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Embedded contact homology and Seiberg–Witten Floer cohomology V

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This is a sequel to four earlier papers by the author that construct an isomorphism between the embedded contact homology and Seiberg–Witten Floer cohomology of a compact 3–manifold with a given contact 1–form. These respective homology/cohomology theories carry additional structure; this sequel proves that the isomorphism that is constructed in the first four papers is compatible with this extra structure.

57R17; 57R57

1 Introduction

The first four papers [10; 11; 12; 13] in this series construct an isomorphism between Hutchings' embedded contact homology [3; 4; 5] and the Seiberg–Witten Floer cohomology as defined by Kronheimer and Mrowka in [7]. Both embedded contact homology and Seiberg–Witten Floer cohomology admit a canonical endomorphism that reduce degree by 2. In addition, embedded contact homology and Seiberg–Witten Floer cohomology come with canonical endomorphisms that are defined using the homology of the ambient 3–manifold. This paper proves that these auxiliary endomorphisms are intertwined by the isomorphism that is constructed in [10; 11; 12; 13]. A formal assertion to this effect is given in Theorem 1.1 in Section 1.c to come. Both embedded contact homology and Seiberg–Witten Floer cohomology have respective canonical *contact* elements, and Theorem 1.1 asserts that the isomorphism in [10; 11; 12; 13] maps one to the other.

Embedded contact homology and Seiberg–Witten Floer cohomology also have refinements which define modules over the group ring of the second homology of the ambient 3–manifold. Theorem 1.1 in the upcoming Section 1.c asserts that these refined versions of embedded contact homology and Seiberg–Witten Floer cohomology are isomorphic.

What follows directly in the intervening subsections sets the stage for the statement of Theorem 1.1. Note in advance that the notation borrows heavily from the first four papers of this series.

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1.a Auxiliary structures on embedded contact homology

This first subsection briefly describes the various auxiliary structures on embedded contact homology. There are four parts to what follows. The first part summarizes the definition of embedded contact homology given by Hutchings in [3] (see also the paper of Hutchings and the author [5]). A somewhat longer summary is provided in [10, Section 2].

Part 1 Let M denote the 3–manifold in question and let a denote the contact 1–form. The manifold M is oriented using the volume form $a \wedge da$. The contact form a is chosen from a certain residual set, this denoted by $\mathcal{N}_M$ and described in [11, Part 3 of Section 1.a]. The 2–plane bundle kernel(a) $\subset TM$ is denoted by K^{-1}; it is oriented using da. With the choice of a compatible, bundle complex structure, the dual bundle K will be viewed as a complex line bundle over M. Its first Chern class is denoted by $c_1(K)$. The vector field v that generates kernel(da) and pairs with a to give 1 is called the Reeb vector field, and its closed integral curves are called Reeb orbits. These curves are oriented by the restriction of a. If γ is a Reeb orbit, then the integral of a along γ is its *length*; this positive number is denoted by ℓ_γ.

Fix a homology class Γ in $H_1(M;\mathbb{Z})$. Such a choice is needed to define embedded contact homology. With Γ chosen, let $\mathcal{Z}_{\text{ech}}$ denote the set defined as follows: An element $\Theta \in \mathcal{Z}_{\text{ech}}$ consists of a finite set of pairs of the form (γ, m) with γ a Reeb orbit and m a positive integer, but constrained to equal 1 when γ is hyperbolic. Require that distinct elements from Θ have distinct Reeb orbit components, and require that the $\sum_{(\gamma,m)\in\Theta} m\gamma$ define the class Γ. Given $L \geq 1$, use $\mathcal{Z}_{\text{ech}}^L$ to denote the subset consisting of those Θ with $\sum_{(\gamma,m)} m\ell_\gamma \leq L$. The assumption that $a \in \mathcal{N}_M$ guarantees that $\mathcal{Z}_{\text{ech}}^L$ is a finite set. Among other virtues, all Reeb orbits of a contact form from $\mathcal{N}_M$ are *nondegenerate* in the sense that the associated linear return map [10, Equation (2-4)] has neither 1 nor -1 as an eigenvalue. Moreover, no fractional root of unity is an eigenvalue.

The embedded contact homology for the chosen class Γ is computed using a differential on the free $\mathbb{Z}$ module generated by equivalence classes of pairs of the form $(\Theta, \mathfrak{o})$ where $\Theta \in \mathcal{Z}_{\text{ech}}$ and where $\mathfrak{o}$ is an ordering of those pairs in Θ whose first component is a positive hyperbolic Reeb orbit. (A hyperbolic Reeb orbit is either positive or negative. It is positive when the eigenvalues of the associated linear return map are positive.) The equivalence relation on $\mathbb{Z}\mathcal{Z}_{\text{ech}}$ identifies pairs $(\Theta, \mathfrak{o})$ with $\pm(\Theta', \mathfrak{o}')$ when $\Theta = \Theta'$ and when $\mathfrak{o}$ differs from $\mathfrak{o}'$ by a permutation. The sign, $+$ or $-$, is the parity of this permutation. The embedded contact homology $\mathbb{Z}$ module is denoted in what follows by $\mathcal{C}_{\text{ech}}$. Given $L \geq 1$, use $\mathcal{C}_{\text{ech}}^L \subset \mathcal{C}_{\text{ech}}$ to denote the submodule that is generated by equivalence classes of pairs $(\Theta, \mathfrak{o})$ with $\Theta \in \mathcal{Z}_{\text{ech}}^L$.

The definition of the differential on $\mathcal{C}_{\mathrm{ech}}$ requires the choice of a complex structure on the oriented 2–plane bundle $K^{-1} = \mathrm{kernel}(a)$. Such a choice endows the $\mathbb{R} \times M$ with an $\mathbb{R}$–invariant almost complex structure, J. The latter maps the tangent vector $\partial/\partial s$ along the $\mathbb{R}$ factor to the Reeb vector field, and acts on the kernel of a as the chosen bundle complex structure. The differential for embedded contact homology can be defined using an almost complex structure of this sort that is suitably generic as described in [10, Section 1.c]. In particular, there is a residual set of allowed almost complex structures – this denoted by $\mathcal{J}_a$.

Choose $J \in \mathcal{J}_a$, and use J to define the notion of a pseudoholomorphic subvariety in $\mathbb{R} \times M$. The differential on $\mathcal{C}_{\mathrm{ech}}$ is denoted by δ; it is described briefly in [10, Section 1.c]. In particular, it can be written as follows: If $(\Theta_+, \mathfrak{o}_+)$ is any given generator, then $\delta(\Theta_+, \mathfrak{o}_+) = \sum_{(\Theta_-, \mathfrak{o}_-) \in \mathcal{C}_{\mathrm{ech}}} \sigma(\Theta_-, \Theta_+)(\Theta_-, \mathfrak{o}_-)$, where $\sigma(\Theta_-, \Theta_+)$ is a count, weighted by ± 1, of the components of a moduli space, $\mathcal{M}_1(\Theta_-, \Theta_+)$, of pseudoholomorphic subvarieties with positive integer weights. The sets Θ_- and Θ_+ determine the asymptotics of the subvarieties that comprise any given element in this moduli space. To elaborate, introduce $H_2(M, \Theta_-, \Theta_+)$ to denote the set of relative homology classes of 2–chains $z \subset M$ with boundary $\sum_{(\gamma,m) \in \Theta_+} m\gamma - \sum_{(\gamma,m) \in \Theta_-} m\gamma$. Chains z and z' define the same class in $H_2(M, \Theta_-, \Theta_+)$ when the closed cycle $z - z'$ is the boundary of a 3–cycle in M. This set $H_2(M, \Theta_-, \Theta_+)$ is an affine space modeled on $H_2(M; \mathbb{Z})$. An element $\Sigma \in \mathcal{M}_1(\Theta_-, \Theta_+)$ is a finite set of pairs where each pair has the form (C, m) with $C \subset \mathbb{R} \times M$ being a pseudoholomorphic subvariety and m a positive integer. These are constrained by Θ_- and Θ_+ as follows: Let $\pi(C) \subset M$ denote the image of C via the projection from $\mathbb{R} \times M$ to M. Then the 2–cycle $\sum_{(C,m) \in \Sigma} m\pi(C)$ is a cycle in $H_2(M, \Theta_-, \Theta_+)$. Other salient properties of $\mathcal{M}_1(\Theta_-, \Theta_+)$ are listed in [10, Section 2.c and Equation (2-10)].

The differential maps $\mathcal{C}^L_{\mathrm{ech}}$ to itself. Let P: $H_1(M; \mathbb{Z}) \to H^2(M; \mathbb{Z})$ denote the Poincaré duality isomorphism, and use p to denote the greatest positive integer divisor of the image in $H^2(M; \mathbb{Z})/\mathrm{Torsion}$ of the cohomology class $-c_1(K) + 2\mathrm{P}(\Gamma)$. Hutchings [3; 2] explains how $\mathcal{C}_{\mathrm{ech}}$ can be given a relative $\mathbb{Z}/p\mathbb{Z}$ grading such that δ decreases by 1 this degree. The definition of this degree is briefly summarized in [10, Section 1.c] and also [12, Part 2 of Section 1.a].

Part 2 This part describes the refined version of embedded contact homology, which Hutchings calls *twisted* embedded contact homology. What follows summarizes mostly from [3, Section 11] where this refinement is described.

To start, the refinement requires the choice of a “reference cycle”, ρ, an oriented, 1–dimensional submanifold in M whose homology class is that of Γ. Suppose that

$\Theta = \{(\gamma, m)\} \in \mathcal{Z}_{\mathrm{ech}}$. Let $H_2(M, \rho, \Theta)$ denote the set that consists of the relative homology classes of 2–chains $z \subset M$ with $\partial z = \sum_{(\gamma,m)\in\Theta} m\gamma - \rho$.

Define $\mathcal{C}_{\mathrm{ech},\rho}$ to be the free $\mathbb{Z}$ module generated by pairs $((\Theta, \mathfrak{o}), W)$ where $(\Theta, \mathfrak{o})$ is a generator of $\mathcal{C}_{\mathrm{ech}}$ and where $W \in H_2(M, \rho, \Theta)$. To define the differential on $\mathcal{C}_{\mathrm{ech},\rho}$, remember that if Θ_- and Θ_+ are both in $\mathcal{Z}_{\mathrm{ech}}$, then any given $\Sigma \in \mathcal{M}_1(\Theta_-\Theta_+)$ defines, via the projection from $\mathbb{R} \times M$ to M, an element in $H_2(M, \Theta_-, \Theta_+)$. This understood, define the integer $\sigma(\Theta_-, \Theta_+, W)$ to be the contribution to $\sigma(\Theta_-, \Theta_+)$ from the components in $\mathcal{M}_1(\Theta_-, \Theta_+)$ that define the class W. The differential on $\mathcal{C}_{\mathrm{ech},\rho}$ sends a given generator $((\Theta_+, \mathfrak{o}_+), W_+)$ to

$$\begin{aligned}\delta\big((\Theta_+, \mathfrak{o}_+), W_+\big)\\ = \textstyle\sum_{(\Theta_-,\mathfrak{o}_-)} \sum_{W\in H_2(M,\Theta_-,\Theta_+)} \sigma(\Theta_-, \Theta_+, W)\big((O_-, \mathfrak{o}_-), W_+ + W\big).\end{aligned} \tag{1-1}$$

The homology of the resulting chain complex is a module over the group ring $\mathbb{Z}[H^1(M;\mathbb{Z})]$. Indeed, this module structure is defined as follows: Let $\mathrm{z} \in \mathbb{Z}$ and let $x \in H^1(M;\mathbb{Z})$. Then $\mathrm{z}x$ sends a given generator $((\Theta, \mathfrak{o}), W)$ to $\mathrm{z}((\Theta, \mathfrak{o}), W + W_x)$ where W_x can be any closed cycle in M that represents the Poincaré dual to x in $H^2(M \cdot \mathbb{Z})$.

Note that the homology as defined using a different reference cycle, ρ', is isomorphic to that defined by ρ, but not canonically isomorphic. The isomorphism becomes canonical with the choice of an element in $H_2(M, \rho, \rho')$.

Given $L \geq 1$, the chain complex $\mathcal{C}_{\mathrm{ech},\rho}$ has its corresponding subcomplex $\mathcal{C}^L_{\mathrm{ech},\rho}$; this defined in the same manner as $\mathcal{C}^L_{\mathrm{ech}}$. This subcomplex is preserved by δ.

Part 3 This part describes the additional structure on the embedded contact homology. What follows summarizes what is said in [3, Section 12] and, with regards to the "U–map", what is said in [5, Section 2.5].

The contact element This is the class defined when $\Gamma = 0$ by the empty set $\varnothing \in \mathcal{C}_{\mathrm{ech}}$. Note that $\delta\varnothing = 0$ by virtue of the fact that the coordinate s is unbounded from above on every pseudoholomorphic curve in $\mathbb{R} \times M$.

The action of $H_1(M)$/Torsion An element $\zeta \in H_1(M)$/Torsion defines a degree–1 map on the embedded contact homology. This map is denoted here by Δ_ζ. It is induced by a map from $\mathcal{C}_{\mathrm{ech},\rho}$ to itself that is defined as follows: To start, assign to each generator $(\Theta, \mathfrak{o})$ of $\mathcal{C}_{\mathrm{ech}}$ an element $W_\Theta \in H_2(M, \rho, \Theta)$. Let $\mathcal{W} = \{W_\Theta\}_{\Theta\in\mathcal{Z}_{\mathrm{ech}}}$. Now let Θ_- and Θ_+, denote given elements in $\mathcal{Z}_{\mathrm{ech}}$, and let $W \in H_2(M, \Theta_-, \Theta_+)$. Then $W_{\Theta_-} + W - W_{\Theta_+}$ defines a class in $H_2(M;\mathbb{Z})$. Let $\zeta_{\mathcal{W}}(W)$ denote the Poincaré duality pairing of this class with ζ. Define $\Delta_{\zeta,\mathcal{W}}\colon \mathcal{C}_{\mathrm{ech},\rho} \to \mathcal{C}_{\mathrm{ech},\rho}$ by demanding that it act on any given generator $((\Theta_+, \mathfrak{o}_+), W_+)$ as

$$\begin{aligned} \Delta_{\zeta,\mathcal{W}}\big((\Theta_+, \mathfrak{o}_+), W_+\big) \\ = \textstyle\sum_{(\Theta_-,\mathfrak{o}_-)} \sum_{W\in H_2(M,\Theta_-,\Theta_+)} \zeta_{\mathcal{W}}(W)\sigma(\Theta_-, \Theta_+, W)\big((O_-, \mathfrak{o}_-), W_+ - W\big). \end{aligned} \tag{1-2}$$

The definition guarantees that $\delta\Delta_{\zeta,\mathcal{W}} + \Delta_{\zeta,\mathcal{W}}\delta = 0$ and so $\Delta_{\zeta,\mathcal{W}}$ defines a homomorphism on embedded contact homology. Note that $\Delta_{\zeta,\mathcal{W}}$ depends on $\mathcal{W}$. Even so, a different choice adds at most a boundary to any given closed cycle in $\mathcal{C}_{\mathrm{ech},\rho}$, and so the induced homomorphism on embedded contact homology is, in fact, independent of the choices that comprise a given version of $\mathcal{W}$. This induced homomorphism is denoted by Δ_ζ. Note that Δ_ζ defines a homomorphism on the cohomology of each $L \geq 1$ version $\mathcal{C}^L_{\mathrm{ech},\rho}$.

The U–map This is a degree -2 map on the embedded contact homology; it is defined in [3, Section 12.1.4]. What follows briefly describes this map. To set the stage for definition, let Θ_- and Θ_+ again denote elements in $\mathcal{Z}_{\mathrm{ech}}$. Hutchings introduces a moduli space, $\mathcal{M}_2(\Theta_-, \Theta_+)$, with the properties listed next. First, $\mathcal{M}_2(\Theta_-, \Theta_+) = \varnothing$ unless the degree of Θ_+ is two less than that of Θ_-. Second, any given element $\Sigma \in \mathcal{M}_2(\Theta_-, \Theta_+)$ is a finite set whose elements are pairs of the form (C, m) where $C \subset \mathbb{R} \times M$ is a pseudoholomorphic subvariety and m is a positive integer. These pairs are constrained so that distinct pairs have distinct subvariety components and such that $\sum_{(C,m)\in\Sigma} m\pi(C) \in H_2(M, \Theta_-, \Theta_+)$. Hutchings proves that this $\mathcal{M}_2(\Theta_-, \Theta_+)$ has the structure of a smooth, 2–dimensional manifold.

Now fix a point $p \in M$ that does not lie on any Reeb orbit. Let $\mathcal{M}_2(\Theta_-, \Theta_+)^p \subset \mathcal{M}_2(\Theta_-, \Theta_+)$ denote the subset of elements which contain a pair whose subvariety component goes through the point $(0, p) \in \mathbb{R} \times M$. If J is suitably generic, this is a finite set. Moreover, respective cyclic orderings, $\mathfrak{o}_-$ and $\mathfrak{o}_+$, for the positive hyperbolic Reeb orbits from Θ_- and Θ_+ give each element in this set an associated weight, either $+1$ or -1. More is said about these ± 1 weights momentarily. If $\mathcal{M}_2(\Theta_-, \Theta_+) \neq \varnothing$, set $\sigma_p(\Theta_-, \Theta_+)$ to equal the sum of the ± 1 weights that are associated to the elements in $\mathcal{M}_2(\Theta_-, \Theta_+)$. Set $\sigma_p(\Theta_-, \Theta_+) = 0$ otherwise.

Define the automorphism U_p on $\mathcal{C}_{\mathrm{ech}}$ by declaring its action on a given generator $(\Theta_+, \mathfrak{o}_+)$ to be

$$U_p(\Theta_+, \mathfrak{o}_+) = \textstyle\sum_{(\Theta_-,\mathfrak{o}_-)} \sigma_p(\Theta_-, \Theta_+)(\Theta_-, \mathfrak{o}_-). \tag{1-3}$$

This descends to give an automorphism of the associated embedded contact homology (see [5] for a proof). The latter automorphism is called the *U–map*. The map U_p

preserves any $L \geq 1$ version of the subcomplex $\mathcal{C}_{\text{ech}}^{L}$ and so defines an automorphism, U, of the latter's cohomology.

