorem 1.2. $\qquad\square$

7   Parabolics

It is a well established fact that most theorems in the deformation theory of Kleinian groups that hold in the absence of parabolics hold also, in some probably weaker form, in the presence of parabolics. It is also well known that proofs in the case with

parabolics are much more cumbersome and technical but follow the same arguments as the proofs in the purely hyperbolic case. For the sake of clarity and transparency of exposition, we decided to prove Theorem 1.2 only in the absence of parabolics. We state now the general results and discuss what changes have to be made in the arguments given above.

Throughout this section we assume that the reader is familiar with basic geometric facts about hyperbolic manifolds with cusps [28].

As mentioned in the introduction, Evans [20] obtained the following extension of Theorem 1.1.

Theorem 7.1 (Evans [20]) *Assume* Γ *is a finitely generated group and that* (ρ_i) *is a sequence of faithful representations in* $\mathcal{D}(\Gamma)$ *converging algebraically to some representation* ρ. *If the convergence* $\rho_i \to \rho$ *is weakly type preserving, then* ρ_i *converges geometrically to* $\rho(\Gamma)$.

An algebraically convergent sequence $\rho_i \to \rho$ is *weakly type preserving* if for every $\gamma \in \Gamma$ such that $\rho(\gamma)$ is parabolic, there is some $i_\gamma \in \mathbb{N}$ such that $\rho_i(\gamma)$ is parabolic for all $i \geq i_\gamma$.

Theorem 1.4 can be extended to this setting as follows:

Theorem 7.2 *Let* Γ *be a finitely generated group, and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$ *converging algebraically to a representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$. *If*

- *the convergence* $\rho_i \to \rho$ *is weakly type preserving, and*
- *there is no* $\gamma \in G \setminus \rho(\Gamma)$ *with* $\gamma^k \in \rho(\Gamma)$ *for some* $k \geq 2$,

then $G = \rho(\Gamma)$.

As before, we will actually prove a stronger statement.

Theorem 7.3 *Let* Γ *be a finitely generated group, and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$ *converging algebraically to a representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$. *If*

- *the convergence* $\rho_i \to \rho$ *is weakly type preserving, and*
- *every degenerate NP–end of* $\mathbb{H}^3/\rho(\Gamma)$ *has a neighborhood that embeds under the covering* $\mathbb{H}^3/\rho(\Gamma) \to \mathbb{H}^3/G$,

then $G = \rho(\Gamma)$.

Recall that an NP–end of a hyperbolic 3–manifold M is an end of the manifold obtained from M by removing neighborhoods of all of its cusps. The full statement of Canary's Covering Theorem [18] shows that if a covering $M \to N$ of infinite volume hyperbolic 3–manifolds is not one-to-one on some degenerate NP–end, then there is some $\gamma \in G \setminus \rho(\Gamma)$ with $\gamma^k \in \rho(\Gamma)$ for some $k \geq 2$. One can see [28] for more details on the Covering Theorem and NP–ends.

Proof Let $M_i = \mathbb{H}^3/\rho_i(\Gamma)$, $M_A = \mathbb{H}^3/\rho(\Gamma)$, $M_G = \mathbb{H}^3/G$ and let $\pi\colon M_A \to M_G$ be the covering induced by the inclusion $\rho(\Gamma) \subset G$. We first recall the basic idea of the proof in the case without cusps. First, there are arbitrarily large compact cores C of M_A such that the components of ∂C facing convex cocompact ends of M_A are strictly convex. Using our second hypothesis above, we can also arrange that the components of these cores facing degenerate ends embed under the covering $\pi\colon M_A \to M_G$. Composing the restriction $\pi|_C$ with the almost isometric embeddings $\pi(C) \to M_i$ supplied by geometric convergence, we obtain maps $f_i\colon C \to M_i$ such that

   (1) if a component S of ∂C faces a degenerate end of M_A then $f_i|_S$ is an embedding for all large i.

