Rigidity of certain solvable actions on the sphere

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An analog of the Baumslag–Solitar group $BS(1,k)$ naturally acts on the sphere by conformal transformations. The action is not locally rigid in higher dimensions, but exhibits a weak form of local rigidity. More precisely, any perturbation preserves a smooth conformal structure.

1 Introduction

Over the last two decades, it has been found that many smooth actions of discrete groups exhibit local rigidity. Most known examples are classified into two classes:

(1) Anosov or partially hyperbolic $\mathbb{Z}^n$–actions and homogeneous actions of cocompact lattices related to Anosov or partially hyperbolic $\mathbb{R}^n$–actions with $n \geq 2$ (see Damjanović and Katok [4], Katok and Lewis [12], Katok and Spatzier [13] and Niţică and Török [16]).

(2) Isometric or quasiaffine actions of lattices or groups with Property (T) (see Benveniste [1], Fisher and Margulis [7; 8] and Zimmer [19]).

See Fisher's survey [6] for more related results.

One of the exceptions is an action of the Baumslag–Solitar group $BS(1,k)$ on the circle. For $k \geq 2$, the *Baumslag–Solitar group* $BS(1,k)$ is a finitely presented solvable group defined by $BS(1,k) = \langle a, b \mid aba^{-1} = b^k \rangle$. It is isomorphic to a group generated by two affine transformations of the real line: $f(x) = kx$ and $g(x) = x + c$ with $c \neq 0$. The natural extensions of f and g to $S^1 = \mathbb{R} \cup \{\infty\}$ define a real analytic action ρ_c of $BS(1,k)$ on S^1. Note ρ_c is conjugate to ρ_1 by a diffeomorphism $h(x) = c^{-1}x$.

Theorem 1.1 (Burslem and Wilkinson [2]) *Any real analytic action of $BS(1,k)$ on the circle is locally rigid. In particular, the action ρ_c is locally rigid.*

In the same paper, Burslem and Wilkinson also gave a smooth classification of C^r actions of $BS(1, k)$ on S^1 by using Navas' complete topological classification of C^2 solvable actions on one-dimensional manifolds [15]. Guelman and Liousse [10] extended the classification by Burslem and Wilkinson to C^1 actions by using Cantwell and Conlon's work [3] on C^1 actions of $BS(1, k)$ on the circle or a closed interval and Rivas' work [17] on C^0 action of $BS(1, k)$ on the real line.

Recently, some people have studied actions of Baumslag–Solitar like groups on higher dimensional manifolds. McCarthy [14] proved the rigidity of trivial actions of a large class of abelian-by-cyclic groups on an arbitrary dimensional closed manifold. Guelman and Liousse [9] studied actions of $BS(1, k)$ on surfaces and gave a C^∞ faithful action on the 2–torus which is not locally rigid even in a topological sense.

In this paper, we study a natural higher dimensional analog of the standard $BS(1, k)$–action ρ_c. For $n \geq 1$ and $k \geq 2$, we define a finitely generated solvable group $\Gamma_{k,n}$ by

$$\Gamma_{n,k} = \langle a, b_1, \ldots, b_n \mid ab_i a^{-1} = b_i^k, \ b_i b_j = b_j b_i \text{ for any } i, j = 1, \ldots, n \rangle.$$

The group $\Gamma_{n,k}$ admits a natural action on the n–dimensional sphere S^n. We identify S^n with $\mathbb{R}^n \cup \{\infty\}$ by the stereographic projection. For any basis $B = (v_1, \ldots, v_n)$ of $\mathbb{R}^n$, define a $\Gamma_{n,k}$–action ρ_B on S^n by

- $\rho_B^a(x) = kx$ and $\rho_B^{b_i}(x) = x + v_i$ for $x \in \mathbb{R}^n = S^n \setminus \{\infty\}$,
- $\rho_B^a(\infty) = \rho_B^{b_i}(\infty) = \infty$.

The action ρ_B preserves the standard conformal structure on S^n and we call it *the standard action* associated to B. For $n = 1$ and $v_1 = c \neq 0$, the group $\Gamma_{1,k}$ is the Baumslag–Solitar group $BS(1, k)$ and the action ρ_B is the standard action ρ_c. Therefore, ρ_B is locally rigid by Theorem 1.1 if $n = 1$. On the other hand, ρ_B is not locally rigid for *any* basis B if $n \geq 2$ (see Proposition 3.1). Hence, a direct analog of Theorem 1.1 does not hold.