Hutchings in [3] defines a refined version of U that gives an automorphism of the twisted embedded contact homology. To say more, fix $W \in H_2(M, \Theta_-, \Theta_+)$ and let $\sigma_p(\Theta_-, \Theta_+, W)$ denote the contribution to $\sigma_p(\Theta_-, \Theta_+)$ from those elements in $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ that define the class W. Define U_p on $\mathcal{C}_{\text{ech},\rho}$ by declaring that its action on a given generator $(\Theta_+, \mathfrak{o}_+, W_+)$ be given by the right hand side of (1-1) with $\sigma_p(\Theta_-, \Theta_+, W)$ replacing $\sigma(\Theta_-, \Theta_+, W)$.

Part 4 This part of the subsection elaborates on the definition of the signs that are used to define the U–map. To start, remark that the data used to define the signs for the differential on $\mathcal{C}_{\text{ech}}$ also orients the moduli space $\mathcal{M}_2(\Theta_-, \Theta_+)$ for any given pair of generators $\{(\Theta_-, \mathfrak{o}_-), (\Theta_+, \mathfrak{o}_+)\}$; this is done using ideas of Quillen that concern determinant line bundles for families of Fredholm operators. The details are described in [12, Part 1 of Section 3.b].

Now let $\Sigma \subset \mathcal{M}_2(\Theta_-, \Theta_+)^p$. As explained to the author by Mike Hutchings, the following is a consequence of the index inequalities in [1]:

(1-4) There is precisely one pair $(C, m) \in \Sigma$ with C not an $\mathbb{R}$–invariant cylinder. Moreover, this pair has $m = 1$. Finally, C is embedded and it does not intersect any $\mathbb{R}$–invariant cylinder from a pair in Σ.

Since p is not on any Reeb orbit, it follows that $(0, p) \in C$. What follows is now a consequence of (1-4): The tangent space to $\mathcal{M}_2(\Theta_-, \Theta_+)$ at Σ is canonically identified with the L^2_1 kernel of C's version of the operator $\mathcal{D}_C$ that is depicted in [10, (2-8)]. To say more, introduce $\pi\colon N \to C$ to denote C's normal bundle. The operator $\mathcal{D}_C$ is an $\mathbb{R}$–linear first order operator that sends a section of N to one of $N \otimes T^{0,1}C$; it is defined so as to send a given section ζ of N to

$$\mathcal{D}_C\zeta = \bar{\partial}\zeta + \nu_C\zeta + \mu_C\bar{\zeta}, \tag{1-5}$$

where ν_C is a certain section of $T^{1,0}C$ and μ_C a section of $N^2 \otimes T^{0,1}C$, these defined by the 1–jet along C of the almost complex structure. This gives a bounded, Fredholm operator from $L^2_1(C; N)$ to $L^2(C; N \otimes T^{0,1}C)$. Its kernel in this guise is denoted kernel$(\mathcal{D}_C)$; this a 2–dimensional vector space of sections of N. Implicit in the assertion that p is generic is the condition that the restriction map kernel$(\mathcal{D}_C) \to N|_p$ is an isomorphism. This understood, then the orientation of $\mathcal{M}_2(\Theta_-, \Theta_+)$ at p also orients $N|_p$. This orientation either agrees or disagrees with the complex orientation of $N|_p$. If it agrees, then Σ contributes $+1$ to $\sigma_p(\Theta_-, \Theta_+)$. If it disagrees, then Σ contributes -1 to $\sigma_p(\Theta_-, \Theta_+)$.

1.b Auxiliary structures on the Seiberg–Witten Floer cohomology

This subsection briefly describes the various auxiliary structures on the Seiberg–Witten Floer cohomology. A detailed account of these structures is given in Kronheimer and Mrowka's book [7]. There are also four parts to this subsection.

Part 1 What follows here is a brief summary of those parts of the definition of the Seiberg–Witten Floer cohomology that are needed for subsequent definitions. Much of what is said in what follows paraphrases material in [10, Section 3]. To start, fix a $\mathrm{Spin}_{\mathbb{C}}$ structure for M and then write the corresponding spinor bundle $\mathbb{S}$ as $E \oplus EK^{-1}$ with E the $+i$ eigenbundle for Clifford multiplication by the contact form a. To relate things to the embedded contact homology, agree to choose the $\mathrm{Spin}_{\mathbb{C}}$ structure so that E's first Chern class is Poincaré dual to the class Γ.

Fix $r \geq 1$ and then choose a suitably generic, small normed function $\mathfrak{g}$ from the Banach space $\mathcal{P}$ of [10, Section 3.d]. In particular, choose $\mathfrak{g}$ so that the gauge equivalence classes of solutions to [10, (3-5)] can be used as generators of the Seiberg–Witten Floer cochain complex, $\mathcal{C}^{\mathrm{SW}}$; and so that the instanton solutions to [10, (3-6)] can be used to define the differential on this same cochain complex. An additional constraint on $\mathfrak{g}$ is given below.

Part 2 This part describes the refined version of the Seiberg–Witten Floer cohomology; this will correspond to the homology of $\mathcal{C}_{\mathrm{ech},\rho}$. The generators for this refinement are supplied in part by the irreducible solutions to the large r version of [10, (3-5)] and in part by any reducible solutions. As it turns out, the reducible solutions play no role in the proof of Theorem 1.1, and so they are not discussed further. The generators that are defined using the irreducible solutions are their equivalence classes with respect to the equivalence relation that identifies $\mathfrak{c}$ and $\mathfrak{c}'$ when $\mathfrak{c}' = u\mathfrak{c}$ with u a smooth, but null-homotopic map from M to S^1. The resulting $\mathbb{Z}$–module is denoted by $\mathcal{C}^{\mathrm{SW}*}$. This $\mathbb{Z}$ module admits a free action of $H^1(M;\mathbb{Z})$ whose quotient is $\mathcal{C}^{\mathrm{SW}}$. The differential on $\mathcal{C}^{\mathrm{SW}*}$ is defined on the irreducible generators via a modified version of [10, (3-4)]. This modification replaces any given version of the weight $\sigma(\mathfrak{c}',\mathfrak{c})$ that appears in [10, (3-4)] with the integer, $\sigma_*(\mathfrak{c}',\mathfrak{c})$, that gives the contribution to $\sigma(\mathfrak{c}',\mathfrak{c})$ from the subset of components in $\mathcal{M}_1(\mathfrak{c}',\mathfrak{c})$ that contain instantons with $s \to -\infty$ limit $\mathfrak{c}'$ and $s \to \infty$ limit $u\mathfrak{c}$ where u is a null-homotopic map from M to S^1. Here, as in [10, Section 3], the space $\mathcal{M}_1(\mathfrak{c}',\mathfrak{c})$ is defined to be the space of instanton solutions to [10, (3-6)] with the following two properties: First, if $\mathfrak{d}$ is in this space, then $\lim_{s\to-\infty}\mathfrak{d}(s) = \mathfrak{c}'$ and $\lim_{s\to\infty}\mathfrak{d}(s) = u\mathfrak{c}$ with $u \in C^\infty(M;S^1)$. The second property refers to the index of $\mathfrak{d}$'s version of operator that is depicted in [10, (3-9)]: The L^2_1 index of this operator must equal 1.

This refined version of Seiberg–Witten Floer cohomology can be viewed as a module over the group ring $\mathbb{Z}[H^1(M;\mathbb{Z})]$. The action of this group ring on a given generator of $\mathcal{C}^{\mathrm{SW}*}$ is defined as follows: Let $\mathfrak{c}$ denote the generator. Fix $\mathrm{z} \in \mathbb{Z}$ and fix $x \in H^1(M;\mathbb{Z})$. Then $\mathrm{z}x(\mathfrak{c}) = \mathrm{z}(u_x\mathfrak{c})$ where u_x can be any smooth map from M to S^1 that defines the class x.

Part 3 This part describes the additional structure on the Seiberg–Witten Floer cohomology.

The contact element This class is defined in all but name by Kronheimer and Mrowka in [6]. In any event, it is the class in the Seiberg–Witten Floer cohomology for the $\mathrm{Spin}_{\mathbb{C}}$ structure with $c_1(E) = 0$ that is described in [9, Theorem 4.1].

The action of $H_1(M)/\mathrm{Torsion}$ This action increases degree by 1. In order to define the action on the irreducible solutions, it is necessary to first make some additional choices. To this end, suppose that $\mathfrak{c}$ is a gauge equivalence class of some irreducible solution to [10, (3-5)]. Fix a generator, $\mathfrak{c}_*$, of the module $\mathcal{C}^{\mathrm{SW}*}$ that projects to $\mathfrak{c}$. Thus, $\mathfrak{c}$ is the orbit of $\mathfrak{c}_*$ under the action of $H^1(M;\mathbb{Z})$ on $\mathcal{C}^{\mathrm{SW}*}$. Use χ to denote the set of such choices. A given $\zeta \in H_1(M;\mathbb{Z})/\mathrm{Torsion}$ defines a homomorphism

$$\Delta_{\zeta,\chi}\colon \mathcal{C}^{\mathrm{SW}*} \to \mathcal{C}^{\mathrm{SW}*} \tag{1-6}$$

by its action on the generators. To specify this action, consider the action on a generator $\mathfrak{c}_+$ from the set χ. Let $\mathfrak{c}_-$ denote another generator of $\mathcal{C}^{\mathrm{SW}*}$ and let $\mathfrak{d}$ denote an instanton solution to [10, (3-6)] with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $u\mathfrak{c}_+$ where u is a null-homotopic map from M to S^1. Write $\mathfrak{c}_+ = u_+\mathfrak{c}_{+*}$ and $\mathfrak{c}_- = u_-\mathfrak{c}_{-*}$ with $\mathrm{u}_\pm$ maps from M to S^1. These define respective classes x_- and x_+ in $H^1(M;\mathbb{Z})$. Set

$$\Delta_{\zeta,\chi}(\mathfrak{c}_+) = \textstyle\sum_{\mathfrak{c}_- \in \mathcal{C}^{\mathrm{SW}*}} \zeta(x_- - x_+)\sigma_*(\mathfrak{c}_-, \mathfrak{c}_+)\mathfrak{c}_-, \tag{1-7}$$

where the notation $\zeta(\cdot)$ invokes the identification

$$H_1(M;\mathbb{Z})/\mathrm{Torsion} = \mathrm{Hom}(H^1(M;\mathbb{Z});\mathbb{Z}).$$

This automorphism of $\mathcal{C}^{\mathrm{SW}*}$ anticommutes with the coboundary endomorphism and so descends to an automorphism of the cohomology of $\mathcal{C}^{\mathrm{SW}*}$. It defines an automorphism of the cohomology of $\mathcal{C}^{\mathrm{SW}}$ by virtue of the fact that it commutes with the $H^1(M;\mathbb{Z})$ action on $\mathcal{C}^{\mathrm{SW}*}$. Neither of these cohomology actions depend on the choices that comprise χ. This understood, these cohomology actions of ζ are denoted in what follows by Δ_ζ.

The U–map This is a degree -2 map on the cohomology of $\mathcal{C}^{\mathrm{SW}*}$ and $\mathcal{C}^{\mathrm{SW}}$. What follows is equivalent to the definition given in [7, Chapters 3.2 and 25.3]. The automorphism U is defined via a map on the generators of $\mathcal{C}^{\mathrm{SW}}$ and $\mathcal{C}^{\mathrm{SW}*}$, and only the irreducible generators are discussed in what follows. To this end, first fix a point $p \in M$. Now suppose that $\mathfrak{c}_+$ and $\mathfrak{c}_-$ are irreducible solutions to [10, (3-5)]. Introduce the moduli space $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ of instanton solutions to [10, (3-6)] with the following two properties: First, the $s \to -\infty$ limit is $\mathfrak{c}_-$ and the $s \to \infty$ limit is $u\mathfrak{c}_+$ where u is a smooth map from M to S^1. Second, the corresponding version of the operator $\mathfrak{D}$ in [10, (3-9)] has Fredholm index equal to 2.

Write a given element $\mathfrak{d} \in \mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ as $(A, \psi = (\alpha, \beta))$, and let $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ denote the subset of elements in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ with the property that α vanishes at the point $(0, p) \in \mathbb{R} \times M$. If the perturbation function $\mathfrak{g}$ is chosen from a certain residual subset of $\mathcal{P}$, then this is a finite set. Moreover, the choices that are needed to define the differential for the Seiberg–Witten Floer cohomology can be used to canonically associate a weight, either $+1$ or -1, to each element in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$. The sum of these ± 1 weights is denoted by $\sigma_p(\mathfrak{c}_-, \mathfrak{c}_+)$. This ± 1 weight assignment is described in more detail momentarily.

These signs can be used to define an automorphism, U_p, on $\mathcal{C}^{\mathrm{SW}}$:

$$U_p \mathfrak{c}_+ = \textstyle\sum_{\mathfrak{c}_- \in \mathcal{C}^{\mathrm{SW}*}} \sigma_p(\mathfrak{c}_-, \mathfrak{c}_+)\mathfrak{c}_-. \tag{1-8}$$

This map commutes with the differential and so descends to a degree -2 homomorphism on the cohomology of $\mathcal{C}^{\mathrm{SW}}$. The latter is independent of the point p, and is denoted by U in what follows.

The definition of U on the cohomology of $\mathcal{C}^{\mathrm{SW}*}$ is defined using the analog of (1-8) that has $\sigma_p(\mathfrak{c}_-, \mathfrak{c}_+)$ replaced by just the contribution to this sum from those instantons with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $u\mathfrak{c}_+$ with u a null-homotopic map from M to S^1.

Part 4 What follows says more about the ± 1 weights that are used to define the sum on the right hand side of (1-8). To start the story, associate the point $\alpha|_{s=0}(p) \in E|_p$ to any given instanton solution $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ of [10, (3-6)]. This association can be viewed as a map from the moduli space of instanton solutions to [10, (3-6)] to a 1–dimensional, complex vector space. For example, this map can be restricted to any given version of the moduli space $\mathcal{M}_2(\cdot, \cdot)$ and likewise to any given version of $\mathcal{M}_1(\cdot, \cdot)$. This map can also be evaluated on any constant map from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$. If the perturbation function $\mathfrak{g}$ is chosen in a suitably generic fashion, then this map has purely transverse zeros on all versions of $\mathcal{M}_2(\cdot, \cdot)$, it lacks zeros on all versions of $\mathcal{M}_1(\cdot, \cdot)$, and it is nonzero on any constant map from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$. As a consequence, there will be but a finite number of zeros in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ for any given pair $\mathfrak{c}_-$ and $\mathfrak{c}_+$.

The data used to orient any given version of $\mathcal{M}_1(\cdot\,,\,\cdot)$ using Quillen's notion of the determinant line bundle also serves to orient any given version of $\mathcal{M}_2(\cdot\,,\,\cdot)$. This is described in [7] and summarized in [12, Part 1 of Section 3.b]. Given that $\mathcal{M}_2(\cdot\,,\,\cdot)$ is oriented, then each transversal zero of a map from $\mathcal{M}_2(\cdot\,,\,\cdot)$ to $\mathbb{C}$ has a well defined local Euler number, this either $+1$ or -1. This local Euler number is the ± 1 weight that is assigned to any given element in $\mathcal{M}_2(\cdot\,,\,\cdot)^p$.

1.c The equivalence

The theorem that follows gives the formal statement of equivalence between the corresponding auxiliary structures for embedded contact homology and Seiberg–Witten Floer cohomology.

Theorem 1.1 *Fix a class $\Gamma \in H_1(M;\mathbb{Z})$ and fix the* $\mathrm{Spin}_{\mathbb{C}}$ *structure whose spinor bundle $\mathbb{S}$ splits as $E \oplus EK^{-1}$ where $c_1(E)$ is Poincaré dual to Γ. Then the isomorphism between the corresponding embedded contact homology and the Seiberg–Witten Floer cohomology given in [10] can be assumed to have the following properties:*

- *It identifies the respective contact elements in the case when $c_1(E) = 0$.*
- *It intertwines the respective actions of $H_1(M;\mathbb{Z})/\mathrm{Torsion}$.*
- *It intertwines the respective U–maps.*

In addition, this isomorphism is induced by an isomorphism between the respective $\mathbb{Z}[H^1(M;\mathbb{Z})]$ refinements of the embedded contact homology and Seiberg–Witten Floer cohomology. The latter isomorphism intertwines the actions of $H_1(M;\mathbb{Z})/\mathrm{Torsion}$ and the U–map.

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2 The proof of Theorem 1.1

The subsections that follow prove Theorem 1.1 but for some technical points that concern the U–map. The latter are proved in Section 3 and Section 4 to come.

The arguments in this section and the subsequent sections refer to the solutions to certain special versions of [10, (3-5) and (3-6)]. These versions are defined using a metric with $|a| = 1$ and $*da = 2a$. Note that a metric of this sort restricts to the kernel of a as the bilinear form $da(\cdot\,, J(\cdot))$ where J is such that this bilinear form is symmetric and positive definite. These special versions are defined using a function $\mathfrak{g}$ used in [10, (3-5), (3-6)] that has the form $\mathfrak{e}_\mu$ with $\mathfrak{e}_\mu$ given in [10, (3-11)] using a coclosed 1–form μ from the Banach space of Ω that is defined in [10, Section 3.d]. With the metric and $\mathfrak{g}$ just described, what is written in [10, (3-5)] is

(2-1)
- $B_A - r(\psi^\dagger \tau \psi - ia) - i * d\mu + \frac{1}{2} B_{A_K} = 0.$
- $D_A \psi = 0.$

The notation here is from [10, Section 3]. In particular, $\psi^\dagger \tau \psi$ is defined in [10, Section 3.a] and the Dirac operator D_A in [10, Section 3.c]. What is denoted by $*$ is the metric's Hodge star operator. The generators of the Seiberg–Witten Floer cochain complex are defined using the solutions to (2-1).

