Geometry & Topology 15 (2011) 2135–2180

Asymptotics of the colored Jones function of a knot

STAVROS GAROUFALIDIS
THANG T Q LÊ

To a knot in 3–space, one can associate a sequence of Laurent polynomials, whose n–th term is the n–th colored Jones polynomial. The paper is concerned with the asymptotic behavior of the value of the n–th colored Jones polynomial at $e^{\alpha/n}$, when α is a fixed complex number and n tends to infinity. We analyze this asymptotic behavior to all orders in $1/n$ when α is a sufficiently small complex number. In addition, we give upper bounds for the coefficients and degree of the n–th colored Jones polynomial, with applications to upper bounds in the Generalized Volume Conjecture. Work of Agol, Dunfield, Storm and W Thurston implies that our bounds are asymptotically optimal. Moreover, we give results for the Generalized Volume Conjecture when α is near $2\pi i$. Our proofs use crucially the cyclotomic expansion of the colored Jones function, due to Habiro.

57N10; 57M25

Dedicated to Louis Kauffman on the occasion of his 60th birthday

1 Introduction

1.1 Asymptotics of the colored Jones function of a knot

To a knot K in 3–space, one can associate a sequence of Laurent polynomials

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$$

for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. The first polynomial $J_{K,1}(q) = 1$, the second $J_{K,2}(q)$ is the famous *Jones polynomial* [17] of K, and $J_{K,n}(q)$ are roughly speaking the Jones polynomials of $(n-1)$–parallels of the knot. More precisely, $J_{K,n}(q)$ is the *quantum group* invariant of K using the n–dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation, normalized by $J_{\text{unknot},n}(q) = 1$ for all n; see Reshetikhin and Turaev [28] and Turaev [33]. The sequence $\{J_{K,n}(q)\}_n$ is often called the *colored Jones function* of the knot K.

The paper is concerned with the asymptotic growth of the colored Jones function. More precisely, fix a knot K and consider the sequence of holomorphic functions

$$f_{K,n}\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad f_{K,n}(z) := J_{K,n}(e^{z/n})$$

Published: 28 October 2011 DOI: 10.2140/gt.2011.15.2135

for $n \in \mathbb{N}$. In other words, we are evaluating the n–th polynomial $J_{K,n}(q)$ at a complex n–th root of e^z. We will be concerned with strong and weak convergence of the sequence $f_{K,n}$, for $n \in \mathbb{N}$. Let us explain what we mean by that. Fix an open subset U of $\mathbb{C}$ containing 0.

Definition 1.1 (a) A sequence of holomorphic functions $f_n \colon U \longrightarrow \mathbb{C}$ *strongly converges in* U to a holomorphic function $f \colon U \longrightarrow \mathbb{C}$ (and write $\mathrm{slim}_{n\to\infty} f_n(z) = f(z)$) if $f_n(z)$ converges to $f(z)$ uniformly on any compact subset of U.

(b) A sequence of holomorphic functions $f_n \colon U \longrightarrow \mathbb{C}$ *weakly converges* to a holomorphic function $f \colon U \longrightarrow \mathbb{C}$ (and write $\mathrm{wlim}_{n\to\infty} f_n(z) = f(z)$) if the Taylor series of $f_n(z)$ at $z = 0$ coefficient-wise converges to the Taylor series of $f(z)$. In other words, for every $k \geq 0$, we have

$$\lim_{n\to\infty} \frac{d^k f_n}{dz^k}\Big|_{z=0} = \frac{d^k f}{dz^k}\Big|_{z=0}.$$

It is easy to see that strong convergence of holomorphic functions implies weak convergence. The converse is not true (see however, Lemma 2.1 below).

The Melvin–Morton–Rozansky (MMR, in short) Conjecture, which was settled by Bar-Natan and the first author in [3], compares the function $f_{K,n}$ of a knot K with the *Alexander polynomial* Δ_K of K, normalized by $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

Theorem 1.2 (MMR conjecture [3]) *For every knot K we have*

$$\mathrm{wlim}_{n\to\infty} \; f_{K,n}(z) = \frac{1}{\Delta_K(e^z)}.$$

Our sample result is the following analytic form of the MMR Conjecture, which has application in the Generalized Volume Conjecture.

Theorem 1.3 (Proof in Section 2.1) *For every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$ such that in U_K, we have*

$$\mathrm{slim}_{n\to\infty} f_{K,n}(z) = \frac{1}{\Delta_K(e^z)}.$$

Given Theorem 1.3 one may ask for a full asymptotic expansion of $f_{K,n}(z)$ in terms of powers of $1/n$. In order to formulate our results, let us introduce the notion of strong and weak asymptotic expansions.

Definition 1.4 Fix an open set U of $\mathbb{C}$, and holomorphic functions $f_n\colon U \longrightarrow \mathbb{C}$ and $R_n\colon U \longrightarrow \mathbb{C}$.

(a) We will say that the sequence f_n is *strongly asymptotic in* U to the series $\sum_{k=0}^{\infty} R_k(z)(\frac{z}{n})^k$, and write

$$f_n(z) \sim^{s}_{n\to\infty} \sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k \tag{1}$$

if for every $N \geq 0$ we have

$$\operatorname{slim}_{n\to\infty}\left(\frac{n}{z}\right)^N \left(f_n(z) - \sum_{k=0}^{N-1} R_k(z)\left(\frac{z}{n}\right)^k\right) = R_N(z). \tag{2}$$

(b) Likewise, we will say that the sequence f_n is *weakly asymptotic in* U to the series $\sum_{k=0}^{\infty} R_k(z)(\frac{z}{n})^k$, and write

$$f_n(z) \sim^{w}_{n\to\infty} \sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k \tag{3}$$

if for every $N \geq 0$ we have

$$\operatorname{wlim}_{n\to\infty}\left(\frac{n}{z}\right)^N \left(f_n(z) - \sum_{k=0}^{N-1} R_k(z)\left(\frac{z}{n}\right)^k\right) = R_N(z). \tag{4}$$

Usually, sequences of holomorphic functions $f_n(z)$ do not have asymptotic expansions (or even a limit, as $n \to \infty$). However, sequences that appear in perturbative expansions of *Quantum Field Theory* are generally expected to have asymptotic expansions. In fact asymptotic expansions are generally easier to define (via *Feynman diagram* techniques) than the partition functions $f_{K,n}(z)$ themselves. Even when the partition functions can be defined, the asymptotic expansions is a numerically useful way to approximate them.

In [30], Rozansky discovered that the sequence $f_{K,n}(z)$ has a weak asymptotic expansion, where the terms are rational functions in the variable e^z. More precisely, Rozansky proved the following result.

Theorem 1.5 [30] *For every knot K there exists a sequence $P_{K,k}(q) \in \mathbb{Q}[q^{\pm 1}]$ of Laurent polynomials with $P_{K,0}(q) = 1$ such that*

$$f_{K,n}(z) \sim^{w}_{n\to\infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}}\left(\frac{z}{n}\right)^k. \tag{5}$$

A different proof, valid for all simple Lie groups, was given by the first author in [7], using work of the first author with Kricker [9]. Our result is strong version of Theorem 1.5.

Theorem 1.6 (Proof in Section 5.2) *For every knot K there exists an open neighborhood $\widetilde{U}_K$ of $0 \in \mathbb{C}$ such that in $\widetilde{U}_K$, we have*

$$f_{K,n}(z) \sim^s_{n\to\infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}} \left(\frac{z}{n}\right)^k. \tag{6}$$

1.2 The Generalized Volume Conjecture

In this section we state some new information about the Volume Conjecture; the latter connects two very different approaches to knot theory, namely Topological Quantum Field Theory and Riemannian (mostly hyperbolic) geometry.

Conjecture 1.7 (Kashaev [18]; Murakami–Murakami [25]) *For every hyperbolic knot K in S^3 we have*

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} = \frac{1}{2\pi} \operatorname{vol}(\rho_{2\pi i}),$$

where $\operatorname{vol}(\rho_{2\pi i})$ *is the* hyperbolic volume *of the knot complement $S^3 - K$.*

In other words, the sequence $f_{K,n}(2\pi i)$ of complex numbers grows exponentially with respect to n, and the exponential growth-rate is proportional to the volume of a hyperbolic knot.

One can define the volume function $\operatorname{vol}(\rho)$ of every representation $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_2(2,\mathbb{C})$ (see Dunfield [5], Cooper et al [4] and Thurston [32]), and $\operatorname{vol}(\rho_{2\pi i})$ is exactly the value of this volume function with $\rho_{2\pi i}$ being the discrete faithful representation of the knot group.

The idea of the Generalized Volume Conjecture (formulated in part by Gukov in [13]) is that we should use other representations of the knot complement in $\mathrm{SL}(2,\mathbb{C})$. For α nearby $2\pi i$, in a small neighborhood of $\rho_{2\pi i}$ there is a unique (up to conjugation) representation

$$\rho_\alpha\colon \pi_1(S^3 - K) \longrightarrow \mathrm{SL}(2,\mathbb{C})$$

which satisfies

$$\rho_\alpha(\text{meridian}) = \begin{pmatrix} e^\alpha & \star \\ 0 & e^{-\alpha} \end{pmatrix}. \tag{7}$$

Alas, there is an additional difficulty. Namely, when $\alpha/(2\pi i)$ is rational, we should distinguish two cases: $\alpha/(2\pi i) = 1$ or $\alpha/(2\pi i) \neq 1$. The Generalized Volume Conjecture for α sufficiently close to $2\pi i$ may now be stated as follows.

Conjecture 1.8 *If $\alpha/(2\pi i) \in (\mathbb{R}-\mathbb{Q}) \cup \{1\}$ is sufficiently close to 1 then*

$$\lim_{n\to\infty} \frac{\log|f_{K,n}(\alpha)|}{n} = c_\alpha \ \mathrm{vol}(\rho_\alpha), \tag{8}$$

and if $\alpha/(2\pi i) \in \mathbb{Q}-\{1\}$, then

$$\limsup_{n\to\infty} \frac{\log|f_{K,n}(\alpha)|}{n} = c_\alpha \ \mathrm{vol}(\rho_\alpha),$$
$$\liminf_{n\to\infty} \frac{\log|f_{K,n}(\alpha)|}{n} = 0,$$

where $c_\alpha \neq 0$ are some nonzero constants.

The distinction of $\alpha/(2\pi i)$ being rational or not is a bit with odds with the notion of *hyperbolic Dehn surgery* developed by Thurston in [32]. When $\alpha/(2\pi i)$ is a rational number, the hyperbolic Dehn surgery theorem associates an orbifold filling to the knot complement whose volume is $\mathrm{vol}(\rho_\alpha)$. Orbifolds are mild generalizations of manifolds. On the other hand, when $\alpha/(2\pi i)$ is irrational, hyperbolic Dehn surgery associates a space which is topologically a 1–point compactification of the knot complement, with volume $\mathrm{vol}(\rho_\alpha)$. In the following, we will refer to the parameter α in the Generalized Volume Conjecture as the *angle*, making contact with standard terminology from hyperbolic geometry.

There are two rather independent parts in the Volume Conjecture:

(a) to show that the limit exists in (8),

(b) to identify the limit with the volume of the corresponding Dehn filling.

At the moment, the Generalized Volume Conjecture is known only for the 4_1 knot and certain values of α; see Murakami [24].

One may further ask what happens to the Generalized Volume Conjecture when the angle α is small. For $\alpha=0$, it is natural to define ρ_0 to be the *trivial* representation. Then for α small enough, there is a unique (up to conjugation) *abelian* $\mathrm{SL}_2(\mathbb{C})$ representation ρ_α that satisfies (7). Abelian representations have 0 volume (see eg Cooper et al [4]). On the other hand, for small enough α, we have $\Delta_K(e^\alpha) \sim \Delta_K(1) = 1$. Thus Theorem 1.3 implies:

Theorem 1.9 *For every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$, such that for $\alpha \in U_K$, we have*

$$\lim_{n\to\infty} \frac{\log|f_{K,n}(\alpha)|}{n} = 0 = \mathrm{vol}(\rho_\alpha).$$

In other words, Theorem 1.3 settles the Generalized Volume Conjecture for small complex angles.

1.3 The Generalized Volume Conjecture near $2\pi i$

Our next result states that the volume conjecture can only be barely true.

Theorem 1.10 (Proof in Section 9) *For every knot K and every fixed integer $m \neq 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \log |J_{K,n+m}(\exp(2\pi i/n))| = 0.$$

It follows that the *double-scaling limit*

$$\lim_{n,k} \frac{1}{n} \log |J_{K,n}(\exp(2\pi i/k))|$$

when $n, k \to \infty$ and $n/k \to 1$ does not exist, or equals to 0; with the latter case in contradiction with the Volume Conjecture. Our next result confirms the strange behavior in the Generalized Volume Conjecture when $\alpha/(2\pi i)$ is rational, not equal to 1.

Theorem 1.11 (Proof in Section 9) *For every knot K there exists a neighborhood V_K of $1 \in \mathbb{C}$ such that when $\alpha/(2\pi i) \in V_K$ is rational and not equal to 1, then*

$$\liminf_{n\to\infty} \frac{|f_{K,n}(\alpha)|}{n} = 0.$$

1.4 Upper bounds for the Generalized Volume Conjecture

Our next theorem is an upper bound for the Generalized Volume Conjecture. Let $\Re(\alpha)$ denote the real part of α.

Theorem 1.12 (Proof in Section 6.3) *For every knot K with $c+2$ crossings and every $\alpha \in \mathbb{C}$, we have*

$$\limsup_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} \leq c \log 4 + \frac{c+2}{2} |\Re(\alpha)|.$$

1.5 Relation with hyperbolic geometry and asymptotically sharp bounds

When $\alpha = 2\pi i$, the upper bound in Theorem 1.12 is not optimal, and does not reveal any relationship between the $\limsup$ and hyperbolic geometry. Our next theorem fills this gap.

