Ricci flow on open 3–manifolds and positive scalar curvature

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We show that an orientable 3–dimensional manifold M admits a complete riemannian metric of bounded geometry and uniformly positive scalar curvature if and only if there exists a finite collection $\mathcal{F}$ of spherical space-forms such that M is a (possibly infinite) connected sum where each summand is diffeomorphic to $S^2 \times S^1$ or to some member of $\mathcal{F}$. This result generalises G Perelman's classification theorem for compact 3–manifolds of positive scalar curvature. The main tool is a variant of Perelman's surgery construction for Ricci flow.

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1 Introduction

Thanks to G Perelman's proof [20; 22; 21] of W Thurston's Geometrisation Conjecture, the topological structure of compact 3–manifolds is now well understood in terms of the canonical geometric decomposition. The first step of this decomposition, which goes back to H Kneser [16], consists in splitting such a manifold as a connected sum of *prime* 3–manifolds, ie 3–manifolds which are not nontrivial connected sums themselves.

It has been known since early work of J H C Whitehead [30] that the topology of *open* 3–manifolds is much more complicated. Directly relevant to the present paper are counterexamples of P Scott [26] and the third author [18] which show that Kneser's theorem fails to generalise to open manifolds, even if one allows infinite connected sums.

Of course, we need to explain what we mean by a possibly infinite connected sum. If $\mathcal{X}$ is a class of closed 3–manifolds, we will say that a 3–manifold M is a *connected sum* of members of $\mathcal{X}$ if there exists a locally finite graph G and a map $v \mapsto X_v$ which associates to each vertex of G a copy of some manifold in $\mathcal{X}$, such that by removing from each X_v as many 3–balls as vertices incident to v and gluing the thus punctured X_v's to each other along the edges of G, one obtains a 3–manifold diffeomorphic to M. This is equivalent to the requirement that M should contain

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a locally finite collection of pairwise disjoint embedded 2–spheres $\mathcal{S}$ such that the operation of cutting M along $\mathcal{S}$ and capping-off 3–balls yields a disjoint union of 3–manifolds which are diffeomorphic to members of $\mathcal{X}$.[1]

Note that restricting this definition to finite graphs and compact manifolds yields a slightly nonstandard definition of a connected sum. In the usual definition of a finite connected sum, one has a *tree* rather than a graph. It is well-known, however, that the graph of a finite connected sum (in the sense of the previous paragraph) can be made into a tree at the expense of adding extra $S^2 \times S^1$ factors. The more general definition we have chosen for this paper seems more natural in view of the surgery theory for Ricci flow. It can be shown that the two definitions are equivalent even when the graph is infinite; however, having a tree rather than a graph is only important for issues of *uniqueness*, which will not be tackled here.

The above-cited articles [26; 18] provide examples of badly behaved open 3–manifolds, which are not connected sums of prime 3–manifolds. From the point of view of Riemannian geometry, it is natural to look for sufficient conditions for a riemannian metric on an open 3–manifold M that rule out such exotic behaviour. One such condition was given by the third author in the paper [17]. Here we shall consider riemannian manifolds of positive scalar curvature. This class of manifolds has been extensively studied since the seminal work of A Lichnerowicz, M Gromov, B Lawson, R Schoen, S-T Yau and others (see eg the survey articles of Gromov [9] and J Rosenberg [23]; see also the recent paper of S Chang, S Weinberger and G Yu [2] which contains results closely related to ours).

Let (M, g) be a riemannian manifold. We denote by $R_{\min}(g)$ the infimum of the scalar curvature of g. We say that g has *uniformly positive scalar curvature* if $R_{\min}(g) > 0$. Of course, if M is compact, then this amounts to insisting that g should have positive scalar curvature at each point of M.

A 3–manifold is *spherical* if it admits a metric of positive constant sectional curvature. M Gromov and B Lawson [10] have shown that any compact, orientable 3–manifold which is a connected sum of spherical manifolds and copies of $S^2 \times S^1$ carries a metric of positive scalar curvature. Perelman [22], completing pioneering work of Schoen and Yau [24] and Gromov and Lawson [11], proved the converse.

In this paper, we are mostly interested in the noncompact case. We say that a riemannian metric g on M has *bounded geometry* if it has bounded sectional curvature and injectivity radius bounded away from zero. It follows from the Gromov–Lawson construction that if M is a (possibly infinite) connected sum of spherical manifolds and

[1] See below for the precise definition of *capping-off*.

copies of $S^2 \times S^1$ such that there are finitely many summands up to diffeomorphism, then M admits a complete metric of bounded geometry and uniformly positive scalar curvature. We show that the converse holds, generalising Perelman's theorem:

Theorem 1.1 *Let M be a connected, orientable 3–manifold which carries a complete riemannian metric of bounded geometry and uniformly positive scalar curvature. Then there is a finite collection $\mathcal{F}$ of spherical manifolds such that M is a connected sum of copies of $S^2 \times S^1$ or members of $\mathcal{F}$.*

In fact, the collection $\mathcal{F}$ depends only on bounds on the geometry and a lower bound for the scalar curvature (cf Corollary 11.1).

Our main tool is R Hamilton's Ricci flow. Let us give a brief review of the analytic theory of Ricci flow on complete manifolds. The basic short time existence result is due to W-X Shi [27]: if M is a 3–manifold and g_0 is a complete riemannian metric on M which has bounded sectional curvature, then there exists $\varepsilon > 0$ and a Ricci flow $g(\cdot)$ defined on $[0, \varepsilon)$ such that $g(0) = g_0$, and for each t, $g(t)$ is also complete of bounded sectional curvature.

For brevity, we say that a Ricci flow $g(\cdot)$ has a given property $\mathcal{P}$ if for each time t, the riemannian metric $g(t)$ has property $\mathcal{P}$. Hence the solutions constructed by Shi are complete Ricci flows with bounded sectional curvature. This seems to be a natural setting for the analytical theory of Ricci flow.[2]

Uniqueness of complete Ricci flows with bounded sectional curvature is due to B-L Chen and X-P Zhu [5].

We shall provide a variant of Perelman's surgery construction for Ricci flow, which has the advantage of being suitable for generalisations to open manifolds. Perelman's construction can be summarised as follows. Let M be a closed, orientable 3–manifold. Start with an arbitrary metric g_0 on M. Consider a maximal Ricci flow solution $\{g(t)\}_{t \in [0, T_{\max})}$ with initial condition g_0. If $T_{\max} = +\infty$, there is nothing to do. Otherwise, one analyses the behaviour of $g(t)$ as t goes to $T_{\max}$ and finds an open subset $\Omega \subset M$ where a limiting metric can be obtained. If Ω is empty, then the construction stops. Otherwise the ends of Ω have a special geometry: they are so-called ε–horns. Removing neighbourhoods of those ends and capping-off 3–balls with nearly standard geometry, one obtains a new closed, possibly disconnected riemannian 3–manifold. Then one restarts Ricci flow from this new metric and iterates the construction. In order to prove that finitely many surgeries occur in any compact time interval, Perelman

[2] However, there have been attempts to generalise the theory beyond this framework; see eg Xu [31] and Simon [28].

makes crucial use of the finiteness of the volume of the various riemannian manifolds involved.

When trying to generalise this construction to open manifolds, one encounters several difficulties. First, the above-mentioned volume argument breaks down. Second, a singularity with $\Omega = M$ could occur, ie there may exist a complete Ricci flow with bounded sectional curvature defined on some interval $[0, T)$ and maximal *among complete Ricci flows with bounded sectional curvature*, such that when t tends to T, $g(t)$ converges to, say, a metric of unbounded curvature $\overline{g}$. Then it is not known whether Ricci flow with initial condition $\overline{g}$ exists at all, and even if it does, the usual tools like the maximum principle may no longer be available. One can imagine, for example, an infinite sequence of spheres of the same radius glued together by necks whose curvature is going to infinity. In this situation $(M, \overline{g})$ would have no horns to do surgery on.

In order to avoid those difficulties, we shall perform surgery *before* a singularity appears. To this end, we introduce a new parameter Θ, which determines when surgery must be done (namely when the supremum $R_{\max}$ of the scalar curvature reaches Θ). We do surgery on tubes rather than horns. Furthermore, we replace the volume argument for nonaccumulation of surgeries by a *curvature* argument: the key point is that at each surgery time, $R_{\max}$ drops by a definite factor (which for convenience we choose equal to $1/2$). This, together with an estimate on the rate of curvature blow-up, is sufficient to bound from below the elapsed time between two consecutive surgeries.

The idea of doing surgery before singularity time is not new: it was introduced by R Hamilton in his paper [12] on 4–manifolds of positive isotropic curvature. Our construction should also be compared with that of G Huisken and C Sinestrari [14] for Mean Curvature Flow, where in particular there is a similar argument for nonaccumulation of surgeries. Needless to say, we rely heavily on Perelman's work, in particular the notions of κ–noncollapsing and canonical neighbourhoods.

Our construction should have other applications. In fact, it has already been adapted by H Huang [13] to complete 4–dimensional manifolds of positive isotropic curvature, using work of B-L Chen, S-H Tang and X-P Zhu [4] in the compact case.

Remaining informal for the moment, we provisionally define a *surgical solution* as a sequence of Ricci flow solutions $\{(M_i, g_i(t))\}_{t\in[t_i,t_{i+1}]}$, with $0 = t_0 < \cdots < t_i < \cdots \leqslant +\infty$ discrete in $\mathbf{R}$, such that M_{i+1} is obtained from M_i by splitting along a locally finite collection of pairwise disjoint embedded 2–spheres, capping-off 3–balls and removing components which are spherical or diffeomorphic to $\mathbf{R}^3$, $S^2 \times S^1$, $S^2 \times \mathbf{R}$, $\mathbf{R}P^3 \# \mathbf{R}P^3$ or a punctured $\mathbf{R}P^3$. If M_{i+1} is nonempty, we further require that $R_{\min}(g_{i+1}) \geqslant R_{\min}(g_i)$ at time t_{i+1}. The formal definition of surgical solutions will be given in Section 2.

The components that are removed at time t_{i+1} are said to *disappear*. If all components disappear, that is if $M_{i+1} = \varnothing$, we shall say that the surgical solution becomes *extinct* at time t_{i+1}. In that case, it is straightforward to reconstruct the topology of the original manifold M_0 as a connected sum of the disappearing components (cf Proposition 2.6 below). Since $\mathbf{R}^3$, $S^2 \times \mathbf{R}$ and punctured $\mathbf{R}P^3$'s are themselves connected sums of spherical manifolds (in fact infinite copies of S^3 and $\mathbf{R}P^3$), the upshot is that M_0 is a connected sum of spherical manifolds and copies of $S^2 \times S^1$.

A simplified version of our main technical result follows.

Theorem 1.2 *Let M be an orientable 3–manifold. Let g_0 be a complete riemannian metric on M which has bounded geometry. Then there exists a complete surgical solution of bounded geometry defined on $[0, +\infty)$, with initial condition (M, g_0).*

When in addition we assume that g_0 has uniformly positive scalar curvature, we get (from the maximum principle and the condition that surgeries do not decrease $R_{\min}$) an a priori lower bound for $R_{\min}$ which goes to infinity in finite time. This implies that surgical solutions given by Theorem 1.2 are automatically extinct. As a consequence, any 3–manifold satisfying the hypotheses of Theorem 1.1 is a connected sum of spherical manifolds and copies of $S^2 \times S^1$. However, we also need to prove finiteness of the summands up to diffeomorphism. Below we state a more precise result, which will suffice for our needs.

Definition 1.3 We say that a riemannian metric g_1 is *ε–homothetic* to some riemannian metric g_2 if there exists $\lambda > 0$ such that λg_1 is ε–close to g_2 in the $C^{[\varepsilon^{-1}]}$–topology. A riemannian metric which is ε–homothetic to a round metric (ie a metric of constant sectional curvature 1) is said to be *ε–round*.

Theorem 1.4 *For all $\rho_0, T > 0$ there exists $Q, \rho > 0$ such that if (M_0, g_0) is a complete riemannian orientable 3–manifold which has sectional curvature bounded in absolute value by 1 and injectivity radius greater than or equal to ρ_0, then there exists a complete surgical solution defined on $[0, T]$, with initial condition (M_0, g_0), sectional curvature bounded in absolute value by Q and injectivity radius greater than or equal to ρ and such that all spherical disappearing components have scalar curvature at least 1, and are 10^{-3}–round or diffeomorphic to S^3 or $\mathbf{R}P^3$.*

Let us explain why this stronger conclusion implies that there are only finitely many disappearing components up to diffeomorphism. By definition, nonspherical disappearing components belong to a finite number of diffeomorphism classes. Now by the

Bonnet–Myers theorem, 10^{-3}–round components with scalar curvature at least 1 have diameter bounded above by some universal constant. Putting this together with the bounds on sectional curvatures and injectivity radius, the assertion then follows from Cheeger's finiteness theorem [3].

Remark 1.5 There is an apparent discrepancy between Theorems 1.2 and 1.4 in that in the former, the surgical solution is defined on $[0, +\infty)$ whereas in the latter it is only defined on a compact interval. However, Theorem 1.2 can be formally deduced from Theorem 1.4 (cf Remark 2.12 at the end of Section 2).

Throughout the paper, we use the following convention:

All 3–manifolds considered here are orientable.

Here is a concise description of the content of the paper: In Section 2, we give some definitions, in particular the formal definition of surgical solutions, and show how to deduce Theorem 1.1 from Theorem 1.4. The remainder of the article (except the last section) is devoted to the proof of Theorem 1.4. In Section 3, we discuss Hamilton–Ivey curvature pinching, the standard solution and prove the Metric Surgery Theorem, which allows to perform surgery. In Section 4, we recall some definitions and results on κ–noncollapsing, κ–solutions and canonical neighbourhoods and fix some constants that will appear throughout the rest of the proof.

In Section 5, we introduce the important notion of (r, δ, κ)–surgical solutions. These are special surgical solutions satisfying various estimates, and with surgery performed in a special way, according to the construction of Section 3. We state an existence theorem for those solutions, Theorem 5.6, which implies Theorem 1.4. Then we reduce Theorem 5.6 to three propositions, called A, B, and C. Sections 6 through 10 are devoted to the proofs of Propositions A, B, C, together with some technical results that are needed in these proofs. For the sake of brevity, we omit proofs or parts of proofs which are very close to the compact, irreducible case tackled in the monograph of the authors with M Boileau and J Porti [1] and focus on the differences.

Section 11 deals with generalisations of Theorem 1.4. One of them is an equivariant version, Theorem 11.3, which implies a classification of 3–manifolds admitting metrics of uniformly positive scalar curvature whose universal cover has bounded geometry. We note that equivariant Ricci flow with surgery in the case of finite group actions on closed 3–manifolds has been studied by J Dinkelbach and B Leeb [8]. We follow in part their discussion; however, things are much simpler in our case, since we are mainly interested in the case of *free* actions. We also give a version of Theorem 5.6 with extra information on the long time behaviour. This may be useful for later applications.

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2 Surgical solutions

Let M be a possibly noncompact, possibly disconnected (orientable) 3–manifold.

2.1 Definitions

Definition 2.1 Let $I \subset \mathbf{R}$ be an interval. An *evolving Riemannian manifold* is a pair $\{(M(t), g(t))\}_{t\in I}$ where for each t, $M(t)$ is a (possibly empty, possibly disconnected) manifold and $g(t)$ a riemannian metric on $M(t)$. We say that it is *piecewise $\mathcal{C}^1$–smooth* if there exists $J \subset I$, which is discrete as a subset of $\mathbf{R}$, such that the following conditions are satisfied:

(i) On each connected component of $I \setminus J$, $t \mapsto M(t)$ is constant, and $t \mapsto g(t)$ is $\mathcal{C}^1$–smooth.

(ii) For each $t_0 \in J$, $M(t_0) = M(t)$ for any $t < t_0$ sufficiently close to t_0 and $t \to g(t)$ is left continuous at t_0.

(iii) For each $t_0 \in J \setminus \{\sup I\}$, $t \to (M(t), g(t))$ has a right limit at t_0, denoted $(M_+(t_0), g_+(t_0))$.

A time $t \in I$ is *regular* if t has a neighbourhood in I where $M(\cdot)$ is constant and $g(\cdot)$ is $\mathcal{C}^1$–smooth. Otherwise it is *singular*.

Definition 2.2 We say a piecewise $\mathcal{C}^1$–smooth evolving Riemannian 3–manifold $\{(M(t), g(t))\}_{t\in I}$ is a *surgical solution* of the Ricci Flow equation

$$\frac{dg}{dt} = -2\,\mathrm{Ric}_{g(t)} \tag{1}$$

if the following statements hold:

(i) Equation (1) is satisfied at all regular times.

(ii) For each singular time t we have $R_{\min}(g_+(t)) \geq R_{\min}(g(t))$.

(iii) For each singular time t there is a locally finite collection $\mathcal{S}$ of disjoint embedded 2–spheres in $M(t)$ and a manifold M' such that:

(a) M' is obtained from $M(t) \setminus \mathcal{S}$ by capping-off 3–balls;

(b) $M_+(t)$ is a union of connected components of M' and $g(t) = g_+(t)$ on $M(t) \cap M_+(t)$;

(c) Each component of $M' \setminus M_+(t)$ is spherical or diffeomorphic to $\mathbf{R}^3$, $S^2 \times S^1$, $S^2 \times \mathbf{R}$, $\mathbf{R}P^3 \# \mathbf{R}P^3$ or a punctured $\mathbf{R}P^3$.

A component of $M' \setminus M_+(t)$ is said to *disappear at time t*. See Figure 1. The surgical solution is *extinct* if $M(t) = \varnothing$ for some t.