With $\mu \in \Omega$ fixed, and $r \geq 1$ chosen, $\mathcal{M}^r$ henceforth denotes the space of gauge equivalence classes of solutions to (2-1). The corresponding version of [10, (3-6)], the equation for instantons, reads

(2-2)
- $\frac{\partial}{\partial s} A + B_A - r(\psi^\dagger \tau \psi - ia) - i * d\mu + \frac{1}{2} B_{A_K} = 0.$
- $\frac{\partial}{\partial s} \psi + D_A \psi = 0.$

Note that (2-1) and (2-2) appear in [10] as (3-17) and (3-18), respectively.

There is also a corresponding version of the operator that is depicted in [10, (3-9)]: This is the operator on sections over $\mathbb{R} \times M$ of $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$ that sends a given section (b, η, ϕ) to one whose respective iT^*M, $\mathbb{S}$ and $i\mathbb{R}$ components are

(2-3)
- $\frac{\partial}{\partial s} b + *db - d\phi - 2^{-1/2} r^{1/2} (\psi^\dagger \tau \eta + \eta^\dagger \tau \psi).$
- $\frac{\partial}{\partial s} \eta + D_A \eta + 2^{1/2} r^{1/2} (\mathrm{cl}(b)\psi + \phi\psi).$
- $\frac{\partial}{\partial s} \phi + *d * b - 2^{-1/2} r^{1/2} (\eta^\dagger \psi - \psi^\dagger \eta).$

The notation here is that used in [10, (3-9)]. In particular, $\mathrm{cl}(\cdot)$ denotes the Clifford multiplication endomorphism as defined in [10, Section 3.a].

Keep in mind the following notational convention: In any given appearance, c_0 denotes a constant that is greater than 1. Moreover, its value can be assumed to increase between subsequent appearances. It is always independent of r and other significant parameters.

2.a The contact elements

What with [11, Proposition 3.1], the fact that the respective contact elements in embedded contact homology and Seiberg–Witten Floer cohomology coincide follows directly from [9, Proposition 4.3] and what is said in its proof.

2.b The $\mathbb{Z}[H^1(M;\mathbb{Z})]$ module structure

This subsection is concerned with the following proposition:

Proposition 2.1 *Fix a class $\Gamma \in H_1(M;\mathbb{Z})$ and fix the* $\mathrm{Spin}_{\mathbb{C}}$ *structure whose spinor bundle $\mathbb{S}$ splits as $E \oplus EK^{-1}$ where $c_1(E)$ is Poincaré dual to Γ. The isomorphism that is given by [10, Theorem 1] is induced by an isomorphism between the respective $\mathbb{Z}[H^1(M;\mathbb{Z})]$ refinements of the embedded contact homology and Seiberg–Witten Floer cohomology.*

Proof of Proposition 2.1 The proof of this proposition has five parts.

Part 1 Fix $L \gg 1$ and suppose that $(\hat{a}, \hat{J})$ is a pair of contact structure from the set $\mathcal{N}_M$ and almost complex structure from $\mathcal{J}_{\hat{a}}$ that obey the conditions in [10, (4-1)]. Define $\mathcal{Z}_{\mathrm{ech}}^{L}$ as in Part 1 of Section 1.a using this pair. Fix a 1–form μ with small $\mathcal{P}$–norm from the Banach space Ω that is described in [10, Section 3.d]. Now take r large enough to invoke [10, Theorem 4.2] for a given choice of 1–form μ. The latter describes a bijection, Φ^r, from $\mathcal{Z}_{\mathrm{ech}}^{L}$ to the set $\mathcal{M}^r$ of gauge equivalence classes of solutions to (1-5) for which the function E given in [10, (3-13)] has value less than $2\pi L$.

Part 2 Fix $r > 0$. The following is a consequence of what is said in [11, Lemma 3.10]: If r is sufficiently large, and if $\mathfrak{c} = (A, \psi = (\alpha, \beta))$ is a solution to the corresponding version of (2-1) with gauge equivalence class in the image of Φ^r, then the connection $\hat{A}$ given by [10, (4-7)] is flat on the complement of the radius r tubular neighborhood of any Reeb orbit in M with length less than L. To elaborate, [10, Lemma 3.10] finds that $|\alpha|$ has distance at most $1/1000$ from 1 on the complement of the radius $c_0 r^{1/2}$ tubular neighborhood of any Reeb orbit in M with length less than L.

With the preceding understood, choose r so that each point on the reference loop ρ has distance at least $3r$ from any Reeb orbit in M with length less than L. Fix a flat connection, θ, on E that admits a nonzero, covariantly constant section on the complement of the radius r tubular neighborhood of ρ. Given $\mathfrak{c}$ as above, write the corresponding connection $\hat{A}$ as $\theta + a_{\mathfrak{c}}$. Suppose that υ is an embedded, oriented loop in M that has distance at least r from ρ and from the Reeb orbits with length less than L. Define

$$\mathrm{x}_{\upsilon}(\mathfrak{c}) = \frac{1}{2\pi i}\int_{\upsilon} a\mathfrak{c}. \tag{2-4}$$

This $\mathrm{x}_{\upsilon}(\mathfrak{c})$ is necessarily an integer. Note as well that $\mathrm{x}_{\upsilon}(u\mathfrak{c}) = \mathrm{x}_{\upsilon}(\mathfrak{c})$ when u is a null-homotopic map from M to S^1. In general, $\mathrm{x}_{\upsilon}(u\mathfrak{c})$ is obtained $\mathrm{x}_{\upsilon}(\mathfrak{c})$ by adding the integer that is obtained by evaluating the class defined by u in $H^1(M;\mathbb{Z})$ on the class defined by υ in $H_1(M;\mathbb{Z})$.

Part 3 This part of the discussion describes a lift of Φ^r so as to give a $1-1$ correspondence between the set $\mathcal{Z}_{\mathrm{ech},\rho} = \{(\Theta, W) : \Theta \in \mathcal{Z}^L_{\mathrm{ech}}$ *and* $W \in H_2(M, \rho, \Theta)\}$ and the set of generators of $\mathcal{C}^{\mathrm{SW}*}$ that have $\mathrm{E} < 2\pi L$. To define this lift, fix a set, Λ, of oriented, embedded loops in $M - \bigcup_{(\gamma,m)\in\Theta}\gamma$ that generate $H_1(M;\mathbb{Z})/\mathrm{Torsion}$ and are such that each point in each loop from Λ has distance at least $3r$ from ρ and from each Reeb orbit in M with length less than L. Any given element $(\Theta, W) \in \mathcal{Z}^L_{\mathrm{ech}}$ defines a map, $\mathrm{N}_{(\Theta,W)}\colon \Lambda \to \mathbb{Z}$; its value on a given loop is the loop's algebraic intersection number with W.

A bijection between the set $\mathcal{Z}^L_{\mathrm{ech},\rho}$ and the set of generators of $\mathcal{C}^{\mathrm{SW}*}$ with $\mathrm{E} < 2\pi L$ is defined as follows: The bijection sends a given $(\Theta, W) \in \mathcal{Z}^L_{\mathrm{ech},\rho}$ to the orbit under the identity component of $C^\infty(M;S^1)$ of a solution, $\mathfrak{c}_{(\Theta,W)}$, to (2-1) with the following properties: First, its orbit under all of $C^\infty(M;S^1)$ is $\Phi^r(\Theta)$. Second, $\mathrm{X}_\upsilon(\mathfrak{c}_{(\Theta,W)}) = \mathrm{N}_{(\Theta,W)}(\upsilon)$ for all $\upsilon \in \Lambda$.

To see that the map just define is a bijection, it is enough to prove that it is equivariant with respect to the actions of $H^1(M;\mathbb{Z})$ on $\mathcal{Z}^L_{\mathrm{ech}}$ and on the set of generators for $\mathcal{C}^{\mathrm{SW}*}$. To this end, suppose that $z \in H^1(M;\mathbb{Z})$ and that W_z is a cycle in M that represents the Poincaré dual of z in $H_2(M;\mathbb{Z})$. Let υ denote an embedded loop in M with distance at least $3r$ from ρ and from any Reeb orbit with length less than L. Then $\mathrm{N}_{(\Theta,W+W_z)}(\upsilon) = \mathrm{N}_{(\Theta,W)}(\upsilon) + z(\upsilon)$ where $z(\upsilon)$ represents the value of the cocycle z on υ. This understood, let $u_z\colon M \to S^1$ denote a smooth map that defines the class z. Then $\mathrm{X}_\upsilon(u_z\mathfrak{c}_{(\Theta,W)}) = \mathrm{X}_\upsilon(\mathfrak{c}_{(\Theta,W)}) + z(\upsilon)$. Taking υ from the set Λ, it follows that $\mathfrak{c}_{(\Theta,W+W_z)} = u_z\mathfrak{c}_{(\Theta,W)}$ as desired.

The bijection just defined is denoted in what follows by Φ^{r*}.

Part 4 Fix a pair Θ_- and Θ_+ from $\mathcal{Z}^L_{\mathrm{ech}}$ and $\Sigma \in \mathcal{M}_1(\Theta_-,\Theta_+)$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote respective solutions to (2-1) that define $\Phi^r(\Theta_-)$ and $\Phi^r(\Theta_+)$. [10, Theorem 4.3] associates to Σ an instanton $\mathfrak{d} \in \mathfrak{M}_1(\mathfrak{c}_-,\mathfrak{c}_+)$. Among other things, $\mathfrak{d}$ is a smooth map from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(M;\mathbb{S})$ such that $\lim_{s\to-\infty}\mathfrak{d}(s) = \mathfrak{c}_-$ and $\lim_{s\to\infty}\mathfrak{d}(s) = u\mathfrak{c}_+$ with u here a smooth map from M to S^1. □

Lemma 2.2 *Fix a pair of generators $(\Theta_-, \mathfrak{o}_-)$ and $(\Theta_+, \mathfrak{o}_+)$ of $\mathcal{C}^L_{\mathrm{ech}}$, an element $W_- \in H_2(M,\rho,\Theta_-)$, an element, W, of $H_2(M,\Theta_-,\Theta_+)$, and an element $\Sigma \in \mathcal{M}_1(\Theta_-,\Theta_+)$ that defines W. Let $\mathfrak{d}$ denote the instanton solution to* (2-2) *with $\lim_{s\to-\infty}\mathfrak{d}(s) = \mathfrak{c}_{(\Theta_-,W_-)}$ that is obtained from Σ using [10, Theorem 4.3]. Then $\lim_{s\to\infty}\mathfrak{d}(s) = u\mathfrak{c}_{(\Theta_+,W_-+W)}$ where u is a homotopically trivial map from M to S^1.*

Proof of Lemma 2.2 Let υ denote a given loop in Λ, and let $\mathrm{N}_W(\upsilon)$ denote its algebraic intersection number with W. Thus,

$$\mathrm{N}_{(\Theta_+,W_-+W)}(\upsilon) = \mathrm{N}_{(\Theta_+,W_-)}(\upsilon) + \mathrm{N}_W(\upsilon). \tag{2-5}$$

Given that Σ defines the element W, the number $\mathrm{N}_W(\upsilon)$ can be viewed as the algebraic intersection number between $\mathbb{R}\times\upsilon$ and the cycle $\sum_{(C,m)\in\Theta} mC$. Note in this regard that there exists $R \gg 1$ such that all intersections occur where $s \in [-R, R]\times M$.

To continue, let $\pi\colon \mathbb{R}\times M \to M$ denote the projection, and use π to view E as a bundle over $\mathbb{R}\times M$. As a pullback, the restriction of E to any given fiber of π has a canonical connection. Now write $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ and view the connection A as a connection on this pullback of E to $\mathbb{R}\times M$. Note that A's restriction to any given fiber of π is identical to the canonical connection. View it as a section of this pullback bundle. Take this view so as to define $\nabla_A\alpha$ as a section of $T*(\mathbb{R}\times M)\otimes E$. Now use [10, (4-7)] to define the connection $\widehat{A}$ from A and α. Note that $\widehat{A}$ as opposed to A need not restrict to any given fiber of π as the canonical connection.

This understood, let $F_{\widehat{A}} \in C^\infty(\mathbb{R}\times M, iT*(\mathbb{R}\times M))$ denote the curvature of $\widehat{A}$. It follows from the construction given in [11, Sections 4–7] that $(1/(2\pi i))F_A$ represents the Poincaré dual of the 2–chain defined by the formal sum $\sum_{(C,m)\in\Theta} mC$. In particular, this has the following consequence: If $\upsilon \in \Lambda$, then

$$\mathrm{N}_W(\upsilon) = \frac{1}{2\pi i}\int_{\mathbb{R}\times\upsilon} F_{\widehat{A}}. \tag{2-6}$$

Now write $\widehat{A} = \theta + a_\Sigma$ where $a_\Sigma \in C^\infty(\mathbb{R}\times M; iT*(\mathbb{R}\times M))$. It follows from (2-6) using Stokes theorem that

$$\mathrm{N}_W(\upsilon) = -\mathrm{X}_\upsilon(\mathfrak{c}_{(\Theta,W_-)}) + \mathrm{X}_\upsilon(u\mathfrak{c}_{(\Theta_+,W_-+W)}). \tag{2-7}$$

What with (2-5), this implies that $\mathrm{N}_{(\Theta_+,W_-+W)}(\upsilon) = \mathrm{X}_\upsilon(u\mathfrak{c}_{(\Theta_+,W_+W)})$. Given that the preceding holds for all $\upsilon \in \Lambda$, it follows that u must be homotopically trivial.

Part 5 Let $\Theta \in \mathcal{Z}^L_{\mathrm{ech}}$. Order the subset of pairs $(\gamma, 1) \in \Theta$ for which γ is positive hyperbolic (its linear return map has positive eigenvalues). Doing so for all such Θ identifies $\mathcal{Z}^L_{\mathrm{ech}}$ with a set of generators of $\mathcal{C}^L_{\mathrm{ech}}$. This identification extends by linearity to define an isomorphism between $\mathcal{C}^L_{\mathrm{ech}}$ and the submodule of $\mathcal{C}^{\mathrm{SW}}$ whose generators are such that $\mathrm{E} < 2\pi L$. [10, Theorem 4.3] describes an isomorphism of complexes of the sort just described. Let T_Φ denote the latter.

The data used to define T_Φ also identifies the set $\mathcal{Z}^L_{\mathrm{ech},\rho}$ with the set of generators of $\mathcal{C}^L_{\mathrm{ech},\rho}$. Given such an identification, the map Φ^{r*} defines a $1-1$ correspondence between the set of generators of $\mathcal{C}^L_{\mathrm{ech},\rho}$ and the set of generators of $\mathcal{C}^{\mathrm{SW}*}$ with $\mathrm{E} < 2\pi L$. This extends by linearity to give an isomorphism between $\mathcal{C}^L_{\mathrm{ech},\rho}$ and the submodule of $\mathcal{C}^{\mathrm{SW}*}$ that is generated by the equivalence classes with $\mathrm{E} < 2\pi L$. Use $T_{\Phi*}$ to denote this last isomorphism. What with [10, Theorem 4.3] and Lemma 2.2 above, it follows that $T_{\Phi*}$ intertwines the embedded contact homology differential with the differential on $\mathcal{C}^{\mathrm{SW}*}$.

Granted the preceding, the arguments given [10, Section 4.c] for [10, Theorem 1] can be repeated with only notational changes to complete the proof of Proposition 2.1. □

2.c The action of $H_1(M;\mathbb{Z})/\mathrm{Torsion}$

The purpose of what follows is to prove that the isomorphism that is described in Proposition 2.1 intertwines the actions of $H_1(M;\mathbb{Z})/\mathrm{Torsion}$. To start this proof, fix $\zeta \in H_1(M;\mathbb{Z})/\mathrm{Torsion}$ and introduce a $\mathcal{W} = \{W_\Theta\}_{\Theta\in\mathcal{Z}_{\mathrm{ech}}}$ so as to define the map $\Delta_{\zeta,\mathcal{W}}$ as in (1-2). The definition of the corresponding automorphism of $\mathcal{C}^{\mathrm{SW}*}$ requires first the specification of a set $\mathcal{X}$, this a lift to $\mathcal{C}^{\mathrm{SW}*}$ of the set of generators of $\mathcal{C}^{\mathrm{SW}}$. Given any value for $L \gg 1$ and then r very large, take this set $\mathcal{X}$ to have the following property: If $\Theta \in \mathcal{Z}_{\mathrm{ech}}^L$, then the lift of $\Phi^r(\Theta)$ is $\Phi^{r*}((\Theta, W_\Theta)) = \mathfrak{c}_{(\Theta,W_\Theta)}$. Granted such a set $\mathcal{X}$, define $\Delta_{\zeta,\mathcal{X}}$ as in (1-6). Note in this regard that if $r' > r$, then it follows from the third bullet of [10, Theorem 4.2] that the lifts in the version of $\mathcal{X}$ defined by r of the elements in $\Phi^r(\mathcal{Z}_{\mathrm{ech}}^L)$ enjoy a canonical identification with those of their $\Phi^{r'}$ counterparts in the r' version of $\mathcal{X}$.

Now let Θ_- and Θ_+ denote given elements in $\mathcal{Z}_{\mathrm{ech}}^L$, and let $W \in H_2(M,\Theta_-,\Theta_+)$. The isomorphism used by Proposition 2.1 is constructed so that the integer $\sigma(\Theta_-,\Theta_+,W)$ that appears in (1-2) is the same as the integer $\sigma_*(\mathfrak{c}_{(\Theta_-,W_-)},\mathfrak{c}_{(\Theta_+,W_++W)})$ that appears in (1-7). This understood, it is enough to prove that the class $W_{\Theta_-} + W - W_{\Theta_+}$ in $H_2(M;\mathbb{Z})$ is Poincaré dual to the class in $H^1(M;\mathbb{Z})$ that is defined by any map $u\colon M \to S^1$ with the property that $\mathfrak{c}_{(\Theta_+,W_{\Theta_-}+W)} = u\,\mathfrak{c}_{(\Theta_+,W_{\Theta_+})}$. This last property follows from the definition of the map Φ^{r*}. It follows from what is said a the end of the preceding paragraph that the isomorphism so defined by a given large choice for r is suitably independent of r, and thus consistently defined on the length filtration of $\mathcal{Z}_{\mathrm{ech},\rho}$ given by the inclusion of a given L version of $\mathcal{Z}_{\mathrm{ech},\rho}^L$ into an $L' > L$ version.

