Trees of cylinders and canonical splittings

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Let T be a tree with an action of a finitely generated group G. Given a suitable equivalence relation on the set of edge stabilizers of T (such as commensurability, coelementarity in a relatively hyperbolic group, or commutation in a commutative transitive group), we define a tree of cylinders T_c. This tree only depends on the deformation space of T; in particular, it is invariant under automorphisms of G if T is a JSJ splitting. We thus obtain $\mathrm{Out}(G)$–invariant cyclic or abelian JSJ splittings. Furthermore, T_c has very strong compatibility properties (two trees are compatible if they have a common refinement).

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1 Introduction

In group theory, a JSJ splitting of a group G is a splitting of G (as a graph of groups) in which one can *read* any splitting of G and which is *maximal* for this property; see Sela [28], Rips and Sela [25], Dunwoody and Sageev [8], Fujiwara and Papasoglu [11], Bowditch [1], and the authors' papers [12; 15; 17]. One needs to place restrictions on allowed edge groups, for instance one defines an abelian JSJ splitting by restricting to splittings over abelian groups. One sometimes considers *relative* JSJ splittings, by restricting to splittings in which certain subgroups are required to be elliptic.

In general, JSJ splittings are not unique, and there is a whole space of JSJ splittings called the *JSJ deformation space*; see Forester [10] and the authors' paper [17]. A *deformation space* [9; 14] is the set of all splittings whose elliptic subgroups are prescribed (one usually also adds constraints on edge groups). A typical example of a deformation space is Culler and Vogtmann's Outer Space [7]. Splittings in the same deformation space are related by a finite sequence of simple moves [9; 14; 6].

JSJ deformation spaces being canonical, they are endowed with a natural action of $\mathrm{Out}(G)$. As they are contractible [5; 14], this usually gives homological information about $\mathrm{Out}(G)$ (see eg [7]).

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However, deformation spaces of splittings, including JSJ deformation spaces, may also have some bad behaviour. For instance the action of $\mathrm{Out}(G)$ may fail to be cocompact; the deformation space may even fail to be finite dimensional.

Much more satisfying is the case when one has a *canonical splitting* rather than just a canonical deformation space. Such a splitting is invariant under automorphisms; in other words, it is a fixed point for the action of $\mathrm{Out}(G)$ on the deformation space containing it. A typical example is the virtually cyclic JSJ splitting of a one-ended hyperbolic group G constructed by Bowditch [1] from the topology of the boundary of G. Having such a splitting gives much more precise information about $\mathrm{Out}(G)$ (see Sela [28] and the second author's paper [20]).

The goal of this paper is to introduce a general construction producing a canonical splitting (called the *tree of cylinders*) from a deformation space $\mathcal{D}$. Rather than splittings (or graphs of groups), we think in terms of trees equipped with an action of G. We always assume that G is finitely generated.

The construction starts with a class $\mathcal{E}$ of allowed edge stabilizers, endowed with an *admissible* equivalence relation (see Definition 3.1). The main examples are commensurability, coelementarity and commutation (see Examples A, B and C below, and Sections 3.1 to 3.7). All trees are assumed to have edge stabilizers in $\mathcal{E}$.

Given a tree $T \in \mathcal{D}$, the equivalence relation on edge stabilizers partitions edges of T into *cylinders*. An essential feature of an admissible relation is that cylinders are connected (they are subtrees of T). By definition, the *tree of cylinders* of T is the tree T_c dual to the covering of T by its cylinders (see Definition 4.3).

Theorem 1 *The tree of cylinders T_c depends only on the deformation space $\mathcal{D}$ containing T.*

Moreover, the assignment $T \mapsto T_c$ is functorial: any equivariant map $T \to T'$ induces a natural cellular map $T_c \to T'_c$ (mapping an edge to an edge or a vertex).

We often say that T_c is the *tree of cylinders of the deformation space*. It is $\mathrm{Out}(G)$–invariant if $\mathcal{D}$ is.

Examples Consider the graph of groups pictured on the left of Figure 1 (edge groups are infinite cyclic and attached to the boundary of punctured tori). Its fundamental group G is hyperbolic, and the splitting depicted is a cyclic JSJ splitting. Its tree of cylinders is the splitting pictured on the right. It belongs to the same deformation space, but it is $\mathrm{Out}(G)$–invariant; in particular, it has a symmetry of order 3 (it is the splitting constructed by Bowditch [1]).

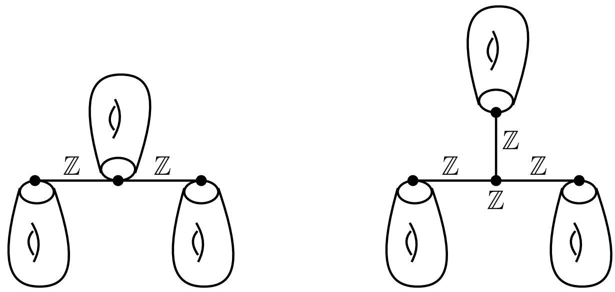

Figure 1: A JSJ splitting of a hyperbolic group and its tree of cylinders

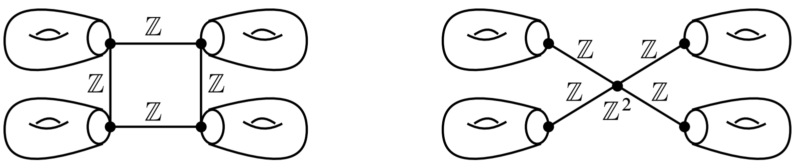

Figure 2: A JSJ splitting of a toral relatively hyperbolic group and its tree of cylinders

Now consider the graph of groups Γ_1 pictured on the left of Figure 2. Its fundamental group G is a torsion-free toral relatively hyperbolic group. The splitting depicted is again a cyclic JSJ splitting, but no splitting in its deformation space $\mathcal{D}$ is $\mathrm{Out}(G)$–invariant (there exist elements t in the $\mathbb{Z}^2$ vertex group of the splitting Γ_2 pictured on the right such that twisting by t around an edge of Γ_2 defines an automorphism having no fixed point in $\mathcal{D}$). The tree of cylinders of $\mathcal{D}$, which is $\mathrm{Out}(G)$–invariant, is the Bass–Serre tree of Γ_2. It does not belong to the same deformation space, because of the vertex with group $\mathbb{Z}^2$.

A basic property of T_c is that it is *dominated* by T: every subgroup which is elliptic (ie fixes a point) in T is elliptic in T_c. But, as in the case of Culler and Vogtmann's Outer Space, it may happen that T_c is trivial. For the construction to be useful, one has to be able to control how far T_c is from T, the best situation being when T_c and T belong to the same deformation space (ie T_c dominates T). One must also control edge stabilizers of T_c, as they may fail to be in $\mathcal{E}$.

Here are the main examples where this control is possible.

Example A G is a hyperbolic group, $\mathcal{E}$ is the family of two-ended subgroups of G and $\sim$ is the *commensurability* relation.

More generally, G is hyperbolic relative to parabolic subgroups $H_1, \dots, H_n$, the class $\mathcal{E}$ consists of the infinite elementary subgroups, and $\sim$ is the *coelementarity* relation ($A \subset G$ is *elementary* if A is virtually cyclic or is contained in a conjugate of some H_i, and $A \sim B$ if and only if $\langle A, B \rangle$ is elementary); one only considers trees in which each H_i is elliptic.

Example B G is torsion-free and CSA (centralizers of nontrivial elements are abelian and malnormal; for instance, the group considered in Figure 2 is CSA), $\mathcal{E}$ is the class of nontrivial abelian subgroups and $\sim$ is the *commutation* relation ($A \sim B$ if and only if $\langle A, B \rangle$ is abelian).

Example C G is torsion-free and commutative transitive (centralizers of nontrivial elements are abelian), $\mathcal{E}$ is the set of infinite cyclic subgroups of G and $\sim$ is the *commensurability* relation.

Theorem 2 *Let G and $\mathcal{E}$ be as in Example A (resp. B). Let T be a tree with edge stabilizers in $\mathcal{E}$, and assume that parabolic subgroups (resp. noncyclic abelian subgroups) are elliptic in T. Then*

(1) *T_c has edge stabilizers in $\mathcal{E}$;*

(2) *T_c belongs to the same deformation space as T;*

(3) *T_c is almost 2–acylindrical.*

A tree is *k–acylindrical* [27] if any segment I of length $> k$ has trivial stabilizer. If G has torsion, we use the notion of *almost acylindricity*: the stabilizer of I is finite.

In general, edge stabilizers of T_c may fail to be in $\mathcal{E}$. In this case, we also consider the *collapsed tree of cylinders* T_c^*, obtained from T_c by collapsing all edges whose stabilizer is not in $\mathcal{E}$.

Theorem 3 *Let G and $\mathcal{E}$ be as in Example C. Let T be a tree with infinite cyclic edge stabilizers, such that solvable Baumslag–Solitar subgroups of G are elliptic in T.*

Then T_c^*, T_c and T lie in the same deformation space, and T_c^* is 2–acylindrical.

Parabolic, abelian or Baumslag–Solitar subgroups as they appear in the hypotheses of Theorems 2 and 3 are always elliptic in T_c. If one does not assume that they are elliptic in T, they are the only way in which the deformation spaces of T and T_c^* differ (see Section 6 for precise statements).

In general, the deformation space of T_c^* may be characterized by the following maximality property: T_c^* dominates any tree T' such that T dominates T' and cylinders of T' are bounded (Proposition 5.12). In many situations, one may replace boundedness of cylinders by acylindricity in the previous maximality statement (see Section 6).

Our results of Section 6 may then be interpreted as describing which subgroups must be made elliptic in order to make T acylindrical. This is used in [18] to construct (under suitable hypotheses) JSJ splittings of finitely generated groups, using acylindrical accessibility.

Theorems 2 and 3 produce a canonical element in the deformation space of T. In particular, this provides canonical $\mathrm{Out}(G)$–invariant JSJ splittings.

Theorem 4 *Let G be hyperbolic relative to $H_1, \dots, H_n$. Assume that G is one-ended relative to $H_1, \dots, H_n$. There is an elementary (resp. virtually cyclic) JSJ tree relative to $H_1, \dots, H_n$ which is invariant under the subgroup of $\mathrm{Out}(G)$ preserving the conjugacy classes of the H_i's.*

The group G is one-ended relative to $H_1, \dots, H_n$ if there is no nontrivial tree with finite edge stabilizers in which each H_i is elliptic.

When G is a one-ended hyperbolic group, Theorem 4 yields the tree constructed by Bowditch [1].

Theorem 5 *Let G be a one-ended torsion-free CSA group. There exists an abelian (resp. cyclic) JSJ tree of G relative to all noncyclic abelian subgroups, which is $\mathrm{Out}(G)$–invariant.*

In particular, one gets canonical cyclic and abelian JSJ splittings of toral relatively hyperbolic groups, including limit groups. See Bowditch [1], Bumagin, Kharlampovich and Miasnikov [2], Paulin [24] Scott and Swarup [26] and Papasoglu and Swenson [23] for other constructions of canonical JSJ splittings. As evidenced by the example on Figure 2, Theorems 4 and 5 do not hold for nonrelative JSJ splittings.