The aim of this paper is to show that the action ρ_B exhibits a weak form of local rigidity for $n \geq 2$.

To state the main theorem, we recall basic concepts on rigidity of group actions. Let Γ be a discrete group and G a topological group. By $\mathrm{Hom}(\Gamma, G)$, we denote the set of homomorphisms from Γ to G. For $\rho \in \mathrm{Hom}(\Gamma, G)$ and $\gamma \in \Gamma$, we put $\rho^\gamma = \rho(\gamma)$. The set $\mathrm{Hom}(\Gamma, G)$ is naturally identified with a subset of the power set G^Γ. The product topology on G^Γ induces a topology on $\mathrm{Hom}(\Gamma, G)$. When G is Hausdorff, a sequence $(\rho_m)_{m \geq 1}$ in $\mathrm{Hom}(\Gamma, G)$ converges to ρ if and only if ρ_m^γ converges to ρ^γ for any $\gamma \in \Gamma$.

Let M be a smooth closed manifold. Below, all smooth maps and diffeomorphisms are of class C^∞. By $\mathrm{Diff}(M)$, we denote the group of diffeomorphisms of M. It naturally becomes a topological group by the C^∞–topology. For a discrete group Γ, a smooth left Γ–action on M is just a homomorphism from Γ to $\mathrm{Diff}(M)$. Hence, $\mathrm{Hom}(\Gamma, \mathrm{Diff}(M))$ is identified with the space of (smooth left) Γ–actions on M. We say that two actions $\rho_1 \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(M_1))$ and $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(M_2))$ are *smoothly conjugate* if there exists a diffeomorphism $h\colon M_1 \to M_2$ such that $\rho_2^\gamma \circ h = h \circ \rho_1^\gamma$ for any $\gamma \in \Gamma$. We also say that an action $\rho_0 \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(M))$ is *locally rigid* if there exists a neighborhood $\mathcal{U}$ of ρ_0 in $\mathrm{Hom}(\Gamma, \mathrm{Diff}(M))$ such that any action ρ in $\mathcal{U}$ is smoothly conjugate to ρ_0.

Now, we are ready to state the main theorem of this paper.

Main Theorem *Suppose $n, k \geq 2$. Let ρ_B be the standard $\Gamma_{n,k}$–action on S^n associated to a basis B of $\mathbb{R}^n$. Then, there exists a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma_{n,k}, \mathrm{Diff}(S^n))$ of ρ_B such that any $\rho \in \mathcal{U}$ is smoothly conjugate to $\rho_{B'}$ for some basis $B' = B'(\rho)$ of $\mathbb{R}^n$. In particular, any action in $\mathcal{U}$ preserves a C^∞ conformal structure of S^n.*

The proof is divided into three steps: First, we show a local version of the main theorem, ie, rigidity of ρ_B as a local action at ∞. This is the main step of the proof. Second, we prove that any perturbation of ρ_B admits a global fixed point near ∞. Finally, we extend the local conjugacy obtained in the first step to a global one.

The strategy for the first step is close to Burslem and Wilkinson's one in [2]. However, there is an essential difference from their case; the action ρ_B admits nontrivial deformation. The difficulty is that there seems no direct way to find a basis $B' = B'(\rho)$ such that ρ is conjugate to $\rho_{B'}$ for a given perturbation ρ of ρ_B. To overcome it, we follow Weil's idea in [18], where he controlled deformation of lattices of Lie groups by the first cohomology of a deformation complex. Note that Benveniste [1] and Fisher [5] proved local rigidity of isometric actions by applying Weil's idea to $\mathrm{Hom}(\Gamma, \mathrm{Diff}(M))$. In their cases, the deformation complex is infinite dimensional, and hence, they needed Hamilton's Implicit Function Theorem for tame maps between Fréchet spaces. In our case, we reduce the deformation complex to a finite dimensional one and Weil's Implicit Function Theorem is sufficient.

In [2], Burslem and Wilkinson gave another proof of the first step above for $\mathrm{BS}(1,k)$–actions on S^1. They showed the existence of an invariant projective structure on a neighborhood of the global fixed point by using the Schwarzian derivative. The author does not know whether there is an analogous proof for higher dimensional case. Finding it seems an interesting problem.