2.d The U–map

The assertions in Theorem 1.1 about the U–map are proved here modulo the proofs of the upcoming Proposition 2.5 and Theorem 2.6. The argument that follows has three parts.

Part 1 This part gives an analog of [10, (2-10)] for the subvarieties that contribute to the U–map. To set the stage, fix a point $p \in M$ that is not on any Reeb orbit.

Lemma 2.3 *A residual set, $\mathcal{J}_{a,p}$, of almost complex structures can be chosen so as to have the following additional property: Fix $J \in \mathcal{J}_{a,p}$ and fix a pair, Θ_- and Θ_+, in $\mathcal{Z}_{\mathrm{ech}}$. Then the following are true:*

- *No element from $\mathcal{M}_1(\Theta_-, \Theta_+)$ has a pair (C, m) with $p \in C$.*
- *The space $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ has a finite set of elements.*
- *Let $\Sigma \in \mathcal{M}_2(\Theta_-, \Theta_+)^p$.*
  (1) *The subvariety $\bigcup_{(C,m)\in\Sigma} C$ is embedded.*
  (2) *There is exactly one element $(C, m) \in \Sigma$ where C is not an $\mathbb{R}$–invariant cylinder. This element has $m = 1$.*
  (3) *Let $(C, 1) \in \Sigma$ be such that C is not an $\mathbb{R}$–invariant cylinder. The kernel of the corresponding version of the operator $\mathcal{D}_C$ from* (1-5) *is 2–dimensional and the restriction of this kernel to $p \in C$ defines an isomorphism with the normal bundle to C at p.*
  (4) *Let $\mathcal{E} \subset \Sigma$ denote an end. Let $q_{\mathcal{E}}, \mathrm{div}_{\mathcal{E}}$ and $\zeta_{q_{\mathcal{E}}}$ denote the data that appear in $\mathcal{E}$'s version of [10, (2-6)]. Then $\mathrm{div}_{\mathcal{E}} = \{q_{\mathcal{E}}\}$ and $\zeta_{q_{\mathcal{E}}} \neq 0$.*
  (5) *Let $\mathcal{E}$ and $\mathcal{E}'$ denote distinct pair of either positive or negative ends of Σ such that $\gamma_{\mathcal{E}} = \gamma_{\mathcal{E}'}$ and $q_{\mathcal{E}} = q_{\mathcal{E}'}$ Let $\zeta_{q_{\mathcal{E}}}$ and $\zeta_{q_{\mathcal{E}'}}$ denote the $2\pi q$–periodic eigenvector that appears in the respective $\mathcal{E}$ and $\mathcal{E}'$ versions of [10, (2-6)]. Then $\zeta_{q_{\mathcal{E}}}|_t \neq \zeta_{q_{\mathcal{E}'}}|_{t+2\pi k}$ for any $t \in S^1$ and $k \in \mathbb{Z}$.*

Proof of Lemma 2.3 As explained to the author by Hutchings, the existence of a version of $\mathcal{J}_a$ where Assertions (1) and (2) of the third bullet hold follow directly from the index inequalities asserted by [1, Lemma 9.5]. Given Assertions (1) and (2) of the third bullet, then Assertion (3) of the third bullet follows using the Sard–Smale theorem in a manner that is standard fare in the curve counting business. Similar arguments using the Sard–Smale theorem prove what is asserted by the first bullet and they prove that the space $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ has at worst countable set of elements, and that each is nondegenerate. The fact that this space is finite is a compactness assertion and this assertion follows from [2, Theorem 1.8]. Assertions (4) and (5) of the third bullet follow using essentially the same arguments as those given in [5, Section 3].

Part 2 This part discusses a slight generalization of the notion of a (δ, L) approximation to any given pair of contact form a and compatible almost complex structure. To set the stage, suppose that a is from the residual set described in [10, Lemma 2.1], that $p \in M$ has been chosen, and that J is from $\mathcal{J}_{a,p}$. Fix $L > 1$ and $\delta > 0$. The definition of a (δ, L) approximation to (a, J) is given in the discussion that surrounds [10, (2-11)]. What follows is the needed generalization of this definition.

A pair $(\hat{a}, \hat{J})$ of contact structure on M and compatible almost complex structure on $\mathbb{R} \times M$ is said to be a (δ, L, p)*-approximation* for (a, J) when the following is true: There is a smooth, 1–parameter family $\{(a_\tau, J_\tau)\}_{\tau \in [0,1]}$ of pairs of contact structure and compatible almost complex structure with $(a_0, J_0) = (a, J)$ and $(a_1, J_1) = (\hat{a}, \hat{J})$; and such that

(1) For each $\tau \in [0, 1]$, the respective sets of a and a_τ Reeb orbits with symplectic action less than L are identical.

(2) Let γ denote a Reeb orbit for a with $\ell_\gamma < L$. If γ is elliptic or hyperbolic as defined using a, then it is respectively elliptic or hyperbolic as defined using any $\tau \in [0, 1]$ version of a_τ and the rotation number is τ–independent.

(3) Let Θ_- and Θ_+ denote generators of $\mathcal{C}^L_{\mathrm{ech}}$. For each $\tau \in [0, 1]$, there exists

   (a) A 1–1 correspondence between the components of the respective J and J_τ versions of the space $\mathcal{M}_1(\Theta_-, \Theta_+)$ such that partnered components contribute the same sign to the respective J and J_τ versions of $\sigma(\Theta_-, \Theta_+)$.

(2-8)   (b) A 1–1 correspondence between the components of the respective J and J_τ versions of the space $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ such that partnered components contribute the same sign to the respective J and J_τ versions of $\sigma_p(\Theta_-, \Theta_+)$.

(4) Let γ denote a Reeb orbit with $\ell_\gamma < L$. There is a coordinate embedding $\varphi\colon S^1 \times D \to M$ of the sort described in the preceding with the following property: If γ is hyperbolic with rotation number k, then the $\hat{a}$–version of the pair (ν, μ) is equal to $(\frac{1}{2}k, i\epsilon e^{-ikt})$ for some $\epsilon \in (0, \delta)$. If γ is elliptic with rotation number R, then

   (i) $(2\pi/\ell_\gamma)\varphi^*\hat{a} = (1 - \mathrm{R}|z|^2)dt + (i/2)(zd\bar{z} - \bar{z}dz)$.

   (ii) The φ^*–pull back of the $\hat{J}$–version of $T^{1,0}(\mathbb{R} \times M)$ is spanned by the forms $ds + i\hat{a}$ and $(\ell_\gamma/(2\pi))(dz - i\mathrm{R}z\,dt)$.

(5) The contact structure $\hat{a}$ comes from the residual set given in [10, Lemma 2.1] and the almost complex structure $\hat{J}$ comes from the set $\mathcal{J}_{\hat{a},p}$.

The proposition that follows gives the analog of [10, Proposition 2.4] for (δ, L, p)–approximations. It refers to the contact 1–forms from the residual set given in [10, Lemma 2.1]. By way of a reminder, if a is a contact form from this set, then the linear return map for any given Reeb orbit lacks eigenvalues that are fractional roots of unity. □

Proposition 2.4 *Let a denote a contact 1–form the residual set given in [10, Lemma 2.1]. Fix a point $p \in M$ and let J denote a complex structure from $\mathcal{J}_{a,p}$. Fix $\delta > 0$ and $L \geq 1$ such that there is no generator Θ of $\mathcal{C}_{\mathrm{ech}}$ with $\sum_{(\gamma,m)\in\Theta} m\ell_\gamma = L$. Suppose that $(\hat{a}, \hat{J})$ is a (δ, L, p) approximation to the given pair (a, J). Then the identification provided by the first item in (2-8) between the Reeb orbits with symplectic action less than L induces a degree preserving isomorphism between the a and $\hat{a}$ versions of $\mathcal{C}^L_{\mathrm{ech}}$ that intertwines the respective differentials and the U–map. Thus, it induces an isomorphism between the respective (a, J) and $(\hat{a}, \hat{J})$ versions of the cohomology of the chain complex $\mathbb{Z}(\mathcal{Z}^L_{\mathrm{ech}})$ that intertwines the respective U–maps.*

Proof of Proposition 2.4 The assertions are a direct consequence of (2-8). □

The next proposition asserts the existence of (δ, L, p) approximations. It is the analog of [10, Proposition 2.5].

Proposition 2.5 *Let a denote a 1–form from the residual set described by [10, Lemma 2.1]. Fix $p \in M$ and let $J \in \mathcal{J}_{a,p}$. Fix $\delta > 0$ and $L \geq 1$ such that there is no element $\Theta \in \mathcal{Z}_{\mathrm{ech}}$ with $\sum_{(\gamma,m)\in\Theta} m\ell_\gamma = L$. Then there exists a (δ, L, p) approximation to (a, J).*

The proof of this proposition is given in the next section.

The existence of a (δ, L, p) approximation makes it possible to use the constructions in [11; 12; 13] to directly compare the respective counts that define the embedded contact homology and Seiberg–Witten Floer cohomology versions of the U–map.

Part 3 To set the stage for what is to follow, fix $L \geq 1$ and $p \in M$. Suppose that $\hat{a}$ is a contact 1–form on M from the residual set described in [10, Lemma 2.1] and that $\hat{J} \in \mathcal{J}_{\hat{a},p}$ with $\hat{a}$ are such that the conditions in [10, (4-1)] are obeyed. The upcoming Theorem 2.6 is a generalization of [10, Theorem 4.3].

Before stating the theorem, reintroduce as notation $\mathcal{M}^r$ to denote the set of gauge equivalence classes of irreducible solutions to (2-1). Given a pair, $\mathfrak{c}_-$ and $\mathfrak{c}_+$, of irreducible solutions to (2-1), introduce $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ to denote the space of instanton solutions to (2-2) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $u\mathfrak{c}_+$ where u can be any smooth map from M to S^1. Associate to each $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ the operator, $\mathfrak{D}_\mathfrak{d}$, that is depicted in (2-3). This operator gives a Fredholm map from

$L^2_1(\mathbb{R}\times M; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ to the space $L^2(\mathbb{R}\times M; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ if both the $\mathfrak{c}_-$ or $\mathfrak{c}_+$ version of the operator that is depicted in [11, (3-8)] has a trivial kernel. In this case, $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+) = \bigcup_{k\in\mathbb{Z}} \mathcal{M}_k(\mathfrak{c}_-, \mathfrak{c}_+)$ where any given $k \in \mathbb{Z}$ version of $\mathcal{M}_k(\mathfrak{c}_-, \mathfrak{c}_+)$ consists of the instantons whose corresponding version of $\mathfrak{D}_{(\cdot)}$ has index k.

Assume now that neither the $\mathfrak{c}_-$ nor the $\mathfrak{c}_+$ version of [11, (3-8)] has trivial kernel. Given that the latter condition holds, a given instanton $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ is said to be a *smooth point* of $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ when the cokernel of $\mathfrak{D}_{\mathfrak{d}}$ is trivial. This terminology is meant to reflect the following fact: Suppose that $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ is a smooth point. Then $\mathfrak{d}$ has a neighborhood in $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ with the structure of a smooth manifold whose dimension is that of the kernel of $\mathfrak{D}_{\mathfrak{d}}$. In particular, there is a ball about the origin in this kernel and a smooth diffeomorphism of this ball onto such a neighborhood of $\mathfrak{d}$ in $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$.

Theorem 2.6 *Fix $L \geq 1$, a point $p \in M$, and a pair $(\hat{a}, \hat{J})$ just described. There exists $\kappa \geq 1$ with the following significance: Define $\mathcal{M}^r$ using $r \geq \kappa$ and a 1–form $\mu \in \Omega$ with $\mathcal{P}$ norm bounded by 1. Then the conclusions of [10, Theorems 4.2–4.5] hold, and the following in addition:*

- *Let Θ_- and Θ_+ denote any two elements in $\mathcal{Z}^L_{\mathrm{ech}}$. Use $\mathfrak{c}_-$ and $\mathfrak{c}_+$ to denote solutions to* (2-1) *whose gauge equivalences classes are the respective images of Θ_- and Θ_+ via the map Φ^r from [10, Theorem 4.2].*
  - (i) *If $\mathfrak{d} = (A, \psi = (\alpha, \beta)) \in \mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$, then $\alpha|_{(0,p)} \neq 0$.*
  - (ii) *The set $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ is finite.*
  - (iii) *The space $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ is smooth near $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$.*
  - (iv) *The assignment to any given $\mathfrak{d} = (A, \psi = (\alpha, \beta)) \in \mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ of $\alpha|_{(0,p)}$ defines a map to a complex line that is smooth on the smooth part of $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ and that vanishes transversely at each point in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$.*
- *There is a bijection, $\Psi^{r,p}$, from $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ to $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ with the following property: The contribution of any given element in $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ to the integer $\sigma_p(\Theta_-, \Theta_+)$ that appears in* (1-3) *is the same as the contribution of its partner in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ to the integer $\sigma_p(\mathfrak{c}_-, \mathfrak{c}_+)$ that appears in* (1-8).

Given Proposition 2.5 and Theorem 2.6, the assertions in Theorem 1.1 about the U–map are proved using the same strategy used in [10, Section 4.c] to prove [10, Theorem 1]. Indeed, the arguments from this point differ only cosmetically from those given in [10, Section 4.c]. Note in this regard that the isomorphism that intertwines the U–maps for the respective $\mathbb{Z}[H^1(M;\mathbb{Z})]$–module refinements of embedded contact homology and Seiberg–Witten Floer cohomology is that used to prove Proposition 2.1.

3 Proof of Proposition 2.5

As is explained in what follows, the same sort of homotopy used in [10, Appendix A] to prove the latter's Proposition 2.4 serve for Proposition 2.5. The explanation for why this is true has five parts.

Part 1 Fix $Q \gg 1$ so as to construct inductively the sequence $\{(a_k, J_k)\}_{k=0,1,2,\ldots,Q}$ as done in [10, Appendix A]. Here, $(a_0, J_0) = (a, J)$; and for each $k \in \{0, \ldots, Q\}$, all but the fourth item of (2-8) are satisfied if $\hat{a} = a_k$ and $\hat{J} = J_k$. The fourth item of (2-8) is replaced by [10, (A-1)]. The construction of (a_k, J_k) from (a_{k-1}, J_{k-1}) proceeds inductively just as in [10, Appendix A]. Given what is done in [10, Appendix A], the only new point is that of guaranteeing Condition (3)(b) in (2-8) for (a_k, J_k) given that it is satisfied for (a_{k-1}, J_{k-1}).

The guarantee for Condition (3)(b) in (2-8) requires a generalization of what is said in [10, (A-4)]. To state the latter, let Θ_- and Θ_+ denote elements in $\mathcal{Z}^L_{\mathrm{ech}}$ and let Σ denote an element in either $\mathcal{M}_1(\Theta_-, \Theta_+)$ or $\mathcal{M}_2(\Theta_-, \Theta_+)^p$. As noted in [10, (2-10)] and in Lemma 2.3 above, there is a unique pair in Σ whose submanifold component is not an $\mathbb{R}$–invariant cylinder. Let $C \subset \mathbb{R} \times M$ denote this pseudoholomorphic submanifold. Let $\mathcal{D}_C$ denote C's version of the operator in (1-5). This operator has trivial kernel as a map from $L^2_1(C; N)$ to $L^2(C; N \otimes T^{0,1}C)$ where $N \to C$ denotes C's normal bundle. and thus it has a well defined inverse, this denoted by $\mathcal{D}_C^{-1}$ The latter is a bounded map to the L^2 orthogonal complement of the kernel of $\mathcal{D}_C$. There is a bound to the norm of $\mathcal{D}_C^{-1}$ that is independent of C, Θ_- and Θ_+. As in the Appendix to [10], this bound is denoted by σ_0.

Let L_0 denote the smallest of the lengths of the Reeb orbits for the original Reeb vector field v. In the case when γ is a Reeb orbit from $\mathcal{R}_L$, and q is a positive integer less than $L/L_0 + 1$, there is a version of the operator $\mathcal{D}_C$ for $C = \mathbb{R} \times \gamma$ whose domain consists of the space L^2_1 sections of the normal bundle of C that are $2\pi q$–periodic around the constant $s \in \mathbb{R}$ slices of $\mathbb{R} \times \gamma$. The latter version of $\mathcal{D}_C$ is also invertible, and σ_0 can be chosen so as to be greater than the norm of its inverse. The existence of this bound σ_0 is a consequence of the fact that there are a finite set of Reeb orbits and integers under consideration.

Part 2 If $k < Q$ and (a_k, J_k) has been constructed so that all but Condition (4) in (2-8) is obeyed, then any given element Σ from the (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ contains just one pair whose submanifold component is not $\mathbb{R}$–invariant. This submanifold, C, is such that $\mathcal{D}_C\colon L^2_1(C; N) \to L^2(C; N \otimes T^{0,1}C)$ is invertible. Here, as before, $\mathcal{D}_C^{-1}$ is a map from $L^2(C; N \otimes T^{0,1}C)$ to the L^2 orthogonal complement in $L^2_1(C;N)$ of kernel$(\mathcal{D}_C)$. There is a single upper bound for the norms of all such versions of $\mathcal{D}_C^{-1}$ because there are at most a finite set of subvarieties under consideration.