Scott and Swarup have constructed in [26] a canonical splitting over virtually polycyclic groups. We show in [16] that their splitting coincides (up to subdivision) with the tree of cylinders of the JSJ deformation space (with $\sim$ being commensurability; see Section 3.2).

In a forthcoming paper [19], we will use trees of cylinders to study some automorphism groups. In particular, we will use Theorem 4 to characterize relatively hyperbolic groups with infinite outer automorphism group (see also Carette [3] for the case of hyperbolic groups) and we will show that the splitting of Theorem 4 has the largest modular group.

Another important feature of the tree of cylinders is compatibility. A tree T is a *refinement* of T' if T' can be obtained from T by equivariantly collapsing a set of edges. Two trees are *compatible* if they have a common refinement. Refinement is a notion much more rigid than domination: any two trees in Culler and Vogtmann's Outer Space dominate each other, but they are compatible if and only if they lie in a common simplex.

Theorem 6 *Let G be finitely presented and one-ended. Assume that $\mathcal{E}$ and $\sim$ are as in Examples A, B or C.*

If T is a JSJ tree over subgroups of $\mathcal{E}$, then T_c is compatible with every tree with edge stabilizers in $\mathcal{E}$ (in which parabolic subgroups are elliptic in Example A).

More general situations are investigated in Section 8. We show in [18] that under the hypotheses of Theorem 6 the tree T_c^* is maximal (for domination) among trees which are compatible with every other tree. In other words, T_c^* belongs to the same deformation space as the *JSJ compatibility tree* defined in [18].

The paper is organized as follows. After preliminaries, we define admissible equivalence relations and give examples (Section 3). In Section 4 we define the tree of cylinders. Besides the geometric definition sketched earlier, we give an algebraic one using elliptic subgroups and we show Theorem 1 (Corollary 4.10 and Proposition 4.11). In Section 5 we give basic properties of T_c. In particular, we show that most small subgroups of G which are not virtually cyclic are elliptic in T_c. We also study boundedness of cylinders and acylindricity of T_c^*, we show $(T_c^*)_c^* = T_c^*$, and we prove the maximality property of T_c^*. Section 6 gives a description of the tree of cylinders in the main examples and proves Theorems 2 and 3 (they follow from Propositions 6.1, 6.3, 6.5). Section 7 recalls some material about JSJ splittings and proves the existence of canonical JSJ splittings as in Theorems 4 and 5. In Section 8 we study compatibility properties of the tree of cylinders and we prove Theorem 6 (it follows from Assertions (1) and (3) of Corollary 8.4 and Assertion (2) of Theorem 8.6).

2 Preliminaries

In this paper, G will be a fixed finitely generated group.

Two subgroups A and B are *commensurable* if $A \cap B$ has finite index in both A and B. We denote by A^g the conjugate $g^{-1}Ag$. The *normalizer* $N(A)$ of A is the set of g such that $A^g = A$. Its *commensurator* $\mathrm{Comm}(A)$ is the set of g such that g commensurates A (ie A^g is commensurable to A). The subgroup A is *malnormal* if $A^g \cap A \neq \{1\}$ implies $g \in A$.

If Γ is a graph, we denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of nonoriented edges.

A tree always means a simplicial tree T on which G acts without inversions. Given a family $\mathcal{E}$ of subgroups of G, an *$\mathcal{E}$–tree* is a tree whose edge stabilizers belong to $\mathcal{E}$. We denote by G_v or G_e the stabilizer of a vertex v or an edge e.

A tree T is *nontrivial* if there is no global fixed point, *minimal* if there is no proper G–invariant subtree. An element or a subgroup of G is *elliptic* in T if it has a global fixed point. An element which is not elliptic is *hyperbolic*. It has an axis on which it acts as a translation.

A subgroup A consisting only of elliptic elements fixes a point if it is finitely generated, a point or an end in general. If a finitely generated subgroup A is not elliptic, there is a unique minimal A–invariant subtree.

A group A is *slender* if A and all its subgroups are finitely generated. A slender group acting on a tree fixes a point or leaves a line invariant (setwise); see Dunwoody and Sageev [8].

A subgroup A is *small* if it has no nonabelian free subgroups. As in [18], we could replace smallness by the following weaker property: whenever G acts on a tree, A fixes a point, or an end, or leaves a line invariant.

A tree T *dominates* a tree T' if there is an equivariant map $f\colon T \to T'$. Equivalently, any subgroup which is elliptic in T is also elliptic in T'. Having the same elliptic subgroups is an equivalence relation on the set of trees, the equivalence classes are called *deformation spaces* [9; 14].

An equivariant map $f\colon T \to T'$ between trees *preserves alignment* if $x \in [a,b] \implies f(x) \in [f(a), f(b)]$. Equivalently, f is a *collapse map*: it is obtained by collapsing certain edges to points. In particular, f does not fold.

We say that T is a *refinement* of T' if there is a collapse map $f\colon T \to T'$. Two trees T and T' are *compatible* if they have a common refinement.

3 Admissible relations

Let $\mathcal{E}$ be a class of subgroups of G, stable under conjugation. It should not be stable under taking subgroups (all trees of cylinders are trivial if it is), but it usually is sandwich-closed: if $A \subset H \subset B$ with $A, B \in \mathcal{E}$, then $H \in \mathcal{E}$. Similarly, $\mathcal{E}$ is usually invariant under $\mathrm{Aut}(G)$.

Definition 3.1 An equivalence relation $\sim$ on $\mathcal{E}$ is *admissible* if the following axioms hold for any $A, B \in \mathcal{E}$:

(1) If $A \sim B$, and $g \in G$, then $A^g \sim B^g$.

(2) If $A \subset B$, then $A \sim B$.

(3) Let T be an $\mathcal{E}$–tree. If $A \sim B$, and A, B fix $a, b \in T$ respectively, then for each edge $e \subset [a, b]$ one has $G_e \sim A \sim B$.

More generally, one can require (3) to hold only for certain $\mathcal{E}$–trees. In particular, given subgroups H_i, we say that $\sim$ is *admissible relative to the* H_i*'s* if (3) holds for all $\mathcal{E}$–trees T in which each H_i is elliptic.

Note that, given a subfamily $\mathcal{E}' \subset \mathcal{E}$, stable under conjugation, the restriction of $\sim$ to $\mathcal{E}'$ is admissible.

When proving that a relation is admissible, the only nontrivial part usually is axiom (3). The following criterion will be useful:

Lemma 3.2 *If* $\sim$ *satisfies* (1), (2) *and axiom* (3') *below, then it is admissible.*

(3') *Let* T *be an* $\mathcal{E}$*–tree. If* $A \sim B$*, and* A, B *are elliptic in* T*, then* $\langle A, B\rangle$ *is also elliptic in* T*.*

Proof We show axiom (3). Let c be a vertex fixed by $\langle A, B\rangle$. Since $[a, b] \subset [a, c] \cup [c, b]$, one may assume $e \subset [a, c]$. One has $A \subset G_e$ because A fixes $[a, c]$, so that $G_e \sim A$ by axiom (2). □

The lemma applies in a relative setting, with (3') restricted to trees in which each H_i is elliptic.

Given an admissible relation $\sim$ on $\mathcal{E}$, we shall associate a tree of cylinders T_c to any $\mathcal{E}$–tree T. If $\sim$ is only admissible relative to subgroups H_i, we require that each H_i be elliptic in T.

Here are the main examples to which this will apply.

3.1 Two-ended subgroups

Let $\mathcal{E}$ be the set of two-ended subgroups of G (a group H is two-ended if and only if some finite index subgroup of H is infinite cyclic). Commensurability, defined by $A \sim B$ if $A \cap B$ has finite index in both A and B, is an admissible equivalence relation.

Axiom (1) is clear. If A, B are two-ended subgroups with $A \subset B$, then A has finite index in B, so (2) holds. If $A, B \in \mathcal{E}$ are commensurable and fix points $a, b \in T$, then $A \cap B$ fixes $[a,b]$. If $e \subset [a,b]$, then $A \cap B \subset G_e$ (with finite index), and $G_e \sim A \cap B \sim A$.

Commensurability is admissible also on the set of infinite cyclic subgroups.

3.2 Commensurability

This class of examples generalizes the previous one. We used it in [16].

Let $\mathcal{E}$ be a conjugacy-invariant family of subgroups of G such that

- any subgroup A commensurable with some $B \in \mathcal{E}$ lies in $\mathcal{E}$;
- if $A, B \in \mathcal{E}$ are such that $A \subset B$, then $[B : A] < \infty$.

As in the previous example, one easily checks that commensurability is an admissible equivalence relation on $\mathcal{E}$.

For instance, $\mathcal{E}$ may consist of all subgroups of G which are virtually $\mathbb{Z}^n$ for some fixed n, or all subgroups which are virtually polycyclic of Hirsch length exactly n. The classes of edge groups $\mathcal{ZK}$ considered by Dunwoody and Sageev in [8] also fit into this example.

3.3 Coelementarity (splittings relative to parabolic groups)

Let G be hyperbolic relative to finitely generated subgroups $H_1, \dots, H_n$. Recall that a subgroup of G is *parabolic* if it is contained in a conjugate of an H_i, *elementary* if it is finite, or two-ended, or parabolic. If every H_i is slender (resp. small), the elementary subgroups are the same as the slender (resp. small) subgroups.

We need the following standard fact (see for instance [22, Lemma 2.5, Theorem 4.3]).

Lemma 3.3 *Any infinite elementary subgroup H is contained in a unique maximal one, $E(H)$. This subgroup is two-ended or conjugate to an H_i.* □

Let $\mathcal{E}$ be the class of *infinite* elementary subgroups of G. Let $\sim$ be the coelementarity relation, defined by saying that $A, B \in \mathcal{E}$ are *coelementary* if and only if $\langle A, B\rangle$ is elementary.

Lemma 3.4 *Coelementarity is an equivalence relation on $\mathcal{E}$.*

Proof We have to prove transitivity. If $\langle A, B\rangle$ and $\langle B, C\rangle$ are elementary, since B is infinite, $E(\langle A, B\rangle) = E(\langle B\rangle) = E(\langle B, C\rangle)$, so $\langle A, B, C\rangle \subset E(\langle B\rangle)$ is elementary. □

Coelementarity is admissible relative to the H_i's. Indeed, let us show axiom (3'), assuming that each H_i is elliptic in T. The group $\langle A, B\rangle$ is elementary. It is clearly elliptic if it is parabolic. If it is two-ended, then it contains A with finite index, so is elliptic in T because A is elliptic.

3.4 Coelementarity (arbitrary splittings)

We again assume G is hyperbolic relative to finitely generated subgroups $H_1, \dots, H_n$, and $\mathcal{E}$ is the family of infinite elementary subgroups. If we want to define a tree of cylinders for any $\mathcal{E}$–tree T, without assuming that the H_i's are elliptic in T, we need $\sim$ to be admissible (not just relative to the H_i's).

