The moduli space of stable quotients

ALINA MARIAN

DRAGOS OPREA

RAHUL PANDHARIPANDE

A moduli space of stable quotients of the rank n trivial sheaf on stable curves is introduced. Over nonsingular curves, the moduli space is Grothendieck's Quot scheme. Over nodal curves, a relative construction is made to keep the torsion of the quotient away from the singularities. New compactifications of classical spaces arise naturally: a nonsingular and irreducible compactification of the moduli of maps from genus 1 curves to projective space is obtained. Localization on the moduli of stable quotients leads to new relations in the tautological ring generalizing Brill–Noether constructions.

The moduli space of stable quotients is proven to carry a canonical 2–term obstruction theory and thus a virtual class. The resulting system of descendent invariants is proven to equal the Gromov–Witten theory of the Grassmannian in all genera. Stable quotients can also be used to study Calabi–Yau geometries. The conifold is calculated to agree with stable maps. Several questions about the behavior of stable quotients for arbitrary targets are raised.

Dedicated to William Fulton on the occasion of his 70th birthday

1 Introduction

1.1 Virtual classes

Only a few compact moduli spaces in algebraic geometry carry virtual classes. The conditions placed on the associated deformation theories are rather strong. The principal cases (so far) are that

(i) stable maps to nonsingular varieties (see Behrend and Fantechi [2], Kim, Kresch and Oh [17] and Li and Tian [22]),

(ii) stable sheaves on nonsingular 3–folds (see Pandharipande and Thomas [36] and Thomas [40]),

(iii)   stable sheaves on nonsingular surfaces (see Li and Tian [22]),

(iv)   Grothendieck's Quot scheme on nonsingular curves (see Ciocan-Fontanine and Kim [3] and Marian and Oprea [27]).

Of the above four families, the first three are understood to be related. The correspondences of Maulik, Nekrasov, Okounkov and Pandharipande [28; 29] and Pandharipande and Thomas [36] relate (i) and (ii). The connections in Lee and Parker [21] and Taubes [39] between Gromov–Witten invariants and Donaldson/Seiberg–Witten invariants relate (i) and (iii). For equivalence with (ii) and (iii), the associated Gromov–Witten theories must be considered with domains varying in the moduli of stable curves $\overline{M}_g$.

The construction of the virtual class of the Quot scheme (iv) requires the curve C to be fixed in moduli. In fact, the Quot scheme of a nodal curve does *not* carry a virtual class via the standard deformation theory. In order to fully connect (i) and (iv), new moduli spaces are required.

1.2   Stable quotients

We introduce here a moduli space of *stable quotients*

$$\mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0$$

on m–pointed curves C with (at worst) nodal singularities. Two basic properties are satisfied:

- The quotient sheaf Q is locally free at the nodes and markings of C.
- The moduli of stable quotients is proper over $\overline{M}_{g,m}$.

The first property yields a virtual class, and the second property leads to a system of invariants over $\overline{M}_{g,m}$. Our main result equates the descendent theory of the moduli of stable quotients to the Gromov–Witten theory of the Grassmannian in all genera.

Stable quotients are defined in Section 2. The basic structures of the moduli space (including the virtual class) are discussed in Section 3. The important case of mapping to a point is studied in Section 4. Comparison results with the Gromov–Witten theory of Grassmannians in the strongest equivariant form are stated in Section 5. The construction of the moduli of stable quotients and proofs of the comparison results are presented in Section 6-7.

The intersection theory of the moduli of stable quotients leads to new tautological relations on the moduli of curves. Basic relations generalizing classical Brill–Noether constructions are presented in Section 8.

Stable quotients can also be used to study Calabi–Yau geometries. The most accessible are the local toric cases. The conifold, given by the total space of

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1,$$

is calculated in Section 9 and found to agree exactly with Gromov–Witten theory.

Given a projective embedding of an arbitrary scheme

$$X \subset \mathbb{P}^n,$$

a moduli space of stable quotients associated to X is defined in Section 10. We speculate, at least when X is a nonsingular complete intersection, that the moduli spaces carry virtual classes in all genera. Virtual classes may exist in even greater generality.

Stable quotient invariants in genus 1 for Calabi–Yau hypersurfaces are discussed in Section 10.2. Let

$$M_1(\mathbb{P}^n, d) \subset \overline{M}_1(\mathbb{P}^n, d)$$

be the open locus of the moduli of stable maps with nonsingular irreducible domain curves. Stable quotients provide a nonsingular[1], irreducible, modular compactification

$$M_1(\mathbb{P}^n, d) \subset \overline{Q}_1(\mathbb{P}^n, d).$$

For the Calabi–Yau hypersurface of degree $n + 1$,

$$X_{n+1} \subset \mathbb{P}^n,$$

genus 1 invariants can be defined naturally as an Euler characteristic of a rank $(n+1)d$ vector bundle on $\overline{Q}_1(\mathbb{P}^n, d)$. The relationship to the Gromov–Witten invariants of X_{n+1} is not yet clear, but there will likely be a transformation.

The paper ends with several questions about the behavior of stable quotients. Certainly, our main results carry over to the hyperquot schemes associated to SL_n–flag varieties. Other variants are discussed in Section 10.3. The toric case has been addressed by Ciocan-Fontanine and Kim [4].

1.3 Later work

Tautological relations coming from the stable quotient geometry, similar to those presented in Section 8, are studied by Pandharipande [31; 32] on the moduli spaces $M_{g,n}^c$ of marked curves of compact type. A Wick formalism is developed in order to evaluate the relations explicitly in terms of κ classes. The main results for $n > 0$ are:

[1]Nonsingularity here is as a Deligne–Mumford stack.

(i)   the κ rings $\kappa^*(M_{g,n}^c)$ are generated by κ classes of degree at most $g-1+\lfloor n/2 \rfloor$,

(ii)   there are no relations between the kappa classes below the threshold degree,

(iii)   there is a natural isomorphism

$$\kappa^*(M_{0,2g+n}^c) \xrightarrow{\sim} \kappa^*(M_{g,n}^c) \ .$$

Result (iii) is used to completely determine the κ rings including formulas for their Betti numbers.

A detailed study of the stable quotient relations on M_g is undertaken by Pandharipande and Pixton [35]. The virtual class of the stable quotient space can be viewed as a new object in the classical theory of linear systems on curves. Using the Wick formalism and a series of transformations, the stable quotient relations are recast to prove an elegant set of relations in $R^*(M_g)$ conjectured by Faber and Zagier a decade ago. Whether the Faber–Zagier relations are a complete set for $R^*(M_g)$ is an interesting question. For $g \le 23$, there are no further relations. No further relations have been found in any genus, but by calculations of Faber, the set does not yield a Gorenstein ring in genus 24.

Finally, stable quotients should be considered to lie between stable maps to the Grassmannian and stable sheaves relatively over $\overline{M}_g$ by [33]. Recent wall-crossing methods of Joyce [16] and Kontsevich and Soibelman [20] will likely be relevant to the study. A step in this direction is taken by Toda [42]: a series of moduli spaces is constructed, depending on a stability parameter and interpolating between the stable quotient and the stable map spaces. Several further directions which have stable quotients as their starting point are Ciocan-Fontanine and Kim [4], Cooper [5] and Manolache [26].

2 Stability

2.1 Curves

A *curve* is a reduced and connected scheme over $\mathbb{C}$ of pure dimension 1. Let C be a curve of arithmetic genus

$$g = h^1(C, \mathcal{O}_C)$$

with at worst nodal singularities. Let

$$C^{\mathrm{ns}} \subset C$$

denote the nonsingular locus. The data $(C, p_1, \dots, p_m)$ with distinct markings $p_i \in C^{\mathrm{ns}}$ determine a genus g, m–pointed, *quasistable curve*. A quasistable curve is *stable* if $\omega_C(p_1 + \cdots + p_m)$ is ample.

2.2 Quotients

Let q be a quotient of the trivial bundle on a pointed quasistable curve C,

$$\mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0.$$

If Q is locally free at the nodes and markings of C, q is a *quasistable quotient*. Quasistability of q implies

(i) the torsion subsheaf $\tau(Q) \subset Q$ has support contained in

$$C^{\mathrm{ns}} \setminus \{p_1, \dots, p_m\},$$

(ii) the associated kernel,

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,$$

is a locally free sheaf on C.

Let r denote the rank of S.

Let $(C, p_1, \dots, p_m)$ be a quasistable curve equipped with a quasistable quotient q. The data $(C, p_1, \dots, p_m, q)$ determine a *stable quotient* if the $\mathbb{Q}$–line bundle

$$(1) \qquad \omega_C(p_1 + \cdots + p_m) \otimes (\wedge^r S^*)^{\otimes \epsilon}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies

$$2g - 2 + m \geq 0.$$

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbb{P}^1 \cong P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

Of course, when considering stable quotients in families, flatness over the base is imposed on both the curve C and the quotient sheaf Q.

2.3 Isomorphisms

Let $(C, p_1, \ldots, p_m)$ be a quasistable curve. Two quasistable quotients

$$(2) \qquad\qquad \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0, \quad \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q'} Q' \to 0$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^n \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasistable quotients

$$\phi\colon (C, p_1, \ldots, p_m, q) \to (C', p_1', \ldots, p_m', q')$$

is an isomorphism of curves

$$\phi\colon C \xrightarrow{\sim} C'$$

satisfying

(i) $\phi(p_i) = p_i'$ for $1 \le i \le m$,

(ii) the quotients q and $\phi^*(q')$ are strongly isomorphic.

Quasistable quotients (2) on the same curve C may be isomorphic without being strongly isomorphic.

Theorem 1 *The moduli space of stable quotients $\overline{Q}_{g,m}(\mathbb{G}(r, n), d)$ parameterizing the data*

$$(C, \ p_1, \ldots, p_m, \ 0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0),$$

with $\mathrm{rank}(S) = r$ *and* $\deg(S) = -d$, *is a separated and proper Deligne–Mumford stack of finite type over* $\mathbb{C}$.

Theorem 1 is obtained by mixing the construction of the moduli of stable curves with the Quot scheme. Keeping the torsion of the quotient away from the nodes and markings is a twist motivated by relative geometry. The proof of Theorem 1 is given in Section 6.

2.4 Automorphisms

The automorphism group A_C of a quasistable curve $(C, p_1, \ldots, p_m)$ may be positive dimensional. If the dimension is 0, A_C is finite. Stability of $(C, p_1, \ldots, p_m)$ is well-known to be equivalent to the finiteness of A_C. If $(C, p_1, \ldots, p_m, q)$ is a stable quotient, the ampleness condition (1) implies that the marked curve $(C, p_1, \ldots, p_m)$ is semistable. Then, the connected component of the automorphism group A_C is a torus.[2]

An *automorphism* of a quasistable quotient $(C, p_1, \ldots, p_m, q)$ is a self-isomorphism. The automorphism group A_q of the quasistable quotient q embeds in the automorphism group of the underlying curve

$$A_q \subset A_C.$$

We leave the proof of the following elementary result to the reader.

Lemma 1 *Let $(C, p_1, \ldots, p_m, q)$ be a quasistable quotient such that $(C, p_1, \ldots, p_m)$ is semistable. Then q is stable if and only if A_q is finite.*

2.5 First examples

The simplest examples occur when $d = 0$. Then, stability of the quotient implies the underlying pointed curve is stable. We see

$$\overline{Q}_{g,m}(\mathbb{G}(r, n), 0) = \overline{M}_{g,m} \times \mathbb{G}(r, n)$$

where $\mathbb{G}(r, n)$ denotes the Grassmannian of r–planes in $\mathbb{C}^n$.

A more interesting example is $\overline{Q}_{1,0}(\mathbb{G}(1, n), 1)$. A direct analysis yields

$$\overline{Q}_{1,0}(\mathbb{G}(1, n), 1) = \overline{M}_{1,1} \times \mathbb{P}^{n-1}.$$

Given a 1–pointed stable genus 1 curve (E, p) and an element $\xi \in \mathbb{P}^{n-1}$, the associated stable quotient is

$$0 \to \mathcal{O}_E(-p) \overset{\iota_\xi}{\to} \mathbb{C}^n \otimes \mathcal{O}_E \to Q \to 0,$$

where ι_ξ is the composition of the canonical inclusion

$$0 \to \mathcal{O}_E(-p) \to \mathcal{O}_E$$

with the line in $\mathbb{C}^n$ determined by ξ.

The open locus $Q_{g,0}(\mathbb{G}(r, n), d) \subset \overline{Q}_{g,0}(\mathbb{G}(r, n), d)$, corresponding to nonsingular domains C, is simply the universal Quot scheme over the moduli space of nonsingular curves.

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[2]We assume $(g, m) \neq (1, 0)$.

3 Structures

3.1 Maps

Over the moduli space of stable quotients, there is a universal curve

(3) $$\pi \colon U \to \bar{Q}_{g,m}(\mathbb{G}(r,n), d)$$

with m sections and a universal quotient

$$0 \to S_U \to \mathbb{C}^n \otimes \mathcal{O}_U \xrightarrow{q_U} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\bar{Q}_{g,m}(\mathbb{G}(r,n), d)$ is equipped with two basic types of maps. If $2g - 2 + m > 0$, then the stabilization of $(C, p_1, \ldots, p_m)$ determines a map

$$\nu \colon \bar{Q}_{g,m}(\mathbb{G}(r,n), d) \to \overline{M}_{g,m}$$

by forgetting the quotient. For each marking p_i, the quotient is locally free over p_i, hence it determines an evaluation map

$$\mathrm{ev}_i \colon \bar{Q}_{g,m}(\mathbb{G}(r,n), d) \to \mathbb{G}(r,n).$$

Furthermore, as in Gromov–Witten theory, there are gluing maps

$$\bar{Q}_{g_1,m_1+1}(\mathbb{G}(r,n), d_1) \times_{\mathbb{G}(r,n)} \bar{Q}_{g_2,m_2+1}(\mathbb{G}(r,n), d_2) \to \bar{Q}_{g,m}(\mathbb{G}(r,n), d)$$

whenever $$g = g_1 + g_2, \quad m = m_1 + m_2, \quad d = d_1 + d_2.$$

In contrast with Gromov–Witten theory, the universal curve (3) is *not* isomorphic to $\bar{Q}_{g,m+1}(\mathbb{G}(r,n), d)$. In fact, there does *not* exist a forgetful map of the form

$$\bar{Q}_{g,m+1}(\mathbb{G}(r,n), d) \to \bar{Q}_{g,m}(\mathbb{G}(r,n), d)$$

since there is no canonical way to contract the quotient sequence.

The general linear group $\mathrm{GL}_n(\mathbb{C})$ acts on $\bar{Q}_{g,m}(\mathbb{G}(r,n), d)$ via the standard action on $\mathbb{C}^n \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathrm{GL}_n(\mathbb{C})$–equivariant.

3.2 Obstruction theory

Even if $2g - 2 + m$ is not strictly positive, the moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A \colon \bar{Q}_{g,m}(\mathbb{G}(r,n), d) \to \mathcal{M}_{g,m}.$$

The moduli of stable quotients with fixed underlying curve

$$(C, p_1, \ldots, p_m) \in \mathcal{M}_{g,m}$$

is simply an open set of the Quot scheme. The following result is obtained from the standard deformation theory of the Quot scheme.