Theorem 1.13 (Proof in Section 8.5) *For every knot K with $c+2$ crossings we have*

$$\limsup_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} \leq \frac{v_8}{2\pi} c,$$

where $$v_8 = 8\Lambda(\pi/4) \approx 3.66386237670887606 02\ldots$$

is the volume of the regular ideal octahedron (see Thurston [32]).

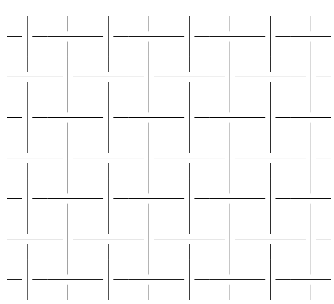
Using an ideal decomposition of a knot complement by placing one octahedron per crossing, it follows that for every knot K with $c+2$ crossings, we have

$$\mathrm{vol}(S^3 - K) \leq v_8 c, \tag{9}$$

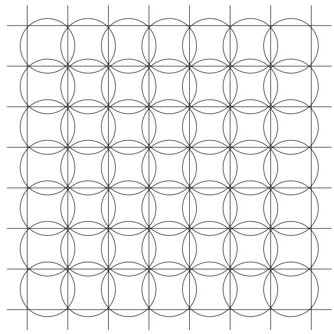
where $\mathrm{vol}(S^3 - K)$ is the hyperbolic volume of the knot complement. On the other hand, if the volume conjecture holds for $\alpha = 2\pi i$, then

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} = \frac{1}{2\pi} \mathrm{vol}(S^3 - K) \leq \frac{v_8}{2\pi} c.$$

One may ask whether (9) (and therefore, whether the bound in Theorem 1.13) is optimal. This may be a little surprising, since it involves all knots (and not just alternating ones) and their number of crossings, an invariant that carries little known geometric information. In conversations with I Agol and D Thurston, it was communicated to us that the upper bound in (9) is indeed optimal. Moreover a class of knots that achieves (in the limit) the optimal ratio of volume to number of crossings is obtained by taking a large chunk of the following *weave*, and closing it up to a knot:



The complement of the weave has a complete hyperbolic structure associated with the *square tessellation* of the Euclidean plane:



Optimality follows along similar lines as the Appendix of Lackenby [20] (by Agol and D Thurston), using a stronger estimate for the lower bound of the volume of Haken manifolds, cut along an incompressible surface: If M is a hyperbolic finite volume 3–manifold containing a properly imbedded orientable, boundary incompressible, incompressible surface S, then

$$\mathrm{vol}(M) \geq \mathrm{vol}(\mathrm{Guts}(M - \mathrm{int}(\mathrm{nbd}(S))),$$

where vol stands for volume, and the Guts terminology are defined by Agol, Dunfield, Storm and W Thurston [1]. The proof of this stronger statement (of [1]) uses, among other things, work of Perelman.

Compare (9) with the following result of Agol, Lackenby and D Thurston [20]: If K is an alternating knot with a planar projection having t twist, then

$$v_3(t-1)/2 < \mathrm{vol}(S^3 - K) < 10v_3(t-1),$$

where $v_3 = 2\Lambda(\pi/3) \approx 1.01494\dots$ is the volume of the regular ideal tetrahedron. Moreover, the class of knots obtained by Dehn filling on the *chain link* has asymptotic ratio of volume by twist number equal to $10v_3$. The corresponding tessellation of the Euclidean plane is given by the *star of David*.

So far, we have formulated a Generalized Volume Conjecture for α near 0 and α near $2\pi i$, using representations near the trivial or near the discrete faithful. How can we connect these choices for other complex angles α? A natural answer to this question requires analyzing asymptotics of solutions of difference equations with a parameter. This is a different subject that we will not discuss here; instead we will refer the curious reader to the paper of the first author with Geronimo [8], and forthcoming work of the first author. For a further discussion, see also Section 11.

1.6 The main ideas and organization of the paper

In Section 2.1 we show that weak convergence plus uniform boundedness implies strong convergence. Thus the strong convergence of Theorem 1.3 and Theorem 1.6 follows from the weak convergence of Theorem 1.2 and Theorem 1.5, plus uniform bounds. Uniform bounds for the colored Jones function require large cancellations. In order to control these cancellations, we use the cyclotomic expansion of the colored Jones function of a knot, which is recalled in Section 3. An important point about this expansion is that its kernel can absorb the exponential bounds of the coefficients of the cyclotomic functions; see Section 4 and Section 5.

Using a state-sum formula for the colored Jones function, we give in Section 6 bounds for the degrees and coefficients of the n–th colored Jones polynomial. The result is also of independent interest. The important point is that the local weights in the state-sum formula (ie, the entries of the R–matrix) are Laurent polynomials, given by some ratio of q–factorials. A priori, the bounds of the n–colored Jones function are not good enough to deduce the bounds for the n–th cyclotomic function. However, in Section 7, we use a lemma on the growth-rate of the number of partitions of an integer, in order to deduce the desired bounds for the cyclotomic function. As a corollary, we can deduce the upper bound of Theorem 1.12.

In the independent Section 8, we give a better bound for the growth-rate of the entries of the R–matrix. The important point is that these entries are ratio of 5 q–factorials, and each q–factorial grows exponentially with rate given by the Lobachevsky function. The q–factorials are arranged in such a way to deduce that the exponential growth-rate of the entries of the R–matrix is given by the volume of an ideal octahedron. Together with our state-sum formulas for the n–th colored Jones polynomial, it results in the upper bound of Theorem 1.13.

We discuss in Section 9 the proof of Theorem 1.10 and Theorem 1.11.

In Section 10 we discuss bounds on the degrees and coefficients of q–holonomic functions. Earlier work of the authors implies that the colored Jones and the cyclotomic functions of a knot are q–holonomic.

In Section 11 we discuss some physics ideas related to the various expansions of the colored Jones function.

Finally, in the Appendices we establish the Volume Conjecture for the Borromean rings using estimates obtained in the proofs of the main results. At the time when the first draft of this paper was written (2004), this was the only hyperbolic link for which the volume conjecture was established. Since then the volume conjecture has been proved for several other hyperbolic links; see eg van der Veen [34].

Note that Theorem 6.3 and Theorem 10.3 are not used in the proofs of our results, and are of independent interest.

The logical dependence of the main theorems is as follows:

$$\begin{tikzcd}
\text{Theorem 1.9} & \text{Theorem 1.10} & \text{Theorem 1.11} & & & \\
& \text{Theorem 1.3} \arrow[ul] \arrow[u] \arrow[ur] & & \text{Theorem 1.6} & & \\
\text{Theorem 1.2} \arrow[ur] & \text{Theorem 2.2} \arrow[u] & \text{Theorem 5.3} \arrow[ur] & \text{Theorem 1.5} \arrow[u] & & \\
& \text{Theorem 4.2} \arrow[u] \arrow[ur] & & \text{Theorem 1.12} & \text{Theorem 1.13} & \\
& & \text{Theorem 6.2} \arrow[ul] \arrow[ur] & \text{Proposition 8.5} \arrow[ur] & & \text{Proposition 8.6} \arrow[ul]
\end{tikzcd}$$

Acknowledgments The authors wish to thank I Agol, D Boyd, N Dunfield, D Thurston and D Zeilberger for many enlightening conversations, and the anonymous referee for his careful reading of our manuscript.

The authors were supported in part by National Science Foundation.

2 Weak versus strong convergence

2.1 A lemma from complex analysis

To prove Theorem 1.3, we need to improve the weak convergence of Theorem 1.2 to the strong convergence. This uses the next lemma on normal families that is sometimes referred to by the name of *Vitali* or *Montel's* theorem. For a reference, see Hille [16] and Schiff [31]. The lemma exhibits the power of holomorphy, coupled with uniform boundedness.

Lemma 2.1 *If*

$$f_n\colon \{z \in \mathbb{C} : |z| < r\} \to \{z \in \mathbb{C} : |z| \leq M\}$$

is a uniformly bounded sequence of holomorphic functions such that for every $m \geq 0$, we have

$$\lim_{n\to\infty} f_n^{(m)}(0) = a_m.$$

Then,

- *the limit* $f(z) = \lim_n f_n(z)$ *exists pointwise for all* z *with* $|z| < r$,
- f *is holomorphic,*
- *the convergence is uniform on compact subsets,*
- *for every* m, $f^{(m)}(0) = a_m$.

In other words, weak convergence and uniform boundedness imply strong convergence.

Proof The sequence $\{f_n\}_n$ is uniformly bounded, so it is a normal family, and contains a convergent subsequence $f_j \to f$. Convergence is uniform on compact sets, and f is holomorphic, and for every $m \geq 0$, $\lim_j f_j^{(m)}(0) = f^{(m)}(0) = a_m$.

If $\{f_n\}_n$ is not convergent (uniformly on compact sets), since it is a normal family, then there exist two subsequences that converge to f and g respectively, with $f \neq g$. Applying the above discussion, it follows that f and g are holomorphic functions with equal derivatives of all orders at 0. Thus, $f = g$, giving a contradiction. □

Theorem 1.3 follows from Lemma 2.1 and the following result, whose proof will be given in Section 4.

Theorem 2.2 (Proof in Section 4.2) *For every knot* K *there exists an open neighborhood* U_K *of* $0 \in \mathbb{C}$ *and a positive number* M *such that for* $\alpha \in U_K$, *and all* $n \geq 1$, *we have*

$$|f_{K,n}(\alpha)| < M.$$

2.2 The main difficulty for uniform bounds

Before we proceed with the proof of Theorem 2.2, let us point out the main difficulty. As we will see later, $J_{K,n}(q)$ is a Laurent polynomial in q whose span (ie, the exponents of its monomials) are $O(n^2)$ and whose coefficients are $e^{O(n)}$. In addition, due to our normalization, $J_{K,n}(1) = 1$. In other words, the $O(n^2)$ many exponentially growing coefficients of $J_{K,n}(q)$ add up to 1. When we evaluate $J_{K,n}(e^{\alpha/n})$, we want to bound the result independent of n. This will happen only if major cancellations occur. How can we control these cancellations? The answer to this is a key cyclotomic expansion of the colored Jones function, which we review next.

3 Two expansions of the colored Jones polynomial

3.1 The loop expansion

With $q = e^h$, one has

$$J_{K,n}(e^h) = \sum_{i=0}^{\infty} a_{K,i}(n)\, h^i \in \mathbb{Q}[\![h]\!].$$

It turns out that $a_{K,i}(n)$ is a polynomial in n with degree less than or equal to i [3]. Hence there are rational numbers $a_{K,i,j}$, depending on the knot K, such that

$$\begin{aligned} J_{K,n}(e^h) &= \sum_{0\le j\le i} a_{K,i,j}\, n^j h^i = \sum_{0\le i, 0\le j\le i} a_{K,i,j} (nh)^j h^{i-j} \\ &= \sum_{0\le j,k} a_{K,j+k,j} (nh)^j h^k. \end{aligned}$$

If we define

$$R_{K,k}(x) = \sum_{0\le j} a_{K,j+k,j}\, x^j \in \mathbb{Q}[\![x]\!],$$

then we have the *loop expansion*

$$J_{K,n}(e^h) = \sum_{k=0}^{\infty} R_{K,k}(nh) h^k \in \mathbb{Q}[\![h]\!]. \tag{10}$$

It turns out that $R_{K,k}(x) \in \mathbb{Q}(e^x)$ are rational functions for all k. In fact, the MMR Conjecture states that

$$R_{K,0}(x) = \frac{1}{\Delta_K(e^x)} \in \mathbb{Q}[\![x]\!].$$

More generally, Rozansky [30] proves there are Laurent polynomials $P_{K,k}(t) \in \mathbb{Q}[t^{\pm 1}]$ such that in $\mathbb{Q}[\![x]\!]$,

$$R_{K,k}(x) = \frac{P_{K,k}(e^x)}{\Delta_K(e^x)^{2k+1}}.$$

Remark 3.1 For every i, j, the function $K \to a_{K,i,j}$ is a finite type invariant of degree i. Although the polynomials $P_{K,k}(t)$ are not finite type invariants (with respect to the usual crossing change of knots), they are finite type invariants with respect to a loop move described by the first author and Rosansky [11]. We will not use these facts in our paper.

3.2 The cyclotomic expansion

Habiro found another interesting expansion of the colored Jones function, known as the cyclotomic expansion. Although the cyclotomic expansion has important arithmetic consequences, we discuss only its algebraic properties here. Let us define:

$$C_{n,k}(q) = \prod_{j=1}^{k}(q^n + q^{-n} - q^j - q^{-j}) \quad \text{with } C_{n,0}(q) := 1. \tag{11}$$

He showed that there exist unique Laurent polynomials $H_{K,k}(q) \in \mathbb{Z}[q^{\pm 1}]$, $k = 0, 1, \dots$ such that

$$J_{K,n}(q) = \sum_{k=0}^{n-1} C_{n,k}(q)\, H_{K,k}(q). \tag{12}$$

For details, see Habiro [15, Section 6]. Note that our $H_{K,n}(q)$ is $J_K(P_n'')$ in Habiro's notation. We will call the expansion (12) the cyclotomic expansion. Since $C_{n,k}(q) = 0$ if $k \geq n$, the summation in (12) can be assumed from 0 to ∞.