Lemma 3.5 *Suppose that each H_i is finitely-ended (ie H_i is finite, virtually cyclic, or one-ended). Then coelementarity is admissible on $\mathcal{E}$.*

Note that H_i is finitely-ended if it is small (does not contain F_2).

Proof We show axiom (3), with T any $\mathcal{E}$–tree (H_i is not required to be elliptic). Let $H = E(\langle A, B\rangle)$ be the maximal elementary subgroup containing both A and B (see Lemma 3.3). It is two-ended or conjugate to an infinite H_i.

If H fixes a point $c \in T$, we argue as in the proof of Lemma 3.2. If H is two-ended, it contains A with finite index and therefore fixes a point in T. Thus, we can assume that H is one-ended and does not fix a point in T. As H is finitely generated, there is a minimal H–invariant subtree $T_H \subset T$.

The segment $[a, b]$ is contained in $[a, a'] \cup [a', b'] \cup [b', b]$ where a' (resp. b') is the projection of a (resp. b) on T_H. If $e \subset [a, a']$, one has $A \subset G_e$ since A fixes $[a, a']$, so that $A \sim G_e$. The same argument applies if $e \subset [b, b']$. Finally, assume $e \subset [a', b'] \subset T_H$. Since H is one-ended, the subgroup of H stabilizing e is infinite, so is in $\mathcal{E}$. Thus $G_e \sim (G_e \cap H) \sim H \sim A$. Axiom (3) follows. □

3.5 Commutation

Recall that G is *commutative transitive* if the commutation relation is a transitive relation on $G \setminus \{1\}$ (ie nontrivial elements have abelian centralizers). For example, torsion-free groups which are hyperbolic relative to abelian subgroups are commutative transitive.

Let G be a commutative transitive group, and let $\mathcal{E}$ be the class of its *nontrivial* abelian subgroups.

Lemma 3.6 *The commutation relation, defined by $A \sim B$ if $\langle A, B\rangle$ is abelian, is admissible.*

Proof It is an equivalence relation because of commutative transitivity. Axioms (1) and (2) are clear. We show axiom (3') (see Lemma 3.2). If $\langle A, B\rangle$ does not fix a point in T, the fixed point sets of A and B are disjoint, and $\langle A, B\rangle$ contains a hyperbolic element g. Being abelian, $\langle A, B\rangle$ then acts by translations on a line (the axis of g). But since A and B are elliptic, $\langle A, B\rangle$ fixes this line pointwise, a contradiction. $\square$

We will also consider a more restricted situation. Note that, in a commutative transitive group, any nontrivial abelian subgroup is contained in a unique maximal abelian subgroup, namely its centralizer.

Definition 3.7 G is CSA if it is commutative transitive, and its maximal abelian subgroups are malnormal.

Since CSA is a closed property in the space of marked groups, Γ–limit groups for Γ torsion-free hyperbolic are CSA (see Sela [29]); also see Champetier [4] for wilder examples.

3.6 Finite groups

Let $q \geq 1$ be an integer. Let $\mathcal{E}$ be the family of all subgroups of G of cardinality q. It is easily checked that equality is an admissible equivalence relation on $\mathcal{E}$. This example will be used in [19].

3.7 The equivalence relation of a deformation space

Lemma 3.8 *Let $\mathcal{E}$ be any conjugacy-invariant family of subgroups of G. The equivalence relation generated by inclusion is admissible relative to $\mathcal{E}$.*

Proof We have to show axiom (3) holds, under the additional hypothesis that all groups of $\mathcal{E}$ are elliptic in T. Since $A \sim B$, we can find subgroups $A_0 = A, A_1, \dots, A_n = B$ in $\mathcal{E}$ where A_i, A_{i+1} are nested (one contains the other). Let $u_0 = a, u_1, \dots, u_n = b$ be points of T fixed by $A_0, \dots, A_n$ respectively. Let i be such that $e \subset [u_i, u_{i+1}]$, and assume for instance that $A_i \subset A_{i+1}$. Then A_i fixes e, so $A_i \subset G_e$ and $G_e \sim A_i \sim A$. $\square$

In particular, let $\mathcal{D}$ be a deformation space (or a restricted deformation space in the sense of Definition 3.12 of [14]). The relation generated by inclusion is admissible on the family $\mathcal{E}$ consisting of generalized edge groups of reduced trees in $\mathcal{D}$ (see Section 4 of [14]).

4 The basic construction

Let $\sim$ be an admissible equivalence relation on $\mathcal{E}$. We now associate a tree of cylinders T_c to any $\mathcal{E}$–tree T. If $\sim$ is only admissible relative to subgroups H_i, we require that each H_i be elliptic in T.

4.1 Cylinders

Definition 4.1 Let T be an $\mathcal{E}$–tree (with each H_i elliptic in T in the relative case).

Define an equivalence relation $\sim_T$ on the set of (nonoriented) edges of T by: $e \sim_T e'$ if and only if $G_e \sim G_{e'}$. A *cylinder* of T is an equivalence class Y. We identify Y with the union of its edges, a subforest of T.

A key feature of cylinders is their connectedness:

Lemma 4.2 *Every cylinder is a subtree.*

Proof Assume that $G_e \sim G_{e'}$. By axiom (3), any edge e'' contained in the arc joining e to e' satisfies $G_e \sim G_{e'} \sim G_{e''}$, thus belongs to the same cylinder as e and e'. □

Two distinct cylinders meet in at most one point. One can then define the tree of cylinders of T as the tree T_c *dual* to the covering of T by its cylinders, as in [13, Definition 4.8]:

Definition 4.3 The *tree of cylinders* of T is the bipartite tree T_c with vertex set $V(T_c) = V_0(T_c) \sqcup V_1(T_c)$ defined as follows:

(1) $V_0(T_c)$ is the set of vertices x of T belonging to (at least) two distinct cylinders.

(2) $V_1(T_c)$ is the set of cylinders Y of T.

(3) There is an edge $\varepsilon = (x, Y)$ between $x \in V_0(T_c)$ and $Y \in V_1(T_c)$ if and only if x (viewed as a vertex of T) belongs to Y (viewed as a subtree of T).

Alternatively, one can define the *boundary* ∂Y of a cylinder Y as the set of vertices of Y belonging to another cylinder, and obtain T_c from T by replacing each cylinder by the cone on its boundary.

It is easy to see that T_c is indeed a tree [13]. Here are a few other simple observations.

If $\mathcal{E}' \subset \mathcal{E}$, and T is an $\mathcal{E}'$–tree, its tree of cylinders as an $\mathcal{E}'$–tree (with respect to the restricted admissible relation) is the same as T_c.

The group G acts on T_c without inversions. It follows from [13, Lemma 4.9] that T_c is minimal if T is minimal. But T_c may be a point, for instance if all edge stabilizers of T are trivial.

Any vertex stabilizer G_v of T fixes a point in T_c: the vertex v of $V_0(T_c)$ if v belongs to two cylinders, the vertex Y of $V_1(T_c)$ if Y is the only cylinder containing v. In other words, T dominates T_c.

A vertex $x \in V_0(T_c)$ may be viewed either as a vertex of T or as a vertex of T_c; its stabilizer in T_c is the same as in T. The stabilizer of a vertex in $V_1(T_c)$ is the (global) stabilizer G_Y of a cylinder $Y \subset T$; it may fail to be elliptic in T (for instance if T_c is a point and T is not), so T_c does not always dominate T. This will be studied in Sections 5 and 6.

Let us now consider edge stabilizers. We note:

Remark 4.4 Edge stabilizers of T_c are elliptic in T, and they always contain a group in $\mathcal{E}$: if $\varepsilon = (x, Y)$ and e is an edge of Y incident on x, then $G_\varepsilon \supset G_e$.

However, edge stabilizers of T_c are not necessarily in $\mathcal{E}$. For this reason, it is convenient to introduce the *collapsed tree of cylinders*:

Definition 4.5 Given an $\mathcal{E}$–tree T, the *collapsed tree of cylinders* T_c^* is the $\mathcal{E}$–tree obtained from T_c by collapsing all edges whose stabilizer is not in $\mathcal{E}$.

4.2 Algebraic interpretation

We give a more algebraic definition of T_c, by viewing it as a subtree of a bipartite graph Z defined algebraically using only information on $\mathcal{E}$, $\sim$, and elliptic subgroups of T. This will make it clear that T_c only depends on the deformation space of T (Corollary 4.10). We motivate the definition of Z by a few observations.

If $Y \subset T$ is a cylinder, all its edges have equivalent stabilizers, and we can associate to Y an equivalence class $\mathcal{C} \in \mathcal{E}/{\sim}$. We record the following for future reference.

Remark 4.6 Given an edge $\varepsilon = (x, Y)$ of T_c, let e be an edge of Y adjacent to x in T. Then G_e is a representative of the class $\mathcal{C}$ and is contained in $G_\varepsilon = G_x \cap G_Y$. If G_ε belongs to $\mathcal{E}$, then it is in $\mathcal{C}$ by axiom (2) of admissible relations. In particular, if all edge stabilizers of T_c are in $\mathcal{E}$, then $(T_c)_c = T_c$. Also note that G_Y represents $\mathcal{C}$ if $G_Y \in \mathcal{E}$.

Depending on the context, it may be convenient to think of a cylinder either as a set of edges of T, or a subtree Y of T, or a vertex of T_c, or an equivalence class $\mathcal{C}$. Similarly, there are several ways to think of $x \in V_0(T_c)$: as a vertex of T, a vertex of T_c, or an elliptic subgroup G_x.

If x is a vertex of T belonging to two cylinders, then its stabilizer G_x is not contained in a group of $\mathcal{E}$: otherwise all edges of T incident to x would have equivalent stabilizers by axiom (2), and x would belong to only one cylinder.

More generally, let v be any vertex of T whose stabilizer is not contained in a group of $\mathcal{E}$. Then G_v fixes v only, and is a maximal elliptic subgroup. Conversely, let H be a subgroup which is elliptic in T, is not contained in a group of $\mathcal{E}$, and is maximal for these properties. Then H fixes a unique vertex v and equals G_v.

Definition 4.7 Given an $\mathcal{E}$–tree T, let Z be the bipartite graph with vertex set $V(Z) = V_0(Z) \sqcup V_1(Z)$ defined as follows:

(1) $V_0(Z)$ is the set of subgroups H which are elliptic in T, not contained in a group of $\mathcal{E}$, and maximal for these properties.
(2) $V_1(Z)$ is the set of equivalence classes $\mathcal{C} \in \mathcal{E}/\sim$.
(3) There is an edge ε between $H \in V_0(Z)$ and $\mathcal{C} \in V_1(Z)$ if and only if H contains a group of $\mathcal{C}$.

As previously observed, $V_0(Z)$ may be viewed as the set of vertices of T whose stabilizer is not contained in a group of $\mathcal{E}$.