2   Proof of Main Theorem

2.1   Local version of the main theorem

Let $M_n(\mathbb{R})$ be the set of real square matrices of size n and $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible matrices in $M_n(\mathbb{R})$. We identify each element of $M_n(\mathbb{R})$ with an n–tuple of column vectors in $\mathbb{R}^n$. Under this identification, $\mathrm{GL}_n(\mathbb{R})$ is the set of bases of $\mathbb{R}^n$. By $\|\cdot\|$, we denote the Euclidean norm of $\mathbb{R}^n$. Let $\mathcal{S}^r(\mathbb{R}^n)$ be the set of symmetric r–multilinear maps from $(\mathbb{R}^n)^r$ to $\mathbb{R}^n$. We define a norm $\|\cdot\|^{(r)}$ on $\mathcal{S}^r(\mathbb{R}^n)$ by

$$\|F\|^{(r)} = \sup\{\|F(\xi_1, \ldots, \xi_r)\| \mid \xi_1, \ldots, \xi_r \in \mathbb{R}^n, \|\xi_i\| \leq 1 \text{ for any } i\}.$$

Note $\|F(\xi_1, \ldots, \xi_r)\| \leq \|F\| \cdot \|\xi_1\| \cdots \|\xi_r\|$ for any $\xi_1, \ldots, \xi_r \in \mathbb{R}^n$ and $\|A\|^{(1)}$ is the operator norm of $A \in M_n(\mathbb{R}) = \mathcal{S}^1(\mathbb{R}^n)$.

Let $\mathcal{D}(\mathbb{R}^n, 0)$ be the group of germs of local diffeomorphisms of $\mathbb{R}^n$ at the origin. For $F \in \mathcal{D}(\mathbb{R}^n, 0)$, we denote the r–th derivative of F at the origin by $D_0^{(r)} F$. It is an element of $\mathcal{S}^r(\mathbb{R}^n)$. For $r \geq 2$, define *the C_{loc}^r–topology* on $\mathcal{D}(\mathbb{R}^n, 0)$ by a pseudodistance $d_{C_{\mathrm{loc}}^r}(F, G) = \sum_{i=1}^r \|D_0^{(i)} F - D_0^{(i)} G\|^{(i)}$. Notice that $d_{C_{\mathrm{loc}}^r}$ is not a distance, and hence, the C_{loc}^r–topology is not Hausdorff.

For a discrete group Γ, the C_{loc}^r–topology on $\mathrm{Hom}(\Gamma, \mathcal{D}(\mathbb{R}^n, 0))$ is naturally introduced as before. We say two local actions $P_1, P_2 \in \mathrm{Hom}(\Gamma, \mathcal{D}(\mathbb{R}^n, 0))$ are *smoothly conjugate* if there exists $H \in \mathcal{D}(\mathbb{R}^n, 0)$ such that $P_2^\gamma \circ H = H \circ P_1^\gamma$ for any $\gamma \in \Gamma$.

Let $\bar{\phi}$ be a diffeomorphism from $S^n \backslash \{0\}$ to $\mathbb{R}^n$ given by

$$\bar{\phi}(x) = \frac{1}{\|x\|^2} \cdot x.$$

For $B \in M_n(\mathbb{R})$, we define a local action $P_B \in \mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ by

$$P_B^\gamma = \bar{\phi} \circ \rho_B^\gamma \circ \bar{\phi}^{-1}.$$

In this subsection, we prove the following local version of the main theorem.

Theorem 2.1   *For $B \in \mathrm{GL}_n(\mathbb{R})$, there exists a C_{loc}^2–neighborhood $\mathcal{U}$ of P_B such that any local action $P \in \mathcal{U}$ is smoothly conjugate to $P_{B'}$ for some $B' = B'(P) \in \mathrm{GL}_n(\mathbb{R})$.*

The proof is divided into several steps. First, we show the stability of the linear part of P^{b_i}. Let $\bar{F}$ be the element of $\mathcal{D}(\mathbb{R}^n, 0)$ given by

$$\bar{F}(x) = k^{-1}x.$$

Notice that $P_B^a = \bar{F}$ and $D_0^{(1)} P_B^{b_i} = I$ for any $B \in M_n(\mathbb{R})$ and $i = 1, \ldots, n$.