It is possible to solve for $H_{K,n}$ from Equation (12). Explicitly, from [15, Lemma 6.1] one has

$$H_{K,n}(q) = \frac{1}{\{2n+2\}!} \sum_{k=1}^{n+1} (-1)^{n+1-k} \{2k\}\{k\} \begin{bmatrix} 2n+2 \\ n+1-k \end{bmatrix} J_{K,k}(q), \tag{13}$$

where we use the following definitions:

$$\{n\} := q^{n/2} - q^{-n/2}, \qquad \{n\}! := \prod_{i=1}^{n} \{i\},$$

$$\{a\}_b := \frac{\{a\}!}{\{a-b\}!} = \prod_{j=a-b+1}^{a} \{j\}, \quad \begin{bmatrix} a \\ b \end{bmatrix} := \frac{\{a\}!}{\{b\}!\{a-b\}!}.$$

3.3 Comparing the cyclotomic and the loop expansion

In the loop expansion, as well as in the cyclotomic expansion, one should treat q^n and q (where n is the color) as two independent variables. Consider two independent variables z (standing for α) and y (standing for α/n). Let us define the following biholomorphic functions

$$c_k(z,y) = \prod_{j=1}^{k} (e^z + e^{-z} - e^{jy} - e^{-jy}),$$
$$h_{K,k}(z,y) = c_k(z,y) H_{K,k}(e^y).$$

The cyclotomic expansion says that for every n we have

$$f_{K,n}(\alpha) = \sum_{k=0}^{\infty} h_{K,k}(\alpha, \alpha/n) \in \mathbb{Q}[\![\alpha]\!]. \tag{14}$$

The loop expansion is a Taylor expansion in α/n, so we will consider the Taylor expansion in y (around 0) of $h_{K,k}(z,y)$:

$$h_{K,k}(z,y) = \sum_{p=0}^{\infty} d_{k,p}(z)\, y^p,$$

where $d_{k,p}(z)$ (which depends on K) is holomorphic for $z \in \mathbb{C}$.

Comparing the loop and the cyclotomic expansion (Equations (10) and (14)), we obtain:

Lemma 3.2 *For every knot K and every $p \in \mathbb{N}$ we have*

$$R_{K,p}(x) = \sum_{k=0}^{\infty} d_{k,p}(x) \in \mathbb{Q}[\![x]\!] \tag{15}$$

as formal power series in x.

4 A reduction of Theorem 2.2 to estimates of the cyclotomic function

4.1 Uniform bounds of the colored Jones function

In this section we will deduce Theorem 2.2 from estimates of the degree and the coefficients of the cyclotomic expansion of the knot. These estimates will be established in Section 6. By definition $f_{K,n}(\alpha) = J_{K,n}(e^{\alpha/n})$, hence Equation (12) gives that

$$f_{K,n}(\alpha) = \sum_{k=0}^{n-1} C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n}). \tag{16}$$

To have upper bounds for $|f_{K,n}(\alpha)|$ we will need bounds for $H_{K,k}(e^{\alpha/n})$ and the "kernel" $C_{n,k}(e^{\alpha/n})$ (the kernel does not depend on the knot K).

Definition 4.1 For a Laurent polynomial $f(q)=\sum_k a_k q^k$, we define its l^1–norm by

$$\|f\|_1=\sum_k |a_k|.$$

The proof of the following Theorem, which gives bounds for the degrees and the l^1–norm of $H_{K,n}$, will be given in Section 7.

Theorem 4.2 (Proof in Section 7) *For every knot K, there are positive constants A_0, A_1 (depending on K) such that for all $n\in\mathbb{N}$ we have*

(a) $H_{K,n}(q)=\displaystyle\sum_{j=-A_0n^2}^{A_0n^2} b_{j,n}q^j$.

(b) $\|H_{K,n}\|_1\le A_1^n$.

The next lemma follows from Theorem 4.2 and an elementary estimate.

Lemma 4.3 *Suppose $|\alpha|<1$.*

(a) *For every knot K there is a constant A_2 such that for every $0\le k\le n$, we have*

$$|H_{K,k}(e^{\alpha/n})|\le (A_2)^k.$$

(b) *There is a constant $A_3>0$ such that every $0\le k\le n$ we have*

$$|C_{n,k}(e^{\alpha/n})|\le (A_3)^k|\alpha|^k.$$

Proof (a) By Theorem 4.2(a),

$$H_{K,k}(e^{\alpha/n})=\sum_{j=-A_0k^2}^{A_0k^2} b_{j,k}e^{j\alpha/n}.$$

From the bounds for j and $k\le n$ one has that $|j/n|\le A_0k$, hence $|e^{j\alpha/n}|\le \exp(A_0\,k\,|\Re(\alpha)|)\le\exp(k\,A_0)$. From the above equation one has

$$|H_{K,k}(e^{\alpha/n})|\le \|H_{K,k}\|_1\,(\exp A_0)^k.$$

Using Theorem 4.2, it is enough to take $A_2=A_1\exp(A_0)$.

(b) By definition,

$$C_{n,k}(e^{\alpha/n})=\prod_{j=1}^{k}(e^{\alpha}+e^{-\alpha}-e^{j\alpha/n}-e^{-j\alpha/n})=\prod_{j=1}^{k}\left(g(\alpha)-g(j\alpha/n)\right),$$

where $g(z) = e^z + e^{-z}$. One has $g'(z) = e^z - e^{-z}$, hence for z on the interval connecting α and $j\alpha/n$, with $0 \le j \le n$, one has $|g'(z)| \le 2\exp(|\alpha|) \le 2e$. By the mean value theorem, we have, for $0 \le j \le k \le n$,

$$|g(\alpha) - g(j\alpha/n)| \le 2e|\alpha - j\alpha/n| \le 2e|\alpha|.$$

It follows that

$$|C_{n,k}(e^{\alpha/n})| \le (2e)^k |\alpha|^k.$$

It is enough to take $A_3 = 2e$. □

4.2 Theorem 4.2 implies Theorem 2.2

It follows from Lemma 4.3 that for $0 \le k \le n$ and $|\alpha| < 1$, we have

$$|C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n})| \le |\alpha A_2 A_3|^k.$$

Let us choose U_K to be the disk centered at the 0, with radius $1/(2A_2A_3 + 1)$, then $|\alpha A_2 A_3| < 1/2$ for $\alpha \in U_K$. Equation (16) and the above estimate imply that for all n and all $\alpha \in U_K$, we have

$$|f_{K,n}(\alpha)| \le \sum_{k=0}^{n-1} |C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n})| \le \sum_{k=0}^{n-1} (1/2)^k < 2,$$

which concludes the proof of Theorem 2.2, assuming Theorem 4.2. □

5 A reduction of Theorem 1.6 to estimates of the cyclotomic function

5.1 Some estimates

The following is a higher order version of Lemma 4.3. The proof is similar.

Lemma 5.1 *Suppose $|\alpha| < 1$.*

(a) *For $1 \le k \le n$, $0 \le l$, and y on the interval from 0 to α/n we have*

$$\left| \frac{\partial^l}{\partial y^l} c_k(\alpha, y) \right| < (A_3)^k |\alpha|^{k-l} k^{2l}.$$

(b) *For any $y \in \mathbb{C}$, $|y| < 1/n$ and $1 \le k \le n$ we have*

$$\left| \frac{\partial^l}{\partial y^l} H_{K,k}(e^y) \right| < (A_2)^k (A_0)^l k^{2l}.$$

Proof (a) We have $c_k(\alpha, y) = \prod_{j=1}^k g_j$, where

$$g_j = e^{\alpha} + e^{-\alpha} - e^{jy} - e^{-jy}.$$

By the Leibniz rule, the l–th derivative (with respect to y) of c_k is the sum

$$\frac{\partial^l}{\partial y^l} c_k(\alpha, y) = \sum_{|\mathbf{l}|=l} \binom{l}{l_1, \dots, l_k} t(\mathbf{l}), \quad \text{where } t(\mathbf{l}) = \prod_{j=1}^k g_j^{(l_j)}. \tag{17}$$

Here $\mathbf{l} = (l_1, \dots, l_k), |\mathbf{l}| := \sum_{j=1}^k l_j$, $l_j \geq 0$. We will estimate each term $t(\mathbf{l})$. Fix $\mathbf{l} = (l_1, \dots, l_k)$. We consider two cases, $l_j = 0$ and $l_j > 0$.

Suppose $l_j = 0$. Then $g_j^{(l_j)} = g_j = e^{\alpha} + e^{-\alpha} - e^{jy} - e^{-jy}$. Since $j \leq k \leq n$, the interval connecting α and jy lies totally in the disk of radius $|\alpha|$ (remember that $|y| \leq |\alpha|/n$). As in the proof of Lemma 4.3, we have

$$|g_j| = |(e^{\alpha} + e^{-\alpha}) - (e^{jy} + e^{-jy})| \leq (2e)|\alpha|. \tag{18}$$

Now suppose $l_j > 0$. Then

$$g_j^{(l_j)} = -e^{jy} j^{l_j} - e^{-jy}(-j)^{l_j}.$$

It is clear $|e^{\pm\alpha}| < e$. Since $|j| \leq |k|$ and $|jy| < |\alpha|$, we have $|e^{\pm jy}(\pm j)^{l_j}| < ek^{l_j}$. Hence

$$|g_j^{(l_j)}| < (2e)\, k^{l_j}. \tag{19}$$

Taking the product over j, using (18), (19) and $\sum l_j = l$, we get

$$\begin{aligned} |t(\mathbf{l})| &< (2e)^k\, k^l\, |\alpha|^{\#\{l_j=0\}} \\ &< (2e)^k\, k^l\, |\alpha|^{k-l} \end{aligned}$$

because $|\alpha| < 1$ and $\#\{l_j = 0\} \geq k - l$. Since

$$\sum_{|\mathbf{l}|=l} \binom{l}{l_1, \dots, l_k} = k^l,$$

from (17) and the above estimate for $t(\mathbf{l})$, we get the result with $A_3 = 2e$.

(b) By Theorem 4.2(a),

$$\frac{\partial^l}{\partial y^l} H_{K,k}(e^y) = \sum_{j=-A_0k^2}^{A_0k^2} b_{j,k} e^{jy} j^l.$$

From the bounds for j and $k \le n$ one has $|e^{jy}| \le \exp(A_0 k)$ and $|j^l| \le (A_0 k^2)^l$. From the above equation one has

$$\left|\frac{\partial^l}{\partial y^l} H_{K,k}(e^y)\right| \le \|H_{K,k}\|_1 \,(\exp A_0)^k (A_0 k^2)^l.$$

Using Theorem 4.2, it is enough to take $A_2 = A_1 \exp(A_0)$. □

Corollary 5.2 *For every knot K there are positive constants A_4, A_5 such that*

(a) *for $0 \le k$, $0 \le N$, and $|\alpha| < 1$ and y on the interval from 0 to α/k, we have*

$$\left|\frac{\partial^N}{\partial y^N} h_{K,k}(\alpha, y)\right| < |\alpha A_4|^{k-N} (A_5 k^2)^N.$$

(b) *for $0 \le k$, $0 \le N$, and $|\alpha| < 1$ and every positive integer n, we have*

$$\left|h_{K,k}(\alpha, \alpha/n) - \sum_{p=0}^{N-1} d_{k,p}(\alpha)\,(\alpha/n)^p\right| < \frac{1}{N!}\left(\frac{\alpha}{n}\right)^N \,|\alpha A_4|^{k-N} (A_5 k^2)^N.$$

Proof (a) The N–th derivative of $h_{K,k}(\alpha, y)$, which is the product of $c_k(\alpha, y)$ and $H_{K,k}(e^y)$, is the sum of 2^N terms, each of the form

$$\frac{\partial^l}{\partial y^l} c_k(\alpha, y)\, \frac{\partial^{N-l}}{\partial y^{N-l}} H_{K,k}(e^y).$$

The absolute value of the above term is bounded by $|\alpha|^{k-l} (A_2 A_3)^k (A_0)^{N-l} k^{2N}$ using Lemma 5.1, which, in turn, is less than $|\alpha|^{k-N} (A_2 A_3)^k (A_0)^N k^{2N}$. Hence, multiplied by 2^N we get

$$\left|\frac{\partial^l}{\partial y^l} h_{K,k}(\alpha, y)\right| < 2^N \times |\alpha|^{k-N} (A_2 A_3)^k (A_0)^N k^{2N} = (\alpha A_2 A_3)^{k-N} (2 A_0 A_2 A_3 k^2)^N.$$

It is enough to take $A_4 = A_2 A_3$ and $A_5 = 2 A_0 A_2 A_3$.

(b) By Taylor's Theorem,

$$\left|h_{K,k}(\alpha, \alpha/n) - \sum_{p=0}^{N-1} d_{k,p}(\alpha)\,(\alpha/n)^p\right| < \frac{1}{N!}\left(\frac{\alpha}{n}\right)^N \max\left|\frac{\partial^N}{\partial y^N} h_{K,k}(\alpha, y)\right|,$$

where max is taken when y is on the interval connecting 0 and α/n. Using the estimate of part (a), we get the result. □

5.2 Theorem 4.2 implies Theorem 1.6

To simplify notation, let us define, for a knot K,

$$f_{K,n}^{[N]}(z) := J_{K,n}(e^{z/n}) - \sum_{k=0}^{N-1} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}} \left(\frac{z}{n}\right)^k . \tag{20}$$

Theorem 1.6 follows from Theorem 1.5, Lemma 2.1 and the following uniform bound.

Theorem 5.3 *For every knot K there exists an open neighborhood $\widetilde{U}_K$ of $0 \in \mathbb{C}$ such that for every $N \geq 0$ there exists a positive number M_N such that for $\alpha \in \widetilde{U}_K$, and all $n \geq 0$, we have*

$$\left| \left(\frac{n}{\alpha}\right)^N f_{K,n}^{(N)}(\alpha) \right| < M_N .$$

Proof of Theorem 5.3, assuming Theorem 4.2 We have the following identities, where the second follows from (14) and (15):

$$\begin{aligned} f_{K,n}^{[N]}(\alpha) &= f_{K,n}(\alpha) - \sum_{p=0}^{N-1} R_{K,p}(\alpha) \left(\frac{\alpha}{n}\right)^p \\ &= \sum_{k=0}^{\infty} h_{K,k}\left(\alpha, \frac{\alpha}{n}\right) - \sum_{p=0}^{N-1} \left(\sum_{k=0}^{\infty} d_{k,p}(\alpha) \right) \left(\frac{\alpha}{n}\right)^p \\ &= \sum_{k=0}^{\infty} \left(h_{K,k}\left(\alpha, \frac{\alpha}{n}\right) - \sum_{p=0}^{N-1} d_{k,p}(\alpha) \left(\frac{\alpha}{n}\right)^p \right) \end{aligned}$$

Using the estimate in Corollary 5.2, we see that

$$\left| \left(\frac{n}{\alpha}\right)^N f_{K,n}^{[N]}(\alpha) \right| < \frac{1}{N!} \sum_{k=0}^{\infty} |\alpha A_4|^{k-N} (A_5 k^2)^N .$$

If $|\alpha A_4| < 1$, the series of the right hand side is absolutely convergent. It is enough to take $\widetilde{U}_K$ to be the disk centered at 0 with radius $1/(2A_4+1)$. This proves Theorem 5.3, assuming Theorem 4.2. □

6 Bounds for the degree and coefficients of the colored Jones function

6.1 Bounds for the degree

In this section we give a bound for the coefficients of the colored Jones polynomial, and deduce Theorem 1.12. This and the next section are logically independent from the previous Section 4 and Section 5.