   (2) if $S \subset \partial C$ faces a convex cocompact end then $f_i|_S$ is a convex immersion.

We then construct for large i a 3–manifold N_i containing C and a covering $\overline{f_i}\colon N_i \to M_i$ with $\overline{f_i}|_C = f_i$. This covering is π_1–surjective and hence a diffeomorphism, so in particular C embeds under the covering $\pi\colon M_A \to M_G$. Since C can be chosen to be arbitrarily large, this proves that the covering π is trivial and hence $G = \rho(\Gamma)$.

If there are parabolics in the algebraic limit, the ends of M_A are more complicated and refinements of the tools above are needed. The natural replacement of the compact core C is a submanifold $C \subset M_A$ with the following properties:

   (1) If a component S of ∂C faces a degenerate NP–end of M_A then S embeds under the covering $M_A \to M_G$.

   (2) If S faces a geometrically finite NP–end then S is strictly convex,

   (3) The complement in C of the μ–cuspidal part $M_A^{\text{cusp}<\mu}$ of M_A is a standard compact core, where μ is the Margulis constant.

The construction of such a submanifold is the same as that used to produce the compact cores above: one takes a large metric neighborhood of the convex core $CC(M_A)$ and deletes standard neighborhoods of the degenerate NP–ends of M_A. In this case, however, the resulting manifold C will contain parts of the cusps of M_A, and will therefore

be noncompact. This is a problem, since $\pi(C) \subset M_G$ will also be noncompact and the almost isometric embeddings from M_G to M_i provided by geometric convergence are only defined on compact sets. However, we still can use the almost isometric embeddings to produce locally bilipschitz maps $f_i \colon C \setminus M_A^{\mathrm{cusp}<\mu} \to M_i$, and we will show that if the convergence $\rho_i \to \rho$ is weakly type preserving then these can be extended to locally bilipschitz maps

$$\widehat{f_i} \colon C \to M_i$$

converging uniformly on compact sets to the restriction $\pi|_C$. The proof of Theorem 7.2 is then word-by-word the same as the proof of Theorem 1.4 with the maps $\widehat{f_i}$ playing the role of f_i.

It remains to construct $\widehat{f_i}$. Consider the maps

$$f_i \colon C \setminus M_A^{\mathrm{cusp}<\mu} \to M_i$$

described above, and let ϵ be a small positive constant. If i is large then f_i is locally $(1+\epsilon)$–bilipschitz; combined with the fact that the convergence $\rho_i \to \rho$ is weakly type preserving and M_A has finitely many cusps, this implies that f_i sends loops homotopic into the cusps of M_A to parabolic loops in M_i with nearly the same length. It follows that $C \cap \partial M_A^{\mathrm{cusp}<\mu}$ is mapped under f_i into a small neighborhood of $\partial M_i^{\mathrm{cusp}<\mu}$. After a small perturbation, we can then arrange that

(1) $f_i(C \cap \partial M_A^{\mathrm{cusp}<\mu}) \subset \partial M_i^{\mathrm{cusp}<\mu}$.

(2) Df_i sends vectors orthogonal to $\partial M_A^{\mathrm{cusp}<\mu}$ to vectors orthogonal to $\partial M_i^{\mathrm{cusp}<\mu}$.

Brock and Bromberg [12, Lemma 6.16] show how to accomplish such a perturbation with a bit of finesse; in particular, their argument shows that we can still assume that f_i is locally bilipschitz.