An evolving riemannian manifold $\{(M(t), g(t))\}_{t \in I}$ is *complete* (resp. *has bounded geometry*) if for each $t \in I$ such that $M(t) \neq \varnothing$, the riemannian manifold $(M(t), g(t))$ is complete (resp. *has bounded geometry*).

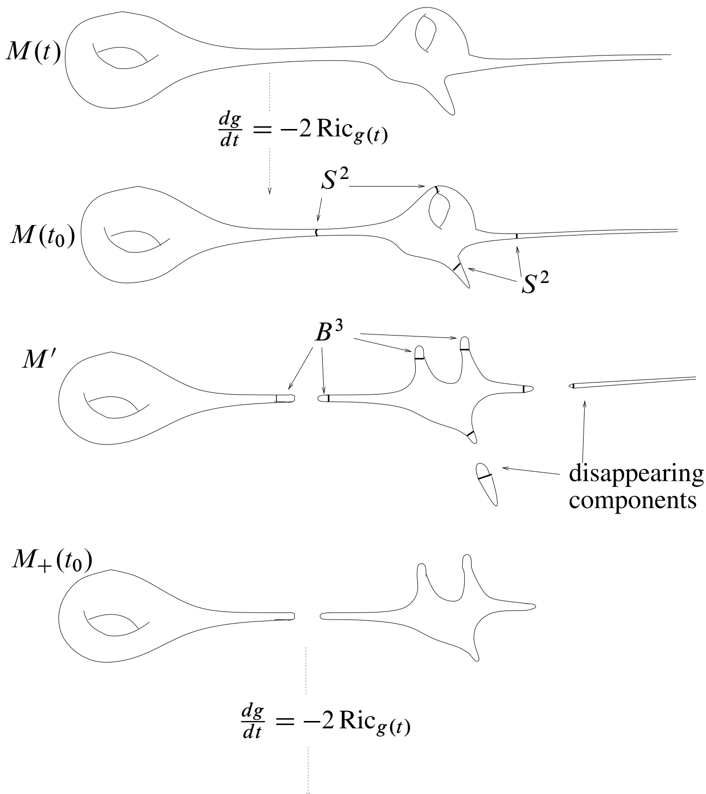

Figure 1

2.2 Deduction of Theorem 1.1 from Theorem 1.4

The purpose of this section is to explain how to deduce Theorem 1.1 from Theorem 1.4. For this, we need a result about the evolution of $R_{\min}$ in a surgical solution. For convenience, we take the convention that $R_{\min}(t)$ is $+\infty$ if $M(t)$ is empty.

Proposition 2.3 *Let $(M(\cdot), g(\cdot))$ be a complete 3–dimensional surgical solution with bounded sectional curvature defined on an interval $[0, T)$. Assume $R_{\min}(0) \geq 0$. Then*

$$R_{\min}(t) \geq \frac{R_{\min}(0)}{1 - 2tR_{\min}(0)/3}.$$

Proof Follows from the evolution equation for scalar curvature, the maximum principle for complete Ricci flows of bounded curvature [7, Corollary 7.45] and the assumption that the minimum of scalar curvature is nondecreasing at singular times of surgical solutions. □

Corollary 2.4 *For every $R_0 > 0$ there exists $T = T(R_0)$ such that the following holds. Let $(M(\cdot), g(\cdot))$ be a complete 3–dimensional surgical solution defined on $[0, T]$, with bounded sectional curvature, and such that $R_{\min}(0) \geq R_0$. Then $(M(\cdot), g(\cdot))$ is extinct.*

We now recall the definition of *capping-off 3–balls* to a 3–manifold.

Definition 2.5 Let M, M' be 3–manifolds. Let $\mathcal{S}$ be a locally finite collection of embedded 2–spheres in M. One says that M' is obtained from $M \setminus \mathcal{S}$ by *capping-off 3–balls* if there exists a collection $\mathcal{B}$ of 3–balls such that M' is the disjoint union

$$M' = (M \setminus \mathcal{S}) \bigsqcup \mathcal{B},$$

where each $S \in \mathcal{S}$ has a tubular neighbourhood $V \subset M$ such that $V \setminus S$ has two connected components V_-, V_+ and there exists $B_-, B_+ \in \mathcal{B}$ such that $V_- \sqcup B_-$ and $V_+ \sqcup B_+$ are 3–balls in M'. Conversely, each $B \in \mathcal{B}$ is included in such a 3–ball of M'.

Note that it is implicit in the above definition that there is an orientation preserving diffeomorphism, say $\phi_-\colon \partial B_- \to S \subset \partial V_-$, such that identifying ∂B_- to the corresponding boundary of V_- one obtains a 3–ball. From Smale [29], the differentiable structure of M' does not depend of the above diffeomorphisms. Moreover, if M' and M'' are obtained from $M \setminus \mathcal{S}$ by capping off 3–balls, one can choose the diffeomorphism from M' to M'' to be the identity on $M \cap M' = M \cap M''$.

We shall need the following topological lemma:

Proposition 2.6 *Let $\mathcal{X}$ be a class of closed 3–manifolds. Let M be a 3–manifold. Suppose that there exists a finite sequence of 3–manifolds $M_0, M_1, \dots, M_p$ such that $M_0 = M$, $M_p = \varnothing$, and for each i, M_i is obtained from M_{i-1} by splitting along a locally finite collection of pairwise disjoint, embedded 2–spheres, capping off 3–balls, and removing some components which are connected sums of members of $\mathcal{X}$. Then each component of M is a connected sum of members of $\mathcal{X}$.*

Proof We prove the result by induction on p. The case $p = 1$ is immediate from the definition of a connected sum.

Supposing that the proposition is true for some p, we consider a sequence $M_0, M_1, \dots, M_{p+1}$ such that $M_0 = M$, $M_{p+1} = \varnothing$, and for each i, M_i is obtained from M_{i-1} by splitting along a locally finite collection of 2–spheres, capping off 3–balls, and removing some components which are connected sums of members of $\mathcal{X}$. By the induction hypothesis, M_1 is a connected sum of members of $\mathcal{X}$.

Let $\mathcal{S}$ be the collection of 2–spheres involved in the process of turning M_0 into M_1. Let $\mathcal{B}$ be the collection of capped-off 3–balls. Let $\mathcal{S}'$ be the collection of 2–spheres involved in the connected sum decomposition of M_1. If $\mathcal{B} \cap \mathcal{S}'$ is empty, then the spheres of $\mathcal{S}'$ actually live in M_0, and the union of $\mathcal{S}$ and $\mathcal{S}'$ splits M_0 into prime summands homeomorphic to members of $\mathcal{X}$. This observation reduces our proof to the following claim:

Claim *$\mathcal{S}'$ can be made disjoint from $\mathcal{B}$ by an ambient isotopy.*

Let us prove the claim. For each component B_i of $\mathcal{B}$, we fix a 3–ball B_i' contained in the interior of B_i and disjoint from $\mathcal{S}'$, and a collar neighbourhood U_i of ∂B_i in $M_1 \setminus B_i$. Since $\{B_i\}$ is locally finite, we may ensure that the U_i's are pairwise disjoint. Choose an ambient isotopy of M_1 which takes B_i' onto B_i and B_i onto $B_i \cup U_i$ for each i. Then after this ambient isotopy, $\mathcal{S}'$ is still locally finite, and is now disjoint from $\mathcal{B}$. □

To see why these results imply Theorem 1.1, take a 3–manifold M and a complete metric g_0 of bounded geometry and uniformly positive scalar curvature on M. By rescaling if necessary we can assume that the bound on the curvature is 1. From the positive lower bound on $R_{\min}(g_0)$ we get an a priori upper bound T for the extinction time of a surgical solution, using Corollary 2.4. Applying Theorem 1.4, we get numbers Q, ρ and a surgical solution $(M(\cdot), g(\cdot))$ with initial condition (M, g_0) defined on $[0, T]$ satisfying the two additional conditions. This solution is extinct, and as we have already explained in the introduction, the disappearing components are connected sums of spherical manifolds and copies of $S^2 \times S^1$, the summands belonging to some finite collection which depends only on the bounds on the geometry. Let $\mathcal{X}$ be the collection of prime factors of the disappearing components. Let $0 = t_0 < t_1 < t_2 < \cdots < t_p = T$ be a set of regular times of $(M(\cdot), g(\cdot))$ such that there is exactly one singular time between each pair of consecutive t_i's. The conclusion of Theorem 1.1 now follows from Proposition 2.6 applied with $M_i = M(t_i)$.

2.3 More definitions

Notation Let $n \geq 2$ be an integer and (M, g) be a riemannian n–manifold.

For any $x \in M$, we denote by $\mathrm{Rm}(x)\colon \Lambda^2 T_x M \to \Lambda^2 T_x M$ the curvature operator, and $|\mathrm{Rm}(x)|$ its norm, which is also the maximum of the absolute values of the sectional curvatures at x. We let $R(x)$ denote the scalar curvature of x. The infimum (resp. supremum) of the scalar curvature of g on M is denoted by $R_{\min}(g)$ (resp. $R_{\max}(g)$).

We write $d\colon M \times M \to [0, \infty)$ for the distance function associated to g. For $r > 0$ we denote by $B(x, r)$ the open ball of radius r around x. Finally, if x, y are points of M, we denote by $[xy]$ a geodesic segment connecting x to y. This is a (common) abuse of notation, since such a segment is not unique in general.

For closeness of metrics we adopt the conventions of [1, Section 2.1].

Definition 2.7 Let $\{(M(t), g(t))\}_{t \in I}$ be a surgical solution and $t \in I$. If t is singular, one sets $M_{\mathrm{reg}}(t) := M(t) \cap M_+(t)$ and denotes by $M_{\mathrm{sing}}(t)$ its complement, ie $M_{\mathrm{sing}}(t) := M(t) \setminus M_{\mathrm{reg}}(t) = M(t) \setminus M_+(t)$. If t is regular, then $M_{\mathrm{reg}}(t) = M(t)$ and $M_{\mathrm{sing}}(t) = \varnothing$.

At a singular time, connected components of $M_{\mathrm{sing}}(t)$ belong to three types:

(i) components of $M(t)$ which are disappearing components of M';

(ii) closures of components of $M(t) \setminus \mathcal{S}$ which give, after being capped-off, disappearing components of M';

(iii) embedded 2–spheres of $\mathcal{S}$.

In particular, the boundary of $M_{\mathrm{sing}}(t)$ is contained in $\mathcal{S}$.

Definition 2.8 We say that a pair $(x, t) \in M \times I$ is *singular* if $x \in M_{\mathrm{sing}}(t)$; otherwise we call (x, t) *regular*.

Definition 2.9 Let t_0 be a time, $[a, b]$ be an interval containing t_0 and X be a subset of $M(t_0)$ such that for every $t \in [a, b)$, we have $X \subset M_{\mathrm{reg}}(t)$. Then the set $X \times [a, b]$ is *unscathed*. Otherwise, we say that X is *scathed*.

Remark 2.10 (1) In the definition of "unscathed", we allow the final time slice to contain singular points, ie we may have $X \cap M_{\text{sing}}(b) \neq \varnothing$. The point is that if $X \times [a,b]$ is unscathed, then $t \mapsto g(t)$ evolves smoothly by the Ricci flow equation on all of $X \times [a,b]$.

(2) Assume that $X \times [a,b]$ is scathed. Then there is $t \in [a,b)$ and $x \in X$ such that $x \notin M_{\text{reg}}(t)$. Assume that t is closest to t_0 with this property. If $t > t_0$ then $x \in M_{\text{sing}}(t)$ and disappears at time t unless $x \in \partial M_{\text{sing}}(t)$ or if the component of $M_{\text{sing}}(t)$ which contains x is a sphere $S \in \mathcal{S}$. If $t < t_0$, then $x \in M_+(t) \setminus M_{\text{reg}}(t)$ is in one of the 3–balls that are added at time t.

Notation For $t \in I$ and $x \in M(t)$ we use the notation $\mathrm{Rm}(x,t)$, $R(x,t)$ to denote the curvature operator and the scalar curvature respectively. For brevity we set $R_{\min}(t) := R_{\min}(g(t))$ and $R_{\max}(t) := R_{\max}(g(t))$.

We use $d_t(\cdot,\cdot)$ for the distance function associated to $g(t)$. The ball of radius ρ around x for $g(t)$ is denoted by $B(x,t,\rho)$.

For the definition of closeness of evolving riemannian manifolds, see [1, Section 2.2].

Definition 2.11 Let $t_0 \in I$ and $Q > 0$. The *parabolic rescaling* with factor Q at time t_0 is the evolving manifold $\{(\bar{M}(t), \bar{g}(t))\}$ where $\bar{M}(t) = M(t_0 + t/Q)$, and

$$\bar{g}(t) = Q\, g\left(t_0 + \frac{t}{Q}\right).$$

Remark 2.12 Theorem 1.2 follows by iteration of Theorem 1.4 via parabolic rescalings. Indeed, one easily obtains from Theorem 1.4, that given $\rho_0, Q_0, T > 0$ there exists $Q, \rho > 0$ such that given $T_0 > 0$ and any complete riemannian orientable 3–manifold (M_0, g_0) with sectional curvature in absolute value $\leqslant Q_0$ and injectivity radius $\geqslant \rho_0$, there exists a complete surgical solution $(M(\cdot), g(\cdot))$ defined on $[T_0, T_0 + T]$, with initial condition $(M(T_0), g(T_0)) = (M_0, g_0)$, and with sectional curvature in absolute value $\leqslant Q$ and injectivity radius $\geqslant \rho$. Applying inductively this finite time existence result proves Theorem Theorems 1.2. In the sequel, we focus on proving Theorem 1.4.

3 Metric surgery

3.1 Curvature pinched toward positive

Let (M,g) be a 3–manifold and $x \in M$ be a point. We denote by $\lambda(x) \geq \mu(x) \geq \nu(x)$ the eigenvalues of the curvature operator $\mathrm{Rm}(x)$. By our definition, all sectional curvatures lie in the interval $[\nu(x),\lambda(x)]$. Moreover, $\lambda(x)$ (resp. $\nu(x)$) is the maximal (resp. minimal) sectional curvature at x. If C is a real number, we sometimes write $\mathrm{Rm}(x)\geq C$ instead of $\nu(x)\geq C$. Likewise, $\mathrm{Rm}(x)\leq C$ means $\lambda(x)\leq C$.

It follows that the eigenvalues of the Ricci tensor are equal to $\lambda+\mu$, $\lambda+\nu$, and $\mu+\nu$; as a consequence, the scalar curvature $R(x)$ is equal to $2(\lambda(x)+\mu(x)+\nu(x))$.

For evolving metrics, we use the notation $\lambda(x,t)$, $\mu(x,t)$, and $\nu(x,t)$, and correspondingly write $\mathrm{Rm}(x,t)\geq C$ for $\nu(x,t)\geq C$, and $\mathrm{Rm}(x,t)\leq C$ for $\lambda(x,t)\leq C$.

Definition 3.1 Let ϕ be a nonnegative function. A metric g on M has *ϕ–almost nonnegative curvature* if $\mathrm{Rm}\geq-\phi(R)$.

Now we consider a family of positive functions $(\phi_t)_{t\geqslant 0}$ defined as follows. Set $s_t:=e^2/(1+t)$ and define $\phi_t\colon[-2s_t,+\infty)\longrightarrow[s_t,+\infty)$ as the reciprocal of the function $s\mapsto 2s(\ln(s)+\ln(1+t)-3)$.

Following J Morgan and G Tian [19], we use the following definition.

Definition 3.2 Let $I\subset[0,\infty)$ be an interval, $t_0\in I$ and $\{g(t)\}_{t\in I}$ be an evolving metric on M. We say that $g(\cdot)$ has *curvature pinched toward positive at time t_0* if for all $x\in M$ we have

$$R(x,t_0)\geqslant-\frac{6}{4t_0+1},\tag{2}$$

$$\mathrm{Rm}(x,t_0)\geqslant-\phi_{t_0}(R(x,t_0)).\tag{3}$$

We say that $g(\cdot)$ has *curvature pinched toward positive* if it has curvature pinched toward positive at each $t\in I$.

Remark that if $|\mathrm{Rm}(g(0))|\leq 1$, then $g(\cdot)$ has curvature pinched toward positive at time 0. Next we state a result due to Hamilton and Ivey in the compact case. For a proof of the general case, see [6, Section 5.1].

Proposition 3.3 (Hamilton–Ivey pinching estimate) *Let a,b be two real numbers such that $0\leq a<b$. Let $(M,\{g(t)\}_{t\in[a,b]})$ be a complete Ricci flow with bounded curvature. If $g(\cdot)$ has curvature pinched toward positive at time a, then $\{g(t)\}_{t\in[a,b]}$ has curvature pinched toward positive.*

3.2 The standard solution

Let us recall the definition of standard initial metric we used in [1]. This metric is the initial condition of the standard solution. The functions f, u below are chosen in [1, Section 7.1].

Definition 3.4 Let $d\theta^2$ denote the round metric of scalar curvature 1 on S^2.

The *standard initial metric* is the riemannian manifold $\mathcal{S}_0 = (\mathbf{R}^3, \overline{g}_0)$, where the metric $\overline{g}_0$ is given in polar coordinates by

$$\overline{g}_0 = e^{-2f(r)} g_u \,,$$

where

$$g_u = dr^2 + u(r)^2 d\theta^2 \,.$$

We also define $\mathcal{S}_u := (\mathbf{R}^3, g_u)$. The origin of $\mathbf{R}^3$, which is also the centre of spherical symmetry, will be denoted by p_0.