Theorem 2 *The deformation theory of the Quot scheme determines a 2–term obstruction theory on $\bar{Q}_{g,m}(\mathbb{G}(r,n), d)$ relative to v^A given by $RHom(S, Q)$.*

An absolute 2–term obstruction theory on $\bar{Q}_{g,m}(\mathbb{G}(r,n), d)$ is obtained from Theorem 2 and the smoothness of $\mathcal{M}_{g,m}$; see Behrend and Fantechi [2] and Getzler and Pandharipande [12]. The analogue of Theorem 2 for the Quot scheme of a *fixed* nonsingular curve was observed in Ciocan-Fontanine and Kim [3] and Marian and Oprea [27].

Proof Let $\mathcal{C} \to \mathcal{M}_{g,m}$ be the universal curve, and let $\mathcal{Q}$ be the relative Quot scheme along its fibers. We write

$$v: \mathcal{Q}' \to \mathcal{M}_{g,m}$$

for the locus of $\mathcal{Q}$ corresponding to locally free subsheaves, and consider the universal sequence

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O} \to Q \to 0$$

over $\mathcal{Q}' \times_{\mathcal{M}_{g,m}} \mathcal{C}$.

We endow $\mathcal{Q}'$ with a relative perfect obstruction theory. Writing π for the projection map

$$\mathcal{Q}' \times_{\mathcal{M}_{g,m}} \mathcal{C} \to \mathcal{Q}',$$

the relative deformation-obstruction theory of the Quot scheme is standardly given by

(4) $$RHom_\pi(S, Q) = R\pi_\star Hom(S, Q).$$

The equality (4) uses the fact that the subsheaf S is locally free.

Finally, we must show that $R\pi_\star Hom(S, Q)$ can be resolved by a two step complex of vector bundles. The proof of Marian and Oprea [27, Theorem 1] applies verbatim to this situation, yielding the claim.[3] □

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[3]The language of Li and Tian [22] is used in [27]. Alternatively, in the Behrend–Fantechi formalism [2], the reduced Atiyah class yields a morphism in the derived category

$$RHom_\pi(S, Q)^\vee \to \mathbb{L}_v$$

to the relative cotangent complex of the morphism v [10].

The $\mathrm{GL}_n(\mathbb{C})$–action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathrm{GL}_n(\mathbb{C})$–equivariant cycle theory,

$$[\bar{Q}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} \in A_*^{\mathrm{GL}_n(\mathbb{C})}(\bar{Q}_{g,m}(\mathbb{G}(r,n),d),\mathbb{Q}).$$

A system of $\mathrm{GL}_n(\mathbb{C})$–equivariant descendent invariants is defined by the brackets

$$\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_m}(\gamma_m)\rangle_{g,d} = \int_{[\bar{Q}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}}} \prod_{i=1}^m \psi_i^{a_i} \cup \mathrm{ev}_i^*(\gamma_i)$$

where $\gamma_i \in A_{\mathrm{GL}_n(\mathbb{C})}^*(\mathbb{G}(r,n),\mathbb{Q})$. The classes ψ_i are obtained from the cotangent lines on the domain (or, equivalently, pulled back from the Artin stack by ν^A).

3.3 Nonsingularity

Let E be a nonsingular curve of genus 1, and let

$$f\colon E \to \mathbb{G}(1,n)$$

be a morphism of degree $d > 0$. The pullback of the tautological sequence on $\mathbb{G}(1,n)$ determines a stable quotient on E. The moduli space of maps is an open[4] subset

(5)
$$M_{1,0}(\mathbb{G}(1,n),d) \subset \bar{Q}_{1,0}(\mathbb{G}(1,n),d)$$

for $d > 0$.

Let (C,q) be a stable quotient parameterized by $\bar{Q}_{1,0}(\mathbb{G}(1,n),d)$. By stability, C is either a nonsingular genus 1 curve or a cycle of rational curves. The associated sheaf S is a line bundle of degree $-d < 0$. The vanishing

$$\mathrm{Ext}^1(S,Q) = 0$$

holds since there are no nonspecial line bundles of positive degree on such curves.

Proposition 1 $\bar{Q}_{1,0}(\mathbb{G}(1,n),d)$ *is a nonsingular irreducible Deligne–Mumford stack of dimension nd for $d > 0$.*

Proof Nonsingularity has already been established. The dimension is obtained from a Riemann–Roch calculation[5] of $\chi(S,Q)$. Irreducibility is clear since $Q_{1,0}(\mathbb{G}(1,n),d)$ is an open set of a projective bundle over the moduli of elliptic curves. $\square$

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[4]If $d > 1$, the subset is nonempty. If $d = 1$, the subset is empty.

[5]The calculation is done in general in Lemma 4 below.

For simplicity, we will denote the moduli space by $\bar{Q}_{1,0}(\mathbb{P}^{n-1}, d)$. Stable quotients provide an efficient compactification (5) of $M_{1,0}(\mathbb{P}^{n-1}, d)$. Instead of desingularizing the moduli of maps by blowing up the closure of

$$M_{1,0}(\mathbb{P}^{n-1}, d) \subset \bar{M}_{1,0}(\mathbb{P}^{n-1}, d)$$

in the moduli of stable maps (see Hu and Li [15] and Vakil and Zinger [43]), the stable quotient space achieves a simple modular desingularization by blowing down.

For large degree d, all line bundles on nonsingular curves are nonspecial. As a result, the following nonsingularity result holds.

Proposition 2 *For $g \geq 2$ and $d \geq 2g - 1$, the forgetful morphism*

$$\nu \colon Q_{g,0}(\mathbb{P}^{n-1}, d) \to M_g$$

is smooth of expected relative dimension.

The result does *not* hold over the boundary or even over the interior if markings are present.

4 Stable quotients for $\mathbb{G}(n, n)$

4.1 $n = 1$

Consider $\bar{Q}_{g,m}(\mathbb{G}(1, 1), d)$ for $d > 0$. The moduli space parameterizes stable quotients

$$0 \to S \to \mathcal{O}_C \to Q \to 0.$$

Hence, S is an ideal sheaf of C.

Let $\bar{M}_{g,m|d}$ be the moduli space of genus g curves with markings

$$\{p_1 \ldots, p_m\} \cup \{\hat{p}_1, \ldots, \hat{p}_d\} \in C^{\mathrm{ns}} \subset C$$

satisfying the conditions

  (i)   the points p_i are distinct,
  (ii) the points $\hat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C \left(\sum_{i=1}^m p_i + \epsilon \sum_{j=1}^d \hat{p}_j \right)$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\hat{p}_j$ and $\hat{p}_{j'}$ to coincide.

The moduli space $\overline{M}_{g,m|d}$ is a nonsingular, irreducible, Deligne–Mumford stack.[6] Given an element

$$[C, p_1, \ldots, p_m, \hat{p}_1, \ldots, \hat{p}_d] \in \overline{M}_{g,m|d} \ ,$$

there is a canonically associated stable quotient

$$(6) \qquad\qquad 0 \to \mathcal{O}_C\left(-\sum_{j=1}^d \hat{p}_j\right) \to \mathcal{O}_C \to Q \to 0.$$

We obtain a morphism

$$\phi\colon \overline{M}_{g,m|d} \to \overline{Q}_{g,m}(\mathbb{G}(1,1), d).$$

The following result is proven by matching the stability conditions.

Proposition 3 *The map ϕ induces an isomorphism of coarse moduli spaces*

$$\overline{M}_{g,m|d}/\mathbb{S}_d \xrightarrow{\sim} \overline{Q}_{g,m}(\mathbb{G}(1,1), d),$$

where the symmetric group $\mathbb{S}_d$ acts by permuting the markings $\hat{p}_j$.

The first example to consider is $\overline{Q}_{0,2}(\mathbb{G}(1,1), d)$ for $d > 0$. The space has a rather simple geometry. For example, the Poincaré polynomial

$$p_d = \sum_{k=0}^{2d-2} B_k t^k$$

where B_k is the k–th Betti number of $\overline{Q}_{0,2}(\mathbb{G}(1,1), d)$, is easily obtained.

Lemma 2 $p_d = (1+t^2)^{d-1}$ for $d > 0$.

Proof Let (C, p_1, p_2, q) be an element of $\overline{Q}_{0,2}(\mathbb{G}(1,1), d)$. By the stability condition, (C, p_1, p_2) must be a simple chain of rational curves with the markings p_1 and p_2 on opposite extremal components. We may stratify $\overline{Q}_{0,2}(\mathbb{G}(1,1), d)$ by the number n of components of C and the distribution of the degree d on these components. The associated quasiprojective strata

$$S_{(d_1,\ldots,d_n)} \subset \overline{Q}_{0,2}(\mathbb{G}(1,1), d)$$

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[6]In fact, $\overline{M}_{g,m|d}$ is a special case of the moduli of pointed curves with weights studied by Hassett [14] and Losev and Manin [25].

are indexed by vectors

$$(d_1, \ldots, d_n), \quad d_i > 0, \quad \sum_{i=1}^{n} d_i = d.$$

Moreover, each stratum is a product,

$$S_{(d_1, \ldots, d_n)} \cong \prod_{i=1}^{n} (\mathrm{Sym}^{d_i}(\mathbb{C}^*)/\mathbb{C}^*).$$

To calculate p_d, we must compute the virtual Poincaré polynomial of the quotient space $\mathrm{Sym}^k(\mathbb{C}^*)/\mathbb{C}^*$ for all $k > 0$. We start with the virtual Poincaré polynomial of $\mathrm{Sym}^k(\mathbb{C})$,

$$p(\mathrm{Sym}^k(\mathbb{C})) = p(\mathbb{C}^k) = t^{2k}.$$

Filtering by the order at $0 \in \mathbb{C}$, we find

$$p(\mathrm{Sym}^k(\mathbb{C})) = \sum_{i=0}^{k} p(\mathrm{Sym}^i(\mathbb{C}^*)).$$

We conclude

$$p(\mathrm{Sym}^k(\mathbb{C}^*)) = t^{2k} - t^{2k-2}$$

for $k > 0$. The quotient by $\mathbb{C}^*$ can be handled simply by dividing by $t^2 - 1$; see [9]. Hence,

$$p(\mathrm{Sym}^k(\mathbb{C}^*)/\mathbb{C}^*) = t^{2k-2}.$$

The Lemma then follows by elementary counting. $\square$

4.2 Classes

There are several basic classes on $\overline{M}_{g,m|d}$. As in the study of the standard moduli space of stable curves, there are strata classes

$$\mathcal{S} \in A^*(\overline{M}_{g,m|d}, \mathbb{Q})$$

given by fixing the topological type of a degeneration. New diagonal classes are defined for every subset $J \subset \{1, \ldots, d\}$ of size at least 2,

$$D_J \in A^{|J|-1}(\overline{M}_{g,m|d}, \mathbb{Q}),$$

corresponding to the locus where the points $\{\hat{p}_j\}_{j \in J}$ are coincident. In fact, the subvariety

$$D_J \subset \overline{M}_{g,m|d}$$

is isomorphic to $\overline{M}_{g,m|(d-|J|+1)}$. The cotangent bundles

$$\mathbb{L}_i \to \overline{M}_{g,m|d}, \quad \hat{\mathbb{L}}_j \to \overline{M}_{g,m|d}$$

corresponding to the two types of markings have respective Chern classes

$$\psi_i = c_1(\mathbb{L}_i), \ \hat{\psi}_j = c_1(\hat{\mathbb{L}}_j) \in A^1(\overline{M}_{g,m|d}, \mathbb{Q}).$$

The Hodge bundle with fiber $H^0(C, \omega_C)$ over the curve $[C] \in \overline{M}_{g,m|d}$,

$$\mathbb{E} \to \overline{M}_{g,m|d},$$

has Chern classes

$$\lambda_i = c_i(\mathbb{E}) \in A^i(\overline{M}_{g,m|d}, \mathbb{Q}).$$

4.3 Cotangent calculus

Assume $2g - 2 + m \geq 0$. Canonical contraction defines a fundamental birational morphism

(7) $$\tau \colon \overline{M}_{g,m+d} \to \overline{M}_{g,m|d}$$

constructed in a more general context by Hassett [14]. By the stability conditions, the contraction map does not affect the points p_i. Therefore, the cotangent lines at the points p_i are unchanged by τ,

$$\tau^*(\psi_i) = \psi_i, \quad 1 \leq i \leq m.$$

However, contraction affects the cotangent line classes at the other points

(8) $$\psi_{m+j} = \tau^*(\hat{\psi}_j) + \Delta_{m+j}.$$

Here, Δ_{m+j} is the sum

$$\Delta_{m+j} = \sum_{J', J''} \Delta_{J', J''},$$

for partitions

$$J' \cup J'' = \{1, \ldots, m+d\}, \quad m+j \in J' \subset \{m+1, \ldots, m+d\}.$$

The boundary divisor $\Delta_{J', J''}$ of $\overline{M}_{g,m+d}$ parameterizes curves

$$C = C' \cup C'', \quad g(C') = 0, \quad g(C'') = g$$

with a single separating node and the markings labelled J' and J'' distributed on C' and C'' respectively.

Let $\prod_{j=1}^{d} \hat{\psi}_j^{y_j}$ be a monomial class on $\overline{M}_{g,m|d}$. Since τ is birational,

$$\tag{9} \tau_* \tau^* \left(\prod_{j=1}^{d} \hat{\psi}_j^{y_j} \right) = \prod_{j=1}^{d} \hat{\psi}_j^{y_j}.$$

After using relations (8) and (9), we see for example

$$\tau_*(\psi_{m+j}) = \hat{\psi}_j + \sum_{j' \neq j} D_{j,j'}.$$

Indeed, the only contributions come from the 2 element sets

$$J' = \{m+j, m+j'\}.$$

For $|J'| > 2$, the pushforward of $\Delta_{J',J''}$ under τ vanishes for dimension reasons. The method proves the following result.

Lemma 3 *There exists a universal formula*

$$\tau_* \left(\prod_{i=1}^{m} \psi_i^{x_i} \prod_{j=1}^{d} \psi_{m+j}^{y_j} \right) = \prod_{i=1}^{m} \psi_i^{x_i} \left(\prod_{j=1}^{d} \hat{\psi}_j^{y_j} + \cdots \right),$$

where the dots are polynomials in the $\hat{\psi}_j$ and D_J classes which are independent of g and m.

4.4 Canonical forms

Let $J, J' \subset \{1, \ldots, d\}$. The cotangent line classes

$$\tag{10} \hat{\psi}_j | D_J = \hat{\psi}_J$$

are all equal for $j \in J$. If J and J' have nontrivial intersection, we obtain

$$\tag{11} D_J \cdot D_{J'} = (-\hat{\psi}_{J \cup J'})^{|J \cap J'|-1} D_{J \cup J'}$$

by examining normal bundles.

If $M(\hat{\psi}_j, D_J)$ is any monomial in the cotangent line and diagonal classes, we can write M in a *canonical form* in two steps:

  (i) Multiply the diagonal classes using (11) until the result is a product of cotangent line classes with $D_{J_1} D_{J_2} \cdots D_{J_l}$ where all the subsets J_i are disjoint.

  (ii) Collect the equal cotangent line classes using (10).

Let M^C denote the resulting *canonical form*.

By extending the operation linearly, we can write any polynomial $P(\widehat{\psi}_j, D_J)$ in canonical form P^C. In particular, the universal formulas of Lemma 3 can be taken to be in canonical form.