Recall that C was constructed by removing standard neighborhoods of the degenerate NP–ends of M_A from its convex core. These neighborhoods can be chosen so that their intersections with $M_A^{\mathrm{cusp}<\mu}$ are foliated by geodesic rays orthogonal to $\partial M_A^{\mathrm{cusp}<\mu}$; the intersection $C \cap M_A^{\mathrm{cusp}<\mu}$ then enjoys the same property. Combining this with (1) and (2) above allows us to extend f_i to a locally bilipschitz map $\widehat{f_i} \colon C \to M_i$ as follows: define $\widehat{f_i}$ to coincide with f_i on $C \setminus M_A^{\mathrm{cusp}<\mu}$ and map geodesic rays in $C \cap M_A^{\mathrm{cusp}<\mu}$ orthogonal to $\partial M_A^{\mathrm{cusp}<\mu}$ isometrically to geodesic rays orthogonal to $\partial M_i^{\mathrm{cusp}<\mu}$. A quick computation in the upper half space model for $\mathbb{H}^3$ verifies that the extension $\widehat{f_i}$ is also locally $(1+\epsilon)$–bilipschitz. It follows that $\widehat{f_i} \to \pi|_C$ uniformly on compact subsets. $\square$

Observe that as in the case without parabolics, we can replace the second hypothesis of Theorem 7.2 with the condition that every degenerate NP–end of M_A has a neighborhood that embeds in M_G.

A version of Proposition 5.1 that applies to manifolds with cusps is also readily established:

Proposition 7.4 *Let M and N be hyperbolic 3–manifolds with infinite volume and let $\tau\colon M \to N$ be a covering. Assume that $\pi_1(M)$ is finitely generated and that M has a degenerate NP–end that does not embed under the covering τ. Then there is a hyperbolic 3–manifold M' with finitely generated fundamental group, with*

$$3|\chi(M')| - \#\{cusps\ in\ M'\} < 3|\chi(M)| - \#\{cusps\ in\ M\}$$

and coverings $\tau'\colon M \to M'$ and $\tau''\colon M' \to N$ with $\tau = \tau'' \circ \tau'$.

The proof is the same as that of Proposition 5.1, except that instead of considering the deformation space $\mathcal{CH}(M)$ of convex-cocompact hyperbolic structures on M one uses geometrically finite metrics whose parabolic loci coincide with that of the original hyperbolic structure on M. The space of such metrics, up to isometries isotopic to the identity map, is by [24, Theorem 8.44] a complex manifold of $\mathbb{C}$–dimension

$$-3\chi(M) - \#\{cusps\ in\ M\} \geq 0.$$

After these observations the proof of Proposition 7.4 is the same as the proof of Proposition 5.1. □

Having provided versions of Theorem 1.4 and Proposition 5.1 that apply to representations with parabolics, we are almost ready to discuss the general form of Theorem 1.2. Before doing so, we need a definition:

Definition Assume that a sequence of subgroups (G_i) of $\mathrm{PSL}_2\,\mathbb{C}$ converges geometrically to a subgroup G. We say that the convergence $G_i \to G$ is *geometrically weakly type preserving* if for every $g \in G$ parabolic there is a sequence (g_i) with $g_i \in G_i$ converging to g and with g_i parabolic for all but finitely many i.

In the terminology of [14], a geometrically convergent sequence of subgroups converges in a geometrically weakly type preserving manner if and only if the associated sequence of pointed manifolds has *uniform length decay*.

The general version of Theorem 1.2 reads now:

Theorem 7.5 *Let* Γ *be a finitely generated group and* (ρ_i) *a sequence in* $\mathcal{D}(\Gamma)$ *that converges algebraically to a representation* ρ *and geometrically to a subgroup* G *of* $\mathrm{PSL}_2\,\mathbb{C}$. *If the convergence* $\rho_i \to G$ *is geometrically weakly type-preserving, then* G *is finitely generated.*