For a Laurent polynomial $f(q) = \sum_{k=m}^{M} a_k q^k$, with $a_m a_M \neq 0$, let us define $\deg_+(f) = M$ and $\deg_-(f) = m$. In [22] the second author showed that there are quadratic bounds for the degrees of the colored Jones polynomial.

Suppose the knot K has a planar projection with $c+2$ crossings. Let ω be the writhe number, ie the number of positive crossing minus the number of negative ones. Then by [22, Proposition 2.1], taking into account the change of variable, the framing, and the normalization, one has the following bounds for the degrees of $J_{K,n}(q)$.

Proposition 6.1 *With the above notation, there are constants $s_\pm$ such that*

$$\deg_+(J_{K,n}) \leq \frac{(c+2)(n-1)^2 + 2(n-1)(s_+-1) - \omega(n^2-1)}{4},$$
$$\deg_-(J_{K,n}) \geq -\frac{(c+2)(n-1)^2 + 2(n-1)(s_--1) + \omega(n^2-1)}{4}.$$

The constants $s_\pm$ have transparent geometric meaning, but we don't need their exact values here.

Another proof of the quadratic bounds, though less as explicit, for the degrees of the colored Jones polynomial using the theory of q–holomorphic functions is given in Section 10.1.

6.2 Bounds for the coefficients

For a Laurent polynomial $f \in \mathbb{Z}[q^{\pm 1/4}]$, we define $\|f\|_1$ as in Definition 4.1, ie $\|f\|_1$ is the sum of the absolute values of its coefficients. Observe that

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1, \quad \|fg\|_1 \leq \|f\|_1 \, \|g\|_1. \tag{21}$$

Since $\|\{j\}\|_1 = 2$, we have, for $k \leq n$

$$\|\{a\}_k\|_1 = \left\| \prod_{j=a-k+1}^{a} \{j\} \right\|_1 \leq 2^k \leq 2^n. \tag{22}$$

It is known that the quantum binomial $\begin{bmatrix} m \\ k \end{bmatrix}$ is a Laurent polynomial in $q^{1/2}$ with *positive* integer coefficients, hence its l^1–norm is obtained by putting $q^{1/2} = 1$, which is the classical binomial $\binom{m}{k}$. One has, if $m \le n$,

$$\left\| \begin{bmatrix} m \\ k \end{bmatrix} \right\|_1 = \binom{m}{k} \le 2^m \le 2^n. \tag{23}$$

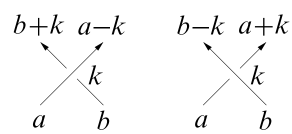
Theorem 6.2 *For every knot K of $c+2$ crossings and every n we have*

$$\| J_{K,n} \|_1 \le n^c 4^{cn}. \tag{24}$$

Proof The proof of the Theorem is easy using the state sum definition of the colored Jones polynomial: The colored Jones polynomial is the sum, over all states, of the weights of the states. There are n^c states, the weight of each is the product of several q–factorials and q–binomial coefficients for which an upper bound can be easily found. Let us now go to the details of the proof.

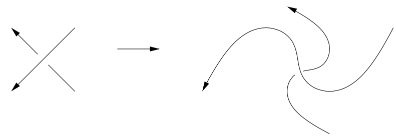
The knot K is the closure of a $(1,1)$–tangle T (or long knot), with orientation given by the direction from the bottom boundary point to the top boundary point. The crossing points of the diagram of T (on the standard 2–plane) break T into $2c+5$ arcs, two of which are *boundary* (ie each contains a boundary point of T). The two crossings adjacent to the boundary arcs are called *boundary crossings*.

To get from $c+2$ to c in the estimate, we will choose the $(1,1)$–tangle T such that (1) when going along T, starting at the bottom boundary point, we must pass the very first crossing (resp. very last crossing) by an overpass (respectively, an underpass) and (2) the two strands at each crossing are pointing upwards, as in the following figure:

(25)


Here is how to get such a $(1,1)$–tangle T. Consider a diagram of K on a 2–sphere S^2. The $c+2$ crossings break the knot diagram into $2c+4$ arcs. At each crossing we have an overpass and an underpass. When we go along the knot starting at some point, following the direction of the orientation, we pass through all these underpasses and overpasses. Hence there must be an arc which starts at an underpass and ends at an overpass, assuming there is at leat one crossing. Remove from S^2 a small disk which is a small neighborhood of a point inside this arc. What is left is a long knot diagram on a disk, which can also be considered as a $(1,1)$–tangle diagram in the strip $\mathbb{R} \times [0,1]$

in the standard 2–plane which satisfies requirement (1). Using the isotopy of the form



which moves crossings (positive or negative) into standard upright position, we get the desired $(1,1)$–tangle.

A state $\mathbf{k}$ is an assignment of numbers, called the colors, to the crossings of the diagram of T, where each color is in $\{0,\dots,n-1\}$. For a fixed state we will color the $2c+5$ arcs as follow. First color the bottom boundary arc by 0. Going along the diagram of T from the bottom boundary point, if we are on an arc of color a and pass a crossing, the next arc will have color $a+k$ or $a-k$, according as the pass is an underpass or an overpass; see (25). Here k is the color of the crossing.

We will only consider states such that the colors of arcs are between 0 and $n-1$ and the color of the top boundary arc is 0. The under/overpass configuration at the two boundary crossings ensures that the two boundary crossings have color 0, otherwise the arcs next to the two boundary arcs would have negative colors. It follows that the number of states is at most n^c.

The weights of the positive crossing (on the left) and negative crossing (on the right in (25)) are

$$R_+(n;a,b,k) = (\text{unit}) \begin{bmatrix} b+k \\ k \end{bmatrix} \{n-1+k-a\}_k, \tag{26}$$

$$R_-(n;a,b,k) = (\text{unit}) \begin{bmatrix} a+k \\ k \end{bmatrix} \{n-1+k-b\}_k, \tag{27}$$

where (unit) stands for $\pm$ a power of $q^{\pm 1/4}$, which does not affect the l^1 norm. Note that both $a+k$ and $b+k$ in the above formulas are between 0 and $n-1$.

The weight of a maximum/minimum point is a also $\pm$ a power of $q^{\pm 1/4}$, whose exact formula is not important for us. Let $F(n,\mathbf{k})$ denote the product of weights of all the crossings and all the extreme points. Then

$$J_{K,n}(q) = \sum_{\mathbf{k}} F(n,\mathbf{k}). \tag{28}$$

Using the estimates (23) and (22), we see that $\|R_\pm(n;a,b,k)\|_1 \le 4^n$. Since the weight of the two boundary crossing is just a unit, the l^1 norm of $F(n,\mathbf{k})$ is less than 4^{cn}. From (28) and the fact that there are n^c states, we get $\|J_{K,n}\|_1 \le n^c 4^{cn}$. $\square$

Since there is a constant b such that $n^c \le b^n$, we have the following.

Theorem 6.3 *For every knot K, there is a constant A_6 such that for every positive integer n,*

$$\| J_{K,n} \|_1 \le (A_6)^n.$$

6.3 Proof of Theorem 1.12

Fix a knot with $c+2$ crossings. The bounds for the degrees of $J_{K,n}$ (see Proposition 6.1) allow us to write

$$J_{K,n}(q) = \sum_j a_{n,j} q^j,$$

where $|j| \le n^2(c+2+|w|)/4 + O(n)$. For such j, we have

$$|e^{j\alpha/n}| = e^{\Re(j\alpha)/n} \le e^{(c+2+|w|)n/4 + O(1))|\Re(\alpha)|}.$$

Using Theorem 6.2 we get

$$|J_{K,n}(e^{\alpha/n})| \le n^c 4^{cn} e^{(c+2+|w|)n/4+O(1))|\Re(\alpha)|}.$$

Thus, $$\frac{1}{n} \log |f_{K,n}(\alpha)| \le c \log 4 + \frac{c+2+|w|}{4} |\Re(\alpha)| + O\left(\frac{\log n}{n}\right).$$

The result follows from the observation that $|\omega| \le c+2$, since $c+2$ is the total number of crossings. □

7 Proof of Theorem 4.2

The goal of this Section is to prove Theorem 4.2.

7.1 The bound for degrees of $H_{K,n}$

Note that

$$\deg_\pm(fg) = \deg_\pm(f) + \deg_\pm(g) \quad \text{and} \quad \deg_+(f+g) \le \max(\deg_+(f), \deg_+(g)).$$

From $\deg_\pm \{k\} = \pm k/2$, we get

$$\deg_\pm(\{k\}!) = \pm k(k+1)/4, \quad \deg_\pm\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \pm k(n-k)/2.$$

From these and Equation (13) we get

$$\begin{aligned}&\deg_+(H_{K,n}(q))\\ &\le \max_{1\le k\le n+1}\left(-\frac{(2n+2)(2n+3)}{4}+k+\frac{k}{2}+\frac{(n+1+k)(n+1-k)}{2}+\deg_+(J_{K,k})\right).\end{aligned}$$

Using Proposition 6.1 for the upper bound of $\deg_+(J_{K,k})$, after a simplification, we get

$$\deg_+(H_{K,n}(q)) \le \max_{1\le k\le n+1}\left(-\frac{n(n+3)}{2}+\frac{c(k-1)^2}{4}+\frac{(k-1)s_+}{2}+\frac{|\omega|(k^2-1)}{4}\right).$$

The right hand side reaches maximum when $k=n+1$. Using $\omega \le c+2$, we have

$$\deg_+(H_{K,n}(q)) \le n^2c/2+n(s_+ + c-1)/2.$$

A similar calculation shows that

$$\deg_-(H_{K,n}(q)) \ge -(n^2c/2+n(s_- + c-1)/2).$$

If we choose A_0 bigger than c and $|s_\pm + c-1|$, then we have $|\deg_\pm(H_{K,n})| \le A_0 n^2$. This proves the first statement of Theorem 4.2.

7.2 The bound for the l^1–norm of $H_{K,n}$

Multiply both sides of (13) by $\{2n+2\}!$, then use (23) and Theorem 6.3, we see that there is a constant A_7 such that

$$\|\{2n+2\}! H_{K,n}(q)\|_1 \le (A_7)^n. \tag{29}$$

The polynomials

$$\widetilde{H}_{K,n}(q) := q^{A_0 n^2} H_{K,n}(q) \quad \text{and} \quad g(q) := \widetilde{H}_{K,n}(q)\prod_{j=1}^{2n+2}(1-q^j)$$

have only nonnegative degrees in q, with $\deg_+(\widetilde{H}_{K,n}) \le 2A_0 n^2$:

$$\widetilde{H}_{K,n}(q) = \sum_{k=0}^{2A_0n^2} a_k q^k \tag{30}$$

Since $g(q)$ is the product of the polynomial on the left hand side of (29) and a power of q, we have

$$\|g(q)\|_1 \le (A_7)^n. \tag{31}$$

There are estimates of l^1–norm using Mahler measure [23]. However, the estimat e (31) is weak: the inequalities of Mahler imply an exponential upper bound on the Mahler measure of $H_{K,n}(q)$, and a doubly exponential upper bound on the l^1–norm of $H_{K,n}(q)$. The following estimate, which does not follow from Mahler measure considerations, was communicated to us by D Boyd. Since

$$\widetilde{H}_{K,n}(q) = g(q)\frac{1}{\prod_{k=j}^{2n+2}(1-q^j)},$$

we have that

$$a_k = \sum_{i=0}^{k} b_i c_{k-i}, \quad \text{where} \quad g(q) = \sum_k b_k q^k \quad \text{and} \quad \frac{1}{\prod_{k=1}^{2n+2}(1-q^k)} = \sum_{k=0}^{\infty} c_k q^k.$$

Note that c_k is the number of partitions of k of length $\leq 2n+2$. Hence $0 \leq c_{k-1} \leq c_k$, and $c_k \leq p_k$, where p_k is the number of partitions of k. Using the growth rate of p_k (see Andrews [2]), we see that there is a constant A_8 such that

$$p_k < (A_8)^{\sqrt{k}}. \tag{32}$$

The crucial part of the above inequality is the exponent $\sqrt{k}$. Now we can easily obtain the desired upper bounds for $\|\widetilde{H}_{K,n}\|_1$. Since $a_k = \sum_{i=0}^{k} b_i c_{k-i}$ we have

$$\begin{aligned} |a_k| \leq \sum_{i=0}^{k} |b_i| c_{k-i} \leq \Bigg(\sum_{i=0}^{k} |b_i|\Bigg) c_k \leq \|g(q)\|_1 c_k \\ \leq (A_7)^n (A_8)^{n\sqrt{2A_0}} \quad \text{by (31), (32) and } k \leq 2A_0 n^2. \end{aligned}$$

It follows that, for $n \geq 1$,

$$\|\widetilde{H}_{K,n}\|_1 \leq \sum_{k=0}^{2A_0 n^2} |a_k| \leq 2A_0 n^2 (A_7)^n (A_8)^{n\sqrt{2A_0}} \leq (A_1)^n,$$

for appropriate A_1. This completes the proof of Theorem 4.2.

