The topology of toric symplectic manifolds

Dusa McDuff

This is a collection of results on the topology of toric symplectic manifolds. Using an idea of Borisov, we show that a closed symplectic manifold supports at most a finite number of toric structures. Further, the product of two projective spaces of complex dimension at least two (and with a standard product symplectic form) has a unique toric structure. We then discuss various constructions, using wedging to build a monotone toric symplectic manifold whose center is not the unique point displaceable by probes, and bundles and blow ups to form manifolds with more than one toric structure. The bundle construction uses the McDuff–Tolman concept of mass linear function. Using Timorin's description of the cohomology algebra via the volume function we develop a cohomological criterion for a function to be mass linear, and explain its relation to Shelukhin's higher codimension barycenters.

1 Introduction

The paper [15] by Masuda and Suh raises many questions about the topology of toric manifolds. One of the most interesting can be loosely stated as:

Question 1.1 To what extent does the cohomology ring $H^*(M)$ determine the toric manifold M or, failing that, the combinatorics of its moment polytope?

Such questions are known under the rubric of *cohomological rigidity*; cf Choi, Panov and Suh [3]. One can interpret them in various contexts, including that of complex manifolds or quasitoric (torus) manifolds. In this paper we work exclusively with closed symplectic manifolds, and refine the above question to ask about the symplectomorphism type of (M, ω). Thus our classification is finer than one that considers only the homeomorphism type of M or the combinatorics of the moment polytope, but coarser than one that considers M as a smooth complex variety with a given symplectic form.

Recall that a closed symplectic $2n$–dimensional manifold (M, ω) is said to be toric if it supports a Hamiltonian action of an n–torus T. This action is generated by a moment map $\Phi\colon M \to \mathfrak{t}^*$ where $\mathfrak{t}^*$ is the dual of the Lie algebra $\mathfrak{t}$ of the torus T.

There is a natural integral lattice $\mathfrak{t}_\mathbb{Z}$ in $\mathfrak{t}$ whose elements H exponentiate to circles Λ_H in T, and hence also a dual lattice $\mathfrak{t}_\mathbb{Z}^*$ in $\mathfrak{t}^*$. The image $\Phi(M)$ is well known to be a convex polytope Δ. It is *simple* (n facets meet at each vertex), *rational* (the conormal vectors $\eta_i \in \mathfrak{t}$ to each facet may be chosen to be primitive and integral), and *smooth* (at each vertex v of Δ the conormals to the n facets meeting at v form a basis for the lattice $\mathfrak{t}_\mathbb{Z}$). Throughout this paper we only consider such polytopes. We write them as

$$(1\text{-}1) \qquad \Delta := \Delta(\kappa) := \big\{ \xi \in \mathfrak{t}^* : \langle \eta_i, \xi \rangle \leq \kappa_i, \ i = 1, \dots, N \big\}.$$

Thus Δ has N facets $F_1, \dots, F_N$ with *outward primitive integral conormals* $\eta_i \in \mathfrak{t}_\mathbb{Z}$ and *support constants* $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{R}^N$. The *faces* of Δ are the intersections $F_I := \bigcap_{i \in I} F_i$, where $I \subset \{1, \dots, N\}$. Given a polytope Δ we usually denote the corresponding symplectic manifold by $(M_\Delta, \omega_\kappa)$. (See Karshon, Kessler and Pinsonnault [9] for more detailed references on this background material.)

We define $\mathcal{C}(\Delta)$ to be the chamber of $\Delta = \Delta(\kappa)$, ie the open connected set of all support constants κ' such that $\Delta(\kappa')$ is analogous to $\Delta(\kappa)$; cf McDuff and Tolman [22]. For $\kappa, \kappa' \in \mathcal{C}(\Delta)$, the symplectic forms ω_κ and $\omega_{\kappa'}$ may be joined by the path $\omega_{t\kappa+(1-t)\kappa'}$, $t \in [0, 1]$, and so are deformation equivalent.

Our first result concerns the question of how many different toric actions can be supported by the same symplectic manifold (M, ω). Here we identify two toric manifolds if there is an equivariant symplectomorphism between them; that is, if their moment polytopes may be identified by an integral affine transformation. Karshon, Kessler and Pinsonnault show in [9] that in dimension $2n = 4$ a given manifold (M, ω) can support at most a finite number of actions. The next theorem gives a cohomological version of this result that is valid in all dimensions. Its proof relies on an argument due to Borisov; the original proof applied only when $[\omega]$ is integral.

Theorem 1.2 (Borisov–McDuff) *Let R be a commutative ring of finite rank with even grading, and write $R_\mathbb{R} := R \otimes_\mathbb{Z} \mathbb{R}$. Suppose given elements $[\omega] \in R_\mathbb{R}$ and $c_1, c_2 \in R$ of degrees $2, 2$ and 4 respectively. Then, up to equivariant symplectomorphism, there are at most finitely many toric symplectic manifolds (M, ω, T) of dimension $2n$ for which there is a ring isomorphism $\Psi\colon H^*(M; \mathbb{Z}) \to R$ that takes the symplectic class and the Chern classes $c_i(M)$, $i = 1, 2$, to the given elements $[\omega] \in R_\mathbb{R}$, $c_i \in R$.*

We prove Theorem 1.2 in Section 3.1.

Remark 1.3 (i) Note that it is crucial to fix the symplectic class $[\omega]$ here. Otherwise, as is shown by the example of the Hirzebruch surfaces, the result is false even for

a ring as simple as $R = H^*(S^2 \times S^2; \mathbb{Z})$. In fact, if $k < \lambda \leq k + 1$ the manifold $S^2 \times S^2$ with product symplectic form $\lambda \operatorname{pr}_1^* \sigma \oplus \operatorname{pr}_2^*(\sigma)$ (where σ is an area form on S^2) supports exactly k different torus actions; cf [9, Example 2.6].

(ii)   One might consider analogous questions for nontoric symplectic manifolds. For example, one might fix the diffeomorphism type of a closed manifold M (rather than its cohomology) and fix a cohomology class $a \in H^2(M; \mathbb{R})$ and ask whether there are only finitely many different (ie nonsymplectomorphic) symplectic structures on M in this class a. The answer here is "no": McDuff [17] constructs an 8–dimensional manifold that supports infinitely many nondiffeomorphic but cohomologous symplectic forms. This paper also shows that the manifold $S^2 \times S^2 \times T^2$ supports infinitely many nonisotopic but cohomologous symplectic forms. In both cases, the class $[\omega]$ is integral and the forms are deformation equivalent, ie they can be joined by a family of (noncohomologous) symplectic forms. Thus they have the same Chern classes. All these examples have nontrivial fundamental group. Work of Ruan [28] and Fintushel and Stern [6] shows that in the simply connected case one can find infinitely many nondeformation equivalent symplectic forms on 6–manifolds of the form $M \times S^2$, for example when the smooth manifold M is homeomorphic to a $K3$ surface. Although these structures have the same Chern classes, it is not clear whether they can be chosen to be cohomologous.

(iii)   The extent to which one needs the hypotheses on the Chern classes is not clear; cf the discussion in Masuda and Suh [15, Section 5]. By Remark 3.2 they are unnecessary if one restricts to integral $[\omega]$.

(iv)   If one asks the same question in the context of T–equivariant cohomology, then Masuda shows in [14] that the equivariant cohomology $H_T^*(M; \mathbb{Z})$, when considered as an algebra over $H^*(BT; \mathbb{Z})$, determines the fan, ie the family of polytopes $\Delta(\kappa)$, $\kappa \in \mathcal{C}(\Delta)$, and hence determines the corresponding toric manifold as a complex variety. To fix the symplectic manifold, one would also have to specify κ, for example by specifying the extension of the symplectic class to $H_T^*(M; \mathbb{R})$.

(v)   Theorem 1.2 implies that the number of conjugacy classes of n–tori in the group $\operatorname{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of (M, ω) is finite, where here we allow conjugation by elements of the full group $\operatorname{Symp}(M, \omega)$ of symplectomorphisms of (M, ω). Since the orbits of any Hamiltonian action of a torus are isotropic, each such torus is maximal in $\operatorname{Ham}(M, \omega)$. However, there might be other maximal tori of smaller dimension. In [27], Pinsonnault shows that in dimension $2n = 4$ there are only finitely many symplectic conjugacy classes of such maximal tori. Again it is important to allow conjugation by elements of $\operatorname{Symp}(M, \omega)$; cf [27, Theorem 1.3].

Manifolds with many toric structures: Blow ups and bundles Now consider the question of which symplectic manifolds support more than one toric structure (up to equivariant symplectomorphism). One easy way to get examples is by blowing up points or other symplectic submanifolds of (M, ω). (In the combinatorial context the blow up procedure at a point is called vertex cutting; cf Choi, Panov and Suh [3, Example 1.1].)

We prove the following result in Section 3.2. Here the weight of a blow up is the symplectic area of the line in the exceptional divisor.

Proposition 1.4 *Suppose that Δ is not a product of simplices, and let $(M_\Delta, \omega_\kappa)$ be the corresponding symplectic manifold. Then, for generic choice of $\kappa \in \mathcal{C}(\Delta)$, there is $\varepsilon_0 > 0$ such that any one point toric blow up $(\widetilde{M}, \omega_{\kappa,\varepsilon})$ of $(M_\Delta, \omega_\kappa)$ with weight $\varepsilon < \varepsilon_0$ has at least two toric structures.*

Remark 1.5 (i) The above result is false when Δ is any product of two simplices other than $\Delta_1 \times \Delta_1$. This follows because Proposition 1.8 below implies that when $\Delta \neq \Delta_1 \times \Delta_1$ there is an open nonempty set of $\kappa \in \mathcal{C}(\Delta)$ such that $(M_\Delta, \omega_\kappa)$ has a unique toric structure, namely that of the product. Since all the vertices of such a product are equivalent in the sense of Definition 3.5, its (small) one point toric blow ups also have a unique structure by Lemma 3.3.

(ii) To see that one must restrict to generic κ here, consider the polygon obtained from the 2–simplex Δ_2 by blowing up each of its three vertices in such a way that all sides of the resulting polygon have equal affine length. (This polygon corresponds to the monotone three point blow up of $\mathbb{C}P^2$.) Then all its vertices are equivalent, so all its one point toric blow ups are the same.

(iii) We show during the course of the proof of Proposition 1.4 that if the vertices of $\Delta(\kappa)$ are all equivalent for generic κ, then Δ is a product of simplices.

Another natural class of examples is provided by product manifolds of the form $M \times S^2$. The idea is this. Each $H \in \mathfrak{t}_{\mathbb{Z}} \smallsetminus \{0\}$ exponentiates to a circle Λ_H in T. Denote by M_H the total space of the associated Hamiltonian bundle[1] $(M, \omega) \to M_H \xrightarrow{\pi} S^2$. One can realize M_H as the quotient $S^3 \times_{S^1} M$ where S^1 acts diagonally on $S^3 \subset \mathbb{C}^2$ and via Λ_H on M. Consider the 1–form

$$\alpha := \frac{i}{4\pi} \Big(\sum_{j=1,2} z_j \, d\bar{z}_j - \bar{z}_j \, dz_j \Big)$$

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[1] A smooth bundle $E \to B$ with fiber F is Hamiltonian if its structural group reduces to the Hamiltonian group $\mathrm{Ham}(F, \sigma)$ of some symplectic form σ on F. Often, as here, σ is given. When $\pi_1(B) = 0$ this is equivalent to saying that the fiberwise symplectic form σ extends to a closed form on the total space.

on S^3. (The form α is the standard contact form normalized so that the integral of $d\alpha$ over the unit disc $\{(z_1, 0) : |z_1| < 1\}$ is 1.) Then the form $\mathrm{pr}^*(\omega) + d\big((\lambda - H)\alpha\big)$, where $\lambda \in \mathbb{R}$ and $\mathrm{pr}\colon S^3 \times M \to M$ is the projection, descends to the quotient M_H and defines a symplectic form Ω_λ there provided that the function $\lambda - H$ is positive on M. Moreover, because the action of Λ_H on M commutes with T the manifold (M_H, Ω_λ) supports an action of $T_H := T^{n+1}$. Thus (M_H, Ω_λ) is toric. Moreover the bundle

(1-2)
$$M \overset{\iota}{\hookrightarrow} M_H \overset{\pi}{\to} S^2$$

is toric in the sense that there is a group homomorphism $\rho\colon T_H \to S^1$ such that the projection $\pi\colon M_H \to S^2$ intertwines the action of T_H on M_H with the action of $\rho(T_H) = S^1$ on S^2; ie

$$\pi(t \cdot x) = \rho(t) \cdot \pi(x), \quad x \in M_H, \ t \in T_H.$$

Suppose now that Λ_H, considered as a loop in the Hamiltonian group $\mathrm{Ham}(M, \omega)$, is contractible. Then the bundle $(M, \omega) \to M_H \to S^2$ is trivial as a Hamiltonian bundle. This readily implies[2] that (M_H, Ω_λ) is symplectomorphic to the product $(S^2 \times M, \sigma \oplus \omega)$ for suitable area form σ on S^2. But we will see in Remark 2.11 (i) that the moment polytope Δ_H of (M_H, Ω_H, T_H) is not affine equivalent to a product when $H \neq 0$. This proves the following result.

Lemma 1.6 *If (M, ω, T) is such that the loop Λ_H contracts in $\mathrm{Ham}(M, \omega)$ for some nonzero $H \in \mathfrak{t}$ then there is $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the product $(S^2 \times M, \lambda\sigma \oplus \omega)$ supports more than one toric structure.*

Remark 1.7 (i) The moment polytope Δ_H of M_H is always combinatorially equivalent to a product. Hence the examples in Lemma 1.6 are not distinguished in papers such as [3].

(ii) One could, of course, also consider the (toric) bundle $S^{2k+1} \times_{S^1} M \to \mathbb{C}P^k$ corresponding to the loop Λ_H for $k > 1$. However, even if Λ_H contracts in $\mathrm{Ham}(M, \omega)$, this bundle is *never* trivial as a Hamiltonian bundle when $H \neq 0$; cf Remark 2.13.

The next question is: when do such loops exist? The paper of McDuff and Tolman [22] analyses this question in great detail. The easiest case is when the loop Λ_H (or one of its finite multiples Λ_{mH}) contracts in the maximal compact subgroup $\mathrm{Isom}_0(M)$ of $\mathrm{Ham}(M, \omega)$, consisting of symplectomorphisms that preserve the natural Kähler metric on (M, ω).[3] Such elements $H \in \mathfrak{t}_{\mathbb{Z}}$ were called *inessential* in [22], and exist

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[2] for example by adapting the proof of Proposition 9.7.2 (ii) on page 341 of McDuff and Salamon [20].
[3] This group is described in slightly different language in Masuda [13].

when the moment polytope Δ of M satisfies some very natural geometric conditions. In particular, by [22, Proposition 3.17] if they exist the polytope Δ must either be a bundle over a simplex or an expansion (wedge). Correspondingly M is either the total space of a toric bundle over $\mathbb{C}P^k$ or is the total space of smooth Lefschetz pencil with axis of (real) codimension 4. (The last statement is explained in [22, Remark 5.4].) Generalized Bott towers, which are iterated bundles formed from projective spaces, are well known examples. Since any wedge and any bundle over $\mathbb{C}P^k$ has a nontrivial inessential function H, many product toric manifolds $M \times S^2$ have more than one toric structure. For further discussion of this issue, see Theorem 1.14 and Theorem 1.17 below.

Manifolds with unique toric structures   Next, one might wonder which symplectic manifolds have just one toric structure. We prove the following result in Section 2.4 by a cohomological argument. We denote by ω_n the usual symplectic form on $\mathbb{C}P^n$ that integrates over a line to 1. Thus $(\mathbb{C}P^n, \omega_n)$ is a toric manifold with moment polytope equal to the standard unit simplex

$$\Delta_n = \{x_1 \geq 0, \dots, x_n \geq 0, \ \sum x_i \leq 1\} \subset \mathbb{R}^n.$$

Proposition 1.8   *Let* $(M, \omega) = (\mathbb{C}P^k \times \mathbb{C}P^m, \omega_k + \lambda\omega_m)$, *where* $\lambda > 0$. *If* $k \geq m \geq 2$ *then* (M, ω) *has a unique toric structure. If* $k > m = 1$, *this remains true provided that* $\lambda \leq 1$, *while if* $k = m = 1$ *we require* $\lambda = 1$.