4.5 Example

The cotangent class intersections on $\overline{M}_{0,2|d}$,

$$(12) \qquad \int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_d^{y_d},$$

for $d > 0$ are straightforward to calculate. Since the dimension of $\overline{M}_{0,2|d}$ is $d - 1$, at least one of the y_j must vanish. After permuting the indices, we may take $y_d = 0$. By studying the geometry of the map

$$\pi \colon \overline{M}_{0,2|d} \to \overline{M}_{0,2|d-1}$$

forgetting $\widehat{p}_d$ in case $d > 1$, we deduce the identities

$$\psi_i = \pi^\star \psi_i + \Delta_i, \ 1 \le i \le 2, \quad \text{and} \quad \widehat{\psi}_j = \pi^\star \widehat{\psi}_j, \ 1 \le j \le d-1.$$

Here, $\Delta_i \cong \overline{M}_{0,2|d-1}$ denotes the canonical section of π corresponding to nodal domains with p_i and $\widehat{p}_d$ on a common rational tail. Since $\Delta_i \cdot \psi_i = 0$, we see

$$\psi_i^{x_i} = \pi^\star \psi_i^{x_i} + \pi^\star \psi_i^{x_i-1} \cdot \Delta_i$$

for $1 \le i \le 2$. In turn, these identities imply

$$\int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}}$$
$$= \int_{\overline{M}_{0,2|d-1}} \psi_1^{x_1-1} \psi_2^{x_2} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}} + \int_{\overline{M}_{0,2|d-1}} \psi_1^{x_1} \psi_2^{x_2-1} \widehat{\psi}_1^{y_1} \cdots \widehat{\psi}_{d-1}^{y_{d-1}}.$$

Solving the recurrence, we conclude (12) vanishes unless all $y_j = 0$ and

$$\int_{\overline{M}_{0,2|d}} \psi_1^{x_1} \psi_2^{x_2} = \binom{d-1}{x_1, x_2}.$$

4.6 Tautological complexes

Consider the universal curve

$$\pi \colon U \to \overline{M}_{g,m|d}$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (6). The complex $R\pi_*(S_U^*) \in D_{\mathrm{coh}}^b(\overline{M}_{g,m|d})$ will arise naturally in localization calculations on the moduli of stable quotients. Base change of the complex to

$$[C, p_1, \ldots, p_m, \hat{p}_1, \ldots, \hat{p}_d] \in \overline{M}_{g,m|d}$$

computes the cohomology groups

$$H^0\left(C, \mathcal{O}_C\left(\sum_{j=1}^d \hat{p}_j\right)\right), \quad H^1\left(C, \mathcal{O}_C\left(\sum_{j=1}^d \hat{p}_j\right)\right)$$

with varying ranks.

A canonical resolution by vector bundles of $R\pi_*(S_U^*)$ is easily obtained from the sequence

$$(13) \qquad 0 \to \mathcal{O}_C \to \mathcal{O}_C\left(\sum_{j=1}^d \hat{p}_j\right) \to \mathcal{O}_C\left(\sum_{j=1}^d \hat{p}_j\right)\Big|_{\sum_{j=1}^d \hat{p}_j} \to 0.$$

The rank d bundle

$$\mathbb{B}_d \to \overline{M}_{g,m|d}$$

with fiber

$$H^0\left(C, \mathcal{O}_C\left(\sum_{j=1}^d \hat{p}_j\right)\Big|_{\sum_{j=1}^d \hat{p}_j}\right)$$

is obtained from the geometry of the points $\hat{p}_j$. The Chern classes of $\mathbb{B}_d$ are universal polynomials in the $\hat{\psi}_j$ and D_J classes, which do not depend on g and m. A precise expression for the Chern classes will be obtained in Section 8.3. Up to a rank 1 trivial factor, $R\pi_*(S_U^*)$ is equivalent to the complex

$$\mathbb{B}_d \to \mathbb{E}^*$$

obtained from the derived pushforward of (13).

4.7 General n

While the moduli space

$$\overline{Q}_{g,m}(\mathbb{G}(1,1), d) \to \overline{M}_{g,m}$$

may be viewed simply as a compactification of the symmetric product of the universal curve over $\overline{M}_{g,m}$, the moduli space $\overline{Q}_{g,m}(\mathbb{G}(n,n), d)$ is more difficult to describe

since the stable subbundles have higher rank. Nevertheless, since $\mathrm{Ext}^1(S, Q)$ always vanishes, we obtain the following result.

Proposition 4 *The moduli space* $\bar{Q}_{g,m}(\mathbb{G}(n, n), d)$ *is nonsingular of expected dimension* $3g - 3 + m + nd$.

5 Gromov–Witten comparison

5.1 Dimensions

The moduli space of stable maps $\bar{M}_{g,m}(\mathbb{G}(r, n), d)$ also carries a perfect obstruction theory and a virtual class. In order to compare with the moduli space of stable quotients, we will always assume $2g - 2 + m \geq 0$ and $0 < r < n$.

Lemma 4 *The virtual dimensions of the moduli spaces* $\bar{M}_{g,m}(\mathbb{G}(r, n), d)$ *and* $\bar{Q}_{g,m}(\mathbb{G}(r, n), d)$ *are equal.*

Proof The virtual dimension of the moduli space of stable maps is

$$\int_\beta c_1(T) + (\dim_{\mathbb{C}} \mathbb{G}(r, n) - 3)(1 - g) + m = nd + (r(n - r) - 3)(1 - g) + m.$$

where β is the degree d curve class and T is the tangent bundle of $\mathbb{G}(r, n)$. Similarly, the virtual dimension of the moduli of stable quotients is

$$\chi(S, Q) + 3g - 3 + m = nd + r(n - r)(1 - g) + 3g - 3 + m,$$

by Riemann–Roch, which agrees. □

5.2 Stable maps to stable quotients

There exists a natural morphism

$$c \colon \bar{M}_{g,m}(\mathbb{G}(1, n), d) \to \bar{Q}_{g,m}(\mathbb{G}(1, n), d).$$

Given a stable map

$$f \colon (C, p_1, \ldots, p_m) \to \mathbb{G}(1, n)$$

of degree d, the image $c([f]) \in \bar{Q}_{g,m}(\mathbb{G}(1, n), d)$ is obtained by the following construction.

The first step is to consider the minimal contraction

$$\kappa \colon C \to \hat{C}$$

of rational components yielding a quasistable curve $(\hat{C}, p_1, \ldots, p_m)$ with the automorphism group of each component of dimension at most 1. The minimal contraction κ is unique – the exceptional curves of κ are the maximal connected trees $T \subset C$ of rational curves which

  (i)   contain no markings,
  (ii) meet $\overline{C \setminus T}$ in a single point.

Let $T_1, \ldots, T_t$ be the set of maximal trees satisfying (i) and (ii). Then,

$$\hat{C} = \overline{C \setminus \bigcup_i T_i}$$

is canonically a subcurve of C. Let $x_1, \ldots, x_t \in \hat{C}^{\mathrm{ns}}$ be the points of incidence with the trees $T_1, \ldots, T_t$ respectively.

Let d_i be the degree of the restriction of f to T_i. Let

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\hat{C}} \to Q \to 0$$

be the pullback by the restriction of f to $\hat{C}$ of the tautological sequence on $\mathbb{G}(1, n)$. The canonical inclusion

$$0 \to S\left(-\sum_{i=1}^t d_i x_i\right) \to S$$

yields a new quotient

$$0 \to S\left(-\sum_{i=1}^t d_i x_i\right) \to \mathbb{C}^n \otimes \mathcal{O}_{\hat{C}} \xrightarrow{\hat{q}} \hat{Q} \to 0.$$

Stability of the map f implies $(\hat{C}, p_1, \ldots, p_m, \hat{q})$ is a stable quotient. We define

$$c([f]) = (\hat{C}, p_1, \ldots, p_m, \hat{q}) \in \overline{Q}_{g,m}(\mathbb{G}(1, n), d).$$

The morphism c has been studied earlier for genus 0 curves in the linear sigma model constructions of Givental [11]. The morphism c is considered for the Quot scheme of a fixed nonsingular curve of arbitrary genus in Popa and Roth [38]. We have described the morphism c pointwise – we refer the reader to Lian, Liu and Yau [23], Manolache [26] and Popa and Roth [38] for a scheme theoretic construction.

5.3 Equivalence

The strongest possible comparison result holds for $\mathbb{G}(1, n)$.

Theorem 3 $c_*[\overline{M}_{g,m}(\mathbb{G}(1, n), d)]^{\mathrm{vir}} = [\overline{Q}_{g,m}(\mathbb{G}(1, n), d)]^{\mathrm{vir}}.$

If $r > 1$, a morphism c for $\mathbb{G}(r, n)$ does not in general exist. However, the following construction provides a substitute. Recall the Plücker embedding

$$\iota\colon \mathbb{G}(r, n) \to \mathbb{G}(1, N) \quad \text{for } N = \binom{n}{r}.$$

The Plücker embedding induces canonical maps

$$\iota_M\colon \overline{M}_{g,m}(\mathbb{G}(r, n), d) \to \overline{M}_{g,m}(\mathbb{G}(1, N), d),$$

$$\iota_Q\colon \overline{Q}_{g,m}(\mathbb{G}(r, n), d) \to \overline{Q}_{g,m}(\mathbb{G}(1, N), d).$$

The morphism ι_M is obtained by composing stable maps with ι. The morphism ι_Q is obtained by associating the subsheaf

$$0 \to \wedge^r S \to \wedge^r \mathbb{C}^n \otimes \mathcal{O}_C$$

to the subsheaf $0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C$.

Theorem 4 *For $0 < r < n$ and all classes $\gamma_i \in A^*_{\mathrm{GL}_n(\mathbb{C})}(\mathbb{G}(r, n), \mathbb{Q})$,*

$$c_* \iota_{M*}\left(\prod_{i=1}^m \mathrm{ev}_i^*(\gamma_i) \cap [\overline{M}_{g,m}(\mathbb{G}(r, n), d)]^{\mathrm{vir}} \right)$$

$$= \iota_{Q*}\left(\prod_{i=1}^m \mathrm{ev}_i^*(\gamma_i) \cap [\overline{Q}_{g,m}(\mathbb{G}(r, n), d)]^{\mathrm{vir}} \right).$$

Since descendent classes in both cases are easily seen to be pulled-back via $c \circ \iota_M$ and ι_Q respectively, there is no need to include them in the statement of Theorem 4. In particular, Theorem 4 implies the fully equivariant stable map and stable quotient brackets (and CoFT) are equal.

5.4 Example

To see Theorem 3 and Theorem 4 are not purely formal, we can study the case of genus 1 maps to $\mathbb{P}^{n-1}$ of degree 1 for $n \geq 2$. Let

$$I \subset \mathbb{P}^{n-1} \times \mathbb{G}(2, n)$$

be the incidence correspondence consisting of points and lines (p, L) with $p \in L$. First, the moduli space of stable maps is

$$\overline{M}_{1,0}(\mathbb{P}^{n-1}, 1) = \overline{M}_{1,1} \times I.$$

We will denote elements of the moduli space of stable maps by (E, p, L) where $(E, p) \in \overline{M}_{1,1}$ and $(p, L) \in I$. The triple (E, p, L) specifies a stable map whose domain is the union of two components E and L joined together at p with the genus

one component E contracted. Second, we have already seen

$$\bar{Q}_{1,0}(\mathbb{P}^{n-1},1) = \bar{M}_{1,1} \times \mathbb{P}^{n-1}.$$

The morphism

$$c\colon \bar{M}_{1,0}(\mathbb{P}^{n-1},1) \to \bar{Q}_{1,0}(\mathbb{P}^{n-1},1)$$

is given by

$$(E, p, L) \to (E, p).$$

The virtual class of the moduli space of stable maps is easily computed from deformation theory,

$$[\bar{M}_{1,0}(\mathbb{P}^{n-1},1)]^{\text{vir}} = c_{n-2}(\text{Obs}) \cap [\bar{M}_{1,0}(\mathbb{P}^{n-1},1)],$$

where the rank $n-2$ obstruction bundle is

$$\text{Obs}_{(E,p,L)} = \frac{\mathbb{E}^* \otimes T_p(\mathbb{P}^{n-1})}{\Psi_p^* \otimes T_p(L)} = \mathbb{E}^* \otimes N_p(\mathbb{P}^{n-1}/L).$$

Here, $\mathbb{E}$ is the Hodge bundle on $\bar{M}_{1,1}$, Ψ_p is the cotangent line, and $N_p(\mathbb{P}^{n-1}/L)$ is the normal space to $L \subset \mathbb{P}^{n-1}$ at p. We see

$$c_{n-2}(\text{Obs}) = c_{n-2}(N_p(\mathbb{P}^{n-1}/L)) - \lambda c_{n-3}(N_p(\mathbb{P}^{n-1}/L))$$
$$+ \lambda^2 c_{n-4}(N_p(\mathbb{P}^{n-1}/L)) + \cdots$$

where $\lambda = c_1(\mathbb{E})$. Since $I \to \mathbb{P}^{n-1}$ is a $\mathbb{P}^{n-2}$–bundle,

$$c_*[\bar{M}_{1,0}(\mathbb{P}^{n-1},1)]^{\text{vir}} = c_*(c_{n-2}(N_p(\mathbb{P}^{n-1}/L)) \cap [\bar{M}_{1,0}(\mathbb{P}^{n-1},1)])$$
$$= [\bar{Q}_{1,0}(\mathbb{P}^{n-1},1)]$$
$$= [\bar{Q}_{1,0}(\mathbb{P}^{n-1},1)]^{\text{vir}}.$$

For the second equality, we use the elementary projective geometry calculation

$$c_{n-2}(Q) = 1,$$

where Q is universal rank $n-2$ quotient on the projective space of lines in $\mathbb{C}^{n-1}$. The last equality follows since the moduli space of stable quotients is nonsingular of expected dimension.

6 Construction

6.1 Quotient presentation

Let g, m, and d satisfy

$$2g - 2 + m + \epsilon d > 0$$

for all $\epsilon > 0$. We will exhibit the moduli space $\bar{Q}_{g,m}(\mathbb{G}(r,n),d)$ as a quotient stack.

To begin, fix a stable quotient $(C, p_1, \ldots, p_m, q)$ where

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0.$$

By assumption, the line bundle

$$\mathcal{L}_\epsilon = \omega_C(p_1 + \cdots + p_m) \otimes (\Lambda^r S^*)^\epsilon$$

is ample for all $\epsilon > 0$. The genus 0 components of C must contain at least 2 nodes or markings with strict inequality for components of degree 0. As a consequence, ampleness of $\mathcal{L}_\epsilon$ for $\epsilon = 1/(d+1)$ is enough to ensure the stability of a degree d quotient. We will fix $\epsilon = 1/(d+1)$ throughout.