Lemma 2.2 *Let m be a positive integer and P_* a local action in $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^m, 0))$. Suppose that $D_0^{(1)} P_*^a = k^{-1}I$ and $D_0^{(1)} P_*^{b_i} = I$ for any $i = 1, \ldots, n$. Then, there exists a C_0^1–neighborhood $\mathcal{U}$ of P_* in $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^m, 0))$ such that $D_0^{(1)} P^{b_i} = I$ for any $P \in \mathcal{U}$ and $i = 1, \ldots, n$.*

Proof Put $c_{kj} = k!/(j!(k-j)!)$. There exists $\delta > 0$ such that

$$\delta \cdot \left(k + k^2 + \sum_{j=2}^{k} c_{kj} \delta^{j-2}(1 + k\delta) \right) \leq \frac{1}{2}.$$

Take a C_{loc}^1–neighborhood $\mathcal{U}$ of P_* such that $\| D_0^{(1)} P^\gamma - D_0^{(1)} P_*^\gamma \| < \delta$ for any $P \in \mathcal{U}$ and $\gamma = a, b_1, \ldots, b_m$. Fix $P \in \mathcal{U}$ and $i = 1, \ldots, n$. Let $A = D_0^{(1)} P^a - D_0^{(1)} P_*^a$ and let $B = D_0^{(1)} P^{b_i} - D_0^{(1)} P_*^{b_i}$. We need to show that $B = 0$. By the equalities $D_0^{(1)} P_*^a = k^{-1}I$, $D_0^{(1)} P_*^{b_i} = I$ and $P^a \circ P^{b_i} = P^{b_i^k} \circ P^a$, we have $(k^{-1}I + A)(I + B) = (I + B)^k(k^{-1}I + A)$. Hence,

$$(k-1)\|B\|^{(1)} = \left\| kAB - k^2 BA - \sum_{j=2}^{k} c_{kj} B^j (I + kA) \right\|^{(1)}$$

$$\leq \left(k\delta + k^2\delta + \sum_{j=2}^{k} c_{kj} \delta^{j-1}(1 + k\delta)) \right) \cdot \|B\|^{(1)}$$

$$\leq \frac{1}{2} \cdot \|B\|^{(1)}.$$

Since $k \geq 2$, we obtain that $B = 0$. $\qquad\square$

Second, we show the stability of the linear part of P^a. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product of $\mathbb{R}^n$. For $v \in \mathbb{R}^n$, we define $Q_v \in \mathcal{S}^2(\mathbb{R}^n)$ by

(1) $$Q_v(\xi, \eta) = \langle \xi, \eta \rangle \cdot v - \langle \xi, v \rangle \cdot \eta - \langle \eta, v \rangle \cdot \xi.$$

By a direct calculation, we can check that

$$D_0^{(2)} P_B^{b_i} = 2Q_{v_i}$$

for any $B = (v_1, \ldots, v_n) \in M_n(\mathbb{R})$ and $i = 1, \ldots, n$.

Lemma 2.3 *For any given $B \in \mathrm{GL}_n(\mathbb{R})$, there exists a C_0^2–neighborhood $\mathcal{U}$ of P_B in $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ such that $D_0^{(1)} P^a = k^{-1} I$ for any $P \in \mathcal{U}$.*

Proof For any $F, G \in \mathcal{D}(\mathbb{R}^n, 0)$ with $D_0^{(1)} G = I$, it is easy to see that

$$D_0^{(2)}(F \circ G) = D_0^{(2)} F + D_0^{(1)} F \circ D_0^{(2)} G,$$
$$D_0^{(2)}(G^k \circ F) = D_0^{(2)} F + k \cdot D_0^{(2)} G \circ (D_0^{(1)} F, D_0^{(1)} F).$$