Remark 1.9   (i)   At first glance, this result is somewhat surprising, since one might well imagine that there are analogs of Hirzebruch structures on products such as $\mathbb{C}P^3 \times \mathbb{C}P^2$. As pointed out in Remark 2.13, the explanation for this lies in the characteristic classes constructed by Kędra and McDuff [10].

(ii)   If $k \geq m = 1$ and $\lambda > 1$, then there are nontrivial toric $\mathbb{C}P^k$ bundles over $\mathbb{C}P^1$ that are symplectomorphic to products for large λ, as one can see by arguments similar to those that prove Lemma 1.6. However, even in the case $k = m = 1$, when we get the Hirzebruch surfaces, the proof that these manifolds are symplectomorphic to products for all relevant λ is nontrivial; see McDuff [17] or McDuff and Salamon [20, Proposition 9.7.2]. Nevertheless, this proof should generalize to show that uniqueness fails whenever λ does not satisfy the conditions in Proposition 1.8.

(iii)   Proposition 1.8 extends work by Choi, Masuda and Suh, who show in [2] that if M is a toric $\mathbb{C}P^k$–bundle over $\mathbb{C}P^m$ then it is diffeomorphic to the product of its base and fiber exactly if its integral cohomology ring is isomorphic to that of the product.

Monotone polytopes   Another natural class of manifolds that might have unique toric structures is that of monotone manifolds. Recall that a symplectic manifold (M, ω) is said to be *monotone* if there is $\lambda > 0$ such that $[\omega] = \lambda c_1(M)$. In this paper, we shall always normalize ω so that $\lambda = 1$. Thus, in the toric case, the moment polytope is scaled so that the affine length of each edge ϵ is precisely $\int_{\Phi^{-1}(\epsilon)} \omega$.

The moment image of a monotone toric manifold is called a monotone polytope.[4] Since rather little seems to be known in general about their structure, we begin our discussion by describing some elementary constructions.

The most interesting of these is that of wedge (called expansion in [22]). It was used by Haase and Melnikov [8] to show that *every* smooth integral polytope is the face of some monotone polytope. We adapt it here to answer some questions raised in [22]. Let us say that a facet F of a polytope is *pervasive* if it meets all other facets and is *powerful* if there is a edge between F and every vertex of Δ not on F. We showed in [22, Theorem A.6] that in dimension ≤ 4 the only polytopes with all facets powerful are combinatorially equivalent to products of simplices. This is not true in higher dimensions, even if one restricts to the monotone case. For every face of a product of simplices is also a product of simplices, while, by Lemma 2.4, a monotone polytope has faces of arbitrary shape.

The next result is an immediate consequence of Lemma 2.4.

Proposition 1.10   *Let Δ' be any smooth polytope with integral vertices. Then some multiple $k\Delta', k \in \mathbb{Z}$, is integrally affine equivalent to a face in a monotone polytope Δ all of whose facets are pervasive and powerful.*

Further, in Lemma 2.6 we use the wedge construction to describe an example found by Paffenholz of a monotone polytope that fails the star-Ewald condition of [16]. As we explain in Section 2.1, this is related to the work of Fukaya, Oh, Ohta and Ono [7] on the Floer homology of toric fibers.

Despite the existence of this rather versatile construction, I do not know the answer to the following question.

Question 1.11   Is there a monotone toric manifold (M, ω) with more than one toric structure?

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[4] These are also known as smooth reflexive polytopes. Note that much of the literature about them is written in terms of their dual polytopes $P \subset \mathfrak{t}$ (which are simplicial) rather than the moment polytopes considered here.

It is not clear whether one can obtain such an example by blowing up a point (vertex cutting). However, the next result shows that one cannot get examples by the bundle construction used in Lemma 1.6 above.

We shall say that two bundles $M \to M_{H_i} \to S^2$, $i = 1, 2$, are bundle isomorphic if there is a commutative diagram

$$
\begin{array}{ccc}
M & \to M_{H_1} & \to S^2 \\
\text{id} \downarrow & \phi \downarrow & \text{id} \downarrow \\
M & \to M_{H_2} & \to S^2
\end{array}
$$

where ϕ is a diffeomorphism. Thus we assume that ϕ is the identity map on the distinguished fiber. However, it need not preserve the symplectic forms on the total spaces.

Definition 1.12 We say that two facets F_i, F_j of Δ are *equivalent*, and write $F_i \sim F_j$, if there is a vector $\xi \in \mathfrak{t}^*$ that is parallel to all other facets of Δ.

It is shown in [22, Lemma 3.4] that $F_i \sim F_j$ precisely if there is a robust[5] affine reflection of $\mathfrak{t}^*$ that takes Δ to itself and interchanges the facets F_i, F_j, fixing all others. Because it is robust, this affine reflection lifts to a symplectomorphism of $(M_\Delta, \omega_\Delta)$ that lies in the maximal compact subgroup $\text{Isom}_0(M_\Delta)$ of $\text{Ham}(M_\Delta, \omega)$; in particular it is isotopic to the identity. It also follows from the Stanley–Reisner presentation of $H^*(M)$ (cf Equation (2-4)) that $F_i \sim F_j$ exactly if the hypersurfaces $\Phi^{-1}(F_i)$ and $\Phi^{-1}(F_j)$ represent the same element in $H_{2n-2}(M)$.

We prove the following result in Section 2.3.

Proposition 1.13 Suppose that (M_H, ω_H, T_H) is a monotone toric manifold with moment polytope Δ_H that is the total space of a toric bundle with fiber (M, ω, T) and base $\mathbb{C}P^1$. Then the following hold.

(i) Either there is a facet F_j of the moment polytope Δ of M such that $H = \eta_j$, or $H = 0$ and Δ_H is affine equivalent to the product $\Delta_1 \times \Delta$.

(ii) If $H = \eta_j$, then the loop Λ_H does not contract in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$ and (M_H, ω_H) is not symplectomorphic to a product $(M \times S^2, \omega \oplus \sigma)$.

(iii) Two of the bundles in (ii) are bundle isomorphic only if they are generated by elements $H_j = \eta_j$, $j = 1, 2$, that correspond to equivalent facets of Δ. In this case, the loops Λ_{H_j} are conjugate in $\text{Ham}(M_\Delta, \omega_\Delta)$.

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[5]This means that the reflection persists as one perturbs κ a little.

Mass linearity Our final set of results again concerns the question of which toric manifolds (M, ω) have nontrivial loops Λ_H that contract in $\mathrm{Ham}(M, \omega)$. Above we discussed inessential H.[6] The papers [22; 23] discuss a more interesting class of functions H called *mass linear functions*. These are functions on Δ whose value $H(B_n)$ at the barycenter $B_n(\kappa)$ of the moment polytope $\Delta = \Delta(\kappa)$ is a linear function of the support numbers $\kappa = (\kappa_1, \dots, \kappa_N)$ of its facets. By [22, Proposition 1.17] every inessential function is mass linear. However, even when $n = 3$ there are pairs (Δ, H) where H is mass linear but is not inessential; in this case we say that H is *essential*. By [22, Theorem 1.4], in 3 dimensions there is precisely one such family (Δ, H) that we describe in Lemma 4.11 below. In these examples, the underlying polytope Δ is a Δ_2–bundle over Δ_1, where Δ_k denotes the standard k–simplex.[7]

We showed in [22, Proposition 1.22] that if a loop Λ_H contracts in $\mathrm{Ham}(M, \omega)$ then H is mass linear. There the argument was based on Weinstein's action homomorphism of $\pi_1(\mathrm{Ham}(M, \omega))$; in Section 4.4 below we explain an alternative argument due to Shelukhin that uses some other homomorphisms. Conversely, one can ask if the mass linearity of H implies that the loop Λ_{mH} contracts in $\mathrm{Ham}(M, \omega)$ for some m. (Proof that this is true in some nontrivial cases is the subject of ongoing research.) Our next result establishes a cohomological version of this statement.

We prove the following result in Section 4, using Timorin's very interesting description of the real cohomology algebra of (M, ω_κ) in terms of the function $V(\kappa)$ that gives the volume of the moment polytope in terms of the support numbers κ.

Theorem 1.14 *Let (M, ω, T) be a toric manifold with moment polytope Δ, and let $H \in \mathfrak{t} \smallsetminus \{0\}$. Let $M \to M_H \to S^2$ be the corresponding bundle. Then the element $H \in \mathfrak{t}_{\mathbb{Z}}$ is mass linear if and only if there is an algebra isomorphism*

$$\Psi \colon H^*(S^2; \mathbb{Q}) \otimes H^*(M; \mathbb{Q}) \xrightarrow{\cong} H^*(M_H; \mathbb{Q})$$

that is compatible with the fibration structure in the sense that it fits into a commutative diagram

$$
\begin{array}{ccccc}
H^*(M) & \leftarrow & H^*(S^2) \otimes H^*(M) & \leftarrow & H^*(S^2) \\
\mathrm{id} \downarrow & & \Psi \downarrow & & \mathrm{id} \downarrow \\
H^*(M) & \leftarrow & H^*(M_H) & \leftarrow & H^*(S^2).
\end{array}
$$

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[6] By slight abuse of language, we often call H a function, thinking of it as a function on the moment polytope Δ. Note also that the moment map for the circle action Λ_H is the composite $x \mapsto \langle H, \Phi(x) \rangle$, of Φ with the projection $\mathfrak{t}^* \to \mathbb{R}$ given by inner product with H.

[7] Unless explicit mention is made to the contrary, we allow the standard simplex to have any size, ie we do not fix κ.

Remark 1.15 If one writes Ψ in terms of a basis for the integral cohomology, then its coefficients give information about the order of the loop Λ_H in $\pi_1(\mathrm{Ham}(M, \omega))$. Indeed, if this order is $m < \infty$ then these coefficients must lie in $\frac{1}{m}\mathbb{Z}$; cf Remark 4.16.

Theorem 4.17 below sharpens Theorem 1.14, using Shelukhin's concept of full mass linearity. He considers all the barycenters B_k, $k = 0, \ldots, n$, of Δ, defining B_k to be the barycenter of the union of the k–dimensional faces of Δ. For example, B_0 is the average of the vertices of Δ. He showed that the numbers $H(B_k) - H(B_n)$ are the values of some natural characteristic classes on toric loops Λ_H, hence proving the following result.

Proposition 1.16 (Shelukhin [29]) *The loop Λ_H contracts in $\mathrm{Ham}(M_\Delta, \omega_\kappa)$ only if $H(B_n) = H(B_k)$ for all $k = 0, \ldots, n - 1$.*

We will say that H is *fully mass linear* if $H(B_k) = H(B_n)$ for $0 \le k \le n - 1$. Theorem 4.17 gives a cohomological interpretation of the full mass linearity condition. In Section 4 we also sharpen some of the combinatorial results of [22], obtaining the following results.

Theorem 1.17 (i) *An element $H \in \mathfrak{t}_\mathbb{Z}$ is mass linear if and only if $H(B_n) = H(B_0)$. Moreover, in this case, $H(B_{n-1}) = H(B_n)$.*

(ii) *Every mass linear function on a polytope of dimension at most 3 is fully mass linear.*

Organization of the paper We begin the proofs by discussing the structure of monotone manifolds, since this will allow us to introduce some of the main constructions. Theorem 1.2 is proved in Section 3.1; the argument does not use Section 2. Mass linearity is discussed in Section 4. This section is essentially independent of the other two.

2 Monotone polytopes

We begin with a general remark about normalizations. The moment polytope $\Delta \subset \mathfrak{t}^* \cong \mathbb{R}^n$ of a toric manifold (M, ω, T) is determined as a subset of $\mathbb{R}^n$ up to the action of the integral affine group $\mathrm{Aff}(n; \mathbb{Z})$. Because the conormals at any vertex form a lattice basis, we may therefore always choose coordinates on $\mathbb{R}^n$ so that the conormals at any chosen vertex v are $-e_1, \ldots, -e_n$, ie the negatives of the standard basis vectors. Then the polytope lies in a translate of the positive quadrant $x_i \geq 0$, $i = 1, \ldots, n$. Sometimes we normalize so that $v = 0$, but often (as in the monotone case considered below) we set $v = (-1, \ldots, -1)$ so that the center point of Δ is at $\{0\}$.

Recall that the symplectic manifold (M, ω) is monotone if $[\omega] = \lambda c_1(M)$ for some $\lambda > 0$. Throughout we will normalize monotone manifolds so that $\lambda = 1$. There are several possible ways of characterizing the moment image of a monotone toric manifold. The following well-known lemma is proved in [16, Lemma 3.3].

Lemma 2.1 *A simple smooth polytope Δ is monotone if and only if it satisfies the following conditions:*

(i) *Δ is an integral (or lattice) polytope in $\mathbb{R}^n$ with a unique interior integral point u_0,*

(ii) *Δ satisfies the* vertex-Fano *condition: for each vertex v_j we have*

$$v_j + \sum_i e_{ij} = u_0,$$

where e_{ij}, $1 \leq i \leq n$, are the primitive integral vectors from v_j pointing along the edges of Δ.

Remark 2.2 (i) If the conditions in Lemma 2.1 are satisfied, then the affine distance[8] $\ell_j(u_0) := \kappa_j - \langle \eta_j, u_0 \rangle$ from u_0 to the facet F_j equals 1 for all j. Hence if we translate Δ so that $u_0 = \{0\}$ the structure constants κ_i in the formula (1-1) are all equal to 1. Conversely, any integral polytope with $\kappa_i = 1$ for all i satisfies conditions (i) and (ii) in Lemma 2.1 with $u_0 = \{0\}$ and so is monotone.

(ii) Another closely related notion is that of Fano polytope. Usually one defines this in terms of the dual $P \subset \mathfrak{t}$ to the moment polytope (namely the fan), and calls P Fano if one can choose support constants κ' for the moment polytope Δ that make it monotone. However, the constants κ' are not specified. Correspondingly, a *Fano toric symplectic manifold* (M, ω_κ, T) is one that may not be monotone but where there is $\kappa' \in \mathcal{C}_\Delta$ such that $(M, \omega_{\kappa'}, T)$ is monotone.

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[8] See [16, Section 2] for a general explanation of how to measure affine distance.

2.1 The wedge construction

This is a very useful construction that appeared in [22] because of our result that any polytope with a nontrivial robust[9] symmetry is either a bundle over a simplex or is an expansion; cf [22, Proposition 3.15]. Moreover, a polytope has such a symmetry exactly if the identity component of its Kähler isometry group (with respect to the natural Kähler metric) is larger than the torus T^n; cf [22, Proposition 5.5]. We called this construction an *expansion*. However, it is known in the combinatorial literature as a *wedge*.

Here is the definition.

Definition 2.3 Suppose that $\Delta \subset \mathbb{R}^n$ is described by the inequalities

$$(2\text{-}1) \qquad\qquad \langle \eta_i, x \rangle \leq \kappa_i, \quad x \in \mathbb{R}^n, \ i \in \{1, \ldots, N\},$$

where $\kappa_i > 0$ so that $\{0\}$ lies in its interior. Its *wedge* (or *expansion*) Δ' along the facet F_k lies in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and is given by the above inequalities for $i \neq k$ (where we identify $\eta_i \in \mathbb{R}^n$ with $(\eta_i, 0) \in \mathbb{R}^{n+1}$) together with

$$x_{n+1} \geq -1, \quad \langle \eta_k, x \rangle + x_{n+1} \leq \kappa_k - 1.$$

Thus we replace the conormal η_k by the two conormals $\eta_k' = (\eta_k, 1)$ and $\eta_{N+1}' = (0, \ldots, 0, -1)$. The original polytope Δ is now the facet F_{N+1}' of the wedge Δ'. In fact, Δ' is made from the product $\Delta \times [-1, \infty)$ by adding a new "top" facet F_k' with conormal $\eta_k' = (1, \eta_k)$ that intersects the "bottom" facet $F_{N+1}' := \{x_{n+1} = -1\}$ in the facet F_k of Δ. The corresponding toric manifold $M_{\Delta'}$ is the total space of a smooth Lefschetz pencil with pages M_Δ and axis (of complex codimension 2) $F_k' \cap F_{N+1} \cong F_k$; cf [22, Remark 5.4].

