Parallelogram decompositions and generic surfaces in $\mathcal{H}^{\text{hyp}}(4)$

DUC-MANH NGUYEN

The space $\mathcal{H}^{\text{hyp}}(4)$ is the moduli space of pairs (M,ω), where M is a hyperelliptic Riemann surface of genus 3 and ω is a holomorphic 1–form having only one zero. In this paper, we first show that *every* surface in $\mathcal{H}^{\text{hyp}}(4)$ admits a decomposition into parallelograms and simple cylinders following a unique model. We then show that if this decomposition satisfies some irrational condition, then the $\mathrm{GL}^+(2,\mathbb{R})$–orbit of the surface is dense in $\mathcal{H}^{\text{hyp}}(4)$; such surfaces are called *generic*. Using this criterion, we prove that there are generic surfaces in $\mathcal{H}^{\text{hyp}}(4)$ with coordinates in any quadratic field, and there are Thurston–Veech surfaces with trace field of degree three over $\mathbb{Q}$ which are generic.

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1 Introduction

Translation surfaces are flat surfaces with conical singularities and trivial linear holonomy, that is, the holonomy of any closed curve is a translation in $\mathbb{R}^2$. The space of translation surfaces together with an oriented parallel line field is identified with the space of holomorphic 1–forms on Riemann surfaces, which is stratified by the orders of the zeros of the 1–form. Fixing $g \geqslant 2$, if $k_1,\dots,k_n$ are positive integers such that $k_1+\cdots+k_n = 2g-2$, we denote by $\mathcal{H}(k_1,\dots,k_n)$ the moduli space of holomorphic 1–forms on Riemann surfaces of genus g which have exactly n zeros with orders $(k_1,\dots,k_n)$. By a result of Kontsevich and Zorich [9], we know that $\mathcal{H}(k_1,\dots,k_n)$ has at most 3 connected components. We denote by $\mathcal{H}_1(k_1,\dots,k_n)$ the subset of $\mathcal{H}(k_1,\dots,k_n)$ consisting of surfaces of unit area. On $\mathcal{H}(k_1,\dots,k_n)$, one has a volume form which is the pullback of the Lebesgue measure in the local charts defined by the *period mappings*. This volume form induces naturally a volume form on $\mathcal{H}_1(k_1,\dots,k_n)$. By results of Masur [10] and Veech [15] (see also Eskin and Okounkov [4]), we know that the volume of $\mathcal{H}_1(k_1,\dots,k_n)$ is finite.

There is an action of $\mathrm{GL}^+(2,\mathbb{R})$ on the space $\mathcal{H}(k_1,\dots,k_n)$. The action of the subgroup $\mathrm{SL}(2,\mathbb{R})$ leaves invariant the Lebesgue measure and preserves the set $\mathcal{H}_1(k_1,\dots,k_n)$. It is now a classical fact, also due to Masur and Veech, that the

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$\mathrm{SL}(2,\mathbb{R})$ action is ergodic in each component of $\mathcal{H}_1(k_1,\dots,k_n)$. A surface of unit area whose $\mathrm{SL}(2,\mathbb{R})$–orbit is dense in a component of $\mathcal{H}_1(k_1,\dots,k_n)$ (or equivalently, whose $\mathrm{GL}^+(2,\mathbb{R})$–orbit is dense in a component of $\mathcal{H}(k_1,\dots,k_n)$) is called *generic*.

Knowledge about the $\mathrm{SL}(2,\mathbb{R})$–orbit closures is useful since it is strongly related to asymptotic behaviors of the linear flow on individual surfaces (see Zorich [16, Section 3.6]). Despite the fact that almost all surfaces are generic, the problem of determining whether the orbit of a particular surface is dense in its component is wide open. We only have a complete classification (due to McMullen [12] and Calta [1]) for the case of genus 2, where we have two strata, $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$, each of which has a single connected component. Recall that the Veech group of a translation surface is the stabilizer subgroup for the action of $\mathrm{SL}(2,\mathbb{R})$. It is a well-known fact (see Masur and Tabachnikov [11], Zorich [16] and McMullen [12]) that the $\mathrm{SL}(2,\mathbb{R})$–orbit of a surface is a closed subset in its stratum if and only if its Veech group is a lattice of $\mathrm{SL}(2,\mathbb{R})$. It turns out from the work of McMullen that, for translation surfaces of genus two, if the Veech group contains a hyperbolic element, then the $\mathrm{SL}(2,\mathbb{R})$–orbit cannot be dense in the corresponding stratum.

More recently, Hubert, Lanneau and Möller [6; 5] give some results on generic surfaces in the hyperelliptic locus $\mathcal{L}$ in $\mathcal{H}^{\mathrm{odd}}(2,2)$, which is one of the two components of $\mathcal{H}(2,2)$. They show that, in contrast with the case of genus 2, there are generic surfaces in $\mathcal{L}$, that is, the $\mathrm{SL}(2,\mathbb{R})$–orbit is dense in $\mathcal{L}$, whose Veech group contains hyperbolic elements. Note that $\mathcal{L}$ is a closed, $\mathrm{SL}(2,\mathbb{R})$–invariant subset of $\mathcal{H}^{\mathrm{odd}}(2,2)$, therefore, the closure of any $\mathrm{SL}(2,\mathbb{R})$–orbit in $\mathcal{L}$ cannot exceed $\mathcal{L}$. The Thurston–Veech construction [13; 14] provides us with translation surfaces which are stabilized by some hyperbolic elements in $\mathrm{SL}(2,\mathbb{R})$, these hyperbolic elements arise as products of parabolic ones. Hubert, Lanneau and Möller also show that there are surfaces in $\mathcal{L}$ obtained from the Thurston–Veech construction whose $\mathrm{SL}(2,\mathbb{R})$–orbit is dense in $\mathcal{L}$.

The stratum $\mathcal{H}(4)$ is the space of holomorphic 1–form on Riemann surfaces of genus 3 which have only one zero (the order is necessarily 4). We have $\dim_{\mathbb{C}} \mathcal{H}(4) = 6$, and $\mathcal{H}(4)$ has two connected components $\mathcal{H}^{\mathrm{hyp}}(4)$ and $\mathcal{H}^{\mathrm{odd}}(4)$ (see Kontsevich and Zorich [9]). In this paper, we will be focusing on the connected component $\mathcal{H}^{\mathrm{hyp}}(4)$ which consists of holomorphic 1–forms defined on hyperelliptic Riemann surfaces. Note that the unique zero of the 1–form is necessarily a Weierstrass point. Equivalently, we can consider $\mathcal{H}^{\mathrm{hyp}}(4)$ as the space of translation surfaces of genus 3 having only one singularity, such that there exists an isometric involution which has exactly 8 fixed points, and acts by $-\mathrm{Id}$ on the homology.

Before stating the main results of this paper, let us recall some basic definitions. On a translation surface, a *saddle connection* is a geodesic segment whose endpoints are

singularities of the surface, which may coincide. For surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$, a saddle connection is then a geodesic loop joining the unique singularity to itself. If γ is a saddle connection, we denote its length by $|\gamma|$. We can also associate to γ together with a choice of orientation a vector $V(\gamma) \in \mathbb{R}^2$ which is the integral of the holomorphic 1–form defining the flat metric along γ. In fact, the integral gives us a complex number that we regard as a vector in $\mathbb{R}^2$ by the standard identification $\mathbb{C} = \mathbb{R} \oplus \imath\mathbb{R}$.

Given a translation surface Σ, a *cylinder* in Σ is an open subset which is isometric to the quotient $\mathbb{R}\times]0; h[/\mathbb{Z}$, where $\mathbb{Z}$ is the cyclic group generated by $(x, y) \mapsto (x+\ell, y)$, and maximal with respect to this property. We will call h the *height*, and ℓ the *width* of C, and define the modulus of C to be the ratio h/ℓ. Note that none of the parameters h, ℓ, m are invariant under $\mathrm{SL}(2,\mathbb{R})$. By definition, we have a map from $\mathbb{R}\times]0; h[$ to Σ, which is locally isometric, with image C. This map can be extended by continuity to a map from $\mathbb{R}\times[0; h]$ to Σ. We call the images of $\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\{h\}$ under this map the *boundary components* of C. Each boundary component of C is a concatenation of saddle connections, and freely homotopic to the simple closed geodesics in C. Observe that the two boundary components of C are, in general, not disjoint subsets of Σ – they can even coincide. We call C a *simple cylinder* when each of its boundary components consists of only one saddle connection.

A direction θ in $\mathbb{S}^1$ is said to be *completely periodic* if Σ is the union of the closures of the cylinders in this direction, in other words, any trajectory of the flow in this direction, is either a closed geodesic or a saddle connection.

Theorem 1.1 *On* every *surface in $\mathcal{H}^{\mathrm{hyp}}(4)$, there always exist four pairs of homologous saddle connections $\delta_i^{\pm}$, $i = 1, \dots, 4$, such that*

- *$\delta_1^{\pm}$ bound a simple cylinder,*
- *$\delta_i^{\pm}$ and $\delta_{i+1}^{\pm}$ bound a topological disk which is isometric to a parallelogram in $\mathbb{R}^2$ for $i = 1, 2, 3$,*
- *$\delta_4^{\pm}$ bound a simple cylinder.*

The configuration of $\delta_1^{\pm}, \dots, \delta_4^{\pm}$ is shown in Figure 1.

Let Σ_0 be a surface in $\mathcal{H}_1^{\mathrm{hyp}}(4)$, and $\delta_i^{\pm}$, $i = 1, \dots, 4$, be as in Theorem 1.1. Cutting Σ along $\delta_3^{\pm}$, we get two connected components whose boundary consists of two geodesic segments. Gluing those geodesic segments together, we then get a flat torus, which will be denoted by Σ', and a surface in $\mathcal{H}(2)$. On the torus Σ', we denote the geodesic segment corresponding to $\delta_3^{\pm}$ by δ_3. As a subsurface of Σ, the torus Σ' inherits a parallel line field, therefore we can view it as a pair (M, ω), where M is a Riemann

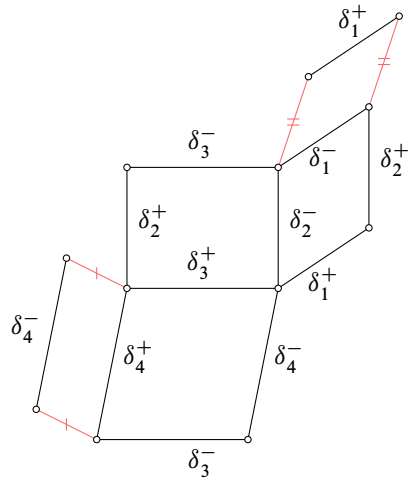

Figure 1: Decomposition into parallelograms and simple cylinders

surface of genus one, and ω is a nonzero holomorphic 1–form on M. Equivalently, we can identify Σ' with the quotient $\mathbb{R}^2/\Lambda$, where Λ is a lattice in $\mathbb{R}^2$, which is the image of the map $H_1(M,\mathbb{Z}) \longrightarrow \mathbb{C} \simeq \mathbb{R}^2$: $c \mapsto \int_c \omega$. A vector in $\mathbb{R}^2$ is said to be *generic* with respect to Λ if it is not collinear with any vector in Λ. We have

Theorem 1.2 *Suppose that $\delta_1^\pm$ and $\delta_3^\pm$ are parallel, that is $V(\delta_1^\pm)$ and $V(\delta_3^\pm)$ are colinear, and $V(\delta_3) = V(\delta_3^\pm)$ is generic with respect to Λ. Then $\mathrm{SL}(2,\mathbb{R})\cdot\Sigma_0$ is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$.*

Using this result, one can show that:

Corollary 1.3 *Let Σ_0 be a surface in $\mathcal{H}_1^{\mathrm{hyp}}(4)$. Suppose that the horizontal direction is completely periodic for Σ_0, and that Σ_0 is decomposed into three horizontal cylinders whose moduli are independent over $\mathbb{Q}$. Then $\mathrm{SL}(2,\mathbb{R})\cdot\Sigma_0$ is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$.*

Remark Since the condition that $V(\delta_1^\pm)$ and $V(\delta_3^\pm)$ are colinear are not generic, the set of surfaces in $\mathcal{H}_1^{\mathrm{hyp}}(4)$ satisfying the hypothesis of Theorem 1.2 has measure zero.

The proof of Theorem 1.1 relies on the action of the hyperelliptic involution on the surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$. The key ingredient of the proof is Lemma 2.1, which says that, on a translation surface of genus one or two, any saddle connection invariant under the distinguished involution of the surface is contained in a simple cylinder.

To prove Theorem 1.2, we will show that the orbit closure contains all the surfaces admitting a splitting as in Theorem 1.1 with $\delta_2^\pm$ parallel to $\delta_3^\pm$. Consequently, the orbit

closure contains all the Veech surfaces, and in particular all the square-tiled surfaces. Since the set of square-tiled surfaces is dense in $\mathcal{H}^{\text{hyp}}_1(4)$ (see for example Douady and Hubbard [3] or Kontsevich and Zorich [9]), we deduce that the orbit closure is the whole component. The proof of Theorem 1.2 uses a theorem of Ratner on action of unipotent subgroups on homogeneous spaces.

To prove Corollary 1.3, we prove that one can find in the $\mathrm{SL}(2,\mathbb{R})$–orbit closure of Σ_0 a surface which satisfies the condition of Theorem 1.2. Using Theorem 1.2, and Corollary 1.3, we can find numerous interesting examples of generic surfaces in $\mathcal{H}^{\text{hyp}}(4)$, in particular, surfaces with coordinates in any quadratic field over $\mathbb{Q}$. Therefore, we have an affirmative answer to a question of Hubert, Lanneau and Möller [7] (see Section 9). We will also construct explicitly some Thurston–Veech surfaces with trace field of degree three over $\mathbb{Q}$ which satisfy the hypothesis of Corollary 1.3.

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2 Simple cylinder invariant under the involution

2.1 Translation surfaces of genus one

Translation surfaces of genus one are simply flat tori. We denote by $\mathcal{H}(0)$ (resp. $\mathcal{H}(0,0)$) the space of triples (M,ω,p) (resp. quadruplet (M,ω,p_1,p_2)), where M is a Riemann surface of genus one (an elliptic curve), ω is a nonzero holomorphic 1–form on M, and p (resp. p_1 and p_2) is a marked point (resp. are marked points) of M. In both cases, we will call the lattice in $\mathbb{R}^2$ obtained by integrating ω along elements of $H_1(M,\mathbb{Z})$ the *associated lattice* of the considered translation surface. If Σ is an element of $\mathcal{H}(0)$ or $\mathcal{H}(0,0)$, we denote by $\Lambda(\Sigma)$ the lattice associated to Σ.

Note that the holomorphic 1–form determines a flat metric structure together with a choice of vertical direction at every point of the surface. For each surface in $\mathcal{H}(0)$, and $\mathcal{H}(0,0)$, we have a distinguished isometric involution which acts like $-\mathrm{Id}$ on the homology of the surface, and either fixes the unique marked point in the case of $\mathcal{H}(0)$, or exchanges the two marked points in the case of $\mathcal{H}(0,0)$. As usual, we will call a geodesic segment joining marked points a *saddle connection*. In the case of $\mathcal{H}(0)$, a saddle connection is just a simple closed geodesic passing through the marked point.