In particular, $(B(0,5), \overline{g}_0)$ has positive sectional curvatures (see [1, Lemma 7.1.2]), and the complement of $B(0,5)$ is isometric to $S^2 \times [0, +\infty)$.

Ricci flow with initial condition $\mathcal{S}_0$ has a maximal solution defined on $[0,1)$, which is unique among complete flows of bounded sectional curvature [22]. This solution is called the *standard solution*.

3.3 The metric surgery theorem

The *standard ε–neck* is the riemannian product $S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1})$, where the S^2 factor is round of scalar curvature 1. Its metric is denoted by g_{cyl}. We fix a basepoint $*$ in $S^2 \times \{0\}$.

Definition 3.5 Let (M, g) be a riemannian 3–manifold and x be a point of M. A neighbourhood $N \subset M$ of x is called an *ε–neck centred at x* if (N, g, x) is ε–homothetic to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), g_{\mathrm{cyl}}, *)$.

If N is an ε–neck and $\psi\colon N_\varepsilon \to N$ is a *parametrisation*, ie a diffeomorphism such that some rescaling of $\psi^*(g)$ is ε–close to g_{cyl}, then the sphere $\psi(S^2 \times \{0\})$ is called a *middle sphere* of N.

We recall a lemma from [1] which allows to fix a universal constant ε_0.

Lemma 3.6 [1, Lemma 3.2.2] *There exists $\varepsilon_0 > 0$ such that the following holds. Let $\varepsilon \in (0, 2\varepsilon_0]$. Let (M, g) be a riemannian 3–manifold. Let y_1, y_2 be points of M. Let $U_1 \subset M$ be an ε–neck centred at y_1 with parametrisation $\psi_1\colon S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to U_1$ and middle sphere S_1. Let $U_2 \subset M$ be a 10ε–neck centred at y_2 with middle sphere S_2. Call $\pi\colon U_1 \to (-\varepsilon^{-1}, \varepsilon^{-1})$ the composition of ψ_1^{-1} with the projection of $S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1})$ onto its second factor.*

Assume that $y_2 \in U_1$ and $|\pi(y_2)| \leq (2\varepsilon)^{-1}$. Then the following conclusions hold:

(i) *U_2 is contained in U_1.*

(ii) *The boundary components of $\partial\overline{U_2}$ can be denoted by S_+, S_- in such a way that*
$$\pi(S_-) \subset [\pi(y_2) - (10\varepsilon)^{-1} - 10, \pi(y_2) - (10\varepsilon)^{-1} + 10],$$
$$\pi(S_+) \subset [\pi(y_2) + (10\varepsilon)^{-1} - 10, \pi(y_2) + (10\varepsilon)^{-1} + 10].$$

(iii) *The spheres S_1, S_2 are isotopic in U_1.*

Definition 3.7 Let δ, δ' be positive numbers. Let g be a riemannian metric on M. Let (U, V, p, y) be a 4–tuple such that U is an open subset of M, V is a compact subset of U, $p \in \operatorname{Int} V$, $y \in \partial V$. Then (U, V, p, y) is called a *marked (δ, δ')–almost standard cap* if there exists a δ'–isometry $\psi\colon B(p_0, 5 + \delta^{-1}) \to (U, R(y)g)$, sending $B(p_0, 5)$ to $\operatorname{Int} V$ and p_0 to p. One calls V the *core* and p the tip.

Theorem 3.8 (Metric surgery) *There exist $\delta_0 > 0$ and a function $\delta'\colon (0, \delta_0] \ni \delta \mapsto \delta'(\delta) \in (0, \varepsilon_0/10]$ tending to zero as $\delta \to 0$, with the following property:*

Let ϕ be a nondecreasing, positive function; let $\delta \leq \delta_0$; let (M, g) be a riemannian 3–manifold with ϕ–almost nonnegative curvature, and $\{N_i\}$ be a locally finite collection of pairwise disjoint δ–necks in M. Let M' be a manifold obtained by cutting M along the middle spheres of the N_i's and capping off 3–balls.

Then there exists a riemannian metric g_+ on M' such that:

(i) *$g_+ = g$ on $M' \cap M$;*

(ii) *For each connected component B of $M' \setminus M$, there exist $p \in \operatorname{Int} B$ and $y \in \partial B$ such that $(N' \cup B, B, p, y)$ is a marked $(\delta, \delta'(\delta))$–almost standard cap with respect to g_+, where N' is the "half" of N adjacent to B in M';*

(iii) *g_+ has ϕ–almost nonnegative curvature.*

Remark 3.9 In the application of the above theorem, M_+ will be a submanifold of M'.

Proof On $M' \cap M$ we set $g_+ := g$. On the added 3–balls we define g_+ as follows. Let $N \subset M$ be one of the δ–necks of the collection, and let S be its middle sphere. By definition there exists a diffeomorphism $\psi\colon S^2 \times (-\delta^{-1}, \delta^{-1}) \longrightarrow N$ and a real number $\lambda > 0$ such that $\|\psi^* \lambda g - g_{\mathrm{cyl}}\| < \delta$ in the $C^{[\delta^{-1}]}$–norm (see [1] for the details). Note that for each $y \in N$ we have that $\lambda / R(y)$ is δ'–close to 1 for some universal $\delta'(\delta)$ tending to zero with δ.

Define $N_+ := \psi(S^2 \times [0, \delta^{-1}))$, ie N_+ is the right half of the neck. Let $\Sigma \subset M' \setminus M$ be the 3–ball that is capped-off to it and $\Phi\colon \partial\Sigma \longrightarrow \partial N_+$ be the corresponding diffeomorphism. Our goal is to define g_+ on Σ in such a way that $(N_+ \cup_\Phi \Sigma, g_+)$ is a $(\delta, \delta'(\delta))$–almost standard cap with ϕ–almost nonnegative curvature.

Let us introduce more notation. For $0 \le r_1 \le r_2$, we let $C[r_1, r_2]$ denote the annular region of $\mathbf{R}^3$ defined by the inequations $r_1 \le r \le r_2$ in polar coordinates. Observe that for all $3 \le r_1 < r_2$, the restriction of g_u to $C[r_1, r_2]$ is isometric to the cylinder $S^2 \times [r_1, r_2]$ with scalar curvature 1. We consider $B := B(0, 5) \subset \mathcal{S}_u$.

Set $V_- := \psi(S^2 \times (-2, 0])$ and $V_+ := \psi(S^2 \times [0, 2))$. Restrict ψ on $S^2 \times (-2, \delta^{-1})$ to $V_- \cup N_+$, where $S^2 \times (-2, \delta^{-1})$ is now considered as the annulus $C(3, 5 + \delta^{-1}) \subset \mathcal{S}_u$. See Figure 2.

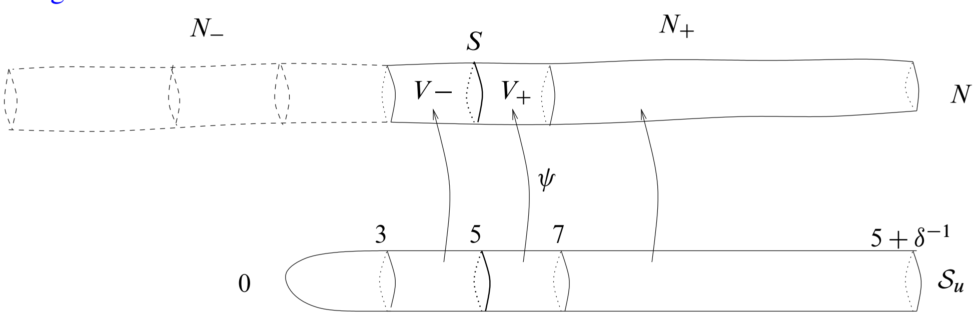

Figure 2

Let $\overline{g} := \psi^*(\lambda g)$ be the pulled-back rescaled metric on $C(3, 5 + \delta^{-1})$. Note that $\|\overline{g} - g_u\| < \delta$ on this set and that $\overline{g}$ has ϕ–almost nonnegative curvature.

On $B(0, 5 + \delta^{-1})$ we define in polar coordinates (see Figure 3)

$$\overline{g}_+ := e^{-2f}\,(\chi g_u + (1-\chi)\overline{g}) = \chi \overline{g}_0 + (1-\chi) e^{-2f}\,\overline{g}$$

where $\chi\colon [0, 5+\delta^{-1}] \to [0, 1]$ is a smooth function satisfying

$$\begin{cases} \chi \equiv 1 & \text{on } [0, 3], \\ \chi' < 0 & \text{on } (3, 4), \\ \chi \equiv 0 & \text{on } [4, 5+\delta^{-1}]. \end{cases}$$

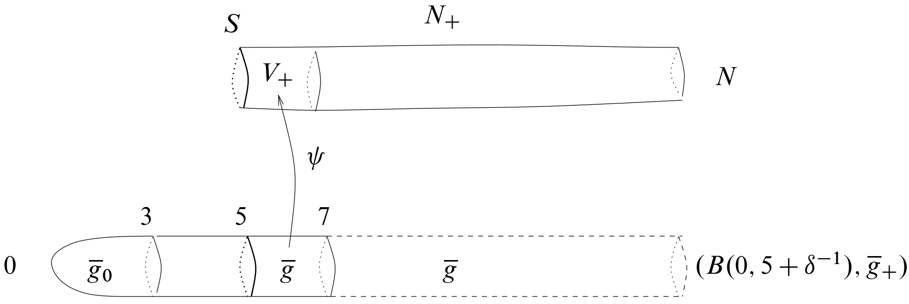

Figure 3

Note that

$$\begin{cases} \overline{g}_+ = \overline{g}_0 & \text{on } B(0,3), \\ \overline{g}_+ = e^{-2f}\,\overline{g} & \text{on } C[4,5], \\ \overline{g}_+ = \overline{g} & \text{on } C[5,5+\delta^{-1}). \end{cases}$$

Finally set

$$\begin{cases} g_+ := (\psi^{-1})^*(\lambda^{-1}\overline{g}) = g & \text{on } N_+, \\ g_+ := \lambda^{-1}\overline{g}_+ & \text{on } B(0,5). \end{cases}$$

Let p be the origin and y be an arbitrary point of ∂B. There remains to show that $((N_+ \cup_{\psi_{|\partial B}} B, B, p, y), g_+)$ is a $(\delta, \delta'(\delta))$–almost standard cap, and has ϕ–almost nonnegative curvature. It suffices clearly to consider g_+ on B, or $\overline{g}_+$ on $B(0,5)$. This is tackled by the following proposition [1, Proposition 7.2.2], applied to $\overline{g}$ and the rescaled pinching function $s \mapsto \lambda^{-1}\phi(\lambda s)$:

Proposition 3.10 *There exists $\delta_1 > 0$ and a function $\delta'\colon (0,\delta_1] \longrightarrow (0, \varepsilon_0/10]$ with limit zero at zero, having the following property: let ϕ be a nondecreasing positive function, $0 < \delta \le \delta_1$ and $\overline{g}$ be a metric on $C(3,5) \subset \mathbf{R}^3$, with ϕ–almost nonnegative curvature, such that $\|\overline{g} - g_u\|_{\mathcal{C}^{[\delta^{-1}]}} < \delta$ on $C(3,5)$. Then the metric*

$$\overline{g}_+ = e^{-2f}\,(\chi g_u + (1-\chi)\overline{g})$$

has ϕ–almost nonnegative curvature, and is $\delta'(\delta)$–close to $\overline{g}_0$ on $B(0,5)$.

Setting $\delta_0 := \delta_1$ completes the proof of Theorem 3.8. □

4 κ–Noncollapsing and canonical neighbourhoods

4.1 κ–Noncollapsing

Let $\{(M(t), g(t))\}_{t\in I}$ be an evolving riemannian manifold. We say that a pair (x,t) is a *point in spacetime* if $t \in I$ and $x \in M(t)$. For convenience we denote by $\mathcal{M}$

the set of all such points. A (backward) *parabolic neighbourhood* of a point (x,t) in spacetime is a set of the form

$$P(x,t,r,-\Delta t) := \{(x',t') \in \mathcal{M} \mid x' \in B(x,t,r), t' \in [t-\Delta t, t]\}.$$

In particular, the set $P(x,t,r,-r^2)$ is called a *parabolic ball* of radius r.

A parabolic neighbourhood $P(x,t,r,-\Delta t)$ is *unscathed* if $B(x,t,r)\times[t-\Delta t,t]$ is unscathed. In this case $P(x,t,r,-\Delta t) = B(x,t,r)\times[t-\Delta t,t]$.

Definition 4.1 Fix $\kappa, r > 0$. We say that $(M(\cdot), g(\cdot))$ is *κ–collapsed* at (x,t) on the scale r if for all $(x',t') \in P(x,t,r,-r^2)$ one has $|\mathrm{Rm}(x',t')| \leq r^{-2}$, and $\mathrm{vol}\, B(x,t,r) < \kappa r^n$. Otherwise, $(M(\cdot), g(\cdot))$ is *κ–noncollapsed* at (x,t) on the scale r.

We say that $(M(\cdot), g(\cdot))$ is *κ–noncollapsed* on the scale r if it is κ–noncollapsed on this scale at every point of $\mathcal{M}$.

4.2 Canonical neighbourhoods

Definition 4.2 Let (M,g) be a riemannian 3–manifold and x be a point of M. We say that U is an *ε–cap centred at x* if U is the union of two sets V, W such that $x \in \mathrm{Int}\, V$, V is diffeomorphic to B^3 or $\mathbf{R}P^3 \setminus B^3$, $\overline{W} \cap V = \partial V$, and W is an ε–neck. A subset V as above is called a *core* of U.

Let $\varepsilon > 0$, $C >> \varepsilon^{-1}$, $\{(M(t), g(t))\}_{t\in I}$ be an evolving riemannian manifold and (x_0,t_0) be a point in spacetime.

Definition 4.3 We call *cylindrical flow* the manifold $S^2\times\mathbf{R}$ together with the product flow on $(-\infty, 0]$, where the first factor is round, normalised so that the scalar curvature at time 0 is identically 1. We denote this evolving metric by $g_{\mathrm{cyl}}(t)$.

Definition 4.4 An open subset $N \subset M(t_0)$ is called a *strong ε–neck*[3] centred at (x_0,t_0) if there is a number $Q > 0$ such that $(N, \{g(t)\}_{t\in[t_0-Q^{-1},t_0]}, x_0)$ is unscathed and after parabolic rescaling with factor Q at time t_0, N is ε–close to $(S^2\times(-\varepsilon^{-1},\varepsilon^{-1}), \{g_{\mathrm{cyl}}(t)\}_{t\in[-1,0]}, *)$.

Remark 4.5 Let $Q > 0$, and consider the flow $(S^2\times\mathbf{R}, Qg_{\mathrm{cyl}}(tQ^{-1}))$ restricted to $(-Q, 0]$. Then for every $x \in S^2\times\mathbf{R}$ and every $\varepsilon > 0$, the point $(x,0)$ is the centre of a strong ε–neck.

[3]We use "strong neck" to denote a subset of $M(t_0)$, rather than a subset of spacetime as other authors do.

We recall that ε*–round* means ε–homothetic to a metric of positive constant sectional curvature (cf Definition 1.3).

Definition 4.6 Let $U \subset M(t)$ be an open subset and $x \in U$ such that $R(x) := R(x,t) > 0$. One says that U is an (ε, C)*–canonical neighbourhood* centred at (x,t) if:

(A) U is of a strong ε–neck centred at (x,t), or

(B) U is an ε–cap centred at x for $g(t)$, or

(C) $(U, g(t))$ has sectional curvatures $> C^{-1}R(x)$ and is diffeomorphic to S^3 or $\mathbf{R}P^3$, or

(D) $(U, g(t))$ is ε–round,

and if moreover, the following estimates hold in cases (A), (B), (C) for $(U, g(t))$: There exists $r \in (C^{-1}R(x)^{-1/2}, CR(x)^{-1/2})$ such that:

(i) $\overline{B(x,r)} \subset U \subset B(x,2r)$;

(ii) The scalar curvature function restricted to U has values in a compact subinterval of $(C^{-1}R(x), CR(x))$;

(iii) If $B(y,r) \subset U$ and if $|\mathrm{Rm}| \leq r^{-2}$ on $B(y,r)$ then

$$C^{-1} < \frac{\operatorname{vol} B(y,r)}{r^3}\,; \tag{4}$$

(iv) $|\nabla R(x)| < CR(x)^{3/2}$;

(v) $|\Delta R(x) + 2|\mathrm{Ric}(x)|^2| < CR(x)^2$;

(vi) $|\nabla \mathrm{Rm}(x)| < C|\mathrm{Rm}(x)|^{3/2}$.

Remark 4.7 In case (D), Estimates (i)–(vi) hold except maybe (iii) (consider eg lens spaces).

(i) implies that diam U is at most $4r$, which in turn is bounded above by a function of C and $R(x)$.

(iii) implies that vol U is bounded from below by $C^{-1}R(x)^{-3/2}$.

Estimate (v) implies the following scale-invariant bound on the time-derivative of R (at a regular time):

$$\left|\frac{\partial R}{\partial t}(x,t)\right| < CR(x,t)^2. \tag{5}$$

We call (ε, C)*–cap* any ε–cap of (M,g) which satisfies (i)–(vi).

In cases (C) and (D), U is diffeomorphic to a spherical manifold.

Cases (C) and (D) are not mutually exclusive.