It also follows from the previous observations that there is a natural embedding of bipartite graphs $j\colon T_c \to Z$: for $v \in V_0(T_c)$, we define $j(v) = G_v \in V_0(Z)$; for $Y \in V_1(T_c)$, with associated equivalence class $\mathcal{C}$, we define $j(Y) = \mathcal{C} \in V_1(Z)$. Note that j is well defined on $E(T_c)$ since adjacent vertices of T_c have adjacent images in Z by Remark 4.6.

A vertex $H \in V_0(Z)$ is a stabilizer G_v for a unique $v \in T$. It is in $j(T_c)$ if and only if v belongs to two cylinders. A vertex $\mathcal{C} \in V_1(Z)$ is in $j(T_c)$ if and only if some representative of $\mathcal{C}$ fixes an edge of T.

Proposition 4.8 *Assume that the action of G on T is minimal and nontrivial, and that T_c is not a point. Then $j(T_c)$ is the set of edges and vertices of Z which are contained in the central edge of a segment of length 5 of Z.*

Proof The action of G on T_c is minimal (see above) and nontrivial, so any edge of T_c and of $j(T_c)$ is the central edge of a segment of length 5. The converse is an immediate consequence of items (1), (3), (4) of the following lemma. □

Lemma 4.9 (1) *A vertex $v \in V_1(Z)$ belongs to $j(T_c)$ if and only if v has valence at least 2 in Z.*

(2) *If $x \in T$ and $G_x \in V_0(Z)$ is adjacent to $j(Y) \in j(T_c)$, then $x \in Y$.*

(3) *An element of $V_0(Z)$ belongs to $j(T_c)$ if and only if it has at least 2 neighbours in $j(T_c)$.*

(4) *An edge of Z lies in $j(E(T_c))$ if and only if its endpoints are in $j(V(T_c))$.*

Proof In statements (1), (3) and (4), the direct implications are clear.

(1) Consider $\mathcal{C} \in V_1(Z)$ of valence at least 2. It has representatives $A \subset G_x$ and $B \subset G_{x'}$ with x, x' distinct points of T. By axiom (3) of admissible relations, all edges in $[x, x']$ have stabilizer in $\mathcal{C}$. This shows $\mathcal{C} \in j(V_1(T_c))$.

(2) Consider $G_x \in V_0(Z)$ adjacent to $\mathcal{C} = j(Y) \in V_1(Z)$ for some cylinder Y of T. Then $\mathcal{C}$ has a representative $A \subset G_x$. Let e be an edge of Y, so that $A \sim G_e$. By axiom (3), the cylinder Y contains the arc joining e to x, so $x \in Y$.

(3) If $G_x \in V_0(Z)$ is adjacent to $j(Y)$ and $j(Y')$, then $x \in Y \cap Y'$ by statement (2), so $x \in V_0(T_c)$, and $G_x \in j(V_0(T_c))$.

(4) Let $j(x)$ and $j(Y)$ be vertices of $j(T_c)$ joined by an edge of Z. Then $x \in Y$ by statement (2), so x and Y are joined by an edge in T_c. □

Corollary 4.10 *If T, T' are minimal nontrivial $\mathcal{E}$–trees belonging to the same deformation space, there is a canonical equivariant isomorphism between their trees of cylinders.*

Proof The key observation is that the graph Z is defined purely in terms of the elliptic subgroups, which are the same for T and T'. The corollary is trivial if T_c, T'_c are both points. If T_c is not a point, then $V_1(Z)$ contains infinitely many vertices of valence ≥ 2. As shown above (statement (1) of Lemma 4.9), this implies that T'_c is not a point, and the result follows directly from Proposition 4.8. □

4.3 Functoriality

There are at least two other ways of proving that T_c only depends on the deformation space of T. One is based on the fact that any two trees in the same deformation space are connected by a finite sequence of elementary expansions and collapses [9]. One checks that these moves do not change T_c.

Another approach is to study the effect of an equivariant map $f\colon T \to T'$ on the trees of cylinders. We always assume that f maps a vertex to a vertex, and an edge to a point or an edge path. If T' is minimal, any f is surjective. We say that f is *cellular* if it maps an edge to an edge or a vertex.

Proposition 4.11 *Let T, T' be minimal $\mathcal{E}$–trees, and $f\colon T \to T'$ an equivariant map. Let T_c, T'_c be the trees of cylinders of T and T'. Then f induces a cellular equivariant map $f_c\colon T_c \to T'_c$. This map does not depend on f, and is functorial in the sense that $(f \circ g)_c = f_c \circ g_c$.*

Corollary 4.10 easily follows from this proposition. The proposition may be proved by considering the bipartite graph Z, but we give a geometric argument.

Proof We may assume that T'_c is not a point.

Lemma 4.12 *Consider an equivariant map $f\colon T \to T'$. For each cylinder $Y \subset T$, the image $f(Y)$ is either a cylinder Y' of T' or a point $p' \in T'$.*

Proof If an edge e' of T' is contained in the image of an edge $e \subset Y$, then $G_{e'}$ contains G_e, hence is equivalent to G_e by axiom (2). This shows that $f(Y)$, if not a point, is contained in a unique cylinder Y'. Conversely, if e' is an edge of Y', then any edge e such that $e' \subset f(e)$ satisfies $G_e \subset G_{e'}$ hence is in Y. Thus $Y' = f(Y)$. □

We say that a cylinder Y is *collapsed* if $f(Y)$ is a point $p' \in T'$. We claim that such a p' belongs to two distinct cylinders of T', so represents an element of $V_0(T'_c)$. Consider the union of all collapsed cylinders, and the component containing Y. It is not the whole of T, so by minimality of T it has at least two boundary points. These points belong to distinct noncollapsed cylinders whose images are the required cylinders containing p'.

Also note that, if $x \in T$ belongs to two cylinders, so does $f(x)$. This is clear if x belongs to no collapsed cylinder, and follows from the previous fact if it does.

This allows us to define f_c on vertices of T_c, by sending $x \in V_0(T_c)$ to $f(x) \in V_0(T'_c)$, and $Y \in V_1(T_c)$ to $Y' \in V_1(T'_c)$ or $p' \in V_0(T'_c)$. If (x, Y) is an edge of T_c, then $f_c(x)$ and $f_c(Y)$ are equal or adjacent in T'_c.

We may describe f_c without referring to f, as follows. The image of $x \in V_0(T_c)$ is the unique point of T' fixed by G_x. The image of Y is the unique cylinder whose edge stabilizers are equivalent to those of Y, or the unique point of T' fixed by stabilizers of edges of Y. Functoriality is easy to check. □

Remark 4.13 Note that, if two edges (x_1, Y_1) and (x_2, Y_2) of T_c are mapped by f_c onto the same edge of T'_c, then $Y_1 = Y_2$. In particular, if the restriction of f to each cylinder is either constant or injective, then f_c preserves alignment.

5 General properties

5.1 The deformation space of T_c

We fix $\mathcal{E}$ and an admissible relation $\sim$. We have seen that T always dominates T_c: any vertex stabilizer of T fixes a point in T_c. Conversely, T_c has two types of vertex stabilizers. If $v \in V_0(T_c)$, then its stabilizer is a vertex stabilizer of T, and $G_v \notin \mathcal{E}$ (see Section 4.2). On the other hand, the stabilizer G_Y of a vertex $Y \in V_1(T_c)$ may fail to be elliptic in T. This means that T_c is not necessarily in the same deformation space as T.

There are various ways to think of G_Y. It consists of those $g \in G$ that map the cylinder Y to itself. If e is any edge in Y, then G_Y is the set of $g \in G$ such that $gG_eg^{-1} \sim G_e$. If $\mathcal{C}$ is the equivalence class associated to Y, then G_Y is the stabilizer of $\mathcal{C}$ for the action of G by conjugation on $\mathcal{E}/\sim$.

In Sections 3.1 and 3.2, the group G_Y is the commensurator of G_e, for any $e \subset Y$. In Sections 3.3 and 3.4, it is the maximal elementary subgroup containing G_e. In Section 3.5, it is the normalizer of the maximal abelian subgroup A containing G_e (it equals A if G is CSA). In Section 3.6, it is the normalizer of G_e.

We first note:

Lemma 5.1 *If T is minimal, then G_Y acts on Y with finitely many orbits of edges.*

Proof By minimality, there are finitely many G–orbits of edges in T. If two edges of Y are in the same orbit under some $g \in G$, then g preserves Y, so they are in the same orbit under G_Y. □

Proposition 5.2 *Given T, the following statements are equivalent:*

(1) *T_c belongs to the same deformation space as T.*
(2) *Every stabilizer G_Y is elliptic in T.*
(3) *Every cylinder $Y \subset T$ is bounded.*
(4) *No cylinder contains the axis of a hyperbolic element of G.*

Proof We have seen (1)$\Leftrightarrow$(2). If Y is bounded, then G_Y fixes a point of T (the "center" of Y). If G_Y fixes a point, then Y is bounded by Lemma 5.1. This shows (2)$\Leftrightarrow$(3).

The implication (3)$\Rightarrow$(4) is clear. For the converse, assume that Y is an unbounded cylinder. We know that G_Y does not fix a point. If all its elements are elliptic, then

G_Y fixes an end of Y. Any ray going out to that end maps injectively to Y/G_Y, contradicting Lemma 5.1. Thus G_Y contains a hyperbolic element g. Being g–invariant, Y contains the axis of g. □

Proposition 5.3 *Assume that any two groups of $\mathcal{E}$ whose intersection is infinite are equivalent. Let H be a subgroup of G which is not virtually cyclic and is not an infinite, locally finite, torsion group.*

If H is small, or commensurates an infinite small subgroup H_0, then H fixes a point in T_c.

The hypothesis on $\mathcal{E}$ is satisfied in Sections 3.1 to 3.6, with the exception of 3.2. Conversely, we will see in Section 6 that, in many examples, groups which are elliptic in T_c but not in T are small.

Besides small groups, the proposition applies to groups H which act on locally finite trees with small infinite stabilizers, for instance generalized Baumslag–Solitar groups. It also applies to groups with a small infinite normal subgroup, such as fundamental groups of Seifert fibered spaces.

Proof The result is clear if H fixes a point of T. If not, we show that H preserves a subtree Y_0 contained in a cylinder. This cylinder will be H–invariant.

First suppose that H is small. If H preserves a line ℓ, then $G_e \cap H$ is the same for all edges in that line. If $G_e \cap H$ is infinite, then ℓ is contained in a cylinder and one can take $Y_0 = \ell$. If $G_e \cap H$ is finite, then H is virtually cyclic, which is ruled out. By smallness, the only remaining possibility is that H fixes a unique end of T.

If $G_e \cap H$ is infinite for some edge e, we let ρ be the ray joining e to the fixed end. The assumption on $\mathcal{E}$ implies that ρ is contained in a cylinder Y_0. This cylinder is H–invariant since $h\rho \cap \rho$ is a ray for any $h \in H$.