Note that all the structural constants κ_j remain the same, except for κ_k which decreases by 1. Moreover $\kappa_{N+1} = 1$. Haase and Melnikov point out in [8, Proposition 2.2] that by repeating this construction until each $\kappa_j = 1$ one finds that every integral polytope with an interior integral point (which we can assume to be at $\{0\}$) is integrally affine equivalent to the face of some monotone polytope. Here is a slight refinement of their result. Recall that a facet F is called *pervasive* if it meets all other facets and *powerful* if there is a edge between F and every vertex of Δ not on F.

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[9] A nontrivial affine transformation of $\Delta(\kappa)$ is called robust if it persists when one perturbs κ; for a more precise definition see [22, Definition 1.11]. These symmetries make up the group $\mathrm{Aff}_0(\Delta)$ of Definition 3.5 below.

Lemma 2.4 *Suppose that Δ is a smooth integral polytope with $\{0\}$ in its interior and with all structural constants $\kappa_i \geq 2$. Then Δ is a face in a monotone polytope for which all facets are both pervasive and powerful.*

Proof The new facets F_{N+1} (the bottom) and F'_k (the top) of any wedge are pervasive. Moreover, any pervasive facet of Δ remains pervasive in Δ'. Similar remarks apply to the concept of powerful since all vertices in Δ' lie either on the top or bottom facet of Δ'. The hypothesis that $\kappa_i \geq 2$ implies that we must wedge at least once along each facet to get a monotone polytope. The result follows. □

In [22] we were interested in polytopes for which all facets are both pervasive and powerful because we were trying to understand mass linear functions H on polytopes Δ. Our basic question was: is it always true that after subtracting an inessential function H_0, the resulting mass linear function has a symmetric facet?[10] Equivalently, is there an inessential H_0 such that $H - H_0 = \sum \gamma_i \kappa_i$ where $\gamma_i = 0$ for some i? The answer would be yes, if every polytope with all facets powerful and pervasive has at least two equivalent facets; cf [22, Lemma 3.19]. Therefore, it would be relevant to know the answer to the following question.

Question 2.5 Is there a smooth polytope whose facets are powerful and pervasive and have the property that no two facets are equivalent?

Of course, to construct such a polytope one cannot use wedging, since the top and bottom facets of a wedge are always equivalent.

We end this subsection by using wedges to construct an example of a smooth monotone polytope Δ that does not satisfy the star-Ewald condition of [16, Definition 3.5] at one of its vertices. This is a condition on each face f of Δ that is designed so that it fails at f exactly if there is a point in the interior of the cone $C(f, 0)$ spanned by f and $\{0\}$ that cannot be displaced by a probe; cf the proof of [16, Theorem 1.2]. Therefore the corresponding Lagrangian toric fibers $L(u)$ in M_Δ may perhaps be nondisplaceable by Hamiltonian isotopies, even though, according to [7], their Floer homology vanishes.

This example is due to Paffenholz [24]. By using the program *Polymake* he shows that all polytopes of dimensions less than 6 do satisfy the star-Ewald condition. However, he found three 6–dimensional examples where the condition fails, and many more 7–dimensional ones. In all but one case the condition failed at a vertex or an edge, but there is one 7–dimensional example (se.7d.02 on his list) where it fails on a nonconvex

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[10] A facet F_j is called *symmetric* (resp. *asymmetric*) if, when we write $H(B_n) = \sum \gamma_i \kappa_i$, the coefficient γ_j vanishes (resp. $\gamma_j \neq 0$).

set consisting of two edges.[11] All of his examples are wedges. We shall explain the easiest one, which is a repeated wedge of the polygon in Figure 1.
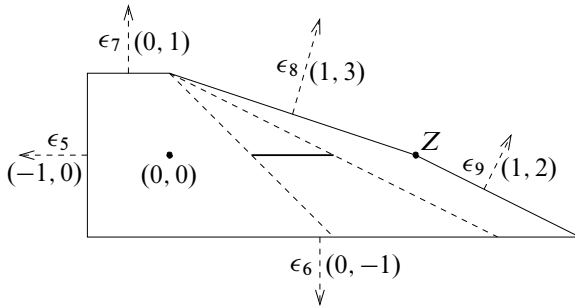

Figure 1: The polygon $\widetilde{\Delta}$. The heavy line segment in the middle are the points that are nondisplaceable by probes. The conormals of $\widetilde{\Delta}$ are the vectors $v_5, \ldots, v_9$ in Equation (2-2) with $\kappa = (1, 1, 1, 3, 3)$.

Consider the monotone 6–dimensional polytope Δ with conormals

$$\eta_i = -e_i, \ i = 1, \ldots, 6, \quad \eta_7 = e_6, \quad \eta_8 = (1, 1, 0, 0, 1, 3), \quad \eta_9 = (0, 0, 1, 1, 1, 2),$$

where e_i are the standard basis in $\mathbb{R}^6$ and we set $\kappa_i = 1$ for all i. Further let $\widetilde{\Delta}$ be the polygon with conormals

$$(2\text{-}2) \quad v_5 = (-1, 0), \quad v_6 = (0, -1), \quad v_7 = (0, 1), \quad v_8 = (1, 3), \quad v_9 = (1, 2),$$

and with $\kappa = (1, 1, 1, 3, 3)$ as in Figure 1. Then $\widetilde{\Delta}$ can be identified with the facet F_{0123} of Δ. Further Δ is obtained from $\widetilde{\Delta}$ by making twice repeated expansions in the edges $\epsilon_8 := F_{12348}, \epsilon_9 := F_{12349}$ of $\widetilde{\Delta}$ (or, more precisely, in the facets corresponding to these edges). The facets F_1, F_2 come from the expansion along ϵ_8 and the facets F_3, F_4 from the expansion along ϵ_9.

Given an integral polytope Δ with $\{0\}$ in its interior, consider the set

$$\mathcal{S}(\Delta) = \{v \in \mathbb{Z}^n \cap \Delta : -v \in \Delta\} \smallsetminus \{0\}$$

of all integral symmetric points in Δ. The *star-Ewald condition* for a vertex z says that there is a point $w \in \mathcal{S}(\Delta)$ that lies in precisely one of the facets through z and is such that $-w$ lies on no facet through z. As mentioned above, this condition is satisfied at z exactly if all the points on the open line segment $C(z, 0)$ from z to $\{0\}$

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[11] If you look at the file, the first edge together with its data is listed first, and the data on the second edge occurs about half way through.

can be displaced by probes.[12] In particular, if Δ is a wedge with top and bottom facets F_T, F_B, then to satisfy the star-Ewald condition at $z \in F_T \cap F_B$ the integer point $-w$ must lie on one of the other facets. Because the union $F_T \cup F_B$ contains all the vertices of Δ and many of its integer points, this condition is quite restrictive, and, as we now see, can fail to hold.

Lemma 2.6 *Let Δ be as in Equation (2-2). Then Δ does not satisfy the star-Ewald condition at the vertex $z = F_{123489}$.*

Proof Because the points $\mathbf{x} = (x_1, \dots, x_6)$ in Δ all satisfy the inequalities $x_i \geq -1$, the coordinates of every point in $\mathcal{S}(\Delta)$ line in the set $\{0, \pm 1\}$. Suppose that $w = (w_1, \dots, w_6) \in \mathcal{S}(\Delta)$ lies in just one facet through z while $-w$ lies in none of them. Then at most one of $w_1, \dots, w_4$ is -1 and none is 1. If they are all 0, then w must lie on F_8 or F_9 so that precisely one of the equations $w_5 + 3w_6 = 1$ and $w_5 + 2w_6 = 1$ holds. Since $w_i \in \{0, \pm 1\}$, we must have $(w_5, w_6) = (-1, 1)$. But then $w = (0, 0, 0, 0, -1, 1)$ does not lie in Δ because $w_5 + 3w_6 > 1$. Therefore by symmetry we just need to consider the cases

$$\text{(a)} \quad w = (-1, 0, 0, 0, w_5, w_6) \quad \text{and} \quad \text{(b)} \quad w = (0, 0, -1, 0, w_5, w_6).$$

In case (a), since $\pm w \in \Delta \smallsetminus F_8$ we need $-1 + w_5 + 3w_6 < 1$ and $1 - w_5 - 3w_6 < 1$. This has the solution $(w_5, w_6) = (1, 0)$. But then $w \in F_9$, which is not allowed. A similar argument applies to case (b). $\qquad \square$

2.2 Symplectic cutting

Another useful way of constructing polytopes is by blow up. As we show in more detail in [23, Section 3], blowing up along a face $f = F_I$ of codimension $k = |I| \geq 2$ adds a new face F_0 to the polytope with conormal $\eta_0 = \sum_{i \in I} \eta_i$ and constant $\kappa_0 = \sum_{i \in I} \kappa_i - \varepsilon$. One can always do this for small $\varepsilon > 0$. However, if Δ is monotone and one wants the blow up Δ' also to be monotone, then, because we need all the $\kappa_j = 1$, one must take $\varepsilon = k - 1$. In this case, the new facet F_0 is a Δ_{k-1}–bundle over f whose fiber edges have affine length $k - 1$, which is precisely the first Chern class of a line in the corresponding exceptional divisor. In particular, there is a monotone blow up of a vertex of a monotone polytope only if all edges through this point have affine length at least n.

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[12] To understand the conditions on w, notice that the probes used to displace the points of the line $C(z, 0)$ have base along the line $C(z, w)$ (which by hypothesis is contained in the interior of a facet F_w through z) and direction $-w$. The condition on $-w$ implies that the interior of the line $C(z, -w)$ lies in the *interior* of Δ so that the probes meet $C(z, 0)$ *before* their halfway point.

In dimension 2, blow ups have size 1, and it is possible to make several such blowups on one polytope. Indeed, one can blow up the triangle (the moment polytope of $\mathbb{C}P^2$) at all three of its vertices to obtain a monotone polytope. Similarly, in dimension 3 one can make several disjoint monotone blowups provided they are along edges. But in dimension 3 it is not possible to blow up two points simultaneously and in a monotone way.[13] For example, the monotone 3–simplex has edges of length 4, while monotone blow ups (of vertices) have size 2. Thus if one did any two such blow ups one would create at least one singular (ie non simple) vertex.

Question 2.7   Is there a monotone polytope Δ of dimension $d > 2$ for which one can make at least two monotone and disjoint blow ups of points, or, more generally, of any two faces of codimension > 2?

In dimension 4 is it not clear exactly what geometric constructions are needed to form all the monotone polytopes. Obviously one can use bundles, or wedges of lower dimensional (nonmonotone) polytopes. Here is a monotone polytope formed by a different construction, that I again owe to Paffenholz [24].

Example 2.8   Let Δ be the 4–dimensional cube $\{\mathbf{x} \in \mathbb{R}^4 : -1 \leq x_i \leq 1\}$. Add the new facet $\sum x_i \geq -1$. The new conormal $\eta_0 = (-1, \ldots, -1)$ is parallel to the exceptional divisor that one would obtain by blowing Δ up at its vertex $(-1, \ldots, -1)$. However, we take $\varepsilon = 3$ to make a monotone blow up. This means that we have cut out some of the vertices and edges of Δ, though none of its facets. The resulting polytope Δ' is smooth because none of the vertices of Δ lie on the new facet. Really one should think of Δ' not as a blow up but as the result of symplectic cutting; cf Lerman [12]. As Paffenholz pointed out, Δ' is not a wedge because its vertices do not all lie on two facets, and it is not a bundle because it is not combinatorially equivalent to a product – its facet F_0 has more vertices than any other.

Let us say that a polytope is *elementary* if removing any of its facets (ie deleting the corresponding inequality from the description given in Equation (1-1) of Δ) results either in a non simple or in an unbounded polytope. Clearly any polytope can be obtained from an elementary one by adding facets. Adding a facet is the most general possible cutting operation, where we no longer restrict the direction of the cut to $\sum_{i \in I} \eta_i$ for some face F_I. If one wants to understand the structure of monotone polytopes one might begin by asking about elementary ones.

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[13] One can check this by listing the possibilities. By [1; 31], there are only eighteen monotone polytopes in dimension 3; they are all blow ups of polytopes obtained from simplices by forming suitable bundles.

Question 2.9 What are the shapes of elementary monotone polytopes? For example, is there any such polytope that is not a bundle or wedge?

2.3 Bundles

The general definition of bundle in the context of moment polytopes is rather complicated (see [22, Definition 3.10]), but that of bundle over the k–simplex Δ_k is easy since the structure is determined by one "slanted" facet F_{N+k+1}. Note that in the following definition we put the base coordinates last, since this is slightly more convenient and follows [22].

Definition 2.10 Write Δ as in Equation (2-1), and normalize by assuming $\eta_i = -e_i$, $i = 1, \ldots, n$, where e_i, $i = 1, \ldots, n$, forms the standard basis of $\mathfrak{t} = \mathbb{R}^n$. Then the bundle Δ' with fiber Δ and base Δ_k is determined (up to integral affine equivalence) by an integral n–vector $A := (a_1, \ldots, a_n)$ and constant $h := \kappa_{N+k} + \kappa_{N+k+1}$ as follows. The polytope Δ' lies in $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ and has conormals

$$(2\text{-}3) \qquad \eta_i' = (\eta_i, 0, \ldots, 0), \ 1 \le i \le N, \quad \eta_{N+i}' = -e_{n+i}, \ i = 1, \ldots, k,$$

$$\eta_{N+k+1}' = (a_1, \ldots, a_n, 1, \ldots, 1) = \sum_{i=1}^{k} \eta_{N+i} + A,$$

where e_{n+i}, $1 \le i \le k$, form the rest of the standard basis in $\mathbb{R}^{n+k}$. The constants κ_i, $1 \le n \le N$, are as before, we take $\kappa_{N+i} = 1$, $1 \le i \le k$, and choose $\kappa_{N+k+1} := h-1$ large enough that the polytope is combinatorially a product of Δ with Δ_k.

Remark 2.11 (i) A bundle over Δ_1 is formed from the product $\Delta \times \mathbb{R}$ by making two slices. The bottom slice F_{N+1} can be normalized to have equation $x_{n+1} = -1$, while the top F_{N+2} is slanted by the vector A. If $M := M_\Delta$, the toric manifold $M_{\Delta'}$ is precisely M_H where $H := A = \sum_{i=1}^{n} a_i e_i$; see [22, Example 5.3]. (As always, a formula like this involves some sign conventions; here we follow the sign choices in [22].)

(ii) If the total space of a bundle (over an arbitrary base $\widehat{\Delta}$) is monotone so are the fiber and base. Even the meaning of the second part of this statement needs clarification; see [16, Lemma 5.2]. However, the first part is straightforward. To prove it, notice that $\Delta \subset \mathbb{R}^n$ can be identified with the face

$$f' := \Delta' \cap \left(\mathbb{R}^n \times \{(-1, \ldots, -1)\} \right) = \bigcap_{1 \le i \le k} F_{N+i}'$$

of Δ'. Therefore the unit edge vectors from a vertex $V \in f'$ divide into two groups, the first group consisting of the edge vectors e'_{Vj}, $j = 1, \ldots, n$ corresponding to the edge vectors e_{Vj} in Δ and the second group given by $e'_{Vj} := e_j$, $j = n+1, \ldots, n+k$. Thus, using the notation of Lemma 2.1, we have

$$V + \sum_j e'_{Vj} = u_0 \in \mathbb{R}^n \iff V + \sum_{j \leq n} e_{Vj} + \sum_i e_{n+i} = (u_0, 0, \ldots, 0) \in R^{n+k}.$$

Therefore the vertex-Fano condition at the vertex V of Δ' readily implies this condition at the corresponding vertex V of Δ.