Put $B = (v_1, \ldots, v_n)$. Since $B = (v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$, there is a constant $\epsilon > 0$ such that $\max_{i=1,\ldots,n}\{\|A'v_i\|\} \geq \epsilon \|A'\|^{(1)}$ for any $A' \in M_n(\mathbb{R})$. By Lemma 2.2, there exists a C_{loc}^1–neighborhood $\mathcal{U}_1$ of P_B such that $D_0^{(1)} P^{b_i} = I$ for any $P \in \mathcal{U}_1$. Let $\mathcal{U}$ be a C_{loc}^2–open neighborhood of P_B consisting of $P \in \mathcal{U}_1$ such that

$$\max_{i=1,\ldots,n} \left\{ 3\|D_0^{(2)} P^{b_i} - 2Q_{v_i}\|^{(2)} + \|k \cdot D_0^{(1)} P^a - I\| \cdot \|D_0^{(2)} P^{b_i}\|^{(2)} \right\} < \epsilon.$$

Fix $P \in \mathcal{U}_1$ and put

$$A = k \cdot D_0^{(1)} P^a - I,$$
$$B_i = D_0^{(2)} P^{b_i} - D_0^{(2)} P_B^{b_i} = D_0^{(2)} P^{b_i} - 2Q_{v_i},$$
$$C_i = A \circ Q_{v_i} - 2Q_{v_i} \circ (A, I).$$

We will show that $A = 0$. Since $P^a \circ P^{b_i} = P^{b_i^k} \circ P^a$, we have

$$k^{-1}(I + A) \circ (2Q_{v_i} + B_i) = k \cdot (2Q_{v_i} + B_i) \circ (k^{-1}(I + A), k^{-1}(I + A)).$$

It implies that

$$\begin{aligned}
2\|C_i\|^{(2)} &= \|A \circ B_i - 2B_i \circ (A, I) - (2Q_{v_i} + B_i) \circ (A, A)\|^{(2)} \\
&\leq \|A\|^{(1)} \cdot (3\|B_i\|^{(2)} + \|A\|^{(1)} \cdot \|D_0^{(2)} P^{b_i}\|^{(2)}) \\
&\leq \epsilon \|A\|^{(1)}
\end{aligned}$$

for any $i = 1, \ldots, n$. The definition of Q_v also implies $C_i(v_i, v_i) = \|v_i\|^2 \cdot Av_i$, and hence, $\|C_i\|^{(2)} \geq \|Av_i\|$. Therefore, we obtain

$$2\epsilon \|A\|^{(1)} \leq 2 \max_{i=1,\ldots,n} \|Av_i\| \leq \epsilon \|A\|^{(1)}.$$

It implies that $A = 0$, and hence, $D_0^{(1)} P^a = k^{-1} \cdot I$. $\square$

Let $\mathcal{M}_1'$ be the subset of $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ consisting of local actions P such that $P^a = \bar{F}$ and $D_0^{(1)} P^{b_i} = I$ for any $i = 1, \ldots, n$. Notice that P_B is an element of $\mathcal{M}_1'$ for any $B \in \mathrm{GL}_n(\mathbb{R})$.

Proposition 2.4 Let B be an element of $\mathrm{GL}_n(\mathbb{R})$. For any C_{loc}^2–neighborhood $\mathcal{U}_0$ of P_B in $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$, there exists another C_{loc}^2–neighborhood $\mathcal{U}$ of P_B such that any $P \in \mathcal{U}$ is smoothly conjugate to a local action in $\mathcal{U}_0 \cap \mathcal{M}_1'$.

Proof By Lemmas 2.2 and 2.3, there exists a C_{loc}^2–neighborhood $\mathcal{U}_1$ of P_B in $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ such that $D_0^{(1)} P^a = k^{-1} \cdot I$ and $D_0^{(1)} P^{b_i} = I$ for any $P \in \mathcal{U}_1$ and $i = 1, \ldots, n$. Fix $P \in \mathcal{U}_1$. It is known that if a local diffeomorphism $F \in \mathcal{D}(\mathbb{R}^n, 0)$ satisfies $D_0^{(1)} F = \alpha I$ for some $0 < \alpha < 1$, then it is smoothly linearizable (see [11, Theorem 6.6.6]). Hence, there exists $H \in \mathcal{D}(\mathbb{R}^n, 0)$ such that $D_0^{(1)} H = I$ and $\bar{F} = H \circ P^a \circ H^{-1}$. We define a local action $P^H \in \mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ by $(P^H)^\gamma = H \circ P^\gamma \circ H^{-1}$. Since $D_0^{(1)} (P^H)^{b_i} = D_0^{(1)} P^{b_i} = I$, the local action P^H is contained in $\mathcal{M}_1'$. From the equation $D_0^{(2)} (H \circ \bar{F}) = D_0^{(2)} (P^a \circ H)$, we obtain $(k-1) D_0^{(2)} H = k^2 D_0^{(2)} P^a$. Hence, there exists a small C_{loc}^2–neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of P_B such that $P^H \in \mathcal{U}_0$ for any $P \in \mathcal{U}$. $\square$