If all groups $G_e \cap H$ are finite, there are two cases. If H contains a hyperbolic element h, the action of H on its minimal subtree T_H is an ascending HNN extension with finite edge groups. It follows that T_H is a line and H is virtually cyclic. If every element of H is elliptic, consider any finitely generated subgroup $H_0 \subset H$. It fixes both an end and a point, so it fixes an edge. We conclude that H_0 is finite, so H is locally finite. This is ruled out.

Now suppose that H commensurates a small subgroup H_0. If H_0 preserves a unique line or fixes a unique end, the same is true for H and we argue as before. If H_0 fixes a point $x \in T$, let Y_0 be the convex hull of the orbit $H.x$. Any segment $I \subset Y_0$ is contained in a segment $[hx, h'x]$ with $h, h' \in H$, and its stabilizer contains $hH_0h^{-1} \cap h'H_0h'^{-1}$ which is commensurable to H_0 hence infinite. The assumption on $\mathcal{E}$ implies that Y_0 is contained in a cylinder. □

Remark 5.4 Assume that there exists C such that any two groups of $\mathcal{E}$ whose intersection has order $> C$ are equivalent. The same proof shows that locally finite subgroups are elliptic in T_c.

5.2 The collapsed tree of cylinders T_c^*

Recall that the collapsed tree of cylinders T_c^* is the tree obtained from T_c by collapsing all edges whose stabilizer is not in $\mathcal{E}$ (Definition 4.5).

Proposition 5.5 *Cylinders of T_c^* have diameter at most 2.*

Proof This follows from Remark 4.6: if in $\mathcal{E}$, the stabilizer of an edge (x, Y) of T_c belongs to the equivalence class $\mathcal{C}$ associated to Y. □

We say that $\mathcal{E}$ is *sandwich-closed* if $A \subset H \subset B$ with $A, B \in \mathcal{E}$ implies $H \in \mathcal{E}$. All families considered in Sections 3.1 through 3.6 have this property.

Sandwich-closedness has the following consequence. If ε is an edge of T_c such that G_ε is contained in a group of $\mathcal{E}$, then $G_\varepsilon \in \mathcal{E}$. This follows from Remark 4.4, which asserts that G_ε contains a group of $\mathcal{E}$.

Lemma 5.6 *Assume that $\mathcal{E}$ is sandwich-closed. Given an equivariant map $f\colon T \to T'$, the cellular map $f_c\colon T_c \to T'_c$ of Proposition 4.11 induces a cellular map $f_c^*\colon T_c^* \to T_c'^*$.*

Proof If ε is an edge of T_c which is collapsed in T_c^*, its image by f_c is a point or an edge ε' with $G_\varepsilon \subset G_{\varepsilon'}$. The group G_ε is not in $\mathcal{E}$, but by Remark 4.4 it contains an element of $\mathcal{E}$. Sandwich-closedness implies $G_{\varepsilon'} \notin \mathcal{E}$, so ε' is collapsed in $T_c'^*$. This shows that the natural map $T_c \to T_c'^*$ factors through T_c^*. □

A subtree $X \subset T$ of diameter exactly 2 has a *center* $v \in V(T)$, and all its edges contain v. We say that X is *complete* if it contains all edges around v, *incomplete* otherwise.

Proposition 5.7 *Assume that $\mathcal{E}$ is sandwich-closed. Let T be a minimal $\mathcal{E}$–tree.*

(1) *Every cylinder of T_c^* has diameter exactly 2. No stabilizer of an incomplete cylinder of T_c^* lies in $\mathcal{E}$.*

(2) *Conversely, assume that all cylinders of T have diameter exactly 2, and $G_Y \notin \mathcal{E}$ for all incomplete cylinders $Y \subset T$. Then $T_c^* = T$.*

Proof It follows from Proposition 5.5 that any cylinder Z of T_c^* has diameter at most 2, and is obtained from the ball of radius one around some $Y \in V_1(T_c)$ by collapsing all edges with stabilizer outside $\mathcal{E}$. It is incomplete if and only if at least one edge is collapsed. Note that the cylinders $Y \subset T$ and $Z \subset T_c^*$ have the same stabilizer $G_Y = G_Z$.

We show that Z has diameter exactly 2. Otherwise Z consists of a single edge. The corresponding edge $\varepsilon = (v, Y)$ of T_c is the unique edge incident on Y with $G_\varepsilon \in \mathcal{E}$, so $G_\varepsilon = G_Y$ (otherwise, one would obtain other edges by applying elements of $G_Y \setminus G_\varepsilon$). By minimality of T_c, there exist other edges ε' incident on Y. They satisfy $G_{\varepsilon'} \notin \mathcal{E}$, and $G_{\varepsilon'} \subset G_Y \in \mathcal{E}$ contradicts sandwich-closedness.

If Z is an incomplete cylinder of T_c^*, at least one edge ε of T_c incident on Y is collapsed in T_c^*, so $G_\varepsilon \notin \mathcal{E}$. As above, sandwich-closedness implies $G_Y \notin \mathcal{E}$. This proves (1).

To prove (2), we shall define an isomorphism $g\colon T_c^* \to T$. We denote by v_Y the center of a cylinder Y of T. Let $f\colon T_c \to T$ be the map sending $Y \in V_1(T_c)$ to $v_Y \in T$, sending $v \in V_0(T_c)$ to $v \in T$, and mapping the edge $\varepsilon = (v, Y)$ to $[v, v_Y]$. Note that $[v, v_Y]$ is an edge if $v \neq v_Y$, and is reduced to a point otherwise.

We first prove that an edge ε of T_c is collapsed by f (ie $v = v_Y$) if and only if $G_\varepsilon \notin \mathcal{E}$. If $G_\varepsilon \notin \mathcal{E}$, sandwich-closedness implies that ε is collapsed, since T is an $\mathcal{E}$–tree. Conversely, if $\varepsilon = (v, Y)$ is collapsed, then $v_Y = v \in V_0(T)$, so v_Y lies in several cylinders of T. This implies that Y is incomplete, so $G_Y \notin \mathcal{E}$. Since G_Y is contained in $G_{v_Y} = G_v$, one has $G_\varepsilon = G_Y \notin \mathcal{E}$.

It follows that f factors through a map $g\colon T_c^* \to T$ which maps edge to edge (without collapse). By minimality, g is onto. There remains to prove that g does not fold. If two edges of T_c^* have the same image, they belong to the same cylinder. But g is injective on each cylinder since (v, Y) is mapped to $[v, v_Y]$. □

Corollary 5.8 *Let $\mathcal{E}$ be sandwich-closed. For any minimal $\mathcal{E}$–tree T, one has $(T_c^*)_c^* = T_c^*$.* □

Proposition 5.9 *Assume that G_Y fixes a point of T whenever there is an edge $\varepsilon = (x, Y)$ of T_c whose stabilizer is not in $\mathcal{E}$. Then T_c and T_c^* belong to the same deformation space. Moreover, given Y, at most one edge $\varepsilon = (x, Y)$ of T_c is collapsed in T_c^*; it satisfies $G_\varepsilon = G_Y$.*

Proof Let $\varepsilon = (x, Y)$ be an edge of T_c such that $G_\varepsilon \notin \mathcal{E}$. It suffices to prove that $G_\varepsilon = G_Y$ and that $G_{\varepsilon'} \in \mathcal{E}$ for every edge $\varepsilon' = (x', Y)$ with $x' \neq x$.

By assumption, G_Y fixes a point in T, hence a point $v \in Y$. If $x \neq v$, let e be the initial edge of the segment $[x, v]$. Then $G_\varepsilon = G_x \cap G_Y \subset G_x \cap G_v \subset G_e$. On the other hand, G_e fixes x and leaves Y invariant, so $G_e \subset G_\varepsilon$. We conclude $G_\varepsilon = G_e \in \mathcal{E}$, a contradiction. Thus $x = v$, and $\varepsilon = (x, Y)$ is the only edge incident to Y with stabilizer not in $\mathcal{E}$. Moreover, since G_Y fixes x, we have $G_\varepsilon = G_Y \cap G_x = G_Y$. □

Corollary 5.10 *If T_c is in the same deformation space as T, then so is T_c^*, and therefore $(T_c^*)_c = T_c$ by Corollary 4.10.* □

Remark 5.11 Suppose that $\mathcal{E}$ is sandwich-closed and that, for any $A \in \mathcal{E}$, any subgroup containing A with index 2 lies in $\mathcal{E}$. Then the hypothesis of Proposition 5.9 is always satisfied when G_Y is small. To see this, we suppose that G_Y is not elliptic in T and we show $G_\varepsilon \in \mathcal{E}$. If G_Y fixes an end of T, its subgroup G_ε, being elliptic, fixes an edge. Since G_ε contains a group in $\mathcal{E}$, sandwich-closedness implies $G_\varepsilon \in \mathcal{E}$. If G_Y acts dihedrally on a line, some subgroup of G_ε of index at most 2 fixes an edge, so $G_\varepsilon \in \mathcal{E}$.

Recall that cylinders of T_c^* have diameter at most 2. We show that T_c^* is maximal for this property.

Proposition 5.12 *Assume that $\mathcal{E}$ is sandwich-closed. If T' is any $\mathcal{E}$–tree dominated by T, and cylinders of T' are bounded, then T' is dominated by T_c^*.*

Proof By Proposition 5.2 and Corollary 5.10, the tree $T_c'^*$ belongs to the same deformation space as T'. Lemma 5.6 shows that T_c^* dominates $T_c'^*$, hence T'. □

5.3 Acylindricity

We now consider acylindricity in the sense of Sela. Recall [27] that a tree is k–*acylindrical* if the stabilizer of any segment of length $> k$ is trivial. It is acylindrical if it is k–acylindrical for some k. To handle groups with torsion, we say that T is *almost k–acylindrical* if the stabilizer of any segment of length $> k$ is finite.

Proposition 5.13 *Assume that any two groups of $\mathcal{E}$ whose intersection is infinite are equivalent. Let T be any $\mathcal{E}$–tree.*

(1) *The tree T_c^* is almost 2–acylindrical.*

(2) *If cylinders of T are bounded (resp. of diameter $\leq k$), then T is almost acylindrical (resp. almost k–acylindrical).*

Recall that the hypothesis on $\mathcal{E}$ is satisfied in Sections 3.1 to 3.6, with the exception of 3.2. The next section will provide examples where the converse to Assertion (2) holds.

Proof The first statement follows from the second one and Proposition 5.5.

Since there are only finitely many orbits of cylinders, consider k such that cylinders have diameter at most k. Any segment I of length $k+1$ contains edges in distinct cylinders. By the assumption on $\mathcal{E}$, the stabilizer of I is finite. □

We also note:

Lemma 5.14 *Let H be small, not virtually cyclic, not locally finite. Then H is elliptic in any almost acylindrical tree T.*

Proof The hypotheses on H are the same as in Proposition 5.3, and the proof is similar. If H is not elliptic in a tree T, it acts on a line with infinite edge stabilizers, or it fixes a unique end and some edge stabilizer is infinite. Both are impossible if T is almost acylindrical. □

6 Examples

We now study specific examples. In most cases, we show that T_c^* is equal to T_c (or at least in the same deformation space), and we describe how far the deformation space of T_c is from that of T.