Proof of Proposition 1.13 We are given a monotone manifold of the form (M_H, ω_H) where $H \in \mathfrak{t}$ generates a loop Λ_H of symplectomorphisms of the toric manifold (M, ω, T). To prove (i) we must show that if $H \neq 0$ then $H = \eta_j$, the conormal to one of facets of the moment polytope Δ of M. By Remark 2.11(ii) the manifold (M, ω, T) is monotone. Moreover, if we identify Δ with the facet $F_{N+1} = \{x_{n+1} = -1\}$ of $\Delta' = \Delta_H$, each vertex V of Δ lies on a unique edge ϵ_V that is parallel to e_{n+1}. Choose $V_0 \in \Delta$ such that this edge is shortest and then choose affine coordinates so that $V_0 = (-1, \ldots, -1)$, and the facets at V_0 have conormals $-e_i$, $i = 1, \ldots, n+1$. Since the center point of Δ_H is $\{0\}$, all $\kappa_j = 1$.

As in Remark 2.11(i), the top facet of Δ_H is given by an equation of the form $\sum_{i \leq n} a_i x_i + x_{n+1} = \kappa_{N+2} = 1$, where $H = (a_1, \ldots, a_n)$. Therefore if $W = (w_1, \ldots, w_n, -1)$ is a vertex in $\Delta \equiv F_{N+1}$, the second endpoint of the edge ϵ_W has last coordinate equal to $1 - \sum_{i \leq n} a_i w_i$. Hence ϵ_W has length

$$\ell(\epsilon_W) = 2 - \sum_{i \leq n} a_i w_i \geq \ell(\epsilon_{V_0}) = 2 + \sum_{i \leq n} a_i.$$

But for each i there is a vertex W_i along the edge from V_0 in the direction of the i–th coordinate axis. Thus $W_i = (-1, \ldots, -1, w_i, -1, \ldots, -1)$ where $w_i > -1$. Substituting $W = W_i$ in the above inequality, we find that $a_i \leq 0$ for all i.

Now consider the vertex-Fano condition at the point

$$W = \epsilon_{V_0} \cap F_{N+2} = (-1, \ldots, -1, x_{n+1})$$

on the top facet. Because ϵ_{V_0} is the shortest vertical edge, all edges through W except for $-\epsilon_{V_0}$ point in directions whose last coordinate is nonnegative. (In fact, one can check as above that the unit vectors along these edges are $e_i - a_i e_{n+1}$.) Hence, because $x_{n+1} \geq 0$, we must have $x_{n+1} = 0$ or $x_{n+1} = 1$. In the latter case all these edges have zero last coordinate, which implies that $H = (a_1, \ldots, a_n) = 0$. Hence F_{N+2} is parallel to F_{N+1} and Δ is a product. In the former case exactly one a_i is nonzero, and

the vertex-Fano condition $-1 - \sum a_i = 0$ shows that $a_i = -1$ Hence $H = -e_i = \eta_i$ for some $i \leq n$. Thus Λ_H is the loop given by a rotation that fixes all the points in the facet F_i. This proves (i).

To prove (ii) we must show that the bundle formed from $H = \eta_i$ is never trivial. Thus we must show that such a loop Λ_H is never contractible. One way to prove this is to consider the Seidel representation $\mathcal{S}$ of the group $\pi_1(\mathrm{Ham}(M, \omega))$ in the group $QH^\times$ of degree $2n$ units in the quantum homology ring of (M, ω). (For a definition of $\mathcal{S}$ in the toric context see [21, Section 2.3].) In the Fano case, it is easy to see that if $H = \eta_i$, we have $\mathcal{S}(\Lambda_H) = [F_i] \otimes \lambda$, where λ is some unit in the Novikov coefficient ring of quantum homology, and $[F_i]$ denotes the homology class of $\Phi^{-1}(F_i)$, the maximal[14] fixed point set of the loop Λ_H; see for example [21, Theorem 1.9]. Since $\mathcal{S}(\lambda_H) \neq [M]$ (the unit in $QH^\times$), the loop Λ_H cannot be contractible.

Similarly, if the loops Λ_{H_i} and Λ_{H_j} are homotopic they must have equal images under $\mathcal{S}$ so that $[F_i] = [F_j]$. But it is well known that the additive relations on $H_{2n-2}(M)$ have the form

$$\sum_i \langle \eta_i, \xi \rangle [F_i] = 0, \quad \xi \in \mathfrak{t}^*.$$

(See [30] for example.) Hence $[F_i] = [F_j]$ if and only if these two facets are equivalent in the sense used here. $\qquad \square$

We end with a question.

Question 2.12 Is there a monotone polytope Δ that supports an essential mass linear function?

Note that our constructions for monotone polytopes tend to destroy essential mass linear functions. For example, if H is an essential mass linear function on Δ and Δ' is the wedge of Δ along some facet then H does not in general induce an essential mass linear function on Δ'. A similar statement is true for bundles; if $\Delta' \to \widehat{\Delta}$ is a bundle with fiber Δ, then essential mass linear functions on Δ do not usually extend to mass linear functions on Δ': explicit examples are given in [23, Section 3].

2.4 An example of uniqueness

We now prove Proposition 1.8. This states that there is a unique toric structure on the product $(M, \omega) := (\mathbb{C}P^k \times \mathbb{C}P^m, \omega_k \oplus \lambda \omega_m)$, if $k \geq m \geq 2$, or if $k > m = 1$ and

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[14] ie the fixed point set on which the moment map $H \circ \Phi$ takes its maximum.

$\lambda \leq 1$, or if $k = m = 1$ and $\lambda = 1$. Notice that all monotone products of projective spaces satisfy these conditions. (Recall that we have normalized ω_k so that its integral over a line is 1.)

Suppose that Δ is the moment polytope for some toric structure on $M = \mathbb{C}P^k \times \mathbb{C}P^m$. Then Δ has precisely $k + m + 2 = \dim \Delta + \operatorname{rank} H^2(M)$ facets. Hence by Timorin [30, Proposition 1.1.1], Δ is combinatorially equivalent to a product of two simplices. Therefore, because Δ is smooth, [22, Lemma 4.10] implies that Δ is a Δ_r–bundle over Δ_s for some r, s. Therefore M is a $\mathbb{C}P^r$–bundle over $\mathbb{C}P^s$. Hence $H^2(M; \mathbb{Z})$ contains an element α such that $\alpha^{s+1} = 0$ while $\alpha^s \neq 0$. It follows that $s = k$ or $s = m$.

Let us now suppose that Δ is not the trivial bundle, ie some $a_i \neq 0$ in the presentation described in Definition 2.10 for a bundle over Δ_s. For each vertex V of the fiber Δ_r, there is an s–dimensional face f_V of Δ that is affine equivalent to $V \times \mu_V \Delta_s$ for some scaling constant μ_V. Choose V so that μ_V is minimal. As at the beginning of Section 2, choose coordinates on $\mathbb{R}^{r+s}$ so that $V = (-1, \ldots, -1)$ and so that the edges from V point in the directions of the coordinate axes. Then, as in the proof of Proposition 1.13, each $a_i \leq 0$.

Now let us calculate $H^*(M; \mathbb{Z})$ using the Stanley–Reisner presentation

$$(2\text{-}4) \qquad \mathbb{Z}[x_1, \ldots, x_{r+s+2}]/\bigl(P(\Delta) + S(\Delta)\bigr),$$

where the additive relations $P(\Delta)$ are $\sum_i \langle \eta_i, e_j \rangle x_i = 0$ (where $e_1, \ldots, e_n$ is a basis for $\mathfrak{t}^*$), and the set $S(\Delta)$ of multiplicative relations is $\prod_{i \in I} x_i = 0$, where I ranges over all minimal subsets $I \subset \{1, \ldots, N\}$ such that the intersection $F_I := \bigcap_{i \in I} F_i$ is empty. Thus in the case at hand there are two multiplicative relations, one from the conormals $(\eta_i, 0)$, where $1 \leq i \leq N = r + 1$, and the other from the conormals η_{N+i}, $1 \leq i \leq s + 1$. The relations for the facets F_{N+i}, $1 \leq i \leq s + 1$, show that $x_{r+2} = \cdots = x_{r+s+1} = \alpha$, say, and $\alpha^{s+1} = 0$. Similarly for the facets F_i, $1 \leq i \leq r + 1$, we find

$$-x_i + x_{r+1} + a_i \alpha = 0, \ i = 1, \ldots, r, \qquad \prod_{i=1}^{r+1} x_i = 0.$$

Thus, if we write $x_{r+1} := \beta$ and define $a_{r+1} := 0$, we find

$$(2\text{-}5) \qquad 0 = \prod_{i=1}^{r+1} (\beta + a_i \alpha) = \beta^{r+1} + \sigma_1 \beta^r \alpha + \cdots + \sigma_r \beta \alpha^r,$$

where $\sigma_1 := \sum_i a_i$, and, more generally, σ_k is the value of the k–th elementary symmetric polynomial on $(a_1, \ldots, a_r, 0)$.

By assumption, there are generators $\alpha_0, \beta_0 \in H^2(M; \mathbb{Z})$ so that $\alpha_0^{s+1} = 0 = \beta_0^{r+1}$. Therefore, for some $A, B, C, D \in \mathbb{Z}$ with $AD - BC = 1$, we must have

$$(A\alpha + B\beta)^{s+1} = 0 = (C\alpha + D\beta)^{r+1}.$$

We now divide into cases, and show in each case if some a_i is nonzero then the conditions in [Proposition 1.8](#) must hold.

Case 1 $1 < s < r$.

In this case, the ring $H^*(M)$ is freely generated by α, β in degrees $\leq 2s$ and there are two relations of degree $2s + 2$, namely

$$\alpha^{s+1} = 0, \quad (A\alpha + B\beta)^{s+1} = 0.$$

If $B \neq 0$, these relations are different so that $H^{2s+2}(M)$ has rank s instead of $s + 1$. Therefore $B = 0$ and $A = \pm 1$. By changing the sign of α_0 we may suppose that $A = 1$, so that $D = 1$. Then we have $(C\alpha + \beta)^{r+1} = 0$. Again, this must agree term by term with [Equation (2-5)](#), once we substitute $\alpha^{s+1} = 0$. Equating coefficients for $\beta^{r-i+1}\alpha^i$ with $i = 1, 2$, we need $\sum_{i \leq r} a_i = (r + 1)C$ and $\sum_{i \neq j} a_i a_j = (r + 1)rC^2$. Hence we also need $\sum a_i^2 = (r + 1)C^2$. But the last inequality together with Cauchy–Schwartz gives

$$\left| \sum a_i \right| \leq \sqrt{r} \sqrt{\sum_i a_i^2} \leq \sqrt{r}\sqrt{r+1}|C|,$$

a contradiction. Therefore this case does not occur.

Case 2 $s > r$.

In this case, the relation $(C\alpha + D\beta)^{r+1}$ must be a nonzero multiple of [Equation (2-5)](#). But, if $C \neq 0$, the coefficient of α^{r+1} is nonzero in the first equation, while it vanishes in [(2-5)](#). Therefore $C = 0$, so that $\sigma_1 = \sum a_i = 0$. But each $a_i \leq 0$ by construction. Hence we must have $a_i = 0$ for all i. Hence, again this case does not occur.

Case 3 $s = r > 1$.

In this case we have four relations in degree $2r + 2 \geq 6$, namely [Equation (2-5)](#) and

$$\alpha^{r+1} = 0, \quad (A\alpha + B\beta)^{r+1} = 0, \quad (C\alpha + D\beta)^{r+1} = 0$$

that must impose just two linearly independent conditions. Since $\alpha^{r+1} = 0$ is independent from [(2-5)](#), the other two equations must be combinations of these. By permuting α_0, β_0 if necessary, we can suppose that $A \neq 0, D \neq 0$. But then if we put $\alpha^{r+1} = 0$ in the relation $(C\alpha + D\beta)^{r+1} = 0$, we must get D^{r+1} times the [Equation (2-5)](#). Comparing coefficients of $\beta^r\alpha$ and $\beta^{r-1}\alpha^2$ we find

$$(r + 1)C = D\sigma_1, \quad \frac{r(r + 1)}{2}C^2 = D^2\sigma_2,$$

which, as in Case 1, is impossible unless all $a_i = 0$. Therefore this case also does not occur.

Case 4 $r \geq s = 1$.

Let us go back to the polytope Δ and look at the face $f_V \cong \mu_V \Delta^s$ at our chosen vertex V. Every edge ϵ in f_V has first Chern class given by[15]

$$(2\text{-}6) \qquad c_1(\epsilon) = s + 1 + \sum a_i.$$

Now observe that the submanifold $\Phi^{-1}(f_V)$ is a section of the bundle $M \to \mathbb{C}P^s$, so that the 2–sphere $\Phi^{-1}(\epsilon)$ lies in a homology class of the form $qL_r + L_s$, where L_i denotes the line in $\mathbb{C}P^i$.

Now observe that if $r > s = 1$ we must have $r = k$ and $s = m$, while if $r = s = 1$ we may assume that $r = k$ and $s = m$. Then, in both cases, we have $\omega(L_r) = 1$ and $\omega(L_s) = \lambda$. Hence $\omega(qL_r + L_s) = q + \lambda > 0$. On the other hand if some $a_i < 0$ then $c_1(\epsilon) < s + 1$ so that $q < 0$. Therefore this case does not occur when $\lambda \leq 1$.

Finally note that if $r = s = 1$ we can interchange the roles of r and s, replacing λ by $1/\lambda$. Therefore, when $k = m = 1$ our argument rules out the existence of nontrivial bundles only in the case $\lambda = 1$.

This completes the proof.

Remark 2.13 As is clear from Definition 2.10, toric bundles over Δ_k and with fiber $\widetilde{\Delta}$ of dimension r are determined by one vector $H = -(a_1, \ldots, a_r)$ that generates a circle action Λ_H on the fiber $(\widetilde{M}, \omega) := (M_{\widetilde{\Delta}}, \omega_\kappa)$. It is tempting to think that this bundle is trivial as long as this circle contracts in $\mathrm{Ham}(\widetilde{M}, \omega)$. But as we saw above, this clearly need not be so when $k > 1$. For example, if $\widetilde{\Delta} = \Delta_r$ then $H = (1, 0, \ldots, 0)$ generates a circle Λ_H that lies in $\mathrm{SU}(r+1)$. Since $\pi_1(\mathrm{SU}(r+1)) \cong \mathbb{Z}/(r+1)\mathbb{Z}$, we find that $\Lambda_{(r+1)H}$ contracts in $\mathrm{SU}(r+1)$ and hence in $\mathrm{Ham}(\mathbb{C}P^r)$. On the other hand, by Proposition 1.8 the bundle is nontrivial when $k > 1$.

To understand this notice that, if Λ_H contracts, then the classifying map $\mathbb{C}P^k \to B\mathrm{Ham}(\widetilde{M}, \omega))$ induces the null map on the 2–skeleton $\mathbb{C}P^1 \subset \mathbb{C}P^k$. When we contract this 2–sphere, we get further obstructions to the null homotopy of the whole map. These obstructions are explained by Kędra and McDuff [10, Theorem 1.1]. We show there that the existence of the contractible circle Λ_H in $\mathrm{Ham}(M, \omega)$ creates a nonzero

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[15] Here $c_1(\epsilon)$ is more correctly described as the first Chern class of the restriction of the tangent bundle TM to the 2–sphere $\Phi^{-1}(\epsilon)$. The paper [9] describes how to calculate $c_1(\epsilon)$ when $n = 2$. See also Remark 3.2 below.

element[16] (a kind of Samelson product) in $\pi_3(\mathrm{Ham}(M, \omega)) \cong \pi_4(B\mathrm{Ham}(M, \omega))$, that has nonzero pullback under the classifying map $\mathbb{C}P^k \to B\mathrm{Ham}(M, \omega)$ of this bundle. Thus the bundle is nontrivial. This makes it unlikely that the total space could ever be diffeomorphic to a product, though it does not completely rule it out without further argument.