Following Weil's idea, we reduce Theorem 2.1 to exactness of a linear complex. Put

$$\mathcal{M}_0 = \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}),$$

$$\mathcal{M}_1 = \{(G_i)_{1 \leq i \leq n} \in \mathcal{D}(\mathbb{R}^n, 0)^n \mid D_0^{(1)} G_i = I, \bar{F} \circ G_i = G_i^k \circ \bar{F} \text{ for any } i\},$$

$$\mathcal{M}_2 = \{(C_{ij})_{1 \leq i < j \leq n} \mid C_{ij} \in \mathcal{S}^3(\mathbb{R}^n)\} = (\mathcal{S}^3(\mathbb{R}^n))^{n(n-1)/2}.$$

Define maps $\Phi \colon \mathcal{M}_0 \to \mathcal{M}_1$ and $\Psi \colon \mathcal{M}_1 \to \mathcal{M}_2$ by

$$\Phi(A, B) = (A \circ P_B^{b_i} \circ A^{-1})_{1 \leq i \leq n},$$

$$\Psi((G_i)_{1 \leq i \leq n}) = \left(\frac{1}{4} \left(D_0^{(3)} (G_i \circ G_j) - D_0^{(3)} (G_j \circ G_i) \right) \right)_{1 \leq i < j \leq n}.$$

By $O_{\mathcal{M}_2}$, we denote the zero element of $\mathcal{M}_2 = S^3(\mathbb{R}^n)^{n(n-1)/2}$. Then,

$$\Psi \circ \Phi(A, B) = O_{\mathcal{M}_2}, \quad \Psi(P^{b_1}, \ldots, P^{b_n}) = O_{\mathcal{M}_2}$$

for any $(A, B) \in \mathcal{M}_0$ and $P \in \mathcal{M}_1'$. Moreover, if $\Phi(A, B) = (P^{b_1}, \ldots, P^{b_n})$, then P is smoothly conjugate to P_B by the linear map A.

The following is a direct corollary of Proposition 2.4.

Corollary 2.5 To prove Theorem 2.1, it is sufficient to show the existence of a C_{loc}^2–neighborhood $\mathcal{V}_*$ of $(P_B^{b_i})_{1 \leq i \leq n}$ in $\mathcal{M}_1$ such that

$$\Psi^{-1}(O_{\mathcal{M}_2}) \cap \mathcal{V}_* = \mathrm{Im}\, \Phi \cap \mathcal{V}_*$$

for any given $B \in \mathrm{GL}_n(\mathbb{R})$. $\square$

Let us recall Weil's Implicit Function Theorem.

Theorem 2.6 (Weil [18]) *Let* $\Phi_0\colon M_0 \to M_1$ *and* $\Phi_1\colon M_1 \to M_2$ *be smooth maps between manifolds* M_0, M_1, *and* M_2. *Suppose that* $\Phi_1 \circ \Phi_0$ *is a constant map with value* $x_2 \in M_2$. *If* $\mathrm{Ker}(D\Phi_1)_{x_1} = \mathrm{Im}(D\Phi_0)_{x_0}$ *for* $x_0 \in M_0$ *and* $x_1 = \Phi_0(x_0) \in M_1$, *then there exists a neighborhood* U *of* x_1 *such that* $\mathrm{Im}\,\Phi_0 \cap U = \Phi_1^{-1}(x_2) \cap U$.