Being the centre of an (ε, C)–canonical neighbourhood is an open property in spacetime: if $U \subset M(t)$ is unscathed on $(t-\alpha, t+\alpha)$ for some $\alpha > 0$, then there exists a neighbourhood Ω of (x,t) such that any $(x',t') \in \Omega$ is the centre of an (ε, C)–canonical neighbourhood. In case (A), one can use the same set $N = U$ and factor Q, but change the parametrisation so that the basepoint $*$ is sent to x rather than x_0. Case (B) is similar. Cases (C), (D) are obvious.

The same argument shows that being the centre of an (ε, C)–canonical neighbourhood is also an open property with respect to a change of metric in the $C^{[\varepsilon^{-1}]}$–topology.

4.3 Fixing the constants

In order to fix the constants, we recall some results of Perelman on κ–solutions and the standard solution.

Theorem 4.8 *For all $\varepsilon > 0$ there exists $C_{\mathrm{sol}} = C_{\mathrm{sol}}(\varepsilon)$ such that if $(M, \{g(t)\}_{t\in(-\infty,0]})$ is a 3–dimensional κ–solution, then every $(x,t) \in M \times (-\infty, 0]$ is the centre of an $(\varepsilon, C_{\mathrm{sol}})$–canonical neighbourhood.*

Proposition 4.9 *There exists $\kappa_{\mathrm{st}} > 0$ such that the standard solution is κ_{st}–noncollapsed on all scales.*

Proposition 4.10 *For every $\varepsilon > 0$ there exists $C_{\mathrm{st}}(\varepsilon) > 0$ such that if (x,t) is a point in the standard solution such that $t > 3/4$ or $x \notin B(p_0, 0, \varepsilon^{-1})$, then (x,t) has an $(\varepsilon, C_{\mathrm{st}})$–canonical neighbourhood. Moreover there is an estimate $R_{\min}(t) \geqslant \mathrm{const}_{\mathrm{st}}(1-t)^{-1}$ for some constant $\mathrm{const}_{\mathrm{st}} > 0$.*

Let K_{st} be the supremum of the sectional curvatures of the standard solution on $[0, 4/5]$. As in [1, Lemma 4.3.5.] we have:

Lemma 4.11 *For all $\varepsilon \in (0, 10^{-4})$ there exists $\beta = \beta(\varepsilon) \in (0,1)$ such that the following holds.*

Let a, b be real numbers satisfying $a < b < 0$ and $|b| \le 3/4$, let $(M, g(\cdot))$ be a surgical solution defined on $(a, 0]$, and $x \in M$ be a point such that:

- *$R(x,b) = 1$;*
- *(x,b) is the centre of a strong $\beta\varepsilon$–neck;*
- *$P(x, b, (\beta\varepsilon)^{-1}, |b|)$ is unscathed and satisfies $|\mathrm{Rm}| \le 2K_{\mathrm{st}}$.*

Then $(x, 0)$ is the centre of a strong ε–neck.

Now we can define our constants. Recall that the constant ε_0 has been fixed thanks to Lemma 3.6. Let $\beta := \beta(\varepsilon_0)$ be the constant given by Lemma 4.11. Finally, define $C_0 := \max(100, 2C_{\mathrm{sol}}(\varepsilon_0/2), 2C_{\mathrm{st}}(\beta\varepsilon_0/2))$.

Definition 4.12 Let $r > 0$. An evolving riemannian manifold $\{(M(t), g(t))\}_{t\in I}$ has property $(\mathrm{CN})_r$ if for all $(x,t) \in \mathcal{M}$, if $R(x,t) \geq r^{-2}$, then (x,t) admits an (ε_0, C_0)–canonical neighbourhood.

Definition 4.13 Let $\kappa > 0$. An evolving riemannian manifold $\{(M(t), g(t))\}_{t\in I}$ has property $(\mathrm{NC})_\kappa$ if it is κ–noncollapsed on all scales less than 1.

5 (r, δ, κ)–Surgical solutions

5.1 Cutoff parameters and (r, δ)–surgery

Theorem 5.1 (Cutoff parameters) *For all $r, \delta > 0$, there exist $h \in (0, \delta r)$ and $D > 10$ such that if $(M(\cdot), g(\cdot))$ is a complete surgical solution of bounded curvature defined on an interval $[a,b]$, with curvature pinched toward positive and satisfying $(\mathrm{CN})_r$, then the following holds:*

Let $t \in [a,b]$ and $x, y, z \in M(t)$ such that $R(x,t) \leq 2/r^2$, $R(y,t) = h^{-2}$, and $R(z,t) \geq D/h^2$. Assume there is a curve γ connecting x to z via y, such that each point of γ with scalar curvature in $[2C_0 r^{-2}, C_0{}^{-1} D h^{-2}]$ is the centre of an ε_0–neck. Then (y,t) is the centre of a strong δ–neck.

This will be proved in Section 6. In the sequel we fix functions $(r,\delta) \mapsto h(r,\delta)$ and $(r,\delta) \mapsto D(r,\delta)$ with the above property. We set $\Theta := 2Dh^{-2}$. This number will be used as a curvature threshold for the surgery process.

Definition 5.2 We say that two real numbers r, δ are *surgery parameters* if $0 < r < 10^{-3}$ and $0 < \delta < \min(\varepsilon_0, \delta_0)$ (where δ_0 is the constant from Theorem 3.8). The *associated cutoff parameters* are $h := h(r,\delta)$, $D := D(r,\delta)$ and $\Theta := 2Dh^{-2}$.

From now on, we fix a function $\delta' \colon (0, \delta_0] \longrightarrow (0, \varepsilon_0/10]$ as in the metric surgery theorem. A marked $(\delta, \delta'(\delta))$–almost standard cap will be simply called a δ–almost standard cap. An open subset U of M is called a δ–almost standard cap if there exist V, p and y such that (U, V, p, y) is a δ–almost standard cap.

Definition 5.3 Fix surgery parameters r, δ and let h, D, Θ be the associated cutoff parameters. Let $\{(M(t), g(t))\}_{t \in I}$ be an evolving riemannian manifold. Let $t_0 \in I$ and (M_+, g_+) be a (possibly empty) riemannian manifold. We say that (M_+, g_+) *is obtained from* $(M(\cdot), g(\cdot))$ *by* (r, δ)*–surgery at time* t_0 if the following conditions are satisfied:

(i) M_+ is obtained from $M(t_0)$ by cutting along a locally finite collection of disjoint 2–spheres, capping off 3–balls, and possibly removing some components that are spherical or diffeomorphic to $\mathbf{R}^3$, $S^2 \times \mathbf{R}$, $\mathbf{R}P^3 \setminus \{\mathrm{pt}\}$, $\mathbf{R}P^3 \# \mathbf{R}P^3$, $S^2 \times S^1$. In addition, a spherical manifold U can only be removed if it is contained in $M(t_0)$, and $(U, g(t_0))$ is ε–round and satisfies $R \geq 1$.

(ii) For all $x \in M_+ \setminus M(t_0)$, there exists a δ–almost standard cap (U, V, p, y) in M_+, such that:
   (a) $x \in V$;
   (b) $y \in M(t_0)$;
   (c) $R(y, t_0) = h^{-2}$;
   (d) (y, t_0) is the centre of a strong δ–neck.

(iii) $R_{\max}(g(t_0)) = \Theta$, and if $M_+ \neq \varnothing$, then $R_{\max}(g_+) \leq \Theta/2$.

Definition 5.4 Fix surgery parameters r, δ and let h, D, Θ be the associated cutoff parameters. Let $I \subset [0, \infty)$ be an interval and $\{(M(t), g(t))\}_{t \in I}$ be a surgical solution. We say that it is an (r, δ)*–surgical solution* if it has the following properties:

(i) It has curvature pinched toward positive and satisfies $R(x, t) \leq \Theta$ for all $(x, t) \in \mathcal{M}$.

(ii) For every singular time $t_0 \in I$, $(M_+(t_0), g_+(t_0))$ is obtained from $(M(\cdot), g(\cdot))$ by (r, δ)–surgery at time t_0.

(iii) Condition $(\mathrm{CN})_r$ holds.

Let $\kappa > 0$. An (r, δ)–surgical solution which in addition satisfies Condition $(\mathrm{NC})_\kappa$ will be called an (r, δ, κ)*–surgical solution.*

5.2 Existence theorem for (r, δ, κ)–surgical solutions

Theorem 1.4 is implied by the following result:

Theorem 5.5 *For every $\rho_0 > 0$ and $T \geq 0$, there exist $r, \delta, \kappa > 0$ such that for any complete riemannian 3–manifold (M_0, g_0) with $|\mathrm{Rm}| \leq 1$ and injectivity radius at least ρ_0, there exists an (r, δ, κ)–surgical solution defined on $[0, T]$ satisfying the initial condition $(M(0), g(0)) = (M_0, g_0)$.*

Theorem 5.5 is itself a special case of the following result, which has the advantage of being suitable for iteration:

Theorem 5.6 *For every $Q_0, \rho_0 > 0$ and all $0 \le T_A < T_\Omega$, there exist $r, \delta, \kappa > 0$ such that for any complete riemannian 3–manifold (M_0, g_0) which satisfies $|\mathrm{Rm}| \le Q_0$, has injectivity radius at least ρ_0, has curvature pinched toward positive at time T_A, there exists an (r, δ, κ)–surgical solution defined on $[T_A, T_\Omega]$ satisfying the initial condition $(M(T_A), g(T_A)) = (M_0, g_0)$.*

Note that in the statement of Theorem 5.5 the assumption of almost nonnegative curvature is not necessary since it is automatic. We shall prove Theorem 5.6 directly.

Our next aim is to reduce Theorem 5.6 to three results, called Propositions A, B, C, which are independent of one another.

Proposition A *There exists a universal constant $\overline{\delta}_A > 0$ having the following property: let r, δ be surgery parameters, a, b be positive numbers with $a < b$, and $\{(M(t), g(t))\}_{t \in (a,b]}$ be an (r, δ)–surgical solution. Suppose $\delta \le \overline{\delta}_A$ and $R_{\max}(b) = \Theta$.*

Then there exists a riemannian manifold (M_+, g_+) obtained from $(M(\cdot), g(\cdot))$ by (r, δ)–surgery at time b, and in addition satisfies:

(i) *g_+ has ϕ_b–almost nonnegative curvature;*

(ii) *$R_{\min}(g_+) \ge R_{\min}(g(b))$.*

Remark 5.7 The manifold M_+ may be empty. In this case, the second assertion in the conclusion follows from the convention $R_{\min}(\varnothing) = +\infty$.

Proposition B *For all $Q_0, \rho_0, \kappa > 0$ there exist $r = r(Q_0, \rho_0, \kappa) < 10^{-3}$ and $\overline{\delta}_B = \overline{\delta}_B(Q_0, \rho_0, \kappa) > 0$ with the following property: let $\delta \le \overline{\delta}_B$, $0 \le T_A < b$ and $(M(\cdot), g(\cdot))$ be a surgical solution defined on $[T_A, b]$ such that $g(T_A)$ satisfies $|\mathrm{Rm}| \le Q_0$ and has injectivity radius at least ρ_0.*

Assume that $(M(\cdot), g(\cdot))$ satisfies Condition $(\mathrm{NC})_{\kappa/16}$, has curvature pinched toward positive, and that for each singular time t_0, $(M_+(t_0), g_+(t_0))$ is obtained from $(M(\cdot), g(\cdot))$ by (r, δ)–surgery at time t_0.

Then $(M(\cdot), g(\cdot))$ satisfies Condition $(\mathrm{CN})_r$.

Proposition C *For all $Q_0, \rho_0 > 0$ and all $0 \le T_A < T_\Omega$ there exists $\kappa = \kappa(Q_0, \rho_0, T_A, T_\Omega)$ such that for all $0 < r < 10^{-3}$ there exists $\overline{\delta}_C = \overline{\delta}_C(Q_0, \rho_0, T_A, T_\Omega, r) > 0$ such that the following holds.*

For all $0 < \delta \leq \overline{\delta}_C$ and $b \in (T_A, T_\Omega]$, every (r,δ)–surgical solution defined on $[T_A, b)$ such that $g(T_A)$ satisfies $|\mathrm{Rm}| \leq Q_0$ and has injectivity radius at least ρ_0, satisfies $(\mathrm{NC})_\kappa$.

Remark 5.8 The formulation of Proposition B, and its use below, are somewhat different in [1]. In the compact case, it is fairly easy to prove that the $(\mathrm{CN})_r$ property is open with respect to time (see [1, Lemma 5.3.2]). This is not the case here.

Proof of Theorem 5.6 assuming Propositions A, B, C We start with two easy lemmas. The first one allows to control the density of surgery times by the surgery parameters.

Lemma 5.9 *Let r, δ be surgery parameters. Let $\{(M(t), g(t))\}_{t\in I}$ be an (r,δ)–surgical solution. Let $t_1 < t_2$ be two singular times. Then $t_2 - t_1$ is bounded from below by a positive number depending only on r, δ.*

Proof We can suppose that $M(\cdot)$ is constant and $g(\cdot)$ is smooth on $(t_1, t_2]$. Since $R_{\max}(t_2) = \Theta$ we can choose a point $x \in M(t_2)$ such that $R(x, t_2) \geq R_{\max}(t_2) - 1$. Since $R_{\max}(g_+(t_1)) \leq \Theta/2$, there exists $t_+ \in [t_1, t_2]$ maximal such that we have $\lim_{t\to t_+, t>t_+} R(x,t) = \Theta/2$. In particular, (x,t) admits an (ε_0, C_0)–canonical neighbourhood for all $t \in (t_+, t_2]$. Integrating inequality (5) on $(t_+, t_2]$ gives a positive lower bound for $t_2 - t_+$ depending only on Θ, hence only on r, δ. □

The second one says that $(\mathrm{NC})_\kappa$ is a closed condition:

Lemma 5.10 *Let $(M(\cdot), g(\cdot))$ be a surgical solution defined on an interval $(a, b]$, $x \in M(b)$ and $r, \kappa > 0$. Suppose that for all $t \in (a,b)$, $x \in M(t)$ and $(M(\cdot), g(\cdot))$ is κ–noncollapsed at (x,t) on all scales less than or equal to r. Then it is κ–noncollapsed at (x,b) on the scale r.*

Its proof is identical as that of [1, Lemma 4.1.4.].

Let $Q_0, \rho_0 > 0$ and $0 \leq T_A < T_\Omega$. Proposition C gives a constant $\kappa = \kappa(Q_0, \rho_0, T_A, T_\Omega)$. Proposition B gives constants $r, \overline{\delta}_B$ depending on κ. We can assume $r^{-2} > 12 Q_0$. Then apply Proposition C again to get a constant $\overline{\delta}_C$. Set $\delta = \min(\overline{\delta}_A, \overline{\delta}_B, \overline{\delta}_C)$. Without loss of generality, we assume that $\kappa \leq \kappa_{\mathrm{st}}$.

From r, δ we get the cutoff parameters h, D, Θ.

Let (M_0, g_0) be a riemannian manifold which has ϕ_A–almost nonnegative curvature, satisfies $R_{\min}(g_0) \geq -6/(4T_A+1)$, $|\mathrm{Rm}| \leq Q_0$, and has injectivity radius at least ρ_0.

Let $\mathcal{X}$ be the set of ordered pairs $(b, \{(M(t), g(t))\}_{t\in[T_A,b)})$ consisting of a number $b \in (T_A, T_\Omega]$ and an (r,δ,κ)–surgical solution such that $(M(T_A), g(T_A)) = (M_0, g_0)$. We first show that $\mathcal{X}$ is nonempty. By standard results on the Ricci flow (see eg [7, Lemma 6.1]) there exists a complete Ricci flow solution $g(\cdot)$ defined on $[T_A, T_A + (16Q_0)^{-1}]$, such that $g(T_A) = g_0$ and $|\mathrm{Rm}| \leqslant 2Q_0$ on the interval. By Proposition 3.3, $g(\cdot)$ has curvature pinched toward positive on $[T_A, T_A + (16Q_0)^{-1}]$. As $R \leqslant 12Q_0 < r^{-2} < \Theta$, $g(\cdot)$ satisfies Property (i) of an (r,δ)–surgical solution, and Properties (ii), (iii) are vacuously satisfied. By Proposition C, it satisfies $(\mathrm{NC})_\kappa$ on the interval. Hence $g(\cdot)$ is a (r,δ,κ)–surgical solution on $[T_A, T_A + (16Q_0)^{-1}]$.

The set $\mathcal{X}$ has a partial ordering defined by $(b_1, (M_1(\cdot), g_1(\cdot))) \le (b_2, (M_2(\cdot), g_2(\cdot)))$ if $b_1 \le b_2$ and $(M_2(\cdot), g_2(\cdot))$ is an extension of $(M_1(\cdot), g_1(\cdot))$.

We want to use Zorn's lemma to prove existence of a maximal element in $\mathcal{X}$. In order to do this, we consider an *infinite chain,* i. e. an infinite sequence of numbers $T_A < b_1 < b_2 < \cdots b_n < \cdots < T_\Omega$ and of (r,δ,κ)–surgical solutions defined on the intervals $[T_A, b_n)$, and which extend one another. In this way we get an evolving manifold $\{(M(t), g(t))\}$ defined on $[T_A, b_\infty)$, where b_∞ is the supremum of the b_n's. By Lemma 5.9, the set of singular times is a discrete subset of $\mathbf{R}$, so $\{(M(t), g(t))\}_{t\in[T_A,b_\infty)}$ is an (r,δ,κ)–surgical solution, thus a majorant of the increasing sequence.