2.2 Saddle connection preserved by the involution

Let us first prove the following lemma, which is the key ingredient for the proof of Theorem 1.1:

Lemma 2.1 *Let γ be a saddle connection on a translation surface Σ which belongs to one of the following strata $\mathcal{H}(0), \mathcal{H}(0,0), \mathcal{H}(2), \mathcal{H}(1,1)$. Suppose that γ is invariant under the distinguished involution in the cases $\mathcal{H}(0)$ and $\mathcal{H}(0,0)$, or under the hyperelliptic involution in the cases $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$. Then there exists a pair of saddle connections (η^+, η^-) which bound a simple cylinder C containing γ. In the case $\mathcal{H}(0)$, we have $\eta^+ \equiv \eta^-$, in all other cases η^+ and η^- are distinct saddle connections.*

Proof We will prove this lemma case by case.

Case $\mathcal{H}(0)$ In this case γ is a simple closed geodesic passing through the marked point. Let η be any simple closed geodesic which meets γ only at the marked point. Then we can take $\eta^+ = \eta^- = \eta$.

Case $\mathcal{H}(0,0)$ In this case γ is a geodesic segment joining two marked points p_1, p_2 of Σ. Let τ denote the distinguished involution of Σ. If η^+ is a simple closed geodesic which meets γ only at p_1, then $\eta^- = \tau(\eta^+)$ is also a simple closed geodesic freely homotopic to η^+, and $\eta^+ \cup \eta^-$ is boundary of a simple cylinder containing γ which is invariant under τ.

Using the action of $\mathrm{SL}(2,\mathbb{R})$, we can assume that γ is horizontal. To see that there always exists a simple closed geodesic which meets γ only at p_1, we consider the vertical flow $\Psi_t, t \in \mathbb{R}$, on Σ. There exists a minimal value $t_0 > 0$ such that $\Psi_{t_0}(\gamma)$ intersects γ. Observe that $\Psi_{t_0}(\gamma)$, which is a segment of the same length and parallel to γ, must contain one endpoint of γ, without loss of generality, we can assume that p_1 is contained in $\Psi_{t_0}(\gamma) \cap \gamma$. By the definition of t_0, we have an isometric immersion Φ from the rectangle $R = [0; |\gamma|] \times [0; t_0] \subset \mathbb{R}^2$ into Σ whose restriction into $\mathrm{int}(R)$ is an embedding. We can suppose that Φ maps the lower side of R onto γ. Then $\Phi^{-1}(\{p_1\}) = \{\tilde{p}_1', \tilde{p}_1''\}$, where $\tilde{p}_1'$ is an endpoint of the lower side of R, and $\tilde{p}_1''$ is a point in the upper side of R. It is easy to see that Φ sends the segment in R joining $\tilde{p}_1'$ and $\tilde{p}_1''$ to a simple closed geodesic which meets γ only at p_1, and the lemma follows.

Case $\mathcal{H}(2)$ Let p denote the unique singularity of Σ, and τ the hyperelliptic involution of Σ. In this case γ is a geodesic segment joining p to itself, and invariant under τ. Note that τ reverse the orientation of γ, and since $\tau(p) = p$, it also fixes the midpoint of γ.

We can assume that γ is horizontal. As before, let $\Psi_t, t \in \mathbb{R}$, denote the vertical flow on Σ. The same argument as in the previous case shows that we have an immersion Φ from a rectangle $R \subset \mathbb{R}^2$ into Σ such that $\Phi_{|\,\text{int}(R)}$ is an embedding, Φ maps the lower side of R onto γ, and there exists a point $\widetilde{p}$ in the upper side of R which is mapped to p. Let $\widetilde{\Delta}$ denote the triangle whose vertices are $\widetilde{P}$ and the two endpoints of the lower side of R. Since $\widetilde{\Delta}$ is contained in R, the restriction $\Phi|_{\text{int}(\widetilde{\Delta})}$ is an embedding, moreover the images of the sides of $\widetilde{\Delta}$ by Φ are three distinct saddle connections, which meet one another only at p. Therefore, $\Delta = \Phi(\widetilde{\Delta})$ is an embedded triangle in Σ whose vertices coincide with p. By construction, γ is a side of Δ, let σ_1, σ_2 denote the two other sides. Let $\Delta', \sigma_1', \sigma_2'$ denote the images of $\Delta, \sigma_1, \sigma_2$ under τ respectively. Observe that Δ' is also an embedded triangle in Σ, and γ is a common side of Δ' and Δ. Here, we have two possibilities:

(1) Δ and Δ' have another common side other than γ, that is, either $\sigma_1' = \sigma_1$, or $\sigma_2 = \sigma_2'$. In this case $\Delta \cup \Delta'$ is a simple cylinder, and we are done.

(2) γ is the only common side of Δ and Δ'. In this case, $\Delta \cup \Delta'$ is an embedded parallelogram in Σ. Let us show that σ_1 and σ_1' bound a cylinder disjoint from $\Delta \cup \Delta'$. Recall the cone angle at p is 6π, and the action of τ at p is the rotation of angle 3π. Fix an orientation for γ, consider γ as a part of $\partial\Delta$ (resp. $\partial\Delta'$), we then have an orientation for σ_1, σ_2 (resp. σ_1', σ_2') subsequently. Consider a small disk D centered at p. The intersection of any oriented saddle connection with D is the union of an outgoing ray, and an incoming ray. These two rays specify a pair of angles at p, since Σ is a translation surface, this pair of angles is either $(\pi, 5\pi)$, or $(3\pi, 3\pi)$. Since γ is invariant under τ, the pair of angle specified by γ is $(3\pi, 3\pi)$, meanwhile the pair of angles specified by σ_1 is $(\pi, 5\pi)$ since $\sigma_1' = \tau(\sigma_1) \neq \sigma_1$.

We claim that the outgoing and the incoming rays of σ_1 are contained in the same half disk cut out by the outgoing and the incoming rays of γ. Indeed, suppose that the outgoing and the incoming rays of σ_1 do not belong to the same half disk (see Figure 2, Case (a)). Then by considering the sum of the angles in Δ, we see that the pair of angles specified by σ_2 is $(3\pi, 3\pi)$, which means that $\sigma_2 = \tau(\sigma_2) = \sigma_2'$, but this is excluded by the hypothesis.

We know that the action of τ on $H_1(\Sigma, \mathbb{Z})$ is $-\text{Id}$, which implies that $\sigma_1 - \sigma_1' = 0$ in $H_1(\Sigma, \mathbb{Z})$. It follows that σ_1 and σ_1' cut Σ into two connected components, each of which is equipped with a flat metric structure with piecewise geodesic boundary. Consider the connected component which does not contain γ. This component does not contain any singularity in its interior, and since the angle between the two rays of σ_1 at P measured inside this component is π, we deduce that there is no singularities in its boundary. The only flat surface with two geodesic boundary components with

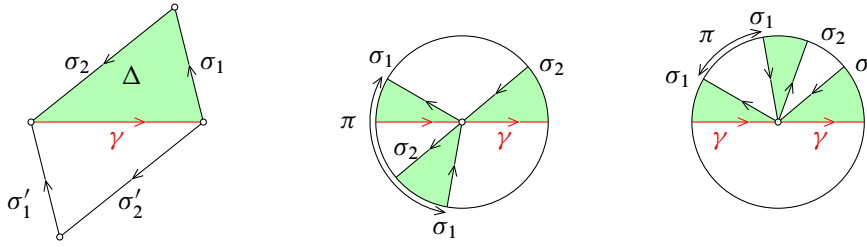


Figure 2: Configurations of geodesic rays at p

no singularities is a cylinder. Therefore, we can conclude that σ_1 and σ_1' bound a cylinder C disjoint from $\Delta \cup \Delta'$.

Consider the subsurface $\Sigma' = \Delta \cup \Delta' \cup C$ of Σ. We first observe that Σ' is invariant under τ. Topologically, Σ' is the complement in a torus of two open disks whose boundaries meet at one point. We can construct Σ' by gluing two parallelograms so that the restriction of τ into Σ' is realized by the central symmetries in both parallelograms. Elementary geometry shows that one can find a saddle connection η^+ in $\Delta \cup C$ which

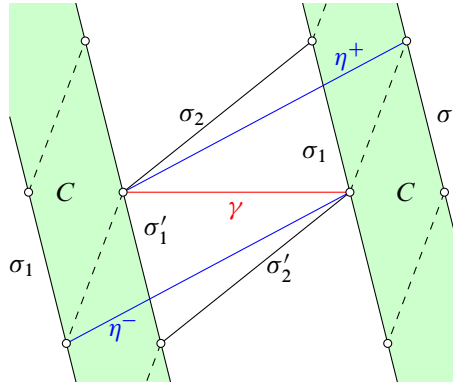


Figure 3: Existence of $\eta^{\pm}$

crosses σ_1 once (see Figure 3). Let η^- denote the image of η^+ under τ. Then η^+ and η^- bound a simple cylinder containing γ. The lemma is then proved for this case.

Case $\mathcal{H}(1,1)$ Let $\{p_1, p_2\}$ denote the singularities of Σ, the cone angles at both p_1 and p_2 are 4π. Recall that in this case, the hyperelliptic involution τ exchanges p_1 and p_2, therefore γ must be a saddle connection joining p_1 to p_2. Without loss of generality, we can assume that γ is horizontal. As we have seen in the previous cases, there exists an embedded triangle Δ in Σ bounded by γ and two other saddle

connections σ_1 and σ_2. Since there are two singularities, one of the two sides σ_1 and σ_2 must joint a singularity to itself, therefore we can assume that σ_1 joins p_1 to itself.

Let $\Delta', \sigma_1', \sigma_2'$ denote the images of $\Delta, \sigma_1, \sigma_2$ under τ respectively. Since τ exchanges p_1 and p_2, σ_1 and σ_1' are two distinct saddle connections. We choose the orientation for γ to be from p_1 to p_2, and choose the orientation of σ_1 and σ_2 (resp. σ_1' and σ_2') subsequently to get an orientation for the boundary of Δ (resp. Δ'). Consider two small disks D_1, D_2 centered at p_1, p_2 respectively. The intersection of σ_1 with D_1 consists of an outgoing ray and an incoming ray, while the intersection of γ with D_1 consists of only an out going ray. Let θ be the angle between the outgoing and the incoming rays of σ_1 measured along the sector of D_1 that does not meet γ (see Figure 4).

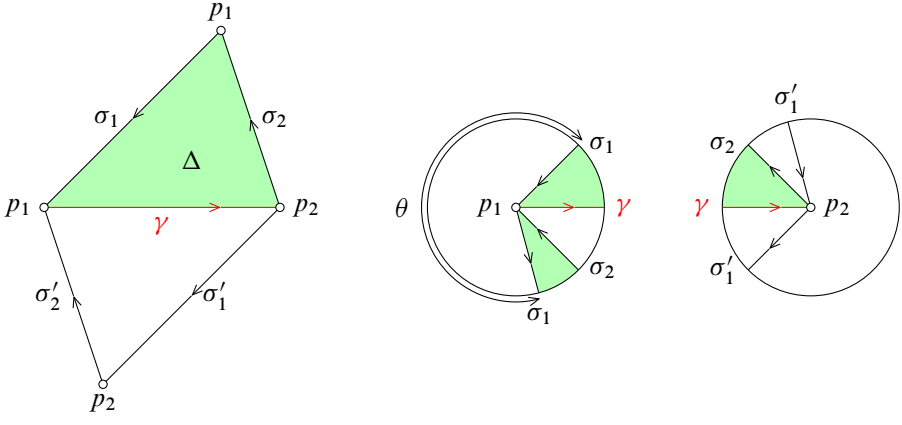


Figure 4: Configuration of geodesic rays at p_1 and p_2

We have two cases:

$\theta = 3\pi$: In this case, the angle between the two rays of σ_1 measured along the sector of D_1 which meets γ is π. A simple computation of angles shows that we must have $\sigma_2 = \sigma_2'$ as subset of Σ, which implies that $\Delta \cup \Delta'$ is actually a cylinder invariant under τ and bounded by σ_1 and σ_1', and the lemma follows immediately.

$\theta = \pi$: Since $\sigma_1 - \sigma_1' = 0$ in $H_1(\Sigma, \mathbb{Z})$, by cutting Σ along σ_1 and σ_1', we obtain two flat surfaces with piecewise geodesic boundary. Observe that the component which does not contain γ has no singularities in the interior, and since the angle between the two rays of σ_1 measured inside this surface is π, we see that it has no singularities in the boundary. It follows that this component is a cylinder C bounded by σ_1 and σ_1'. Now, using the same argument as in the case $\mathcal{H}(2)$, we see that there exists a pair of saddle connections $\eta^{\pm}$ in $\Delta \cup \Delta' \cup C$, which are exchanged by τ, and bound a simple cylinder containing γ. The proof of the lemma is now complete. □

3 Proof of Theorem 1.1

3.1 Existence of simple cylinder on hyperelliptic translation surfaces

To prove Theorem 1.1, we first show:

Lemma 3.1 *For any $g \geqslant 2$, on every surface of the stratum $\mathcal{H}^{\text{hyp}}(2g-2)$, there always exists a simple cylinder which is invariant under the hyperelliptic involution.*

Proof Let Σ be a surface in the stratum $\mathcal{H}^{\text{hyp}}(2g-2)$. A construction due to Veech [14] (see also [7]) allows us to construct Σ from a $2g$–gon P in $\mathbb{R}^2$ centered at the origin, and invariant under the central symmetry of $\mathbb{R}^2$. The polygon P is not necessarily convex, however it has a horizontal diagonal d which passes through the origin contained in the interior. Let A_0, B_0 denote the left and right endpoints of d respectively. We denote by $A_1, \ldots, A_{2g-1}$ (resp. $B_1, \ldots, B_{2g-1}$) the vertices of P above (resp. below) the diagonal d in the counterclockwise order. We consider by convention that $A_{2g} = B_0$, and $B_{2g} = A_0$. The surface Σ is obtained by identifying the opposite sides of P (see Figure 5).

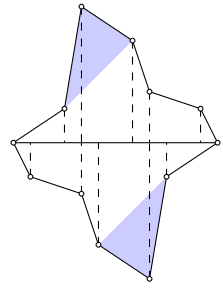


Figure 5: Construction of a hyperelliptic surface from a symmetric polygon

Let $y\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote the vertical coordinate function of $\mathbb{R}^2$. Let i_0 be the smallest index in $\{0, 1, \ldots, 2g-1\}$ so that $y(A_{i_0}) = \max\{y(A_1), \ldots, y(A_{2g-1})\}$. By the choice of i_0, we see that the diagonal $\overline{A_{i_0-1}A_{i_0+1}}$ is contained inside P. By symmetry, the diagonal $\overline{B_{i_0-1}B_{i_0+1}}$ is also contained inside P. Since the sides $\overline{A_{i_0-1}A_{i_0}}$ and $\overline{A_{i_0}A_{i_0+1}}$ are identified with $\overline{B_{i_0-1}B_{i_0}}$ and $\overline{B_{i_0}B_{i_0+1}}$ respectively, it follows that the union of the two triangles $\Delta^u = (A_{i_0-1}A_{i_0}A_{i_0+1})$ and $\Delta^l = (B_{i_0-1}B_{i_0}B_{i_0+1})$ is projected to a simple cylinder C of Σ. Now, the hyperelliptic involution of Σ corresponds to the central symmetry at the origin, which interchanges the two triangles Δ^u and Δ^l, therefore the hyperelliptic involution preserves C, and exchanges its boundary components. □

Remark This lemma is also true for surfaces in $\mathcal{H}^{\text{hyp}}(g-1, g-1)$.