8 Growth rates of R–matrices and the Lobachevsky function

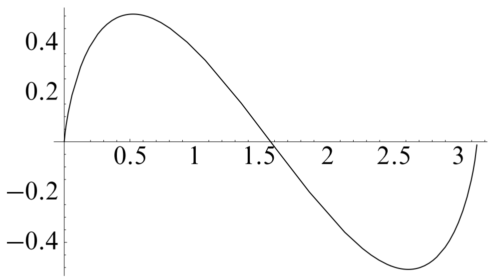
8.1 The Lobachevsky function

In Section 6 we got a simple but crude estimate for the l^1–norm of the R–matrices, which are a ratio of five quantum factorials. In this largely independent section we will give refined (and optimal) estimates for the growth rate of the R–matrices. These estimates reveal the close relationship between hyperbolic geometry and the asymptotics of the quantum factorials.

Recall that the *Lobachevsky function* is given by

$$\Lambda(z) = -\int_0^z \log|2\sin x|\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2nz)}{n^2}.$$

The Lobachevsky function is an odd, periodic function with period π. Its graph for $z \in [0,\pi]$ is:



Definition 8.1 If $f(q) \in \mathbb{Z}[q^{\pm 1/4}]$, let us denote by $\mathrm{ev}_n(f)$ the *evaluation* of f at $q^{1/4} = e^{\pi i/(2n)}$.

For $0 \le k \le n$ we have

$$\mathrm{ev}_n|\{k\}| = |e^{k\pi i/n} - e^{-k\pi i/n}| = 2\sin(k\pi/n),$$

hence,

$$\log(\mathrm{ev}_n|(\{j\}!|)|) = \sum_{k=1}^{j}\log|2\sin(k\pi/n)|,$$

which is very closely related to a Riemann sum of the integral in the definition of the Lobachevsky function. It is not surprising to have the following.

Proposition 8.2 *For every $\alpha \in (0,1)$ we have*

$$\log|\mathrm{ev}_n(\{\lfloor \alpha n\rfloor\}!)| = -\frac{n}{\pi}\Lambda(\pi\alpha) + O(\log n).$$

Here $O(\log n)$ is a term which is bounded by $C\log n$ for some constant C independent *of α.*

Remark 8.3 The proof reveals an asymptotic expansion of the form

$$\mathrm{ev}_n(\{\lfloor \alpha n\rfloor\}!) \sim n^{\theta}\exp\Big(-\frac{n}{\pi}\Lambda(\pi\alpha)\Big)\Big(C_0 + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\Big).$$

for explicitly computable constants C_i and θ.

Proof Recall the *Euler–MacLaurin summation formula* with error term (see for example, Olver [26, Chapter 8])

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{m-1} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(b) - f^{(2k-1)}(a)) + R_m(a,b,f),$$

where B_k is the k–th *Bernoulli number* and the error term has an estimate

$$|R_m(a,b,f)| \le (2-2^{1-2m})\frac{|B_{2m}|}{(2m)!}\int_a^b |f^{(2m)}(x)|\,dx.$$

Applying the above formula for $m = 1$ to $f(x) = \log(2\sin x\pi/n)$, we have

$$\begin{aligned}
\log\Bigg(\prod_{k=1}^{\lfloor \alpha n\rfloor} 2\sin(k\pi/n)\Bigg)
&= \frac{1}{2}(f(1) + f(\lfloor \alpha n\rfloor)) + \int_1^{\lfloor \alpha n\rfloor} \log(2\sin(t\pi/n))\,dt + R_1(1,\lfloor \alpha n\rfloor, f)\\
&= \frac{1}{2}(f(1) + f(\alpha n) + \int_1^{\alpha n} \log(2\sin(t\pi/n))\,dt + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n)\\
&= \frac{1}{2}(f(1) + f(\alpha n)) + \frac{n}{\pi}\int_{\pi/n}^{\pi\alpha} \log|2\sin(u)|u + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n)\\
&= \frac{1}{2}(f(1) + f(\alpha n)) + \frac{n}{\pi}\left(-\Lambda(\pi\alpha) + \Lambda\left(\frac{\pi}{n}\right)\right) + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n).
\end{aligned}$$

Here $\epsilon(\alpha,n)$ comes from adjusting the boundary of integration and satisfies $|\epsilon(\alpha,n)| = O(1)$. Note that

$$\frac{1}{2}|f(1) + f(\alpha n)| = O(\log n).$$

Moreover, $f''(x) = \frac{\pi^2}{n^2}(\csc(\pi x/n))^2 > 0$. Hence

$$\int_1^{\lfloor \alpha n\rfloor} |f''(x)|\,dx = \int_1^{\lfloor \alpha n\rfloor} f''(x)\,dx \le \int_1^{\alpha n} f''(x)\,dx = \frac{\pi}{n}\left(\cot(\alpha\pi) - \cot\left(\frac{\pi}{n}\right)\right).$$

It follows easily that

$$|R_1(1,\lfloor \alpha n\rfloor, f)| = O(1).$$

Furthermore, using L'Hospital's rule, one can see that

$$\frac{n}{\pi}\left|\Lambda\left(\frac{\pi}{n}\right)\right| = O(\log n).$$

The result follows. □

Corollary 8.4 *For every $\alpha \in (0,1)$ and any fixed number d we have*

$$\log|\mathrm{ev}_n(\{\lfloor \alpha n + d \rfloor\}!)| = -\frac{n}{\pi}\Lambda(\pi\alpha) + O(\log n).$$

Proof There is $\varepsilon > 0$ such that for big enough n, we have $\varepsilon \le x/n \le 1-\varepsilon$ for every integer x between $\lfloor \alpha n \rfloor$ and $\lfloor \alpha n + d \rfloor$. For such x, we have $0 < 2\sin\varepsilon\pi < 2\sin(x\pi/n) < 2$, so there is a constant M such that $|\log 2\sin(x\pi/n)| < M$. There are at most $|d|+1$ such values of x. Hence the difference between $\log|\mathrm{ev}_n(\{\lfloor \alpha n + d \rfloor\}!)|$ and $\log|\mathrm{ev}_n(\{\lfloor \alpha n \rfloor\}!)|$ by absolute value is less than $(|d|+1)M$, a constant. The result follows. □

8.2 Asymptotics of the R–matrix using ideal octahedra

Since the entries of the R–matrix are given by ratios of five quantum factorials (see Equation (26)), Proposition 8.2 gives a formula for the asymptotic behavior of the entries of the R–matrix when evaluated at $e^{2\pi i/n}$.

Proposition 8.5 (a) *Suppose that α, β, κ are real numbers that satisfy the inequalities*

$$\alpha, \beta, \kappa \in [0,1] \quad 0 \le \beta + \kappa \le 1, \quad 0 \le \alpha - \kappa \le 1. \tag{33}$$

Then the limit

$$r_+(\alpha,\beta,\kappa) := \lim_{n\to\infty} \frac{1}{n} \log|\mathrm{ev}_n(R_+(n; \lfloor n\alpha \rfloor, \lfloor n\beta \rfloor, \lfloor n\kappa \rfloor))|, \tag{34}$$

exists and is equal to

$$r_+(\alpha,\beta,\kappa) = [-\Lambda(\pi(\beta+\kappa)) + \Lambda(\pi\beta) + \Lambda(\pi\kappa) - \Lambda(\pi\alpha) + \Lambda(\pi(\alpha-\kappa))]/\pi. \tag{35}$$

(b) *The quantity $r_+(\alpha,\beta,\kappa)$ equals to $1/(2\pi)$ times the volume of an ideal octahedron with vertices*

$$(0, 1, \infty, z_\kappa, (z_\beta z_\kappa - 1)/(z_\beta - 1), z_\alpha) \in (\mathbb{C} \setminus \{0,1\})^6, \tag{36}$$

where $(z_\alpha, z_\beta, z_\kappa) = (e^{2\pi i\alpha}, e^{2\pi i\beta}, e^{2\pi i\kappa})$.

(c) *Suppose that α, β, κ are real numbers that satisfy*

$$\alpha, \beta, \kappa \in [0,1] \quad 0 \le \alpha + \kappa \le 1, \quad 0 \le \beta - \kappa \le 1.$$

Then the limit

$$r_-(\alpha,\beta,\kappa) := \lim_{n\to\infty} \frac{1}{n} \log|\mathrm{ev}_n(R_-(n; \lfloor n\alpha \rfloor, \lfloor n\beta \rfloor, \lfloor n\kappa \rfloor))|.$$

exists and is equal to

$$r_-(\alpha,\beta,\kappa) = r_+(\beta,\alpha,\kappa). \tag{37}$$

Proof (a) Observe that

$$|\mathrm{ev}_n(\{j\})| = |\mathrm{ev}_n(\{n-j\})| = 2\sin(j\pi/n) \text{ and } |\mathrm{ev}_n(\{n-1\}!)| = \prod_{j=1}^{n-1} 2\sin(j\pi/n) = n.$$

From these, we have that

$$|\mathrm{ev}_n(\{j\}!)| = \frac{n}{|\mathrm{ev}_n(\{n-1-j\}!)|}. \tag{38}$$

Using (26) and then (38), we have

$$\begin{aligned} |\mathrm{ev}_n(R_+(n;a,b,k))| &= \frac{|\mathrm{ev}_n(\{b+k\}!)|\,|\mathrm{ev}_n(\{n-1+k-a\}!)|}{|\mathrm{ev}_n(\{b\}!)|\,|\mathrm{ev}_n(\{k\}!)|\,|\mathrm{ev}_n(\{n-1-a\}!)|} \\ &= \frac{|\mathrm{ev}_n(\{b+k\}!)|\,|\mathrm{ev}_n(\{a\}!)|}{|\mathrm{ev}_n(\{b\}!)|\,|\mathrm{ev}_n(\{k\}!)|\,|\mathrm{ev}_n(\{a-k\}!)|}. \end{aligned} \tag{39}$$

Proposition 8.2 concludes the proof of (a).

(b) This was pointed out to us by D Thurston. Although this fact is not used in the proof of Proposition 8.6 nor in the proof of Theorem 1.13, it is an interesting geometric fact. To prove it, recall that the boundary of 3–dimensional hyperbolic space is $\mathbb{C} \cup \{\infty\}$. Let T_z denote the *regular ideal tetrahedron* of shape $z \in \mathbb{C} - \{0,1\}$. T_z is isometric to the ideal tetrahedron with ordered vertices at $0, 1, \infty$ and z in the boundary of 3–dimensional hyperbolic space. For $z \in \mathbb{C} \setminus \{0,1\}$, the ideal octahedron T_z is isometric to $T_{1/(1-z)}$ and $T_{z/(z-1)}$ by an orientation-preserving isometry, and isometric to $T_{1/z}$, $T_{(1-z)/z}$ and $T_{(z-1)/z}$ by an orientation-reversing isometry. Thus, when $z \in \mathbb{C} \setminus \{0,1\}$, we have

$$\begin{aligned} &\mathrm{vol}(T_z) = \mathrm{vol}(T_{1/(1-z)}) = \mathrm{vol}(T_{z/(z-1)}) \\ &\qquad = -\mathrm{vol}(T_{1/z}) = -\mathrm{vol}(T_{(1-z)/z}) = -\mathrm{vol}(T_{(z-1)/z}). \end{aligned} \tag{40}$$

The shape of the ideal tetrahedron with distinct ordered vertices (z_0, z_1, z_2, z_3) in $\mathbb{C} \cup \{\infty\}$ is given by the *cross-ratio*

$$[z_0 : z_1 : z_2 : z_3] = \frac{(z_0-z_3)(z_1-z_2)}{(z_0-z_2)(z_1-z_3)} \tag{41}$$

following the convention of Dupont and Zicker [6, Equation 1.4]. If $(\alpha_1, \alpha_2, \alpha_3)$ denote the three dihedral angles of T_z at opposite pairs of edges, then the volume $\mathrm{vol}(T_z)$ is given by Ratcliffe [27, Theorem 10.4.10]:

$$\mathrm{vol}(T_z) = \Lambda(\alpha_1) + \Lambda(\alpha_2) + \Lambda(\alpha_3).$$

If the shape parameter $z = e^{i\theta}$ is a complex number of magnitude 1, then the dihedral angles of T_z coincide with the angles of an isosceles triangle with angles

$(\theta, (\pi-\theta)/2, (\pi-\theta)/2)$. In that case, we have

$$\mathrm{vol}(T_{e^{i\theta}}) = \Lambda\,(\theta) + \Lambda\left(\frac{\pi-\theta}{2}\right) + \Lambda\left(\frac{\pi-\theta}{2}\right).$$

The following symmetries of the Lobachevsky function [27, Theorem 10.4.3,10.4.4]

$$\Lambda(-\theta) = -\Lambda(\theta), \quad \frac{1}{2}\Lambda(2\theta) = \Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{2}\right)$$

imply that

$$\mathrm{vol}(T_{e^{i\theta}}) = 2\Lambda\left(\frac{\theta}{2}\right). \tag{42}$$

Now, we return to the proof of part (b). Fix α, β, κ as in (33) and consider the complex numbers of magnitude 1:

$$(z_\alpha, z_\beta, z_\kappa) = (e^{2\pi i\alpha}, e^{2\pi i\beta}, e^{2\pi i\kappa})$$

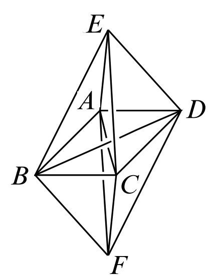
Consider five ideal tetrahedra with shapes

$$(z_\beta z_\kappa)^{-1}, \quad z_\beta, \quad z_\kappa, \quad z_\alpha^{-1}, \quad z_\alpha z_\kappa^{-1}. \tag{43}$$

Equations (35) and (42) imply that the sum of their volumes is given by $2\pi\, r_+(\alpha, \beta, \kappa)$. Now consider the ideal octahedron with vertices

$$(A, B, C, D, E, F) = (0, 1, \infty, z_\kappa, (z_\beta z_\kappa - 1)/(z_\beta - 1), z_\alpha)$$

drawn as follows:



It can be triangulated into five ideal tetrahedra $ABDE$, $BDCE$, $ABCD$, $ABCF$ and $ACDF$ with ordered vertices and with shape parameters

$$\left\{1 - \frac{1}{z_\beta z_\kappa}, \frac{z_\beta}{z_\beta - 1}, \frac{z_\kappa}{z_\kappa - 1}, \frac{z_\alpha}{z_\alpha - 1}, \frac{z_\alpha}{z_\kappa}\right\}$$

computed according to Equation (41). Adding up the volumes of these tetrahedra, with proper orientations concludes the proof of (b). (c) is analogous to (a). We thank the referee for correcting the vertices of the octahedron in an earlier version of this paper. □

8.3 The maximum of the growth rate of the R–matrix

In this section we determine the maximum of $r_{\pm}(\alpha, \beta, \kappa)$.