The plan for what follows is to exhibit a k–independent constant Q_* such that if $Q > Q_*$, then (a_{k+1}, J_{k+1}) can be constructed so that

(3-1)

- $(\hat{a} = a_{k+1}, \hat{J} = J_{k+1})$ obeys all but the fourth item of (2-8).
- (a_{k+1}, J_{k+1}) obeys instead the index $k+1$ version of [10, (A-1)].
- If Θ_- and Θ_+ are in $\mathcal{Z}^L_{\mathrm{ech}}$ and C is a subvariety $\mathcal{M}_2(\Theta_-, \Theta_+)^p$, then the operator $\mathcal{D}_C$ is invertible as a from the orthogonal complement in $L^2_1(C;N)$ of kernel$(\mathcal{D}_C)$ to $L^2(C; N \otimes T^{0,1}C)$.
- The evaluation map e_p: kernel$(\mathcal{D}_C) \to N|_p$ is an isomorphism.

The pair (a_{k+1}, J_{k+1}) is constructed as in the Appendix of [10]. By way of a reminder, this requires the choice of a positive constant, ρ. There is an upper bound for the choice of ρ that is determined by k, but there is no positive lower bound for the choice. The construction starts with the pair, $(a_{k+1,\rho}, J_{k+1,\rho})$ which are described in Parts 1 and 2 of the Appendix A to [10]. The desired pair (a_{k+1}, J_{k+1}) will be a small perturbation of $(a_{k+1,\rho}, J_{k+1,\rho})$. To verify (3-1), it is necessary to first consider the $(a_{k+1,\rho}, J_{k+1,\rho})$ versions of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$.

Part 3 Take $\rho > 0$ and very small. Fix any given Θ_- and Θ_+ from $\mathcal{Z}_{\mathrm{ech}}$. This done, each Σ in the (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ is assigned a partner in the $(a_{k+1,\rho}, J_{k+1,\rho})$ version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$. This is done as follows. Write the partner as Σ'. If $(C, m) \in \Sigma$ and C is $\mathbb{R}$–invariant, then $(C, m) \in \Sigma'$. If C is not $\mathbb{R}$–invariant, then the pair $(C, 1)$ is replaced by $(C', 1)$ where C' is obtained from C by solving [10, (A-16)] for a suitable, small normed section of C's normal bundle. What follows describes how this is done.

To start, introduce the Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ from Step 2 in Part 3 of Appendix A to [10]. Note in particular that elements in $\mathcal{B}_1$ are in $L^2_1(C;N) \cap C^0(C;N)$. Introduce the linear operators $\mathfrak{p}_1$ and $\mathfrak{p}_2 \cdot \nabla_C$ that appear in [10, (A-16)]. As noted in Lemma A.2 from the Appendix to [10], there exists Q_* which is independent of k, C, Θ_- and Θ_+ with the following significance: Let $\mathcal{B}_1^\perp$ denote the L^2–orthogonal complement to the kernel of $\mathcal{D}_C$. If $Q > Q_*$, then $\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1$ is invertible as a map from $\mathcal{B}_1^\perp$ to $\mathcal{B}_0$. Note that the norm of this inverse can depend on C.

Choose $\sigma > 0$ so that elements in $\mathcal{B}_1$ with norm $\|\cdot\|_* < 2\sigma$ define sections of the disk bundle $N_1 \subset N$ that is described in Step 1 of Part 3 of [10, Appendix A]. Note in particular that this bundle is the domain of an exponential map, e_C. This is a map from N_1 to $\mathbb{R}\times M$ that restricts to C as the identity; it is locally a diffeomorphism that embeds each normal disk. Let η denote a section of N_1 over Σ which has $|s|\to\infty$ limit equal to zero. Then $C' = e_C\circ\eta$ is a deformation of C in $\mathbb{R}\times M$. In particular, this deformation is $J_{k+1,\rho}$–pseudoholomorphic if and only if it obeys Equation (A-16) in [10].

Define a map, $\mathfrak{y}\colon \mathcal{B}_1^{\perp}\to\mathcal{B}_1$ so as to send any given $\eta\in\mathcal{B}_1^{\perp}$ to $\mathfrak{y}(\eta)=\eta-e_p^{-1}(\eta|_p)$. Said differently, the image of $\mathfrak{y}$ is the graph of the map $\eta\to -e_p^{-1}(\eta|_p)$ from $\mathcal{B}_1^{\perp}$ to kernel$(\mathcal{D}_C)$. By construction, $\mathfrak{y}(\eta)|_p=0$. For $\sigma'\in(0,\sigma)$, use $\mathcal{U}_{\sigma'}\subset\mathcal{B}_1^{\perp}$ to denote the ball of radius σ'. Define $\mathcal{T}_p\colon\mathcal{U}_{\sigma'}\to\mathcal{B}_1^{\perp}$ by sending any given element η to

$$\begin{aligned}\mathcal{T}_p(\eta) = -(\mathcal{D}_C+\mathfrak{p}_2\cdot\nabla_C+\mathfrak{p}_1)^{-1}\big((\mathfrak{p}_2\cdot\nabla_C+\mathfrak{p}_1)e_p^{-1}(\eta|_p)\\ +\Re_1(\mathfrak{y}(\eta))\cdot\nabla_C\mathfrak{y}(\eta)+\mathcal{R}_0\mathfrak{y}(\eta)+\mathfrak{p}_0\big).\end{aligned}\tag{3-2}$$

Here is the reason for this definition: If $\mathcal{T}_p(\eta)=\eta$, then the composition $e_C\circ\mathfrak{y}(\eta)$ is a pseudoholomorphic deformation of $\mathbb{R}\times M$ that contains the point p.

The following lemma is the analog in the present context of [10, Lemma A.3]:

Lemma 3.1 *There exists $\sigma'\in(0,\sigma)$ such that if ρ is sufficiently small then the following is true: Suppose that Σ is an element in the J_k version of $\mathcal{M}_1(\Theta_-,\Theta_+)$ or $\mathcal{M}_2(\Theta_-,\Theta_+)^p$ and that C is the submanifold from Σ that is not $\mathbb{R}$–invariant. Then $\mathcal{T}_p$ defines a contraction mapping from $\mathcal{U}_{\sigma'}$, to itself. For such ρ, the map $\mathcal{T}_p$ has a unique fixed point in $\mathcal{U}_{\sigma'}$. Moreover, this fixed point has $\|\cdot\|_*$ norm bounded by $c_C\rho$, and it is a smooth section of N that obeys [10, (A-16)]. Here, c_C is independent of ρ.*

Proof of Lemma 3.1 Note first that

$$\|\mathcal{T}_p(\eta)\|_*\le c_C\big(\rho\|\eta\|_*+\|\eta\|_*^2+\rho^2\big).\tag{3-3}$$

Here, c_C depends on C. Without the term $(\mathfrak{p}_2\cdot\nabla_C+\mathfrak{p}_1)e_p^{-1}(\eta|_p))$, the bound in (3-3) lacks the term $\rho\|\eta\|_*$ on the right and follows using [10, (A-13) and (A-17)]. The additional term $c_C\rho\|\eta\|_*$ is the bound for $-(\mathcal{D}_C+\mathfrak{p}_2\cdot\nabla_C+\mathfrak{p}_1)^{-1}((\mathfrak{p}_2\cdot\nabla_C+\mathfrak{p}_1)e_p^{-1}(\eta|_p))$. It is obtained by using the fact that the norm of any element in the kernel of $\mathcal{D}_C$ with unit L^2 norm is bounded along C by a C–dependent constant times $e^{-|s|/c_0}$. As a parenthetical remark, note that the constant c_0 that appears in this exponential can be taken to be independent of k and C. The reason is as follows: This exponential decay is determined by the smallest of the absolute values of those versions of the operator from (1-5) that are defined using the Reeb orbits from $\Theta_-\cup\Theta_+$ with their corresponding pairs $(\nu_{\tau=k/Q}, \mu_{\tau=k/Q})$. Similar considerations find that

$$\|\mathcal{T}_p(\eta) - \mathcal{T}_p(\eta')\|_* \le c_{C2}\big(\rho + \|\eta\|_* + \|\eta'\|_*\big)\|\eta - \eta'\|_*. \tag{3-4}$$

The lemma follows from (3-3) and (3-4) using the contraction mapping theorem. □

The result asserted by Lemma 3.1 can be formalized to say that there is an injective map from the (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ to the $(a_{k+1,\rho}, J_{k+1,\rho})$ version when ρ is small. This map, $\mathcal{F}_2$, is defined in the following manner: Let Σ denote any given element in the (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ and let Σ' denote its partner. The pairs in Σ and Σ' with $\mathbb{R}$–invariant submanifold components agree. Let $(C, 1) \in \Sigma$ denote the one additional element. Use C to define the map $\mathcal{T}_p$ and let η denote the fixed point given by Lemma 3.1. Take $C' \subset \mathbb{R} \times M$ to be the image of C via the composition $e_C \circ \mathfrak{y}(\eta)$.

Part 4 The next lemma states some relevant properties of the elements that lie in $\mathcal{F}_2$'s image when ρ is small. These properties are the analogs of those asserted by [10, Lemmas A.4].

Lemma 3.2 *Suppose that ρ is very small. Let Σ denote an element in the (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ and let Σ' denote its partner in the $(a_{k+1,\rho}, J_{k+1,\rho})$ version as defined by the map $\mathcal{F}_2$. Let Σ' denote the submanifold from Σ' that is not $\mathbb{R}$–invariant and let N' denote the normal bundle to C'.*

- *The associated operator $\mathcal{D}_{C'}$, has trivial cokernel.*
- *The evaluation map e_p: $\mathrm{kernel}(\mathcal{D}_{C'}) \to N'|_p$ is an isomorphism.*
- *The signs that Σ and Σ' would contribute to the respective (a_k, J_k) and $(a_{k+1,\rho}, J_{k+1,\rho})$ versions of the U–map are identical.*

Proof of Lemma 3.2 The proof that $\mathcal{D}_{C'}$, has trivial cokernel differs in no essential way from the proof of [10, Lemma A.4]. Recall that this is proved as follows: First, $\mathcal{D}_{C'}$ is written as an operator from $L^2_1(C; N)$ to $L^2(C; N \otimes T^{0,1}C)$, this an operator that appears as $\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r}$. The operator $\mathfrak{r}$ is seen to define a bounded operator from $L^2_1(C; N)$ to $L^2(C; N \otimes T^{0,1}C)$ with norm bounded by $c_0\|\eta\|_*$, and thus by $c_C\rho$. Meanwhile, [10, Lemma A.2] proves that the small ρ versions of $\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1$ are invertible. □

To prove the second assertion, use what was just said to define an isomorphism between the kernel($\mathcal{D}_C$) and kernel($\mathcal{D}_{C'}$) as follows: The map sends any given element λ to

$\lambda + \mathfrak{k}(\lambda)$ where $\mathfrak{k}(\lambda) = -(\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r})^{-1}((\mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r})\lambda)$. As is argued momentarily, this map $\mathfrak{k}(\cdot)$ has the following property:

(3-5) There exist a decreasing function $\rho \to \epsilon(\rho)$ with $\epsilon(0) = 0$ and such that if $\lambda \in \text{kernel}(\mathcal{D}_C)$ has unit L^2 norm, then $|\mathfrak{k}(\lambda)| \le c_C \epsilon(\rho)$ at each point in C.

This implies the assertion made by the second bullet since the identification of $\mathcal{D}_{C'}$ with $\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r}$ is compatible with the identifications between the normal bundles of C and C' at p given by the exponential map e_C.

To obtain (3-5), note first that $\mathfrak{r}$ has operator norm as a map from $\mathcal{B}_1$ to $\mathcal{B}_0$ that is bounded by $c_C \|\eta\|_* \le c_C \rho$. Here, c_C is a constant that is independent of ρ. Indeed, this follows using [8, Theorem 3.5.2 and Lemma 5.4.1] because the first order terms in $\mathfrak{r}$ have coefficients bounded by $c_C|\eta|$ and the coefficients of the zeroth order terms are bounded by $c_C|\nabla\eta|$. Meanwhile, the operator $\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1$ is invertible; its inverse maps from the space $\mathcal{B}_0$ to $\mathcal{B}_1^{\perp}$ with norm independent of ρ when ρ is small. This implies that $(\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r})$ is invertible when ρ is small; and that its inverse maps $\mathcal{B}_0$ to $\mathcal{B}_1^{\perp}$ with norm independent of ρ when ρ is small. Note that these observations prove that $\mathfrak{k}(\lambda)$ is in $\mathcal{B}_1^{\perp}$. What with [8, Theorem 3.5.2], bound on the size of its $\|\cdot\|_*$ norm gives a bound on its pointwise norm.

To bound $\|\mathfrak{k}(\lambda)\|_*$, fix $R \gg 1$ and take $\rho > \rho_R$ where ρ_R is such that the support of $\mathfrak{p}_1$ is described in [10, (A-14)]. Write $(\mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + r)\lambda$ as $(\mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1' + \mathfrak{r})\lambda + \mathfrak{p}_+\lambda$ where $\mathfrak{p}_1'$ is the part of $\mathfrak{p}_1$ whose support is in the disks with radius R^{-1} about the points where C intersects the Reeb orbits from $\mathcal{R}_L$. As noted previously, elements in the kernel of $\mathcal{D}_C$ with unit L^2 norm have pointwise norm bounded by $c_C e^{-|s|/c_0}$. This being the case, and given that $\mathfrak{p}_+$ has support where $|s| > R$, it follows that $\|(\mathcal{D}_C + \mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1 + \mathfrak{r})^{-1}\mathfrak{p}_+\lambda\|_* \le \epsilon(p)$ where $\epsilon(\cdot)$ can be assumed to be a continuous function with $\lim_{\rho\to 0}\epsilon(\rho) = 0$. In particular. the contribution to $\mathfrak{k}(\lambda)|_p$ from $\mathfrak{p}_+\lambda$ has norm bounded by $c_C\epsilon(p)$. Meanwhile, the contribution to $\mathfrak{k}(\lambda)$ from $(\mathfrak{p}_2 \cdot \nabla_C + \mathfrak{p}_1' + \mathfrak{r})\lambda$ has $\mathcal{B}_0$ norm bounded by $c_C R^{-1}$ as can be seen using what is said above about $\mathfrak{r}$, and what is said in the Appendix of [10] about $\mathfrak{p}_2$ and $\mathfrak{p}_1'$.

Given what just said about the norm of $\mathfrak{k}(\lambda)$ at p, the argument for the final bullet of Lemma 3.2 differs only cosmetically from those used to prove the final assertion of [10, Lemma A.4]; as such, this last argument is left to the reader.

Part 5 Take ρ very small, and in particular, so that the conclusions of Lemma 3.2 are valid. Perturb $(a_{k+1,\rho}, J_{k+1,\rho})$ in the manner that is described in Part 5 of Appendix A to [10] with the following added constraint: It should be made so that the new version of the almost complex structure also agrees with the original on all pseudoholomorphic curves that appear in elements from $\mathcal{F}_2$'s image. Moreover, the 2–jets of the new and original versions should also agree on the latter. Let $(a_{k+1,\rho}, J_{k+1,\rho})$ now denote the result of such a perturbation.

By construction, the perturbation to the new version of $(a_{k+1,\rho}, J_{k+1,\rho})$ does not affect the pseudoholomorphic subvarieties that are obtained using the map $\mathcal{F}_2$. Thus, $\mathcal{F}_2$ still gives an injective map from the (a_k, J_k) version $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ to the $(a_{k+1,\rho}, J_{k+1,\rho})$ version.

Lemma 3.3 *There exists $\rho_k > 0$ such that if $\rho < \rho_k$, then the map $\mathcal{F}_2$ just described is a bijection from (a_k, J_k) version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ to the $(a_{k+1,\rho}, J_{k+1,\rho})$ version of $\mathcal{M}_2(\Theta_-, \Theta_+)^p$.*

Proof of Lemma 3.3 The argument used in Part 6 of Appendix A of [10] can be repeated almost verbatim to prove Lemma 3.3. □

Lemma 3.3 has the following happy consequence: Any sufficiently small ρ version of $(a_{k+1,\rho}, J_{k+1,\rho})$ obeys now all but item (4) of (2-5) with the latter replaced by [10, (A-1)]. Thus, any sufficiently small ρ version of this pair can serve for (a_{k+1}, J_{k+1}).

4 Theorem 2.6

This section contains the proof of Theorem 2.6. Note in this regard that the proof uses much the same technology that is used in [11; 12; 13] to prove [10, Theorem 4.3]. This being the case, much of what follows refers to sections in [11; 12; 13] and their notation.

4.a L^∞–Norms

This section constitutes a digression to derive some inequalities that are used in the subsequent parts of the proof. To set the stage for what is to follow, let $\Sigma \subset \mathbb{R} \times M$ denote a pseudoholomorphic submanifold that obeys the five constraints in [11, Section 4.b]. [11, Propositions 6.4, 7.1 and 7.6] associate to Σ certain instanton solutions to any sufficiently large r version of (2-2). To say more, introduce the ball $\mathcal{B}$ from [11, Proposition 6.4]. Let V_0 denote the vector space that appears in [11, Proposition 7.1] and introduce the map $q\colon \mathcal{B} \to V_0$ as in [11, Proposition 7.1]. Fix $\lambda \in q(\mathcal{B})$ and let $\xi^\lambda \in \mathcal{B} \cap q^{-1}(\lambda)$ denote the element given by this same [11, Proposition 7.1]. Let $\mathfrak{h} = \mathfrak{h}(\xi^\lambda)$ and $\mathfrak{q} = \mathfrak{q}(\xi^\lambda)$ be as given in [11, Proposition 6.4]. Lemma 4.1 to come provides, among other things, an L^∞ bound on both $\mathfrak{q}$ and $\mathfrak{h}$.

As λ varies in $q(B)$, the assignments $\lambda \to \mathfrak{h}(\xi^\lambda) - \mathfrak{h}(\xi^0)$ and $\lambda \to \mathfrak{q}(\xi^\lambda)$ give smooth maps from $q(\mathcal{B})$ to the Hilbert space $\mathbb{H}$ from [11, Section 6.a]. Lemma 4.1 also gives an L^∞ bound on the derivatives of these maps. Note in this regard that the lemma uses $\mathfrak{h}'$ and $\mathfrak{q}'$ to denote the respective the directional derivative of $\mathfrak{h}$ and $\mathfrak{q}$ in the direction of any given unit normed vector in V_0.