Lemma 5 *There exists a sufficiently large and divisible integer f such that the line bundle $\mathcal{L}^f$ is very ample with no higher cohomology*

$$H^1(C, \mathcal{L}^f) = 0.$$

Proof We will show that for all $k \geq 5$, the choice

$$f = k(d+1)$$

suffices. Then,

$$\mathcal{L}^f = \left(\omega_C \left(\sum_{i=1}^m p_i \right) \right)^{k(d+1)} \otimes (\Lambda^r S^*)^k.$$

To check very ampleness, we verify

$$(14) \qquad\qquad H^1(C, \mathcal{L}^f \otimes I_{q_1} I_{q_2}) = 0$$

for all pairs of (not necessarily distinct) points $q_1, q_2 \in C$. By duality, the vanishing (14) is equivalent to

$$\mathrm{Ext}^0(I_{q_1} I_{q_2}, \omega_C \otimes \mathcal{L}^{-f}) = 0.$$

If $q_1, q_2 \in C^{\mathrm{ns}}$, we can check instead

$$H^0(C, \omega_C(q_1 + q_2) \otimes \mathcal{L}^{-f}) = 0,$$

which is clear since the line bundle has negative degree on each component. The following three cases also need to be taken into account:

   (i) q_1 is node and $q_2 \in C^{\mathrm{ns}}$,

  (ii) q_1 and q_2 are distinct nodes,

 (iii) $q_1 = q_2$ are coincident nodes.

Cases (i)–(iii) can be easily handled. For instance, to prove (i), consider the normalization at q_1,

$$\pi \colon \tilde{C} \to C,$$

and let $\pi^{-1}(q_1) = \{q_1', q_1''\}$. We have

$$\text{Ext}^0(I_{q_1} I_{q_2}, \omega_C \otimes \mathcal{L}^{-f}) = H^0(\tilde{C}, \omega_{\tilde{C}} \otimes \pi^* \mathcal{L}^{-f}(q_1' + q_1'' + q_2))$$

which vanishes since the line bundle on the right has negative degree on each component. Indeed, for a component of genus h, degree e and ℓ special points (nodes or markings), the maximum possible degree is

$$3 + (-k(d+1) + 1)(2h - 2) - k(d+1)\ell - ek < 0.$$

The last inequality follows immediately when $h \geq 2$, while the remaining cases are checked by hand using that $k \geq 5$. For instance, it may be useful to split the analysis according to the following 4 situations:

- $h = 0$, $\ell > 2$,
- $h = 0$, $\ell = 2$, $e > 0$,
- $h = 1$, $\ell > 0$,
- $h = 1$, $\ell = 0$, $e > 0$.

Next, for case (ii), we use that

$$\text{Ext}^0(I_{q_1} I_{q_2}, \omega_C \otimes \mathcal{L}^{-f}) = H^0(\tilde{C}, \omega_{\tilde{C}} \otimes \pi^\star \mathcal{L}^{-f}(q_1' + q_1'' + q_2' + q_2'')) = 0,$$

while for (iii), we have

$$\text{Ext}^0(I_{q_1}^2, \omega_C \otimes \mathcal{L}^{-f}) = H^0(\tilde{C}, \omega_{\tilde{C}} \otimes \pi^\star \mathcal{L}^{-f}(2q_1' + 2q_1'')) = 0.$$

In both cases, the vanishing follows since the degree is negative on every component of $\tilde{C}$. $\qquad\square$

By the vanishing of the higher cohomology, the dimension

$$(15) \qquad h^0(C, \mathcal{L}^f) = 1 - g + k(d+1)(2g - 2 + m) + kd$$

is independent of the choice of stable quotient. Let V be a vector space of dimension (15). Given an identification

$$H^0(C, \mathcal{L}^f) \cong \mathsf{V}^*,$$

we obtain an embedding

$$i : C \hookrightarrow \mathbb{P}(\mathsf{V}),$$

well-defined up to the action of the group $\mathrm{PGL}(V)$. Let Hilb denote the Hilbert scheme of curves in $\mathbb{P}(V)$ of genus g and degree

$$k(d+1)(2g-2+m)+kd\,,$$

equal to the degree of $\mathcal{L}^f$. Each stable quotient gives rise to a point in

$$\mathcal{H} = \mathsf{Hilb} \times \mathbb{P}(V)^m,$$

where the last factors record the locations of the markings $p_1, \ldots, p_m$.

Elements of $\mathcal{H}$ are tuples $(C, p_1, \ldots, p_m)$. A quasiprojective subscheme $\mathcal{H}' \subset \mathcal{H}$ is defined by requiring

(i) the points $p_1, \ldots, p_m$ are contained in C,

(ii) the curve $(C, p_1, \ldots, p_m)$ is quasistable.

We denote the universal curve over $\mathcal{H}'$ by

$$\pi \colon \mathcal{C}' \to \mathcal{H}'.$$

Next, we construct the π–relative Quot scheme

$$\mathsf{Quot}(n-r, d) \to \mathcal{H}'$$

parametrizing rank $n-r$ degree d quotients

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0$$

on the fibers of π. A locally closed subscheme

$$\mathcal{Q}' \subset \mathsf{Quot}(n-r, d)$$

is further singled out by requiring

(iii) Q is locally free at the nodes and markings of C,

(iv) the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to C agrees with the line bundle

$$\left(\omega\left(\sum p_i\right)\right)^{k(d+1)} \otimes (\Lambda^r S^*)^k.$$

The action of $\mathrm{PGL}(V)$ extends to $\mathcal{H}'$ and $\mathcal{Q}'$. A $\mathrm{PGL}(V)$–orbit in $\mathcal{Q}'$ corresponds to a stable quotient up to isomorphism. By stability, each orbit has finite stabilizers. The moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n), d)$ is the stack quotient $[\mathcal{Q}'/\mathrm{PGL}(V)]$.

6.2 Separatedness

We prove the moduli stack $\bar{Q}_{g,m}(\mathbb{G}(r,n),d)$ is separated by the valuative criterion.

Let $(\Delta, 0)$ be a nonsingular pointed curve with complement

$$\Delta^0 = \Delta \setminus \{0\}.$$

We consider two flat families of quasistable pointed curves

$$\mathcal{X}_i \to \Delta, \quad p_1^i, \ldots, p_m^i \colon \Delta \to \mathcal{X}_i,$$

and two flat families of stable quotients

$$0 \to S_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}_i} \to Q_i \to 0,$$

for $1 \le i \le 2$. We assume the two families are isomorphic away from the central fiber. We will show the isomorphism extends over 0. In fact, by the separatedness of the Quot functor, we only need to show that the isomorphism extends to the families of curves $\mathcal{X}_i \to \Delta$ in a manner preserving the sections.

By the semistable reduction theorem, discussed for instance in Harris and Morrison [13], we derive that possibly after base change ramified over 0, there exists a third family

$$\mathcal{Y} \to \Delta, \quad p_1, \ldots, p_m \colon \Delta \to \mathcal{Y}$$

of quasistable pointed curves and dominant morphisms

$$\pi_i \colon \mathcal{Y} \to \mathcal{X}_i$$

compatible with the sections. We may assume that π_i restricts to an isomorphism away from the nodes of $(\mathcal{X}_i)_0$.

After pullback, we obtain exact sequences of quotients

(16) $$0 \to \pi_i^* S_i \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{Y}} \to \pi_i^* Q_i \to 0$$

on $\mathcal{Y}$ of the same degree and rank. Exactness holds after pullback since the quotient Q_i is locally free at the nodes of $(\mathcal{X}_i)_0$.

The two pullback sequences (16) must agree on the central fiber by the separatedness of the Quot functor. We claim the central fiber $\mathcal{Y}_0$ cannot contain components which are contracted over the nodes of $(\mathcal{X}_1)_0$ but uncontracted over the nodes of $(\mathcal{X}_2)_0$. Indeed, if such a component E existed, the quotient $\pi_1^* Q_1$ would be trivial on E, whereas by stability, since E is rational, the quotient $\pi_2^* Q_2$ could not be trivial. We conclude the families $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic. $\qquad\square$

6.3 Properness

We prove the moduli stack $\bar{Q}_{g,m}(\mathbb{G}(r,n),d)$ is proper by the valuative criterion. Let

$$\pi^0 \colon \mathcal{X}^0 \to \Delta^0, \quad p_1, \ldots, p_m \colon \Delta^0 \to \mathcal{X}^0$$

carry a flat family of stable quotients

(17) $$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}^0} \to Q \to 0$$

which we must extend over Δ, possibly after base-change. By standard reductions, after base change and normalization, we may assume the fibers of π^0 are nonsingular and irreducible curves, possibly after adding the preimages of the nodes to the marking set. The original family is reconstructed by gluing stable quotients on different components via the evaluation maps at the nodes [7].

Once the general fiber of π^0 is assumed to be nonsingular, we construct an extension

$$\pi \colon \mathcal{X} \to \Delta, \quad p_1, \ldots, p_m \colon \Delta \to \mathcal{X}$$

with central fiber an m–pointed stable curve [13].[8] After resolving the possible singularities of the total space at the nodes of $\mathcal{X}_0$ by blowups, we may take $\mathcal{X}$ to be a nonsingular surface. Using the properness of the relative Quot functor, we complete the family of quotients across the central fiber:

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}} \to Q \to 0.$$

The extension may fail to be a quasistable quotient since Q may not be locally free at the nodes or the markings of the central fiber. This will be corrected by further blowups.

We will first treat the case when S has rank 1. As explained by Okonek, Schneider and Spindler [30, Lemma 1.1.10], the sheaf S^* is reflexive over the nonsingular surface $\mathcal{X}$, hence locally free. Consider the image T of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to S^*$$

which can be written as

$$T = S^* \otimes I_Z$$

---

[7] The gluing maps were noted in Section 3.1.

[8] There are exactly two cases where the central fiber can not be taken to stable,

$$(g, m) = (0, 2) \quad \text{or} \quad (1, 0).$$

In both cases, the central fiber can be taken to be irreducible and nodal. The argument afterwards is the same. We leave the details to the reader.

for a subscheme $Z \subset \mathcal{X}$. The quotient Q will have torsion supported on Z. By the flatness of Q, the subscheme Z is not supported on any components of the central fiber.

We consider a point $\xi \in \mathcal{X}$ which is a node or marking of the central fiber. After restriction to an open set containing ξ, we may assume all components of Z pass through ξ. After a sequence of blowups

$$\mu \colon \tilde{\mathcal{X}} \to \mathcal{X},$$

we may take

$$\tilde{Z} = \mu^{-1}(Z) = \sum_i m_i E_i + \sum_j n_j D_j,$$

where the $E_i \subset \mathcal{X}$ are the exceptional curves of μ and the D_j intersect the E_i away from the nodes and markings. Since we are only interested in constraining the behavior of $\tilde{Z}$ at the nodes or markings over ξ, the morphism μ can be achieved by repeatedly blowing up only nodes or markings of the fiber over ξ.

On the blowup, the image of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to \mu^* S^*$$

factors though $\mu^* S^*(-\tilde{Z})$. Setting

$$\tilde{S} = \mu^* S \left(\sum_i m_i E_i \right) \hookrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\tilde{\mathcal{X}}},$$

we obtain a flat family

$$(18) \qquad\qquad 0 \to \tilde{S} \to \mathbb{C}^n \otimes \mathcal{O}_{\tilde{\mathcal{X}}} \to \tilde{Q} \to 0$$

on $\tilde{\mathcal{X}}$ where the quotient $\tilde{Q}$ is locally free at the nodes or the markings of the (reduced) central fiber.

Unfortunately, the above blowup process yields a family

$$\tilde{\mathcal{X}} \to \Delta$$

with possible nonreduced components occurring in chains over nodes and markings of $\mathcal{X}_0$. The multiple components can be removed by base change and normalization,

$$\mathcal{X}' \to \tilde{\mathcal{X}},$$

with the nodes and markings of $\mathcal{X}'_0$ mapping to the nodes and markings of $\tilde{\mathcal{X}}_0^{\mathrm{red}}$.

The pullback of (18) to $\mathcal{X}'$ yields a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}'} \to Q' \to 0.$$

The quotient Q' is certainly locally free (and hence flat) over the nodes and markings of $\mathcal{X}_0'$. The quotient Q' may fail to be flat over finitely many nonsingular points of $\mathcal{X}_0'$. A flat limit

(19) $$0 \to S'' \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}'} \to Q'' \to 0$$

can then be found after altering Q' only at the latter points. Since Q'' is locally free over the nodes and markings of $\mathcal{X}_0'$, we have constructed a quasistable quotient. However (19) may fail to be stable because of possible unstable genus 0 components in the central fiber.

By the economical choice of blowups (occurring only at nodes and markings over ξ), all unstable genus 0 curves P carry exactly 2 special points and

$$S''|_P \cong \mathcal{O}_P.$$

All such unstable components are contracted by the line bundle

$$\mathcal{L} = \omega_C(p_1 + \cdots + p_m)^{d+1} \otimes \Lambda^r(S'')^*.$$

Indeed, $\mathcal{L}^k$ is π'–relatively[9] basepoint free for $k \geq 2$ and trivial over the unstable genus 0 curves. As a consequence, $\mathcal{L}^k$ determines a morphism

$$q \colon \mathcal{X}' \to \mathcal{Y} = \mathrm{Proj}\left(\bigoplus_m \mathcal{L}^{km} \right).$$

The pushforward

$$0 \to q_* S'' \to \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{Y}} \to q_* Q'' \to 0$$

is stable. We have constructed the limit of the original family (17) of stable quotients over Δ^0.

The case when the subsheaf S has arbitrary rank is similar. The cokernel K of the map

$$(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* \to S^*$$

has support of dimension at most 1. The initial Fitting ideal of K, denoted $\mathcal{F}_0(K)$, endows the support of K with a natural scheme structure. After a suitable composition of blowups

$$\mu \colon \widetilde{\mathcal{X}} \to \mathcal{X},$$

we may take

$$\mathcal{F}_0(p^* K) = p^* \mathcal{F}_0(K)$$

---

[9]Here, $\pi' \colon \mathcal{X}' \to \Delta$.

to be divisorial with only exceptional components passing through the nodes and markings of the central fiber. Let V be the exceptional part of $\mathcal{F}_0(p^*K)$. We set

$$K' = \mu^* K \otimes \mathcal{O}_V,$$

and define the sheaves $\widetilde{K}$ and $\widetilde{S}$ by the diagram

$$
\begin{array}{ccccc}
(\mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}})^* & \longrightarrow & \widetilde{S}^* & \longrightarrow & \widetilde{K} \\
\| & & \downarrow & & \downarrow \\
(\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{X}})^* & \longrightarrow & \mu^* S^* & \longrightarrow & \mu^* K \\
& & \downarrow & & \downarrow \\
& & K' & =\!=\!= & K'.
\end{array}
$$

The Fitting ideal $\mathcal{F}_0(\widetilde{K})$ does not vanish on exceptional divisors of μ. Therefore, the quotient

$$0 \to \widetilde{S} \to \mathbb{C}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}} \to \widetilde{Q} \to 0$$

is locally free at the nodes or the markings of the (reduced) central fiber. The remaining steps exactly follow the rank 1 case. $\qquad\square$

7 Proofs of Theorem 3 and Theorem 4

7.1 Localization

Since Theorem 3 is a special case of Theorem 4, we will restrict our attention to the latter.

The idea is to proceed by localization with respect to the maximal torus $\mathbf{T} \subset \mathrm{GL}_n(\mathbb{C})$ acting with diagonal weights $\mathsf{w}_1, \ldots, \mathsf{w}_n$. By the usual splitting principle, the torus calculation is enough for the full equivariant result. Localization formulas for the virtual classes of the moduli of stable maps and stable quotients[10] are both covered by the results of Graber and Pandharipande [12]. We will compare fixed point residues pushed forward via the Plücker embedding to

$$\overline{Q}_{g,m}(\mathbb{G}(1, N), d),$$

for $N = \binom{n}{r}$.