Proof   As in the proof of Theorem 1.2, we choose a finitely generated subgroup $H \subset G$ containing $\rho(\Gamma)$ and minimizing the quantity

$$3|\chi(\mathbb{H}^3/H)| - 2\,\#\{\text{cusps in } \mathbb{H}^3/H\} \geq 0.$$

We claim that $G = H$. By Proposition 2.9, there are for large i representations $\sigma_i\colon H \to \mathrm{PSL}_2\,\mathbb{C}$ with $\sigma_i(H) = \rho_i(\Gamma_i)$ converging algebraically to the inclusion of H in G. In particular, G is the geometric limit of the groups $\sigma_i(H)$. The assumption that the convergence $\sigma_i(H) = \rho_i(\Gamma) \to G$ is geometrically weakly type-preserving implies that the algebraic convergence of σ_i to the inclusion of H into G is (algebraically) weakly type preserving. In particular we deduce from Theorem 7.3 that either $G = H$ or some degenerate NP–end of $\mathbb{H}^3/H$ has no neighborhood that embeds under the cover $\mathbb{H}^3/H \to \mathbb{H}^3/G$. The second possibility is ruled out by the choice of H and Proposition 7.4, so $G = H$ and therefore is finitely generated. $\square$

Appendix A   Some assorted results

As mentioned in the introduction, we discuss here to which extent some other results concerning faithful representations remain true if the condition of faithfulness is relaxed.

Definition   A sequence (ρ_i) of representations is *eventually faithful* if for all $\gamma \in \Gamma\setminus\{1\}$ there is some i_γ such that $\gamma \notin \mathrm{Ker}(\rho_i)$ for all $i \geq i_\gamma$.

Many algebraically convergent sequences determine an eventually faithful sequence. The following well-known lemma formalizes this; its proof is a simple application of the Jørgensen inequality.

Lemma A.1   *Let* (ρ_i) *be a sequence of representations in* $\mathcal{D}(\Gamma)$ *that converges algebraically to a representation* ρ. *Then given any* $\gamma \in \Gamma$, *we have that* $\gamma \in \ker\rho$ *if and only if there exists* $i_0 \in \mathbb{N}$ *such that* $\gamma \in \ker\rho_i$ *for all* $i \geq i_0$.

Therefore, if $\ker\rho$ is finitely normally generated then for large i the representations ρ_i factor through $\Gamma/\ker(\rho)$; the resulting sequence of representations of $\Gamma/\ker(\rho)$ will be eventually faithful.

Proof One direction of the implication is clear. For the other direction, we assume that $\gamma \in \ker \rho$ and try to prove that $\gamma \in \ker \rho_i$ for large i.

As $\mathcal{D}(\Gamma)$ is closed the representation ρ is nonelementary, so there are two elements $\alpha_1, \alpha_2 \in \Gamma$ such that $\rho(\alpha_1)$ and $\rho(\alpha_2)$ are isometries of $\mathbb{H}^3$ of hyperbolic type and have distinct axis. Since

$$\rho_i(\gamma) \longrightarrow \rho(\gamma) = \mathrm{Id},$$

we have for sufficiently large i that each of the two pairs $\{\rho_i(\gamma), \rho_i(\alpha_k)\}$ violates the Jørgensen inequality (see [28, Theorem 2.17]). Therefore, both groups $\langle \rho_i(\gamma), \rho_i(\alpha_k) \rangle$, $k = 1, 2$, are abelian. But since $\rho_i(\alpha_1)$ and $\rho_i(\alpha_2)$ have different axes the only way that both of these groups can be abelian is if $\rho_i(\gamma)$ is elliptic or trivial. This is a contradiction, since it is nontrivial by assumption and cannot be elliptic since ρ_i is discrete and torsion-free. $\square$

We assume from now on that (ρ_i) is an eventually faithful sequence of representations in $\mathcal{D}(\Gamma)$. If $S \subset \Gamma$ is a finite generating set, then each representation ρ_i determines a convex function

$$d_{\rho_i} \colon \mathbb{H}^3 \to \mathbb{R}, \quad d_{\rho_i}(x) = \sum_{\gamma \in S} d_{\mathbb{H}^3}(x, \rho_i(\gamma)x).$$