3.2 Proof of Theorem 1.1

Let Σ be a surface in $\mathcal{H}^{\text{hyp}}(4)$, we denote by τ the hyperelliptic involution of Σ. By Lemma 3.1, we know that there exists a simple cylinder C_1 in Σ bounded by a pair of saddle connections (δ_1^+, δ_1^-) such that $\tau(C_1) = C_1$ and $\tau(\delta_1^+) = \delta_1^-$. Cutting off C_1 from Σ, we then get a surface whose boundary is a figure eight, ie the union of two circles meeting at one point. Splitting the common point of the two circles into two points, we get a surface whose boundary is the union of two geodesic segments corresponding to the pair (δ_1^+, δ_1^-). Gluing these two segments together, we then get a surface Σ' in $\mathcal{H}(1,1)$ with a marked saddle connection which will be denoted by δ_1.

We consider Σ' as a subsurface of Σ. Since τ preserves C_1, its restriction τ' to Σ' is the hyperelliptic involution of Σ'. Observe that δ_1 is invariant under τ'. By Lemma 2.1, we know that there exists a pair of saddle connections (δ_2^+, δ_2^-) which bound a simple cylinder C_2 containing δ_1. Again, we have that τ' preserves C_2 and exchanges δ_2^+ and δ_2^-. Note that since δ_2^+ and δ_2^- meet δ_1 at only the endpoints of δ_1, which are the singularities of Σ', we deduce that δ_2^+ and δ_2^- correspond to a pair of homologous saddle connections in the initial surface Σ.

Now, cut off C_2 from Σ', what is left is a surface with two boundary components corresponding to δ_2^+ and δ_2^-. Gluing the two boundary components so that the two singularities are identified, we get a surface in $\mathcal{H}(2)$ with a marked saddle connection, which is invariant by the hyperelliptic involution. Lemma 2.1 then allows us to continue the procedure until we are left with a simple cylinder. Since in each step, we cut out a simple cylinder, a simple computation on Euler character shows that we get to this situation after four steps. Thus, this procedure provides us with four pairs of homologous saddle connections (δ_i^+, δ_i^-), $i = 1, \dots, 4$, in Σ which satisfy the properties asserted in the statement of the theorem. □

Corollary 3.2 *There exists on* any *surface Σ in $\mathcal{H}^{\text{hyp}}(4)$ a pair of homologous saddle connections which are exchanged by the hyperelliptic involution, and decompose Σ into a union of a surface in $\mathcal{H}(2)$, and a surface in $\mathcal{H}(0,0)$. In both components of this decomposition, this pair of saddle connections corresponds to a saddle connection invariant under the distinguished involution.*

Proof Let (δ_i^+, δ_i^-), $i = 1, \dots, 4$, be the saddle connections in Σ satisfying the properties in Theorem 1.1. One can easily check that both pairs (δ_2^+, δ_2^-) and (δ_3^+, δ_3^-) satisfy the property asserted in the corollary. □

4 Splitting of surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$

4.1 Flat torus with a marked geodesic segment

Throughout this paper, by a "flat torus" we will mean a Riemann surface of genus one together with a nonzero holomorphic 1–form. Equivalently, we identify a flat torus with the quotient $\mathbb{C}/\Lambda$, where Λ is a lattice isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Using this identification, we can associate to any oriented geodesic s in the torus a vector $V(s)\in\mathbb{R}^2$. If $u=(x_1,y_1)$ and $v=(x_2,y_2)$ are two vectors in $\mathbb{R}^2$, we set $u\wedge v=x_1y_2-x_2y_1$. The following lemma is elementary, but will be useful for us in the sequel.

Lemma 4.1 *Let T be a flat torus, and s be a geodesic segment joining two distinct points p_1,p_2 in T. Let c be a simple closed geodesic passing through p_1, not parallel to s. Then p_1 is the unique intersection point of c and s if and only if $|V(s)\wedge V(c)|<\mathrm{Area}(T)$.*

Proof Using $\mathrm{SL}(2,\mathbb{R})$, we can assume that $V(c)$ is horizontal and $V(s)$ is vertical. Cutting T along c, we then get a cylinder C. Let h be the height of C. The fact that p_1 is the unique intersection point of s and c is equivalent to $|V(s)|<h$. Since

$$|V(s)\wedge V(c)|=|V(s)||V(c)|<h|V(c)|=\mathrm{Area}(C)=\mathrm{Area}(T),$$

the lemma follows immediately. □

4.2 The space of splittings

Let Σ be a surface in $\mathcal{H}^{\mathrm{hyp}}(4)$. We denote by p the unique singularity of Σ. Let $\delta_i^{\pm}$, $i=1,\dots,4$, be four pairs of saddle connections in Σ as in Theorem 1.1. Cutting Σ along (δ_1^+,δ_1^-) and (δ_3^+,δ_3^-), we get three following components:

- C_1 is a cylinder bounded by δ_1^+ and δ_1^-. Gluing δ_1^+ and δ_1^- together so that the two points corresponding to p are identified, we then get a surface in $\mathcal{H}(0)$ with a marked saddle connection.
- C_2 is an annulus equipped with a flat metric structure with piecewise geodesic boundaries, each boundary component of C_2 consists of two geodesic segments (corresponding to $\delta_1^+\cup\delta_3^+$, and $\delta_1^-\cup\delta_3^-$). Gluing δ_1^+ and δ_3^+ to δ_1^- and δ_3^- respectively, we then get an element of $\mathcal{H}(0,0)$, together with two saddle connections whose union is a simple closed curve.
- C_3 is a one holed flat torus, the boundary of C_3 is connected and consists of two geodesic segments corresponding to δ_3^+ and δ_3^-. Gluing these two segments together, we then get an element in $\mathcal{H}(0,0)$ together with a marked saddle connection.

Remark We get a similar decomposition of Σ by cutting along the pairs (δ_2^+, δ_2^-) and (δ_4^+, δ_4^-).

Let **Spl** denote the set of $(T_1, T_2, T_3, v_1, v_2)$, where $T_1 \in \mathcal{H}(0)$, $T_2, T_3 \in \mathcal{H}(0,0)$ and $v_i \in \mathbb{R}^2$, $i = 1, 2$, satisfy the following:

(a) There are a saddle connection in T_1 and a saddle connection in T_2 both have associated vector equal to v_1.

(b) There are a saddle connection in T_2, and a saddle connection in T_3 both have associated vector equal to v_2.

(c) $v = v_1 + v_2$ is a primitive vector of the lattice $\Lambda_2 = \Lambda(T_2)$ associated to T_2, and there exists another primitive vector w such that $\Lambda_2 = \mathbb{Z}v \oplus \mathbb{Z}w$ and $0 < |v_i \wedge w| < \text{Area}(T_2)$, $i = 1, 2$.

We denote by $\mathbf{Spl}_1$ the subset of **Spl** consisting of elements $(T_1, T_2, T_3, v_1, v_2)$ such that $\text{Area}(T_1) + \text{Area}(T_2) + \text{Area}(T_3) = 1$.

Remark

- We have a natural action of $\text{GL}^+(2, \mathbb{R})$ on **Spl**.
- Condition (c) implies in particular that, if we denote by s_1 and s_2 the saddle connections in T_2 corresponding v_1 and v_2 respectively, then there exists a pair of simple closed geodesics c^+, c^- passing through the two marked points such that $V(c^+) = V(c^-)$, and $c^\pm$ cut T_2 into two cylinders, each of which contains one of the two saddle connections s_1, s_2.

Given an element $(T_1, T_2, T_3, v_1, v_2)$ in **Spl**, we construct a surface in $\mathcal{H}^{\text{hyp}}(4)$ as follows:

- Cutting T_1 along the saddle connection corresponding to v_1, we get a cylinder C_1.
- Let s_1, s_2 be the saddle connections in T_2 corresponding to v_1 and v_2 respectively. Since $v_1 + v_2$ is a primitive vector in $\Lambda(T_2)$, we see that $s_1 \cup s_2$ is a simple closed curve in T_2. Cutting T_2 along s_1 and s_2, we then get a cylinder with piecewise geodesic boundary, which will be denoted by C_2.
- Slitting open T_3 along saddle connection corresponding to v_2, we get a one holed torus which will be denoted by C_3.
- We can now glue C_1, C_2, C_3 together following the model shown in Figure 1 so that all the marked points are identified, we then get a surface in $\mathcal{H}^{\text{hyp}}(4)$.

This construction provides us with a map Ψ: $\mathbf{Spl} \longrightarrow \mathcal{H}^{\mathrm{hyp}}(4)$. A direct consequence of Theorem 1.1 is the following:

Proposition 4.2 *The map Ψ is surjective, locally homeomorphic, and $\mathrm{GL}^+(2,\mathbb{R})$–equivariant.*

4.3 Special splitting

Let $X = (T_1, T_2, T_3, v_1, v_2)$ be an element of **Spl**, we say that X is a *special splitting* if v_1 and v_2 are parallel (collinear). We denote by **SSpl** the set of special splittings in **Spl**, and by $\mathbf{SSpl}_1$ the intersection $\mathbf{SSpl} \cap \mathbf{Spl}_1$.

Consider a point $X = (T_1, T_2, T_3, v_1, v_2)$ in **SSpl**, we denote by Λ_i, $i = 1, 2, 3$, the lattices associated to T_i. Let C_1 (resp. C_2) denote the cylinder obtained by cutting T_1 (resp. T_2) along the saddle connection corresponding to v_1 (resp. along the union of the saddle connections corresponding to v_1 and v_2). Let m_i, $i = 1, 2$, denote the modulus of C_i, we will call m_1 (resp. m_2) the modulus of the pair (T_1, v_1) (resp. of the pair $(T_2, v_1 + v_2)$). By construction, C_1 and C_2 are isometric to two cylinders in the direction v_1 on the surface $\Sigma = \Psi(X)$. Set

$$\alpha = \frac{|v_2|}{|v_1|} \quad \text{and} \quad \bar{m} = \frac{m_1}{m_2}.$$

Observe that we have the following relation between $\bar{m}$ and α:

$$\bar{m} = \frac{\mathrm{Area}(T_1)}{\mathrm{Area}(T_2)}(1+\alpha)^2.$$

Since α and the areas of T_i are $\mathrm{SL}(2,\mathbb{R})$–invariant, so is $\bar{m}$. We will call $\bar{m}$ the moduli ratio of $(T_1, T_2, T_3, v_1, v_2)$.

Using $\mathrm{SO}(2,\mathbb{R})$, we can assume that C_1 and C_2 are horizontal. We can also define the twists for C_1 and C_2 as follows: let $w_1 = (w_1^x, w_1^y)$ (resp. $w_2 = (w_2^x, w_2^y)$) be a primitive vector in Λ_1 (resp. Λ_2) such that $\Lambda_1 = \mathbb{Z}v_1 \oplus \mathbb{Z}w_1$ (resp. $\Lambda_2 = \mathbb{Z}(v_1 + v_2) \oplus \mathbb{Z}w_2$). We define the twists t_1, t_2 of C_1 and C_2 respectively to be

$$t_1 = \frac{w_1^x}{|v_1|} \mod \mathbb{Z} \quad \text{and} \quad t_2 = \frac{w_2^x}{|v_1| + |v_2|} \mod \mathbb{Z}.$$

We also call t_1 (resp. t_2) the twist of the pair (T_1, v_1) (resp. of the pair $(T_2, v_1 + v_2)$).

Recall that a vector w in $\mathbb{R}^2$ is generic with respect to a lattice $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$ if w is not parallel to any vector in Λ. We first prove a slightly weaker version of Theorem 1.2, where we suppose in addition that the moduli ratio is irrational. As we will see, this condition can be relaxed, and we obtain Theorem 1.2 as a consequence of this theorem.

Theorem 4.3 *Let $X_0 = (T_1^0, T_2^0, T_3^0, v_1^0, v_2^0)$ be an element in $\mathbf{SSpl}_1$. Let Λ_i^0, $i = 1, 2, 3$, denote the lattice associated to T_i^0, and $\bar{m}_0$ denote the moduli ratio of X_0. Suppose that*

- *$\bar{m}_0 \notin \mathbb{Q}$,*
- *v_2^0 is generic with respect to Λ_3^0.*

Then $\mathcal{O} := \mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)$ is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$.

4.4 Ratner's Theorem

The first important ingredient of the proof of Theorem 4.3 is a consequence of the famous theorem of Ratner on action of unipotent subgroups on homogeneous spaces. Before stating this theorem, let us first recall some basic notions. Let G be a Lie group, and $\mathfrak{g}$ be its Lie algebra. An element g of G is *unipotent* if $\mathrm{Ad}_g - \mathrm{Id}$ is nilpotent in $\mathrm{End}(\mathfrak{g})$. Let λ denote the right Haar measure of G, G is called *unimodular* if the left Haar measure equals the right Haar measure, or equivalently if $|\det \mathrm{Ad}_g| = 1$ for all g in G. A discrete subgroup Γ of G is called a *lattice* if we have $\lambda(G/\Gamma) < \infty$. If G has a lattice then it is unimodular. It is well-known that $\mathrm{SL}(2,\mathbb{R})$ is unimodular, but its subgroup consisting upper triangular matrices is not.

Theorem 4.4 (Ratner) *Let G be a finite dimensional Lie group, Γ be a lattice in G, and $X = G/\Gamma$. Let U be a connected subgroup of G generated by unipotent element. Then for any x in X, the closure $\overline{U \cdot x}$ of the U–orbit of x is a homogeneous space of finite volume, that is there exists a closed unimodular subgroup $H \subset G$ containing U such that*

- *$\overline{U \cdot x} = H \cdot x$,*
- *$x\Gamma x^{-1} \cap H$ is a lattice in H.*

Put $G = \mathbb{R} \times \mathbb{R} \times \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \mathrm{SL}(2,\mathbb{Z})$. Then Γ is a lattice in G. An element of G/Γ is a triple $(\theta_1, \theta_2, \Lambda)$, where $\theta_i \in \mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$, and $\Lambda \simeq \mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$ such that $\mathrm{Vol}(\mathbb{R}^2/\Lambda) = 1$. Let m_1, m_2 be two positive real numbers. We set

$$U = U_{m_1,m_2} = \left\{ \left(m_1 t, m_2 t, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right),\ t \in \mathbb{R} \right\},$$

then U is a unipotent subgroup of G. As we will see, the space $X = G/\Gamma$ can be used to parametrize the subspace of space of special splittings consisting of splittings such that $v_1 = (1, 0)$, $v_2 = (\alpha, 0)$, α is a constant, and the areas of the tori are fixed.

In this parametrization, the action of U_{m_1,m_2} defined above corresponds to the action of the unipotent subgroup

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},$$

of $\mathrm{SL}(2,\mathbb{R})$. Applying Theorem 4.4 in this case, we have:

Corollary 4.5 *Suppose that $m_1/m_2 \notin \mathbb{Q}$. Let Λ be a lattice in $\mathbb{R}^2$ which contains no horizontal vectors. Then for any $(\theta_1,\theta_2) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, we have*

$$\overline{U \cdot (\theta_1,\theta_2,\Lambda)} = G/\Gamma.$$

Proof By Ratner Theorem, we know that $\overline{U \cdot (\theta_1,\theta_2,\Lambda)} = H \cdot (\theta_1,\theta_2,\Lambda)$, where H is connected, unimodular subgroup of G. All we need to show is that $H = G$.