Lemma 4.1 *There is a constant $\kappa \geq 1$ with the following significance: If $r \geq \kappa$, then [11, Propositions 6.4, 7.1 and 7.6] can be invoked to construct $\mathfrak{h}(\xi^\lambda)$ and $\mathfrak{q}(\xi^\lambda)$ from any given $\lambda \in V_0$ with norm $|\lambda| \leq \kappa^{-1}$. Then the following is true:*

- *Both $|\mathfrak{h}(\xi^\lambda)|$ and $|\mathfrak{q}(\xi^\lambda)|$ are bounded pointwise by $\kappa r^{-1/\kappa}$.*
- *The norms of their directional derivatives, $|\mathfrak{h}'|$ and $|\mathfrak{q}'|$, are also bounded by $\kappa r^{-1/\kappa}$.*

Proof of Lemma 4.1 The proof has six steps.

Step 1 It follows from what is said in [11, Lemma 3.10] that the $\xi = 0$ version of $|\mathfrak{h}(\xi)|$ is bounded by r^{-1/c_0}. Meanwhile, it follows from the description of $\mathfrak{h}(\xi)$ in [11, Section 5.b] and the bound $|\xi^\lambda| \leq c_0(r^{-1/c_0} + |\lambda|)$ from Proposition 7.1 that $|\mathfrak{h}(\xi^\lambda)|$ is also bounded by r^{-1/c_0}.

The desired bound on the $|\mathfrak{h}'|$ follows directly from the description of $\mathfrak{h}$ in [11, Section 5.b] given a suitable bound on the $\mathcal{K}$–norm of the derivative of the map $\lambda \to \xi^\lambda$. To say more about this, let $\xi' \in \mathcal{K}$ denote the directional derivative of this map in the direction of a given unit length vector in V_0. It follows from what is said in [11, Section 5.b] that $|\mathfrak{h}'| \leq r^{-1/c_0}\|\xi'\|_{\mathcal{K}}$.

Step 2 To bound $\|\xi'\|_{\mathcal{K}}$, first recall that [11, Proposition 7.2] and [11, Section 7.a] define ξ^λ to be the fixed point in $\mathcal{B}$ of the map $\chi_\lambda \colon \mathcal{B} \to \mathcal{K}$ that sends any given element ξ to $-(\mathcal{F})^{-1}(\mathcal{T}_0 + \mathcal{T}_2(\xi), -\lambda)$ where $\mathcal{F}$ is the map $(\mathcal{T}_1, q)$ from $\mathcal{K}$ to $\mathcal{L} \times V_0$. This said, its directional derivative, ξ', in the direction of a given unit vector $\lambda' \in V_0$ obeys

$$\xi' = -(\mathcal{F})^{-1}(\mathcal{T}_{2*}|_{\xi_\lambda} \cdot \xi', -\lambda'), \tag{4-1}$$

where $\mathcal{T}_{2*}$ denotes the differential of the map $\mathcal{T}_2$. Introduce the map $q^* \colon V_0 \to \mathcal{K}$ as defined at the end of [11, Section 7.f]. It is a consequence of what is said in the second and third bullets of [11, Proposition 7.2] that

$$\big\|\xi' - q^*(\lambda')\big\|_{\mathcal{K}} \leq c_0(r^{-1/2+8\sigma} + |\lambda|). \tag{4-2}$$

This implies that $\|\xi'\|_{\mathcal{K}} \leq c_0$.

Step 3 A bound for $|\mathfrak{q}|$ is derived in this step and Steps 4–6. Introduce the pair (A^*, ψ^*) from [11, Section 5.a]. The bundle E can be trivialized over any ball in $\mathbb{R}\times M$ of radius $100r^{-1/2}$ so that the following is true: Let θ denote the product connection given by this trivialization and write A^* in this ball as $\theta+(2r)^{1/2}a$. Then $|a|\le c_0$. Moreover, given an integer $k\ge 1$, there exists an r–independent constant c_k such that the derivatives to order k of a and ψ in this ball are bounded by $c_k r^{k/2}$.

These last observation have the following consequence: Let $\mathfrak{D}_*$ denote the operator on $C^\infty(\mathbb{R}\times M; iT^*M\oplus\mathbb{S}\oplus i\mathbb{R})$ that is given by using (A^*,ψ^*) in [11, (6-4)]. Suppose that $U\subset\mathbb{R}\times M$ is a ball of radius c_0^{-1}, and that $\mathfrak{f}$ and $\mathfrak{w}$ are L^2_1 sections over U of $iT^*M\oplus\mathbb{S}\oplus i\mathbb{R}$ that obey $\mathfrak{D}_*\mathfrak{f}=\mathfrak{w}$. Let $U'\subset U$ denote the concentric ball with half the radius of U and let $U''\subset U$ denote the concentric ball with one fourth the radius of U. If $r\le c_0^{-1}$, then

$$\sup_{x\in U''}|\mathfrak{f}|\le c_0\sup_{x\in U'}\int_{\mathrm{dist}(x,\cdot)\le 1/\sqrt{r}}\frac{1}{\mathrm{dist}(x,\,\cdot)^3}|\mathfrak{w}|+c_0r^2\sup_{x\in U'}\int_{\mathrm{dist}(x,\cdot)\le 1/\sqrt{r}}|\mathfrak{f}|. \tag{4-3}$$

This is proved using standard parametrix arguments.

Step 4 Set $\xi=\xi^\lambda$ and reintroduce the pair (A^ξ,ψ^ξ) from [11, Section 5.b]. Write $A^\xi=A^*+(2r)^{1/2}(b_{*\xi}+\phi_{*\xi}ds)$ and $\psi^\xi=\psi^*+\eta_{*\xi}$ so as to define $b_{*\xi}=(b_{*\xi},\eta_{*\xi},\phi_{*\xi})$, this a section over $\mathbb{R}\times M$ of $iT^*M\oplus\mathbb{S}\oplus i\mathbb{R}$. It follows from what is said in [11, Proposition 7.1 and Lemma 3.10] that $|\mathfrak{b}_{*\xi}|\le c_0(r^{-1/4}+|\lambda|)$. This has the following consequence: The (A^ξ,ψ^ξ) version of [11, (6-4)] differs from $\mathfrak{D}_*$ by a zeroth order term whose L^∞ norm is bounded by $c_0r^{1/2}|\mathfrak{b}_{*\zeta}|\le c_0r^{1/2}(r^{-1/4}+|\lambda|)$. Thus, the top line of [11, (6-1)] can be written as an equation for $\mathfrak{b}=\mathfrak{b}(\xi^\lambda)$ that has the schematic form $\mathfrak{D}_*\mathfrak{b}=\mathfrak{w}$ where

$$|\mathfrak{w}|\le c_0r^{1/2}(r^{-1/c_0}+|\lambda|)|\mathfrak{b}|+c_0r^{1/2}|\mathfrak{b}|^2+c_0r^{-1/2}|\mathfrak{v}|. \tag{4-4}$$

This understood, the plan for what follows is to use (4-3) with $\mathfrak{f}=\mathfrak{b}$ and with $\mathfrak{w}$ just described to bound $\|\mathfrak{b}\|_\infty$ by r^{-1/c_0}. Given that $|\mathfrak{h}|\le r^{-1/c_0}$, this will supply the promised bound for $|\mathfrak{q}|$.

Step 5 To start this plan, observe that the contribution to the left most integral on the right hand side of (4-3) from $r^{1/2}(r^{-1/c_0}+|\lambda|)|\mathfrak{b}|$ is at most $c_0(r^{-1/c_0}+|\lambda|)\|\mathfrak{b}\|_\infty$. Note in particular that this is less than $\frac{1}{100}\|\mathfrak{b}\|_\infty$ if $|\lambda|\le c_0^{-1}$ and $r\ge c_0$.

The contribution to the left most integral on the right hand side of (4-3) from $r^{1/2}|\mathfrak{b}|^2$ is no greater than

$$c_0r^{1/2}\|\mathfrak{b}\|_\infty\sup_{x\in\mathbb{R}\times M}\int_{\mathrm{dist}(x,\cdot)<1/\sqrt{r}}\frac{1}{\mathrm{dist}(x,\,\cdot)^3}|\mathfrak{b}|. \tag{4-5}$$

Introduce the norm $\|\cdot\|_*$ as defined in [11, (6-51)]. The integral in this last expression is bounded by $c_0 r^{-\upsilon/4}\|\mathfrak{b}\|_*$ as can be seen by breaking it into a sum indexed by the positive integers where the k–th term is the contribution to the integral from the region where the distance to x lies in the interval $(2^{-k}r^{-1/2}, 2^{-k+1}r^{-1/2})$. Given this bound, and given the fifth bullet of [11, Proposition 7.1], it follows that what is written in (4-5) is bounded also by $c_0(r^{-1/c_0} + |\lambda|)\|\mathfrak{b}\|_\infty$. As noted in the preceding paragraph, this is less than $\frac{1}{100}\|\mathfrak{b}\|_\infty$ if $|\lambda|$ is less than c_0^{-1} and r is greater than c_0.

Consider next the contribution to the left most integral on the right hand side of (4-3) from $r^{-1/2}|\mathfrak{v}|$. To this end, reintroduce the norm $\|\cdot\|_{\mathcal{K}*}$ as defined in [11, (2-27)], and note that this norm has the following property: Let ζ and C be as in [11, (2-27)]. Let $\rho < 1$ be given. Then

$$\sup_{u\in C}\int_{\mathrm{dist}(\cdot,u)<\rho}\frac{1}{\mathrm{dist}(u,\,\cdot)}|\nabla\zeta| \le c_0\rho^{\upsilon}\|\zeta\|_{\mathcal{K}*}. \tag{4-6}$$

This can be seen by writing the integrand as a sum indexed by the positive integers where the k–th term gives the contribution to the integral from the region where the distance to u lies in the interval $(2^{-k}\rho, 2^{-k+1}\rho)$.

What with (4-6), it follows from the description of $\mathfrak{v}$ in [11, Section 6.d], especially [11, (6-32), (6-38)–(6-40) and (6-49)], that the contribution to the left most integral on the right hand side of (4-3) from $r^{-1/2}|\mathfrak{v}|$ is at most $c_0 r^{-1/c_0}$.

Step 6 The observations made in Step 5 imply that the left most integral on the right hand side of (4-3) is no greater than

$$\frac{1}{50}\|\mathfrak{b}\|_\infty + r^{-1/c_0} \tag{4-7}$$

if $r \ge c_0$ and $|\lambda| \le c_0^{-1}$. Meanwhile, the $\mathfrak{f} = \mathfrak{b}$ version of the right most integral on the right hand side of (4-3) is no greater than $r^{-\upsilon/2}r^{1/2}\|\mathfrak{b}\|_*$. In particular, this is bounded curtesy of [11, Propositions 7.1 and 6.4] by $r^{-\upsilon/2}(r^{-1/4} + |\lambda|)$. This being the case, the $\mathfrak{f} = \mathfrak{b}$ version of (4-3) implies that $\|\mathfrak{b}\|_\infty \le r^{-1/c_0}$ when $r \ge c_0$ and $|\lambda| \le c_0^{-1}$.

Step 7 This step explains very briefly how to derive the desired bound for $|\mathfrak{q}'|$. The strategy is very much the same as that used to bound $|\mathfrak{q}|$. Here is a sketch of what is done: Differentiate the top equation in [11, (6-1)] so as to derive an equation of $\mathfrak{b}' = \mathfrak{h}' + \mathfrak{q}'$ that has the schematic form $\mathfrak{D}_*\mathfrak{b}' = \mathfrak{w}'$. Equation (4-3) is used with $\mathfrak{f} = \mathfrak{b}'$ and with $\mathfrak{w}'$ in lieu of $\mathfrak{w}$ to bound $\|\mathfrak{b}'\|_\infty$. The latter with the bound from Steps 1 and 2 on $\|\mathfrak{h}'\|_\infty$ gives the promised bound for $\|\mathfrak{q}'\|_\infty$.

The details of the derivation differ only cosmetically from what was done in Steps 3–6 to obtain the bound for $\|\mathfrak{q}\|_\infty$, especially so given (4-6), given Step 2's bound on $\|\xi'\|_{\mathcal{K}}$, and given the bound from [11, Proposition 6.4] on $\|\mathfrak{q}'\|_*$. $\square$

4.b The map $\Psi^{r,p}$

The purpose of this subsection is to define the map $\Psi^{r,p}$. To prepare for the definition, fix $\vartheta \in \mathcal{M}_2(\Theta_-, \Theta_+)^p$. This element ϑ obeys Constraints 1–4 of [11, Section 4.b]. Fix a constant translation of ϑ along the $\mathbb{R}$ factor of $\mathbb{R} \times M$ so that the resulting submanifold, Σ, obeys all five of the constraints in [11, Section 4.b]. Let $s_* \in \mathbb{R}$ denote the translation amount. This is to say that any given point $(s, x) \in \mathbb{R} \times M$ lies in Σ if and only if $(s - s_*, x)$ lies in ϑ. In particular, the point $q = (s_*, p)$ is in Σ. The translation by s_* is needed to satisfy the fifth constraint in [11, Section 4.b].

If $r \geq c_0$, then the map Φ^r given in [11, Theorem 4.2] is defined on both Θ_- and Θ_+. This is assumed in what follows. In addition, [11, Propositions 6.4, 7.1 and 7.6] associate to Σ a family of instanton solutions to (2-2) that are parametrized by a small radius ball in the vector space V_0 from [11, Proposition 7.1].

Proposition 4.2 *There is a constant $\kappa \geq 1$ with the following significance: Fix $r \geq \kappa$. Let $B \subset V_0$ denote the ball of radius κ^{-1}. [11, Propositions 6.4, 7.1 and 7.6] can be invoked using any given element in B to obtain an instanton solution to* (2-2). *This defines a map, $\Psi_B\colon B \to \mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$. Given $\lambda \in B$, write the spinor component of the instanton $\Psi_B(\lambda)$ as $\psi_{(\lambda)} = (\alpha_{(\lambda)}, \beta_{(\lambda)})$.*

- *The image of Ψ_B consists of smooth points, and Ψ_B is an embedding.*
- *There is a unique element, $\lambda_p \in B$ with $\alpha_{(\lambda_p)} = 0$ at q. This element λ_p is such that $|\lambda_p| \leq r^{-1/\kappa}$.*
- *The assignment to $\lambda \in B$ of $\alpha_{(\lambda)}|_q$ defines a map from B to a complex line that vanishes transversally at λ_p.*

This proposition is proved momentarily.

The map $\Psi^{r,p}$ is defined as follows: Fix $r \geq c_0$ so that the map Φ^r from [10, Theorem 4.2] is defined, and so that Proposition 4.2 can be invoked for any given element in $\mathcal{M}_2(\Theta_-, \Theta_+)$. Let $\vartheta \in \mathcal{M}_2(\Theta_-, \Theta_+)^p$ and let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to (2-1) that define the respective gauge equivalence class of the image of Θ_- and Θ_+ via the map Φ^r of [10, Theorem 4.2]. Introduce V_0, B, the map $\Psi_B\colon B \to \mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$, and the element $\lambda_p \in B$ as in Proposition 4.2. Define $\Psi^{r,p}(\vartheta)$ to be the translate of the instanton $\Psi_B(\lambda_p)$ by $-s_*$ along the $\mathbb{R}$ factor of $\mathbb{R} \times M$.

4.c Proof of Proposition 4.2

This proof has six parts.

Part 1 [12, Proposition 3.1] guarantees that the image of Ψ_B consists of smooth points in the subset $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ of $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ when B has radius c_0^{-1} or less. The rest of this part with Parts 2 and 3 prove that Ψ_B has injective differential when B has radius c_0^{-1} or less.

To start, remark that a tangent space at a given smooth point $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ can be viewed as the kernel of the operator that is defined by $\mathfrak{d}$ via (2-3). By way of reminder, the kernel here and in what follows refers to the version whose domain and range are the respective spaces L^2_1 and L^2 sections over $\mathbb{R} \times M$ of the bundle $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$. When $\mathfrak{d}$ has the form $\Psi_B(\lambda)$, this operator is gauge equivalent to an operator on the Hilbert space $\mathbb{H}$ from [11, Part 1 in Section 6.a]. To describe this gauge equivalent operator, introduce $\xi = \xi^\lambda \in \mathcal{B} \cap q^{-1}(\lambda)$ to denote the element given by [11, Proposition 7.1] from λ. Define $\mathfrak{b}(\xi^\lambda)$ from ξ^λ as described by [11, Proposition 6.4] and then use $\mathfrak{b}(\xi^\lambda)$ in [11, (5-19)] to define the pair $(A, \psi) = (A_{(\lambda)}, \psi_{(\lambda)})$. Define $\mathfrak{D}_{(\lambda)}$ by [11, (6-4)] using this version of (A, ψ). This $\mathfrak{D}_{(\lambda)}$ is gauge equivalent to $\mathfrak{d}$'s version of [12, (1-12)]. Thus, the tangent space to $\mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ at $\Psi_B(\lambda)$ is isomorphic to kernel$(\mathfrak{D}_{(\lambda)}) \subset \mathbb{H}$ and the differential of Ψ_B at λ can be viewed as a linear map from V_0 to kernel$(\mathfrak{D}_{(\lambda)})$. This map to kernel$(\mathfrak{D}_{(\lambda)})$ is written in what follows as the composition $\wp\cdot\mathcal{Y}$ where $\mathcal{Y}$ is a linear map from V_0 into $\mathbb{H}$ and $\wp$ is the L^2–orthogonal projection to the complement to a certain linear subspace.