---

[10] An analogous localization computation for the virtual class of the Quot scheme of a fixed nonsingular curve was carried out in [27]. In particular, the fixed loci and their contributions were explicitly determined. The localization for the stable quotient space is conceptually similar.

To state the localization theorem, we write

$$j\colon \overline{M}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}} \hookrightarrow \overline{M}_{g,m}(\mathbb{G}(r,n),d)$$

for the inclusion of the fixed point locus. The restriction of the deformation-obstruction theory of $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ to the fixed point locus determines

(i) a virtual class of $\overline{M}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}$ obtained from the $\mathbf{T}$–invariant part,

(ii) a virtual normal bundle N_M^{vir} obtained from the moving part.

Then,

$$[\overline{M}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} = j_\star\big(\mathrm{e}(-N_M^{\mathrm{vir}}) \cap [\overline{M}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}]^{\mathrm{vir}}\big).$$

We wrote e for the $\mathbf{T}$–equivariant Euler class in K–theory. In parallel, on the stable quotient side, we have

$$[\overline{Q}_{g,m}(\mathbb{G}(r,n),d)]^{\mathrm{vir}} = j_\star\big(\mathrm{e}(-N_Q^{\mathrm{vir}}) \cap [\overline{Q}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}]^{\mathrm{vir}}\big).$$

We will use the Plücker diagram

$$
\begin{array}{ccc}
\overline{M}_{g,m}(\mathbb{G}(r,n),d) & \xrightarrow{\iota_M} & \overline{M}_{g,m}(\mathbb{G}(1,N),d) \\
& & \downarrow{c} \\
\overline{Q}_{g,m}(\mathbb{G}(r,n),d) & \xrightarrow{\iota_Q} & \overline{Q}_{g,m}(\mathbb{G}(1,N),d).
\end{array}
$$

By equivariance, we obtain the diagram of fixed loci

$$
\begin{array}{ccc}
\overline{M}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}} & \xrightarrow{\iota_M^{\mathbf{T}}} & \overline{M}_{g,m}(\mathbb{G}(1,N),d)^{\mathbf{T}} \\
& & \downarrow{c^{\mathbf{T}}} \\
\overline{Q}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}} & \xrightarrow{\iota_Q^{\mathbf{T}}} & \overline{Q}_{g,m}(\mathbb{G}(1,N),d)^{\mathbf{T}}.
\end{array}
$$

The fixed loci will be described in the next subsection, and their virtual fundamental classes will be shown to agree with the usual fundamental classes. Finally, we will prove the required equality

$$c_\star^{\mathbf{T}} \iota_{M\star}^{\mathbf{T}} (\mathrm{e}(-N_M^{\mathrm{vir}}) \cap [\overline{M}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}]^{\mathrm{vir}})$$

$$= \iota_{Q\star}^{\mathbf{T}}\big(\mathrm{e}(-N_Q^{\mathrm{vir}}) \cap [\overline{Q}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}]^{\mathrm{vir}}\big).$$

In fact, the identity will be verified over each connected component of the fixed point locus $\overline{Q}_{g,m}(\mathbb{G}(1,N),d)^{\mathbf{T}}$.

7.2 T–fixed loci for stable maps

The **T**–fixed loci of the moduli space $\overline{M}_{g,m}(\mathbb{G}(r,n),d)$ are described in detail in [12]. We briefly recall here that the fixed loci are indexed by decorated graphs

$$(\Gamma, \nu, \gamma, \epsilon, \delta, \mu),$$

where

(i) $\Gamma = (V, E)$ such that V is the vertex set and E is the edge set (with no self-edges),

(ii) $\nu\colon V \to \mathbb{G}(r,n)^{\mathbf{T}}$ is an assignment of a **T**–fixed point $\nu(v)$ to each element $v \in V$,

(iii) $\gamma\colon V \to \mathbb{Z}_{\geq 0}$ is a genus assignment,

(iv) ϵ is an assignment to each $e \in E$ of a **T**–invariant curve $\epsilon(e) \cong \mathbb{P}^1$ of $\mathbb{G}(r,n)$ together with a covering number $\delta(e) \geq 1$,

(v) μ is a distribution of the m markings to the vertices V.

The graph Γ is required to be connected. The two vertices incident to the edge $e \in E$ must correspond via ν to the two **T**–fixed points incident to $\epsilon(e)$. The sum of γ over V together with $h^1(\Gamma)$ must equal g. The sum of δ over E must equal d.

The **T**–fixed locus corresponding to a given graph is, up to automorphisms, the product

$$\prod_v \overline{M}_{\gamma(v),\mathrm{val}(v)},$$

where $\mathrm{val}(v)$ counts all incident edges and markings. The stable maps in the **T**–fixed locus are easily described. If the condition

$$2\gamma(v) - 2 + \mathrm{val}(v) > 0$$

holds, then the vertex v is said to be *nondegenerate*. Such a vertex corresponds to a collapsed curve varying in $\overline{M}_{\gamma(v),\mathrm{val}(v)}$. Moreover, each edge e gives a degree $\delta(e)$ covering of the invariant curve $\epsilon(e) \cong \mathbb{P}^1$, ramified only over the two torus fixed points. The stable map is obtained by gluing along the graph incidences. The factor $\overline{M}_{\gamma(v),\mathrm{val}(v)}$ corresponding to a *degenerate* vertex in the product above is interpreted as a point.

7.3 T–fixed loci for stable quotients

7.3.1 The indexing set The $\mathbf{T}$–fixed loci of $\bar{Q}_{g,m}(\mathbb{G}(r,n),d)$ are similarly indexed by decorated graphs

$$(\Gamma, \nu, \gamma, s, \epsilon, \delta, \mu)$$

where

(i) $\Gamma = (V, E)$ such that V is the vertex set and E is the edge set (no self-edges are allowed),

(ii) $\nu: V \to \mathbb{G}(r,n)^{\mathbf{T}}$ is an assignment of a $\mathbf{T}$–fixed point $\nu(v)$ to each element $v \in V$,

(iii) $\gamma: V \to \mathbb{Z}_{\geq 0}$ is a genus assignment,

(iv) $s(v) = (s_1(v), \ldots, s_r(v))$ is an assignment of a tuple of nonnegative integers with $\mathbf{s}(v) = \sum_{i=1}^{r} s_i(v)$ together with an inclusion

$$\iota_s: \{1, \ldots, r\} \to \{1, \ldots, n\},$$

(v) ϵ is an assignment to each $e \in E$ of a $\mathbf{T}$–invariant curve $\epsilon(e)$ of $\mathbb{G}(r,n)$ together with a covering number $\delta(e) \geq 1$,

(vi) μ is a distribution of the markings to the vertices V.

The graph Γ is required to be connected. The two vertices incident to the edge $e \in E$ must correspond via ν to the two $\mathbf{T}$–fixed points incident to $\epsilon(e)$. The sum of γ over V together with $h^1(\Gamma)$ must equal g. The assignment s determines the splitting type of the subsheaf over the vertex v. The inclusion ι_s determines r trivial factors of $\mathbb{C}^n \otimes \mathcal{O}_C$ in which the subsheaf S injects. The inclusion ι_s must be compatible with $\nu(v)$. The sum of $\mathbf{s}(v)$ over V together with the sum of δ over E must equal d.

A vertex v occurring in stable quotient graphs is *degenerate* if

$$\gamma(v) = 0, \quad \mathrm{val}(v) = 2, \quad \mathbf{s}(v) = 0.$$

For *nondegenerate* vertices, the stability condition

$$2\gamma(v) - 2 + \mathrm{val}(v) + \epsilon \cdot \mathbf{s}(v) > 0$$

holds for every strictly positive $\epsilon \in \mathbb{Q}$. The valence of v, as before, counts all incident edges and markings.

7.3.2 Mixed pointed spaces The **T**–fixed loci for the stable quotients are described in terms of mixed pointed spaces. Let $s = (s_1, \ldots, s_r)$ be a tuple of nonnegative integers. Let $\overline{M}_{g,A|s}$ be the moduli space of genus g curves with markings

$$\{p_1, \ldots, p_A\} \cup \bigcup_{j=1}^{r} \{\hat{p}_{j1}, \ldots, \hat{p}_{js_j}\} \in C^{\mathrm{ns}} \subset C$$

satisfying the conditions

  (i) the points p_i are distinct,

  (ii) the points $\hat{p}_{jk}$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C \left(\sum_{i=1}^{A} p_i + \epsilon \sum_{j,k} \hat{p}_{jk} \right)$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\hat{p}_{jk}$ and $\hat{p}_{j'k'}$ to coincide. If

$$\mathbf{s} = \sum_{j=1}^{r} s_j,$$

then $\overline{M}_{g,A|s} = \overline{M}_{g,A|\mathbf{s}}$ defined in [Section 4.1](#).

7.3.3 Torus fixed quotients Fix a decorated graph $(\Gamma, \nu, \gamma, s, \epsilon, \delta, \mu)$ indexing a **T**–fixed locus of the moduli space $\overline{Q}_{g,m}(\mathbb{G}(r,n), d)$. The corresponding **T**–fixed locus is, up to a finite map, the product of mixed pointed spaces

$$\prod_{v \in V} \overline{M}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)}.$$

As usual, the factors corresponding to degenerate vertices are treated as points.

The corresponding **T**–fixed stable quotients can be described explicitly. For each vertex v of the graph, pick a curve C_v in the mixed moduli space with markings

$$\{p_1, \ldots, p_{\mathrm{val}(v)}\} \cup \bigcup_{j=1}^{r} \{\hat{p}_{j1}, \ldots, \hat{p}_{js_j(v)}\}.$$

For each edge e, pick a rational curve C_e. A pointed curve C is obtained by gluing the curves C_v and C_e via the graph incidences, and distributing the markings on the domain via the assignment μ.

(i) On the component C_v, the stable quotient is given by the exact sequence

$$0 \to \bigoplus_{j=1}^{r} \mathcal{O}_{C_v}\left(-\sum_{k=1}^{s_j(v)} \hat{p}_{jk}\right) \to \mathbb{C}^n \otimes \mathcal{O}_{C_v} \to Q \to 0.$$

The first inclusion is the composition of

$$\bigoplus_{j=1}^{r} \mathcal{O}_{C_v}\left(-\sum_{k=1}^{s_j(v)} \hat{p}_{jk}\right) \to \mathbb{C}^r \otimes \mathcal{O}_{C_v}$$

with the r–plane $\mathbb{C}^r \otimes \mathcal{O}_{C_v} \to \mathbb{C}^n \otimes \mathcal{O}_{C_v}$ determined by i_s.

(ii) For each edge e, consider the degree δ_e covering of the **T**–invariant curve $\epsilon(e) \cong \mathbb{P}^1$ in the Grassmannian $\mathbb{G}(r, n)$:

$$f_e \colon C_e \to \epsilon(e)$$

ramified only over the two torus fixed points. The stable quotient is obtained pulling back the tautological sequence of $\mathbb{G}(r, n)$ to C_e.

The gluing of stable quotients on different components via the maps described in Section 3.1 is made possible by the compatibility of i_s, ν and ϵ.

7.4 Contributions

Consider a graph indexing a fixed locus in the stable map space. When lifted to

$$\prod_v \overline{M}_{\gamma(v),\mathrm{val}(v)},$$

the contribution of the normal bundle N_M^{vir} to the localization formula takes the product form

$$\prod_v \mathsf{MapCont}(v) \prod_e \mathsf{MapCont}(e) \prod_{(v,e)} \mathsf{MapCont}(v, e)$$

for vertices v, edges e and flags (v, e). Here,

$$\mathsf{MapCont}(v) \in A^{\star}_{\mathbf{T},\mathrm{loc}}(\overline{M}_{\gamma(v),\mathrm{val}(v)})$$

are cohomology classes involving also the *localized* equivariant parameter, while $\mathsf{MapCont}(e)$ and $\mathsf{MapCont}(v, e)$ (and the contributions of degenerate vertices) are pulled back from $A^{\star}_{\mathbf{T},\mathrm{loc}}(\mathrm{Spec}(\mathbb{C}))$. This factorization is obtained in [12].

Similarly, the stable quotient contributions will be written in the product form

$$\prod_v \mathsf{QuotCont}(v) \prod_e \mathsf{QuotCont}(e) \prod_{(v,e)} \mathsf{QuotCont}(v,e),$$

where $\qquad\qquad \mathsf{QuotCont}(v) \in A^\star_{\mathbf{T},\mathrm{loc}}(\overline{M}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)}),$

and the edge and flag contributions are in $A^\star_{\mathbf{T},\mathrm{loc}}(\mathrm{Spec}(\mathbb{C}))$. We now describe the exact expressions.

7.4.1 Vertices for stable maps Consider the case of a nondegenerate vertex v occurring in a graph for stable maps. The vertex corresponds to the moduli space[11] $\overline{M}_{\gamma(v),\mathrm{val}(v)}$. The vertex contribution is computed in [12][12]

$$(20) \qquad \mathsf{MapCont}(v) = \frac{e(\mathbb{E}^* \otimes T_{\nu(v)})}{e(T_{\nu(v)})} \frac{1}{\prod_e w(e)/\delta(e) - \psi_e}.$$

Here, e denotes the Euler class, $T_{\nu(v)}$ is the **T**–representation on the tangent space of $\mathbb{G}(r,n)$ at $\nu(v)$, and

$$\mathbb{E} \to \overline{M}_{\gamma(v),\mathrm{val}(v)},$$

is the Hodge bundle. Finally, the product in the denominator is over all edges incident to v. The factor $w(e)$ denotes the **T**–weight of the tangent representation $T_{\nu(v)}\epsilon(v)$ along the corresponding **T**–fixed edge, and ψ_e denotes the cotangent line at the corresponding marking of $\overline{M}_{\gamma(v),\mathrm{val}(v)}$.

7.4.2 Vertices for stable quotients Next, let v be a nondegenerate vertex occurring in a graph for stable quotients. For simplicity, assume

$$\iota_s(j) = j, \quad 1 \le j \le r.$$

The vertex corresponds to the moduli space[13] $\overline{M}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)}$ where the subsheaf is given by

$$0 \to S = \bigoplus_{j=1}^r \mathcal{O}_C\left(-\sum_{k=1}^{s_j(v)} \hat{p}_{jk}\right) \xrightarrow{\iota_s} \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0.$$

The vertex contribution, determined by the *moving* part of $\mathrm{RHom}(S,Q)$ and the moving part of the deformation space of the underlying curve, is

$$(21) \qquad \mathsf{QuotCont}(v) = \frac{e(\mathrm{Ext}^1(S,Q)^{\mathrm{m}})}{e(\mathrm{Ext}^0(S,Q)^{\mathrm{m}})} \frac{1}{\prod_e w(e)/\delta(e) - \psi_e}.$$

---

[11] As usual we order all issues and quotient by the overcounting.

[12] We deviate from the conventions of [12] slightly. Some of the flag contributions in [12] are absorbed here by the vertices.

[13] Again, we order all issues and quotient by the overcounting.

Since the Ext spaces are not separately of constant rank, a better form is needed for (21).