Conjugating our representations by $\mathrm{PSL}_2\,\mathbb{C}$ if necessary, we may assume that some base point $0 \in \mathbb{H}^3$ is the unique minimum of each d_{ρ_i}. In this case we say that the sequence (ρ_i) consists of *normalized representations* and we set $d_{\rho_i} = d_{\rho_i}(0)$. It is well-known that the sequence (ρ_i) contains an algebraically convergent subsequence if

$$\liminf d_{\rho_i} < \infty.$$

Otherwise, the sequence of actions of Γ via ρ_i on the scaled hyperbolic spaces $(1/d_{\rho_i})\mathbb{H}^3$ contains a subsequence which converges in the equivariant Gromov–Hausdorff topology to a nontrivial action $\Gamma \curvearrowright T$ on some real tree T. Recall that an action on a tree is called trivial if it has global fixed points. Morgan and Shalen [33], Paulin [37] and Bestvina [7] proved that if the representations ρ_i are faithful the action $\Gamma \curvearrowright T$ is *small*, meaning that the stabilizers of nondegenerate segments in T are virtually abelian. Their arguments still apply if the sequence is only eventually faithful; see Hossein and Souto [34]:

Theorem A.2 *Every eventually faithful sequence of normalized representations in $\mathcal{D}(\Gamma)$ has a subsequence that either converges algebraically in $\mathcal{D}(\Gamma)$ or converges in the Gromov–Hausdorff topology to a nontrivial small action $\Gamma \curvearrowright T$ on a $\mathbb{R}$–tree T.* $\square$

It is a theorem of Morgan and Shalen [33] that if the fundamental group of a compact, irreducible and atoroidal 3–manifold M admits a nontrivial small action on a real tree, then either ∂M is compressible or there are essential properly embedded annuli in $(M, \partial M)$. Combining this fact with Theorem A.2 we obtain the following result, essentially due to Thurston:

Corollary A.3 *Assume that* Γ *is the fundamental group of a compact 3–manifold with incompressible and acylindrical boundary. Then every eventually faithful sequence of normalized representations in* $\mathcal{D}(\Gamma)$ *contains a convergent subsequence.* □

Many other results, for instance Thurston's double limit theorem [35] or the results in [26], ensuring the existence of convergent subsequences of sequences of faithful representations can be reduced to the nonexistence of certain actions of groups on trees; variants of all these results still hold for eventually faithful sequences. It may be therefore surprising that some convergence results for sequences of faithful representations completely fail in our more general setting.

In [39], Thurston proved a generalization of Corollary A.3 to the case that Γ is the fundamental group of a compact 3–manifold M with incompressible boundary. More precisely, the so-called *only-windows-break theorem* asserts that whenever (ρ_i) is a sequence of faithful representations of $\Gamma = \pi_1(M)$ and $N \subset M$ is a component of the complement of the characteristic manifold, then the sequence $(\rho_i|_{\pi_1(N)})$ has, up to conjugacy, a convergent subsequence. Leaving the interested reader to consult [21] for information about the characteristic manifold, we limit ourselves to the following concrete example.

Let H be a handlebody of genus 2 and $\gamma \subset \partial H$ a simple closed curve on the boundary of H such that $\partial H \setminus \gamma$ is incompressible and acylindrical; for instance, γ can be taken in the Masur domain of H [26]. We consider the manifold N obtained by doubling H along $\mathcal{N}(\gamma)$, where $\mathcal{N}(\gamma)$ is a regular neighborhood of γ in ∂H. The choice of γ ensures that N has incompressible boundary and that there is a unique (up to isotopy) properly embedded essential annulus $A \subset N$, the annulus along which we have glued. The annulus A cuts N open into two copies H_1 and H_2 of H.