To define the map $\mathcal{Y}$, note first that the map $\lambda \to \mathfrak{b}(\xi^\lambda) - \mathfrak{b}(0)$ defines a smooth map from V_0 to $\mathbb{H}$. Let λ' denote a unit normed element in V_0 and introduce $\mathfrak{b}'$ to denote the directional derivative of $\mathfrak{b}(\xi^{(\cdot)}) - \mathfrak{b}(0)$ at λ in the direction λ'. This $\mathfrak{b}'$ is one of two contributions to $\mathcal{Y}\lambda'$. The second contribution comes via the chain rule from the derivative at $\xi = \xi^\lambda$ of the assignment $\xi \to (A^\xi, \psi^\xi)$ with the latter as defined in [11, Section 5.b]. Denote this second contribution by $\mathfrak{t}'$. Thus, $\mathcal{Y}\lambda' = \mathfrak{b}' + \mathfrak{t}'$.

To say more about $\wp\cdot\mathcal{Y}$, note that the $iT^*M \oplus \mathbb{S}$ components of $\mathfrak{D}_{(\lambda)}(\mathcal{Y}\lambda')$ are zero. This can be seen by differentiating the equation in the top bullet of [11, (6-6)]. The effect of $\wp$ is to change $\mathcal{Y}\lambda'$ by adding an element in $\mathbb{H}$ that has the form

$$\left(-d\chi, 2r^{1/2}\chi\psi_{(\lambda)}, -\frac{\partial}{\partial s}\chi\right) \tag{4-8}$$

where χ is a certain $i\mathbb{R}$–valued, L^2_2 function on $\mathbb{R} \times M$. As a parenthetical remark, note that (4-8) can be written as $(\mathfrak{D}_{(\lambda)})^\dagger \mathfrak{u}_\chi$ where $\mathfrak{u}_\chi = (0, 0, \chi)$ and where $(\mathfrak{D}_{(\lambda)})^\dagger$ denotes the formal L^2 adjoint of $\mathfrak{D}_{(\lambda)}$. No matter the choice of χ, the addition of

(4-8) to $\mathcal{Y}\lambda'$ gives a section, $\mathfrak{f}_\chi$, of $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$ over $\mathbb{R} \times M$ with the property that $\mathfrak{D}\mathfrak{f}_\chi$ has zero iT^*M and $\mathbb{S}$ components. This element has zero $i\mathbb{R}$ component if and only if χ obeys the equation

$$-\frac{\partial^2}{\partial s^2}\chi + d^\dagger d\chi + r|\psi_{(\lambda)}|^2\chi + \mathfrak{m} = 0 \tag{4-9}$$

where $\mathfrak{m}$ here denotes the $i\mathbb{R}$ component of $\mathfrak{D}_{(\lambda)}(\mathcal{Y}\lambda')$.

Part 2 Introduce $\mathcal{C}$ to denote the set of components of Σ. To find the desired solution to (4-9), note that it is a consequence of Lemma 4.1 that $|\psi_{(\lambda)}| \geq \frac{1}{2}$ at points with distance $c_0 r^{-1/2}$ or greater from $\bigcup_{C\in\mathcal{C}} C$ if the radius of B is less than c_0^{-1} Granted that such is the case, then the bilinear form

$$z \to \mathfrak{h}(\chi) = \int_{\mathbb{R}\times M} \left(\left|\frac{\partial}{\partial s}x\right|^2 + |dx|^2 + r\left|\psi_{(\lambda)}\right|^2|x|^2 \right) \tag{4-10}$$

on $L^2_1(\mathbb{R}\times M; i\mathbb{R})$ obeys $\mathfrak{h}(\chi) \geq c_0 r\|\chi\|_2^2$. This last fact, and the fact that $\mathfrak{m}$ is an L^2 section of $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$ imply via a standard variational argument that (4-9) has a unique L^2_1 solution. Let χ now denote this solution. This $i\mathbb{R}$–valued function is smooth, it obeys

$$\int_{\mathbb{R}\times M} \left(\left|\frac{\partial}{\partial s}x\right|^2 + |dx|^2 + r|x|^2 \right) \leq c_0 r^{-1}\|\mathfrak{m}\|_2^2, \tag{4-11}$$

and what is written in (4-8) defines an element in $\mathbb{H}$.

Part 3 The differential of Ψ_B on λ' is nonzero if the addition of (4-8) to $\mathcal{Y}\lambda'$ is nonzero. To see that such is the case, use [11, (4-2), Proposition 6.4] and the description of $\mathfrak{h}(\cdot)$ given in [11, Section 6.d] to see that the directional derivative of $\mathfrak{b}(\xi^{(\cdot)})$ at λ in the direction of λ' obeys

$$\|\mathfrak{b}'\|_{\mathbb{H}} \leq c_0 r^{-1/2}. \tag{4-12}$$

To say something about $\mathfrak{t}'$, reintroduce from [11, (6-9)] the $\xi = \xi^\lambda$ version of the homomorphism $\mathfrak{t}_\xi$. It then follows from what is said in [11, Section 5.b] that $\mathfrak{t}'$ is an element in $\mathbb{H}$ that differs from $t_{\xi^\lambda}(\lambda')$ by a term whose L^2 norm is bounded by $c_0 r^{-1/c_0}$ and whose $\mathbb{H}$–norm is bounded by $r^{1/2-1/c_0}$. Note in this regard that the L^2 norm of $t_{\xi^\lambda}(\lambda')$ is bounded from below by c_0^{-1} and from above by c_0. This is implied by [12, (3-3)]. Meanwhile, its $\mathbb{H}$–norm is bounded from below $c_0^{-1}r^{1/2}$ and bounded from above by $c_0 r^{1/2}$.

What just said about $\mathfrak{b}'$ and $\mathfrak{t}'$ implies that

(4-13)
- $\|\mathcal{Y}\lambda'\|_2 \geq c_0^{-1}$.
- $\|\mathfrak{m}\|_2 \leq c_0(\|\mathfrak{D}_{(\lambda)}(t_{\xi^\lambda}(\lambda'))\|_2 + r^{1/2-1/c_0})$.

To more about the L^2 norm of $\mathfrak{m}$, introduce the operator $\mathfrak{D}$ that is defined by [11, (6-4)] using for (A, ψ) the $\xi = \xi^\lambda$ version of (A^ξ, ψ^ξ). By way of reminder, this pair is defined in [11, Section 5.b]. It is a consequence of [11, Proposition 6.4] and the bound by $c_0 r^{1/2}$ on $\|t_{\xi^\lambda}(\lambda')\|_{\mathbb{H}}$ that

$$\|\mathfrak{D}_{(\lambda)}(t_{\xi^\lambda}(\lambda'))\|_2 \leq \|\mathfrak{D}(t_{\xi^\lambda}(\lambda'))\|_2 + c_0 r^{1/2}(r^{-1/c_0} + |\lambda|). \tag{4-14}$$

Now use the definition of t_ξ in [11, (6-9)] with the description of $\mathfrak{D}$ in [11, Section 6.a] and what is said about V_0 in [11, Section 7.f] to see that $\|\mathfrak{D}(t_{\xi^\lambda}(\lambda'))\|_2 \leq c_0 r^{1/2}(r^{-1/c_0} + |\lambda|)$. This bound also follows from what is said in the proof of [12, Proposition 3.1]. These last bounds with the second bullet in (4-14) imply that

$$\|\mathfrak{m}\|_2 \leq c_0 r^{1/2}(r^{-1/c_0} + |\lambda|). \tag{4-15}$$

What with (4-15), the bound in (4-11) and the lower bound given by the first bullet in (4-13) imply that $\|\wp \cdot \mathcal{Y}\lambda'\|_2 > c_0^{-1}$ if the radius of B is less than c_0^{-1} Given what was said in Part 1, such a nonzero lower bound implies, in particular, that Ψ_B has injective differential.

Part 4 This part sets the stage for proof that Ψ_B is $1-1$ when B's radius is bounded by c_0^{-1}. To start, let $C \subset \mathbb{R} \times M$ denote the one element from $\mathcal{C}$ that is not $\mathbb{R}$–invariant. Let $\pi \colon N \to C$ denote C's normal bundle. [11, Part 2 of Section 4.a] describes an exponential map, e_C, that maps a fixed radius subbundle $N_1 \subset N$ into $\mathbb{R} \times M$ as an immersion. This map sends the zero section to C, its differential along the zero section is an isometry, and it embeds each fiber disk in N_1 as a pseudoholomorphic disk in $\mathbb{R} \times M$.

The tangent space to $\mathcal{M}_2(\Theta_-, \Theta_+)$ at Σ is isomorphic to the kernel of C's version of the operator that appears in [11, (4-5)]. This operator maps L^2_1 sections of N to L^2 sections of $N \otimes T^{0,1}C$. Its kernel is 2–dimensional; and this kernel is, by definition, the vector space V_0. Thus each element in V_0 is a section of N. By assumption, the differential of e_C along the zero section maps V_0 isomorphically onto $N|_p$.

As constructed, C passes through the point $q = (s_*, p) \in \mathbb{R} \times M$. Fix a unitary identification of $N|_q$ with $\mathbb{C}$. Fix $T \gg 1$ and let $D \subset \mathbb{C} = N|_q$ denote the radius $Tr^{-1/2}$ disk centered on the origin. The exponential map e_C is used implicitly in what follows to identify D with its image via e_C in $\mathbb{R} \times M$. When $\lambda \in B$, write

$\psi_{(\lambda)} = (\alpha_{(\lambda)}, \beta_{(\lambda)})$. What with the $\mathbb{C}$–linear identification between $N|_q$ and $E|_q$ from [11, Section 5.a], the component $\alpha_{(\lambda)}$ on D can be viewed as a section over D of π^*N, thus a map from D to $\mathbb{C}$.

Introduce from [11, Part 1 of Section 1.b] the vortex moduli space $\mathfrak{C}_1$. This space has a unique symmetric vortex. This vortex has a unique lift as a pair (A^C, α^C) of connection on, and section of the product bundle $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that α^C at any given point $z \in \mathbb{C}$ has the form $|\alpha^C| z/|z|$. What with [11, (5-19) and (5-2)], the map $\alpha_{(\lambda)}$ on D can be written in terms of the complex coordinate z on D as

$$\alpha_{(\lambda)}(z) = \alpha^C(r^{1/2}z + \lambda|_q) + \eta_{(\lambda)}(z), \tag{4-16}$$

where $\eta_{(\lambda)}$ is the $E \subset E \oplus EK^{-1}$ component of $\mathfrak{b}(\lambda)$.

Part 5 The assertion that Ψ_B is 1–1 follows as a direct corollary to:

Lemma 4.3 *There exists $\kappa \geq 1$ with the following significance: If $r \geq \kappa$ and if the radius of B is less than κ^{-1}, then each $\lambda \in B$ version of* (4-16) *has a unique zero, z_λ in the subdisk in D of radius $\kappa^{-1}r^{-1/2}$. Moreover, the resulting map $\lambda \to z_\lambda$ maps B diffeomorphically onto its image.*

Proof of Lemma 4.3 Take $r \geq c_0$ and the radius of B less than c_0^{-1} where c_0 here is such that the constructions that lead to (4-16) can be made. Note that $\alpha^C|_D$ vanishes transversely at the origin in D. In particular, its differential at $0 \in D$ is a $\mathbb{C}$–linear isomorphism from $\mathbb{C}$ to itself, this multiplication by a positive, real number. Denote the latter as Δ. Write α^C as $\Delta z + u(z)$ with $|u| \leq c_0|z|^2$ and $|\nabla_{A^C} u| \leq c_0|z|$. This partial Taylor's expansion of α^C can be used to write a zero of $\alpha_{(\lambda)}$ on D as the fixed point of a map, $\mathcal{P}_\lambda$ from $\mathbb{C}$ to $\mathbb{C}$ that sends any given $z \in D$ to

$$\mathcal{P}_\lambda(z) = -r^{-1/2}\lambda|_q - r^{-1/2}\Delta^{-1}\big(\eta_{(\lambda)}(z) + u(r^{1/2}z + \lambda|_q)\big). \tag{4-17}$$

As argued next, the map $\mathcal{P}_\lambda$ has a unique fixed point in the disk of radius $c_0 r^{-1/2}$ about the origin in D. To see why this is, first use the top bullet of Lemma 4.1 to prove

$$|\mathcal{P}_\lambda(z)| \leq r^{-1/2}|\lambda|_q| + c_0 r^{-1/2}\big(r^{-1/c_0} + r|z|^2 + |\lambda|_q|^2\big). \tag{4-18}$$

This bound has the following consequence: There exits c_0 such that if $|\lambda| \leq c_0^{-1}$, then $\mathcal{P}_\lambda$ maps the disk of radius $c_0^{-1}r^{-1/2}$ in $\mathbb{C}$ to itself. The map $\mathcal{P}_\lambda$ is a contraction on such a disk. if $|\nabla_{A_{(\lambda)}}\eta_{(\lambda)}| \leq c_0^{-1}r^{1/2}$ when $r \geq c_0$ and $|\lambda| \leq c_0^{-1}$. That such a bound exists is a direct consequence of [13, Lemmas 3.6 and 4.3]. Indeed, the latter have the following corollary:

(4-19) There exists $c_0 \geq 1$, and given $\epsilon > 0$, there exists an r–independent constant c_ϵ which are such that $|\nabla_{A_{(\lambda)}} \eta_{(\lambda)}| \leq \epsilon r^{1/2}$ when $r \geq c_\epsilon$ and $|\lambda| \leq c_0^{-1}$.

Granted that $\mathcal{P}_\lambda$ is a contraction on the radius $c_0 r^{-1/2}$ disk centered at the origin, the contraction mapping theorem asserts that it has a unique fixed point on this disk. Let z_λ denote this fixed point. This fixed point has norm $|z_\lambda| \leq c_0 r^{-1/2} |\lambda|$.

Now suppose that λ and λ' are two elements in V_0 with norms less than c_0^{-1} with c_0 such that the z_λ and $z_{\lambda'}$ can be defined as above. It then follows from the λ and λ' versions of (4-17) that

$$r^{1/2}(z_\lambda - z_{\lambda'}) = (\lambda - \lambda')|_q + \mathfrak{r}, \tag{4-20}$$

where $|\mathfrak{r}|$ is bounded by c_0 times the sum of the following terms:

(4-21)
- $|\eta_{(\lambda)}(z_{\lambda'}) - \eta_{(\lambda')}(z_{\lambda'})|$.
- $|\nabla_{A_{(\lambda)}} \eta_\lambda||z_\lambda - z_{\lambda'}|$.
- $(r^{1/2}|z_\lambda| + r^{1/2}|z_{\lambda'}| + |\lambda|_q| + |\lambda'|_q|)(|z_\lambda - z_{\lambda'}| + |(\lambda - \lambda')|_q|)$.

It is a consequence of what is said in the second bullet of Lemma 4.1 that the first item in (4-21) is bounded by $c_0 r^{-1/c_0} |(\lambda - \lambda')|_q|$. What is said in (4-19) implies that the following: Given $\epsilon > 0$, there exists $c_\epsilon \geq 1$ such that if $r \geq c_\epsilon$, then the second item in (4-21) is bounded by $\epsilon r^{1/2} |z_\lambda - z_{\lambda'}|$. As for the third item in (4-21), use the equality between z_λ and the right hand side of (4-17) to prove the following: Given $\epsilon > 0$, there exists an r–independent constant $c_\epsilon \geq 1$ with the following significance: The third item in (4-21) is bounded by $\epsilon(r^{1/2}|z_\lambda - z_{\lambda'}| + |(\lambda - \lambda')|_q|)$ if $|\lambda|$ and $|\lambda'|$ are both less than c_ϵ^{-1}.

These last points guarantee that the map $\lambda \to z_\lambda$ separates points in B if $r \geq c_0$ and if B's diameter is less than c_0^{-1}.

These last points also guarantee that the map $z_{(\cdot)}$ is Lipschitz on B. An argument to show that $z_{(\cdot)}$ is smooth uses the implicit function theorem as applied to the map from $D \times B$ to $\mathbb{C}$ that sends any given pair $(z, \lambda) \to z - \mathcal{P}_\lambda(z)$. □

Part 6 This part establishes the claims that are made by the second and third bullets in Proposition 4.2. Consider first the existence of λ_q. To this end, again identify $N|_p$ with $\mathbb{C}$. Let $\mathrm{T}\colon \mathbb{C} = N_p \to V_0$ denote the inverse of the restriction isomorphism. Let $D_0 \subset \mathbb{C}$ denote a disk of radius c_0^{-1} with c_0 such that $\mathrm{T}(D_0) \subset B$. Write the map $z \to \alpha^C(z)$ as $\Delta z + u(z)$, this as in the proof of Lemma 4.3. Let $\zeta \in D_0$. Then $\lambda = \mathrm{T}(\zeta)$

is such that $\alpha_{(\mathrm{T}(\zeta))}(0) = 0$ if and only if ζ is a fixed point of the map $\mathcal{Z}\colon D_0 \to \mathbb{C}$ given by

$$\mathcal{Z}(\zeta) = -\Delta^{-1}\big(\eta_{\mathrm{T}(\zeta)}|_q + u(\zeta)\big). \tag{4-22}$$

As is argued next, this map has a unique fixed point in D_0. This fixed point is λ_p. To start the argument, use the first bullet in Lemma 4.1 to see that $|\mathcal{Z}(\zeta)| \le c_0(r^{-1/c_0} + |\zeta|^2)$. This implies that $\mathcal{Z}$ maps the disk of radius c_0^{-1} to itself if $r \ge c_0$. To see if $\mathcal{Z}$ is a contraction, use the second bullet of Lemma 4.1 to see that

$$|\mathcal{Z}(\zeta) - \mathcal{Z}(\zeta')| \le c_0(r^{-1/c_0} + |\zeta| + |\zeta'|)|\zeta - \zeta'|. \tag{4-23}$$

Thus, $\mathcal{Z}$ is a contraction mapping of the disk of radius c_0^{-1} to itself if $r \ge c_0^{-1}$.