Proposition 8.6 (a) *With α, β, κ satisfying (33), $r_+(\alpha, \beta, \kappa)$ achieves maximum when $\alpha = 3/4, \beta = 1/4$, and $\kappa = 1/2$. Moreover*

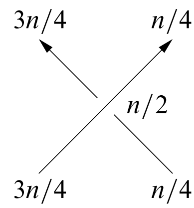
$$r_+(3/4, 1/4, 1/2) = \frac{v_8}{2\pi},$$

where $$v_8 = 8\Lambda(\pi/4) \approx 3.66386237670887606 02\ldots$$

is the volume of the regular hyperbolic ideal octahedron.

(b) *Similarly, $r_-(\alpha, \beta, \kappa)$ reaches maximum when $\alpha = 1/4, \beta = 3/4$, and $k = 1/2$; and its maximum value is the same as that of $r_+(\alpha, \beta, \kappa)$.*

Thus, asymptotically, the winning configuration is given by:



Proof It is enough to consider the case of r_+. The result for r_- follows from (37). Let $\delta = \alpha - \kappa$, we have

$$r_+(\alpha, \beta, \kappa) = -\Lambda(\pi(\beta + \kappa)) + \Lambda(\pi\beta) + \Lambda(\pi\kappa) - \Lambda(\pi(\delta + \kappa)) + \Lambda(\pi(\delta)),$$

with domain $0 \le \beta, \delta, \kappa$, and $\beta + \kappa \le 1, \delta + \kappa \le 1$. Note the symmetry between β and δ.

Using $\Lambda'(x) = -\log(2\sin x)$ for $0 < x < \pi$, one can easily show that the function $-\Lambda(\pi(\beta + \kappa)) + \Lambda(\pi\beta)$, for a fixed $\kappa \in [0, 1]$, achieves maximum at $\beta = (1-\kappa)/2$. It follows that the maximum of $r_+(\alpha, \beta, \kappa)$ is the same as the maximum of

$$g(\kappa) := 2(-\Lambda(\pi(\beta + \kappa)) + \Lambda(\pi\beta)) + \Lambda(\pi\kappa),$$

with $\beta = (1-\kappa)/2$. The domain for g is $\kappa \in [0, 1]$. Using the derivative of g it is easy to show that g achieves maximum when $\kappa = 1/2$. In this case $\alpha = 3/4, \beta = 1/4$. □

Remark 8.7 Another proof is to use part (b) of Proposition 8.5 and the fact that the volume of an ideal octahedron is maximized at a regular ideal octahedron; see [27].

Lemma 8.8 *If z, w are complex numbers that satisfy $|z| = |w| = 1$ and $|1-z| = |1-w|$, then $z = w^{\pm 1}$.*

Proof Let us define

$$C_{u_0,r} := \{u \in \mathbb{C} \mid |u-u_0| = r > 0\}.$$

Then $C_{u_0,r}$ is a circle with center u_0 and radius r. Fixing w, it follows that $z \in C_{0,1} \cap C_{1,|1-w|}$. The intersection of two circles is two points, and since w and $w^{-1} = \overline{w}$ both lie in the intersection, the result follows. □

8.4 The maximum of the R–matrices at roots of unity

Proposition 8.6 gives the maximum of the growth rate of $\mathrm{ev}_n(R_+(n;a,b,k))$ as $n \to \infty$. The following proposition gives the maximum of $\mathrm{ev}_n(R_+(n;a,b,k))$, for a fixed n.

Proposition 8.9 (Proof in Appendix B.2) *The value of $|\mathrm{ev}_n(R_+(n;a,b,k))|$ achieves its maximum at $a = \lfloor 3n/4 \rfloor$, $b = \lfloor (n-1)/4 \rfloor$, and $k = a-b$. The value of $|\mathrm{ev}_n(R_-(n;a,b,k))|$ achieves its maximum at $a = \lfloor (n-1)/4 \rfloor$, $b = \lfloor 3n/4 \rfloor$, and $k = b-a$. The maximum value of $|\mathrm{ev}_n(R_+(n;a,b,k))|$ is the same as that of $|\mathrm{ev}_n(R_-(n;a,b,k))|$.*

Note that for these optimal values in the R_+ case, $|a - 3n/4| \leq 1$, $|b - n/4| \leq 1$ and $|k - n/2| \leq 1$. The proof of this proposition will be given in Appendix B.

From Corollary 8.4 and Propositions 8.5, 8.6 and 8.9, we have the following.

Corollary 8.10 *The growth rate of the maximum of $|\mathrm{ev}_n(R_+(n;a,b,k))|$ is given by*

$$\lim_{n\to\infty} \frac{\max_{a,b,k} \log |\mathrm{ev}_n(R_\pm(n;a,b,k))|}{n} = \frac{v_8}{2\pi}.$$

8.5 Proof of Theorem 1.13

Recall that by (28), the colored Jones function is the sum of n^c summands. Each summand $F(n,\mathbf{k})$ is the product of R–matrices (which are weights of crossing points) and weights of extreme points (which have absolute value 1). There are $c+2$ crossing points, but the weights of the two boundary crossing have absolute value 1. Hence

$$|J_{K,n}(e^{2\pi i/n})| \leq n^c \Big(\max_{a,b,k} |\mathrm{ev}_n(R_\pm(n;a,b,k))| \Big)^c.$$

From the growth rate of $\max_{a,b,k} |\mathrm{ev}_n(R_\pm(n;a,b,k))|$ given by Corollary 8.10 we get the theorem.

9 The Generalized Volume Conjecture near $\alpha = 2\pi i$

In this section we will prove Theorems 1.10 and 1.11 which are concerned with the Generalized Volume Conjecture near $2\pi i$. Our proofs use crucially the well-known *symmetry principle* (see Kirby and Melvin [19] and Lê [21]: Suppose m, m' and n are positive integers with $m \equiv \pm m' \bmod n$, then

$$J_{K,m}(e^{2\pi i/n}) = J_{K,m'}(e^{2\pi i/n}). \tag{44}$$

Note that this fact is also a consequence of the existence of the cyclotomic expansion. However, the case of higher rank Lie algebra requires results from canonical basis theory; see [21].

Proof of Theorem 1.10 The symmetry principle implies that for all $n > m > 0$, we have

$$J_{K,n\pm m}(e^{2\pi i/n}) = J_{K,m}(e^{2\pi i/n}),$$

which implies that

$$\lim_{n\to\infty} J_{K,n\pm m}(e^{2\pi i/n}) = \lim_{n\to\infty} J_{K,m}(e^{2\pi i/n}) = J_{K,m}(1) = 1,$$

from which Theorem 1.10 follows easily. □

Proof of Theorem 1.11 Fix a knot K and consider the neighborhood U_K of 0 as in Theorem 1.3. Define $V_K = 1 + U_K$.

Let us suppose that $\alpha/(2\pi i) \in V_K$ is a rational number not equal to 1. Assume that $\alpha = 2\pi i p/m$ with p, m unequal coprime positive integers. Let $N = np$. Then, the symmetry principle implies that

$$\begin{aligned} f_{K,N}(\alpha) &= J_{K,N}(e^{\alpha/N}) \\ &= J_{K,np}(e^{2\pi i/(nm)}) \\ &= J_{K,n|p-m|}(e^{2\pi i/(nm)}). \end{aligned}$$

Since $n|p-m|/(nm) = |p/m-1| \in U_K$, Theorem 1.3 implies that

$$\lim_{n\to\infty} J_{K,n|p-m|}(e^{2\pi i/(nm)}) = \frac{1}{\Delta(e^{2\pi i(|p/m-1|})}.$$

In other words,

$$\lim_{n\to\infty} f_{K,np}(\alpha) = \frac{1}{\Delta(e^{2\pi i(|p/m-1|})}$$

is bounded. The result follows. □

10 The q–holonomic point of view

10.1 Bounds on l^1–norm of q–holonomic functions

The main result of [10] is that for every knot K, the functions J_K and H_K are q–holonomic. Recall that a sequence $f\colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$ is *q–holonomic* if satisfies a q-linear difference equation. In other words, there exists a natural number d and polynomial $a_j(u,v) \in \mathbb{Q}[u,v]$ for $j = 0, \dots, d$ with $a_d \neq 0$ such that for all $n \in \mathbb{N}$ we have

$$\sum_{j=0}^{d} a_j(q^n, q) f_{n+j}(q) = 0. \tag{45}$$

In this section we observe that q–holonomic functions satisfy a priori upper bounds on their degrees and (under an integrality assumption) on their l^1–norm. As a simple corollary, we obtain another proof of the quadratic bounds in Proposition 6.1, though not as explicit.

Definition 10.1 We say that a sequence $f\colon \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm 1}]$ is *q–integral holonomic* if it satisfies an *q–difference equation* as above with $a_d = 1$.

Question 10.2 Is it true that J_K and C_K are q–integral holonomic for every knot K?

For a partial answer, see the first author's paper [12] with Sun.

Theorem 10.3 (a) *If $f\colon \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm 1}]$ is q–holonomic, then for all n we have*

$$\deg_+(f_n) = O(n^2) \quad \text{and} \quad \deg_-(f_n) = O(n^2).$$

(b) *If f is q–integral holonomic, then for all n we have*

$$\|f_n\|_1 \le C^n$$

for some constant C. In particular,

$$\limsup_{n\to\infty} \frac{\log|f_n(e^{\alpha/n})|}{n} \le C_\alpha$$

for all $\alpha \in \mathbb{C}$.

In other words, integral q–holonomic functions grow at most exponentially.

Proof Suppose f satisfies (45). It is easy to see that for every $a(u,v) \in \mathbb{Q}[u,v]$, there exists a constant C' such that $\deg_+(a(q^n,q)) < C'n$ for every $n \geq 1$. We choose such a common C' for all $a_j(q^n,q)$, $j = 0, 1, \dots, d$, and, in addition, $C' > \deg_+ f(n)$ for $n = 0, 1, \dots, d$.

We will prove by induction on $n \geq 1$ that $\deg_+ f(n) \leq C'n^2$. By assumption, it is true for $n = 1, \dots, d$. For $n \geq 1$, Then, by induction we have

$$\begin{aligned} \deg_+ f_{n+d}(q) &= \deg_+ \big(la_d(q^n,q)\, f_{n+d}(q)\big) - \deg_+ a_d(q^n,q) \\ &= \deg_+\left(-\sum_{j=0}^{d-1} a_j(q^n,q) f_{n+j}(q)\right) - \deg_+ a_d(q^n,q) \\ &< C'n + C'(n+d-1)^2 + C'n \leq C'(n+d)^2. \end{aligned}$$

The second claim in (a) follows similarly.

For (b), let $c_j = \|a_j(Q,q)\|_1$ for $j = 0, \dots, d-1$, and choose C so that

- $C^d \geq c_{d-1}C^{d-1} + \cdots + c_0 C^0$,
- $\|f_n(q)\|_1 \leq C^n$ for $n = 0, \dots, d-1$.

Then, it is easy to see by induction that (b) holds for all n. □

Remark 10.4 It is easy to see that the bounds of Theorem 10.3 are sharp. For example, consider the sequence $f_n(q) = (1+q)(1+q^2)\dots(1+q^n)$.

Theorem 10.3 gives an alternative proof of the quadratic bounds for the degrees of the color Jones polynomial, though not as explicit as in Proposition 6.1. If Question 10.2 has a positive answer then Theorem 10.3 also gives an alternative proof of Theorem 6.2.

10.2 Bounds for higher rank groups

In [10], we considered the colored Jones function

$$J_{\mathfrak{g},K} \colon \Lambda_w \longrightarrow \mathbb{Z}[q^{\pm 1}]$$

of a knot K, where $\mathfrak{g}$ is a *simple Lie algebra* with *weight lattice* Λ_w. In the above reference, the authors proved that $J_{\mathfrak{g},K}$ is a q–holonomic function, at least when $\mathfrak{g}$ is not G_2. For $\mathfrak{g} = \mathfrak{sl}_2$, $J_{\mathfrak{sl}_2,K}$ is the colored Jones function J_K discussed earlier.

In [10] , the authors gave state-sum formulas for $J_{\mathfrak{g},K}$ similar to (28) where the summand takes values in $\mathbb{Z}[q^{\pm 1/D}]$, where D is the size of the center of $\mathfrak{g}$.

The methods of the present paper give an upper bound for the growth-rate of the $\mathfrak{g}$–colored Jones function. More precisely, we have:

Theorem 10.5 *For every simple Lie algebra $\mathfrak{g}$ (other than G_2) and every $\alpha \in \mathbb{C}$, and every $\lambda \in \Lambda_w$, there exists a constant $C_{\mathfrak{g},\alpha,\lambda}$ such that for every knot with $c+2$ crossings, we have*

$$\limsup_{n\to\infty} \frac{\log|J_{\mathfrak{g},K,n\lambda}(e^{\alpha/n})|}{n} \leq C_{\mathfrak{g},\alpha,\lambda} c.$$

The details of the above theorem will be explained in a subsequent publication.