The space $\mathcal{M}_0$ admits a natural smooth structure as an open subset of a finite dimensional vector space $M_n(\mathbb{R})^2$. The space $\mathcal{M}_2 = (\mathcal{S}^3(\mathbb{R}^n))^{n(n-1)/2}$ also does as a finite dimensional vector space. If the maps Φ and Ψ are smooth with respect to some smooth structure on $\mathcal{M}_1$ compatible to the C_{loc}^2–topology and satisfy $\mathrm{Ker}\,D\Psi_{(P_B^{b_1},\ldots,P_B^{b_n})} = \mathrm{Im}\,D\Phi_{(I,B)}$, then Theorem 2.1 follows from Corollary 2.5 and Weil's theorem. To introduce a smooth structure on $\mathcal{M}_1$, we define a map $\Theta\colon \mathcal{M}_1 \to \mathcal{S}^2(\mathbb{R}^n)^n$ by

$$\Theta(G_1,\ldots,G_n) = \frac{1}{2}(D_0^{(2)}G_1,\ldots,D_0^{(2)}G_n).$$

Lemma 2.7 *The map* Θ *is a homeomorphism with respect to the* C_{loc}^2–*topology on* $\mathcal{M}_1$.

Proof Since $D_0^{(1)}G_i = I$ for any $(G_1,\ldots,G_n) \in \mathcal{M}_1$ and any i, the map Θ is continuous by the definition of the C_{loc}^2–topology.

Next, we show that Θ is surjective. Put $(e_1,\ldots,e_n) = I$ and take $Q \in \mathcal{S}^2(\mathbb{R}^n)$. Let $G_Q^t \in \mathcal{D}(\mathbb{R}^n,0)$ be the time-t map of the local flow generated by the quadratic vector field $X_Q(x) = Q(x,x)$. Then, $G_Q^t(0) = 0$, $D_0^{(1)}G_Q^t = I$, and $\bar{F} \circ G_Q^t = G_Q^{kt} \circ \bar{F}$ for any $t \in \mathbb{R}$. Since

$$
\begin{aligned}
\frac{d}{dt}[D_0^{(2)}G_Q^t](e_i,e_j)\Big|_{t=t_0} &= \frac{\partial}{\partial t}\frac{\partial^2}{\partial x_i \partial x_j}G_Q^t(x)\Big|_{(x,t)=(0,t_0)} \\
&= \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial}{\partial t}G_Q^t(x)\Big|_{(x,t)=(0,t_0)} \\
&= \frac{\partial^2}{\partial x_i \partial x_j}Q(G_Q^{t_0}(x),G_Q^{t_0}(x))\Big|_{x=0} \\
&= 2Q(D_0^{(1)}G_Q^{t_0}(e_i),D_0^{(1)}G_Q^{t_0}(e_j)) \\
&= 2Q(e_i,e_j)
\end{aligned}
$$

for any $i,j = 1,\ldots,n$ and $t_0 \in \mathbb{R}$, we have $D_0^{(2)}G_Q^t = 2tQ$ for any t. Therefore, $\Theta(G_{Q_1}^1,\ldots,G_{Q_n}^1) = (Q_1,\ldots,Q_n)$ for any $(Q_1,\ldots,Q_n) \in \mathcal{S}^2(\mathbb{R}^n)^n$.

Finally, we show Θ is injective. Note that the bijectivity of Θ implies that it is an open map. Take $G_1, G_2 \in \mathcal{D}(\mathbb{R}^N, 0)$ such that $D_0^{(1)} G_i = I$ and $\bar{F} \circ G_i = G_i^k \circ \bar{F}$ for $i = 1, 2$, and $D_0^{(2)} G_1 = D_0^{(2)} G_2$. We will show $G_1 = G_2$. For $R > 0$, we put $B_R = \{z \in \mathbb{R}^N \mid \|z\| \leq R\}$. Fix representatives $\tilde{G}_i$ of G_i for $i = 1, 2$. Since $G_i(0) = 0$ and $D_0^{(1)} G_i = I$, there exists $R_0 > 0$ and $1 < c < \sqrt[4]{k}$ such that

- $\tilde{G}_2^m \circ \tilde{G}_1^{m'}$ is well-defined on B_{R_0} for any $m, m' = 1, \ldots, k$,
- $\bar{F} \circ \tilde{G}_i = \tilde{G}_i^k \circ \bar{F}$ on B_{R_0} for $i = 1, 2$,
- $\max\{\|\tilde{G}_1^m(z)\|, \|\tilde{G}_2 \circ \tilde{G}_1^m(z)\|\} \leq c\|z\|$, and $\|\tilde{G}_2^m(z) - \tilde{G}_2^m(z')\| \leq c\|z - z'\|$ for any $z, z' \in B_{R_0}$ and $m = 1, \ldots, k$.