Let x be any element of G which is projected to $(\theta_1,\theta_2,\Lambda)$. Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of H and G respectively. Set

$$\mathbf{a} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have $\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}\mathbf{u}_+ \oplus \mathbb{R}\mathbf{u}_- \oplus \mathbb{R}\mathbf{a}$, and $\mathfrak{g} = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$. Observe that the Lie bracket of $\mathfrak{g}$ is trivial on the $\mathbb{R}$ components, and we have

$$[\mathbf{u}_+,\mathbf{u}_-] = \mathbf{a}, [\mathbf{a},\mathbf{u}_+] = 2\mathbf{u}_+, [\mathbf{a},\mathbf{u}_-] = -2\mathbf{u}_-.$$

Since $U \subset H$, the Lie algebra $\mathfrak{h}$ contains $\mathbf{v}_0 = (m_1,m_2,\mathbf{u}_+)$. Observe that $x\Gamma x^{-1} = \mathbb{Z}\times\mathbb{Z}\times M \cdot \mathrm{SL}(2,\mathbb{Z}) \cdot M^{-1}$, where M is any matrix in $\mathrm{SL}(2,\mathbb{R})$ sending the standard basis of $\mathbb{R}^2$ to a basis of the lattice Λ. We denote by A and N be the following subgroups of $\mathrm{SL}(2,\mathbb{R})$:

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t \in \mathbb{R} \right\}.$$

Let $\mathrm{pr}_2\colon \mathfrak{g} \longrightarrow \mathfrak{sl}(2,\mathbb{R})$ denote the natural projection. The image of $\mathfrak{h}$ under pr_2 is a subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ which contains $\mathbb{R}\mathbf{u}_+$.

Case 1 $\mathrm{pr}_2(\mathfrak{h}) = \mathbb{R}\mathbf{u}_+$. We have three possibilities:

- $\mathfrak{h} = \mathbb{R}\mathbf{v}_0$ implies $H = U$, but by assumption, $U \cap \mathbb{Z}\times\mathbb{Z}\times M\cdot\mathrm{SL}(2,\mathbb{Z})\cdot M^{-1} = \{(0,0,\mathrm{Id})\}$ is not a lattice in U.
- $\mathfrak{h} = \mathbb{R}\mathbf{v}_0 \oplus \mathbb{R}\mathbf{u}_+ = \mathbb{R}\mathbf{u}_+ \oplus \mathbb{R}_{m_1,m_2}$, where $\mathbb{R}_{m_1,m_2} = \mathbb{R}\cdot(m_1,m_2) \subset \mathbb{R}^2$. It follows that $H = \mathbb{R}_{m_1,m_2} \times N$. But again, $H \cap \mathbb{Z}\times\mathbb{Z}\times M\cdot\mathrm{SL}(2,\mathbb{Z})\cdot M^{-1} = \{(0,0,\mathrm{Id})\}$.
- $\mathfrak{h} = \mathbb{R}^2 \oplus \mathbb{R}\mathbf{u}_+ \Longrightarrow H = \mathbb{R}^2 \times N$. But we have $N \cap M\cdot\mathrm{SL}(2,\mathbb{Z})\cdot M^{-1} = \{\mathrm{Id}\}$, therefore, $H \cap \mathbb{Z}\times\mathbb{Z}\times M\cdot\mathrm{SL}(2,\mathbb{Z})\cdot M^{-1}$ is not a lattice.

Case 2 $\mathrm{pr}_2(\mathfrak{h}) = \mathbb{R}\mathbf{u}_+ \oplus \mathbb{R}\mathbf{a}$. Let v be any vector in $\mathfrak{h}$ such that $\mathrm{pr}_2(\mathbf{v}) = \mathbf{a}$. Then we have $[\mathbf{v}, \mathbf{v}_0] = 2\mathbf{u}_+$. Therefore, we see that $\mathfrak{h}$ contains the vectors

- $\mathbf{u}_+$,
- $\mathbf{v}_1 = \mathbf{a} + \mathbf{w}$, with $\mathbf{w} = (k_1, k_2) \in \mathbb{R}^2$,
- $\mathbf{w}_0 = \mathbf{v_0} - \mathbf{u}_+ = (m_1, m_2) \in \mathbb{R}^2$.

Here we have two possibilities:

- $\mathfrak{h} = \mathbb{R}\mathbf{w}_0 \oplus \mathbb{R}\mathbf{u}_+ \oplus \mathbb{R}\mathbf{v}_1 \Longrightarrow H = \mathbb{R} \times A'N$, where
$$A' = \left\{ \left(k_1 t, k_2 t, \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right),\ t \in \mathbb{R} \right\} \subset G.$$
It follows that $H \simeq \mathbb{R} \times AN$. But since AN is not unimodular, neither is H.
- $\mathfrak{h} = \mathbb{R}^2 \oplus \mathbb{R}\mathbf{u}_+ \oplus \mathbb{R}\mathbf{a}$, therefore $H = \mathbb{R}^2 \times AN$, but again H is not unimodular.

Case 3 $\mathrm{pr}_2(\mathfrak{h}) = \mathfrak{sl}(2, \mathbb{R})$. Let $\mathbf{v}$ be a vector in $\mathfrak{h}$ such that $\mathrm{pr}_2(\mathbf{v}) = \mathbf{u}_-$. Then
$$[\mathbf{v}_0, \mathbf{v}] = \mathbf{a},\ [[\mathbf{v}_0, \mathbf{v}], \mathbf{v_0}] = 2\mathbf{u}_+ \text{ and } [[\mathbf{v}_0, \mathbf{v}], \mathbf{v}] = -2\mathbf{u}_-.$$
It follows that $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{h}$. We then have two possibilities:

- $H = \mathbb{R}_{m_1,m_2} \times \mathrm{SL}(2, \mathbb{R})$, in this case $H \cap \mathbb{Z} \times \mathbb{Z} \times M \cdot \mathrm{SL}(2, \mathbb{Z}) \cdot M^{-1} = (0, 0, M \cdot \mathrm{SL}(2, \mathbb{Z}) \cdot M^{-1})$ is not a lattice in H.
- $H = \mathbb{R}^2 \times \mathrm{SL}(2, \mathbb{R})$, this is the only admissible possibility.

We can then conclude that $\overline{U \cdot (\theta_1, \theta_2, \Lambda)} = G/\Gamma$. □

Remark Similar results for $\mathbb{R}^k \times \mathrm{SL}(2, \mathbb{R})^n / \mathbb{Z}^k \times \mathrm{SL}(2, \mathbb{Z})^n$ with small k and n can be found in [7].

For any (A_1, A_2, A_3, α) in $\mathbb{R}^4_{>0}$, let $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)$ denote the subset of **SSpl** consisting of elements $(T_1, T_2, T_3, v_1, v_2)$ such that $\mathrm{Area}(T_i) = A_i$, $i = 1, 2, 3$, and $|v_2|/|v_1| = \alpha$. Using Corollary 4.5, we have the following lemma:

Lemma 4.6 *Let $X = (T_1, T_2, T_3, v_1, v_2)$ be an element in $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)$. If $(A_1/A_2)(\alpha+1)^2$ is irrational, and v_2 is generic with respect to the lattice $\Lambda_3 = \Lambda(T_3)$ then*
$$\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X)}.$$
In particular, let $X_0 = (T_1^0, T_2^0, T_3^0, v_1^0, v_2^0)$ be as in Theorem 4.3. Then
$$\Psi(\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)},$$
where $A_i^0 = \mathrm{Area}(T_i^0)$, $i = 1, 2, 3$, and $\alpha_0 = |v_2^0|/|v_1^0|$.

Proof Let $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}}$ denote the subset of $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)$ consisting of elements $(T_1, T_2, T_3, v_1, v_2)$ with $v_1 = (1, 0)$. Observe that

$$\mathbf{SSpl}(A_1, A_2, A_3, \alpha) = \mathrm{SL}(2, \mathbb{R}) \cdot \mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}}.$$

We have a natural mapping φ: $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}} \longrightarrow G/\Gamma$ which sends a special splitting $(T_1, T_2, T_3, v_1, v_2)$ to an element $(t_1, t_2, \Lambda) \in G/\Gamma$, where t_1 and t_2 are the twists of (T_1, v_1) and $(T_2, v_1 + v_2)$ respectively, and Λ is the lattice associated to T_3 normalized to have covolume one. Remark that φ is a homeomorphism onto its image, and that $\varphi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}})$ is an open dense set of G/Γ, its complement consists of triples $(\theta_1, \theta_2, \Lambda)$ where the lattice Λ contains a horizontal vector of length $\leqslant \alpha/\sqrt{A_3}$ (this set corresponds to the case where $v_2 = \lambda v$ with $v \in \Lambda(T_3)$, and $\lambda \geqslant 1$).

Let m_1 and m_2 denote the moduli of (T_1, v_1) and $(T_2, v_1 + v_2)$ respectively. Recall that we have $m_1/m_2 = (A_1/A_2)(\alpha + 1)^2$. Clearly, the set $\mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}}$ is invariant under the action of the group

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t \in \mathbb{R} \right\}.$$

We define the action of U on G/Γ using the identification $U \simeq U_{m_1, m_2}$. It follows that φ is U–equivariant.

Without loss of generality, we can assume that $X \in \mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}}$. Consider $x = \varphi(X) \in G/\Gamma$. The hypothesis on X implies that x satisfies the conditions of Corollary 4.5, therefore $\overline{U \cdot x} = G/\Gamma$. Since φ is U–equivariant and a local homeomorphism, we deduce that $\overline{U \cdot X} = \varphi^{-1}(\overline{U \cdot x}) = \mathbf{SSpl}(A_1, A_2, A_3, \alpha)_{\text{hor}}$, and the lemma follows. □

5 Surfaces admitting special splitting are contained in the orbit closure

Our aim in this section is to prove the following

Proposition 5.1 *Let $X_0 = (T_1^0, T_2^0, T_3^0, v_1^0, v_2^0)$ be as in Theorem 4.3. We have $\Psi(\mathbf{SSpl}_1) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)}$.*

5.1 Dual splitting

Given $X = (T_1, T_2, T_3, v_1, v_2)$ in $\mathbf{SSpl}$, we will denote both saddle connections in T_1 and T_2 corresponding to v_1 by δ_1, similarly, we denote by δ_2 the two saddle

connections in T_2 and T_3 corresponding to v_2. Recall that the saddle connections δ_i, $i = 1, 2$, give rises to a pair of homologous saddle connections in the surface $\Sigma = \Psi(X)$, which will be denoted by $\delta_i^{\pm}$.

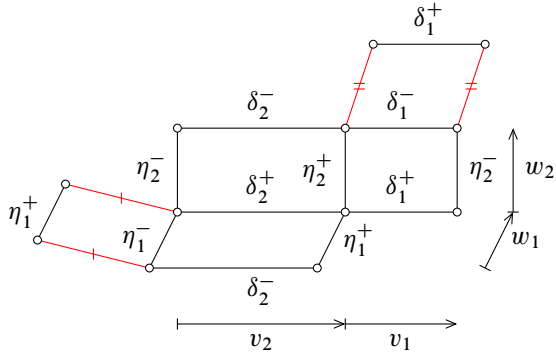


Figure 6: Dual splittings

Let η_1 be a simple closed geodesic in T_3 which meets δ_2 once, and let $\eta_1^{\pm}$ denote the pair of saddle connections parallel to η_1. Similarly, let η_2 be a simple closed geodesics in T_2 which meets $\delta_1 \cup \delta_2$ once, and let $\eta_2^{\pm}$ denote the pair of saddle connections parallel to η_2. Observer that $\eta_1^{\pm}$ (resp. $\eta_2^{\pm}$) are homologous saddle connections in Σ. We choose the orientation of η_1 and η_2 so that $\eta_1^+ * \eta_2^+$ is freely homotopic to a simple closed curve in Σ. Set $w_1 = V(\eta_1), w_2 = V(\eta_2)$.

Cutting Σ along $\eta_1^{\pm}$ and $\eta_2^{\pm}$, we see that the surface Σ is obtained from another element $X^\vee = (T_1^\vee, T_2^\vee, T_3^\vee, w_1, w_2)$ in **Spl**. We will call $X^\vee$ a *dual splitting* of X. Note that $X^\vee$ does not belong to **SSpl** in general, and there are infinitely many splittings dual to a given splitting. We also have

$$\mathrm{Area}(T_3^\vee) = \mathrm{Area}(T_1) + \frac{\mathrm{Area}(T_2)}{1 + |v_2|/|v_1|}. \tag{1}$$

Throughout this section, we set $A_i^0 = \mathrm{Area}(T_i^0)$, $i = 1, 2, 3$, and $\alpha_0 = |v_2^0|/|v_1^0|$.

5.2 Changing splitting

The first step to prove Proposition 5.1 is the following:

Lemma 5.2 *If* $(A_1, A_2, A_3, \alpha) \in \mathbb{R}^4_{>0}$ *satisfies*

$$A_1 + A_2 + A_3 = 1,$$
$$A_1 + \frac{A_2}{1+\alpha} = A_1^0 + \frac{A_2^0}{1+\alpha_0},$$

then $\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$.

Remark We know $\overline{\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X_0)}$ contains $\Psi(\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0))$ by Lemma 4.6. Let $X = (T_1, T_2, T_3, v_1, v_2)$ be an element in $\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)$, and Σ be the surface in $\mathcal{H}^{\mathrm{hyp}}(4)$ constructed from X. Let $\delta_i, \delta_i^{\pm}, \eta_i, \eta_i^{\pm}$, $i = 1, 2$, and $X^\vee = (T_1^\vee, T_2^\vee, T_3^\vee, w_1, w_2)$ be as in the previous subsection, where $X^\vee$ is a dual splitting of X.

Let $\sigma_1^{\pm}$ (resp. $\sigma_2^{\pm}$) be a pair of homologous saddle connections in $T_3^\vee$ (resp. $T_2^\vee$) which bound a simple cylinder containing η_2 (see Figure 7). Viewed as saddle connections of Σ, the pairs $\sigma_1^{\pm}$ and $\sigma_2^{\pm}$ determine a splitting of Σ. If $\sigma_1^{\pm}$ and $\sigma_2^{\pm}$ are parallel, then we have another special splitting of Σ. To prove the lemma, we will show that for any (A_1, A_2, A_3, α) in $\mathbb{R}^4_{>0}$, there exists an element X in $\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)$ for which one can find $\sigma_1^{\pm}, \sigma_2^{\pm}$ determining a special splitting with parameters (A_1, A_2, A_3, α). We can then use Lemma 4.6 to conclude, first, for (A_1, A_2, A_3, α) satisfying the irrational condition of Lemma 4.6, and then for all (A_1, A_2, A_3, α) by continuity.

Proof of Lemma 5.2 Without loss of generality, we can assume that $v_1 = (1, 0)$ and $v_2 = (\alpha_0, 0)$. Let C_1 (resp. C_2) denote the cylinder obtained by slitting T_1 (resp. T_2) along saddle connection δ_1 (resp. along the saddle connections δ_1 and δ_2). Let h_i and t_i denote the height and the twist of C_i, $i = 1, 2$. Note that $h_1 = A_1^0$, and $h_2 = A_2^0/(\alpha_0 + 1)$. We fix $t_2 = 0$, consequently, we can choose $\eta_2^{\pm}$ to be vertical, and therefore $w_2 = V(\eta_2^{\pm}) = (0, h_2)$.

Set $\Lambda_i = \Lambda(T_i)$ and $\Lambda_i^\vee = \Lambda(T_i^\vee)$, $i = 1, 2, 3$. Recall that v_1 is a primitive vector of Λ_1, let $u_1 = (x, h_1)$ be another primitive vector in Λ_1 such that $\Lambda_1 = \mathbb{Z}u_1 \oplus \mathbb{Z}v_1$. Observe that the parameter x can be chosen arbitrarily. Similarly, $w_1 = (y, z)$ is a primitive vector in $\Lambda_1^\vee$, let $\widehat{u}_1$ be another primitive vector such that $\Lambda_1^\vee = \mathbb{Z}w_1 \oplus \mathbb{Z}\widehat{u}_1$. Note that $\Lambda_2^\vee = \mathbb{Z}v_2 \oplus \mathbb{Z}(w_1 + w_2)$ and $\Lambda_3 = \mathbb{Z}w_1 \oplus \mathbb{Z}(v_2 + \widehat{u}_1)$. The parameters $(x, y, z, \widehat{u}_1) \in \mathbb{R}^3 \times \mathbb{R}^2$ uniquely determine the element X in $\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)$. By construction, the parameters $(x, y, z, \widehat{u}_1)$ must satisfy the conditions

$$|v_2 \wedge w_2| < \mathrm{Area}(T_2^\vee) = |v_2 \wedge (w_1 + w_2)| < 1 - \mathrm{Area}(T_3^\vee) = \frac{\alpha_0 A_2^0}{\alpha_0 + 1} + A_3^0, \tag{2}$$

$$|w_1 \wedge (v_2 + \widehat{u}_1)| = A_3^0. \tag{3}$$

Simple computations show that (2) is equivalent to

$$0 < z < \frac{A_3^0}{\alpha_0}. \tag{4}$$

Note that the conditions (2) and (3) are sufficient, that is, if the parameters $(x, y, z, \widehat{u}_1)$ satisfy these two conditions, then they determine an element in $\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)$.