11 Some physics

11.1 A small dose of physics

One does not need to know the relation of the colored Jones function and quantum field theory in order to understand the statement and proof of Theorem 1.6. Nevertheless, we want to add some philosophical comments, for the benefit of the willing reader. According to Witten [35], the Jones polynomial $J_{K,n}$ can be expressed by a partition function of a topological quantum field theory in 3 dimensions—a gauge theory with Chern–Simons Lagrangian. The stationary points of the Lagrangian correspond to $\mathrm{SU}(2)$–flat connections on an ambient manifold, and the observables are knots, colored by the n–dimensional irreducible representation of $\mathrm{SU}(2)$. In case of a knot in S^3, there is only one ambient flat connection, and the corresponding perturbation theory is a formal power series in $h = \log q$.

Rozansky exploited a cut-and-paste property of the Chern–Simons path integral and considered perturbation theory of the knot complement, along an abelian flat connection with monodromy given by (7). In fact, Rozansky calls such an expansion the $U(1)$–*reducible connection contribution* (in short, $U(1)$–RCC) to the Chern–Simons path integral, where $U(1)$ stands for the fact that the flat $\mathrm{SU}(2)$ connections are actually $U(1)$–valued abelian connections. Formal properties of such a perturbative expansion, enabled Rozansky to deduce (in physics terms) the loop expansion of the colored Jones function [29]. Rozansky also proved the existence of the loop expansion using an explicit state-sum description of the colored Jones function [30].

Of course, perturbation theory means studying formal power series that rarely converge. Perturbation theory at the trivial flat connection in a knot complement converges, as it resumes to a Laurent polynomial in e^h; namely the n–th colored Jones polynomial. The volume conjecture for small complex angles is precisely the statement that perturbation theory for abelian flat connections (near the trivial one) does converge.

At the moment, there is no physics (or otherwise) formulation of perturbation theory of the Chern–Simons path integral along a discrete and faithful $\mathrm{SL}_2(\mathbb{C})$ representation. Nor is there an adequate explanation of the relation between $\mathrm{SU}(2)$ gauge theory (valid near $\alpha = 0$) and a complexified $\mathrm{SL}_2(\mathbb{C})$ gauge theory, valid near $\alpha = 2\pi i$. These are important and tantalizing questions, with no answers at present.

11.2 The WKB method

Since we are discussing physics interpretations of Theorem 1.6 let us make some more comments. Obviously, when the angle α is sufficiently big, the asymptotic expansion of Equation (6) may break down. For example, when e^{α} is a complex root of the Alexander polynomial, then the right hand side of (6) does not make sense, even to leading order. In fact, when α is near $2\pi i$, then the solutions are expected to grow exponentially, and not polynomially, according to the Volume Conjecture.

The breakdown and change of rate of asymptotics is a well-documented phenomenon well-known in physics, associated with WKB analysis, after Wentzel–Krammer–Brillouin; see for example [26]. In fact, one may obtain an independent proof of Theorem 1.6 using *WKB analysis*, that is, the study of asymptotics of solutions of difference equations with a small parameter. The key idea is that the sequence of colored Jones functions is a solution of a linear q–difference equation, as was established in [10]. A discussion on WKB analysis of q–difference equations was given by Geronimo and the first author in [8].

The WKB analysis can, in particular, determine *small exponential corrections* of the form $e^{-c_\alpha n}$ to the asymptotic expansion of Theorem 1.6, where c_α depends on α, with $\mathrm{Re}(c_\alpha) < 0$ for α sufficiently small. These small exponential corrections (often associated with instantons) cannot be captured by classical asymptotic analysis (since they vanish to all orders in n), but they are important and dominant (ie, $\mathrm{Re}(c_\alpha) > 0$) when α is near $2\pi i$, according to the volume conjecture. Understanding the change of sign of $\mathrm{Re}(c_\alpha)$ past certain so-called Stokes directions is an important question that WKB addresses.

We will not elaborate or use the WKB analysis in the present paper. Let us only mention that the loop expansion of the colored Jones function can be interpreted as WKB asymptotics on a q–difference equation satisfied by the colored Jones function.

Appendix A The volume conjecture for the Borromean rings

It is well-known that the complement of the Borromean rings B can be geometrically identified by gluing two regular ideal octahedra; see Thurston [32]. As a result, the volume $\mathrm{vol}(S^3 - B)$ of $S^3 - B$ is equal to $2v_8$.

Suppose L is a k–component framed link, and $n_1, \dots, n_k$ are positive integers. The colored Jones polynomial $\widetilde{J}_L(n_1, \dots, n_k) \in \mathbb{Z}[q^{\pm 1/4}]$ is the sl_2–quantum invariant of the link whose components are colored by sl_2–modules of dimensions $n_1, \dots, n_k$ [28; 33]. The normalization is chosen so that for the unknot, $\widetilde{J}_L(n) = [n]$. Define

$$J_{L,n}(q) := \frac{J_L(n, n, \dots, n)}{[n]}.$$

The next theorem confirms the volume conjecture for the Borromean rings.

Theorem A.1 *Let B be the Borromean rings, then*

$$\lim_{n\to\infty} \frac{\log |J_{B,n}(e^{2\pi i/n})|}{n} = \frac{1}{2\pi} \operatorname{vol}(S^3 - B).$$

Proof For an integer j and a positive integer k let $x_j = 2\sin(j\pi/n)$ and $z_k = \prod_{j=1}^{k} x_j$.

Then (see (38))

$$x_j = x_{n-j} = -x_{n+j}, \tag{46}$$

$$z_k = n/z_{n-1-k} \quad \text{for } 1 \le k \le n-1. \tag{47}$$

Using Habiro's formula for $\widetilde{J}_L$ of the Borromean ring [14; 15], one has

$$J_{B,n}(q) = \sum_{l=0}^{n-1} (-1)^l \, \frac{\{n\}^2 \big(\prod_{j=1}^{l} \{n+j\}\{n-j\}\big)^3}{\big(\prod_{j=l+1}^{2l+1} \{j\}\big)^2}.$$

When $q^{1/2} = e^{i\pi/n}$, one has $\{j\} = 2i \sin(j\pi/n)$, which is 0 exactly when j is divisible by n. Hence if $2l+1 < n$, then the denominator of the term in the above sum is never 0, while the numerator is 0, since it has 2 factors $\{n\}$. On the other hand, if $2l+1 > n$, then the denominator has 2 factors $\{n\}$, which would cancel with the 2 same factors of the numerator. Hence at $q^{1/2} = e^{i\pi/n}$ one can assume that $2l+1 \ge n$, or $l > n/2 - 1$:

$$\begin{aligned} J_{B,n}(e^{2\pi i/n}) &= \sum_{n > l > n/2-1} (-1)^l \operatorname{ev}_n \frac{\big(\prod_{j=1}^{l} \{n+j\}\{n-j\}\big)^3}{\big(\prod_{j=l+1}^{n-1} \{j\} \prod_{j=n+1}^{2l+1} \{j\}\big)^2} \\ &= \sum_{n > l > n/2-1} \frac{(z_l)^6}{(z_{n-l-1})^2 \, (z_{2l+1-n})^2} \quad \text{by (46).} \end{aligned}$$

Using (47), which says $z_l = n/z_{n-1-l}$, we have

$$J_{B,n}(e^{2\pi i/n}) = \sum_{n>l>n/2-1} n^2\,(\gamma_l)^2, \quad \text{where } \gamma_l = \frac{(z_l)^2}{(z_{n-1-l})^2\, z_{2l+1-n}}. \tag{48}$$

By (51) below, with $a_l = l$, $b_l = n-1-l$ and $k_l = 2l+1-n$, we have

$$\gamma_l = |\mathrm{ev}_n(R_+(n; a_l, b_l, k_l))|.$$

By Proposition 8.9, $|\mathrm{ev}_n(R_+(n; a_l, b_l, k_l))|$ achieves the maximum at

$$a_{\max} = \lfloor 3n/4 \rfloor, \quad b_{\max} = \lfloor (n-1)/4 \rfloor \quad \text{and} \quad k_{\max} = a_{\max} - b_{\max}.$$

When $l = \lfloor 3n/4 \rfloor$ we have $a_l = a_{\max}$, while $|b_l - b_{\max}| \le 1$ and $|k_l - k_{\max}| \le 1$. It is easy to see that

$$\lim_{n\to\infty} \frac{\log \gamma_{\lfloor (n-1)/4 \rfloor} - \log |\mathrm{ev}_n(R(n; a_{\max}, b_{\max}, k_{\max}))|}{n} = 0. \tag{49}$$

There are less than n summands in the right hand side of (48), and each summand is positive. Hence

$$n^2\,(\gamma_{\lfloor 3n/4 \rfloor})^2 < J_{B,n}(e^{2\pi i/n}) < n^3 |\mathrm{ev}_n(R(n; a_{\max}, b_{\max}, k_{\max}))|^2.$$

From (49) it follows that

$$2\pi \lim_{n\to\infty} \frac{\log |J_{B,n}(e^{2\pi i/n})|}{n} = 2\pi \lim_{n\to\infty} \frac{\log |\mathrm{ev}_n(R(n; a_{\max}, b_{\max}, k_{\max}))|^2}{n},$$

which is equal to $2v_8$, according to Corollary 8.10. □

Appendix B Proof of Proposition 8.9

B.1 Preliminary estimates

Again we denote $x_j = 2\sin(j\pi/n)$. The following is obvious.

Lemma B.1 (a) The x_j, as a function of j, is increasing for $j \in [0, n/2]$ and decreasing for $j \in [n/2, n]$. In particular, for $j \le l \le n-j$ we have $x_j \le x_l$.

(b) For every $1 \le j \le n-1$, one has $2 \ge x_j$. For $n/4 \le j \le 3n/4$, one has $2 \le (x_j)^2$.

Lemma B.2 *For a fixed k, $1 \le k \le n-1$, the value of $y_b(k) := \prod_{j=b+1}^{b+k} x_j$ achieves its maximum at*

$$b = \beta(k) := \lfloor (n-k)/2 \rfloor. \tag{50}$$

Proof We will prove that if $b < \beta(k)$, then $y_b(k) \le y_{b+1}(k)$, while if $b > \beta(k)$ then $y_b(k) \le y_{b-1}(k)$. This will prove the lemma.

Suppose $b < \beta(k)$. Then $b \le \lfloor (n-k)/2 \rfloor - 1 \le (n-k)/2 - 1$. It follows that $(b+1) \le b+k+1 \le n-(b+1)$. From Lemma B.1(a) we get $x_{b+1} \le x_{b+k+1}$. Hence $y_{b+1}(k)/y_b(k) = x_{b+k+1}/x_{b+1} \ge 1$, or $y_{b+1}(k) \ge y_b(k)$.

Suppose now $b \ge 1 + \beta(k)$. If $b \ge n/2$, then $x_b \ge x_{b+k}$ since y_j is deceasing on $[n/2, n]$. If $b < n/2$, then from $b \ge 1 + \lfloor (n-k)/2 \rfloor$ one can easily show that $b+k \ge n-b \ge n/2$. Hence we also have $x_b = x_{n-b} \ge x_{b+k}$. Thus $y_{b-1}(k)/y_b(k) = x_b/x_{b+k} \ge 1$. □

Using (39), with $|\mathrm{ev}_n(\{j\})| = x_j$, we have

(51)
$$|\mathrm{ev}_n R_+(n;a,b,k)| = \frac{y_b(k)\, y_{a-k}(k)}{y_0(k)}, \quad |\mathrm{ev}_n R_-(n;a,b,k)| = \frac{y_a(k)\, y_{b-k}(k)}{y_0(k)}.$$

By Lemma B.2, both $y_b(k)$ and $y_{a-k}(k)$ achieve maximum when $b = a-k = \beta(k)$. Hence

$$\max |\mathrm{ev}_n R_+(n;a,b,k)| = \max_{0\le k\le n} s(k), \quad \text{where } s(k) = \frac{(y_{\beta(k)}(k))^2}{y_0(k)}. \tag{52}$$

Lemma B.3 *One has*

$$\frac{s(k+1)}{s(k)} = \frac{(x_{\beta(k)})^2}{x_{k+1}} \tag{53}$$

with the denominator satisfying

$$x_{k+1} = \begin{cases} x_{2\beta(k)-1} & \text{if } n-k \text{ is even,} \\ x_{2\beta(k)} & \text{if } n-k \text{ is odd.} \end{cases} \tag{54}$$

Proof By definition,

$$s(k) = \frac{(y_{\beta(k)}(k))^2}{y_0(k)} = \frac{\left(\prod_{j=1}^k x_{\beta(k)+j}\right)^2}{\prod_{j=1}^k x_j}. \tag{55}$$

Note that, with $\beta(k) = \lfloor (n-k)/2 \rfloor$, we have

$$\beta(k+1) = \begin{cases} \beta(k) - 1 & \text{if } n-k \text{ is even,} \\ \beta(k) & \text{if } n-k \text{ is odd.} \end{cases} \tag{56}$$

We consider two cases: $n-k$ is even and $n-k$ odd.

(a) $n-k$ is even. Replacing k with $k+1$ in (55), then using $\beta(k+1) = \beta(k)-1$, we get

$$s(k+1) = \frac{\left(\prod_{j=1}^{k+1} x_{\beta(k+1)+j}\right)^2}{\prod_{j=1}^{k+1} x_j} = \frac{\left(\prod_{j=1}^{k+1} x_{b_k-1+j}\right)^2}{\prod_{j=1}^{k+1} x_j} = \frac{(x_{\beta(k)})^2 \left(\prod_{j=1}^{k} x_{\beta(k)+j}\right)^2}{x_{k+1}\prod_{j=1}^{k} x_j}.$$

Dividing by $s(k)$, we get (53). As for the denominator, using $x_j = x_{n-j}$ and $n-k = 2\beta(k)$,

$$x_{k+1} = x_{n-k-1} = x_{2\beta(k)-1}.$$

This proves the lemma when $n-k$ is even.