3 Questions concerning finiteness

3.1 Finite number of toric structures

Proposition 3.1 *Let (M, ω) be a $2n$–dimensional symplectic manifold. Then, the number of distinct toric structures on (M, ω) is finite, where we identify equivariantly symplectomorphic actions.*

Proof Let $\Phi\colon M \to \mathbb{R}^n$ be the moment map of some toric structure on (M, ω) with image $\Phi(M) =: \Delta$. The number N of facets of the polytope Δ is $n + \dim H^2(M; \mathbb{R})$. We first show that Δ is determined by the classes $x_i \in H^2(M; \mathbb{Z})$, $i = 1, \ldots, N$, that are Poincaré dual to the divisors $\Phi^{-1}(F_i)$ corresponding to the facets F_i. Then we will show that these classes x_i lie in a finite subset of $H^2(M; \mathbb{Z})$.

To prove the first statement, number the x_i so that $x_1 x_2 \ldots x_n \neq 0$ and $e_1 := -x_1, \ldots, e_n := -x_n$ form a basis for $H^2(M; \mathbb{Z})$. Then the Stanley–Reisner presentation of $H^*(M)$ (cf [Equation (2-4)](#)) implies that the coordinates of the conormals for the other facets can be read off from the linear relations between the x_i, $i = 1, \ldots, N$. (Recall that we always assume that the conormals are primitive integral vectors, ie that their coefficients have no common factor.)

Therefore it remains to determine the support constants κ_i. Because of the translational invariance of Δ, the first n of these can be chosen at will. Once these are chosen, the other κ_i can be determined by looking at a suitably ordered set of edge lengths. To see this, let us set $\kappa_i = 0, i \leq n$, so that

$$v_0 := \bigcap_{i=1}^n F_i = (0, \ldots, 0).$$

Suppose that v_1 is connected to v_0 by the edge ϵ_j that is transverse to F_j at v_0 for some $j \leq n$. Then $\epsilon_j = \bigcap_{i \in I} F_i$ where $I := \{i \leq n, i \neq j\}$, and its affine length is

$$\ell(\epsilon_j) = \int_{\epsilon_j} [\omega] = \int_M x_I[\omega], \quad \text{where } x_I := \prod_{i \in I} x_i.$$

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[16] It is detected by a characteristic class very similar to those used by Shelukhin; cf [Equation (4-4)](#) below.

If the other endpoint of ϵ_j is transverse to F_k, then κ_k is determined by $\ell(\epsilon_j)$. Proceeding in this way, we can find κ_k first for all facets joined to v_0 by one edge, then for those joined to v_0 by a path consisting of two edges, and so on.

Therefore it suffices to show that there are a finite number of possibilities for these classes $x_i \in H^2(M; \mathbb{Z})$, $i = 1, \ldots, N$. Following a suggestion of Borisov[17], let us look at the Hodge–Riemann form on $H^2(M; \mathbb{R})$ given by

$$\langle \alpha, \beta \rangle := \int_M \alpha \, \beta \, \omega^{n-2}.$$

By the Hodge index theorem, this is nondegenerate of type $(1, -1, \ldots, -1)$; in other words it is negative definite on the orthogonal complement to $[\omega]$. (A nonanalytic proof of this result for toric manifolds may be found in Timorin [30].) Write $x_i = y_i + r_i[\omega]$ where $\langle y_i, \omega \rangle = 0$ and $r_i \in \mathbb{R}$. Then each $r_i > 0$, since

$$r_i \langle \omega, \omega \rangle = \langle x_i, \omega \rangle = \int_{F_i} \omega^{n-1} > 0$$

because it is a positive multiple of the ω–volume of the Kähler submanifold $\Phi^{-1}(F_i)$. Further, because $c_1(M) = \sum x_i$ by Davis and Januszkiewicz [4], we have

$$\sum_i r_i \langle \omega, \omega \rangle = \sum_i \int_M x_i \, \omega^{n-1} = \int_M c_1(M) \, \omega^{n-1} =: C \langle \omega, \omega \rangle.$$

Therefore, each $r_i < C$, so that $\sum r_i^2 < NC^2$. Finally, because $c_1^2 - 2c_2 = \sum x_i^2$ we have

$$A := \int_M (c_1^2 - 2c_2) \, \omega^{n-2} = \sum_i \langle x_i, x_i \rangle = \sum_i r_i^2 + \sum_i \langle y_i, y_i \rangle.$$

Since each $\langle y_i, y_i \rangle \leq 0$ by the Hodge index theorem, we find that

$$0 \leq -\sum_i \langle y_i, y_i \rangle \leq NC^2/V^2 - A.$$

Therefore the integral classes x_i lie in a bounded subset of $H^2(M; \mathbb{R})$. Thus they are all contained in a finite subset of $H^2(M; \mathbb{Z})$. $\qquad\square$

Remark 3.2   There are various elementary proofs of finiteness when $[\omega]$ is integral. Perhaps the simplest is again due to Borisov, who pointed out the following argument. Normalize Δ so that one vertex is at the origin and the edges from it point along the positive coordinate axes. Denote by S_i the $(n - 1)$ simplex in the hyperplane $\xi_i = 0$ with edges of unit length, and suppose that $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ is some vertex of Δ.

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[17] private communication

Then the volume of the cone spanned by S_i and v is $a_i/n!$. Since this cone lies in Δ, the coordinates of v are bounded by the volume V of Δ. Therefore, the vertices lie in a bounded subset of the lattice $\mathbb{Z}^n$ whose size is determined by $V = (1/n!)\int_M \omega^n$.

Another approach is first to note that the number and affine lengths of the edges are bounded by some constants K, L because the sum of their lengths is $\int_M c_{n-1}\omega$ and each edge has length at least 1. It then follows that the geometry of each edge ϵ is bounded. To see this, note that this geometry is determined by the Chern numbers $c_{F_i}(\epsilon)$ of the normal line bundle to F_i along $\Phi^{-1}(\epsilon)$, where $F_1, \ldots, F_{n-1}$ are the facets containing ϵ. Because each edge has length between 1 and L and each 2–face is a convex polygon, we must have $c_{F_i}(\epsilon) \leq L$ for each such i. But $\int_{\Phi^{-1}(\epsilon)} c_1(M) = 2 + \sum_i c_{F_i}(\epsilon)$. It follows that the $c_{F_i}(\epsilon)$ are bounded above and below. Since Δ is made by putting together at most K edges, there are again only finitely many possibilities for Δ.

Proof of Theorem 1.2 The proof of Proposition 3.1 used only cohomological facts about M. The number of facets of Δ is determined by the rank of $H^2(M)$. We also needed to know $\int_M \omega^n$, $\int_M c_1\omega^{n-1}$ and $\int_M (c_1^2 - 2c_2)\,\omega^{n-2}$. But, once one knows the classes c_1, c_2 and $[\omega]$, these integrals are determined by the integral cohomology ring. This holds because there is a unique generator u of $H^{2n}(M;\mathbb{Z})$ such that $\omega^n = \lambda u$ for some $\lambda > 0$, and then an integral such as $\int_M c_1\omega^{n-1}$ is equal to $a \in \mathbb{R}$, where $c_1\omega^{n-1} = a\,u$. This completes the proof. $\qquad\square$

3.2 Manifolds with more than one toric structure: Blow ups

One easy to way to construct different toric structures on a symplectic manifold (M, ω) is by blowing up. Suppose given a toric structure on (M, ω) with moment map $\Phi\colon M \to \Delta$. As we show in more detail in [23, Section 3], blowing up along a face $f = f_I$ of codimension $k = |I| \geq 2$ adds a new facet F_0 to the polytope with conormal $\eta_0 = \sum_i \eta_i$ and constant $\kappa_0 = \sum_i \kappa_i - \varepsilon$. The new moment polytope Δ_f is $\Delta \smallsetminus Y_{f,\varepsilon}$, where

$$Y_{f,\varepsilon} = \big\{\xi \in \Delta : \langle \eta_0, \xi \rangle > \kappa_0 \big\}.$$

This is a smooth moment polytope for small $\varepsilon > 0$. The corresponding symplectic manifold $(\widetilde{M}_f, \widetilde{\omega}_\varepsilon)$ is formed from (M, ω) by excising $\Phi^{-1}(Y_{f,\varepsilon})$ and collapsing the boundary along its characteristic flow. This is an example of symplectic cutting; cf Lerman [12]. If $f = v$ is a vertex, we call the resulting toric manifold a *one point toric blow up of weight ε*. The underlying symplectic manifold is called the[18] one point blow up of (M, ω).

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[18] A subtle point is concealed here. For most manifolds it is not known whether there is $\varepsilon_0 > 0$ such that the space of symplectic embeddings of a ball of size $\varepsilon \leq \varepsilon_0$ into (M, ω) is connected. Since each

Lemma 3.3 Let (M, ω, T) be a toric manifold with moment polytope Δ. Then there is $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ all of its one point toric blow ups of weight ε are symplectomorphic.

Sketch of proof In this case $\Phi^{-1}(Y_{f,\varepsilon})$ is the image of a standard ball $B^{2n}(\varepsilon)$ of radius $\sqrt{\varepsilon/\pi}$. Because the Hamiltonian group acts transitively on M, it is easy to see that given any two symplectic embeddings $B^{2n}(\varepsilon') \to M$ one can find $\varepsilon_0 \in (0, \varepsilon')$ such that their restrictions to $B^{2n}(\varepsilon_0)$ – and hence to any smaller ball – are isotopic. This implies that the corresponding blow up manifolds are symplectomorphic; see for example [19]. To complete the proof, it remains to observe that there are a finite number of toric blow ups. $\qquad\square$

Similarly, if one blows up along faces f, f' for which the inverse images $\Phi^{-1}(f)$ and $\Phi^{-1}(f')$ are Hamiltonian isotopic, the resulting blow ups are symplectomorphic for small enough ε.

Remark 3.4 There are many interesting questions here about exactly how big one can take ε_0 to be; cf the discussion in Pelayo [25, Section 3].

Definition 3.5 Two vertices of Δ are said to be *equivalent* if there is an integral affine self-map of Δ taking one to the other. Further we define $\mathrm{Aff}(\Delta) := \mathrm{Aff}(\Delta(\kappa))$ to be the group of all integral affine self-maps of $\Delta(\kappa)$, and $\mathrm{Aff}_0(\Delta)$ to be the subgroup that is generated by reflections that interchange equivalent facets.

As explained in the discussion after Definition 1.12, the elements of $\mathrm{Aff}_0(\Delta)$ lift to elements in the Hamiltonian group of (M, ω) while the elements in $\mathrm{Aff}(\Delta) \smallsetminus \mathrm{Aff}_0(\Delta)$ lift to sympectomorphisms that are not isotopic to the identity; in fact, because they interchange nonequivalent facets, they act nontrivially on $H^2(M)$. (See also Masuda [13].) Observe that $\mathrm{Aff}(\Delta)$ depends on κ, while $\mathrm{Aff}_0(\Delta)$ does not. (In the terminology of [22], $\mathrm{Aff}_0(\Delta)$ consists of robust transformations.) In particular, the question of which vertices of $\Delta(\kappa)$ are equivalent depends on κ. Explicit examples of this are provided by blow ups of $\mathbb{C}P^2$. Note also that, when κ is generic, the two groups $\mathrm{Aff}(\Delta(\kappa))$ and $\mathrm{Aff}_0(\Delta(\kappa))$ coincide because different homology classes in $H_{2n-2}(M)$ are distinguished by the ω–volume of their representatives.

The next corollary follows by combining these remarks with Lemma 3.3.

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such embedding gives rise to a symplectic blow up (see [19]), it is not known whether all sufficiently small one point blow ups are symplectomorphic. In the toric case, this problem does not arise since we have given a unique way to do such a blow up at each vertex. Even if one allows an ostensibly more general process by using equivariant embeddings, the invariance of the image (see Pelayo [25, Lemma 2.1]) shows that this gives nothing new.

Corollary 3.6 *Let (M, ω, T) be a toric manifold whose moment polytope has k pairwise nonequivalent vertices. Then there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the one point ε–blow up of (M, ω) has at least k different toric structures.*

Proof Let Δ be the moment polytope of (M, ω, T), and let Δ'_ε and Δ''_ε be the polytopes obtained by blowing up Δ at two nonequivalent vertices v' and v'' by some amount $\varepsilon > 0$. Delzant's theorem [5] states that a toric manifold is determined up to equivariant symplectomorphism by the integral affine equivalence class of its moment polytope. Therefore if the toric manifolds corresponding to these two blow ups are equivariantly symplectomorphic there is an integral affine transformation A_ε taking Δ'_ε to Δ''_ε. We may suppose that ε is less than half the length of the shortest edge of Δ. Then the only facet of Δ'_ε with all edges of length ε its the exceptional divisor F'_0. Since a similar statement holds for Δ''_ε, the facet $A_\varepsilon(F'_0)$ must be the exceptional divisor F''_0 of Δ''_ε. But the quantities η_i and κ_i that determine the other facets of Δ'_ε and Δ''_ε are independent of ε and must be permuted by A_ε. Hence A_ε induces a self-map of Δ which is independent of ε and takes v' to v''. Thus v' is equivalent to v'', contrary to the hypothesis. Thus the toric manifolds obtained by blowing up two nonequivalent vertices are not equivariantly symplectomorphic. One the other hand, if $\varepsilon > 0$ is sufficiently small the underlying manifolds are symplectomorphic by Lemma 3.3. □

We begin the proof of Proposition 1.4 by considering the following special case.

Lemma 3.7 *Suppose that Δ is a nontrivial and generic bundle over Δ_1. Then Δ has at least two inequivalent vertices.*

Proof By Definition 2.10 the structure of Δ is determined by the vector $A = -(a_1, \ldots, a_r)$ and the length h. The bottom and top facets F_{N+1} and F_{N+2} are equivalent, and, as explained in the proof of Proposition 1.13, the numbers $h - a_1, \ldots h - a_r$, $h - a_{r+1}$ (where $a_{r+1} := 0$ as in Section 2.4) are the lengths of the vertical edges of Δ, ie those that are parallel to e_{r+1}, going between the bottom and top facets. Since $A \neq 0$, there are two vertices v_1, v_2 on the bottom facet F_{N+1} that are the endpoints of vertical edges of different lengths. Since Δ is generic, we can choose h so that every affine transformation of Δ must preserve the set of vertical edges (possibly changing their orientation), because there are no other edges of precisely these lengths. Hence v_1 and v_2 cannot be equivalent. □

Remark 3.8 It is possible that there are just two equivalence classes of vertices. For instance Δ might be a Δ_2–bundle over Δ_1 with $A = -(0, 1)$.

The following lemma generalizes Theorem 1.20 of [22] which imposes the extra condition that $F_I := \bigcap_{i \in I} F_i = \varnothing$ for all equivalence classes I and concludes that Δ is a product of simplices.

Lemma 3.9 *Suppose that each facet F_i of Δ is equivalent to some other facet F_j. Then, if κ is generic, either $\Delta(\kappa)$ is a product of simplices or $\Delta(\kappa)$ has at least two nonequivalent vertices.*

Proof If Δ has dimension 2, then it must either be Δ_2 or $\Delta_1 \times \Delta_1$. Now assume inductively that the lemma holds for all polytopes of dimension $\leq n - 1$, where $n := \dim \Delta$.