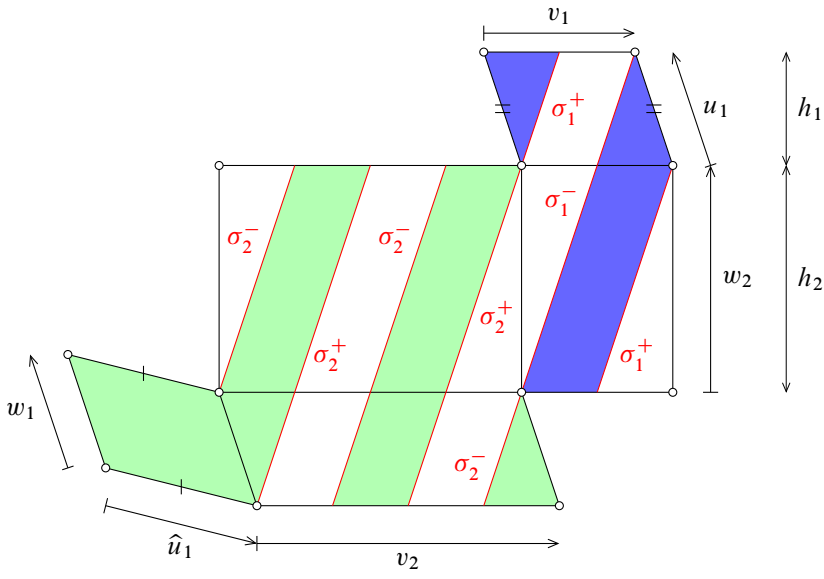


Figure 7: Finding new special splittings

Claim 1 *For any $(A_1, A_2, A_3, \alpha) \in \mathbb{R}^4_{>0}$ satisfying the conditions of the lemma, there exist $(x, y, z) \in \mathbb{R}^3$ with z satisfying (4) such that we can find a primitive vector v_1' in $\Lambda_3^\vee$, and a primitive vector v_2' in $\Lambda_2^\vee$ such that*

(i) $v_2' = \alpha v_1'$,

(ii) $|v_1' \wedge w_2| = A_2/(\alpha+1)$,

(iii) $|v_2' \wedge w_2| < \text{Area}(T_2^\vee)$.

Proof of Claim 1 Recall that, by assumption, we have

$$A_1 + \frac{A_2}{\alpha+1} = A_1^0 + \frac{A_2^0}{\alpha_0+1}.$$

Since $A_1 + A_2 < A_1^0 + A_2^0 + A_3^0 = 1$, it follows that

$$\frac{\alpha A_2}{\alpha+1} < \frac{\alpha_0 A_2^0}{\alpha_0+1} + A_3^0. \tag{5}$$

From (5), we deduce that there exist $p, q \in \mathbb{N}$, $p > 0, q > 0$, such that

$$\max\left\{\frac{\alpha_0 A_2^0/(\alpha_0+1)}{\alpha\alpha_0(h_1+h_2)}, \frac{\alpha A_2/(\alpha+1)}{\alpha\alpha_0(h_1+h_2)}\right\} < \frac{p}{q} < \frac{\alpha_0 A_2^0/(\alpha_0+1) + A_3^0}{\alpha\alpha_0(h_1+h_2)}.$$

Set

$$x=\frac{1}{p}\left(\frac{A_2}{h_2(\alpha+1)}-1\right),\quad y=\frac{1}{q}\left(\frac{\alpha A_2}{h_2(\alpha+1)}-\alpha_0\right),\quad z=\alpha\frac{p}{q}(h_1+h_2)-h_2.$$

By the choice of p,q, it is straight forward to verify that z satisfies (4). We have

$$u_1=(x,h_1)=\left(\frac{A_2}{ph_2(\alpha+1)}-\frac{1}{p},h_1\right),$$
$$w_1=(y,z)=\left(\frac{\alpha A_2}{qh_2(\alpha+1)}-\frac{\alpha_0}{q},\alpha\frac{p}{q}(h_1+h_2)-h_2\right).$$

Set
$$v_1'=v_1+p(u_1+w_2)=\left(\frac{A_2}{h_2(\alpha+1)},p(h_1+h_2)\right)$$
$$v_2'=v_2+q(w_1+w_2)=\alpha\left(\frac{A_2}{h_2(\alpha+1)},p(h_1+h_2)\right).$$

Since $\Lambda_3^\vee$ is generated by v_1 and u_1+w_2, we see that v_1' is a primitive vector in $\Lambda_3^\vee$, similarly, v_2' is a primitive vector in $\Lambda_2^\vee$. Clearly, we have $v_2'=\alpha v_1'$, hence (i) is satisfied. We have

$$|v_1'\wedge w_2|=\begin{vmatrix}A_2/(h_2(\alpha+1)) & 0\\ p(h_1+h_2) & h_2\end{vmatrix}=\frac{A_2}{\alpha+1},$$

therefore (ii) is satisfied. Next, we have

$$\operatorname{Area}(T_2^\vee)=|v_2\wedge(w_1+w_2)|=\alpha\alpha_0\frac{p}{q}(h_1+h_2),$$

and
$$|v_2'\wedge w_2|=\alpha|v_1'\wedge w_2|=\frac{\alpha A_2}{\alpha+1}.$$

By the choice of p,q, we have $|v_2'\wedge w_2|<\operatorname{Area}(T_2^\vee)$, hence (iii) is satisfied. □

Claim 2 *Given (x,y,z) as in Claim 1, there exist $\hat{u}_1$ satisfying (3) such that v_2' is generic with respect to the lattice $\mathbb{Z}w_1\oplus\mathbb{Z}(v_2'+\hat{u}_1)$.*

Proof of Claim 2 Since

$$|v_2'\wedge w_1|=\alpha_0(h_1+h_2)\frac{p}{q}-\frac{A_2}{\alpha+1}>0,$$

we deduce that $\{v_2',w_1\}$ is a basis of $\mathbb{R}^2$. Therefore, we can write $\hat{u}_1=\lambda w_1+\mu v_2'$. Observe that, once w_1 is fixed, the set of $\hat{u}_1$ satisfying (3) is parameterized by $\lambda\in\mathbb{R}$, with fixed μ.

Observe that v_2' is parallel to a vector in $\mathbb{Z}w_1\oplus\mathbb{Z}(v_2'+\hat{u}_1)$ if and only if $\lambda\in\mathbb{Q}$. Indeed, suppose that $v_2'=\lambda'(mw_1+n(v_2'+\hat{u}_1))$, with $m,n\in\mathbb{Z}$. Then we must

have $n \neq 0$, otherwise v_2' and w_1 are colinear, therefore $\widehat{u}_1 = -(m/n)w_1 + \lambda'' v_2'$. It follows immediately that there exist $\widehat{u}_1$ satisfying (3) such that v_2' is generic with respect to $\mathbb{Z}w_1 \oplus \mathbb{Z}(v_2' + \widehat{u}_1)$. □

Let us now show that the lemma will follow from Claims 1 and 2. Choose (x, y, z) as in Claim 1, and choose $\widehat{u}_1$ as in Claim 2. Then the parameters $(x, y, z, \widehat{u}_1)$ give us an element X in $\mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)$. We have $\Sigma = \Psi(X) \in \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$.

Let $\sigma_1^{\pm}$ (resp. $\sigma_2^{\pm}$) be the pair of saddle connections in $T_3^{\vee}$ (resp. in $T_2^{\vee}$) corresponding to v_1' (resp. v_2'). Since

$$|v_1' \wedge w_2| = \frac{A_2}{\alpha + 1} < \mathrm{Area}(T_3^{\vee}) = \frac{A_2^0}{\alpha_0 + 1} + A_1^0,$$

from Lemma 4.1, we deduce that $\sigma_1^{\pm}$ meet η_2 at only one point. Consequently, we see that $\sigma_1^{\pm}$ bound a simple cylinder containing η_2. Similarly, since $\mathrm{Area}(T_2^{\vee}) = |v_2' \wedge w_1| + |v_2' \wedge w_2|$, it follows that $\sigma_2^{\pm}$ cut $T_2^{\vee}$ into two cylinders, one contains η_1, the other contains η_2. Consequently, $\sigma_1^{\pm}$ and $\sigma_2^{\pm}$ give rise to two pairs of homologous saddle connections in Σ which determine a special splitting $X' = (T_1', T_2', T_3', v_1', v_2')$. We have

$$\begin{aligned}
\mathrm{Area}(T_1') &= \mathrm{Area}(T_3^{\vee}) - |v_1' \wedge w_2| = \frac{A_2}{\alpha + 1} + A_1 - \frac{A_2}{\alpha + 1} = A_1.\\
\mathrm{Area}(T_2') &= |(v_1' + v_2') \wedge w_2| = A_2.
\end{aligned}$$

Therefore, $\mathrm{Area}(T_3') = A_3$. Since $\Lambda(T_3') = \mathbb{Z}w_1 \oplus \mathbb{Z}(v_2' + \widehat{u}_1)$, it follows from the choice of $\widehat{u}_1$ that v_2' is generic with respect to $\Lambda(T_3')$. We can then conclude that for any $(A_1, A_2, A_3, \alpha) \in \mathbb{R}^4_{>0}$ such that

$$\begin{aligned}
A_1 + A_2 + A_3 &= 1,\\
A_1 + \frac{A_2}{\alpha + 1} &= A_1^0 + \frac{A_2^0}{\alpha_0 + 1},
\end{aligned}$$

there exist $X' = (T_1', T_2', T_3', v_1', v_2') \in \mathbf{SSpl}(A_1, A_2, A_3, \alpha)$, with v_2' generic with respect to $\Lambda(T_3')$, such that $\Psi(X') \in \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$. We can now complete the proof of Lemma 5.2 as follows: first, for any (A_1, A_2, A_3, α) such that $(A_1/A_2)(\alpha+1)^2 \notin \mathbb{Q}$, it follows from Lemma 4.6 that $\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$. By continuity of Ψ, it follows that $\overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$ contains $\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha))$ for all (A_1, A_2, A_3, α). □

To complete the proof of Proposition 5.1, we need the following:

Lemma 5.3 *For any (A_1, A_2, A_3, α) such that*

$$A_1 + A_2 + A_3 = 1,$$
$$A_1 + \frac{A_2}{\alpha + 1} < 1 - (A_1^0 + \frac{A_2^0}{\alpha_0 + 1}) = \frac{\alpha_0 A_2^0}{\alpha_0 + 1} + A_3^0,$$

we have $\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$.

Proof Since

$$A_1 + \frac{A_2}{\alpha + 1} < 1 - \left(A_1^0 + \frac{A_2^0}{\alpha_0 + 1} \right),$$

we can find $(A_1', A_2', A_3', \alpha') \in \mathbb{R}^4_{>0}$ such that

$$A_1' + A_2' + A_3' = 1,$$
$$A_1' + \frac{A_2'}{\alpha' + 1} = A_1^0 + \frac{A_2^0}{\alpha_0 + 1},$$
$$A_3' = A_1 + \frac{A_2}{\alpha + 1}.$$

From Lemma 5.2, we know that $\Psi(\mathbf{SSpl}(A_1', A_2', A_3', \alpha')) \subset \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Psi(X_0)}$. Consider an element $X = (T_1, T_2, T_3, v_1, v_2) \in \mathbf{SSpl}(A_1', A_2', A_3', \alpha')$, where $v_1 = (1, 0)$, $v_2 = (\alpha', 0)$. Let Λ_i denote the lattice associated to T_i, $i = 1, 2, 3$. Observe that we can choose X such that (see Figure 8)

- Λ_1 contains no vertical vectors,
- Λ_2 contains a vector vertical vector w_2 such that $\Lambda_2 = \mathbb{Z}(v_1 + v_2) \oplus \mathbb{Z}w_2$,
- $\Lambda_3 = \mathbb{Z}v_3 \oplus \mathbb{Z}w_1$, where v_3 is horizontal, and w_1 is vertical.

By assumption, we see that all $\Lambda_1, \Lambda_2, \Lambda_3$ contain horizontal vectors. Let C_i, $i = 1, 2, 3$, denote the horizontal cylinder obtained by slitting T_i along the horizontal saddle connections, which correspond to the primitive horizontal vectors in T_i. Let ℓ_i and h_i denote width and the height of C_i. Note that h_1, h_2 are determined by $(A_1', A_2', A_3', \alpha')$, and ℓ_3 and h_3 must satisfy $\ell_3 > \alpha'$ and $\ell_3 h_3 = A_3'$.

By construction, the surface Σ constructed from X admits another special splitting $X^\vee = (T_1^\vee, T_2^\vee, T_3^\vee, w_1, w_2)$ which is dual to X. Since Λ_1 contains no vertical vectors, the lattice $\Lambda_3^\vee$ does not contain any vertical vector. Let $\overline{m}^\vee$ denote the moduli ratio of $X^\vee$:

$$\overline{m}^\vee = \frac{(\ell_3 - \alpha')(h_2 + h_3)}{\alpha' h_2}.$$

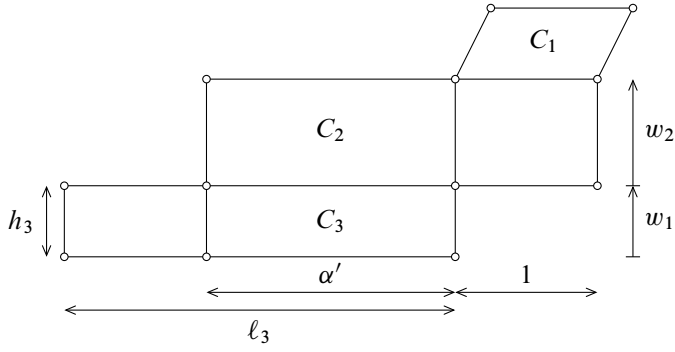


Figure 8: Switching between horizontal and vertical splittings

Since $\ell_3 = A_3'/h_3$, we see that $\bar{m}^\vee$ is a nonconstant rational function of h_3. Therefore, we can find h_3 so that $\bar{m}^\vee \notin \mathbb{Q}$. We deduce that there exists an element X in $\mathbf{SSpl}(A_1', A_2', A_3', \alpha')$ such that the element $X^\vee$ defined above satisfies the conditions of Theorem 4.3. Let $A_i^\vee$, $i = 1, 2, 3$, denote the area of $T_i^\vee$, and $\alpha^\vee = |w_2|/|w_1|$. By construction, we have

$$A_1^\vee + \frac{A_2^\vee}{\alpha^\vee + 1} = A_3' = A_1 + \frac{A_2}{\alpha + 1}.$$

Therefore, it follows from Lemma 5.2 that

$$\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2, R) \cdot \Psi(X^\vee)} \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)}. \qquad \square$$

5.3 Proof of Proposition 5.1

All we need to show is that $\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)}$ for all (A_1, A_2, A_3, α) such that $A_1 + A_2 + A_3 = 1$. Choose $(A_1', A_2', A_3', \alpha')$ in $\mathbb{R}^4_{>0}$ so that

$$A_1' + \frac{A_2'}{\alpha' + 1} < \min\left\{1 - \left(A_1^0 + \frac{A_2^0}{\alpha_0 + 1}\right), 1 - \left(A_1 + \frac{A_2}{\alpha + 1}\right)\right\}.$$

By Lemma 5.3, we know that $\Psi(\mathbf{SSpl}(A_1', A_2', A_3', \alpha')) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)}$. Let X be an element in $\mathbf{SSpl}(A_1', A_2', A_3', \alpha')$ which satisfies the conditions of Theorem 4.3. Since we have

$$A_1 + \frac{A_2}{\alpha + 1} < 1 - \left(A_1' + \frac{A_2'}{\alpha' + 1}\right),$$

by applying Lemma 5.3 with X in the place of X_0, we see that

$$\Psi(\mathbf{SSpl}(A_1, A_2, A_3, \alpha)) \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X)} \subset \overline{\mathrm{SL}(2, \mathbb{R}) \cdot \Psi(X_0)},$$

and the proposition follows. $\square$

6 Proof of Theorem 4.3

By Proposition 5.1, we know that $\overline{\mathcal{O}} = \overline{\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X_0)}$ contains all the surfaces that admit a special splitting. We will show that $\overline{\mathcal{O}}$ contains all the Veech surfaces in $\mathcal{H}_1^{\mathrm{hyp}}(4)$, in particular, $\overline{\mathcal{O}}$ contains all the square-tiled surfaces. Since the set of square-tiled surfaces is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$ (see [9, Lemma 4]), it follows immediately that $\overline{\mathcal{O}} = \mathcal{H}_1^{\mathrm{hyp}}(4)$.