Granted the preceding, the contraction mapping theorem asserts that $\mathcal{Z}$ has a unique fixed point in the radius c_0^{-1} disk if $r \ge c_0$. Let ζ_p denote this fixed point. It follows from (4-22) using the first bullet in Lemma 4.1 that this fixed point has norm no greater than r^{-1/c_0}. Set $\lambda_p = \mathrm{T}(\zeta_p)$. By construction, $\alpha_{(\lambda_p)}$ vanishes at q.

To finish the arguments for the second bullet, fix c_0 so any $\lambda \in B$ with $|\lambda| \le c_0^{-1}$ lies in $\mathrm{T}(D_0)$. Define $B_1 \subset B$ to be the subdisk about the origin with this radius. The fact that ζ_p is unique insures that λ_p is the only element in B' with the property that $\alpha_{(\lambda)}|_q = 0$.

To obtain the assertions of the third bullet of Proposition 4.2, write

$$\alpha_{(\mathrm{T}(\zeta))}|_q = \Delta\zeta + u(\zeta) + \eta_{(\mathrm{T}(\zeta))}|_q. \tag{4-24}$$

This identifies the map $\lambda \to \alpha_{(\lambda)}|_q$ on B_1 as a map from B_1 to $\mathbb{C}$, thus a section of the trivial line bundle. To see that this map vanishes transversely at λ_p, use the second item of Lemma 4.3 to deduce that the differential of what is depicted on the right hand side of (4-24) differs from Δ by at most $c_0(|\zeta| + r^{-1/c_0})$. In particular, this differential is nowhere zero on D_0 if the latter's radius is less than c_0^{-1} and if $r \ge c_0$.

4.d The question of signs

The definition of the U–map for embedded contact homology uses a weight, either $+1$ or -1, that is associated to each element in $\mathcal{M}_2(\Theta_-, \Theta_+)^p$. Meanwhile, the definition of the U–map for the Seiberg–Witten Floer cohomology uses a $+1$ or -1 weight that is associated to each element of $\mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$. What follows in this subsection is a proof that the ± 1 embedded contact homology weight of any given element $\vartheta \subset \mathcal{M}_2(\Theta_-, \Theta_+)^p$ is identical to the Seiberg–Witten Floer cohomology weight of $\Psi^{r,p}(\vartheta) \in \mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$. This proof has four steps. The discussion introduces notation and conventions from [12, Section 3.b].

Step 1 Associated to any generator $(\Theta, \mathfrak{o})$ of $\mathcal{C}_{\mathrm{ech}}$ is a free $\mathbb{Z}/2\mathbb{Z}$ module $\Lambda(\Theta, \mathfrak{o})$. The set of these modules has the following salient feature: Let Θ_- and Θ_+ denote any pair of elements in $\mathcal{Z}_{\mathrm{ech}}$ and let $\Lambda(\Theta_-, \Theta_+)$ denote the orientation sheaf of $\mathcal{M}_1(\Theta_-, \Theta_+)$ and for the union of the components of $\mathcal{M}_2(\Theta_-, \Theta_+)$ that contain points in $\mathcal{M}_2(\Theta_-, \Theta_+)^p$. The sheaf $\Lambda(\Theta_-, \Theta_+)$ is canonically isomorphic to $\Lambda(\Theta_-, \mathfrak{o}_-) \otimes \Lambda(\Theta_+, \mathfrak{o}_+)^*$. As a consequence, a set of the form $\{\mathfrak{o}_{(\Theta,\mathfrak{o})} \in \Lambda(\Theta, \mathfrak{o}) :$ $(\Theta, \mathfrak{o})$ is a generator of $\mathcal{C}_{\mathrm{ech}}\}$ defines an orientation for

$$\begin{aligned}(4\text{-}25)\qquad &\Big(\bigcup\nolimits_{\Theta_-,\Theta_+\in\mathcal{Z}_{\mathrm{ech}}} \mathcal{M}_1(\Theta_-, \Theta_+)\Big) \\ &\qquad\qquad \cup\Big(\bigcup\nolimits_{\Theta_-,\Theta_+\in\mathcal{Z}_{\mathrm{ech}}} \Big(\bigcup\nolimits_{\Sigma\in\mathcal{M}_2(\Theta_-,\Theta_+)^p} T\mathcal{M}_2(\Theta_-, \Theta_+)|_\Sigma\Big).\end{aligned}$$

An orientation of the set in (4-25) of this sort is deemed a *coherent system of orientations*. By way of a reminder, this term signifies that the orientations behave consistently when curves are concatenated end to end. The precise definition of this term is given in [5, Section 9.5].

A coherent system of orientations must be used to define both the differential in embedded contact homology and the U–map for embedded contact homology. The constructions in [12, Section 3.b] describe a particular coherent system of orientations. (The discussion in [12, Section 3.b] proves coherence only for $\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}_{\mathrm{ech}}} \mathcal{M}_1(\Theta_-, \Theta_+)$, but the arguments apply with only notational changes to the whole of (4-25).)

Step 2 Associated to any irreducible generator $\mathfrak{c} \in \mathcal{M}^r$ is a free $\mathbb{Z}/2\mathbb{Z}$ module $\Lambda(\mathfrak{c})$. The set $\{\Lambda(\mathfrak{c})\}_{\mathfrak{c}\in\mathcal{M}^r}$ has the following property: Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote any pair of elements in $\mathcal{M}^r$; and let $\Lambda(\mathfrak{c}_-, \mathfrak{c}_+)$ denote the orientation sheaf for the smooth portion of $\mathfrak{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ and for the union of those components of $\mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ that contain one or more points of $\mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$. Then $\Lambda(\mathfrak{c}_-, \mathfrak{c}_+)$ is canonically isomorphic to $\Lambda(\mathfrak{c}_-) \otimes \Lambda(\mathfrak{c}_+)^*$. This understood, a set $\{o_\mathfrak{c} \in \Lambda(\mathfrak{c})\}_{\mathfrak{c}\in\mathcal{M}^r}$ defines an orientation for

$$(4\text{-}26)\quad \Big(\bigcup\nolimits_{\mathfrak{c}_-,\mathfrak{c}_+\in\mathcal{M}^r} \mathfrak{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)\Big) \cup \Big(\bigcup\nolimits_{\mathfrak{c}_-,\mathfrak{c}_+\in\mathcal{M}^r} \Big(\bigcup\nolimits_{\mathfrak{d}\in\mathfrak{M}_2(\mathfrak{c}_-,\mathfrak{c}_+)^p} T\mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)|_\mathfrak{d}\Big).$$

An orientation for this set of the sort just described is deemed a *coherent system of orientations*. A coherent system of orientations must be used to define both the differential for the Seiberg–Witten Floer cohomology and the U–map. A precise definition of this term is given in [7].

Step 3 Fix $L \gg 1$ and take r very large. [12, Section 3.b] explains how to obtain a coherent system of orientations for

$$\bigcup\nolimits_{\mathfrak{c}_-,\mathfrak{c}_+\in\Phi^r(\mathcal{Z}^L_{\mathrm{ech}})} \mathfrak{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$$

from the coherent system of orientations for

$$\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}^L_{\mathrm{ech}}} \mathcal{M}_1(\Theta_-,\Theta_+)$$

that defines o_{ech}. These orientations are intertwined by the map Ψ^r from [10, Theorem 4.2]. Moreover, the isomorphism between embedded contact homology and Seiberg–Witten Floer cohomology that is described in [10, Section 4], and in Section 2 here, assumes that these are the orientations that are used to define the respective differentials for embedded contact homology and Seiberg–Witten Floer cohomology.

The arguments used in [12, Section 3.b] can be repeated verbatim (save for some minor notation) to extend the preceding correspondence of orientations so as to obtain a coherent system of orientations for

$$\begin{aligned}(4\text{-}27)\qquad &\big(\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}^L_{\mathrm{ech}}} \mathfrak{M}_1(\Phi^r(\Theta_-),\Phi^r(\Theta_+))\big)\\ &\quad\cup\big(\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}^L_{\mathrm{ech}}}\big(\bigcup_{\Sigma\in\mathcal{M}_2(\Theta_-,\Theta_+)^p} T\mathfrak{M}_2(\Phi^r(\Theta_-),\Phi^r(\Theta_+))|_{\Psi^{r,p}(\Sigma)}\big)\big)\end{aligned}$$

from the set $\{o_{(\Theta,\mathfrak{o})}\}_{(\Theta,\mathfrak{o})\text{ is a generator of }\mathcal{C}_{\mathrm{ech}}}$. As noted above, this set defines a coherent system of orientations for

$$\begin{aligned}(4\text{-}28)\qquad &\big(\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}^L_{\mathrm{ech}}} \mathcal{M}_1(\Theta_-,\Theta_+)\big)\\ &\qquad\qquad\cup\big(\bigcup_{\Theta_-,\Theta_+\in\mathcal{Z}_{\mathrm{ech}}}\big(\bigcup_{\Sigma\in\mathcal{M}_2(\Theta_-,\Theta_+)^p} T\mathcal{M}_2(\Theta_-,\Theta_+)|_{\Sigma}\big)\big).\end{aligned}$$

But for the slight notational changes, the arguments from [12, Section 3.b] establish the following with regards to the induced orientations on the $T\mathfrak{M}_2$ part of (4-27): Let Θ_- and Θ_+ denote any two elements in $\mathcal{Z}_{\mathrm{ech}}$ and let Σ denote any given element in $\mathcal{M}_2(\Theta_-,\Theta_+)^p$. Take $r \ge c_0$ so as to define the map Ψ_B as in Proposition 4.2 and the element λ_p in Proposition 4.2's ball B. Let $C\in\mathcal{C}$ denote the element that is not $\mathbb{R}$ invariant, and use the canonical identification between $T\mathcal{M}_2(\Theta_-,\Theta_+)|_\Sigma$ and kernel$(\mathcal{D}_C)$ to identify the former vector space with Proposition 4.2's vector space V_0. Then the differential of Ψ_B at λ_p is orientation preserving.

Step 4 Fix $L>1$ and let $(\Theta_-,\mathfrak{o}_-)$ and $(\Theta_+,\mathfrak{o}_+)$ denote a pair of generators of $\mathcal{C}^L_{\mathrm{ech}}$. Fix $\vartheta\in\mathcal{M}_2(\Theta_-,\Theta_+)$ and let $\chi_{\mathrm{ech}}\in\{\pm1\}$ denote ϑ's contribution to the weight $\sigma_p(\Theta_-,\Theta_+)$ that appears in (1-3). This sign can be computed as follows: Translate ϑ along the $\mathbb{R}$–factor of $\mathbb{R}\times M$ to obtain the centered element Σ. Let $C\in\mathcal{C}$ denote the one element that is not $\mathbb{R}$–invariant, and identify $T\mathcal{M}_2(\Theta_-,\Theta_+)|_\Sigma$ with kernel$(\mathcal{D}_C)$ as done above. Let e_{C*} denote the restriction homomorphism from kernel$(\mathcal{D}_C)$ to $N|_q$. The sign χ_{ech} is equal to 1 if the restriction homomorphism e_{C*} is orientation preserving, and -1 if it reverses orientation.

Fix r very large and let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote respective solutions to (2-1) that generate $\Phi^r(\Theta_-)$ and $\Phi^r(\Theta_+)$. Let $\chi_{\mathrm{SW}}\in\{\pm1\}$ denote the contribution that $\Psi^{r,p}(\vartheta)$ would make to the weight $\sigma_p(\mathfrak{c}_-, \mathfrak{c}_+)$ that appears in (1-8). This sign can be computed as follows: Introduce $\mathfrak{d} \in \mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ to denote the corresponding translate of the instanton $\Psi^{r,p}(\vartheta)$. Thus, $\mathfrak{d} = \Psi_B(\lambda_p)$. A neighborhood of $\mathfrak{d}$ in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)$ is parameterized via the map Ψ_B by the ball $B \subset V_0$. As noted in the previous step, this map is orientation preserving. Write $\Psi_B(\lambda)$ as $(\alpha_{(\lambda)}, \beta_{(\lambda)})$. As argued in the previous steps, the assignment $\lambda \to \alpha_{(\lambda)}|_q$ defines a section of a smooth, complex line bundle near $\lambda_p \in B$ which vanishes transversally at λ_p. Thus, the differential of this section at λ_p is an isomorphism from V_0 to the bundle $E|_q$. The sign χ_{SW} is $+1$ if this isomorphism is orientation preserving and it is -1 otherwise.

To see that $\chi_{\mathrm{ech}} = \chi_{\mathrm{SW}}$, reintroduce the notation from Parts 4 and 5 of the previous subsection. Note in particular that (4-16) implicitly uses the complex linear identification between $N|_q$ with $E|_q$ that comes from the definitions in [11, Section 5.a]. With a $\mathbb{C}$–linear trivialization understood, the assignment $\lambda \to \alpha_{(\lambda)}|_q$ can be viewed as a map from B to $N|_q$. This map sends any given $\lambda \in B$ to

$$\Delta e_{C*}(\lambda) + u(e_{C*}(\lambda)) + \eta_\lambda|_q, \tag{4-29}$$

where Δ, u and η are as in (4-17) and (4-22). Given that $\Delta \in \mathbb{C} - \{0\}$, the conclusion that $\chi_{\mathrm{ech}} = \chi_{\mathrm{SW}}$ follows from (4-23) when r is large. Indeed, (4-23) implies that the differential of (4-29) at λ_p is orientation preserving if and only if e_{C*} is orientation preserving if $r \geq c_0$.

4.e The surjectivity of $\Psi^{r,p}$

Fix $L > 1$ and a pair Θ_- and Θ_+ of elements in $\mathcal{Z}^L_{\mathrm{ech}}$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to (2-1) that define the respective equivalence classes $\Phi^r(\Theta_-)$ and $\Phi^r(\Theta_+)$. The purpose of this subsection is to prove that $\Psi^{r,p}$ maps $\mathcal{M}_2(\Theta_-, \Theta_+)^p$ onto $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ if r is sufficiently large. The proof is practically identical save for notation to that given in Theorem 1.2 of [13]. What follows summarizes these arguments.

As with the proof of [13, Theorem 1.2], the argument begins by assuming to the contrary that there exists a sequence $\{(r_n, (A_n, \psi_n))\}_{n=1,2,\dots}$ such that

(4-30)
- The sequence $\{r_n\}_{n=1,2,\dots} \subset [1, \infty)$ is increasing and unbounded.
- Any given $\mathfrak{d}_n = (A_n, \psi_n)$ is a solution to the $r = r_n$ version of (2-2) in $\mathcal{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$.
- $\mathfrak{d}_n$ is not in the image of $\Psi^{r,p}$.

The derivation of nonsense from this assumption will prove that the large r versions of $\Psi^{r,p}$ are surjective.

The arguments for [13, Lemmas 6.1 and 6.2] can be repeated modulo some small notational changes to prove the following:

Lemma 4.4 *There is an element $\vartheta \subset \mathcal{M}_2(\Theta_-, \Theta_+)^p$ and a subsequence of $\{(A_n, \psi_n)\}_{n=1,2,\dots}$ (hence renumbered consecutively from 1) with the following property: For each n, write ψ_n as a pair (α_n, β_n). The sequence whose n–th element is*

$$r_n^{1/2}\Big(\sup_{z\in\bigcup_{(C,m)\in\vartheta} C} \operatorname{dist}(z, \alpha_n^{-1}(0)) + \sup_{z\in\alpha_n^{-1}(0)} \operatorname{dist}\big(\textstyle\bigcup_{(C,,m)\in\vartheta} C, z\big)\Big)$$

converges with limit zero. In addition, if $I \subset \mathbb{R}$ is an interval of length 1 and υ is a 2–form on $\mathbb{R}\times M$ with $\|\upsilon\|_\infty = 1$ and support on $I \times M$, then the sequence whose n–th element is $(i/(2\pi))\int_{\mathbb{R}\times M} \upsilon \wedge F_{\hat{A}} - \sum_{(C,m)\in\vartheta} m \int_C \upsilon$ also converges with limit zero.

Now translate ϑ by a constant amount, s_*, along the $\mathbb{R}$–factor of $S^1 \times M$ so that the result, Σ, is centered. Translate each large n version of (A_n, ψ_n) by the same amount. So as to avoid excessive notation, use (A_n, ψ_n) henceforth to denote the result of this translation.

The arguments given in [13, Sections 6.b–6.e] can be repeated verbatim now to conclude that all sufficiently large n versions of (A_n, ψ_n) are gauge equivalent on $\mathbb{R}\times M$ to an instanton that is obtained from the $r = r_n$ version of Proposition 4.2's map Ψ_B. In fact, given $\delta_* > 0$, these arguments find that any sufficiently large n version of (A_n, ψ_n) is gauge equivalent to the image, via the $r = r_n$ version of Ψ_B, of a point $\lambda^{(n)}$ with $|\lambda^{(n)}| < \delta_*$. This understood, it follows then from Proposition 4.2 that any sufficiently large n version of (A_n, ψ_n) is gauge equivalent to the translate of $\Psi^{r,p}(\vartheta)$.

This last conclusion directly contradicts what is assumed in (4-30) thus proving that all large r versions of $\Psi^{r,p}$ are surjective.

4.f Proof of Theorem 2.6

The claim made by item (i) of the first bullet follows directly from Lemma 4.1. This is by virtue of the following facts: First, the point $x \in M$ is disjoint from all of the Reeb orbits. Second, there are but a finite number of components of $\mathcal{M}_1(\Theta_-, \Theta_+)$. Finally, if $\Sigma \in \mathcal{M}_1(\Theta_-, \Theta_+)$ and $(C, m) \in \Sigma$, then C avoids the point $(0, x) \in \mathbb{R}\times M$. The assertion made by (ii) of the first bullet, that $\mathfrak{M}_2(\mathfrak{c}_-, \mathfrak{c}_+)^p$ is finite, follows from the fact that $\Psi^{r,p}$ is surjective. The assertion made by (iii) of the first bullet follows from the first bullet of Proposition 4.2. The assertion made by (iv) of the first bullet follows the second and third bullets of Proposition 4.2.

The assertion made by the second bullet follows from Proposition 4.2 with what is said in Section 4.d and Section 4.e.

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