In this example, Thurston's only-windows-break theorem asserts:

Theorem A.4 (Thurston) *Let N be as above and (ρ_i) a sequence of discrete and faithful representations of $\pi_1(N)$ into $\mathrm{PSL}_2 \mathbb{C}$. Then the sequence of restrictions*

$$\rho_i|_{\pi_1(H_1)} \colon \pi_1(H_1) \to \mathrm{PSL}_2 \mathbb{C}$$

has a subsequence that converges up to conjugacy in $\mathrm{PSL}_2 \mathbb{C}$.

We claim that there is an eventually faithful sequence of discrete representations of $\pi_1(N)$ for which the claim of Theorem A.4 fails.

By construction, the manifold N is the double of H along $\mathcal{N}(\gamma)$. Let

$$\tau\colon N \to H$$

be the map given by "folding" along $\mathcal{N}(\gamma)$. Identifying H with H_1, one of the two pieces of N, we choose a base point $p \in \mathcal{N}(\gamma)$ and hence we have a homomorphism

$$\tau_*\colon \pi_1(N, p) \to \pi_1(H, p).$$

We consider also the Dehn-twist

$$\delta\colon N \to N$$

along the annulus A.

Claim   The sequence $\tau_* \circ \delta_*^n\colon \pi_1(N, p) \to \pi_1(H, p)$ is eventually faithful.

It is worth observing that this claim is really a statement about a "twist" automorphism of any amalgamation of a free group with itself over some nontrivial cyclic subgroup.

Proof   The fundamental group $\pi_1(N, p)$ is isomorphic to the amalgamation $\pi_1(H_1, p) \star_{\langle \gamma \rangle} \pi_1(H_2, p)$, where with an abuse of notation we regard γ as an element of both $\pi_1(H_1, p)$ and $\pi_1(H_2, p)$. The folding map $\tau_*\colon \pi_1(N, p) \to \pi_1(H, p)$ is the unique map from the amalgamation to $\pi_1(H, p)$ that restricts to the canonical identifications $\pi_1(H_i, p) \to \pi_1(H, p)$.

Fix a free basis for $\pi_1(H)$. There are then associated free bases for $\pi_1(H_1, p)$ and $\pi_1(H_2, p)$, and after a conjugation any element of $\pi_1(N, p)$ can be written as

$$w = a_1 b_1 a_2 b_2 \cdots a_k b_k,$$

where $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are reduced words in the generators of $\pi_1(H_1, p)$ and $\pi_1(H_2, p)$, respectively, and all are nontrivial except possibly a_1. Regarding γ as a cyclically reduced word in the generators of $\pi_1(H, p)$, we can ensure by choosing k to be minimal that none of the words $a_2, \ldots, a_k, b_1, \ldots, b_k$ correspond to elements of $\pi_1(H, p)$ that are powers of γ. Then the n–fold twist of w has the following expression:

$$\delta_*^n(w) = a_1(\gamma^n b_1 \gamma^{-n}) a_2(\gamma^n b_2 \gamma^{-n}) \cdots a_n(\gamma^n b_n \gamma^{-n}).$$

After applying the folding map τ_*, this becomes

$$\tau_* \circ \delta_*^n(w) = a_1'(\gamma^n b_1' \gamma^{-n}) a_2'(\gamma^n b_2' \gamma^{-n}) \cdots a_n'(\gamma^n b_n' \gamma^{-n}),$$

where $a'_1, b'_1, \ldots, a'_k, b'_k$ are the images of $a_1, b_1, \ldots, a_k, b_k$ under the identifications $\pi_1(H_i, p) \to \pi_1(H, p)$. However, as none of $b'_1, a'_2, b'_2, \ldots, a'_k, b'_k$ are powers of the cyclically reduced word γ, there is only a bounded amount of cancellation when reducing the word above. Therefore, if n is large then $\tau_* \circ \delta^n_*(w)$ is nontrivial. $\square$

Remark Essentially we are saying that $\pi_1(N, p)$ is fully residually free. This fact holds for every group obtained by doubling a free group along a cyclic subgroup.