Recall that T always dominates T_c. They are in the same deformation space if and only if, for every cylinder Y, the group G_Y is elliptic in T. Note that, if all groups in $\mathcal{E}$ are infinite, any virtually cyclic G_Y is elliptic in T because by Remark 4.4 it contains some G_e (with finite index).

We also show that T_c^*, which is almost 2–acylindrical by Proposition 5.13, is maximal for this property: it dominates any almost acylindrical tree which is dominated by T. This is because, in the examples, groups which are elliptic in T_c^* but not in T are small, and Lemma 5.14 applies. Describing the deformation space of T_c^* may thus be interpreted as finding which subgroups must be made elliptic in order to make T almost acylindrical. One may ask in general whether a maximal almost acylindrical tree dominated by a given T always exists.

6.1 Relatively hyperbolic groups

Proposition 6.1 *Let G be hyperbolic relative to $H_1, \dots, H_n$. Let $\sim$ be coelementarity, as in Section 3.3. Let T be a tree with infinite elementary edge stabilizers, such that each H_i is elliptic in T.*

(1) *Edge stabilizers of T_c are infinite elementary, so $T_c^* = T_c$.*

(2) *T_c belongs to the same deformation space as T. In particular, it has the same nonelementary vertex stabilizers as T.*

(3) *T_c is almost 2–acylindrical (and dominates any almost acylindrical tree which is dominated by T).*

Proof Let $\varepsilon = (x, Y)$ be an edge of T_c. Here G_Y is the maximal elementary subgroup containing G_e, for any edge e of Y. This shows that G_ε is elementary. It is infinite because it contains an element of $\mathcal{E}$ (Remark 4.4).

To prove (2), we must show that every G_Y is elliptic in T. If parabolic, G_Y is elliptic by assumption. If virtually cyclic, it is elliptic by a remark made above (it contains some G_e with finite index). Assertion (2) follows (its second half is a general fact [14, Corollary 4.4]).

Assertion (3) now follows from Proposition 5.13 (the parenthesized statement is trivial in this case). □

If we do not assume that H_i is elliptic in T, we get:

Proposition 6.2 *Let G be hyperbolic relative to finitely generated one-ended subgroups $H_1, \dots, H_n$. Let $\sim$ be coelementarity, as in Section 3.4. Let T be a tree with infinite elementary edge stabilizers.*

(1) *Edge stabilizers of T_c are infinite elementary, so $T_c^* = T_c$.*

(2) *T and T_c have the same nonelementary vertex stabilizers. A subgroup is elliptic in T_c if and only if it is elliptic in T, or parabolic. In particular, T_c is in the same deformation space as T if and only if every parabolic subgroup is elliptic in T.*

(3) *T_c is almost 2–acylindrical. If the H_i's are small, then T_c dominates any almost acylindrical tree T' which is dominated by T.*

Proof It is still true that every G_Y is a maximal elementary subgroup (so (1) holds), but a parabolic G_Y may now fail to be elliptic in T. As pointed out at the beginning of the section, every virtually cyclic G_Y is elliptic in T.

If G_v is a nonelementary vertex stabilizer of T, then v belongs to two cylinders (otherwise G_v would be contained in some G_Y), so G_v is a vertex stabilizer of T_c. The converse is clear since a nonelementary vertex stabilizer of T_c fixes a vertex of $V_0(T_c)$, so is a vertex stabilizer of T.

To prove (2), there remains to show that each H_i is elliptic in T_c. If it is not elliptic in T, there is an edge e with $G_e \cap H_i$ infinite (recall that H_i is one-ended). In particular, $G_e \sim H_i$. The associated equivalence class $\mathcal{C}$ is invariant under conjugation by elements of H_i, so H_i preserves the cylinder containing e hence is elliptic in T_c.

Acylindricity again follows from Proposition 5.13. The second part of (3) holds provided every H_i is elliptic in T', in particular if H_i is small by Lemma 5.14. □

6.2 Abelian splittings of CSA groups

Proposition 6.3 *Let G be a torsion-free CSA group. Let $\mathcal{E}$ (nontrivial abelian groups) and $\sim$ (commutation) be as in Section 3.5. Let T be an $\mathcal{E}$–tree.*

(1) *Edge stabilizers of T_c are nontrivial and abelian, so $T_c^* = T_c$.*

(2) *T and T_c have the same nonabelian vertex stabilizers. A subgroup is elliptic in T_c if and only if it is elliptic in T or is a noncyclic abelian group. In particular, T_c is in the same deformation space as T if and only if every noncyclic abelian subgroup of G is elliptic in T.*

(3) *T_c is 2–acylindrical and dominates any acylindrical $\mathcal{E}$–tree T' which is dominated by T.*

Proof If $Y \in V_1(T_c)$ is a cylinder, its stabilizer G_Y is the set of $g \in G$ such that gG_eg^{-1} commutes with G_e, for e any edge of Y. By the CSA property, G_Y is the maximal abelian subgroup containing G_e (if $g \in G_Y$, then gG_eg^{-1} and G_e are contained in the same maximal abelian subgroup A, and $g \in A$ by malnormality). Conversely, a noncyclic abelian subgroup acts on T with nontrivial edge stabilizers and therefore leaves some cylinder invariant. As in the previous proof, nonabelian vertex stabilizers are the same for T and T_c.

Assertions (1) and (2) follow from these observations. The vertex stabilizers of T_c are the nonabelian vertex stabilizers of T, the noncyclic maximal abelian subgroups, and possibly cyclic subgroups which are elliptic in T.

The tree T_c is 2–acylindrical by Proposition 5.13. A group H which is elliptic in T_c but not in T is abelian and noncyclic, hence elliptic in T' by Lemma 5.14. Assertion (3) follows. □

We also note the following result, which gives a converse to the second assertion of Proposition 5.13:

Proposition 6.4 *Let G, $\mathcal{E}$ and T be as in Proposition 6.3. The following are equivalent:*

(1) *Every noncyclic abelian subgroup is elliptic.*

(2) *Cylinders of T are bounded (equivalently, T_c is in the same deformation space as T).*

(3) *T is acylindrical.*

(4) *No nontrivial element of G fixes a line.*

Proof We have just seen (1)$\Leftrightarrow$(2). Proposition 5.13 gives (2)$\Rightarrow$(3), and obviously (3)$\Rightarrow$(4). To prove (4)$\Rightarrow$(2), suppose that Y is an unbounded cylinder. By Proposition 5.2, it contains the axis of a hyperbolic element g. Let e be an edge contained in that axis, and A the maximal abelian subgroup containing G_e. Since gG_eg^{-1} commutes with G_e, the CSA property implies $g \in A$. Thus G_e fixes the axis, contradicting (4). $\square$

6.3 Cyclic splittings of commutative transitive groups

The relation $\sim$ now is commensurability, as in Section 3.1, so G_Y is the commensurator of G_e if e is an edge of a cylinder Y.

For $s \neq 0$, denote by $\mathrm{BS}(1,s)$ the solvable Baumslag–Solitar group $\langle a,t \mid tat^{-1} = a^s \rangle$. It is commutative transitive if and only if $s \neq -1$. Note that $\mathrm{BS}(1,1) = \mathbb{Z}^2$.

Proposition 6.5 *Let G be torsion-free and commutative transitive. Let $\mathcal{E}$ be the class of infinite cyclic subgroups of G, and let $\sim$ be commensurability as in Section 3.1. Let T be an $\mathcal{E}$–tree.*

(1) *T_c^* and T_c are in the same deformation space.*

(2) *Every noncyclic vertex stabilizer of T is a vertex stabilizer of T_c and T_c^*, and every other noncyclic vertex stabilizer of T_c^* is isomorphic to some $\mathrm{BS}(1,s)$. Every subgroup isomorphic to $\mathrm{BS}(1,s)$ is elliptic in T_c and T_c^*. In particular, T_c and T_c^* belong to the same deformation space as T if and only if every subgroup of G isomorphic to a $\mathrm{BS}(1,s)$ is elliptic in T.*

(3) *T_c^* is 2–acylindrical and dominates any acylindrical $\mathcal{E}$–tree which is dominated by T.*

Proof We note the following algebraic facts, whose proof is left to the reader. Let $Z \subset H$ be an infinite cyclic subgroup of a commutative transitive torsion-free group, and let A be the centralizer of Z. Then $Z \subset A \subset \mathrm{Comm}(Z) \subset N(A)$. If $A = Z$, then Z is malnormal.

Consider a cylinder $Y \subset T$, and a vertex $v \in Y$. All edge stabilizers G_e, for $e \subset Y$, are commensurable, hence have the same centralizer A by commutative transitivity. By the previous remark, one has $A \subset G_Y \subset N(A)$ since G_Y is the commensurator of G_e.

Lemma 6.6 *Assume that $G_v \cap G_Y$ is noncyclic. Then G_Y fixes v, and only v. Moreover,*

(1) *if Y is the only cylinder containing v, then $G_Y = G_v$ and no edge of T_c incident to the vertex $Y \in V_1(T_c)$ gets collapsed in T_c^*;*

(2) *if v belongs to two cylinders, the edge $\varepsilon = (v, Y)$ of T_c is collapsed in T_c^* (the vertex Y "disappears" in T_c^*).*

Proof We first show that $G_v \cap A$ is noncyclic. Assume that $G_v \cap A$ is cyclic, necessarily infinite since it contains G_e for e an edge of Y adjacent to v. By the initial note above, $G_v \cap A$ is malnormal in G_v, so $G_v \cap G_Y = G_v \cap A$ is cyclic, a contradiction.

Since $G_v \cap A$ is noncyclic, v is its unique fixed point. It is also the unique fixed point of A (which centralizes $G_v \cap A$), and of $G_Y \subset N(A)$.

The "moreover" is clear: the only collapsible edge of T_c incident to Y is (v, Y), which exists if and only if v belongs to two cylinders. □

By Proposition 5.9, the lemma implies that T_c and T_c^* belong to the same deformation space. Moreover, any vertex stabilizer H of T_c^* which is not a vertex stabilizer of T equals G_Y for some cylinder Y such that $G_v \cap G_Y$ is cyclic for every vertex $v \in Y$. The group G_Y acts on Y with all edge and vertex stabilizers infinite cyclic. Since it is commutative transitive, it is easy to see that G_Y must be isomorphic to $\mathbb{Z}$ or a $\mathrm{BS}(1, s)$ (otherwise G_Y contains $F_2 \times \mathbb{Z}$ or $\langle a, b \mid a^m = b^n \rangle$ with $m, n \geq 2$; such groups are not commutative transitive). Conversely, any $\mathrm{BS}(1, s)$ is elliptic in T_c by Proposition 5.3.