Let Σ be a Veech surface in $\mathcal{H}_1^{\mathrm{hyp}}(4)$. From Corollary 3.2, we know that there exists on Σ a pair of homologous saddle connections $\delta^{\pm}$ such that by cutting along $\delta^{\pm}$, and gluing the two geodesic segments in the boundary of each of the connected component obtained from the cutting, we get a surface in $\mathcal{H}(0,0)$, which will be denoted by Σ', and a surface in $\mathcal{H}(2)$ which will be denoted by Σ''. On both Σ' and Σ'' we have a marked saddle connection corresponding to the pair $\delta^{\pm}$, we denote both of them by δ. Without loss of generality, we can assume that δ is horizontal. Since Σ is a Veech surface, the Veech dichotomy (see [8; 11]) implies that Σ is decomposed into cylinders which are filled with horizontal closed geodesics. In particular, we see that Σ'' is a union of horizontal cylinders. We have two possibilities:

Case 1 Σ'' is the union of two cylinders. In this case, there exists another pair of homologous horizontal saddle connections $\gamma^{\pm}$ in Σ'' which, together with $\delta^{\pm}$, determine a special splitting of Σ. Therefore, $\Sigma \in \overline{\mathcal{O}}$ by Proposition 5.1.

Case 2 Σ'' contains only one horizontal cylinder. In this case, there exist two other horizontal saddle connections γ_1, γ_2 in Σ such that $\delta * \gamma_1 * \gamma_2$ is freely homotopic to a simple closed geodesic. Consequently, Σ'' can be constructed from a single parallelogram P by the gluing as shown in Figure 9. Actually, P is an octagon whose opposite sides are parallel and have the same length. Let $U = (x,0), V_1 = (y,0), V_2 = (z,0)$ with $x > 0, y > 0, z > 0$ be the vectors associated to the saddle connections $\delta, \gamma_1, \gamma_2$ respectively.

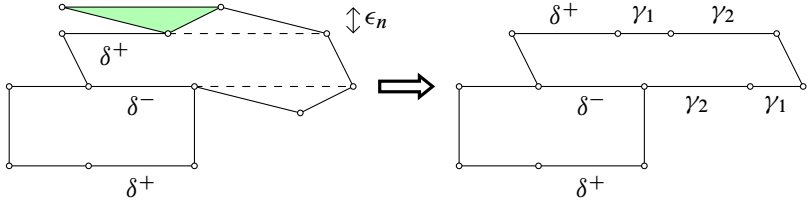


Figure 9: Surfaces with special splitting converging to Σ

Let $\{\epsilon_n\}$ be a sequence of positive real number decreasing to zero. For each ϵ_n, we construct a surface Σ_n in $\mathcal{H}_1^{\mathrm{hyp}}(4)$ as follows: first, we construct a surface Σ_n'' from an

octagon P_n, which is obtained from P by replacing V_1 by the vector $V_1^{(n)} = (y, \epsilon_n)$, and V_2 by the vector $V_2^{(n)} = (z, -\epsilon_n)$. Then we glue Σ_n'' to Σ' along the pair of homologous saddle connections $\delta^{\pm}$, and rescale to get a surface in $\mathcal{H}_1^{\text{hyp}}(4)$. By construction, we see that Σ_n admits a special splitting by horizontal saddle connections, therefore $\Sigma_n \in \overline{\mathcal{O}}$. As $\epsilon_n \longrightarrow 0$, the sequence $\{\Sigma_n\}$ converges to Σ, hence we have $\Sigma \in \overline{\mathcal{O}}$.

The proof of Theorem 4.3 is now complete. □

7 Proof of Theorem 1.2

We can now prove Theorem 1.2 as a consequence of Theorem 4.3. The idea is to show that there exists in the closure of $\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X_0)$ a surface which admits a special splitting satisfying the conditions of Theorem 4.3. As usual, let A_i^0 and Λ_i^0 denote the area and the associated lattice of T_i^0, $i = 1, 2, 3$. We can assume that $v_1^0 = (1, 0)$ and $v_2^0 = (\alpha_0, 0)$. Let t_1^0 and t_2^0 denote the twists of the pairs (T_1^0, v_1^0) and $(T_2^0, v_1^0 + v_2^0)$ respectively (see 4.3). Obviously, we only have to consider the case

$$\bar{m}_0 = m_1^0/m_2^0 = \frac{A_1^0}{A_2^0}(\alpha_0 + 1)^2 \in \mathbb{Q}.$$

Let n_1, n_2 be the integers such that $\gcd(n_1, n_2) = 1$ and $n_1 m_1^0 + n_2 m_2^0 = 0$. Ratner's Theorem for the action of

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t \in \mathbb{R} \right\},$$

implies $\overline{U\cdot\Psi(X_0)} \supset \Psi(X)$ for all $X = (T_1, T_2, T_3, v_1, v_2) \in \mathbf{SSpl}(A_1^0, A_2^0, A_3^0, \alpha_0)_{\text{hor}}$ such that

$$n_1(t_1 - t_1^0) + n_2(t_2 - t_2^0) \in \mathbb{Z},$$

where t_1, t_2 are the twists of the pairs (T_1, v_1) and $(T_2, v_1 + v_2)$ respectively. Consider such an X with $t_2 = 0$. Let Λ_i denote the lattice associated to T_i, $i = 1, 2, 3$. Since $t_2 = 0$, the lattice Λ_2 contains vertical vectors, let w_1 be the primitive vertical vector in Λ_2, and let $\eta_2^{\pm}$ denote the pair of homologous saddle connections in T_2 such that $V(\eta_2^{\pm}) = w_2$. Let w_1 be a primitive vector in the lattice Λ_3 such that $|w_1 \wedge v_2^0| < A_3^0$, and let $\eta_1^{\pm}$ denote the pair of homologous saddle connections in T_3 such that $V(\eta_1^{\pm}) = w_1$. The saddle connections $\eta_1^{\pm}$ and $\eta_2^{\pm}$ determine a splitting $X^{\vee} = (T_1^{\vee}, T_2^{\vee}, T_3^{\vee}, w_1, w_2)$ of the surface $\Sigma = \Psi(X)$. Let $\Lambda_i^{\vee}$ and $A_i^{\vee}$ denote the associated lattice and the area of $T_i^{\vee}$, $i = 1, 2, 3$. Here we have two cases:

Case 1 $t_1 \notin \mathbb{Q}$.

In this case, the lattice $\Lambda_3^\vee$ does not contain any vertical vector. We can choose w_1 to be vertical and $A_1^\vee/A_2^\vee(\alpha^\vee+1)^2 \notin \mathbb{Q}$, where $\alpha^\vee = |w_2|/|w_1|$ (see Lemma 5.3). Hence the splitting $X^\vee$ satisfies the conditions of Theorem 4.3, it follows immediately that $\overline{\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X_0)} = \overline{\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X^\vee)} = \mathcal{H}_1^{\mathrm{hyp}}(4)$.

Case 2 $t_1 \in \mathbb{Q}$.

In this case, $\Lambda_3^\vee$ contains vertical vectors. Let $\widehat{u}_3 = (0, h_3)$, with $h_3 > 0$, be the primitive vertical vector of $\Lambda_3^\vee$, and $\widehat{v}_3 = (\ell, h)$, with $\ell > 0$ and $0 \leqslant h < h_3$, be another primitive vector such that $\Lambda_3^\vee = \mathbb{Z}\widehat{u}_3 \oplus \mathbb{Z}\widehat{v}_3$.

By assumption, we have $w_2 = (0, h_2)$, with $h_2 > 0$. Remark that we must have $h_3 > h_2$. Recall that we are free to choose T_3 and w_1 provided $\mathrm{Area}(T_3) = A_3^0$, and $|w_1 \wedge v_2^0| < A_3^0$. By construction, we have $\Lambda_2^\vee = \mathbb{Z}v_2^0 \oplus \mathbb{Z}(w_1 + w_2)$. The theorem follows from the following observation (see also Lemma 5.2)

Claim *We can choose Λ_3 and w_1 so that there exist a primitive vector v_1' of $\Lambda_3^\vee$, and a primitive vector v_2' of $\Lambda_2^\vee$ such that the surface Σ admits a special splitting $X' = (T_1', T_2', T_3', v_1', v_2')$ dual to $X^\vee$ which satisfies the conditions of Theorem 4.3.*

Proof of Claim Set $A_1 = \ell(h_3 - h_2) > 0$, and choose (A_2, A_3, α) in $\mathbb{R}^3_{>0}$ such that

$$\begin{aligned} &A_1 + A_2 + A_3 = 1,\\ &A_1 + \frac{A_2}{\alpha+1} = A_1^0 + \frac{A_2^0}{\alpha_0+1},\\ &\frac{A_1}{A_2}(\alpha+1)^2 \notin \mathbb{Q}. \end{aligned}$$

Since
$$A_1 + \frac{A_2}{\alpha+1} = A_1^0 + \frac{A_2^0}{\alpha_0+1} = A_3^\vee = \ell h_3$$
and $A_1 = \ell(h_3 - h_2)$, it follows that $A_2/(\alpha+1) = \ell h_2$. Thus

$$\frac{\alpha A_2}{\alpha+1} = \alpha\ell h_2 < 1 - \left(A_1^0 + \frac{A_2^0}{\alpha_0+1}\right) = \frac{\alpha_0 A_2^0}{\alpha_0+1} + A_3^0 =: A_0. \tag{6}$$

Choose $q \in \mathbb{N}$ large enough so that

$$\begin{cases} \dfrac{h}{h_3 q} < \dfrac{\alpha\ell h_2}{\alpha\alpha_0 h_3},\\[2ex] \dfrac{1}{q} < \dfrac{1}{2}\dfrac{A_0 - \alpha\ell h_2}{\alpha\alpha_0 h_3}. \end{cases}$$

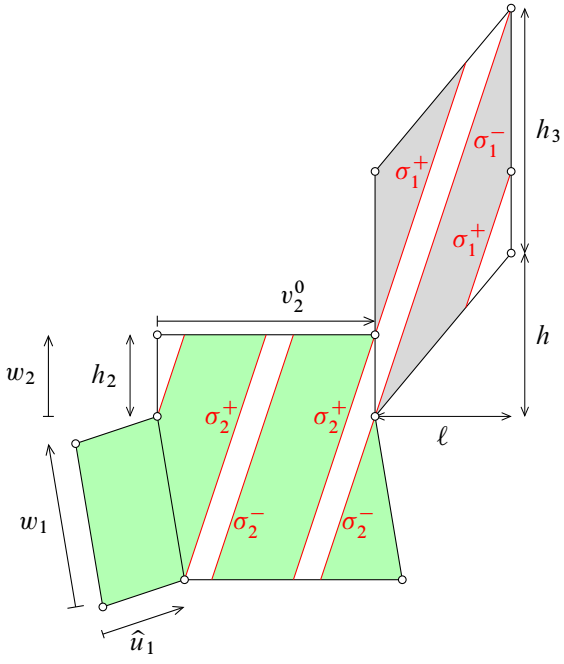


Figure 10: Case $t_1 \in \mathbb{Q}$, ie the lattice $\Lambda_3^\vee$ contains vertical vectors

From (6), it follows that there exists $p \in \mathbb{N}$ such that

$$\frac{\alpha\ell h_2}{\alpha\alpha_0 h_3} - \frac{h}{h_3 q} < \frac{p}{q} < \frac{A_0}{\alpha\alpha_0 h_3} - \frac{h}{h_3 q}.$$

Now, we can take

$$\begin{aligned} v_1' &= p\widehat{u}_3 + \widehat{v}_3 = (\ell, ph_3 + h),\\ w_1 &= \left(\frac{\alpha\ell - \alpha_0}{q}, \alpha h_3\left(\frac{h}{h_3 q} + \frac{p}{q}\right) - h_2\right),\\ v_2' &= v_2^0 + q(w_1 + w_2) = (\alpha\ell, \alpha(ph_3 + h)). \end{aligned}$$

Observe that v_1' and v_2' are primitive vectors in $\Lambda_3^\vee$ and $\Lambda_2^\vee$ respectively. Clearly, we have $v_2' = \alpha v_1'$. By the choice of p, q, we also have

$$\begin{aligned} |v_1' \wedge w_2| &= \ell h_2 < \ell h_3 = A_3^\vee = A_1^0 + \frac{A_2^0}{\alpha_0 + 1},\\ |v_2' \wedge w_2| &= \alpha\ell h_2 < \alpha_0\alpha h_3\left(\frac{h}{h_3 q} + \frac{p}{q}\right) = |v_2^0 \wedge (w_1 + w_2)| = A_2^\vee,\\ A_2^\vee &= \alpha_0\alpha h_3\left(\frac{h}{h_3 q} + \frac{p}{q}\right) < A_0 = 1 - A_3^\vee. \end{aligned}$$

Consequently, the surface Σ admits a special splitting determined by two pairs of homologous saddle connections $\sigma_1^{\pm}$ and $\sigma_2^{\pm}$, where $\sigma_1^{\pm}$ is the pair of saddle connections in $T_3^{\vee}$ corresponding to v_1', and $\sigma_2^{\pm}$ is the pair of saddle connections in $T_2^{\vee}$ corresponding to v_2' (see Figure 10). Let $X' = (T_1', T_2', T_3', v_1', v_2')$ denote this special splitting. Then we have $\mathrm{Area}(T_i') = A_i$, $i = 1, 2, 3$.