(b) $n-k$ is odd. Replacing k with $k+1$ in (55), then using $\beta(k+1) = \beta(k)$, we get

$$\begin{aligned} s(k+1) &= \frac{\left(\prod_{j=1}^{k+1} x_{b_k+j}\right)^2}{\prod_{j=1}^{k+1} x_j} \\ &= \frac{(x_{\beta(k)+k+1})^2 \left(\prod_{j=1}^{k} x_{\beta(k)+j}\right)^2}{x_{k+1}\prod_{j=1}^{k} x_j} = \frac{(x_{\beta(k)+k+1})^2}{x_{k+1}} s(k). \end{aligned} \tag{57}$$

Using $x_j = x_{n-j}$ and $n-\beta(k)-k-1 = \beta(k)$, we have

$$x_{\beta(k)+k+1} = x_{n-\beta(k)-k-1} = x_{\beta(k)},$$

which, together with Equation (57), proves Equation (53). As for the denominator, using $n-k-1 = 2\beta(k)$,

$$x_{k+1} = x_{n-k-1} = x_{2\beta(k)}.$$

This completes the proof of the lemma. □

As k increases from 0 to n, $\beta(k) = \lfloor (n-k)/2 \rfloor$ decreases and covers all integers from $\lfloor n/2 \rfloor$ to 0.

Lemma B.4 (a) *If $n \geq 7$ then $s(k)$ achieves maximum at an integer k such that $\beta(k) = \lfloor (n-1)/4 \rfloor$.*

(b) *$s(k)$ achieves maximum at k which is the smallest integer such that $\beta(k) = \lfloor (n-1)/4 \rfloor$.*

Proof (a) We will show that

(a1) if $\beta(k) > (n-1)/4$ then $s(k+1) \geq s(k)$,

(a2) if $\beta(k) \leq (n-1)/4 - 1$ then $s(k-1) > s(k)$.

This will show that the maximum can be achieved for a k such that $\beta(k) = \lfloor (n-1)/4 \rfloor$.

Proof of (a1) Suppose $\beta(k) > \frac{n-1}{4}$. There is no integer in the interval $(\frac{n-1}{4}, \frac{n}{4})$, because otherwise (by multiplying by 4) there would be an integer in $(n-1, n)$. It follows that $\beta(k) \geq n/4$.

Besides, $\beta(k) = \lfloor (n-k)/2 \rfloor \leq n/2$. Thus $\beta(k) \in [n/4, n/2]$. By Lemma B.1(b), $(x_{\beta(k)})^2 \geq 2 \geq x_j$ for any $1 \leq j \leq n$. It follows that the right hand side of (53) is bigger than or equal to 1, or $s(k+1)/s(k) \geq 1$.

Proof of (a2) Suppose $\beta(k) \leq \frac{n-1}{4} - 1$. Then $\beta(k-1) \leq \frac{n-1}{4}$ since by (56), either $\beta(k-1) = \beta(k)$ or $\beta(k-1) = \beta(k) + 1$.

Since $2\beta(k-1) < n/2$, by Lemma B.1(a), $x_{2\beta(k-1)} \geq x_{2\beta(k-1)-1}$. It follows that x_k, either equal to $x_{2\beta(k-1)}$ or $x_{2\beta(k-1)-1}$ by (54), satisfies

$$x_k \geq x_{2\beta(k-1)-1}.$$

By Lemma B.3 and the above inequality,

$$\frac{s(k)}{s(k-1)} = \frac{(x_{\beta(k-1)})^2}{x_k} \leq \frac{(x_{\beta(k-1)})^2}{x_{2\beta(k-1)-1}} < 1,$$

where the last inequality follows from Lemma B.5 below. This completes the proof of the part (a).

(b) When $n < 7$ the statement is checked by explicit calculation. We will assume $n \geq 7$.

There are two values of k such that $\beta(k) = \lfloor (n-1)/4 \rfloor$. Let κ be the smaller one, then the other one is $\kappa + 1$. Then $n - \kappa$ is odd since otherwise $\beta(\kappa - 1) = \beta(k)$.

Since $n - \kappa$ is odd, by Lemma B.3, we have

$$\frac{s(\kappa+1)}{s(\kappa)} = \frac{(x_{\beta(\kappa)})^2}{x_{2\beta(\kappa)-1}},$$

which is less than 1 by Lemma B.5. This means $s(k)$ achieves maximum at $k = \kappa$. □

B.2 Proof of Proposition 8.9

First consider $R_+(n; a, b, k)$. By (52), Lemmas B.4(b) and B.2, $|R_+(n; a, b, k)|$ achieves maximum when k satisfies the condition in Lemma B.4(b), $b = \lfloor (n-1)/4 \rfloor$, and $a = b + k$. The value of k satisfying the condition in Lemma B.4(b) can be calculated easily:

$$k = \begin{cases} n/2+1 & n = 0 \bmod 4, \\ (n-1)/2 & n = 1 \bmod 4, \\ n/2 & n = 2 \bmod 4, \\ (n+1)/2 & n = 3 \bmod 4. \end{cases}$$

From there one can calculate $a = k + b$. It is easy to check that the values of a, b, k are exactly the ones given in Proposition 8.9.

Now turn to $R_-(n; a, b, k)$. By (51),

$$|R_-(n; a, b, k)| = |R_+(n; b, a, k)|.$$

So $|R_-(n; a, b, k)|$ and $|R_+(n; b, a, k)|$ have the same maximum, and $|R_-(n; a, b, k)|$ achieves maximum when $|R_+(n; b, a, k)|$ achieves maximum.

This completes the proof of Proposition 8.9, modulo the following lemma.

Lemma B.5 *For $1 \le j \le \frac{n-1}{4}$, with $n \ge 7$, one has $x_{2j-1} > x_j^2$.*

Proof With $x_j = 2\sin(j\pi/n)$, the statement is equivalent to

$$\sin((2j-1)\pi/n) > 2\sin^2(j\pi/n),$$

which, using $2\sin^2(x) = 1 - \cos(2x)$, is equivalent to

$$\sin((2j-1)\pi/n) + \cos(2j\pi/n) > 1 \quad \text{for } 1 \le j \le \frac{n-1}{4}. \tag{58}$$

We will prove (58) not only for integer j, but for all real $j \in [1, \frac{n-1}{4}]$.

The function $f(j) = \sin((2j-1)\pi/n) + \cos(2j\pi/n)$ has the second derivative

$$f''(j) = -(2\pi/n)^2 \left(\sin((2j-1)\pi/n) + \cos(2j\pi/n)\right)$$

which is strictly negative on the interval $[1, \frac{n-1}{4}]$. Hence $f(j)$ achieves absolute minimum at one of the end points 1 and $\frac{n-1}{4}$. It is enough to show that the values of f at these two end points are bigger than 1.

At the end point 1, he inequality $f(1) > 1$ is

$$\sin(\pi/n) + \cos(2\pi/n) > 1. \tag{59}$$

The function $f_1(x) = \sin x + \cos(2x)$ has the second derivative

$$f_1''(x) = -\sin x - 4\cos(2x)$$

which is strictly negative on the interval $(0,\pi/6)$. Hence on the closed interval $[0,\pi/6]$ the function $f_1(x)$ achieves the absolute minimum at one of the end points. But $f_1(0)=f_1(\pi/6)=1$. If $n\geq 7$, then $\pi/n\in(0,\pi/6)$. Hence $f_1(\pi/n)>1$, which is (59).

At the end point $\frac{n-1}{4}$, one has

$$\begin{aligned} f\left(\frac{n-1}{4}\right) &= \sin\left(\frac{\pi}{2}-\frac{3\pi}{2n}\right)+\cos\left(\frac{\pi}{2}-\frac{\pi}{2n}\right) \\ &= \cos(3\pi/2n)+\sin(\pi/2n) \quad \text{because } \sin(\pi/2-x)=\cos x. \end{aligned}$$

Hence $f((n-1)/4)>1$ is equivalent to

$$\sin(\pi/2n)+\cos(3\pi/2n)>1. \tag{60}$$

Look at the function $f_2(x)=\sin x+\cos(3x)$ on the interval $[0,\pi/14]$. The second derivative

$$f_2''(x)=-\sin x-9\cos(3x)$$

is strictly negative on the interval $(0,\pi/14)$, and f_2 has values $f_2(0)=1$ and $f_2(\pi/14)=1.004\ldots>1$. It follows that $f_2(x)>1$ for $x\in(0,\pi/14]$. If $n\geq 7$, then $\pi/2n\in(0,\pi/14]$. Hence $f_2(\pi/2n)>1$, which is (60). □

References

[1] **I Agol**, **P A Storm**, **W P Thurston**, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, J. Amer. Math. Soc. 20 (2007) 1053–1077 MR2328715 With an appendix by N Dunfield

[2] **G E Andrews**, *The theory of partitions*, Cambridge Math. Library, Cambridge Univ. Press (1998) MR1634067 Reprint of the 1976 original

[3] **D Bar-Natan**, **S Garoufalidis**, *On the Melvin–Morton–Rozansky conjecture*, Invent. Math. 125 (1996) 103–133 MR1389962

[4] **D Cooper**, **M Culler**, **H Gillet**, **D D Long**, **P B Shalen**, *Plane curves associated to character varieties of 3–manifolds*, Invent. Math. 118 (1994) 47–84 MR1288467

[5] **N Dunfield**, *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds*, Invent. Math. 136 (1999) 623–657 MR1695208

[6] **J L Dupont**, **C K Zickert**, *A dilogarithmic formula for the Cheeger–Chern–Simons class*, Geom. Topol. 10 (2006) 1347–1372 MR2255500

[7] **S Garoufalidis**, *Beads: from Lie algebras to Lie groups* `arXiv:math.GT/0201056`

[8] **S Garoufalidis**, **J S Geronimo**, *Asymptotics of q–difference equations*, from: "Primes and knots", (T Kohno, M Morishita, editors), Contemp. Math. 416, Amer. Math. Soc. (2006) 83–114 MR2276137

[9] **S Garoufalidis**, **A Kricker**, *A rational noncommutative invariant of boundary links*, Geom. Topol. 8 (2004) 115–204 MR2033481

[10] **S Garoufalidis**, **T T Q Lê**, *The colored Jones function is q–holonomic*, Geom. Topol. 9 (2005) 1253–1293 MR2174266

[11] **S Garoufalidis**, **L Rozansky**, *The loop expansion of the Kontsevich integral, the null-move and S–equivalence*, Topology 43 (2004) 1183–1210 MR2080000

[12] **S Garoufalidis**, **X Sun**, *The C–polynomial of a knot*, Algebr. Geom. Topol. 6 (2006) 1623–1653 MR2253459

[13] **S Gukov**, *Three-dimensional quantum gravity, Chern–Simons theory, and the A-polynomial*, Comm. Math. Phys. 255 (2005) 577–627 MR2134725

[14] **K Habiro**, *On the colored Jones polynomials of some simple links*, from: "Recent progress towards the volume conjecture (Japanese) (Kyoto, 2000)", Sūrikaisekikenkyūsho Kōkyūroku 1172 (2000) 34–43 MR1805727

[15] **K Habiro**, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Invent. Math. 171 (2008) 1–81 MR2358055

[16] **E Hille**, *Analytic function theory, Vol. II*, Intro. to Higher Math., Ginn and Co., Boston-New York-Toronto (1962) MR0201608

[17] **V F R Jones**, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) 126 (1987) 335–388 MR908150

[18] **R M Kashaev**, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997) 269–275 MR1434238

[19] **R Kirby**, **P Melvin**, *The 3–manifold invariants of Witten and Reshetikhin–Turaev for* $\mathrm{sl}(2,\mathbf{C})$, Invent. Math. 105 (1991) 473–545 MR1117149

[20] **M Lackenby**, *The volume of hyperbolic alternating link complements*, Proc. London Math. Soc. (3) 88 (2004) 204–224 MR2018964 With an appendix by I Agol and D Thurston

[21] **T T Q Lê**, *Integrality and symmetry of quantum link invariants*, Duke Math. J. 102 (2000) 273–306 MR1749439

[22] **T T Q Lê**, *The colored Jones polynomial and the A–polynomial of knots*, Adv. Math. 207 (2006) 782–804 MR2271986

[23] **K Mahler**, *An application of Jensen's formula to polynomials*, Mathematika 7 (1960) 98–100 MR0124467

[24] **H Murakami**, *Some limits of the colored Jones polynomials of the figure-eight knot*, Kyungpook Math. J. 44 (2004) 369–383 MR2095421

[25] **H Murakami**, **J Murakami**, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. 186 (2001) 85–104 MR1828373

[26] **F W J Olver**, *Asymptotics and special functions*, AKP Classics, A K Peters Ltd., Wellesley, MA (1997) MR1429619 Reprint of the 1974 original

[27] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, second edition, Graduate Texts in Math. 149, Springer, New York (2006) MR2249478

[28] **N Y Reshetikhin**, **V G Turaev**, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990) 1–26 MR1036112

[29] **L Rozansky**, *A universal $U(1)$–rcc invariant of links and rationality conjecture* `arXiv:math.GT/0201139`

[30] **L Rozansky**, *The universal R–matrix, Burau representation, and the Melvin–Morton expansion of the colored Jones polynomial*, Adv. Math. 134 (1998) 1–31 MR1612375

[31] **J L Schiff**, *Normal families*, Universitext, Springer, New York (1993) MR1211641

[32] **W P Thurston**, *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Lecture Notes (1979) Available at `http://msri.org/publications/books/gt3m/`

[33] **V G Turaev**, *Quantum invariants of knots and 3–manifolds*, de Gruyter Studies in Math. 18, de Gruyter, Berlin (1994) MR1292673

[34] **R van der Veen**, *Proof of the volume conjecture for Whitehead chains*, Acta Math. Vietnam. 33 (2008) 421–431 MR2501851

[35] **E Witten**, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989) 351–399 MR990772

School of Mathematics, Georgia Institute of Technology
Atlanta GA 30332-0160, USA

`stavros@math.gatech.edu`, `letu@math.gatech.edu`

`http://people.math.gatech.edu/~stavros/`,
`http://people.math.gatech.edu/~letu/`

Proposed: Vaughan Jones Received: 27 September 2007
Seconded: Tom Mrowka, Joan Birman Revised: 31 August 2011