To prove Assertion (2), there remains to show that any noncyclic vertex stabilizer G_v of T is a vertex stabilizer of T_c and T_c^*. This is clear if v belongs to two cylinders. If it belongs to a unique cylinder Y, the lemma tells us that $G_v = G_Y$ is a vertex stabilizer of T_c and of T_c^*.

Assertion (3) now follows from Proposition 5.13 and Lemma 5.14. □

6.4 Commensurability

In our last examples $\sim$ is again commensurability, but we do not make assumptions on G, so our results are less precise.

Proposition 6.7 *Let $\mathcal{E}$ be the set of two-ended subgroups of G, and $\sim$ be commensurability, as in Section 3.1. Given an $\mathcal{E}$–tree T, the following are equivalent:*

(1) *Cylinders of T are bounded (equivalently, T_c is in the same deformation space as T).*
(2) *The commensurator of each edge stabilizer is elliptic in T.*
(3) *T is almost acylindrical.*
(4) *No element of infinite order fixes a ray.*

Proof (1)$\Leftrightarrow$(2) is clear because $G_Y = \mathrm{Comm}(G_e)$ if $e \subset Y$. (1)$\Rightarrow$(3) follows from Proposition 5.13 and (3)$\Rightarrow$(4) is clear.

To prove (4)$\Rightarrow$(1), assume that some cylinder is unbounded. By Proposition 5.2, it contains the axis A_g of a hyperbolic element g. Let e_0 be an edge of A_g, let $e_i = g^i(e_0)$, and $H_i = G_{e_i}$. If $H_i \subset H_{i\pm 1}$ for some i, then H_i fixes a ray, contradicting (4). If not, we can find $h_0 \in H_0$ and $h_2 \in H_2$ not fixing e_1, and $h = h_0h_2$ is hyperbolic. As H_0 and H_2 are commensurable, there is a finite index subgroup $H \subset H_0 \cap H_2$ which is normal in both H_0 and H_2. Since h normalizes H, the fixed point set of H contains the axis of h, and (4) does not hold. □

Corollary 6.8 *For any $\mathcal{E}$–tree T, the tree T_c^* is almost 2–acylindrical and dominates any almost acylindrical $\mathcal{E}$–tree which is dominated by T.*

Proof This follows from Proposition 5.12 and Proposition 5.13. □

In Section 3.2, we get:

Proposition 6.9 *Let $\sim$ be the commensurability relation, with $\mathcal{E}$ as in Section 3.2. Let T be an $\mathcal{E}$–tree.*

(1) *T_c belongs to the same deformation space as T if and only if there exists k such that any segment I of length $> k$ contains an edge e with the index $[G_e : G_I]$ infinite.*
(2) *Assume that edge stabilizers of T are finitely generated. Then T_c belongs to the same deformation space as T if and only if no group commensurable to an edge stabilizer fixes a ray.*

In particular, T_c belongs to the same deformation space as T when every group in $\mathcal{E}$ is infinite and T is almost acylindrical. But without further hypotheses on $\mathcal{E}$ we cannot claim that T_c^* is almost acylindrical.

Proof (1) follows from the fact that a segment I is contained in a cylinder if and only if $[G_e : G_I]$ is finite for every $e \subset I$. The proof of (2) is fairly similar to that of Proposition 6.7, and left to the reader. Finite generation of edge stabilizers is used to construct the normal subgroup of finite index H. □

7 JSJ splittings

7.1 Generalities

We review basic facts about JSJ splittings and JSJ deformation spaces. See our papers [15; 17] for details.

In order to define JSJ splittings, one needs a family of edge groups which is closed under taking subgroups. Since $\mathcal{E}$ does not always have this property, we introduce the following substitute.

Definition 7.1 The family $\mathcal{E}$ is *substable* if, whenever G splits over a group A contained in a group $B \in \mathcal{E}$, then $A \in \mathcal{E}$.

Remark 7.2 When we work relative to a family of subgroups (like in Section 3.3), the splitting of G in the definition should be relative to this family.

Example 7.3 In Sections 3.1, 3.4, 3.5 (with G torsion-free), $\mathcal{E}$ is substable if and only if G is one-ended. In Section 3.2, $\mathcal{E}$ is substable if and only if G does not split over a group having infinite index in a group of $\mathcal{E}$. In Section 3.3, we restrict to relative splittings, and $\mathcal{E}$ is substable if and only if G is one-ended relative to the H_i's (ie there is no nontrivial tree with finite edge stabilizers in which every H_i is elliptic).

We fix $\mathcal{E}$ and $\sim$, with $\mathcal{E}$ substable. All trees are assumed to be $\mathcal{E}$–trees. Strictly speaking, we consider $\overline{\mathcal{E}}$–trees, where $\overline{\mathcal{E}}$ consists of all groups contained in a group of $\mathcal{E}$. But substability guarantees that every $\overline{\mathcal{E}}$–tree is an $\mathcal{E}$–tree.

A subgroup $H \subset G$ is *universally elliptic* if it is elliptic in every tree. A tree is universally elliptic if all its edge stabilizers are.

A tree is a *JSJ tree over* $\mathcal{E}$ if it is universally elliptic, and maximal for this property: it dominates every universally elliptic tree. When $\overline{\mathcal{E}}$ consists of all groups with a given property (eg abelian, slender, elementary), we use the words abelian JSJ, slender JSJ, elementary JSJ….

JSJ trees always exist when G is finitely presented, and sometimes when G is only finitely generated (in particular in the situations studied below). They belong to the same deformation space, called the *JSJ deformation space over $\mathcal{E}$*. If T_J is a JSJ tree, and T' is any tree, there is a tree $\widehat{T}$ which refines T_J and dominates T'.

All these definitions and facts extend to the relative case: given a collection of subgroups, one only considers trees in which these subgroups are elliptic.

7.2 QH–vertices

A vertex stabilizer of a JSJ tree is *flexible* if it is not universally elliptic, and does not belong to $\overline{\mathcal{E}}$. A key fact of JSJ theory is that flexible vertex stabilizers often have a very special form.

Definition 7.4 A vertex stabilizer G_v is a QH–subgroup (and v is a QH–vertex) if there is an exact sequence $1 \to F \to G_v \overset{\pi}{\to} \Sigma \to 1$, where $\Sigma = \pi_1(S)$ is the fundamental group of a hyperbolic 2–orbifold with boundary. Moreover, each incident edge stabilizer is conjugate to a subgroup of a boundary subgroup $B \subset G_v$, defined as the preimage under π of $\pi_1(C)$, with C a component of ∂S.

In Section 8.3 we will need a description of flexible vertex stabilizers in the following cases (see [18, Sections 11 and 13] for proofs). Assume that G is one-ended.

- G is torsion-free and CSA. A flexible vertex stabilizer G_v of an abelian JSJ tree is a QH–subgroup, with S a surface and F trivial: G_v is isomorphic to $\Sigma = \pi_1(S)$, where S is a compact surface.
- G is hyperbolic relative to slender subgroups. A flexible vertex stabilizer G_v of a slender JSJ tree is a QH–subgroup, with F finite.

In both cases, every incident edge stabilizer is conjugate to a finite index subgroup of a boundary subgroup. Boundary subgroups are two-ended, and maximal among small subgroups of G_v. Every boundary subgroup contains an incident edge stabilizer.

7.3 Canonical JSJ splittings

We now use trees of cylinders to make JSJ splittings canonical (ie we get trees which are invariant under automorphisms). See [18] for a proof that JSJ splittings exist under the stated hypotheses, and a discussion of their flexible vertices.

Theorem 7.5 *Let G be hyperbolic relative to $H_1, \dots, H_n$. Assume that G is one-ended relative to $H_1, \dots, H_n$. There is an elementary (resp. virtually cyclic) JSJ tree relative to $H_1, \dots, H_n$ which is invariant under the subgroup of $\mathrm{Out}(G)$ preserving the conjugacy classes of the H_i's.*

Remark 7.6 When H_i is not slender, this allows nonslender splittings. Still, one can describe flexible subgroups of this JSJ tree as QH–subgroups [18, Section 13].

Proof First consider the case where $\mathcal{E}$ is the family of infinite elementary subgroups, as in Section 3.3. It is substable because G is one-ended relative to the H_i's (see Example 7.3). Let T be an elementary JSJ tree relative to the H_i's, and T_c its tree of cylinders for coelementarity.

By Proposition 6.1, the tree T_c has elementary edge stabilizers and lies in the JSJ deformation space. It is universally elliptic as its edge stabilizers are either parabolic, or are virtually cyclic and contain an edge stabilizer of T with finite index (Remark 4.4). It is invariant under the subgroup of $\mathrm{Out}(G)$ preserving the conjugacy classes of the H_i's because the JSJ deformation space is. The theorem follows.

Now turn to the case where $\mathcal{E}$ is the class of infinite virtually cyclic subgroups, still substable because of one-endedness. We start with a virtually cyclic JSJ tree (relative to the H_i's), we let T_c be its tree of cylinders (for coelementarity, restricted to $\mathcal{E}$, not commensurability), and we consider T_c^* obtained by collapsing edges of T_c whose stabilizer is not virtually cyclic. By Proposition 5.9 and Proposition 6.1, the trees T_c^*, T_c and T lie in the same deformation space. Moreover, T_c^* is universally elliptic because its edge stabilizers contain an edge stabilizer of T with finite index (Remark 4.4). It follows that T_c^* is a canonical JSJ splitting. □

A similar argument, using Proposition 6.3, shows:

Theorem 7.7 *Let G be a one-ended torsion-free CSA group. There exists an abelian (resp. cyclic) JSJ tree of G relative to all noncyclic abelian subgroups, which is $\mathrm{Out}(G)$–invariant.* □

8 Compatibility

Recall that two trees T and T' are *compatible* if they have a common refinement. The goal of this section is to show that T_c is compatible with many splittings. In particular, we show that trees of cylinders of JSJ deformation spaces often are *universally compatible*, that is compatible with every $\mathcal{E}$–tree.

This is proved under two different types of hypotheses: in Section 8.2 we assume that $\sim$ preserves universal ellipticity (this is true in particular when $\sim$ is commensurability), and in Section 8.3 we work in the setting of CSA groups and relative hyperbolic groups. Theorem 6 follows from Corollary 8.4 and Theorem 8.6

8.1 A general compatibility statement

We first prove a general compatibility statement, independent of JSJ theory.

Proposition 8.1 *Let T, T' be minimal $\mathcal{E}$–trees. If T dominates T', then T_c is compatible with T' and T'_c.*

Proof We have to construct a common refinement $\hat{T}$ of T_c and T' (compatibility of T_c with T'_c will follow, by the proposition, since T dominates T'_c). Choose a map $f\colon T \to T'$ as in Section 4.3. For each $p \in V(T_c)$, denote by Y_p the following subset of T: the point p if $p \in V_0(T_c)$, the cylinder defining p if $p \in V_1(T_c)$. Consider $Z_p = f(Y_p) \subset T'$. By Lemma 4.12, it is either a point or a cylinder of T'. Note that a given edge of T' is contained in exactly one Z_p.