Hence we can apply Zorn's lemma. Let $(b_{\max}, (M(\cdot), g(\cdot))) \in \mathcal{X}$ be a maximal element. Its scalar curvature lies between -6 and Θ, so it is bounded independently of t. Its curvature is pinched toward positive so the sectional curvature is also bounded independently of t. Using the Shi estimates, we deduce that all derivatives of the curvature are also bounded at time $b_{\max}$. This allows to take a smooth limit and extend $(M(\cdot), g(\cdot))$ to a surgical solution defined on $[T_A, b_{\max}]$, with $R_{\max}(b_{\max}) \le \Theta$. Condition $(\mathrm{NC})_\kappa$ is still satisfied on this closed interval by Lemma 5.10. Hence we can apply Proposition B, which implies that Property $(\mathrm{CN})_r$ is satisfied on $[T_A, b_{\max}]$. We thus obtain an (r,δ,κ)–surgical solution on the closed interval $[T_A, b_{\max}]$.

To conclude, we prove by contradiction that $b_{\max} = T_\Omega$. Assume that $b_{\max} < T_\Omega$ and consider the following two cases:

Case 1 $R_{\max}(b_{\max}) < \Theta$.

Applying Shi's short time existence theorem for Ricci flow with initial metric $g(b_{\max})$, we can extend $g(\cdot)$ to a complete smooth Ricci flow with bounded curvature defined on an interval $[T_A, b_{\max} + \alpha)$ for some $\alpha > 0$. By Proposition 3.3 the extension satisfies the hypothesis that the curvature is pinched toward positive.

Lemma 5.11 *There exists $\alpha' \in (0,\alpha]$ such that Condition* $(\mathrm{NC})_{\kappa/16}$ *holds for* $\{g(t)\}_{t\in[T_A,b_{\max}+\alpha')}$.

Proof Let $x \in M(b_{\max})$, $t \in (b_{\max}, b_{\max}+\alpha')$ and $\rho \in (0,10^{-3})$ be such that $|\mathrm{Rm}| \le \rho^{-2}$ on $P(x,t,\rho,-\rho^2)$. Choosing α' small enough, we can ensure that $B(x,b_{\max},\rho/2) \subset B(x,t,\rho)$ and moreover, $P(x,b_{\max},\rho/2,-\rho^2/4) \subset P(x,t,\rho,-\rho^2)$. It follows that $|\mathrm{Rm}| \le 4\rho^{-2}$ on $P(x,b_{\max},\rho/2,-\rho^2/4)$. Since $(\mathrm{CN})_\kappa$ is satisfied up to time $b_{\max}$, we deduce that $\mathrm{vol}\, B(x,b_{\max},\rho/2) \ge \kappa(\rho/2)^3$. Again by proper choice of α', $\mathrm{vol}_{g(t)}\, B(x,b_{\max},\rho/2)$ is at least half of $\mathrm{vol}\, B(x,b_{\max},\rho/2)$. Hence

$$\mathrm{vol}\, B(x,t,\rho) \ge \mathrm{vol}_{g(t)}\, B(x,b_{\max},\rho/2) \ge \frac{1}{2}\,\mathrm{vol}\, B(x,b_{\max},\rho/2) \ge \frac{\kappa}{16}\rho^3. \qquad \square$$

Setting $M(t) := M(b_{\max})$ for $t \in (b_{\max}, b_{\max}+\alpha')$, we obtain a surgical solution $\{(M(t),g(t))\}_{t\in[T_A,b_{\max}+\alpha')}$ satisfying assumptions of Proposition B. It follows that Condition $(\mathrm{CN})_r$ is satisfied on $[T_A, b_{\max}+\alpha')$. Integrating inequality (5) on $[b_{\max}, b_{\max}+\alpha')$, one obtains $R_{\max} < \Theta$ on this interval for $\alpha' > 0$ small enough. We thus have that $\{(M(t),g(t))\}_{t\in[T_A,b_{\max}+\alpha')}$ is an (r,δ)–surgical solution. By Proposition C, it is an (r,δ,κ)–surgical solution. This contradicts maximality of $b_{\max}$.

Case 2 $R_{\max}(b_{\max}) = \Theta$.

Proposition A yields a riemannian manifold (M_+, g_+). If M_+ is empty, then the construction stops. Suppose $M_+ \ne \varnothing$. Applying Shi's short time existence theorem for Ricci flow on M_+ with initial metric g_+, we obtain a positive number α and a complete smooth Ricci flow with bounded curvature $\{g(t)\}_{t\in(b_{\max},b_{\max}+\alpha)}$ on M_+ whose limit from the right as t tends to $b_{\max}$ is equal to g_+. By Proposition 3.3 it has curvature pinched toward positive.

Lemma 5.12 *There exists $\alpha' \in (0,\alpha]$ such that Condition* $(\mathrm{NC})_{\kappa/16}$ *holds on the interval* $[T_A, b_{\max}+\alpha')$.

Proof Let $x \in M(b_{\max})$, $t \in (b_{\max}, b_{\max}+\alpha')$ and $\rho \in (0,10^{-1})$ be such that $|\mathrm{Rm}| \le \rho^{-2}$ on $P(x,t,\rho,-\rho^2)$. If $B(x,t,\rho/2)$ is unscathed and stays so until $b_{\max}$, then we can repeat the argument used to prove Lemma 5.11. Otherwise it follows from the assumption $\kappa \le \kappa_{\mathrm{st}}$ and properties of almost standard caps. $\square$

Setting $M(t) := M_+$ for $t \in (b_{\max}, b_{\max}+\alpha')$, we obtain a surgical solution $\{(M(t),g(t))\}_{t\in[T_A,b_{\max}+\alpha')}$ satisfying assumptions of Proposition B. It follows that Condition $(\mathrm{CN})_r$ is satisfied on $[T_A, b_{\max}+\alpha')$. Since $R_{\max}(g_+) \le \Theta/2$, we obtain by integrating (5) on $(b_{\max}, b_{\max}+\alpha')$, that $R_{\max} < \Theta$ on this interval for α' small enough. We thus have that $\{(M(t),g(t))\}_{t\in[T_A,b_{\max}+\alpha')}$ is an (r,δ)–surgical solution. By Proposition C, it is an (r,δ,κ)–surgical solution. Again this contradicts the assumption that $b_{\max}$ should be maximal. $\square$

6 Choosing cutoff parameters

In this section, we give some technical results necessary to prove Theorem 5.1. Their statements are nearly identical to those of the corresponding results of [1, Chapter 6], *surgical solutions* replacing *Ricci flow with bubbling-off*. The proofs are also almost identical, the minor adaptations being made precise below.

6.1 Bounded curvature at bounded distance

We shall need the following well-known consequence of curvature pinching:

Proposition 6.1 *Let $(U_k, g_k(\cdot), *_k)$ be a sequence of pointed evolving metrics defined on intervals $I_k \subset \mathbf{R}_+$, and having curvature pinched toward positive. Let $(x_k, t_k) \in U_k \times I_k$ be a sequence such that $(1+t_k)R(x_k, t_k)$ goes to $+\infty$. Then the sequence $\bar{g}_k := R(x_k, t_k)g(t_k)$ has the following properties:*

(i) *The sequence $R_{\min}(\bar{g}_k)$ tends to 0.*

(ii) *If $(U_k, \bar{g}_k, *_k)$ converges in the pointed $\mathcal{C}^2$ sense, then the limit has nonnegative curvature operator.*

We also recall:

Lemma 6.2 [1, Lemma 3.3.2] *Let $\varepsilon \in (0, 10^{-1}]$. Let (M, g) be a riemannian 3–manifold, $N \subset M$ be an ε–neck, and S be a middle sphere of N. Let $[xy]$ be a geodesic segment such that $x, y \in M \setminus N$ and $[xy] \cap S \neq \varnothing$. Then the intersection number of $[x, y]$ with S is odd. In particular, if S is separating in M, then x, y lie in different components of $M \setminus S$.*

We summarise the conclusion of Lemma 6.2 by saying that N is *traversed* by the segment $[xy]$.

Corollary 6.3 *Let $\varepsilon \in (0, 10^{-1}]$. Let (M, g) be a riemannian 3–manifold, $U \subset M$ be an ε–cap and V be a core of U. Let x, y be points of $M \setminus U$ and $[xy]$ a geodesic segment connecting x to y. Then $[xy] \cap V = \varnothing$.*

As for Ricci flow with bubbling-off, we then have:

Theorem 6.4 (Curvature-distance) *For all $A, C > 0$ and all $\varepsilon \in (0, 2\varepsilon_0]$, there exist $Q = Q(A, \varepsilon, C) > 0$ and $\Lambda = \Lambda(A, \varepsilon, C) > 0$ with the following property. Let $I \subset [0, +\infty)$ be an interval and $\{(M(t), g(t))\}_{t \in I}$ be a surgical solution with curvature pinched toward positive. Let $(x_0, t_0) \in \mathcal{M}$ be such that:*

(i) $R(x_0, t_0) \geq Q$;

(ii) *For each point* $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, *if* $R(y, t) \geq 2R(x_0, t)$, *then* (y, t) *has an* (ε, C)*–canonical neighbourhood.*

Then for all $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, *we have*

$$\frac{R(y, t_0)}{R(x_0, t_0)} \leq \Lambda.$$

Proof It suffices to redo the proof of [1, Theorem 6.1.1], with the following minor differences:

- In Step 1, to control the injectivity radius, one can use property (iii) in the definition of (ε, C)–canonical neighbourhoods, as the canonical neighbourhood considered is not ε–round.
- In Step 2, to prove that $[x'_k y'_k]$ is covered by strong ε–necks, one has to rule out closed canonical neighbourhoods. This is clear by the curvature ratio properties. Then use Corollary 6.3 instead of [1, Lemma 3.3.2]. □

6.2 Existence of cutoff parameters

For the convenience of the reader, we restate Theorem 5.1.

Theorem 6.5 (Cutoff parameters) *For all* $r, \delta > 0$, *there exist* $h \in (0, \delta r)$ *and* $D > 10$ *such that if* $(M(\cdot), g(\cdot))$ *is a complete surgical solution of bounded curvature defined on an interval* $[a, b]$, *with curvature pinched toward positive and satisfying* $(\mathrm{CN})_r$, *then the following holds:*

Let $t \in [a, b]$ *and* $x, y, z \in M(t)$ *such that* $R(x, t) \leq 2/r^2$, $R(y, t) = h^{-2}$, *and* $R(z, t) \geq D/h^2$. *Assume there is a curve* γ *connecting* x *to* z *and containing* y, *such that each point of* γ *with scalar curvature in* $[2C_0 r^{-2}, C_0{}^{-1} D h^{-2}]$ *is the centre of a* ε_0*–neck. Then* (y, t) *is the centre of a strong* δ*–neck.*

Proof The proof is almost the same as for [1, Theorem 6.2.1], arguing by contradiction. We only need to adapt Step 1.

Fix constants $r > 0, \delta > 0$, sequences $h_k \to 0$, $D_k \to +\infty$, a sequence $(M_k(\cdot), g_k(\cdot))$ of surgical solutions satisfying the above hypotheses, and sequences $t_k > 0$, x_k, y_k, $z_k \in M$ such that $R(x_k, t_k) \leq 2r^{-2}$, $R(z_k, t_k) \geq D_k h_k^{-2}$, and $R(y_k, t_k) = h_k^{-2}$. Let γ_k be a curve from x_k to z_k such that $y_k \in \gamma_k$, whose points of scalar curvature in $[2C_0 r^{-2}, C_0{}^{-1} D h^{-2}]$ are centre of ε_0–neck. Finally assume that (y_k, t_k) is not the centre of a strong δ–neck.

Consider the sequence $(\overline{M}_k(\cdot), \bar{g}_k(\cdot))$ defined by the following parabolic rescaling

$$\bar{g}_k(t) = h_k^{-2} g_k(t_k + t h_k^2)\,.$$

In order to clarify notation, we shall put a bar on points when they are involved in geometric quantities computed with respect of the metric $\bar{g}_k$. Thus for instance, we have $R(\bar{y}_k, 0) = 1$. The contradiction will come from extracting a converging subsequence of the pointed sequence $(\overline{M}_k(\cdot), \bar{g}_k(\cdot), \bar{y}_k, 0)$ and showing that the limit is the cylindrical flow on $S^2 \times \mathbf{R}$, which implies that for k large enough, y_k is the centre of some strong δ–neck, contradicting our hypothesis.

Step 1 $(\overline{M}_k(0), \bar{g}_k(0), \bar{y}_k)$ subconverges in the pointed $\mathcal{C}^\infty$ sense to $(S^2 \times \mathbf{R}, g_\infty)$ where g_∞ is a product metric of nonnegative curvature operator and scalar curvature at most 2.

Proof First we control the curvature on balls around $\bar{y}_k$. Since $R(y_k, t_k)$ goes to $+\infty$, Theorem 6.4 implies that for all $\rho > 0$, there exists $\Lambda(\rho) > 0$ and $k_0(\rho) > 0$ such that $\bar{g}_k(0)$ has scalar curvature bounded above by $\Lambda(\rho)$ on $B(\bar{y}_k, \rho)$ for $k \geq k_0(\rho)$. Moreover, by Assumption (iii) of the definition of canonical neighbourhoods, $\bar{g}_k(\cdot)$ is ${C_0}^{-1}$–noncollapsed at $(y_k, 0)$. Indeed $R(y_k, t_k) = {h_k}^{-2} \in [2C_0 r^{-2}, {C_0}^{-1} D h^{-2}]$, hence y_k is the centre of a ε_0–neck. Thus we can apply Gromov's compactness theorem to extract a converging subsequence with regularity $C^{1,\alpha}$.

Let us prove that for large k, the ball $B(\bar{y}_k, \rho)$ is covered by ε_0–necks. Recall that $g_k(t_k)$ satisfies

$$|\nabla R| < C_0 R^{3/2},$$

at points covered by canonical neighbourhoods. Take a point y such that $R(y, t_k) \leq 2C_0 r^{-2}$ and integrate the previous inequality on the portion of $[y_k y]$ where $R \geq 2C_0 r^{-2}$. An easy computation yields

$$d(\bar{y}, \bar{y}_k) \geq \frac{1}{h_k}\frac{2}{C_0}\left(\frac{r}{\sqrt{2C_0}} - h_k\right) \geqslant \rho\,, \tag{6}$$

for k larger than some $k_1(\rho) \geq k_0(\rho)$. It follows that the scalar curvature of $g_k(t_k)$ is at least $2C_0 r^{-2}$ on $B(\bar{y}_k, \rho)$ for every integer $k \geq k_1(\rho)$. It follows that $x_k \notin B(\bar{y}_k, \rho)$ and that $B(\bar{y}_k, \rho)$ is covered by (ε_0, C_0)–canonical neighbourhoods. On the other hand, for k larger than some $k_2(\rho)$, we have $R(\bar{y}, 0) \leqslant \Lambda(\rho) < {C_0}^{-1} D h^{-2}$ for all $\bar{y} \in B(\bar{y}_k, \rho)$. It follows that $\gamma \cap B(\bar{y}_k, \rho)$ is covered by ε_0–necks. As $z_k \notin B(\bar{y}_k, \rho)$, it follows that $B(\bar{y}_k, \rho)$ is contained in the union $U_{\rho,k}$ of these necks: indeed, every segment coming from $\bar{y}_k$ and of length less than ρ lies there.

Now by the $(\mathrm{CN})_r$ assumption, these necks are strong ε_0–necks. The scalar curvature on $B(\bar{y}_k, \rho)$ being less than $\Lambda(\rho)$ for $k \geqslant k_0$, we deduce that on each strong neck,

$\overline{g}_k(t)$ is smoothly defined on $[-1/(2\Lambda(\rho)), 0]$, and has curvature bounded above by $2\Lambda(\rho)$. Hence for each $\rho > 0$, the parabolic balls $P\big(\overline{y}_k, 0, \rho, -1/(2\Lambda(\rho))\big)$ are unscathed, with scalar curvature bounded above by $2\Lambda(\rho)$ for all $k \geq k_2(\rho)$. Since $g_k(\cdot)$ has curvature pinched toward positive, this implies a uniform control of the curvature operator there.

Hence we can apply the local compactness theorem for Ricci flow [1, Theorem C.3.3]. Up to extracting, $(\overline{M}_k(0), \overline{g}_k(0), \overline{y}_k)$ converges to some complete noncompact pointed riemannian 3–manifold $(\overline{M}_\infty, \overline{g}_\infty, \overline{y}_\infty)$. By Proposition 6.1, the limit has nonnegative curvature operator.

Passing to the limit, we get a covering of $\overline{M}_\infty$ by $2\varepsilon_0$–necks. Then Proposition 7.6 shows that $\overline{M}_\infty$ is diffeomorphic to $S^2 \times \mathbf{R}$. In particular, it has two ends, so it contains a line, and Toponogov's theorem implies that it is the metric product of some (possibly nonround) metric on S^2 with $\mathbf{R}$.

As a consequence, the spherical factor of this product must be $2\varepsilon_0$–close to the round metric on S^2 with scalar curvature 1. Hence the scalar curvature is bounded above by 2 everywhere. This finishes the proof of Step 1. □

Henceforth we pass to a subsequence, so that $(\overline{M}_k(0), \overline{g}_k(0))$ satisfies the conclusion of Step 1.

The rest of the proof is the same as for [1, Theorem 6.2.1]. □

7 Proof of Proposition A

7.1 Piecing together necks and caps

Definition 7.1 An *ε–tube* is an open subset $U \subset M$ which is equal to a union of ε–necks, and whose closure in M is diffeomorphic to $S^2 \times I$, $S^2 \times \mathbf{R}$ or $S^2 \times [0, +\infty)$.