Let $\hat{P}_i \subset C$ be the divisor associated to points corresponding to s_i

$$\hat{P}_i = \sum_{j=1}^{s_i(v)} \hat{p}_{ij}.$$

In the quotient sequence above we have

$$S = \bigoplus_{i=1}^{r} \mathcal{O}_C(-\hat{P}_i), \quad Q = (\mathbb{C}^{n-r} \otimes \mathcal{O}_C) \oplus \left(\bigoplus_{i=1}^{r} \mathcal{O}_{\hat{P}_i} \right).$$

We calculate

$$\mathrm{Hom}(S, Q) = S^\vee \otimes \mathbb{C}^{n-r} \oplus \left(\bigoplus_{1 \leq i,j \leq r} \mathcal{O}_C(\hat{P}_j) \otimes \mathcal{O}_{\hat{P}_i} \right).$$

Using the exact sequence

$$0 \to (\mathbb{C}^r)^\vee \otimes \mathcal{O}_C \to S^\vee \to \bigoplus_{i=1}^{r} \mathcal{O}_C(\hat{P}_i)|_{\hat{P}_i} \to 0,$$

we can compute the moving part of $\mathrm{RHom}(S, Q)$. We see that (21) is equivalent to

$$\mathrm{QuotCont}(v) = \frac{e(\mathbb{E}^* \otimes T_{\nu(v)})}{e(T_{\nu(v)})} \frac{1}{\prod_e w(e)/\delta(e) - \psi_e}$$

$$\cdot \frac{1}{\prod_{i \neq j} e(H^0(\mathcal{O}_C(\hat{P}_i)|_{\hat{P}_j}) \otimes [w_j - w_i])}$$

$$\cdot \frac{1}{\prod_{i, j^*} e(H^0(\mathcal{O}_C(\hat{P}_i)|_{\hat{P}_i}) \otimes [w_{j^*} - w_i]))},$$

where the products in the last factors satisfy the following conditions

$$1 \leq i \leq r, \quad 1 \leq j \leq r, \quad r + 1 \leq j^* \leq n.$$

The brackets $[\cdot]$ in the above expression denote the trivial line bundle with the specified weights. Using the remarks of Section 4.6, the second and third line in the above expression give a universal polynomial

$$\mathsf{Q}_v(\hat{\psi}_i, D_J) \in A^\star_{\mathbf{T},\mathrm{loc}}(\overline{M}_{\gamma(v),\mathrm{val}(v)|\mathbf{s}(v)})[\hat{\psi}_i, D_J],$$

independently of $\gamma(v)$ and $\mathrm{val}(v)$.

In a similar fashion, the vertex contribution to the fixed part of $\mathrm{RHom}(S, Q)$ equals

$$\bigoplus_{i=1}^{r} \mathrm{RHom}(\mathcal{O}_C(-\hat{P}_i), \mathcal{O}_{\hat{P}_i}).$$

Hence, the virtual fundamental class of the fixed locus agrees with the usual fundamental class.

While the vertex contributions for stable maps and stable quotients appear quite different, the genus dependent part of the integrand involving the Hodge bundle is the *same*. The differences all involve the local geometry of the points.

7.5 Matching

7.5.1 Genus 0 Theorem 4 is immediate in genus 0 by a geometric argument. Since

$$\overline{M}_{0,m}(\mathbb{G}(r,n), d) \quad \text{and} \quad \overline{Q}_{0,m}(\mathbb{G}(r,n), d)$$

are nonsingular of expected dimension, the virtual class in both cases is the usual fundamental class. Moreover, since the moduli spaces are irreducible[14] and birational, Theorem 4 in the form

$$(22) \qquad c_* \iota_{M*}\big([\overline{M}_{0,m}(\mathbb{G}(r,n), d)]^{\mathrm{vir}}\big) = \iota_{Q*}\big([\overline{Q}_{0,m}(\mathbb{G}(r,n), d)]^{\mathrm{vir}}\big).$$

follows since image of $c \circ \iota_M$ simply coincides with the image of ι_Q.

7.5.2 Arbitrary genus   In arbitrary genus, the argument relies on the localization theorem. The crucial step is to notice Theorem 3 and Theorem 4 are a consequence of a universal calculation in a moduli space of pointed curves. In fact, the universal calculation is genus independent since the genus dependent integrand factors agree.

Matching the stable map and stable quotient localization formulas requires a discussion of the morphisms $\iota_M^{\mathbf{T}}$, $\iota_Q^{\mathbf{T}}$ and $c^{\mathbf{T}}$. The Plücker morphism $\iota_M^{\mathbf{T}}$ takes a fixed locus in $\overline{M}_{g,m}(\mathbb{G}(r,n), d)$ and maps it isomorphically to a fixed locus in $\overline{M}_{g,m}(\mathbb{G}(1,N), d)$. The morphism

$$c^{\mathbf{T}}\colon \overline{M}_{g,m}(\mathbb{G}(1,N), d)^{\mathbf{T}} \to \overline{Q}_{g,m}(\mathbb{G}(1,N), d)^{\mathbf{T}}$$

is more interesting to analyze. For stable maps, the composition $c^{\mathbf{T}}\iota_M^{\mathbf{T}}$ collapses the unmarked genus 0 tails into torsion quotients, the torsion having multiplicity equal to the degree of the collapsed components. Different fixed loci may be mapped nonisomorphically to a single component of $\overline{Q}_{g,m}(\mathbb{G}(1,N), d)^{\mathbf{T}}$. Similarly, the stable

---

[14]See [18; 41].

quotient side $\bar{Q}_{g,m}(\mathbb{G}(r,n),d)^{\mathbf{T}}$ has many splitting types of the subbundle S which are collapsed via the Plücker morphism into the same fixed locus in $\bar{Q}_{g,m}(\mathbb{G}(1,N),d)^{\mathbf{T}}$.

Differences in the localization formulas occur in the nondegenerate vertices. For noncollapsed edges (not occurring in genus 0 tails of the stable map space) and noncollapsed degenerate vertices of valence 2, the edge and vertex contributions exactly coincide, since the moduli spaces are identified as stacks with the same perfect-obstruction theories. Edges corresponding to unmarked rational tails on the Gromov–Witten side are collapsed to nondegenerate vertices – their contributions will be absorbed by these vertices.

Summarizing, the resulting localization formulas for both stable maps and stable quotients take the product forms

$$\prod_v \mathsf{MapCont}(v) \prod_e \mathsf{MapCont}(e) \prod_{(v,e)} \mathsf{MapCont}(v,e)$$

$$\prod_v \mathsf{QuotCont}(v) \prod_e \mathsf{QuotCont}(e) \prod_{(v,e)} \mathsf{QuotCont}(v,e),$$

where the last two factors agree, while

$$\mathsf{MapCont}(v) = \frac{\mathsf{e}(\mathbb{E}^\vee \otimes T_{\nu(v)})}{\mathsf{e}(T_{\nu(v)})} \prod_e \frac{1}{w(e)/\delta(e) - \psi_e} \mathsf{M}_v(\hat{\psi}_j, D_J)$$

$$\mathsf{QuotCont}(v) = \frac{\mathsf{e}(\mathbb{E}^\vee \otimes T_{\nu(v)})}{\mathsf{e}(T_{\nu(v)})} \prod_e \frac{1}{w(e)/\delta(e) - \psi_e} \mathsf{Q}_v(\hat{\psi}_j, D_J).$$

Furthermore,

$$\mathsf{M}_v(\hat{\psi}_j, D_J),\ \mathsf{Q}_v(\hat{\psi}_j, D_J) \in A^\star_{\mathbf{T},\mathrm{loc}}(\bar{M}_{\gamma(v),\mathrm{val}(v)|s(v)})$$

are universal polynomials which do not depend on $\gamma(v)$ and $\mathrm{val}(v)$, but only on $s(v)$ and $\nu(v)$. Lemma 3 is used to obtain the polynomial Q_v. To apply the Lemma, note that $c^{\mathbf{T}}$ essentially coincides with the Hassett contractions (7), composed with diagonal maps which increase the multiplicities of coinciding points according to the degree of the collapsed rational tails. For example, the simplest diagonal morphism

$$\bar{M}_{g,m|1} \to \bar{M}_{g,m|d}$$

is obtained by assigning multiplicity d to the last marking. Such a morphism corresponds to the contraction of a degree d rational tail of a stable map into a degree d torsion quotient with support at the attaching node of the tail.

We will prove the equality

$$M_v(\widehat{\psi}_j, D_J) = Q_v(\widehat{\psi}_j, D_J).$$

We may take the polynomials M_v and Q_v to be in canonical form as defined in Section 4.4. Equality (22) established in genus 0 implies a matching after **T**–equivariant localization. In particular, there is a matching obtained for **T**–equivariant residues on the locus $\overline{M}_{0,\mathrm{val}(v)|s(v)}$. Hence, in genus 0, we have

$$M_v(\widehat{\psi}_j, D_J) = Q_v(\widehat{\psi}_j, D_J)$$

in $A^\star_{\mathbf{T},\mathrm{loc}}(\overline{M}_{0,\mathrm{val}(v)|s(v)})$. By the independence result of Section 7.6, we conclude the much stronger equality

$$M_v = Q_v$$

as abstract *polynomials*. This implies Theorems 3 and 4 for arbitrary genus. □

7.6 Independence

7.6.1 Polynomials Consider variables $\widehat{\psi}_1, \ldots, \widehat{\psi}_d$ and

$$\{ D_J \mid J \subset \{1, \ldots, d\}, \ |J| \geq 2 \}$$

for fixed $d \geq 0$. Given a polynomial $P(\widehat{\psi}_j, D_J)$, we obtain a canonical form P^C in the sense of Section 4.4.

We view P^C in two different ways. First, P^C yields a class

$$(23) \qquad P^C = P(\widehat{\psi}_j, D_J) \in A^*(\overline{M}_{0,m|d}, \mathbb{Q})$$

for every m. We will always take $m \geq 3$ to avoid unstable cases. Second, P^C is an abstract polynomial. If P^C always vanishes in the first sense (23), then we will show that P^C vanishes as an abstract polynomial.

If $P(\widehat{\psi}_j, D_J)$ is symmetric with respect to the $\mathbb{S}_d$–action on the variables, then P^C is also symmetric. The class (23) lies in the $\mathbb{S}_d$–invariant sector,

$$(24) \qquad P^C \in A^*(\overline{M}_{0,m|d}, \mathbb{Q})^{\mathbb{S}_d} = A^*(\overline{M}_{0,m|d}/\mathbb{S}_d, \mathbb{Q}).$$

Hence, for symmetric P, only the vanishing in $A^*(\overline{M}_{0,m|d}/\mathbb{S}_d, \mathbb{Q})$ will be required to show P^C vanishes as an abstract polynomial.

7.6.2 Partitions Fix a codimension k. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell)$$

be a set partition of $\{1, \ldots, d\}$ with $\ell \geq d - k$ nonempty parts,

$$\bigcup_{i=1}^{\ell} \mathcal{P}_i = \{1, \ldots, d\}.$$

The parts $\mathcal{P}_i$ are ordered by lexicographical ordering.[15] Let

$$\tau = (t_1, \ldots, t_\ell), \quad \sum_{i=1}^{\ell} t_i = k - d + \ell$$

be an ordered partition of $k - d + \ell$. The parts t_i *are* allowed to be 0.

Let $\mathsf{P}[d, k]$ be the set of all such pairs $[\mathcal{P}, \tau]$. Let $\mathbf{V}[d, k]$ be a $\mathbb{Q}$–vector space with basis given by the elements of $\mathsf{P}[d, k]$. To each pair $[\mathcal{P}, \tau] \in \mathsf{P}[d, k]$ and integer $m \geq 3$, we associate the class

$$X_m[\mathcal{P}, \tau] = \widehat{\psi}_{\mathcal{P}_1}^{t_1} \ldots \widehat{\psi}_{\mathcal{P}_\ell}^{t_\ell} \cdot D_{\mathcal{P}_1} \cdots D_{\mathcal{P}_\ell} \in A^k(\overline{M}_{0,m|d}) \ .$$

7.6.3 Pairing Let $\mathbf{W}_m$ the $\mathbb{Q}$–vector space with basis given by the symbols $[\mathcal{S}, \mu]$ where $\mathcal{S} \subset \overline{M}_{0,m|d}$ is a stratum of dimension $s \geq k$ and μ is a monomial in the variables

$$\psi_1, \ldots, \psi_m$$

of degree $s - k$.

There is a canonical Poincaré pairing

$$I \colon \mathbf{V}[d, k] \times \mathbf{W}_m \to \mathbb{Q}$$

defined on the bases by

$$I([\mathcal{P}, \tau], [\mathcal{S}, \mu]) = \int_{\mathcal{S}} X_m[\mathcal{P}, \tau] \cup \mu(\psi_1, \ldots, \psi_m).$$

Lemma 6 *A vector $v \in \mathbf{V}[d, k]$ is 0 if and only if v is in the null space of all the pairings I for $m \geq 3$.*

---

[15]The choice of ordering will not play a role in the argument.

Proof To $[\mathcal{P}, \tau] \in \mathrm{P}[d, k]$, we associate $Y_m[\mathcal{P}, \tau] \in \mathbf{W}_m$, where

$$m = 4 + k - d + 2\ell$$

and ℓ is the length of $\mathcal{P}$, by the following construction.

Let $\mathcal{S} \subset \overline{M}_{0,m|d}$ be the stratum consisting of a chain of $\ell + 2$ rational curves attached head to tail,

$$R_0, R_1, R_2, \ldots, R_\ell, R_{\ell+1},$$

with the m markings $p_1, \ldots, p_m$ distributed by the rules

  (i) R_0 and $R_{\ell+1}$ each carry exactly 2 markings,

 (ii)   for $1 \leq i \leq \ell$, R_i carries $t_i + 1$ markings,

(iii)   the marking are distributed in order from left to right.

We write r_i for the minimal label of the markings on R_i. The d markings $\hat{p}_1, \ldots, \hat{p}_d$ are distributed by the rules

 (iv) R_0 and $R_{\ell+1}$ each carry 0 markings,

  (v)   for $1 \leq i \leq \ell$, R_i carries the markings corresponding to $\mathcal{P}_i$.

The dimension of $\mathcal{S}$ is easily calculated:

$$\begin{aligned}
\dim(\mathcal{S}) &= \dim(\overline{M}_{0,m|d}) - \ell - 1 \\
&= 4 + k - d + 2\ell + d - 3 - \ell - 1 \\
&= \ell + k.
\end{aligned}$$

The associated element of $\mathbf{W}_m$ is defined by

$$Y_m[\mathcal{P}, \tau] = [\mathcal{S}, \psi_{r_1} \cdots \psi_{r_\ell}].$$

The next step is to find when the pairing

(25) $$I([\mathcal{P}, \tau], Y_{m'}[\mathcal{P}', \tau'])$$

is nontrivial for

$$[\mathcal{P}, \tau], \ [\mathcal{P}', \tau'] \in \mathrm{P}[d, k],$$

with $m' = 4 + k - d + 2\ell'$. By definition, the pairing (25) equals

(26) $$\int_{\mathcal{S}'} X_{m'}[\mathcal{P}, \tau] \cup \psi_{r'_1} \cdots \psi_{r'_{\ell'}} = \int_{\mathcal{S}'} \psi_{r'_1} \cdots \psi_{r'_{\ell'}} \cdot \hat{\psi}_{\mathcal{P}_1}^{t_1} \ldots \hat{\psi}_{\mathcal{P}_\ell}^{t_\ell} \cdot D_{\mathcal{P}_1} \cdots D_{\mathcal{P}_\ell}.$$

The integral (26) is calculated by distributing the diagonal points corresponding to $D_{\mathcal{P}_j}$ to the components R'_i of curves in $\mathcal{S}'$ in all possible ways. Note that unless there is at

least one diagonal $D_{\mathcal{P}_j}$ distributed to *each* R'_i for $1 \leq i \leq \ell'$, the contribution to the integral (26) vanishes. Hence, nonvanishing implies $\ell \geq \ell'$.