Since H is a handlebody, $\pi_1(H, p)$ is a free group. Hence, we can choose some sequence of faithful representations $(\sigma_n) \in \mathcal{D}(\pi_1(H, p))$ which cannot be conjugated to obtain convergent subsequences. By construction, $\tau_* \circ \delta^n_*$ is the identity on $\pi_1(H, p)$ for all n. So setting

$$\rho_n = \sigma_n \circ \tau_* \circ \delta^n_*\colon \pi_1(N, p) \longrightarrow \mathrm{PSL}(2, \mathbb{C}),$$

we obtain an eventually faithful sequence of representations which does not contain any subsequence whose restriction to $\pi_1(H, p)$ converges up to conjugacy.

Theorem A.5 *The only-windows-break theorem fails for eventually faithful sequences of representations.* $\square$

An alternative statement of Theorem A.5 could be that when one is not absolutely faithful more than the windows can get broken.

References

[1] **I Agol**, *Tameness of hyperbolic* 3*–manifolds* `arXiv:math.GT/0405568`

[2] **J W Anderson**, **R D Canary**, *Algebraic limits of Kleinian groups which rearrange the pages of a book*, Invent. Math. 126 (1996) 205–214 MR1411128

[3] **J W Anderson**, **R D Canary**, *Cores of hyperbolic* 3*–manifolds and limits of Kleinian groups*, Amer. J. Math. 118 (1996) 745–779 MR1400058

[4] **J W Anderson**, **R D Canary**, *Cores of hyperbolic* 3*–manifolds and limits of Kleinian groups. II*, J. London Math. Soc. (2) 61 (2000) 489–505 MR1760675

[5] **J W Anderson**, **R D Canary**, **M Culler**, **P B Shalen**, *Free Kleinian groups and volumes of hyperbolic* 3*–manifolds*, J. Differential Geom. 43 (1996) 738–782 MR1412683

[6] **R Benedetti**, **C Petronio**, *Lectures on hyperbolic geometry*, Universitext, Springer, Berlin (1992) MR1219310

[7] **M Bestvina**, *Degenerations of the hyperbolic space*, Duke Math. J. 56 (1988) 143–161 MR932860

[8]	**F Bonahon**, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) 124 (1986) 71–158 MR847953

[9]	**F Bonahon**, **J-P Otal**, *Variétés hyperboliques à géodésiques arbitrairement courtes*, Bull. London Math. Soc. 20 (1988) 255–261 MR931188

[10]	**J F Brock**, *Continuity of Thurston's length function*, Geom. Funct. Anal. 10 (2000) 741–797 MR1791139

[11]	**J F Brock**, *Iteration of mapping classes and limits of hyperbolic* 3–*manifolds*, Invent. Math. 143 (2001) 523–570 MR1817644

[12]	**J F Brock**, **K W Bromberg**, *On the density of geometrically finite Kleinian groups*, Acta Math. 192 (2004) 33–93 MR2079598

[13]	**J F Brock**, **R D Canary**, **Y N Minsky**, *The classification of Kleinian surface groups II: the ending lamination conjecture*, to appear in Ann. of Math. (2) `arXiv: math.GT/0412006`

[14]	**J F Brock**, **J Souto**, *Algebraic limits of geometrically finite manifolds are tame*, Geom. Funct. Anal. 16 (2006) 1–39 MR2221251

[15]	**D Calegari**, **D Gabai**, *Shrinkwrapping and the taming of hyperbolic* 3–*manifolds*, J. Amer. Math. Soc. 19 (2006) 385–446 MR2188131

[16]	**R D Canary**, *The Poincaré metric and a conformal version of a theorem of Thurston*, Duke Math. J. 64 (1991) 349–359 MR1136380

[17]	**R D Canary**, *Ends of hyperbolic* 3–*manifolds*, J. Amer. Math. Soc. 6 (1993) 1–35 MR1166330