We obtain $\hat{T}$ from T_c by "blowing up" each vertex p to the subtree Z_p. Formally, we define $\hat{T}$ as the tree obtained from $T_1 = \bigsqcup_{p \in V(T_c)} Z_p$ as follows: for each edge pq of T_c, with $Y_p = \{x\}$ and Y_q a cylinder containing x, add an edge to T_1, the endpoints being attached to the two copies of $f(x)$ in Z_p and Z_q respectively.

The tree T_c can be recovered from $\hat{T}$ by collapsing each Z_p to a point. We show that $\hat{T}$ is also a refinement of T'. Let $g\colon \hat{T} \to T'$ be the map defined as being induced by the identity on T_1, and being constant on each added edge. It preserves alignment because a given edge of T' is contained in exactly one Z_p, and g is injective on each Z_p. One therefore recovers T' from $\hat{T}$ by collapsing the added edges. □

8.2 Universal compatibility when $\sim$ preserves universal ellipticity

As before, we fix $\mathcal{E}$ and $\sim$, with $\mathcal{E}$ substable. All trees are assumed to be $\mathcal{E}$–trees. In this subsection, we assume that $\sim$ preserves universal ellipticity in the following sense.

Definition 8.2 The relation $\sim$ *preserves universal ellipticity* if, given $A, B \in \mathcal{E}$ with $A \sim B$, the group A is universally elliptic if and only if B is.

For instance, commensurability always preserves universal ellipticity. In the case of a relatively hyperbolic group G, coelementarity preserves universal ellipticity if one restricts to trees in which each H_i is elliptic (Section 3.3). Similarly, in Section 3.5, one has to restrict to trees in which noncyclic abelian subgroups are elliptic (the next subsection will provide nonrelative results).

Proposition 8.3 *Assume that $\mathcal{E}$ is substable and $\sim$ preserves universal ellipticity. If T_J is a JSJ tree over $\mathcal{E}$, its tree of cylinders is compatible with any $\mathcal{E}$–tree.*

Using Example 7.3, we immediately deduce:

Corollary 8.4 *Let G be finitely presented.*

(1) *Let $\mathcal{E}$ be the class of two-ended subgroups as in Section 3.1. If G is one-ended, the tree of cylinders of the JSJ deformation space over $\mathcal{E}$ is compatible with any tree with two-ended edge stabilizers.*

(2) *More generally, let $\mathcal{E}$ and $\sim$ be as in Section 3.2. If G does not split over a subgroup having infinite index in a group of $\mathcal{E}$, the tree of cylinders of the JSJ deformation space over $\mathcal{E}$ is compatible with any $\mathcal{E}$–tree.*

(3) *Let G be hyperbolic relative to finitely generated subgroups H_i as in Section 3.3, and assume that G is one-ended relative to the H_i's. Then the tree of cylinders of the elementary JSJ deformation space relative to the H_i's is compatible with any $\mathcal{E}$–tree in which each H_i is elliptic.* □

Remark 8.5 Finite presentability of G is required only to know that the JSJ deformation space exists.

Proof of Proposition 8.3 Let T be an $\mathcal{E}$–tree, and let $\widehat{T}$ be a refinement of T_J which dominates T (see Section 7.1). Let X be the tree obtained from $\widehat{T}$ by collapsing all the edges whose stabilizer is not $\sim$–equivalent to an edge stabilizer of T_J. The collapse map from $\widehat{T}$ to T_J factors through the collapse map $p\colon \widehat{T} \to X$. In particular, X dominates T_J.

Since T_J is universally elliptic, and $\sim$ preserves universal ellipticity, X is universally elliptic. By maximality of the JSJ deformation space, X lies in the JSJ deformation space. In particular, X and T_J have the same tree of cylinders X_c. We have to show that X_c is compatible with T.

Let $\widehat{T}_c$ be the tree of cylinders of $\widehat{T}$, which is compatible with T by Proposition 8.1 since $\widehat{T}$ dominates T. Because of the way X was defined, the restriction of $p\colon \widehat{T} \to X$ to any cylinder is either constant or injective. By Remark 4.13, $p_c\colon \widehat{T}_c \to X_c$ is a collapse map, so X_c is compatible with T. □

8.3 Universal compatibility when $\sim$ does not preserve universal ellipticity

Theorem 8.6 *Let G be one-ended.*

(1) *Suppose G is hyperbolic relative to* slender *subgroups $H_1, \dots, H_n$. The tree of cylinders of the slender JSJ deformation space is compatible with every tree whose edge stabilizers are slender.*

(2) *Suppose G is torsion free and CSA. The tree of cylinders of the abelian JSJ deformation space is compatible with every tree whose edge stabilizers are abelian.*

The tree of cylinders is defined with $\mathcal{E}$ as in Sections 3.4 and 3.5: it consists of all infinite slender (resp. abelian) subgroups, and $\sim$ is coelementarity (=co-slenderness) or commutation (note that each H_i is finitely-ended, so $\mathcal{E}$ is admissible by Lemma 3.5). The family $\mathcal{E}$ is substable because G is one-ended, but $\sim$ does not preserve universal ellipticity.

Proof Let T_J be a JSJ tree, and T_c its tree of cylinders. If $x \in V_0(T_c)$, we know that $G_x \notin \mathcal{E}$ (see Section 4.2). On the other hand $G_Y \in \mathcal{E}$ if $Y \in V_1(T_c)$, and edge stabilizers of T_c belong to $\mathcal{E}$ (see Section 6.1 and Section 6.2).

We now show that T_c is universally elliptic. Let $\varepsilon = (x, Y)$ be an edge. Let $e \subset Y$ be an edge of T_J adjacent to x. We have $G_e \subset G_\varepsilon \subset G_x$. If G_x is universally elliptic, so is G_ε. Otherwise, G_x is flexible. It is associated to a 2–orbifold S as described in Section 7, and G_e has finite index in a boundary subgroup B. Since B is the unique maximal small subgroup of G_x containing G_e, it also contains G_ε. Thus G_e has finite index in G_ε, and G_ε is universally elliptic because G_e is.

Given any $\mathcal{E}$–tree T, we now construct a common refinement $\hat{T}$ of T_c and T by blowing up T_c as in the proof of Proposition 8.1. There are several steps.

Step 1 We first define a G_p–invariant subtree $Z_p \subset T$, for p a vertex of T_c.

If $p \in V_0(T_c)$, the group G_p is not in $\mathcal{E}$. Consider its action on T. It fixes a unique point, or it is a flexible vertex group of T_J and has a minimal invariant subtree in T (because it is finitely generated). We define Z_p as that point or subtree.

If $p \in V_1(T_c)$, then G_p belongs to $\mathcal{E}$. The cylinder of T_J defining p corresponds to an equivalence class $\mathcal{C} \in \mathcal{E}/\sim$ (which contains G_p) as in Section 4.2. If this class corresponds to a cylinder of T (ie if there is an edge of T with stabilizer equivalent to G_p), we define Z_p as that cylinder. If not, we now show that G_p fixes a point of T; this point is necessarily unique (otherwise, there would be a cylinder), and we take it as Z_p.

Recall that G_p is in $\mathcal{E}$, hence is abelian or slender. If it does not fix a point in T, it fixes an end or preserves a line. Furthermore, it contains a subgroup G_e, for e an edge of T_J. This subgroup is elliptic in T because T_J is universally elliptic. Being contained in G_p, the group G_e fixes an end or preserves a line in T, so some subgroup of index at most 2 of G_e fixes an edge of T. The stabilizer of this edge is equivalent to G_e, hence to G_p. It yields a cylinder associated to $\mathcal{C}$, a contradiction.

Step 2 We now explain how to attach edges of T_c to $T_1 = \bigsqcup_{p \in V(T_c)} Z_p$. Let $\varepsilon = pq$ be an edge, with $p \in V_0(T_c)$ and $q \in V_1(T_c)$. We show that G_ε fixes a unique point x_ε in Z_p, and this point x_ε belongs to Z_q; we then attach the endpoints of ε to the copies of x_ε in Z_p and Z_q.

Note that G_ε is elliptic in T (because T_c is universally elliptic), and preserves Z_p and Z_q. If Z_p is not a point, then G_p is flexible, so is an extension $F \to G_p \to \Sigma$. As explained above, G_ε is contained in a boundary subgroup $B_0 \subset G_p$ with finite index.

We consider the action of G_p on its minimal subtree $Z_p \subset T$. Every boundary subgroup $B \subset G_p$ contains some G_e with finite index (with e an edge of T_J), hence acts elliptically. Being normal and finite, the group F acts as the identity, so there is an induced action of Σ on Z_p. For that action, boundary subgroups of Σ are elliptic, and edge stabilizers are finite or two-ended because they are slender (resp. abelian). This implies that B_0, hence also G_ε, fixes a unique point x_ε of Z_p (see [21, Theorem III.2.6] for the case of surface groups; the extension to an orbifold group is straightforward, as it contains a surface group with finite index).

We now show $x_\varepsilon \in Z_q$. If not, G_ε fixes the initial edge e of the segment joining x_ε to its projection onto Z_q. The stabilizer of e is equivalent to G_q because $G_e \sim G_\varepsilon \sim G_q$, and Z_q was defined as the cylinder containing e, so it contains x_ε.

Step 3 We can now construct $\widehat{T}$ by gluing edges of T_c to T_1 as in the proof of Proposition 8.1. It refines T_c, and there is a natural map $g\colon \widehat{T} \to T$ which is constant on all the edges corresponding to the edges of T_c, and which is isometric in restriction to each Z_p. To show that it is a collapse map, it suffices to see that Z_p and $Z_{p'}$ (viewed as subtrees of T) cannot have an edge e in common if p, p' are distinct vertices of T_c.

We assume they do, and we reach a contradiction. Let $e_p \subset \widehat{T}$ be the copy of e in Z_p. Then $G_{e_p} \subset G_e$. One has $G_{e_p} \in \mathcal{E}$ because $\mathcal{E}$ is substable, and Axiom (2) implies $G_e \sim G_{e_p}$. Similarly, $G_e \sim G_{e'_p}$. This also shows that $\widehat{T}$ is an $\mathcal{E}$–tree.

Note that p and p' cannot both belong to $V_1(T_c)$, as Z_p and $Z_{p'}$ then are points or distinct cylinders of T. We may therefore assume $p \in V_0(T_c)$. Let ε be the initial edge of the segment $[p, p'] \subset T_c$. By connectedness of cylinders, the segment joining e_p to $e_{p'}$ is contained in a cylinder of $\widehat{T}$. Since this segment contains the edge of $\widehat{T}$ corresponding to ε, we have $G_\varepsilon \sim G_{e_p}$. Thus $\langle G_\varepsilon, G_{e_p} \rangle$ is a small subgroup of G_p, and therefore is contained with finite index in a boundary subgroup B. By [21], G_{e_p} fixes a unique point of Z_p, contradicting the fact that G_{e_p} fixes e_p. $\square$

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