If $\ell = \ell'$, then the distribution rule (v) implies the set theoretic intersection

$$\mathcal{S}' \cap D_{\mathcal{P}_1} \cap \cdots \cap D_{\mathcal{P}_\ell}$$

is empty unless $\mathcal{P} = \mathcal{P}'$. If $\mathcal{P} = \mathcal{P}'$, the only nonvanishing diagonal distribution is given by sending $D_{\mathcal{P}_i}$ to R'_i. The integral (26) is easily seen to be nonzero then if and only if $\tau = \tau'$. Indeed, the contribution of R'_i to the integral is

$$\int_{\overline{M}_{0,t'_i+3||\mathcal{P}_i|}} \psi_{r'_i} \hat{\psi}^{t_i}_{\mathcal{P}_i} \cdot D_{\mathcal{P}_i} = \int_{\overline{M}_{0,t'_i+4}} \psi_{r'_i} \psi^{t_i}_{t'_i+4} = \begin{cases} t_i + 1 & \text{if } t_i = t'_i, \\ 0. & \text{otherwise} \end{cases}$$

The linear functions on $\mathbf{V}[d,k]$ determined by $I(\,\cdot\,, Y_{m'}[\mathcal{P}',\tau'])$ are block lower-triangular with respect to the partial ordering by the length of the set partition. Moreover, the diagonal blocks are themselves diagonal with nonzero entries. □

Following the notation of Section 7.6.1, Lemma 6 proves that if

$$P^C \in A^*(\overline{M}_{0,m|d}, \mathbb{Q})$$

vanishes for all $m \geq 3$, then P^C vanishes as an abstract polynomial. The proofs of Theorem 3 and Theorem 4 are therefore complete.

8 Tautological relations

8.1 Tautological classes

Let $g \geq 2$. The tautological ring of the moduli space of curves

$$R^*(M_g) \subset A^*(M_g, \mathbb{Q})$$

is generated by the classes

$$\kappa_i = \epsilon_*(\psi_1^{i+1}), \quad M_{g,1} \xrightarrow{\epsilon} M_g.$$

Here, $\kappa_0 = 2g - 2$ is a multiple of the unit class. A conjectural description of $R^*(M_g)$ is presented in Faber [6]. The basic vanishing result,

$$R^i(M_g) = 0$$

for $i > g - 2$, has been proven by Looijenga [24].

8.2 Relations

Let $g \geq 2$ and $d \geq 0$. The moduli space

$$M_{g,0|d} \xrightarrow{\epsilon} M_g$$

is simply the d–fold product of the universal curve over M_g. Given an element

$$[C, \hat{p}_1, \ldots, \hat{p}_d] \in M_{g,0|d} ,$$

there is a canonically associated stable quotient

$$(27) \qquad 0 \to \mathcal{O}_C\left(-\sum_{j=1}^{d} \hat{p}_j\right) \to \mathcal{O}_C \to Q \to 0.$$

Consider the universal curve

$$\pi: U \to M_{g,0|d}$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (27). Let

$$\mathbb{F}_d = -R\pi_*(S_U^*) \in K(M_{g,0|d})$$

be the class in K–theory. For example,

$$\mathbb{F}_0 = \mathbb{E}^* - \mathbb{C}$$

is the dual of the Hodge bundle minus a rank 1 trivial bundle.

By Riemann–Roch, the rank of $\mathbb{F}_d$ is

$$r_g(d) = g - d - 1.$$

However, $\mathbb{F}_d$ is not always represented by a bundle. By the derivation of Section 4.6,

$$(28) \qquad \mathbb{F}_d = \mathbb{E}^* - \mathbb{B}_d - \mathbb{C},$$

where $\mathbb{B}_d$ has fiber $H^0(C, \mathcal{O}_C(\sum_{j=1}^{d} \hat{p}_j)|_{\sum_{j=1}^{d} \hat{p}_j})$ over $[C, \hat{p}_1, \ldots, \hat{p}_d]$. Alternatively, $\mathbb{B}_d$ is the ϵ–relative tangent bundle.

Theorem 5 *For every integer $k > 0$,*

$$\epsilon_*\left(c_{r_g(d)+2k}(\mathbb{F}_d)\right) = 0 \in R^*(M_g).$$

Since the morphism ϵ has fibers of dimension d,

$$\epsilon_* \left(c_{r_g(d)+2k}(\mathbb{F}_d)\right) \in R^{g-2d-1+2k}(M_g).$$

By Looijenga's vanishing, Theorem 5 is only nontrivial when

$$0 \le 2d - 2k - 1 \le g - 2.$$

The vanishing of Theorem 5 does not naively extend. We calculate

$$(29) \qquad \epsilon_* \left(c_{r_g(1)+1}(\mathbb{F}_1)\right) = \kappa_{g-2} - \lambda_1 \kappa_{g-3} + \cdots + (-1)^{g-2} \kappa_0 \lambda_{g-2}$$

in $R^{g-2}(M_g)$ by (28). However, the class (29) is known not to vanish by the pairing with $\lambda_g \lambda_{g-1}$ calculated in [34].

Theorem 5 directly yields relations among the generators κ_i of $R^*(M_g)$ by the standard ϵ pushforward rules [6]. The construction is more subtle than the method of [6] as the relations only hold after pushforward. An advantage is that the boundary terms of the relations here can easily be calculated.

8.3 Example

The Chern classes of $\mathbb{F}_d$ can be easily computed. Recall the divisor $D_{i,j}$ where the markings $\hat{p}_i$ and $\hat{p}_j$ coincide. Set

$$\Delta_i = D_{1,i} + \cdots + D_{i-1,i},$$

with the convention $\Delta_1 = 0$. Over $[C, \hat{p}_1, \ldots, \hat{p}_d]$, the virtual bundle $\mathbb{F}_d$ is the formal difference

$$H^1(\mathcal{O}_C(\hat{p}_1 + \cdots + \hat{p}_d)) - H^0(\mathcal{O}_C(\hat{p}_1 + \cdots + \hat{p}_d)).$$

Taking the cohomology of the exact sequence

$$0 \to \mathcal{O}_C(\hat{p}_1 + \cdots + \hat{p}_{d-1}) \to \mathcal{O}_C(\hat{p}_1 + \cdots + \hat{p}_d) \to \mathcal{O}_C(\hat{p}_1 + \cdots + \hat{p}_d)|_{\hat{p}_d} \to 0,$$

we find

$$c(\mathbb{F}_d) = \frac{c(\mathbb{F}_{d-1})}{1 + \Delta_d - \hat{\psi}_d}.$$

Inductively, we obtain

$$(30) \qquad c(\mathbb{F}_d) = \frac{c(\mathbb{E}^*)}{(1 + \Delta_1 - \hat{\psi}_1) \cdots (1 + \Delta_d - \hat{\psi}_d)}.$$

In the $d = 2$ and $k = 1$ case, Theorem 5 gives the vanishing of the class

$$\epsilon_* c_{g-1}(\mathbb{F}_d) = \epsilon_* \left[\frac{c(\mathbb{E}^*)}{(1 - \hat{\psi}_1)(1 + \Delta - \hat{\psi}_2)} \right]^{g-1},$$

where Δ is the divisor of coincident markings on $M_{g,0|2}$. The superscript indicates the degree $g-1$ part of the bracketed expression. Expanding, we obtain

$$(31) \qquad \sum_i (-1)^i \lambda_{g-1-i} \sum_{i_1+i_2=i} \epsilon_* \left(\hat{\psi}_1^{i_1} (\hat{\psi}_2 - \Delta)^{i_2} \right) = 0.$$

We have

$$\epsilon_* \left(\hat{\psi}_1^{i_1} (\hat{\psi}_2 - \Delta)^{i_2} \right) = \sum_m (-1)^{i_2-m} \binom{i_2}{m} \epsilon_* \left(\hat{\psi}_1^{i_1} \hat{\psi}_2^m \Delta^{i_2-m} \right).$$

Using
$$\Delta^2 = -\hat{\psi}_1 \Delta = -\hat{\psi}_2 \Delta$$

and the ϵ–calculus rules in [6], we rewrite the last expression as

$$-\sum_{m \neq i_2} \binom{i_2}{m} \epsilon_* (\hat{\psi}_1^{i_1+i_2-1} \Delta) + \epsilon_* (\hat{\psi}_1^{i_1} \hat{\psi}_2^{i_2}) = -(2^{i_2} - 1) \kappa_{i_1+i_2-2} + \kappa_{i_1-1} \kappa_{i_2-1}.$$

After summing over i_1, i_2 in (31), we arrive at the relation

$$(32) \qquad \sum_{i=2}^{g-1} (-1)^i \lambda_{g-1-i} \left(\left(\sum_{i_1+i_2=i} \kappa_{i_1-1} \kappa_{i_2-1} \right) - (2^{i+1} - i - 2) \kappa_{i-2} \right) = 0$$

in $R^{g-3}(M_g)$.

The λ classes can be expressed in terms of the κ classes by Mumford's Chern character calculation

$$\mathrm{ch}_{2\ell}(\mathbb{E}) = 0, \quad \mathrm{ch}_{2\ell-1}(\mathbb{E}) = \frac{B_{2\ell}}{(2\ell)!} \kappa_{2\ell-1}$$

for $\ell > 0$. From (32), we obtain a relation involving only the tautological generators κ_i. To illustrate, in genus 6, we obtain the relation

$$25 \kappa_1^3 + 15912 \kappa_3 - 1080 \kappa_1 \kappa_2 = 0,$$

which is consistent with the presentation of $R^*(M_6)$ in [6].

8.4 Brill–Noether construction

The $k = 1$ case of Theorem 5 for positive $d \leq g$ admits an alternative derivation via Brill–Noether theory.[16]

To start, consider the rank d bundle,

$$\mathbb{W}_d \to M_{g,0|d} \ ,$$

---

[16]The Brill–Noether connection was suggested by Faber who recognized Equation (32).

with fiber $H^0(C, \omega_C|_{\sum_{j=1}^d \hat{p}_j})$ over $[C, \hat{p}_1, \ldots, \hat{p}_d]$. There is a canonical map of vector bundles on $M_{g,0|d}$,

$$\rho \colon \mathbb{E} \to \mathbb{W}_d ,$$

defined by the restriction $H^0(C, \omega_C) \to H^0(C, \omega_C|_{\sum_{j=1}^d \hat{p}_j})$. After dualizing, we obtain

$$\rho^* \colon \mathbb{W}_d^* \to \mathbb{E}^* .$$

If ρ^* fails to have maximal rank at $[C, \hat{p}_1, \ldots, \hat{p}_d] \in M_{g,0|d}$, the divisor $\hat{p}_1 + \cdots + \hat{p}_d$ must move in a nontrivial linear series. The degeneracy locus of ρ^* precisely defines the Brill–Noether variety [1]

$$G_d^1 \subset M_{g,0|d} ,$$

well-known to be of expected codimension $g - d + 1$. Since

$$\epsilon \colon G_d^1 \to M_g$$

has positive dimensional fibers, certainly

$$\epsilon_*[G_d^1] = 0 \in A^*(M_g).$$

By the Porteous formula [8],

$$[G_d^1] = c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*) .$$

Hence, we obtain the relation

$$(33) \qquad \epsilon_* \left(c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*) \right) = 0 \in R^*(M_g) .$$

Lemma 7 $\mathbb{W}_d \cong \mathbb{B}_d^*$.

Proof Let $\hat{P} \subset C$ be the divisor $\hat{p}_1 + \cdots + \hat{p}_d$. The fiber of $\mathbb{W}_d$ over $[C, \hat{p}_1, \ldots, \hat{p}_d]$ is

$$\text{Ext}^0(\mathcal{O}_C, \omega_C|\hat{P}) \cong \text{Ext}^1(\mathcal{O}_{\hat{P}}, \mathcal{O}_C)^*$$

by Serre duality. Let

$$I = [\mathcal{O}_C(-\hat{P}) \to \mathcal{O}_C]$$

denote the complex of line bundles in grade -1 and 0. Since I is quasi-isomorphic to $\mathcal{O}_{\hat{P}}$, we find

$$\text{Ext}^1(I, \mathcal{O}_C) \cong \text{Ext}^1(\mathcal{O}_{\hat{P}}, \mathcal{O}_C).$$

On the other hand, we have

$$I^* = [\mathcal{O}_C \to \mathcal{O}_C(\hat{P})] \quad \text{and} \quad \text{Ext}^1(\mathcal{O}_C, I^*) \cong \text{Ext}^0(\mathcal{O}_C, \mathcal{O}_{\hat{P}}(\hat{P})).$$

We have hence found a canonical isomorphism

$$\text{Ext}^1(\mathcal{O}_{\hat{P}}, \mathcal{O}_C) \cong \text{Ext}^0(\mathcal{O}_C, \mathcal{O}_{\hat{P}}(\hat{P}))$$

where the latter space is the fiber of $\mathbb{B}_d$. $\qquad\qquad\square$

The $k = 1$ case of Theorem 5 concerns the class

$$c_{g-d+1}(\mathbb{F}_d) = c_{g-d+1}(\mathbb{E}^* - \mathbb{B}_d - \mathbb{C})$$
$$= c_{g-d+1}(\mathbb{E}^* - \mathbb{B}_d)$$
$$= c_{g-d+1}(\mathbb{E}^* - \mathbb{W}_d^*).$$

Hence, the vanishing

$$\epsilon_*(c_{g-d+1}(\mathbb{F}_d)) = 0$$

of Theorem 5 exactly coincides with the Brill–Noether vanishing (33).

Theorem 5 may be viewed as a generalization of Brill–Noether vanishing obtained from the virtual geometry of the moduli of stable quotients.

8.5 Proof of Theorem 5

Consider the proper morphism

$$v\colon Q_g(\mathbb{P}^1, d) \to M_g.$$

The universal curve

$$\pi\colon U \to Q_g(\mathbb{P}^1, d)$$

carries the basic divisor classes

$$s = c_1(S_U^*), \quad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U and the π–relative dualizing sheaf. The class

$$(34) \qquad v_*\big(\pi_*(s^a\omega^b) \cdot 0^c \cap [Q_g(\mathbb{P}^1, d)]^{\mathrm{vir}}\big) \in A^*(M_g, \mathbb{Q}),$$

where 0 is first Chern class of the trivial bundle, certainly vanishes if $c > 0$. Theorem 5 is proven by calculating (34) by localization. We will find Theorem 5 is a subset of a richer family of relations.