Proposition 7.2 *Let $\varepsilon \in (0, 2\varepsilon_0]$. Let (M, g) be a connected, orientable riemannian 3–manifold. Let X be a closed, connected subset of M such that every point of X is the centre of an ε–neck or an ε–cap. Then there exists an open subset $U \subset M$ containing X such that either*

(i) *U is equal to M and diffeomorphic to S^3, $S^2 \times S^1$, $\mathbf{R}P^3$, $\mathbf{R}P^3 \# \mathbf{R}P^3$, $\mathbf{R}^3$, $S^2 \times \mathbf{R}$ or a punctured $\mathbf{R}P^3$, or*

(ii) *U is a 10ε–cap, or*

(iii) *U is a 10ε–tube.*

Proof First we deal with the case where X is covered by ε–necks.

Lemma 7.3 *If every point of X is the centre of an ε–neck, then there exists an open set U containing X such that U is a 10ε–tube, or U is equal to M and diffeomorphic to $S^2 \times S^1$ or $S^2 \times \mathbf{R}$.*

Proof Let x_0 be a point of X and N_0 be a 10ε–neck centred at x_0, contained in an ε–neck U_0, also centred at x_0. If $X \subset N_0$ we are done. Otherwise, since X is connected, we can pick a point $x_1 \in X \cap N_0$ and a 10ε–neck N_1 centred at x_1, with x_1 arbitrarily near the boundary of N_0. By Lemma 3.6, an appropriate choice of x_1 ensures that $N_1 \subset U_0$ and the middle spheres of N_0 and N_1 are isotopic. In particular, the closure of $N_0 \cup N_1$ is diffeomorphic to $S^2 \times I$, so $N_0 \cup N_1$ is a 10ε–tube.

If $X \subset N_0 \cup N_1$ then we can stop. Otherwise, we pick a 10ε–neck N_2 centred at some point x_2 near the boundary component of N_1 that does not lie in N_0, and iterate the construction as long as possible. Three cases may occur.

Case 1 The construction stops with some 10ε–tube $N_0 \cup \cdots \cup N_k$ containing X. Then we are done.

Case 2 The construction stops with some 10ε–tube $N_0 \cup \cdots \cup N_k$ such that adding another neck N_{k+1} does *not* produce a 10ε–tube.

This can only happen if $N_{k+1} \cap N_0$ is non empty. In this case, by adjusting the centre x_{k+1} of N_{k+1}, we can ensure that $N_0, \dots, N_{k+1}$ cover M and that the intersection of N_{k+1} and N_0 is topologically standard. In this case, M fibers over the circle with fiber S^2. Since M is orientable, it follows that M is diffeomorphic to $S^2 \times S^1$.

Case 3 The construction can be iterated forever.

In this case, the union U of all N_k's is a 10ε–tube.

Claim *The frontier of U is connected, equal to the boundary component of $\bar{N}_0$ which does not lie in N_1.*

We prove the claim by contradiction. If it is not true, then we can pick two points $x, y \in X$, each one being close to a distinct component of the frontier of U. Since $U \cap X$ is connected, we can find a path γ connecting x to y in $U \cap X$. Now γ is compact, so it can be covered by finitely many 10ε–necks, each of which is contained in some ε–neck. We thus obtain a finite collection of ε–necks which cover U. Hence U is relatively compact. This shows that the scalar curvature is bounded on U. Hence each N_k has a definite size, and adding each N_k to $N_0, \dots, N_{k-1}$ adds definite volume. It follows that $\operatorname{vol} U$ is infinite, which is a contradiction. This proves the claim.

We continue the proof of Lemma 7.3. If U contains X, then we are done. Otherwise, we pick a point $x_{-1} \in N_0 \cap X$ close to the frontier, and choose a neck N_{-1} centred at x_{-1}. We perform the same iterated construction as before. At each stage, we have a 10ε–tube $N_{-k} \cup \cdots \cup N_{-1} \cup U$ whose frontier is connected. Hence the analogue of Case 2 above cannot occur. If the construction stops, then we have found a 10ε–tube containing X. Otherwise the union V of all N_k's for $k \in \mathbf{Z}$ is a 10ε–tube. Repeating the argument used to prove the claim, we see that the frontier of V is empty. Since M is connected, it follows that $V = M \cong S^2 \times \mathbf{R}$. □

To complete the proof of Proposition 7.2, we need to deal with the case where there is a point $x_0 \in X$ which is the centre of an ε–cap C_0. By definition of a cap, some collar neighbourhood of the boundary of C_0 is an ε–neck U_0. If $X \not\subset C_0$, pick a point x_1 close to the boundary of C_0. If x_1 is the centre of a 10ε–neck N_1, then we apply Lemma 3.6 again to find that $C_1 := C_0 \cup N_1$ is a 10ε–cap. Again we iterate this construction until one of the following things occur:

Case 1 The construction stops with a 10ε–cap containing X.

Case 2 The construction stops with a 10ε–cap $C_k = C_0 \cup \cdots \cup N_k$ and a point x_{k+1} near its frontier, such that x_{k+1} is the centre of a 10ε–cap C whose boundary is contained in C_k. Then $C_k \cup C$ equals M and is diffeomorphic to S^3, $\mathbf{R}P^3$ or $\mathbf{R}P^3 \# \mathbf{R}P^3$.

Case 3 The construction goes on forever. Then the same volume argument as in the proof of the above Claim shows that the union of all $C_k' s$ is M. Thus M itself is a 10ε–cap, diffeomorphic to $\mathbf{R}^3$ or a punctured $\mathbf{R}P^3$. □

Putting $X = M$, we obtain the following corollary:

Theorem 7.4 *Let $\varepsilon \in (0, 2\varepsilon_0]$. Let (M, g) be a connected, orientable riemannian 3–manifold. If every point of M is the centre of an ε–neck or an ε–cap, then M is diffeomorphic to S^3, $S^2 \times S^1$, $\mathbf{R}P^3$, $\mathbf{R}P^3 \# \mathbf{R}P^3$, $\mathbf{R}^3$, $S^2 \times \mathbf{R}$ or a punctured $\mathbf{R}P^3$.*

Here is another consequence of Proposition 7.2:

Corollary 7.5 *Let $\varepsilon \in (0, 2\varepsilon_0]$. Let (M, g) be an orientable riemannian 3–manifold. Let X be a closed submanifold of M such that every point of X is the centre of an ε–neck or an ε–cap. Then one of the following conclusions holds:*

(i) *M is diffeomorphic to S^3, $S^2 \times S^1$, $\mathbf{R}P^3$, $\mathbf{R}P^3 \# \mathbf{R}P^3$, $\mathbf{R}^3$, $S^2 \times \mathbf{R}$ or a punctured $\mathbf{R}P^3$.*

(ii) *There exists a locally finite collection $N_1, \ldots, N_p$ of 10ε–caps and 10ε–tubes with disjoint closures such that $X \subset \bigcup_i N_i$.*

Proof We apply Proposition 7.2 to each connected component of X. If Case (i) of the required conclusion does not hold, then we have found a locally finite collection of 10ε–caps and 10ε–tubes which cover X. By merging some of them if necessary, we can ensure that they have disjoint closures. □

Finally, we have a more precise result when there are just necks:

Proposition 7.6 *Let $\varepsilon \in (0, 2\varepsilon_0]$. Let (M, g) be an open riemannian 3–manifold such that every point of M is the centre of an ε–neck. Then M is diffeomorphic to $S^2 \times \mathbf{R}$.*

This follows immediately from the proof of Lemma 7.3.

7.2 Proof of Proposition A

Recall the statement:

Proposition A *There exists a universal constant $\overline{\delta}_A > 0$ having the following property: let r, δ be surgery parameters, a, b be positive numbers with $a < b$, and $\{(M(t), g(t))\}_{t \in (a,b]}$ be an (r, δ)–surgical solution. Suppose $\delta \le \overline{\delta}_A$ and $R_{\max}(b) = \Theta$.*

Then there exists a riemannian manifold (M_+, g_+) obtained from $(M(\cdot), g(\cdot))$ by (r, δ)–surgery at time b, and in addition satisfies:

(i) *g_+ has ϕ_b–almost nonnegative curvature;*

(ii) *$R_{\min}(g_+) \ge R_{\min}(g(b))$.*

Throughout this section we shall work in the riemannian manifold $(M(b), g(b))$. In particular all curvatures and distances are taken with respect to this metric.

Let $\mathcal{G}$ (resp. $\mathcal{O}$, resp. $\mathcal{R}$) be the set of points of $M(b)$ of scalar curvature less than $2r^{-2}$ (resp. $\in [2r^{-2}, \Theta/2)$, resp. $\ge \Theta/2$).

For brevity, we call *cutoff neck* a strong δ–neck centred at some point of scalar curvature h^{-2}. Note that cutoff necks are contained in $\mathcal{O}$, and have diameter and volume bounded by functions of h, δ–alone.

Lemma 7.7 *There exists a locally finite collection $\{N_i\}$ of pairwise disjoint cutoff necks such that any connected component of $M(b)\setminus\bigcup_i N_i$ is contained in either $\mathcal{G}\cup\mathcal{O}$ or $\mathcal{R}\cup\mathcal{O}$.*

Proof By Zorn's Lemma, there exists a maximal collection $\{N_i\}$ of pairwise disjoint cutoff necks. Such a collection is automatically locally finite, eg because if K is any compact subset, all cutoff necks that meet K are contained in the $h(2\delta^{-1}+1)$–neighbourhood of K, which has finite volume.

Suppose that some component X of $M(b)\setminus\bigcup_i N_i$ contains at least one point $x\in\mathcal{G}$ and one point $z\in\mathcal{R}$. Since X is a closed subset of $M(b)$, there exists a geodesic path γ in X connecting x to z.

Claim *The intersection of γ and ∂X is empty.*

Proof of the claim First observe that if y is a point of ∂X, then y has a canonical neighbourhood U. This neighbourhood cannot be a cap, because then U would contain the whole of X, which would imply that $X\subset\mathcal{O}$. Hence U is a neck.

If such a point y belonged to γ, then by Lemma 6.2 the neck U would be traversed by γ. This contradicts the fact that $\gamma\subset X$. □

In order to apply Theorem 5.1, one has to prove the following:

Claim *Each point of γ with scalar curvature in $[2C_0r^{-2}, {C_0}^{-1}Dh^{-2}]$ is the centre of some ε_0–neck.*

Proof of the claim Let $y\in\gamma$ be such a point. By the curvature assumptions, y is the centre of a (ε_0, C_0)–canonical neighbourhood U, disjoint from x and z. Hence U cannot be a closed manifold. It remains to rule out the (ε_0, C_0)–cap case. We argue by contradiction. Assume that U is an (ε_0, C_0)–cap. Then $U = N\cup C$, where N is a ε_0–neck, $N\cap C=\varnothing$, $\bar{N}\cap C=\partial C$ and $y\in\operatorname{Int} C$. For simplicity dilate the metric by a factor such that the scalar curvature of N is close to 1. Denote by S the middle sphere of N. The curve γ is clearly not minimizing in U. In particular if x' (resp. z') is an intersection point of γ with S lying between x and y (resp. y and z), then $d(x',z')\le\operatorname{diam}(S) << 2{\varepsilon_0}^{-1} < d(x',y)+d(y,z')$. The geodesic segment $[x'z']\subset U$ is not contained in X, otherwise this would contradict the minimality of γ in X. Hence there exists $p\in[x',z']\cap\partial X$. By definition of X, the corresponding component of ∂X is a boundary component of some neck N_i. Let us prove that γ intersects N_i, which is a contradiction. Denote by ${S_i}^+$ the above boundary component of N_i, and note that $d({S_i}^+, S) < \mathrm{diam}(S)$. Let N' be a $10\varepsilon_0$–subneck of N_i which admits ${S_i}^+$ as a boundary component, and $p' \in N'$ be its centre. Then $d(p', S) < \mathrm{diam}(S) + (10\varepsilon_0)^{-1} < (4\varepsilon_0)^{-1}$. It follows from Lemma 3.6 that S' is isotopic to S in N. In particular γ intersects S'. □

Let y be a point of γ of scalar curvature h^{-2}. Theorem 5.1 yields a cutoff neck N centred at y. Any δ–neck meeting N has to be traversed by γ, so N is disjoint from the N_i's. This contradicts maximality of $\{N_i\}$. □

Having established Lemma 7.7, we prove Proposition A. Let $\{N_i\}$ be a collection of cutoff necks given by that lemma. Applying Theorem 3.8, we obtain a Riemannian manifold (M', g_+). By construction, the components of M' fall into two types. Either they have curvature less than $\Theta/2$, or they are covered by canonical neighbourhoods. Applying Theorem 7.4, we may safely throw away the components of the second type, obtaining the manifold (M_+, g_+). We remark that the operation cannot decrease $R_{\min}$ (in fact $R_{\min}(g_+)$ is equal to $R_{\min}(g(b))$ unless M_+ is empty, in which case it is equal to $+\infty$). Thus the proof of Proposition A is complete.

8 Persistence

Notation If $(M(\cdot), g(\cdot))$ is a piecewise $\mathcal{C}^1$ evolving manifold defined on some interval $I \subset \mathbf{R}$ and $[a,b] \subset I$, we call *restriction of* g to $[a,b]$ the evolving manifold

$$t \mapsto \begin{cases} (M_+(a), g_+(a)) & \text{if } t = a, \\ (M(t), g(t)) & \text{if } t \in (a,b]. \end{cases}$$

We shall still denote by $g(\cdot)$ the restriction. Given $(x,t) \in \mathcal{M}$, $r > 0$ and $\Delta t > 0$ we define the forward parabolic neighbourhood $P(x,t,r,\Delta t)$ as the set

$$P(x,t,r,\Delta t) = \{(x',t') \in \mathcal{M} \mid x' \in B(x,t,r), t \leqslant t' \leqslant t + \Delta t\}.$$

When we consider a restriction of $g(\cdot)$ to some $[a,b] \subset I$, the parabolic neighbourhood $P(x,a,r,\Delta t)$ will be defined using the ball $B(x,a,r)$ of radius r with respect to the metric $g_+(a)$.

A parabolic neighbourhood $P(x,t,r,\Delta t)$ is said to be *unscathed* if $x' \in M_{\mathrm{reg}}(t')$ for all $x' \in B(x,t,r)$ and $t' \in [t, t+\Delta t)$. Otherwise it is *scathed*.

Given two surgical solutions $(M(\cdot),, g(\cdot))$ and $(M_0(\cdot), g_0(\cdot))$, we say that an unscathed parabolic neighbourhood $P(x,t,r,\Delta t)$ of $(M(\cdot), g(\cdot))$ is ε*–close* to another unscathed parabolic neighbourhood $P(x_0,t,r_0,\Delta t)$ of $(M_0(\cdot), g_0(\cdot))$ if

$(B(x,t,r), g(\cdot))$ is ε–close to $(B(x_0,t,r_0), g_0(\cdot))$ on $[t, t+\Delta t]$. We say that $P(x,t,r,\Delta t)$ is *ε–homothetic* to $P(x_0,t_0,r_0,\lambda\Delta t))$ if it is ε–close after a parabolic rescaling by λ.

The goal of this section is to prove the following technical theorem:

Theorem 8.1 (Persistence of almost standard caps) *For all $A>0$, $\theta \in [0,1)$ and $\hat{r}>0$, there exists $\overline{\delta} = \overline{\delta}_{\mathrm{per}}(A,\theta,\hat{r})$ with the following property. Let $(M(\cdot), g(\cdot))$ be a surgical solution defined on some interval $[a,b]$, which is a (r,δ)–surgical solution on $[a,b)$, with $r \geqslant \hat{r}$ and $\delta \geqslant \overline{\delta}$. Let $t_0 \in [a,b)$ be a singular time and consider the restriction of $(M(\cdot), g(\cdot))$ to $[t_0,b]$. Let $p \in (M(t_0), g(t_0))$ be the tip of some δ–almost standard cap of scale h. Let $t_1 \leq \min(b, t_0+\theta h^2)$ be maximal such that $P(p,t_0,Ah,t_1-t_0)$ is unscathed. Then the following holds:*

(i) *The parabolic neighbourhood $P(p,t_0,Ah,t_1-t_0)$ is A^{-1}–homothetic to the neighbourhood $P(p_0,0,A,(t_1-t_0)h^{-2})$.*
(ii) *If $t_1 < \min(b, t_0+\theta h^2)$, then $B(p,t_0,Ah) \subset M_{\mathrm{sing}}(t_1)$ disappear at time t_1.*

Remark 8.2 In [1], the conclusion in the last case was that $B(p,t_0,Ah) \subset M_{\mathrm{sing}}(t_1)$.

Before giving the proof of the theorem, we summarize some technical results from [1, Chapter 8].

Let T_0 be a positive real number. Let $\mathcal{M}_0 = (X_0, g_0(\cdot), p_0)$ (the model) be a complete 3–dimensional Ricci flow defined on $[0,T_0]$ such that the quantity

$$\Lambda(N) := \max_{X_0\times[0,T_0]} \{ |\nabla^p \mathrm{Rm}| \mid 0 \leq p \leq N \}$$

is finite for all $N \in \mathbf{N}$.

Corollary 8.3 (Persistence of the model in dimension 3) *Let $A>0$, there exists $\rho = \rho(\mathcal{M}_0, A) > A$ with the following property. Let $\{(M(t), g(t))\}_{t\in[0,T]}$ be a surgical solution with $T \leq T_0$. Suppose that:*

(a) *$(M(\cdot), g(\cdot))$ has curvature pinched toward positive;*
(b) *$|\partial R/\partial t| \leq C_0 R^2$ at any (x,t) with $R(x,t) \geqslant 1$.*

Let $p \in M(0)$ and $t \in (0,T]$ be such that:

(c) *$B(p,0,\rho)$ is ρ^{-1}–close to $B(p_0,0,\rho) \subset X_0$;*
(d) *$P(p,0,\rho,t)$ is unscathed and $|\mathrm{Rm}| \leqslant \Lambda([A]+1)$ there.*

Then $P(p,0,A,t)$ is A^{-1}–close to $P(p_0,0,A,t)$.