Suppose first that there is some equivalence class I with $|I| \geq 3$ and renumber the facets so that $\{1, 2\} \subset I$. Then by Proposition 3.17 of [22], Δ is the 1–fold expansion of the facet F_2 along its facet F_{12}. In particular F_2 is pervasive, ie meets all other facets. Lemma 3.27 in [22] states that, when F_2 is pervasive, two facets F_j, F_k, where $j, k \neq 2$, are equivalent in Δ exactly if the facets F_{2j}, F_{2k} are equivalent in F_2. Because $|I| \geq 3$, this implies that there is $j \in I \smallsetminus \{1, 2\}$ such that $F_{21} \sim F_{2j}$. Hence, the equivalence classes of facets of F_2 all have more than one element. Therefore, by the inductive hypothesis F_2 is either a product of simplices or has at least two nonequivalent vertices.

In the former case, Δ is the expansion of $F_2 = \Delta_{k_1} \times \cdots \times \Delta_{k_p}$ along a facet F_{12} that we may assume to have the form $F \times \Delta_{k_2} \times \cdots \times \Delta_{k_p}$ for some facet F of Δ_{k_1}. It is now easy to check from the definition of expansion that

$$\Delta = \Delta_{k_1+1} \times \cdots \times \Delta_{k_p}.$$

In the latter case, there are at least two vertices v_1, v_2 of F_2 that are not equivalent under $\mathrm{Aff}(F_2) = \mathrm{Aff}_0(F_2)$. It suffices to show that they are not equivalent under $\mathrm{Aff}_0(\Delta)$. Suppose not, and let $\phi \in \mathrm{Aff}_0(\Delta)$ be such that $\phi(v_1) = v_2$. Then $\phi(F_2) \neq F_2$. But because $\phi \in \mathrm{Aff}_0(\Delta)$ we must have $\phi(F_2) \sim F_2$. Let $\alpha \in \mathrm{Aff}_0(\Delta)$ be the reflection that interchanges the facets $\phi(F_2)$ and F_2. Then $\alpha \circ \phi(F_2) = F_2$. But $v_2 \in F_2 \cap \phi(F_2)$ is fixed by α. Hence v_1 and v_2 are equivalent in F_2, contrary to hypothesis. This completes the proof when there is some equivalence class with > 2 facets.

It remains to consider the case when all equivalence classes have two elements. Suppose there is such an equivalence class $I = \{1, 2\}$ with $F_{12} = \varnothing$. Then again each equivalence class of facets of F_2 has at least 2 elements, and [22, Proposition 3.17] implies that Δ is an F_2–bundle over Δ_1. If this bundle is nontrivial, then Lemma 3.7 implies that Δ has at least 2 nonequivalent vertices. If it is trivial, then either F_2 (and

hence also Δ) is a product of simplices, or we can use the two nonequivalent vertices of F_2 supplied by the inductive hypothesis to find two such vertices of Δ.

The remaining possibility is that each equivalence class consists of precisely two intersecting facets F_i, F_i', $1 \leq i \leq \ell$. In this case, the proof is completed by Lemma 3.10 below. $\qquad\square$

Lemma 3.10 *Suppose that Δ is a polytope such that each facet is equivalent to at most one other. Suppose further that each pair of equivalent facets intersects. Then, for generic κ, the polytope $\Delta(\kappa)$ has at least five nonequivalent vertices.*

Proof Pick one facet F_i, $1 \leq i \leq \ell$, from each equivalence class with more than one element, and denote the other facets in these equivalence classes by F_i' where $F_i \sim F_i'$. We first claim that the face $f := F_1 \cap \cdots \cap F_\ell$ has the following properties:

- (a) $f \neq \varnothing$.
- (b) Δ is made from f by expanding once along each of the facets $F_i' \cap f$.
- (c) No two facets of f are equivalent.
- (d) f has at least 5 vertices.

We prove this by induction on ℓ. If $\ell = 1$, (a) is clear and (b) holds by [22, Proposition 3.17] (which states that when $F_1 \cap F_1' \neq \varnothing$, the polytope Δ is the expansion of F_1 along $F_1 \cap F_1'$). Therefore F_1 is pervasive, so that we can deduce (c) by applying the result [22, Lemma 3.27] which is quoted above. Finally note that the only polytopes with ≤ 4 vertices are the simplices Δ_k, $k \leq 3$, and the trapezoid. Since these all fail condition (c), (d) must hold. Thus these claims hold when $\ell = 1$. If $\ell > 1$, apply the inductive hypothesis to F_1 and use [22, Proposition 3.17].

Now consider the face f as a polytope in its own right, and pick any two distinct vertices v_1, v_2 of f. Because κ is generic, every self-equivalence of f (resp. Δ) acts trivially on homology and so belongs to $\mathrm{Aff}_0(f)$ (resp. $\mathrm{Aff}_0(\Delta)$). Hence condition (c) implies that no two vertices of f are equivalent as vertices of f. It follows easily that they cannot be equivalent in Δ. For because the self-equivalences ϕ of Δ belong to $\mathrm{Aff}_0(\Delta)$ they are products of the commuting reflections ρ_i, where ρ_i interchanges the pair F_i, F_i' and acts as the identity on all other facets. If v_1, v_2 are equivalent in Δ, we may choose $\phi \in \mathrm{Aff}_0(\Delta)$ which is a product of a minimal number of the ρ_i so that $\phi(v_1) = v_2$. If ϕ interchanges F_i, F_i' then, as above, both v_1 and v_2 lie in $F_i \cap F_i'$. But then $\rho_i \circ \phi$ is a shorter product that takes v_1 to v_2, a contradiction. Thus Δ has at least 5 inequivalent vertices. $\qquad\square$

Proof of Proposition 1.4 We must show that the one point blow up of the toric manifold (M, ω) has at least two toric structures, provided that ω is generic and M is not a product of projective spaces with the product toric structure. By Corollary 3.6, it suffices to show that if all vertices of $\Delta(\kappa)$ are equivalent for some generic κ then Δ is a product of simplices. This will follow from Lemma 3.9 if we show that each facet of Δ is equivalent to at least one other facet. But given a facet F and a vertex $v' \notin F$, by hypothesis v' is equivalent to every vertex $v \in F$. Hence there is $\phi \in \mathrm{Aff}_0(\Delta)$ that takes v to v'. Therefore $\phi(F)$, which contains v', cannot equal F. Therefore F is equivalent to at least one other facet, namely $\phi(F)$. $\qquad\square$

4 Full mass linearity

We begin by improving some results from [22] and then introduce the idea of full mass linearity.

4.1 Some properties of mass linear functions

Let M be a toric $2n$–dimensional manifold with moment polytope Δ, where

$$\Delta = \{\xi \in \mathfrak{t}^* : \langle \eta_i, \xi \rangle \leq \kappa_i, \ i = 1, \ldots, N\}.$$

Consider the volume $V(\kappa)$ of Δ as a function of its support numbers κ_i, $i = 1, \ldots, N$. The results of Timorin [30] show that the algebra $H^*(M; \mathbb{R})$ is isomorphic to $\mathbb{R}[\partial_1, \ldots, \partial_N]/I(V)$ where we interpret ∂_i as the differential operator $\partial/\partial\kappa_i$ and $I(V)$ consists of all differential operators with constant coefficients that annihilate the polynomial V; cf the discussion at the beginning of Section 2.6 in [30]. His argument is the following. He observes that the translational invariance of V implies that $\sum \langle \xi, \eta_j \rangle \partial_j V = 0$ for all $\xi \in \mathfrak{t}^*$. Further, he shows that $\partial_I V =: V_I$ is the volume

$$\mathrm{vol}\, F_I := \frac{1}{k!} \int_{\mathrm{PD}(F_I)} \omega^k$$

of the Kähler submanifold $\Phi^{-1}(F_I)$ of the face F_I. (Here M is equipped with the natural symplectic form $\omega = \omega(\kappa)$ whose integral over the 2–sphere corresponding to each edge is the affine length of that edge.) It follows that $\partial_I V = 0$ whenever $F_I = \varnothing$. He then shows in [30, Theorem 2.6.2] that these relations generate $I(V)$. It follows immediately that his algebra is isomorphic to the Stanley–Reisner presentation for $H^*(M)$ described in Equation (2-4).

Note that this isomorphism takes ∂_i to the Poincaré dual of the facet F_i. Hence the first Chern class of M is represented by the operator $\sum_i \partial_i$. More generally, the k–th Chern class is represented by the operator $\sum_{|I|=k} \partial_I$.

Consider an element $H \in \mathfrak{t}$. By taking inner products, we get an induced function, also denoted H, from $\Delta(\kappa) \to \mathbb{R}$. This is said to be *mass linear* if $H(B_n)$ is a linear function of the κ_i, where B_n is the barycenter of $\Delta(\kappa)$. Thus there are constants $\gamma_i \in \mathbb{R}$ such that $H(B_n) = \sum \gamma_i \kappa_i$. It is proved in [22, Lemma 3.19] that in this situation the vector $H \in \mathfrak{t}$ is precisely $\sum \gamma_i \eta_i$. Thus, if H is mass linear, there are constants γ_i such that

$$(4\text{-}1) \qquad H(B_n) = \sum \gamma_i \kappa_i \quad \text{and} \quad H = \sum \gamma_i \eta_i.$$

If μ denotes the moment $\int_\Delta H \, d\text{Vol}$ of H we have $\mu = H(B_n)V$. Generalizing Timorin's ideas, we proved the following result in [22, Proposition 2.2].

Lemma 4.1 *For any $H \in \mathfrak{t}$ the face F_I has volume $V_I := \partial_I V$ and H–moment $\mu_I = \partial_I \mu$.*

Therefore, in the mass linear case we have

$$\mu_I = H(B_n)V_I + \sum \gamma_i V_{I \smallsetminus i}.$$

The following combinatorial result improves some of the conclusions of [22]. We denote by ϵ_{vj} the directed edge that starts at the vertex v and ends transversely to F_j.

Proposition 4.2 *Suppose that $H \in \mathfrak{t}$ is mass linear and $H(B_n) = \sum \gamma_j \kappa_j$. Then:*

(i) $\sum_j \gamma_j = 0$.

(ii) $\sum \ell(\epsilon_{vj}) \gamma_j = 0$ where the sum is over all directed edges ϵ_{vj}, and $\ell(\epsilon)$ denotes the affine length of ϵ.

Proof (i) Fix a vertex $v := F_I$. For each $i \in I$ there is a unique edge $_i\epsilon_v = F_{I \smallsetminus i}$ that starts at v transversely to F_i. Its other endpoint is transverse to a unique facet F_j where $j \notin I$. (Thus $_i\epsilon_v = \epsilon_{vj}$ in the previous notation.) For each $j \notin I$ define $I(j) \subset I$ to be the (possibly empty) set of i such that the second endpoint of $_i\epsilon$ is transverse to F_j. Then the sets $I(j)$, $j \notin I$, form a partition of I. Correspondingly, the sets $J^v(j) := I(j) \cup \{j\}$, $j \notin I$, form a partition of $\{1, \ldots, N\}$. Therefore, (i) will follow if we show that

$$\sum_{i \in J^v(j)} \gamma_i = 0 \quad \text{for all } j \notin I.$$

But this holds by the following calculation. Fix $j \notin I$ and let $K(j) := I \cup \{j\}$. We first claim that for each $k \in K$, the vertex $F_{K(j) \smallsetminus k}$ is nonempty exactly if $k \in J^v(j)$. This is clear if $k = j$. Otherwise $k \in I$ and $F_{K \smallsetminus k} = F_{I \smallsetminus k} \cap F_j$ is nonempty exactly

if the second endpoint of $_k\epsilon_v$ lies on F_j, in other words exactly if $k \in I(j)$. Using Lemma 4.1 and the fact that the intersection of every set of $n+1$ facets is empty, we now find that

$$0 = \mu_{K(j)} = \partial_K \left(\sum \gamma_i \kappa_i \, V \right) = \sum_{i \in K} \gamma_i \, V_{K \smallsetminus i} = \sum_{i \in J^v(j)} \gamma_i.$$

(ii)   Given $K \subset \{1, \dots, N\}$ with $|K| = n+1$, define $\mathcal{E}(K)$ to be the set of all edges F_L, where $L := L_{s,t} := K \smallsetminus \{s,t\}$ is an edge with endpoints $w_s := F_{K \smallsetminus \{s\}}$ and $w_t := F_{K \smallsetminus \{t\}}$. These sets partition the set of all edges of Δ, since for any edge ϵ the set $K(\epsilon) := \{i : F_i \cap \epsilon \neq \varnothing\}$ has precisely $n+1$ elements, and $\epsilon \in \mathcal{E}(K(\epsilon))$.

If $\mathcal{E}(K) \neq \varnothing$, pick any directed edge $\epsilon_{vj} \in \mathcal{E}(K)$. Then $K = I \cup \{j\}$, in the language of (i). Consider any edge $L_{s,t} \in \mathcal{E}(K)$. If $s = j$ then the edge has endpoints $v = w_j$ and $w_t \in F_j$, and so is the edge previously called $_s\epsilon_v$. Otherwise $w_s, w_t \in F_j$. Observe that w_s and w_t are joined by the edge $F_{(I \smallsetminus \{s,t\}) \cup \{j\}} \in \mathcal{E}(K)$. It follows that the edges $L_{s,t}$ in $\mathcal{E}(K)$ form a complete graph. Moreover these are the edges of the dimension m face $f := \bigcap_{s \in I \smallsetminus I(j)} F_s$, where $m = |I(j)|$. Hence this face is a simplex, so that all its edges have the same length λ.

We need to calculate the sum of $\ell(\epsilon_{vj}) \gamma_j$ over directed edges. But this equals the sum of $\ell(\epsilon)(\gamma_s + \gamma_t)$ over unoriented edges, where ϵ joins F_s to F_t. We proved in (i) that

$$\sum_{L_{s,t} \in \mathcal{E}(K(j))} (\gamma_s + \gamma_t) = 0.$$

Since the sets $\mathcal{E}(K(j))$ partition the edges of Δ, this proves (ii). □

Remark 4.3   The above proof shows that the coefficients γ_i of a mass linear function satisfy many enumerative identities, that is, identities that depend only on the combinatorics of Δ. Thus the existence of a mass linear function imposes many restrictions on the combinatorics of Δ. For example, if there is no edge from the vertex v to F_j, then the equivalence class $J(j) = I(j) \cup \{j\}$ in (i) consists only of $\{j\}$, and we conclude that $\gamma_j = 0$. This reproves the result in [22, Proposition A.2] that every asymmetric facet is powerful. Using this, one can immediately deduce that many polytopes have no nonzero mass linear functions H. For example, no polygon with more than four edges has such an H.

As another example, suppose that Δ is any polytope other than a simplex and blow it up at one of its vertices v_0 to obtain Δ'. Then, because Δ has $> n+1$ vertices, there is a vertex w in Δ that is not connected to v_0 by an edge and so is not connected to the exceptional divisor F_0' of Δ' by an edge. Therefore if H is a nonzero mass linear function on Δ', the coefficient γ_0 in the expression $H(B_n(\Delta'))$ must vanish; in other

words the exceptional divisor F_0' is symmetric. A similar argument shows that every H–asymmetric facet F_i' of Δ' (ie one with $\gamma_i \neq 0$) must meet F_0'. For otherwise, the corresponding facet F_i of Δ does not meet v_0. Since Δ is not a simplex there is an edge ϵ from v_0 which does not meet F_i. In the blow up, this edge meets F_0' in a vertex v' which is not joined to F_i' by an edge. (There is only one edge from v' that does not lie in F_0', namely the blow up of ϵ.) Hence F_i' is not powerful, contradicting our previous results.

This discussion is taken much further in the papers [22; 23], that classify all mass linear functions on polytopes of dimensions ≤ 4. However, rather than focussing on combinatorial identities these papers analyze the properties of the symmetric and asymmetric facets.

4.2 Full mass linearity

The following condition was suggested by the work of Shelukhin which is discussed further in Section 4.4 below.

Definition 4.4 Let $H \in \mathfrak{t}$. For each $s = 0, 1, \ldots, n$, let

$$V^s := \sum_{|I|=n-s} V_I$$

be the sum of the volumes of the faces of dimension s, and let

$$\mu^s := \sum_{|I|=n-s} \mu_I$$

be the sum of the corresponding H–moments. Define B_s to be the center of mass of the facets $F^s := \bigcup_{|I|=n-s} F_I$. Thus B_n is the usual center of mass and B_0 is the average of the vertices. Then we say that H is *fully mass linear* if $H(B_s) = H(B_n)$ for all $s = 0, \ldots, n-1$.