[18]	**R D Canary**, *A covering theorem for hyperbolic* 3–*manifolds and its applications*, Topology 35 (1996) 751–778 MR1396777

[19]	**T D Comar**, *Hyperbolic Dehn surgery and convergence of Kleinian groups*, PhD thesis, University of Michigan (1996) Available at `http://proquest.umi.com/ pqdweb?did=739567211&Fmt=7&RQT=309`

[20]	**R Evans**, *Weakly type-preserving sequences and strong convergence*, Geom. Dedicata 108 (2004) 71–92 MR2112665

[21]	**W Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Math. 43, Amer. Math. Soc. (1980) MR565450

[22]	**T Jørgensen**, *On cyclic groups of Möbius transformations*, Math. Scand. 33 (1973) 250–260 (1974) MR0348103

[23]	**T Jørgensen**, *On discrete groups of Möbius transformations*, Amer. J. Math. 98 (1976) 739–749 MR0427627

[24]	**M Kapovich**, *Hyperbolic manifolds and discrete groups*, Progress in Math. 183, Birkhäuser, Boston (2001) MR1792613

[25]	**S P Kerckhoff**, **W P Thurston**, *Noncontinuity of the action of the modular group at Bers' boundary of Teichmüller space*, Invent. Math. 100 (1990) 25–47 MR1037141

[26]　**G Kleineidam**, **J Souto**, *Algebraic convergence of function groups*, Comment. Math. Helv. 77 (2002) 244–269　MR1915041

[27]　**A Marden**, *The geometry of finitely generated kleinian groups*, Ann. of Math. (2) 99 (1974) 383–462　MR0349992

[28]　**K Matsuzaki**, **M Taniguchi**, *Hyperbolic manifolds and Kleinian groups*, Oxford Math. Monogr., Oxford Science Publ., The Clarendon Press, Oxford Univ. Press, New York (1998)　MR1638795

[29]　**C T McMullen**, *Cusps are dense*, Ann. of Math. (2) 133 (1991) 217–247　MR1087348

[30]　**C T McMullen**, *Renormalization and* 3*–manifolds which fiber over the circle*, Annals of Math. Studies 142, Princeton Univ. Press (1996)　MR1401347

[31]　**C T McMullen**, *Hausdorff dimension and conformal dynamics. I. Strong convergence of Kleinian groups*, J. Differential Geom. 51 (1999) 471–515　MR1726737

[32]　**Y Minsky**, *The classification of Kleinian surface groups. I. Models and bounds*, Ann. of Math. (2) 171 (2010) 1–107　MR2630036

[33]　**J W Morgan**, **P B Shalen**, *Valuations, trees, and degenerations of hyperbolic structures. I*, Ann. of Math. (2) 120 (1984) 401–476　MR769158

[34]　**H Namazi**, **J Souto**, *Heegaard splittings and pseudo-Anosov maps*, Geom. Funct. Anal. 19 (2009) 1195–1228　MR2570321

[35]　**J-P Otal**, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque 235, Soc. Math. France (1996)　MR1402300

[36]　**J-P Otal**, *Thurston's hyperbolization of Haken manifolds*, from: "Surveys in differential geometry, Vol. III (Cambridge, MA, 1996)", (C-C Hsiung, S-T Yau, editors), Int. Press, Boston (1998) 77–194　MR1677888

[37]　**F Paulin**, *Topologie de Gromov équivariante, structures hyperboliques et arbres réels*, Invent. Math. 94 (1988) 53–80　MR958589

[38]　**W P Thurston**, *Hyperbolic structures on* 3*–manifolds II: Surface groups and* 3*– manifolds which fiber over the circle*　`arXiv:math.GT/9801045`

[39]　**W P Thurston**, *Hyperbolic structures on* 3*–manifolds. I. Deformation of acylindrical manifolds*, Ann. of Math. (2) 124 (1986) 203–246　MR855294

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