Proof The proofs of [1, Corollary 8.2.2 and 8.2.4] work for surgical solutions. □

Proof of Theorem 8.1 Consider as model the standard solution $\mathcal{X}_0 := (\mathcal{S}_0, g_0(\cdot), p_0)$ restricted to $[0,\theta]$. Let us assume for simplicity that $T \geq t_0+\theta h^2$, so that $t_1 = t_0+\theta h^2$. For any nonnegative integer N, we now set

$$\Lambda_N = \max_{\mathcal{S}_0\times[0,\theta]}\{\,|\nabla^p \operatorname{Rm}|, |R|;\ 0\leq p\leq N\}.$$

In the sequel we consider the restriction of $(M(\cdot), g(\cdot))$ to $[t_0, b]$ and we define

$$\overline{g}(t) := h^{-2} g(t_0+th^2) \quad \text{for } t\in[0,\min\{\theta,(b-t_0)h^{-2}\}].$$

Note that $\overline{g}(\cdot)$ satisfies Assumptions (a) and (b) of Corollary 8.3. Indeed, it is readily checked that the curvature pinched toward positive property is preserved by the parabolic rescaling, since $t_0 \geqslant 0$ and $h^{-2}\geqslant 1$. On the other hand, if $g(\cdot)$ satisfies $(\mathrm{CN})_r$ on $[0,b)$, it follows easily by a continuity argument that any (x,b) with $R(x,b) \geqslant 2r(b)^{-2}$ satisfies the estimate $|\partial R/\partial t| \leqslant C_0 R^2$ at (x,b). After rescaling by $h(b)^{-2} >> 2r^{-2}$, this property holds at points with scalar curvature above 1.

Fix $A>0$ and set $\rho := \rho(\mathcal{M}_0, A)$. By the definition of an δ–almost standard cap, the ball $B_{\overline{g}}(p,0,5+\delta^{-1})$ is δ'–close to $B(p_0,0,5+\delta^{-1})\subset\mathcal{S}_0$. Let $T_{\max}\in[0,\theta]$ be the maximal time such that $P_{\overline{g}}(p,0,A,T_{\max})$ is unscathed. By closeness at time 0 one has $|R_{\overline{g}}|\leq 2\Lambda_0$ on $B_{\overline{g}}(p,0,\delta^{-1})$.

Now for $t\in[0,\min((4\Lambda C_0)^{-1},\theta)]$ such that $P_{\overline{g}}(p,0,\rho,t)$ is unscathed, we have $|R_{\overline{g}}|\leq 4\Lambda_0$ on $P_{\overline{g}}(p,0,\rho,t)$ by the time derivative estimate on the scalar curvature. Using the pinching assumptions, we deduce $|\operatorname{Rm}_{\overline{g}}|\leq \Lambda_0'$ on the same neighbourhood.

Set $T_{\times 2} := \min(\theta, (4\Lambda_0 C_0)^{-1}, (4\Lambda_0')^{-1})$. The above curvature bound gives, for $t\leq \min(\theta, T_{\times 2})$,

$$\frac{1}{2C}\leq \frac{\overline{g}(t)}{\overline{g}(0)}\leq 2C$$

on $B_{\overline{g}}(p,0,\delta^{-1})$, for some $C=C(\theta)$. In particular, for all $x\in B_{\overline{g}}(p,0,\rho)$ and all $0<t\leq\min(\theta,T_{\times 2})$, the point (x,t) is not the centre of a δ–neck because $d_{\overline{g}(t)}(x,p)\leq 2C d_{\overline{g}(0)}(x,p)\leq 2C\rho$ and the length of a δ–neck at time t is larger than $1/2\delta^{-1}R(x,t)^{-1/2}\geq(4\delta\Lambda_0)^{-1}>4C\rho$, if δ is small enough.

This implies that $P_{\overline{g}}(p,0,\rho,t)$ is unscathed if $t\leq\min(T_{\times 2},T_{\max})$. Indeed, if not, there exists $t'<t$ such that $P_{\overline{g}}(p,0,\rho,t')$ is unscathed but $B_{\overline{g}}(p,0,\rho)\cap M_{\text{sing}}(t')\neq\varnothing$. If $B_{\overline{g}}(p,0,\rho)$ is not contained in $M_{\text{sing}}(t')$, then it must intersect a surgery sphere of $\mathcal{S}(t')$, which is the middle sphere of a δ–neck centred at (x,t'). The above estimate rules out this possibility. Hence $B_{\overline{g}}(p,0,A)\subset B_{\overline{g}}(p,0,\rho)\subset M_{\text{sing}}(t')$ for $t'<T_{\max}$.

This is impossible by assumption. Remark that if $B(p,0,\rho)\cap M_{\mathrm{sing}}(t)\neq\varnothing$ the same arguments shows that $B(p,0,\rho)$ is contained in $M_{\mathrm{sing}}(t)$ and disappears at time t.

We can now apply Corollary 8.3, so $P_{\bar g}(p,0,\rho_1,t)$ is ρ_1^{-1}–close to $P(p_0,0,\rho_1,t)$ for all $t\le\min(T_{\times 2},T_{\max})$. If $T_{\times 2}\ge T_{\max}$ we are done. Otherwise by closeness we have that $|\mathrm{Rm}_{\bar g}|$ and $|R_{\bar g}|$ are no greater than $2\Lambda_0$ on $P_{\bar g}(p,0,\rho_1,T_{\times 2})$. Then $|R_{\bar g}|\le 4\Lambda_0$ on $P_{\bar g}(p,0,\rho_1,\min(2T_{\times 2},T_{\max}))$ where $\rho_1=\rho(\mathcal{M}_0,\rho_2)$ and $\rho_2>0$. We then iterate the above argument, which terminates in finitely many steps, as in [1, Corollary 8.2.2]. If $T_{\max}<\theta$, then Conclusion (ii) follows from the previous remark. This finishes the proof of Theorem 8.1. □

9 Proof of Proposition B

Let us recall the statement of Proposition B:

Proposition B *For all $Q_0,\rho_0,\kappa>0$ there exist $r=r(Q_0,\rho_0,\kappa)<10^{-3}$ and $\bar\delta_B=\bar\delta_B(Q_0,\rho_0,\kappa)>0$ with the following property: let $\delta\le\bar\delta_B$, $0\le T_A<b$ and $(M(\cdot),g(\cdot))$ be a surgical solution defined on $[T_A,b]$ such that $g(T_A)$ satisfies $|\mathrm{Rm}|\le Q_0$ and has injectivity radius at least ρ_0.*

Assume that $(M(\cdot),g(\cdot))$ satisfies Condition $(\mathrm{NC})_{\kappa/16}$, has curvature pinched toward positive, and that for each singular time t_0, $(M_+(t_0),g_+(t_0))$ is obtained from $(M(\cdot),g(\cdot))$ by (r,δ)–surgery at time t_0.

Then $(M(\cdot),g(\cdot))$ satisfies Condition $(\mathrm{CN})_r$.

In order to prove Proposition B, we argue by contradiction. Suppose that some fixed numbers $Q_0,\rho_0,\kappa>0$ have the property that for all $r\in(0,10^{-3})$ and $\bar\delta_B>0$ there exist counterexamples. Then we can consider sequences $r_k\to 0$, $\delta_k\to 0$, and a sequence of (r_k,δ_k,κ)–surgical solutions $(M_k(\cdot),g_k(\cdot))$ on $[0,b)$ which satisfy Condition $(\mathrm{NC})_{\kappa/16}$, have curvature pinched toward positive, and such that for each singular time t_0, $(M_{k,+}(t_0),g_{k,+}(t_0))$ is obtained from $(M(\cdot),g(\cdot))$ by (r,δ)–surgery at time t_0, but $(\mathrm{CN})_{r_k}$ fails for some t_k. This last assertion means that there exists $x_k\in M_k(t_k)$ such that
$$Q_k:=R(x_k,t_k)\ge r_k^{-2},$$
and yet (x_k,t_k) does not have a (ε_0,C_0)–canonical neighbourhood.

By a standard point-picking argument (see [15, Lemma 52.5]), we may choose the sequence of bad points (x_k,t_k) and $H_k\to+\infty$ such that for all $t\in[t_k-H_kQ_k^{-1},t_k]$ and $x\in M_k(t)$, if $R(x,t)\ge 2Q_k$ then (x,t) has a (ε_0,C_0)–canonical neighbourhood.

Without loss of generality, we assume that

$$\delta_k \le \overline{\delta}\left(k, 1-\frac{1}{k}, r_k\right)$$

(the right-hand side being the parameter given by the Persistence Theorem 8.1). We need a preliminary lemma.

Lemma 9.1 (Parabolic balls of bounded curvature are unscathed) *For all $K>0$, $\rho>0$ and $\tau>0$ there exists an integer $k_0=k_0(K,\rho,\tau)$ such that for all $k \geqslant k_0$, if $|\mathrm{Rm}| \leqslant K$ on $B_{\overline{g}_k}(x_k,0,\rho)\times(-\tau,0]$ then $P_{\overline{g}_k}(x_k,0,\rho,-\tau)$ is unscathed.*

Proof Arguing by contradiction, fix k and assume that there exist $z_k \in B_{\overline{g}_k}(x_k,0,\rho)$ and $s_k \in [-\tau,0)$ such that $z_k \notin M_{\mathrm{reg}}(s_k)$. As z_k exists after s_k, there is an added cap V in $M_+(s_k)$ such that $z_k \in V$. We can take s_k to be maximal satisfying this property, ie the set $B_{\overline{g}_k}(x_k,0,\rho)\times(s_k,0]$ is unscathed. Then we argue as in the proof of [1, Lemma 9.2.1], using the persistence Theorem 8.1. □

From this point on, the proof of Proposition B is almost identical to that of the corresponding proposition in [1] and hence omitted.

10 Proof of Proposition C

We recall the statement:

Proposition C *For all $Q_0,\rho_0>0$ and all $0\le T_A<T_\Omega$ there exists $\kappa=\kappa(Q_0,\rho_0,T_A,T_\Omega)$ such that for all $0<r<10^{-3}$ there exists $\overline{\delta}_C=\overline{\delta}_C(Q_0,\rho_0,T_A,T_\Omega,r)>0$ such that the following holds.*

Let $0<\delta\leqslant\overline{\delta}_C$ and $b\in(T_A,T_\Omega]$, and $(M(\cdot),g(\cdot))$ be a (r,δ)–surgical solution defined on $[T_A,b)$ such that $g(T_A)$ satisfies $|\mathrm{Rm}|\leqslant Q_0$, has injectivity radius at least ρ_0, ϕ_A–almost nonnegative curvature and satisfies $R_{\min}(g_0)\ge -6/(4T_A+1)$. Then $g(\cdot)$ satisfies $(\mathrm{NC})_\kappa$.

Standard estimates on Ricci flow (see eg [7, Lemma 6.1]) imply that the solution is smooth on $[T_A, T_A+\min\{b,2^{-4}Q_0^{-1}\}]$ with $|\mathrm{Rm}|\leqslant 2Q_0$ (if δ is small enough compared to Q_0). By usual comparison arguments, one deduces that there exists a constant κ_{norm} depending only on the normalisation of the initial condition, ie Q_0,ρ_0, such that $(M(\cdot),g(\cdot))$ satisfies $(\mathrm{NC})_{\kappa_{\mathrm{norm}}}$ on $[T_A,T_A+2^{-4}Q_0^{-1}]$.

We set $\kappa_0:=\min(\kappa_{\mathrm{norm}},\kappa_{\mathrm{sol}}/2,\kappa_{\mathrm{st}}/2)$.

10.1 Preliminaries

Let $v_k(\rho)$ denote the volume of a ball of radius ρ in the model space of constant sectional curvature k and dimension 3.

Let $\kappa > 0$. One says that a Riemannian ball $B(x,\rho)$ is κ–noncollapsed if $|\operatorname{Rm}| \leqslant \rho^{-2}$ on $B(x,\rho)$ and if $\operatorname{vol} B(x,\rho) \geqslant \kappa\rho^3$. Similarly, a parabolic ball $P(x,t,\rho,-\rho^{-2})$ is κ–noncollapsed if $|\operatorname{Rm}| \leqslant \rho^{-2}$ on $P(x,t,\rho,-\rho^{-2})$ and $\operatorname{vol} B(x,t,\rho) \geqslant \kappa\rho^3$.

We recall the following elementary lemma from [1, Section 10.1].

Lemma 10.1 (i) *If $B(x,\rho)$ is κ–noncollapsed, then for every $\rho' \in (0,\rho)$, $B(x,\rho')$ is $C\kappa$–noncollapsed, where $C := v_0(1)/v_{-1}(1)$. The same property holds for $P(x,t,\rho',-\rho'^2) \subset P(x,t,\rho,-\rho^2)$.*

(ii) *Let r,δ be surgery parameters and $g(\cdot)$ be an (r,δ)–surgical solution. Assume that $P_0 = P(x_0,t_0,\rho_0,-\rho_0^2)$ is a scathed parabolic neighbourhood such that $|\operatorname{Rm}| \leqslant \rho_0^{-2}$ on P_0. Then P_0 is $e^{-12}\kappa_{\mathrm{st}}/2$–noncollapsed.*

Remark 10.2 (1) From (i), we deduce that in order to establish noncollapsing at some point (x,t) on all scales ≤ 1, it suffices to do it on the maximal scale $\rho \leq 1$ such that $|\operatorname{Rm}| \leq \rho^{-2}$ on $P(x,t,\rho,\rho^{-2})$. This observation will be useful later.

(2) If some metric ball $B(y,\rho)$ is contained in a (ε,C_0)–canonical neighbourhood *which is not ε_0–round* and satisfies $|\operatorname{Rm}| \leqslant \rho^{-2}$, then $B(y,\rho)$ is C_0^{-1}–noncollapsed on the scale ρ by inequality (4) (cf Definition 4.6)

10.2 The proof

We turn to the proof of Proposition C. Since it is similar in many ways to that of Proposition C of [1], we omit some details. The main differences lie in the formalism since we work with surgical solutions rather than Ricci flows with bubbling-off. The possible noncompactness of the manifold does not create any additional difficulty. An important technical point is that the list of possible canonical neighbourhoods is larger than in [1], which accounts for some differences near the end of the proof.

Let M_{reg} be the set of regular points in spacetime. This is an open, arcwise connected 4–manifold. Likewise we let M_{sing} be the set of singular points in spacetime. Let $\gamma\colon [t_0,t_1] \to \bigcup_t M(t)$ be a map such that $\gamma(t) \in M(t)$ for every t. Let $\bar{t} \in [t_0,t_1]$. Here we adopt the convention that $M_+(t) = M(t)$ if t is regular.

Definition 10.3 One says that γ is *continuous at* $\overline{t}$ if there is $\sigma > 0$ such that:

(1) $t \to \gamma(t) \in M(\overline{t})$ on $[\overline{t}-\sigma, \overline{t})$ and is left continuous at $\overline{t}$;

(2) $t \to \gamma(t) \in M_+(\overline{t})$ on $]\overline{t}, \overline{t}+\sigma]$ and has a right limit at $\overline{t}$ denoted $\gamma_+(\overline{t})$;

(3) Assume $\overline{t} < t_1$. If $\gamma(\overline{t}) \in M_{\mathrm{reg}}(\overline{t})$, then $\gamma(\overline{t}) = \gamma_+(\overline{t})$ under the identification of $M_{\mathrm{reg}}(\overline{t})$ and $M(\overline{t}) \cap M_+(\overline{t})$; if $\gamma(\overline{t}) \in S \subset \mathcal{S}$, then $\gamma(\overline{t}) = \gamma_+(\overline{t})$ under the identification of S and the corresponding component of $\partial M \cap M_+(\overline{t})$.

In particular, if $\gamma(\overline{t}) \in M_{\mathrm{sing}}(\overline{t}) \setminus \mathcal{S}$ for $\overline{t} < t_1$, it is not continuous at $\overline{t}$. Indeed, $\gamma(\overline{t})$ disappears at time $\overline{t}$.

We say that γ is *unscathed* if $\gamma(t) \in M_{\mathrm{reg}}(t)$ for all $t \in [t_0, t_1)$. Otherwise γ is *scathed.*

Let (x_0, t_0) be a point, and $\rho_0 \in (0, 1]$ be a radius such that $|\mathrm{Rm}| \le \rho_0^{-2}$ on $P_0 := P(x_0, t_0, \rho_0, -\rho_0^2)$.

We make the following reduction: by Lemma 10.1 and 10.2(1), we assume that $\rho_0 \le 1$ is maximal with the above property, and that P_0 is unscathed.

As before we set $B_0 := B(x_0, t_0, \rho_0)$.