Note the following points.

• Since B_0 is clearly a linear function of the support numbers κ_i, every fully mass linear function is mass linear.

• Every inessential function is fully mass linear since the barycenters B_s must lie on all planes of symmetry of Δ, ie they are invariant under the action of elements in $\mathrm{Aff}_0(\Delta)$.

• We explain in Section 4.4 Shelukhin's argument that the quantities $H(B_s) - H(B_n)$ are values of certain real-valued characteristic classes for Hamiltonian bundles with

fiber $(M_\Delta, \omega_\kappa)$. It follows that *the function H is fully mass linear whenever Λ_H has finite order in $\pi_1\big(\mathrm{Ham}(M_\Delta, \omega_\kappa)\big)$.* In fact, there is one such characteristic class I_β for each product c_β of Chern classes on M. However, as we show in Corollary 4.18, the vanishing of these classes I_β gives no new information.

We continue our discussion by explaining precisely what full mass linearity means.

Lemma 4.5 Let $H \in \mathfrak{t}$ be mass linear with $H(B_n) = \sum \gamma_i \kappa_i$. Then $H(B_{n-r}) = H(B_n)$ exactly if the identity

$$(*_r) \qquad\qquad \sum_{i, J : i \in J, |J| = r-1} \gamma_i V_J = 0,$$

holds, where we interpret $(*_1)$ to be the identity $\sum \gamma_i = 0$. In particular, H is fully mass linear exactly if $(*_r)$ holds for $r = 1, \dots, n$.

Proof First consider μ^{n-1}. By Timorin [30] we have $\mu^{n-1} = \sum_i \partial_i \mu$. Therefore, because H is mass linear,

$$
\begin{aligned}
H(B_{n-1}) V^{n-1} = \mu^{n-1} &= \sum_i \partial_i \mu \\
&= \sum_i H(B_n) V_i + \sum_i \partial_i \Big(\sum \gamma_j \kappa_j\Big) V \\
&= H(B_n) V^{n-1} + \Big(\sum \gamma_i\Big) V.
\end{aligned}
$$

This proves the case $r = 1$ of the first statement. Note also that because $\sum \gamma_i = 0$ by Proposition 4.2 (i), we always have $H(B_{n-1}) = H(B_n)$.

More generally, since $\sum \gamma_i = 0$, we have

$$
\begin{aligned}
H(B_{n-r}) V^{n-r} = \mu^{n-r} &= \sum_{|I|=r} \partial_I \Big(\sum \gamma_j \kappa_j V\Big) \\
&= H(B_n) V^{n-r} + \sum_{i \notin J, |J| = r-1} \gamma_i \partial_J V \\
&= H(B_n) V^{n-r} + \Big(\sum_i \gamma_i\Big) \sum_{|J| = r-1} \partial_J V - \sum_{i \in J, |J| = r-1} \gamma_i \partial_J V \\
&= H(B_n) V^{n-r} - \sum_{i \in J, |J| = r-1} \gamma_i V_J.
\end{aligned}
$$

Therefore we see that the identity $H(B_{n-r}) = H(B_n)$ holds exactly if $(*_r)$ holds. This proves the first statement. The second is clear. □

Remark 4.6 The identity $(*_{n+1})$ is $\sum_{i\in J,|J|=n} \gamma_i V_J = 0$, which is equivalent to saying that $\sum_i N_i \gamma_i = 0$ where N_i is the number of vertices in the facet F_i. But this holds for all mass linear functions, as one can see by computing $0 = \sum_{|I|=n+1} \partial_I \left(\sum \gamma_j \kappa_j V \right)$ as above. Another way to calculate this is to think of it as a sum over directed edges, namely $\sum_{\epsilon_{vj}} \gamma_j$. We proved that this sum vanishes in the course of proving part (ii) of Proposition 4.2 since the lengths turned out to be irrelevant.

Proposition 4.7 *The following conditions are equivalent:*

- *H is mass linear.*
- *$H(B_n) = H(B_0)$.*
- *$H(B_n) = H(B_{n-1}) = H(B_0)$.*

Proof If H is mass linear, then we saw in the proof of Proposition 4.7 that $H(B_n) = H(B_{n-1})$ because $\sum \gamma_i = 0$. Further, the difference between $\mu^0 = H(B_0)V^0$ and $H(B_n)V^0$ is

$$\sum_{i\in J,|J|=n-1} \gamma_i V_J = \sum_{i\in J,|J|=n-1} \gamma_j \ell(F_J).$$

But we saw in Proposition 4.2(ii) that this sum vanishes. Hence the first condition implies the second and third.

But we noted earlier that $H(B_0)$ is a linear function of the κ_j. Hence the second condition implies the first. $\square$

Remark 4.8 (i) This argument shows that the identity $H(B_n) = H(B_0)$ implies $H(B_n) = H(B_{n-1})$. Thus if H is mass linear these three points always lie on the same level set of H. In contrast, Shelukhin [29] showed in the monotone case that the three points B_n, B_{n-1} and B_0 are collinear.

(ii) The r–th equation in Lemma 4.5 corresponds to a condition on μ^s, where $s = n - r$ that we calculate assuming that H is mass linear. Therefore this equation is not equivalent to the fact that $H(B_{n-r}) = H(B_n)$. In fact, we give an example in Remark 4.10 (ii) below showing that the identities $(*_r)$, $r = 1, \ldots, n$ do not by themselves imply mass linearity.

Corollary 4.9 *Suppose that the mass linear function H has coefficients γ_i as in Equation (4-1). Then H is fully mass linear exactly if $\sum \gamma_i \partial_i^k V = 0$ for all $k = 1, \ldots, n$.*

Proof These identities are equivalent to $(*_r)$, $r = 1, \ldots, n$ because the functions $\sum x_i^k$ form a basis for the symmetric polynomials over $\mathbb{Q}$. $\square$

Remark 4.10 (Geometric interpretation of equations $(*_2)$) (i) Inessential mass linear functions H are generated by vectors $\xi_H \in \mathfrak{t}^*$ with the property that the facets

$$\{F_i : \langle \eta_i, \xi_H \rangle \neq 0\}$$

are all equivalent. In this case, we saw that

$$H(B_n(\kappa)) = \sum \langle \eta_i, \xi_H \rangle \kappa_i, \quad \text{as well as} \quad H = \sum \langle \eta_i, \xi_H \rangle \eta_i.$$

Moreover, if H is elementary, ie of the form $\eta_i - \eta_j$, then by [22, Lemma 3.4] there is an affine reflection symmetry of Δ_H that interchanges the two facets F_i and F_j preserving the transverse vector ξ_H.

We claim that a very similar statement holds for mass linear functions H that satisfy $(*_2)$. In other words, for each such function there is a vector $\xi_H \in \mathfrak{t}^*$ such that

(4-2) $$\qquad\qquad \gamma_i = \langle \eta_i, \xi_H \rangle \quad \text{where } H(B_n(\kappa)) = \sum \gamma_i \kappa_i.$$

To see this, observe that equation $(*_2)$, $\sum_i \gamma_i V_i = 0$, says that the operator $\sum \gamma_i \partial_i$ is in the annihilator $I(V)$. Timorin showed that $I(V)$ is generated by additive relations of the form $\sum \langle \eta_i, \xi \rangle \partial_i = 0$ where $\xi \in \mathfrak{t}^*$, as well as some multiplicative relations $\partial_I = 0$. Since $\sum \gamma_i \partial_i$ is linear, it has to correspond to some vector $\xi_H \in \mathfrak{t}^*$. Note that the first part of Equation (4-2) shows that ξ_H must be parallel to all symmetric facets. However, it is not clear whether there is further geometric significance to this vector.

This observation explains the condition $\sum \gamma_i a_i = 0$ in Lemma 4.11 below. For in this case $\xi_H = -(\gamma_1, \dots, \gamma_k, 0) \in \mathbb{R}^{k+1} \equiv \mathfrak{t}^*$ while the two facets with conormals $\eta_{n+1} = (0, \dots, 0, 1)$ and $\eta_{n+2} = (-a_1, \dots, -a_k, 1)$ are symmetric.

(ii) In general, one cannot reduce the mass linearity condition for $H := \sum_i \langle \eta_i, \xi_H \rangle \eta_i$ to any obvious condition on ξ_H. Consider for example the $\Delta_1 \times \Delta_1$–bundle over Δ_1 with conormals

$$\eta_1 = -e_1, \quad \eta_2 = -e_2, \quad \eta_3 = e_1, \quad \eta_4 = e_2, \quad \eta_5 := -e_3, \quad \eta_6 := e_3 - v,$$

where $v = (a_1, a_2, 0)$ as in Lemma 4.11. For generic (a_1, a_2) (ie $a_1 a_2 \neq 0$, and $a_1 - a_2 \neq 0$), this has just one pair of equivalent facets, namely the base facets F_5, F_6. Since the other facets are neither pervasive nor flat, [22, Theorem 1.10] implies that Δ has no mass linear functions for which F_5, F_6 are symmetric. On the other hand, if $\xi_H := (-a_2, a_1, 0)$ we get $H = a_2(\eta_1 - \eta_3) - a_1(\eta_2 - \eta_4)$. So this H satisfies $(*_1)$, and it satisfies $(*_2)$ by construction. One can easily check that $(*_3)$ holds. Thus, by Proposition 4.2, H satisfies all the identities in Lemma 4.5, but it is not mass linear.

4.3 Examples

We now describe one of the basic examples from [22; 23]. Suppose that $\Delta \subset \mathbb{R}^k \times \mathbb{R}$ is a Δ^k–bundle over Δ^1 with conormals

(4-3)
$$\eta_i = -e_i, \ i = 1, \dots, k, \quad \eta_{k+1} = \sum_{i=1}^{k} e_i,$$

$$\eta_{k+2} = -e_{k+1}, \qquad \qquad \eta_{k+3} = e_{k+1} + \sum_{i+1}^{k} a_i e_i$$

(cf Definition 2.10). Thus Δ is determined by the vector $A := (a_1, \dots, a_k)$. For convenience we later set $a_{k+1} := 0$.

Lemma 4.11 *With Δ as in Equation (4-3), the function $H = \sum_{i=1}^{k+1} \gamma_i \eta_i$ is fully mass linear exactly if it is mass linear, which happens exactly if*

$$\sum \gamma_i = 0 \quad \text{and} \quad \sum \gamma_i a_i = 0.$$

Proof It is easy to check that the volume function of Δ is

$$V(\kappa) = \frac{1}{k!} h \lambda^k - \frac{1}{(k+1)!} \left(\sum a_i \right) \lambda^{k+1},$$

where
$$h = \kappa_{k+2} + \kappa_{k+3} + \sum_{i \leq k+1} a_i \kappa_i, \quad \lambda = \sum_{i=1}^{k+1} \kappa_i.$$

Moreover, one can show by direct calculation that $H = \sum_{i=1}^{k+1} \gamma_i \eta_i$ is mass linear on Δ exactly if $\sum \gamma_i = 0$ and $\sum \gamma_i a_i = 0$. The case $k = 3$ is worked out in detail in [22, Proposition 4.6]. The general case is similar; details will appear in [23, Section 4].

Therefore we need to show that these two conditions imply that H is fully mass linear. By Corollary 4.9 it suffices to see that $\sum \gamma_i \partial_i^m V = 0$ for all m. This is an easy calculation. Note also that when $m = 1$ this condition says that $\sum \gamma_i a_i = 0$ and is equivalent to the statement that $H(B_{n-2}) = H(B_n)$. □

Corollary 4.12 *Every mass linear function on a polytope of dimension $d \leq 3$ is fully mass linear.*

Proof This is an immediate consequence of Proposition 4.7 when $d = 2$, and is anyway clear because all mass linear functions in 2 dimensions are inessential, and hence fully mass linear. We showed in [22] that when $d = 3$ the only essential mass

linear H occur on polytopes that are Δ_2 bundles over Δ_1, and (modulo adding an inessential function) are of the form considered in Lemma 4.11. Hence the result follows from Lemma 4.11. □

Remark 4.13 In dimension 4, it is easy to check that a mass linear function H is fully mass linear if:

- it is geometrically generated; ie there is vector $\xi_H \in \mathfrak{t}^*$ such that $\gamma_i = \langle \eta_i, \xi_H \rangle$ for all i; and

- $\sum_{i \neq j, i, j \in \mathcal{A}} \gamma_i V_{ij} = 0$ where $\mathcal{A} = \{i : F_i$ is asymmetric$\} = \{i : \gamma_i \neq 0\}$.

The pairs (Δ, H) where Δ has dimension 4 and H is essential are classified in [23]. It appears that in all cases H is fully mass linear. It would be interesting to find a more conceptual proof; the classification in [23] is too complicated to transfer easily to higher dimensions.

4.4 Mass linearity and characteristic classes

We now explain Shelukhin's approach to mass linearity. Every Hamiltonian bundle $P \to S^2$ with fiber (M, ω) carries a canonical extension $u \in H^2(P; \mathbb{R})$ of the class of the symplectic form on M called the *coupling class*.[19] One also considers the vertical Chern classes $c_{n-s}^{\mathrm{Vert}} \in H^*(P)$, which are just the ordinary Chern classes of the tangent bundle to the fibers of $P \to S^2$. Using this data one can define a homomorphism $\pi_1(\mathrm{Ham}(M, \omega)) \to \mathbb{R}$ by integrating a product of some vertical Chern classes with a suitable power of u over P. For example, we define I_s by integrating $c_{n-s}^{\mathrm{Vert}} u^{s+1}$.[20]

If the element $\Lambda_H \in \pi_1(\mathrm{Ham}(M, \omega))$ is toric, then as we saw above M_H is toric. Moreover, for each $s = 0, \ldots, n-1$, the class c_{n-s}^{Vert} is Poincaré dual to F_H^s, the union of the faces of Δ_H of dimension $s+1$ and transverse to the fiber, ie the union of the prolongations to Δ_H of all faces of Δ of dimension s.

Shelukhin showed in [29, Theorem 4] that H is fully mass linear if and only if the corresponding loop $\Lambda_H \in \pi_1(\mathrm{Ham}(M, \omega))$ is in the kernel of the homomorphisms I_s, for $0 \leq s < n$. In fact, by finding a nice representative for the coupling class u in terms of the normalized Hamiltonian $H - H(B_n)$, he showed that

$$(4\text{-}4) \qquad I_s(\Lambda_H) = \mathrm{const} \int_{F^s} \left(H - H(B_n) \right) d\mathrm{Vol} = \mathrm{const} \left(H(B_s) - H(B_n) \right) V^s.$$

---

[19]This is the unique extension such that $\int_P u^{n+1} = 0$.

[20] These characteristic classes were first defined by Lalonde, McDuff and Polterovich [11]; see also Kȩdra and McDuff [10, Section 3].

This motivated Definition 4.4: since our work on mass linear functions is primarily aimed at understanding the kernel of the map $\pi_1(T) \to \pi_1(\mathrm{Ham}(M, \omega))$, fully mass linear functions are really more relevant to us than mass linear ones. However, the examples in the previous section show that mass linearity seems to be the most crucial part of the full condition.

More generally, given any tuple $\beta := (\beta_1, \ldots, \beta_n)$ with $|\beta| := \sum i\beta_i \leq n + 1$, set $c_\beta^{\mathrm{Vert}} := \prod (c_i^{\mathrm{Vert}})^{\beta_i}$ and define

$$(4\text{-}5) \qquad\qquad I_\beta(H) = \int_{M_H} c_\beta^{\mathrm{Vert}} u^{n+1-\beta}.$$

Shelukhin also observed that $I_\beta(H)$ must vanish if Λ_H has finite order in $\pi_1(\mathrm{Ham})$. If $c_\beta := \prod (c_i)^{\beta_i}$ is represented by the weighted sum $\sum_{|I|=|\beta|} m_I F_I$ of faces of Δ, then as above

$$(4\text{-}6) \qquad\qquad I_\beta(H) = \mathrm{const} \sum m_I \big(H(B_{F_I}) - H(B_n) \big),$$

where B_{F_I} is the barycenter of F_I.