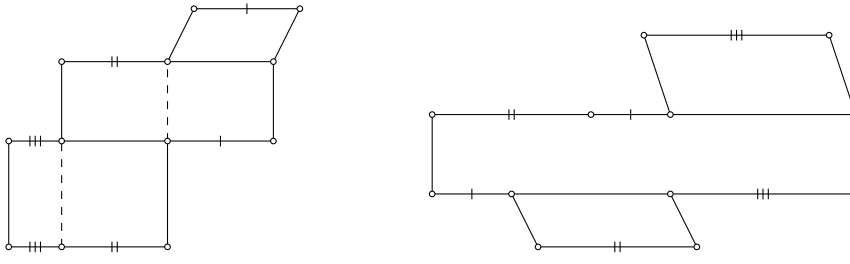
By construction, w_1 is a primitive vector of $\Lambda_1^{\vee}$. Let $\widehat{u}_1$ be another primitive such that $\Lambda_1^{\vee} = \mathbb{Z}w_1 \oplus \mathbb{Z}\widehat{u}_1$. Recall that we can choose T_3 arbitrarily provided $\mathrm{Area}(T_3) = A_3^0$, therefore we are free to choose $\widehat{u}_1$, provided $|\widehat{u}_1 \wedge w_1| = A_1^{\vee} = 1 - (A_2^{\vee} + A_3^{\vee})$. It is easy to check that we can choose such a $\widehat{u}_1$ so that the vector v_2' is generic with respect to $\Lambda(T_3') = \mathbb{Z}w_1 \oplus \mathbb{Z}(v_2' + \widehat{u}_1)$. With this choice, we see that splitting X' satisfies the conditions of Theorem 4.3, and the claim is then proved. □

By Theorem 4.3, we know that $\overline{\mathrm{SL}(2,\mathbb{R})\cdot\Sigma} = \overline{\mathrm{SL}(2,\mathbb{R})\cdot\Psi(X')} = \mathcal{H}_1^{\mathrm{hyp}}(4)$. Since $\Sigma \in \overline{\mathrm{SL}(2,\mathbb{R})\cdot\Sigma_0}$, it follows $\overline{\mathrm{SL}(2,\mathbb{R})\cdot\Sigma_0} = \mathcal{H}_1^{\mathrm{hyp}}(4)$, and Theorem 1.2 is proven. □

8 Surfaces admitting completely periodic directions with three cylinders

8.1 Two models of decomposition into three cylinders

Lemma 8.1 *Let Σ be a surface in $\mathcal{H}^{\mathrm{hyp}}(4)$. Assume that Σ is decomposed into three horizontal cylinders, that is, the horizontal direction is completely periodic for Σ with three cylinders. Then the surface Σ can be reconstructed from three (horizontal) cylinders by one of the gluing models in Figure 11.*



Case I Case II

Figure 11: Two models of constructing surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$ from three cylinders

Proof First, observe that Σ has exactly 5 horizontal saddle connections, since the angle at the unique singular point of Σ is 10π. Let C_1, C_2, C_3 denote the three

horizontal cylinders. Since each of the horizontal saddle connections is contained in the lower boundary component of a unique cylinder, we then have a partition of set of horizontal saddle connections into three subsets, there are only two such partitions corresponding to two ways of writing 5 as the sum of three positive integers: $5 = 1+1+3 = 1+2+2$.

Next, let us show that the hyperelliptic involution τ of Σ preserves each of the cylinders C_i, $i = 1, 2, 3$. Consider a simple closed geodesic c_i in C_i close to its lower boundary. Since $\tau(c_i) + c_i = 0$ in $H_1(\Sigma, \mathbb{Z})$, we deduce that c_i and $\tau(c_i)$ cut Σ into two connected components, each of which is equipped with a flat metric with geodesic boundaries. Since Σ has only one singularity, one of the two components has no singularities in the interior, and must be a cylinder. Therefore, c_i and $\tau(c_i)$ are contained in the same cylinder C_i. As a consequence, we see that τ maps the lower boundary of each cylinder to its upper boundary. In particular, the upper boundary and the lower boundary of each cylinder contain the same number of saddles connections, and moreover, each saddle connection in the lower boundary is paired up with a saddle connection in the upper boundary, which is its image under τ.

From these two observations, it is now easy to check that there are only two ways to construct Σ from three cylinders, which are shown in Figure 11. □

Remark The fact that the hyperelliptic involution preserves each of the cylinders is already known to Kontsevich and Zorich [9, Lemma 8].

8.2 Proof of Corollary 1.3

8.2.1 Proof of Corollary 1.3, Case (I) In this case, let C_1^0 denote the unique simple cylinder of the decomposition, C_2^0 denote the cylinder adjacent to C_1^0, and C_3^0 the remaining cylinder. Let ℓ_i^0, h_i^0, m_i^0 denote respectively the width, the height, and the modulus of C_i^0. Since m_1^0, m_2^0, m_3^0 are independent over $\mathbb{Q}$, by applying Ratner's Theorem for the action of U, we deduce that $\overline{U \cdot \Sigma_0}$ contains all the surfaces Σ which are constructed from 3 horizontal cylinders C_1, C_2, C_3 by the same gluing model, whenever C_i has the same width and height as C_i^0.

On each boundary component of C_1 we have a marked point which corresponds to the unique singularity of Σ. Let $v_1 = (v_1^x, v_1^y)$ be the associated vector of any geodesic segment joining the marked point in the lower boundary to the marked point in the upper boundary. We then define the twist t_1 of C_1 to be $v_1^x/\ell_1^0 \mod \mathbb{Z}$. On each boundary component of C_2 (resp. C_3), we have two marked points, therefore each boundary component is the union of two geodesic segments. From Lemma 8.1, we see that each segment in the upper boundary of C_2 is paired up with a segment in the

lower boundary component by the hyperelliptic involution. Consider such a pair of segments. Let $v_2 = (v_2^x, v_2^y)$ be the vector associated to a segment joining the two left endpoints, we then define the twist t_2 of C_2 to be $t_2 = v_2^x/\ell_2^0 \mod \mathbb{Z}$. The twist t_3 of C_3 is defined in the same manner.

Observe that any value of (t_1, t_2, t_3) gives us a unique surface Σ in $\overline{U \cdot \Sigma_0}$. Consider the case $t_2 = t_3 = 0$, in that case Σ admits a special splitting by two pairs of vertical homologous saddle connections. It is easy to see that if t_1 is not in $\mathbb{Q}$ then this splitting satisfies the condition of Theorem 1.2, tt follows immediately that $\overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Sigma_0} = \overline{\mathrm{SL}(2,\mathbb{R}) \cdot \Sigma} = \mathcal{H}_1^{\mathrm{hyp}}(4)$.

8.2.2 Proof of Corollary 1.3 Case (II) In this case, we have two simple cylinders, which will be denoted by C_1^0 and C_2^0, the remaining cylinder has 3 saddle connections in each boundary component, and will be denoted by C_3^0. Let $\gamma_1^+, \gamma_2^+, \gamma_3^+$ denote the saddle connections contained in the upper boundary of C_3^0 such that γ_1^+ (resp. γ_2^+) is also the lower boundary of C_1^0 (resp. C_2^0). Let γ_i^- denote the image of γ_i^+ under τ. Note that the lower boundary of C_3^0 is the union of $\gamma_1^-, \gamma_2^-, \gamma_3^-$, and in fact $\gamma_3^+ = \gamma_3^-$ (see Figure 12).

Let ℓ_i, h_i denote respectively the width and the height of C_i^0, $i = 1, 2, 3$. Since the cylinders C_1^0 and C_2^0 are simple, we define their twists t_1^0, t_2^0 as in Case (I). Let δ^+ (resp. δ^-) denote a pair of homologous saddle connections in C_3^0 which joins the left (resp. right) endpoints of γ_1^- and γ_1^+. Using the action of U, we can assume that $\delta^\pm$ are vertical.

Applying the Ratner's Theorem, we see that $\overline{U \cdot \Sigma_0}$ contains all surfaces obtained from three cylinders (C_1, C_2, C_3) by the same gluing model as (C_1^0, C_2^0, C_3^0), provided C_i has the same width and height as C_i^0. In particular, $\overline{U \cdot \Sigma_0}$ contains all surfaces constructed from three cylinders (C_1, C_2, C_3) with $C_3 = C_3^0$, and, for $i = 1, 2$, the twist t_i of C_i can be chosen arbitrarily. Let Σ be such a surface. Cut Σ along $\delta^\pm$, then glue the geodesic segments corresponding to $\delta^\pm$ on each component together, we get a surface in $\mathcal{H}(0,0)$ containing C_1, which will be denoted by Σ', and a surface in $\mathcal{H}(2)$ containing C_2, which will be denoted by Σ''. In both of Σ' and Σ'', we have a marked saddle connection corresponding to $\delta^\pm$, we denote both of them by δ.

In Σ', for any $t_1 \in]0,1[$, we have a pair of homologous saddle connections $\sigma_1^\pm$ such that $V(\sigma_1^\pm) = w_1 = (t_1\ell_1, h_1 + h_3)$. This pair of saddle connections cut Σ' into two cylinders, one of which contains δ.

Suppose that $t_2 \in]0,1[$. Then there exists a pair of homologous saddle connections $\sigma_2^\pm$ in Σ'' such that $V(\sigma_2^\pm) = w_2 = (t_2\ell_2, h_2 + h_3)$, which bound a simple cylinder containing δ. If we cut off the simple cylinder bounded by $\sigma_2^\pm$ from Σ'', and then glue

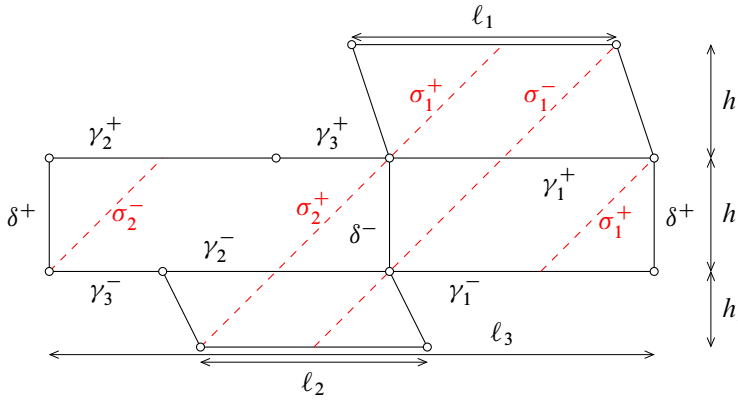


Figure 12: Finding new special splittings

the geodesic segments corresponding to $\sigma^{\pm}$, we obtain a torus T in $\mathcal{H}(0,0)$ together with a marked saddle connection σ_2. Let Λ denote the lattice in $\mathbb{R}^2$ associated to T. Then Λ is generated by $u=(\ell_2,h_3)$ and $v=(|\gamma_3^-|+(1-t_2)\ell_2,-h_2)$. Note that u is independent of t_2.

Recall that w_2 is parallel to a vector in $\Lambda=\mathbb{Z}u\oplus\mathbb{Z}v$ if and only if we can write $v=\lambda u+\mu w_2$ with $\lambda\in\mathbb{Q}$. A direct computation shows that

$$\lambda = 1+\frac{|\gamma_3^-|(h_2+h_3)}{\ell_2(h_2+h_3)-t_2\ell_2h_3}.$$

As t_2 varies in $]0,1[$, we see that the set of t_2 for which w_2 is parallel to a vector in Λ is countable, which means that, given any $\epsilon>0$, we can find $t_2\in]0,\epsilon[$ such that w_2 is not parallel to any vector in Λ. Therefore, we can find

$$t_2\in\Big]0,\frac{\ell_1(h_1+h_3)}{\ell_2(h_2+h_3)}\Big[,$$

such that w_2 is not parallel to any vector in Λ. Now, take

$$t_1=\frac{\ell_2(h_2+h_3)}{\ell_1(h_1+h_3)}t_2,$$

we have $t_1\in]0,1[$, hence we can find $\sigma_1^{\pm}$ as above. By the choice of t_1 and t_2, w_1 and w_2 are parallel. Reconstruct Σ from Σ' and Σ'', we see that $\sigma_1^{\pm}$ and $\sigma_2^{\pm}$ determine a special splitting of Σ, which satisfies the condition of Theorem 1.2. Since $\Sigma\in\overline{U\cdot\Sigma_0}$, the corollary follows. □

9 Applications

9.1 Generic surfaces with coordinates in a quadratic field

In [7], Hubert, Lanneau and Möller raise the following question: does there exist a generic translation surface of genus g with all coordinates in a number field K such that $[K:\mathbb{Q}] < g$? Theorem 1.2 provides us with an affirmative answer to this question for the case $\mathcal{H}^{\text{hyp}}(4)$. For every quadratic field K, one can easily construct a surface in $\mathcal{H}^{\text{hyp}}_1(4)$ with all coordinates in K which satisfies the condition of Theorem 1.2. Such a surface with coordinates in $\mathbb{Q}[\sqrt{2}]$ is shown in Figure 13.

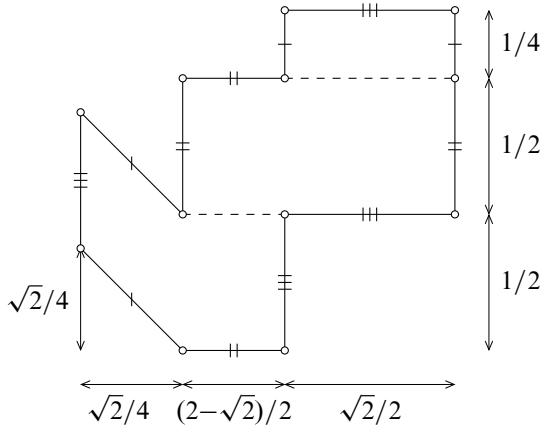


Figure 13: A generic surface in $\mathcal{H}^{\text{hyp}}_1(4)$ with coordinates in $\mathbb{Q}[\sqrt{2}]$

9.2 Thurston–Veech surface with cubic trace field

Surfaces obtained by the Thurston–Veech construction have large Veech group, which contains infinitely many hyperbolic elements (see [7] for definition and further detail on Thurston–Veech construction). Recall that the trace field of a translation surface is the field generated over $\mathbb{Q}$ by the traces of the matrices in its Veech group. If K is the trace field of a translation surface of genus g then $[K:\mathbb{Q}] \leqslant g$. For $g = 2$, McMullen [12] shows that if $[K:\mathbb{Q}] = 2$ then the $\mathrm{SL}(2,\mathbb{R})$–orbit of the surface can not be dense in its stratum. However, for $g = 3$, Hubert, Lanneau and Möller [6; 5] show that there exist surfaces in the hyperelliptic locus $\mathcal{L}$ of $\mathcal{H}^{\text{odd}}(2,2)$ obtained by the Thurston–Veech construction with trace field of degree 3 such that the $\mathrm{SL}(2,\mathbb{R})$–orbit is dense in $\mathcal{L}$. Note that $\mathcal{L}$ is a closed $\mathrm{SL}(2,\mathbb{R})$–invariant subset of $\mathcal{H}^{\text{odd}}(2,2)$, therefore these surfaces can be viewed as generic.

The surfaces obtained from Thurston–Veech construction are completely algebraically periodic in the sense of Calta and Smillie [2]. In particular, if such a surface admits a special splitting $(T_1, T_2, T_3, v_1, v_2)$, then v_2 must be parallel to a vector in the lattice associated to T_3. Therefore, a Thurston–Veech surface can never satisfy the condition of Theorem 1.2. However, if the trace field is of degree 3 over $\mathbb{Q}$, one can find examples of Thurston–Veech surfaces which admit decompositions into three parallel cylinders whose moduli are independent over $\mathbb{Q}$. By Corollary 1.3, it follows that the $\mathrm{SL}(2,\mathbb{R})$–orbits of such surfaces are dense in $\mathcal{H}^{\text{hyp}}_1(4)$. Here below, we will give the explicit construction of some of such surfaces.

We construct surfaces in $\mathcal{H}^{\text{hyp}}(4)$ for which the horizontal and vertical directions are completely periodic with three cylinders. To get such a surface, we glue three horizontal cylinders C_1, C_2, C_3 as shown in Figure 14.