10.3 The case $\rho_0 \geqslant r/100$

Lemma 10.4 *Let $\hat{r}, \Delta, \Lambda$ be positive numbers. Then there exists $\overline{\delta} = \overline{\delta}(\hat{r}, \Delta, \Lambda) > 0$ with the following property. Let $(M(\cdot), g(\cdot))$ be an (r, δ)–surgical solution on an interval $I = [a, a+\Delta]$ with $\delta \leqslant \overline{\delta}$ and $r \geqslant \hat{r}$ on I. Let $(x_0, t_0) \in M \times I$ and $\rho_0 \geqslant \hat{r}$ be such that $P_0 := P(x_0, t_0, \rho_0, -\rho_0^2) \subset M \times I$ is unscathed and $|\mathrm{Rm}\,| \le \rho_0^{-2}$ on P_0. Let γ be a continuous spacetime curve defined on $[t_1, t_0]$ with $t_1 \in [0, t_0]$ and such that $\gamma(t_0) = x_0$ and γ is scathed. Then $\mathcal{L}_{t_0-t_1}(\gamma) \ge \Lambda$.*

Here $\mathcal{L}_{t_0-t_1}$ denotes the $\mathcal{L}$–length based at (x_0, t_0), that is

$$\mathcal{L}_{t_0-t_1}(\gamma) = \int_{t_1}^{t_0} \sqrt{t_0 - t}\big(R(\gamma(t), t) + |\dot{\gamma}(t)|^2_{g(t)}\big)\, dt.$$

Proof See the proof of [1, Lemma 10.3.1]. □

A consequence of the previous lemma is the following result (see [15, Lemmas 78.3 and 78.6] or [1] for more details).

Lemma 10.5 *Let $\hat{r}, \Delta, \Lambda$ be positive numbers. There exists $\overline{\delta} := \overline{\delta}(\hat{r}, \Delta, \Lambda)$ with the following property. Let $g(\cdot)$ be a (r, δ)–surgical solution defined on $[a, a+\Delta]$ such that $r \geqslant \hat{r}$ and $\delta \le \overline{\delta}$. Let (x_0, t_0) and $\rho_0 \geqslant \hat{r}$ be such that $P_0 := P(x_0, t_0, \rho_0, -\rho_0^2)$ is unscathed and $|\mathrm{Rm}\,| \le \rho_0^{-2}$ on P_0. Then:*

(i) *For all* $(q,t) \in M \times [a, a+\Delta]$, *if* $\ell(q, t_0 - t) < \Lambda$, *then there is an unscathed minimising* $\mathcal{L}$*–geodesic* γ *connecting* x_0 *to* q*;*

(ii) *For all* $\tau > 0$, $\min_q \ell(q,\tau) \leq 3/2$ *and is attained.*

We come back to the proof of Proposition C. Recall that P_0 is unscathed, and satisfies $|\mathrm{Rm}| \leq \rho_0^{-2}$. The arguments of [20, Section 7] apply to unscathed minimising $\mathcal{L}$–geodesics. In particular, if $\gamma(\tau) = \mathcal{L}_\tau \exp_{(x_0,t_0)}(v)$ is minimising and unscathed on $[0, \tau_0]$, then $\tau^{-3/2} e^{-\ell(v,\tau)} J(v,\tau)$ is nonincreasing on $[0, \tau_0]$.

Define

$$Y(\tau) := \{v \in T_{x_0}M \mid \mathcal{L}\exp(v)\colon [0,\tau] \longrightarrow M \text{ is minimising and unscathed}\}.$$

It is easy to check that $\tau \leq \tau' \Rightarrow Y(\tau) \supset Y(\tau')$. Then we set

$$\widetilde{V}_{\mathrm{reg}}(\tau) := \int_{Y(\tau)} \tau^{-3/2} e^{-\ell(v,\tau)} J(v,\tau)\, dv.$$

This function is nondecreasing on $[0, t_0]$. One then adapts the proof of κ–noncollapsing in the smooth case, replacing $\widetilde{V}$ by $\widetilde{V}_{\mathrm{reg}}$. The reader is referred to [1, Section 10.3.2] for details.

10.4 The case $\rho_0 \leqslant r/100$

Since $\rho_0 < 1$, the reduction made at the beginning of the proof implies that there exists $(y,t) \in \overline{P}_0$ such that $|\mathrm{Rm}(y,t)| = \rho_0^{-2}$. Hence we have

$$|\mathrm{Rm}(y,t)| \geqslant r^{-2} \geqslant 10^6.$$

Since $\{g(t)\}$ has curvature pinched toward positive,

$$R(y,t) \geqslant |\mathrm{Rm}(y,t)| = \rho_0^{-2} \geqslant 10000 r(t)^{-2}.$$

Hence (y,t) has an (ε_0, C_0)–canonical neighbourhood U.

If U is not ε_0–round, it gives C_0–noncollapsing at (y,t) (cf 10.2(2)). In the other case we will need an extra argument.

Case 1 U is not ε_0–round.

Let us show that $B(x_0, t, e^{-2}\rho_0) \subset U$. This is clear if U is closed, so we only have to deal with the cases of necks and caps.

By the curvature bounds on $\bar{P}_0$ we have $d_t(x_0,y) \leqslant e^2\rho_0$ and $B(x_0,t,e^{-2}\rho_0) \subset B(x_0,t_0,\rho_0)$. If U is an ε_0–neck, then $d_t(y,\partial\bar{U}) \geqslant (2\varepsilon_0)^{-1}R(y,t)^{-1/2}$. Since $R(y,t) \leqslant 6\rho_0^{-2}$, we get

$$d_t(y,\partial\bar{U}) \geqslant (2\sqrt{6}\varepsilon_0)^{-1}\rho_0 \geqslant (e^2+e^{-2})\rho_0\,,$$

hence $B(x_0,t,e^{-2}\rho_0) \subset U$.

If U is an (ε_0,C_0)–cap, write it $U=V\cap W$ where V is a core. Let $\gamma\colon [0,1]\to\bar{B}_0$ be a minimising $g(t_0)$–geodesic connecting y to x_0. If $x_0\notin V$, let $s\in[0,1]$ be maximal such that $\gamma(s)\in\partial V$. Since $\gamma(s)\in B_0$, we have $R(\gamma(s),t)\geqslant 6\rho_0^{-2}$ and we deduce that $d(\gamma(s),\partial\bar{U}) \geq (\sqrt{6}\varepsilon_0)^{-1}\rho_0$. As $d_t(\gamma(s),x_0)\leqslant e^2\rho_0$ we get $B(x_0,t,e^{-2}\rho_0)\subset U$.

Comparing this to inequality (4) in Definition 4.6, we see that

$$\mathrm{vol}_{g(t)}\, B(x_0,t,e^{-2}\rho_0) \geqslant C_0^{-1}(e^{-2}\rho_0)^3\,.$$

By estimates on distortion of distances and volume as in the proof of Lemma 10.1, we conclude that

$$\mathrm{vol}_{g(t_0)}\, B_0 \geqslant C_0^{-1}e^{-18}\rho_0^3\,.$$

Case 2 U is ε_0–round.

Note that the method of Case 1 applies equally well if U is homeomorphic to S^3 or $\mathbf{R}P^3$, so we assume it is not the case.

The only thing we have to do is to prove that there are only finitely many possible topologies for U. For simplicity of notation we assume $(x_0,t_0)=(y,t)$, ie the point (x_0,t_0) has an ε_0–round canonical neighbourhood U, and $|\mathrm{Rm}(x_0,t_0)|\geq 1000r^{-2}$.

Lemma 10.6 *There exists $t_0'<t_0$ such that:*

- *U is unscathed on $[t_0',t_0]$;*
- *for every $t\in[t_0',t_0]$ $(U,g(t))$ is ε_0–round;*
- *letting ρ_0' be defined at (x_0,t_0') in the obvious way, we have $2r\geq\rho_0'\geq r/2$.*

Proof Let $t_0''<t_0$ be minimal such that U is unscathed and for every $t\in[t_0'',t_0]$, $(U,g(t))$ is ε_0–round and $R\geq r^{-2}$ on $(U,g(t))$. We claim that $R_{\min}=r^{-2}$ on $(U,g(t_0''))$. Indeed by continuity $R\geq r^{-2}$ on $(U,g(t_0''))$. Hence (x_0,t_0'') has a canonical neighbourhood V. By continuity, $(U,g(t_0''))$ is $2\varepsilon_0$–round, so $V=U$; since we have excluded S^3 and $\mathbf{R}P^3$, we deduce that V is in fact ε_0–round. Since ε_0–roundness is an open property, it follows that if $R_{\min}>r^{-2}$ on $(U,g(t_0''))$ then t_0'' is not minimal. This proves the claim.

By ε_0–roundness, $R(\cdot,t_0'') \approx r^{-2}$ on U and $|\mathrm{Rm}(\cdot,t_0'')| \approx r^{-2}/6$. Therefore we can find $t_0' \in (t_0'', t_0)$ such that $|\mathrm{Rm}(\cdot,t_0')| \approx r^{-2}$ on U and $|\mathrm{Rm}| \le r^{-2}$ on $P(x_0, t_0', r, -r^{-2})$ (comparing with the evolving round metric one can find t_0' close to $t_0'' + \frac{5}{4}r^2$). It follows that the maximal radius ρ_0' such that $|\mathrm{Rm}| \le {\rho_0'}^{-2}$ on $P(x_0, t_0', \rho_0', -{\rho_0'}^2) =: P_0$ with P_0 unscathed, is close to r. □

Since $\rho_0' \ge r/2$ we can argue as in Section 10.3 to get uniform noncollapsing at (x_0, t_0') on the unit scale. As $(U, g(t_0'))$ is ε_0–homothetic to $(U, g(t_0))$ and $\rho_0' \le 2r < 1$, we also have uniform noncollapsing at (x_0, t_0) on the unit scale.

11 Generalisations and open questions

11.1 Consequences and generalisations

First we state a finiteness result which follows immediately from Theorem 5.5 and Corollary 2.4.

Corollary 11.1 *Let R_0, Q, ρ be positive numbers. Then the class of prime 3–manifolds admitting complete riemannian metrics of scalar curvature greater than R_0, sectional curvature bounded in absolute value by Q, and injectivity radius greater than ρ is finite up to diffeomorphism.*

Remark that the primeness hypothesis is necessary: otherwise, one could have, say, a connected sum of arbitrarily many copies of the same manifold. The key point is that the geometric bounds considered here do *not* imply any diameter bound (nor compactness of the manifold for that matter). Hence Corollary 11.1 is not a purely geometric finiteness theorem, but rather a mixed geometric-topological finiteness theorem.

Next we discuss an equivariant version of our main technical theorem.

Definition 11.2 Let $(M(\cdot), g(\cdot))$ be a surgical solution defined on some interval I. Let Γ be a group endowed with an action on each $M(t)$ for $t \in I$, which is constant in between singular times. We say that $(M(\cdot), g(\cdot))$ is Γ*–equivariant* if for each t, the action of Γ on $M(t)$ is isometric, and for each singular time t, the union of all 2–spheres along which surgery is performed is Γ–invariant.

Theorem 11.3 *Let M be an orientable 3–manifold. Let g_0 be a complete riemannian metric on M which has bounded geometry. Let Γ be a group acting properly discontinuously on M by isometries for g_0. Then there exists a complete surgical solution $(M(\cdot), g(\cdot))$ of bounded geometry defined on $[0, +\infty)$, with initial condition*

$(M(0), g(0)) = (M, g_0)$, and such that there is for each t a properly discontinuous action of Γ on $M(t)$ such that $(M(\cdot), g(\cdot))$ is Γ–equivariant, and such that if t is a singular time and x a point belonging to some disappearing component, then (x, t) has an (ε_0, C_0)–canonical neighbourhood. Furthermore, if the action of Γ on M is free, then one can ensure that for each t, the action of Γ on $M(t)$ is also free.

Proof We repeat the proof of Theorem 5.6, paying attention to equivariance with respect to the group Γ. By the Chen–Zhu uniqueness theorem [5], Ricci flow automatically preserves the symmetries of the original metric, so the only thing to check is that surgery can be done equivariantly. For this we can apply [8, Lemma 3.9]. Note that the constant ϵ appearing in that paper is a priori smaller than our ε_0. However, it is easy to check that if we replace ε_0 by some smaller positive number ε'_0 in the proof of Theorem 5.6 and subsequently the constants β_0 and C_0 by the appropriate constants β'_0 and C'_0, then the proof goes through without changes.

For the addendum where it is assumed that the action of Γ is free, there is an additional point to check: that surgery can be done so that the action of Γ on the post-surgery manifold is still free. For simplicity, we are going to explain this in a riemannian setting, ignoring the issue of strong necks, which is irrelevant here.

Let $(X, \widetilde{g})$ be a 3–manifold with an isometric, free, properly discontinuous action of Γ and $\{N_i\}$ be a Γ–invariant, locally finite collection of pairwise disjoint δ–necks in X. Let (Y, g) be the quotient riemannian manifold X/Γ.

Suppose first that for each N_i and each nontrivial element $\gamma \in \Gamma$ we have $N_i \cap \gamma N_i = \varnothing$. Then the collection $\{N_i\}$ projects to a locally finite collection of pairwise disjoint δ–necks in Y. Hence we can do metric surgery on Y, obtaining a riemannian manifold (Y_+, g_+). We then lift the construction, getting a riemannian manifold $(X_+, \widetilde{g_+})$ which on the one hand is obtained from $(X, \widetilde{g})$ by metric surgery on $\{N_i\}$, and on the other hand inherits a free, properly discontinuous, isometric action of Γ.

Thus we are done unless there exists i and γ such that $N_i \cap \gamma N_i \neq \varnothing$. In this case, N_i is invariant by γ. Since γ acts freely, it must act on N_i by an involution, so that N_i projects to a cap $C \subset Y$ diffeomorphic to a punctured $\mathbf{R}P^3$. In this case, C contains, say a 4δ–neck whose preimage in X contains two 4δ–necks interchanged by γ. Thus, up to replacing δ by 4δ, we can apply the construction of the first paragraph. □

Corollary 11.4 *Let M be a connected, orientable 3–manifold which carries a complete metric g of uniformly positive scalar curvature. Assume that the riemannian universal cover of (M, g) has bounded geometry. Then M is a connected sum of spherical manifolds and copies of $S^2 \times S^1$.*

Proof We apply Theorem 11.3 to the universal cover of (M, g) endowed with the action of $\Gamma := \pi_1(M)$. Let $(\widetilde{M}(\cdot), \widetilde{g}(\cdot))$ be a surgical solution satisfying the conclusion of that theorem. By Corollary 2.4, this surgical solution must be extinct. Thus M is a connected sum of metric quotients of the disappearing components of $(\widetilde{M}(\cdot), \widetilde{g}(\cdot))$. There remains to check that such quotients are themselves connected sums of spherical manifolds and copies of $S^2 \times S^1$.

We use the fact that the disappearing components are covered by canonical neighbourhoods, and the action of Γ on them is isometric. Let X be such a component. Remark that X is simply connected, since the van Kampen theorem implies that surgery along 2–spheres on a simply connected 3–manifold produces simply connected 3–manifolds. If X is compact, then by Perelman's Geometrisation Theorem, X is diffeomorphic to the 3–sphere, and its quotients are spherical manifolds.

If X is noncompact, then it is diffeomorphic to $S^2 \times \mathbf{R}$ or $\mathbf{R}^3$. In the former case, it is an exercise in topology (cf [25]) to show that the quotient can only be $S^2 \times \mathbf{R}$ itself, a punctured $\mathbf{R}P^3$, $S^2 \times S^1$ or a connected sum of two copies of $\mathbf{R}P^3$.

In the latter case, we obviously need to use the geometry. As in [8, Section 3], we consider the open subset T consisting of all points that are centres of ε_0–necks. Since X is diffeomorphic to $\mathbf{R}^3$, T is an ε_0–tube, and its complement C is the core of an ε_0–cap and diffeomorphic to the 3–ball. By definition, T is automatically invariant by any isometry. Hence C is also invariant by any isometry. Thus by the Brouwer fixed point theorem, X does not admit any nontrivial free isometric group action. □

Here is a more precise theorem which may be useful for subsequent applications:

Theorem 11.5 *There exist sequences $r_k, \delta_k, \kappa_k > 0$ such that for any complete normalised riemannian 3–manifold (M_0, g_0), there exists a surgical solution $(M(\cdot), g(\cdot))$ defined on $[0, +\infty)$, satisfying the initial condition $(M(0), g(0)) = (M_0, g_0)$, and such that for every nonnegative integer k, the restriction of $(M(\cdot), g(\cdot))$ to $[k, k+1]$ is an $(r_k, \delta_k, \kappa_k)$–surgical solution.*

Moreover, if (M_0, g_0) is endowed with a properly discontinuous isometric action of some group Γ, then the surgical solution can be made Γ–equivariant. In addition, if the action of Γ on M_0 is free, then the action on each $M(t)$ can be chosen to be free.

This follows from iteration of Theorem 5.6. Indeed, assuming that the parameters $r_k, \delta_k, \kappa_k > 0$ are known, we deduce from $\Theta_k := \Theta(r_k, \delta_k)$ a bound for the sectional curvature Q_k. From this and the κ_k–noncollapsing property, we deduce a lower bound for volumes of balls of radius at most $Q_k^{-1/2}$. This gives a lower bound ρ_k for the injectivity radius of every metric, in particular the metric $g(k+1)$.

The addendum about equivariance follows as explained in the proof of Theorem 11.3.

11.2 Open questions

The first question asks whether the hypotheses of Corollary 11.4 are necessary.

Question Let M be a connected, orientable 3–manifold which admits a complete riemannian metric of uniformly positive scalar curvature. Is M a connected sum of spherical manifolds and copies of $S^2 \times S^1$?

Next we consider what happens when we relax the hypothesis on the scalar curvature from uniform positivity to positivity. This class is significantly wider, eg it includes $S^1 \times \mathbf{R}^2$. One could even relax the condition further to nonnegativity.

Question (Problem 27 in [32]) Classify 3–manifolds admitting complete riemannian metrics of positive (resp. nonnegative) scalar curvature up to diffeomorphism.

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