Lemma 4.14 *If H is fully mass linear then $I_\beta(H) = 0$ for all β.*

We prove this in Corollary 4.18; it is a consequence of our cohomological description of mass linearity.

Some of these classes always vanish by the standard ABBV localization formula. A particularly easy case is when $\beta = c_1 c_n$. Then

$$I_\beta(H) = \int_{F_H^n} c_1^{\mathrm{Vert}}$$

is the integral of c_1^{Vert} over the edges of Δ_H that do not lie in any fiber. Modulo a constant, this is simply the sum of the isotropy weights of H at the vertices of Δ and so always vanishes.[21] (As explained by Shelukhin, these are special cases of some vanishing results for Futaki invariants.)

---

[21] Here is a brief proof: Because $I_\beta(H)$ is linear in H it is enough to prove this for a set of integral H whose rational span includes $\mathfrak{t}_{\mathbb{Z}}$. Therefore we can assume that the critical points of H are just the vertices of Δ, and that at each vertex the weights are pairwise linearly independent. Then the set of points in M with nontrivial stabilizer is a union of 2–spheres; each has exactly two fixed points with opposite weights. See Pelayo and Tolman [26, Theorem 2, Lemma 13] for a much more precise version of this result that uses the ABBV localization in its proof.

Remark 4.15 Formula (4-6) holds for all ways of representing the class c_β as a sum of facets. This gives yet more identities that have to be satisfied by fully mass linear functions. But many of these will be automatically satisfied. For example, if two facets F_1 and F_2 are homologous, then there is an affine self-map of Δ that interchanges them (cf the discussion after Definition 1.12). Hence $H(B_{F_1})$ is a linear function of κ if and only if $H(B_{F_1 \cup F_2})$ is. Similarly if two faces F_I and $F_{I'}$ are homologous there may well be an affine self-map that interchanges them. However, in the absence of such we might get new information. This could be combined with an analysis of the asymmetric and symmetric facets considered in [22; 23].

4.5 A cohomological interpretation of mass linearity

We saw in Lemma 2.5 of [22] that the set of mass linear functions $H \in \mathfrak{t}$ forms a rational subspace of $\mathfrak{t}$, and hence is generated by elements of the integer lattice $\mathfrak{t}_{\mathbb{Z}}$ of $\mathfrak{t}$. Hence we will restrict attention here to $H \in \mathfrak{t}_{\mathbb{Z}}$. Each such H exponentiates to a circle subgroup Λ_H of the Hamiltonian group of the toric manifold (M_Δ, ω), and as before, we denote by M_H the corresponding fibration over S^2 with fiber M and clutching map Λ_H. In this section we describe what it means for H to be mass linear in terms of the cohomology algebra of M_H.

We now investigate the volume function V^H of Δ_H. Note that $\Delta_H \subset \mathfrak{t}^* \times \mathbb{R}$ has N facets F_j^H corresponding to the F_j in Δ with conormals $(\eta_j, 0)$, and two other facets F_{N+1}, F_{N+2} with conormals $\eta_{N+1} = (0, -1), \eta_{N+2} = (0, 1) + H$; cf Remark 2.11 (i). Thus, because we may write $H = \sum_{i \leq N} \gamma_i \eta_i$ we have

$$(4\text{-}7) \qquad \eta_{N+1} + \eta_{N+2} - \sum_{i \leq N} \gamma_i \eta_i = 0.$$

Further the top facet is given by points $(\xi, t) \in \mathfrak{t}^* \times \mathbb{R}$ such that $t + H(\xi) = \kappa_{N+2}$. The volume V^H of Δ_H is therefore

$$V^H = \int_\Delta \left(\int_{-\kappa_{N+2}}^{\kappa_{N+1} - H(\xi)} dt \right) d\mathrm{Vol}(\xi) = \big(\kappa_{N+1} + \kappa_{N+2} - H(B_n)\big) V,$$

where B_n is the center of gravity of Δ.

By Timorin,

$$H^*(M_H) \cong \mathbb{R}[\partial_1, \dots, \partial_{N+2}] / I(V^H) =: R_H$$

where we interpret ∂_i as the differential operator $\partial / \partial \kappa_i$ and $I(V^H)$ consists of all differential operators that annihilate the polynomial V^H. The multiplicative relations in $I(V^H)$ are $\partial_{N+1} \partial_{N+2} = 0$ together with all multiplicative relations $\partial_I = 0$ for V.

Since there is also a new additive relation $\partial_{N+1} - \partial_{N+2} = 0$, we will from now on set $\kappa_{N+2} = \kappa_{N+1}$ and use the relation $\partial_{N+1}^2 = 0$. Therefore we take V^H to be

(4-8)
$$V^H = \left(2\kappa_{N+1} - H(B_n)\right)V,$$

Remark 4.16 In the next theorem we must be careful about the coefficients. In order for Λ_H to be a circle action, we assumed that $H \in \mathfrak{t}$ is integral. However, in the mass linear case this does not mean that the coefficients γ_i in the expression $H(B_n) = \sum \gamma_i \kappa_i$ are integers. For example, if $\Delta = \Delta_1 \subset \mathfrak{t}^* = \mathbb{R}$ is the 1–simplex with conormals $\eta_1 = -1, \eta_2 = 1$, and if $H = \eta_2 \in \mathfrak{t}$, then $H(B_1) = -\frac{1}{2}\kappa_1 + \frac{1}{2}\kappa_2$. Correspondingly, Λ_H is the rotation of $S^2 = M_\Delta$ by one full turn, with order 2 in $\pi_1(\mathrm{Ham}(S^2, \omega))$. In fact, we prove in [22, Proposition 1.22] that the loop Λ_H contracts in $\mathrm{Ham}(S^2, \omega)$ only if the $\gamma_i \in \mathbb{Z}$. It follows that if Λ_H has finite order m in $\pi_1(\mathrm{Ham}(M, \omega))$, then the numbers $m\gamma_i$ are all integers. Note also that the γ_i are always rational because, as we point out in [22, Remark 2.4], the polynomial functions $V(\kappa)$ and $\mu(\kappa)$ have rational coefficients.

In Theorem 4.17 below, we consider cohomology with coefficients $\mathbb{R}$. However, the isomorphism Ψ (if it exists) is rational, and it induces an isomorphism on integral homology exactly if the coefficients γ_i are integers. (Note that $H^*(M; \mathbb{Z})$ is torsion free when M is a toric symplectic manifold.) Note also that Φ induces the identity map on the cohomology $H^*(M)$ of the fiber.

Theorem 4.17 Let (M, ω, T) be a toric manifold with moment polytope Δ, and let $H \in \mathfrak{t} \smallsetminus \{0\}$. Let $M \to M_H \to S^2$ be the corresponding bundle.

(i) The function H is mass linear on Δ with $H(B_n) = \sum \gamma_i \kappa_i$ if and only if there is an algebra isomorphism

$$\Psi \colon H^*(S^2) \otimes H^*(M) \equiv (\mathbb{R}[z]/z^2) \otimes (\mathbb{R}[\partial_1, \dots, \partial_N]/I(V)) \to H^*(M_H)$$

that is compatible with the fibration structure on $H^*(M_H)$, ie if we identify $H^*(M_H)$ with the algebra $\mathbb{R}[\partial_1, \dots, \partial_{N+1}]/I(V^H)$ as above then there are constants α_i such that

$$\Psi(z) = \partial_{N+1} \quad \text{and} \quad \Psi(\partial_i) = \partial_i' := \partial_i + \alpha_i \partial_{N+1} \in I(V^H).$$

(ii) If H is mass linear, then it is fully mass linear exactly if Ψ takes the Chern classes c_s^M in $H^0(S^2) \otimes H^*(M)$ to the vertical Chern classes c_s^{Vert} in $H^*(M_H)$ for all $s = 1, \dots, n$.

(iii) If H is mass linear then $\Psi(c_1^M) = c_1^{\mathrm{Vert}}$ and $\Psi(c_n^M) = c_n^{\mathrm{Vert}}$.

Proof Suppose first that H is mass linear. Then, by Equation (4-1), there are constants γ_i such that $H = \sum \gamma_i \eta_i$ and $H(B_n) = \sum \gamma_i \kappa_i$. Since Δ_H is combinatorially equivalent to a product, it follows from the Stanley–Reisner presentation for $H^*(M_H) =: R_H$ that this algebra is additively isomorphic to a product.[22] By Equation (4-7) the additive relations for V^H are

$$0 = \sum_{j \leq N} \langle \xi_i, \eta_j \rangle \partial_j + \langle \xi_i, \eta_{N+2} \rangle \partial_{N+1} = \sum_{j \leq N} \langle \xi_i, \eta_j \rangle \big(\partial_j + \gamma_j \partial_{N+1} \big)$$

where ξ_i runs over a basis for $\mathfrak{t}^*$. Therefore, if we take $\alpha_i = \gamma_i$ for all i, the map Ψ defined in (i) is an additive homomorphism. Therefore it remains to check that the relations $\partial_I = 0$ that generate the multiplicative relations in $I(V)$ are taken by Ψ to relations ∂'_I in $I(V^H)$.

To see this, note that $V_I = 0$ if and only if $F_I = \varnothing$, while $F_I = \varnothing$ implies $\mu_I = 0$. Therefore, because $\mu = (\sum \gamma_i \kappa_i) V$, for such I we have

$$0 = \mu_I = H(B_n) V_I + \sum_{i \in I} \gamma_i V_{I \smallsetminus i} = \sum_{i \in I} \gamma_i V_{I \smallsetminus i}.$$

Hence
$$\prod_{i \in I} \partial'_i V^H = \prod_{i \in I} \big(\partial_i + \gamma_i \partial_{N+1} \big) \big(2\kappa_{N+1} - H(B_n) \big) V$$

$$= \big(2\kappa_{N+1} - H(B_n) \big) V_I + \sum_{i \in I} (2\gamma_i - \gamma_i) V_{I \smallsetminus i}$$

$$= \sum_{i \in I} \gamma_i V_{I \smallsetminus i} = 0,$$

as required.

Therefore there is an algebra homomorphism $\Psi \colon H^*(S^2) \otimes H^*(M) \to H^*(M_H)$. By construction, its composition with the restriction map $H^*(M_H) \to H^*(M)$ is surjective. Therefore, by the Leray–Hirsch theorem, it is an isomorphism.

Conversely, suppose that

$$\Phi \colon H^*(S^2) \otimes H^*(M) \to H^*(M_H)$$

is an isomorphism of algebras that is compatible with the fibration, ie its restriction to the fiber $H^*(M)$ is the identity and it takes the generator of $H^2(S^2)$ to the pullback of this class in $H^2(M_H)$. We must show that H is mass linear.

Let us think of the symplectic class $[\omega] = [\omega_\kappa]$ on M as a function of the support numbers κ of the polytope $\Delta = \Delta(\kappa)$. In terms of the chosen isomorphism $H^2(M)$ with

[22] In fact this is true for all Hamiltonian bundles over S^2 by [11; 18].

the degree 2 part of the algebra $\mathbb{R}[\partial_1, \ldots, \partial_N]/I(V)$, we may write $[\omega_\kappa] = \sum \ell_i(\kappa)\partial_i$ where the coefficients $\ell_i(\kappa)$ are linear functions of κ. Similarly, the symplectic class $[\Omega_\kappa]$ (which is determined by positions of the facets of the polytope $\Delta_H(\kappa)$) is a linear function of κ.

Because Φ restricts to the identity on the fiber and is compatible with the identity map on the base, the induced map on H^{2n+2} preserves the integer lattice, and hence preserves the cohomological fundamental class. Therefore there is a well defined the Poincaré dual isomorphism $\Phi_*: H_*(S^2) \otimes H_*(M) \to H_*(M_H)$ that takes the (homology) fundamental class $[S^2 \times M]$ to $[M_H]$. Further, the image $Z := \Phi_*([S^2])$ of the fundamental class of S^2 is independent of κ. Hence the above remarks imply that

$$\int_Z \Omega_\kappa = L(\kappa)$$

is a linear function of κ.

Now observe that the volume V^H is a cohomological invariant of M_H: up to a constant, it is obtained by evaluating $(\Omega_\kappa)^{n+1}$ on the fundamental class in $H_{2n+2}(M_H)$. Thus we can evaluate V^H in the product algebra. But here it is just the product of the area of $[S^2]$ (with respect to $\Phi^{-1}[\Omega_\kappa]$) with the volume V of M. Since the area of $[S^2]$ is $\int_Z[\Omega_\kappa]$ it follows that V^H has the form $L(\kappa_i)V$ where L depends linearly on the κ_i as we saw above. Because, as we noted in [Remark 4.16](), the functions V^H and V have rational coefficients, the coefficients of L must also be rational. But we saw in [Equation (4-8)]() that $V^H = (2\kappa_{N+1} - H(B_n))V$. It follows that $H(B_n)$ is a linear function of the κ_i with rational coefficients. This completes the proof of (i).

Now consider (ii). The Chern classes c_s of M are Poincaré dual to the classes in H_{2n-2s} represented by the face sums $\sum_{|I|=s} F_I =: F^s$. Thus they are represented in the algebra $\mathbb{R}[\partial_1, \ldots, \partial_N]/I(V)$ by the differential operator $\sum_{|I|=s} \partial_I$. These same operators also represent the vertical Chern classes of the trivial bundle $S^2 \times M$.

Next observe that the vertical Chern classes in M_H are represented by similar sums over all faces of Δ^H that are transverse to the fiber. Now the element ∂_i in the algebra $R_H := \mathbb{R}[\partial_1, \ldots, \partial_{N+2}]/I(V^H)$ represents the Poincaré dual to F_i^H, the prolongation of F_i to Δ^H. Hence the operator in R_H that represents c_s^{Vert} is $\sum_{|I|=s, I \subset I_0} \partial_I$, where $I_0 := \{1, \ldots, N\}$.

Therefore we must show that H is fully mass linear if and only if

$$\Psi\left(\sum_{|I|=s, I \subset I_0} \partial_I\right) - \sum_{|I|=s, I \subset I_0} \partial_I \in I(V^H), \quad s = 1, \ldots, n.$$

For simplicity, in the sums below we assume without explicit mention that $I \subset I_0$. Then we have

$$
\Psi\left(\sum_{|I|=s} \partial_I\right) V^H = \sum_{|I|=s} \partial'_I V^H
$$

$$
= \sum_{|I|=s} \prod_{i \in I} (\partial_i + \gamma_i \partial_{N+1}) V^H
$$

$$
= \sum_{|I|=s} \partial_I V^H + \sum_{|I|=s, i \in I} 2\gamma_i \partial_{I \smallsetminus i} V.
$$

We saw in the proof of Lemma 4.5 that the vanishing of the second sum above is equivalent to the identity $H(B_{n-s}) = H(B_n)$. Therefore (ii) holds. Moreover (iii) holds by Proposition 4.7. □

The next result concerns the homomorphisms I_β of Equation (4-5).

Corollary 4.18 *Lemma 4.14 holds.*

Proof Suppose that H is fully mass linear. Since c_β^M is a product of the classes c_i^β and c_β^{Vert} is the corresponding product of the c_i^{Vert}, the isomorphism Ψ above takes c_β^M to the class c_β^{Vert} for all β. Moreover, because the coupling class u is the unique extension of $[\omega]$ such that $u^{n+1} = 0$, Ψ takes the coupling class of the product to that for M^H. Hence we can evaluate the integral $I_\beta(H)$ of (4-5) on the product, where it vanishes. □

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Mathematics Department, Barnard College, Columbia University
MC4410, 3009 Broadway, New York NY 10027, USA

dmcduff@barnard.edu

http://www.barnard.edu/mcduff/index.htm