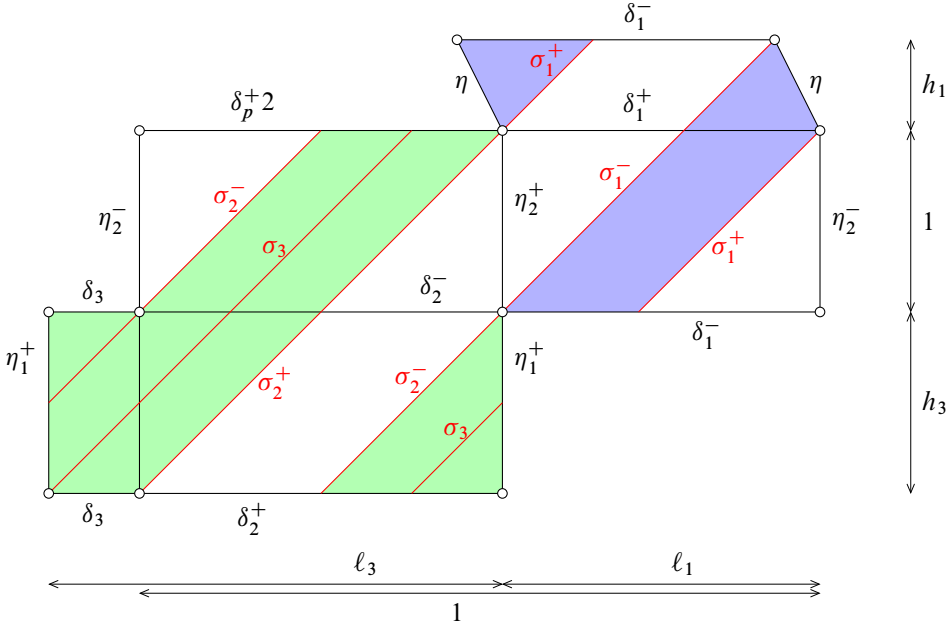


Figure 14: Cubic Thurston–Veech surface with a nonparabolic completely periodic direction

Let ℓ_i and h_i denote the width and the height of C_i, $i = 1, 2, 3$. We define the twists t_i of C_i, $i = 1, 2, 3$, as in Section 8.2.1. In what follows we fix $t_2 = t_3 = 0$, and consider the cases $t_1 = (n-1)/n$, $n \in \mathbb{N}$.

We denote by $\delta_1^{\pm}, \delta_2^{\pm}, \delta_3$ the horizontal saddle connections contained in the boundary of C_1, C_2, C_3 as shown in Figure 14. We choose the orientation for every horizontal saddle

connection to be from the left to the right, and for every vertical saddle connection to be upward. Let η_1^- (resp. η_1^+) denote the vertical saddle connection in C_3 joining the left (resp. right) endpoints of δ_2^+ and δ_2^-. Similarly, let η_2^- (resp. η_2^+) denote the vertical saddle connection in C_2 joining the left (resp. right) endpoints of δ_2^+ and δ_2^-. We see that the surface Σ admits a special spitting $(T_1^\vee, T_2^\vee, T_3^\vee, w_1, w_2)$ determined by $\eta_1^\pm$ and $\eta_2^\pm$ ($w_i = V(\eta_i^\pm)$, $i = 1, 2$). Since $t_1 = (n-1)/n$, there exists a vertical saddle connection in $T_3^\vee$ which crosses δ_1^- $(n-1)$ times, we denote this saddle connection by η_3. It follows that Σ is decomposed into three vertical cylinders, which will be denoted by $C_i^\vee$, $i = 1, 2, 3$, where

- $C_1^\vee$ is bounded by $\eta_i^\pm$,
- $C_2^\vee$ is bounded by $\eta_1^+ \cup \eta_2^+$ and $\eta_1^- \cup \eta_2^-$,
- $C_3^\vee$ is bounded by $\eta_2^+ \cup \eta_3$ and $\eta_2^- \cup \eta_3$ (η_3 bounds $C_3^\vee$ from both sides).

Fix $\ell_2 = h_2 = 1$, and let m_i denote the modulus of C_i, and $m_i^\vee$ denote the modulus of $C_i^\vee$, $i = 1, 2, 3$. Set

$$\begin{aligned}
a &= \frac{m_1}{m_2} = \frac{h_1}{\ell_1},\\
b &= \frac{m_3}{m_2} = \frac{h_3}{\ell_3},\\
c &= \frac{m_2^\vee}{m_1^\vee} = \frac{h_3(1-\ell_1)}{(h_3+1)(\ell_1+\ell_3-1)},\\
d &= \frac{m_2^\vee}{m_3^\vee} = \frac{n^2(h_1+1)(1-\ell_1)}{(h_3+1)\ell_1}.
\end{aligned}$$

Let η denote the saddle connection in C_1 such that $V(\eta) = (-\ell_1/n, h_1)$. We have a pair of homologous saddle connections $\sigma_1^\pm$ in $T_3^\vee$ such that

$$\sigma_1^+ = \sigma_1^- = \delta_1^- + \eta_2^- + \eta$$

in $H_1(\Sigma, \mathbb{Z})$. Note that $\sigma_1^\pm$ bound a cylinder containing η. Similarly, we have in $T_2^\vee$ a pair of homologous saddle connections $\sigma_2^\pm$ such that

$$\sigma_2^+ = \sigma_2^- = \delta_2^+ + \eta_1^+ + \eta_2^+.$$

We have

$$V(\sigma_1^\pm) = \left(\frac{n-1}{n}\ell_1, h_1 + 1\right), V(\sigma_2^\pm) = (1-\ell_1, h_3+1).$$

Claim 1 *$V(\sigma_1^+)$ and $V(\sigma_2^+)$ are colinear if and only if $d = n(n-1)$.*

Proof The fact that $V(\sigma_1^+)$ is parallel to $V(\sigma_2^+)$ is equivalent to

$$\begin{aligned} & \frac{n-1}{n}\frac{\ell_1}{1-\ell_1} = \frac{1+h_1}{1+h_3} \\ \iff \quad & \frac{1-\ell_1}{1+h_3}\frac{1+h_1}{\ell_1} = \frac{n-1}{n} \\ \iff \quad & \frac{d}{n^2} = \frac{n-1}{n} \\ \iff \quad & d = n(n-1). \end{aligned}$$

□

Clearly, the surface Σ is completely determined by the values of $(h_1, \ell_1, h_3, \ell_3)$. We will find some values of $(h_1, \ell_1, h_3, \ell_3)$ such that the vertical and horizontal directions are parabolic, ie a, b, c, d are rational numbers, and Σ admits a decomposition into three cylinders in the direction $V(\sigma_1^{\pm})$ whose moduli are independent over $\mathbb{Q}$. For this purpose, we fix $n \in \mathbb{N}$, a, b, c in $\mathbb{Q}$, and $d = n(n-1)$, then we compute $(h_1, \ell_1, h_3, \ell_3)$ as functions of (n, a, b, c).

First, observe that since $c = m_2^\vee / m_1^\vee$ is a rational number, the vector $V(\sigma_2^+)$ is parallel to a vector in the lattice associated to $T_1^\vee$. It follows that Σ is decomposed into three cylinders in the direction $V(\sigma_1^+)$. We denote these cylinders by C_i', $i = 1, 2, 3$, where

- C_1' is the cylinder containing η, and bounded by $\sigma_1^{\pm}$,
- C_2' is the cylinder containing $\eta_2^{\pm}$, and bounded by $\sigma_1^+ \cup \sigma_2^+$ and $\sigma_1^- \cup \sigma_2^-$,
- C_3' is the complement of $C_1' \cup C_2'$, which is bounded by $\sigma_2^+ \cup \sigma_3$ and $\sigma_2^- \cup \sigma_3$, where σ_3 is a saddle connection parallel to $\sigma_2^{\pm}$, and bounds C_3' from both sides.

Let m_i', $i = 1, 2, 3$, denote the modulus of C_i'.

Claim 2 *Suppose that $c \in \mathbb{N}$. Then we have*

$$\begin{aligned} \frac{m_1'}{m_2'} &= \frac{1}{(n-1)^2}(na\ell_1 + 1)\left(\frac{n}{\ell_1} - 1\right), \\ \frac{m_3'}{m_2'} &= \frac{b}{nc^2}(n - \ell_1). \end{aligned}$$

Proof Let h_i', ℓ_i', $i = 1, 2, 3$, denote the height and the width of C_i'. We have

$$\frac{h_1'}{h_2'} = \frac{\text{Area}(C_1')}{\text{Area}(T_3^\vee) - \text{Area}(C_1')} = \frac{(h_1+1)\ell_1 - ((n-1)/n)\ell_1}{((n-1)/n)\ell_1} = \frac{1+nh_1}{n-1}.$$

Using the vertical projection onto the horizontal axis, we have

$$\frac{\ell_2'}{\ell_1'} = 1 + \frac{1-\ell_1}{((n-1)/n)\ell_1} = \frac{n-\ell_1}{(n-1)\ell_1}.$$

Therefore

$$\frac{m_1'}{m_2'} = \frac{h_1'}{h_2'}\frac{\ell_2'}{\ell_1'} = \frac{1+nh_1}{(n-1)}\frac{n-\ell_1}{(n-1)\ell_1} = \frac{1}{(n-1)^2}(1+na\ell_1)\frac{n-\ell_1}{\ell_1}.$$

Rescaling so that $V(\delta_3)=(1,0)$ and $V(\eta_1)=(0,1)$, that is $m_1^\vee = 1$. Since $m_2^\vee/m_1^\vee = c$, we deduce that $V(\sigma_2^+)$ is collinear with the vector $(c,1)$. When c is an integer, in the standard torus $\mathbb{R}^2/\mathbb{Z}^2$, the number of intersection points of the simple closed geodesics corresponding to the vectors $(c,1)$ and $(0,1)$ is given by $|(c,1)\wedge(0,1)| = c$. Therefore, the saddle connection σ_3 crosses η_1^+ c times. Using the projection along $V(\sigma_1^\pm)$ onto the vertical axis we have $h_3'/h_2' = h_3/c$. Then using the vertical projection onto the horizontal axis, we have $\ell_2'/\ell_3' = (1-\ell_1/n)/(c\ell_3)$. Therefore,

$$\frac{m_3'}{m_2'} = \frac{h_3'}{h_2'}\frac{\ell_2'}{\ell_3'} = \frac{1}{nc^2}\frac{h_3}{\ell_3}(n-\ell_1) = \frac{b}{nc^2}(n-\ell_1). \qquad \square$$

Given $n\in\mathbb{N}, n>1$, $a,b\in\mathbb{Q}, a>0, b>0$, and $c\in\mathbb{N}, c>0$, set

$$P(X) = \frac{nc}{n-1}(X+a)\left(\frac{n}{(n-1)b}(X-1)(X+a) - \frac{X}{b} - X + 1\right) - \frac{n}{n-1}(X-1)(X+a) + X.$$

Claim 3 *Suppose that $P(X)$ is irreducible over $\mathbb{Q}$, and has a real root α satisfying*

$$\alpha > 1,$$
$$\frac{n(\alpha-1)(\alpha+a)}{(n-1)\alpha} > 1.$$

Then by taking

$$\ell_1 = \frac{1}{\alpha}, \quad h_1 = \frac{a}{\alpha}, \quad \ell_3 = \frac{1}{b}\left(\frac{n(\alpha-1)(\alpha+a)}{(n-1)\alpha} - 1\right), \quad h_3 = \frac{n(\alpha-1)(\alpha+a)}{(n-1)\alpha} - 1,$$

the construction above gives us a Thurston–Veech surface with trace field of degree 3 over $\mathbb{Q}$ for which the moduli of the three cylinders in the direction $V(\sigma_1^\pm)$ are independent over $\mathbb{Q}$. Consequently, this surface is generic in $\mathcal{H}^{\mathrm{hyp}}(4)$ by Corollary 1.3.

Proof From the choice of ℓ_1, h_1, ℓ_3, h_3, we only need to check that

(a) $$\ell_1 + \ell_3 - 1 > 0,$$

(b) $$\frac{m_2^\vee}{m_1^\vee} = \frac{1-\ell_1}{1+h_3}\frac{h_3}{\ell_1+\ell_3-1} = c,$$

(c) $$\frac{m_2^\vee}{m_3^\vee} = \frac{n^2(1-\ell_1)(1+h_1)}{\ell_1(1+h_3)} = n(n-1).$$

Condition (a) is satisfied since we have

$$\begin{aligned}\ell_1 + \ell_3 - 1 &= \frac{1}{\alpha} + \frac{n}{b(n-1)}\frac{(\alpha-1)(\alpha+a)}{\alpha} - \frac{1}{b} - 1\\ &= \frac{n-1}{cn(\alpha+a)}\left(\frac{n}{n-1}\frac{(\alpha-1)(\alpha+a)}{\alpha} - 1\right) > 0.\end{aligned}$$

Conditions (b) and (c) follow immediately from the fact that α is a root of P. To see that the trace field of Σ is of degree 3, remark that we have $m_i \in \mathbb{Q}$, $i = 1, 2, 3$, but

$$m_3^\vee = \frac{\ell_1}{n^2(1+h_1)} = \frac{1}{n^2(\alpha+a)}.$$

is an algebraic number of degree 3 over $\mathbb{Q}$. From Claim 2, we have

$$\begin{aligned}\frac{m_1'}{m_2'} &= \frac{1}{(n-1)^2}(1+na\ell_1)\left(\frac{n}{\ell_1} - 1\right)\\ &= \frac{1}{(n-1)^2}\left(1 + \frac{na}{\alpha}\right)(n\alpha - 1)\\ &= \frac{n}{(n-1)^2}\alpha + \frac{n^2a-1}{(n-1)^2} - \frac{na}{(n-1)^2}\frac{1}{\alpha}.\end{aligned}$$

and

$$\begin{aligned}\frac{m_3'}{m_2'} &= \frac{b}{nc^2}(n-\ell_1)\\ &= \frac{b}{c^2} - \frac{b}{nc^2}\frac{1}{\alpha}.\end{aligned}$$

Since $\alpha, 1, 1/\alpha$ are independent over $\mathbb{Q}$, it follows that m_1', m_2', m_3' are independent over $\mathbb{Q}$. □

Below are some explicit examples of Thurston–Veech surfaces obtained from the construction above which satisfy the condition of Corollary 1.3. Here $\widetilde{P}(X)$ is a polynomial proportional to $P(X)$ with coefficients in $\mathbb{Z}$ and values of α, ℓ_1 and ℓ_3 are approximate.

(n,a,b,c)	$\widetilde{P}(X)$	α	ℓ_1	ℓ_3
$(4,1,10,5)$	$8X^3-70X^2-5X+64$	8.716407	0.114726	1.046891
$(5,2,10,3)$	$15X^3-127X^2-152X+260$	9.352026	0.106929	1.167271
$(5,1/5,2,1)$	$25X^3-115X^2+83X+15$	3.643625	0.274452	1.242959
$(5,1/2,5,1)$	$20X^3-176X^2+121X+75$	7.983332	0.125261	1.655175
$(2,1,6,1)$	$2X^3-11X^2+10$	5.323574	0.187844	1.545243
$(2,2,9,2)$	$8X^3-34X^2-53X+76$	5.175414	0.193221	1.175327

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Institut de Mathématiques de Bordeaux, Bat A33, Université Bordeaux 1
351, cours de la Libération, F-33405 Talence Cedex, France

duc-manh.nguyen@math.u-bordeaux1.fr

http://www.math.u-bordeaux1.fr/~nguyendm/

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