Let the 1–dimensional torus $\mathbb{C}^*$ act on a 2–dimensional vector space $V \cong \mathbb{C}^2$ with diagonal weights $[0, 1]$. The $\mathbb{C}^*$–action lifts canonically to the following spaces and sheaves:

$$\mathbb{P}(V), \quad Q_g(\mathbb{P}(V), d), \quad U, \quad S_U \quad \text{and} \quad \omega_\pi.$$

We lift the $\mathbb{C}^*$–action to a rank 1 trivial bundle on $Q_g(\mathbb{P}(V), d)$ by specifying fiber weight 1. The choices determine a $\mathbb{C}^*$–lift of the class

$$\pi_*(s^a \cdot \omega^b) \cdot 0^c \cap [Q_g(\mathbb{P}(V), d)]^{\mathrm{vir}} \in A_{2d+2g-1-a-b-c}(Q_g(\mathbb{P}(V), d), \mathbb{Q}).$$

The pushforward (34) is determined by the virtual localization formula [12]. There are only two $\mathbb{C}^*$–fixed loci. The first corresponds to a vertex lying over $0 \in \mathbb{P}(V)$. The locus is isomorphic to

$$M_{g,0|d}/\mathbb{S}_d$$

and the associated subsheaf (27) lies in the first factor of $V \otimes \mathcal{O}_C$ when considered as a stable quotient in the moduli space $Q_g(\mathbb{P}(V), d)$. Similarly, the second fixed locus corresponds to a vertex lying over $\infty \in \mathbb{P}(V)$.

The localization contribution of the first locus to (34) is

$$\frac{1}{d!}\epsilon_*\big(\pi_*(s^a\omega^b) \cdot c_{g-d-1+c}(\mathbb{F}_d)\big)$$

where s and ω are the corresponding classes on the universal curve over $M_{g,0|d}$. Let $c_-(\mathbb{F}_d)$ denote the total Chern class of $\mathbb{F}_d$ evaluated at -1. The localization contribution of the second locus is

$$\frac{(-1)^{g-d-1}}{d!}\epsilon_*\big[\pi_*((s-1)^a\omega^b) \cdot c_-(\mathbb{F}_d)\big]^{g-d-2+a+b+c},$$

where $[\gamma]^k$ is the part of γ in $A^k(M_{g,0|d}, \mathbb{Q})$.

Both localization contributions are found by straightforward expansion of the vertex formulas of Section 7.4.2. Summing the contributions yields the following result.

Proposition 5 Let $c > 0$. Then

$$\epsilon_*\Big(\pi_*(s^a\omega^b) \cdot c_{g-d-1+c}(\mathbb{F}_d)$$
$$+ (-1)^{g-d-1}\big[\pi_*((s-1)^a\omega^b) \cdot c_-(\mathbb{F}_d)\big]^{g-d-2+a+b+c}\Big) = 0$$

in $R^*(M_g)$.

If $a = 0$ and $b = 1$, the relation of Proposition 5 specializes to Theorem 5 for even $c = 2k$. $\qquad\square$

Question 1 Do the relations obtained from Proposition 5 generate all the relations among the classes κ_i in $R^*(M_g)$?

8.6 Further examples

Let $\sigma_i \in A^1(U, \mathbb{Q})$ be the class of the i–th section of the universal curve

$$\pi\colon U \to M_{g,0|d} \ .$$

The class $s = c_1(S_U^*)$ of Proposition 5 is

$$s = \sigma_1 + \cdots + \sigma_d \in A^1(U, \mathbb{Q}).$$

We calculate
$$\pi_*(s) = d$$
$$\pi_*(\omega) = 2g - 2$$
$$\pi_*(s\,\omega) = \hat{\psi}_1 + \cdots + \hat{\psi}_d$$
$$\pi_*(s^2) = -(\hat{\psi}_1 + \cdots + \hat{\psi}_d) + 2\Delta$$

in $A^*(M_{g,0|d}, \mathbb{Q})$, where

$$\Delta = \sum_{i<j} D_{i,j} \in A^1(M_{g,0|d}, \mathbb{Q})$$

is the symmetric diagonal. The pushforwards $\pi_*(s^a \omega^b)$ are all easily obtained.

Using the above π_* calculations, the $a = 1$, $b = 1$, $c = 2k$ case of Proposition 5 yields

$$\epsilon_*\big(2(\hat{\psi}_1 + \cdots + \hat{\psi}_d) \cdot c_{r_g(d)+2k}(\mathbb{F}_d) + (2g - 2)\, c_{r_g(d)+2k+1}(\mathbb{F}_d)\big) = 0.$$

The $a = 2$, $b = 0$, $c = 2k$ case yields

$$\epsilon_*\big(-2(\hat{\psi}_1 + \cdots + \hat{\psi}_d - 2\Delta) \cdot c_{r_g(d)+2k}(\mathbb{F}_d) + 2d \cdot c_{r_g(d)+2k+1}(\mathbb{F}_d)\big) = 0.$$

Summation yields a third relation,

$$\epsilon_*\big(2\Delta \cdot c_{r_g(d)+2k}(\mathbb{F}_d) + (d + g - 1) \cdot c_{r_g(d)+2k+1}(\mathbb{F}_d)\big) = 0.$$

The relations of Proposition 5 include the classes $c_{r_g(d)+2k+1}(\mathbb{F}_d)$ that were omitted in Theorem 5.

9 Calabi–Yau geometry

The moduli of stable quotients may be used to define counting invariants in the local Calabi–Yau geometries. For example consider the conifold, the total space of

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1.$$

Just as in Gromov–Witten theory, we define

$$(35) \quad N_{g,d} = \frac{1}{d^2} \int_{[\bar{Q}_{g,2}(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(R^1\pi_*(S_U) \oplus R^1\pi_*(S_U)) \cup \mathrm{ev}_1^*(H), \cdot \mathrm{ev}_2^*(H)$$

where S_U is the universal subsheaf on the universal curve

$$\pi: U \to \bar{Q}_{g,2}(\mathbb{P}^1,d)$$

and $H \in H^2(\mathbb{P}^1,\mathbb{Q})$ is the hyperplane class. The two point insertions are required for stability in genus 0. Let

$$F(t) = \sum_{g \geq 1} N_{g,1} t^{2g}.$$

Proposition 6 *The local invariants $N_{g,d}$ are determined by the two equations*

$$N_{g,d} = d^{2g-3} N_{g,1},$$

$$F(t) = \left(\frac{t/2}{\sin(t/2)} \right)^2.$$

Proof We compute the integral $N_{g,d}$ by localization. Let $\mathbb{C}^*$ act on the vector space $V \cong \mathbb{C}^2$ with diagonal weights $[0,1]$. The $\mathbb{C}^*$–action lifts canonically to $\bar{Q}_{g,2}(\mathbb{P}(V),d)$ and S_U. For the first S_U in the integrand (35), we use the canonical lifting of $\mathbb{C}^*$. For the second S_U, we tensor by a trivial line bundle with fiber weights -1 over the two $\mathbb{C}^*$–fixed points of $\mathbb{P}(V)$. The classes H are lifted to the distinct $\mathbb{C}^*$–fixed points on $\mathbb{P}(V)$.

The above choice of $\mathbb{C}^*$–action on the integrand exactly parallels the choice of $\mathbb{C}^*$–action taken in [7] for the analogous Gromov–Witten calculation. The vanishing obtained in [7] also applies for the stable quotient calculation here. The only loci with nonvanishing contribution to the localization sum consist of two vertices of genera

$$g_1 + g_2 = g$$

connected by a single edge of degree d. The moduli spaces at these vertices are $\bar{M}_{g_i,2|0}$ where

(i) the first two points are the respective node and marking,

(ii) there are no markings after the bar by vanishing.

We find that the only nonvanishing contributions occur on $\mathbb{C}^*$–fixed loci where the moduli of stable quotients and the moduli of stable maps are isomorphic. Moreover, on these loci, the bundle $R^1\pi_*(S_U)$ agrees with the analogous Gromov–Witten bundle. Hence, the stable quotient integral $N_{g,d}$ is equal to the Gromov–Witten calculation of the conifold [7]. $\square$

The matching is somewhat of a surprise. While the virtual classes of the stable quotient and stable maps spaces to $\mathbb{P}^1$ are related by Theorem 3, the bundles in the respective integrands for the conifold geometry are *not* compatible. However, the differences happen away from the nonvanishing loci.

If $g \geq 1$, no point insertions are required for stability. The associated conifold integral is more subtle to calculate, but the same result is obtained. We leave the details to the reader.[17]

Proposition 7 *For $g \geq 1$,*

$$N_{g,d} = \int_{[\bar{Q}_{g,0}(\mathbb{P}^1, d)]^{\mathrm{vir}}} e(R^1 \pi_*(S_U) \oplus R^1 \pi_*(S_U)).$$

There are many other well-defined local toric Calabi–Yau geometries to consider for stable quotients both in dimension 3 and higher [19; 37]. The simplest is local $\mathbb{P}^2$.

Question 2 What is the answer for the stable quotient theory for

$$\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2 \ ?$$

10 Other targets

10.1 Virtual classes

Let $X \subset \mathbb{P}^n$ be a projective variety. There is a naturally associated substack

$$(36) \qquad \bar{Q}_{g,m}(X, d) \subset \bar{Q}_{g,m}(\mathbb{P}^n, d)$$

defined by the following principle. Let $I \subset \mathbb{C}[z_0, \ldots, z_n]$ be the homogeneous ideal of X. Given an element

$$(37) \qquad (C, \ p_1, \ldots, p_m, \ 0 \to S \to \mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0)$$

of $\bar{Q}_{g,m}(\mathbb{P}^n, d)$, consider the dual

$$\mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q^*} S^*$$

---

[17]The vanishing, as before, matches the $\mathbb{C}^*$–fixed point loci of the stable quotients and stable maps spaces. However, the two which correspond to a single vertex of genus g are now not obviously equal. The match for these is obtained by redoing the pointed integral (35) with both H classes in the integrand taken to lie over the *same* $\mathbb{C}^*$–fixed point.

as a line bundle with $n + 1$ sections $s_0, \ldots, s_n$. The stable quotient (37) lies in $\bar{Q}_{g,m}(X, d)$ if for every homogeneous degree k polynomial $f_k \in I$,

$$(38) \qquad\qquad f_k(s_0, \ldots, s_n) = 0 \in H^0(C, S^{k*}).$$

Condition (38) is certainly well-defined in families and determines a Deligne–Mumford substack. Local equations for the substack (36) can easily be found.

Question 3 If X is nonsingular, does $\bar{Q}_{g,m}(X, d)$ carry a canonical 2–term perfect obstruction theory?

The moduli space $\bar{Q}_{g,m}(X, d)$ depends upon the projective embedding of X. If $\bar{Q}_{g,m}(X, d)$ does carry a virtual class, the theory will almost certainly differ somewhat from the Gromov–Witten counts.

If $X \subset \mathbb{P}^n$ is nonsingular complete intersection, more definite claims can be made. For simplicity, assume X is a hypersurface defined by a degree k equation F. Given an element

$$(C, \ p_1, \ldots, p_m, \ 0 \to S \to \mathbb{C}^{n+1} \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0)$$

of $\bar{Q}_{g,m}(X, d)$, the pullback to C of the tangent bundle to X may be viewed as the complex

$$(39) \qquad\qquad S^* \otimes Q \xrightarrow{dF} S^{k*}$$

defined by differentiation of the section F on the zero locus. We speculate an obstruction theory on $\bar{Q}_{g,m}(X, d)$ can be defined by the hypercohomology of the sequence (39). The 2–term condition follows from the fact that the map dF has cokernel with dimension 0 support. Many details have to be checked here.

10.2 Elliptic invariants

An interesting example to consider is the moduli space $\bar{Q}_{1,0}(X_{n+1} \subset \mathbb{P}^n, d)$ of stable quotients associated to the Calabi–Yau hypersurfaces $X_{n+1} \subset \mathbb{P}^n$.

By Proposition 1, $\bar{Q}_{1,0}(\mathbb{P}^n, d)$ is a nonsingular space of expected dimension $(n+1)d$. As before, let S_U be the universal subsheaf on the universal curve

$$\pi \colon U \to \bar{Q}_{1,0}(\mathbb{P}^n, d).$$

Since S_U is locally free of rank 1, S_U is a line bundle. By the vanishing used in the proof of Proposition 1,

$$\pi_* S_U^{*(n+1)} \to \bar{Q}_{1,0}(\mathbb{P}^n, d)$$

is locally free of rank $(n+1)d$.

We define the genus 1 stable quotient invariants of $X_{n+1} \subset \mathbb{P}^n$ by the integral

$$(40) \qquad N_{1,d}^{X_{n+1}} = \int_{\overline{Q}_{1,0}(\mathbb{P}^n, d)} e(\pi_* S_U^{*(n+1)}).$$

The definition of $N_{1,d}^{X_{n+1}}$ is compatible with the discussion of the virtual classes of hypersurfaces in Section 10.1.

The genus 1 Gromov–Witten theory of hypersurfaces has recently been solved by Zinger [44]. Substantial work is required to convert the Gromov–Witten calculation to an Euler class on a space of genus 1 maps to projective space. The stable quotient invariants are immediately given by such an Euler class. There is no obstruction to calculating (40) by localization.

Question 4 What is the relationship between the stable quotient and stable map invariants in genus 1 for Calabi–Yau hypersurfaces?

10.3 Variants

There are several variants which can be immediately considered. Let X be a nonsingular projective variety with an ample line bundle L. The stable quotient construction can be carried out over the moduli space of stable maps $\overline{M}_{g,m}(X, \beta)$ instead of the moduli space of curves $\overline{M}_{g,m}$. An object then consists of three pieces of data:

  (i)   a genus g, m–pointed, quasistable curve $(C, p_1, \ldots, p_m)$,
  (ii) a map $f: C \to X$ representing class $\beta \in H_2(X, \mathbb{Z})$,
  (iii) and a quasistable quotient sequence

$$0 \to S \to \mathbb{C}^n \otimes \mathcal{O}_C \to Q \to 0.$$

Stability is defined by the ampleness of

$$\omega_C(p_1 + \cdots + p_m) \otimes f^*(L^3) \otimes (\wedge^r S^*)^{\otimes \epsilon}$$

on C for every strictly positive $\epsilon \in \mathbb{Q}$. We leave the details to the reader. The moduli space is independent of the choice of L.

The moduli space carries a 2–term obstruction theory and a virtual class. The corresponding descendent theory is equivalent to the Gromov–Witten theory of $X \times \mathbb{G}(r, n)$ by straightforward modification of the arguments used to prove Theorem 4.

There is no reason to restrict to the trivial bundle in (iii) above. We may fix a rank n vector bundle

$$B \to X$$

and replace the quasistable quotient sequence by

$$0 \to S \to f^*(B) \to Q \to 0.$$

The corresponding theory is perhaps equivalent to the Gromov–Witten theory of the Grassmannian bundle over X associated to B. As B may not split, a torus action may not be available. The strategy of the proof of Theorem 4 does not directly apply.

A stranger replacement of the trivial bundle can be made even when X is a point. We may choose the quotient sequence to be

$$0 \to S \to H^0(C, \omega_C) \otimes \mathcal{O}_C \to Q \to 0.$$

The middle term is essentially the pullback of the Hodge bundle from the moduli space of curves.

Question 5 What do integrals over the moduli of stable Hodge quotients correspond to in Gromov–Witten theory?

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago
851 S Morgan Street, Chicago IL 60607, USA

Department of Mathematics, University of California San Diego
9500 Gilman Drive #0112, La Jolla CA 92093-0112, USA

Department of Mathematics, Princeton University
406 Fine Hall, Princeton NJ 08544, USA

alina@math.uic.edu, doprea@math.ucsd.edu, rahulp@math.princeton.edu