For $0 < R \leq R_0$, we put

$$\Delta(R) = \sup_{z \in B_R} \frac{\|\tilde{G}_1(z) - \tilde{G}_2(z)\|}{\|z\|^3}.$$

Since $D_0^{(2)} G_1 = D_0^{(2)} G_2$, then we have that $\tilde{G}_1 - \tilde{G}_2$ is of at least third order at the origin. Hence, $\Delta(R)$ is finite. For any $z \in B_{R_0}$ and $m = 1, \ldots, k$, we have $\max\{\|\tilde{G}_1^m(k^{-1}z)\|, \|\tilde{G}_2 \circ \tilde{G}_1^m(k^{-1}z)\|\} \leq (c/k)\|z\| \leq R_0$, and hence,

$$
\begin{aligned}
\|\tilde{G}_1(z) - \tilde{G}_2(z)\| &= k \cdot \|\bar{F} \circ \tilde{G}_1(z) - \bar{F} \circ \tilde{G}_2(z)\| \\
&= k \cdot \|\tilde{G}_1^k(k^{-1}z) - \tilde{G}_2^k(k^{-1}z)\| \\
&\leq k \sum_{m=1}^k \|\tilde{G}_2^{m-1} \circ \tilde{G}_1^{k-m+1}(k^{-1}z) - \tilde{G}_2^m \circ \tilde{G}_1^{k-m}(k^{-1}z)\| \\
&\leq kc \sum_{m=1}^k \|\tilde{G}_1^{k-m+1}(k^{-1}z) - \tilde{G}_2 \circ \tilde{G}_1^{k-m}(k^{-1}z)\|.
\end{aligned}
$$

Since $\|\tilde{G}_1^{k-m}(k^{-1}z)\| \leq (c/k)\|z\| \leq R_0$, it implies that

$$\|\tilde{G}_1(z) - \tilde{G}_2(z)\| \leq k^2 c \cdot \Delta(R_0) \cdot [(c/k) \cdot \|z\|]^3 = (c^4/k) \cdot \Delta(R_0) \cdot \|z\|^3.$$

Therefore, $\Delta(R_0) \leq (c^4/k)\Delta(R_0)$. Since $c < \sqrt[4]{k}$, we have $\Delta(R_0) = 0$, and hence, $\tilde{G}_1 = \tilde{G}_2$ on B_{R_0}. $\qquad\square$

Since

$$D_0^{(2)}(A \circ P_B^{b_i} \circ A^{-1}) = A \circ D_0^{(2)} P_B^{b_i} \circ (A^{-1}, A^{-1}) = 2A \circ Q_{v_i} \circ (A^{-1}, A^{-1})$$

for $A \in \mathrm{GL}_n(\mathbb{R})$ and $B = (v_1, \ldots, v_n) \in \mathrm{GL}_n(\mathbb{R})$, the map $\Theta \circ \Phi$ satisfies

$$(2) \qquad (\Theta \circ \Phi)(A, B) = (A \circ Q_{v_i} \circ (A^{-1}, A^{-1}))_{1 \leq i \leq n}.$$

Hence, $\Theta \circ \Phi$ is smooth. For $Q, Q' \in \mathcal{S}^2(\mathbb{R}^n)$, we define $[Q, Q'] \in \mathcal{S}^3(\mathbb{R}^n)$ by

$$[Q, Q'](\xi, \eta, \theta) = \{Q(\xi, Q'(\eta, \theta)) + Q(\eta, Q'(\theta, \xi)) + Q(\theta, Q'(\xi, \eta))\}$$
$$- \{Q'(\xi, Q(\eta, \theta)) + Q'(\eta, Q(\theta, \xi)) + Q'(\theta, Q(\xi, \eta))\}$$

It can be checked that

$$(3) \qquad [D_0^{(2)}G_1, D_0^{(2)}G_2] = D_0^{(3)}(G_1 \circ G_2) - D_0^{(3)}(G_2 \circ G_1)$$

for any $G_1, G_2 \in \mathcal{D}(\mathbb{R}^N, 0)$ with $D_0^{(1)}G_1 = D_0^{(1)}G_2 = I$. Therefore,

$$(4) \qquad (\Psi \circ \Theta^{-1})(Q_1, \dots, Q_n) = ([Q_i, Q_j])_{1 \le i < j \le n}.$$

Since the bracket is bilinear, the map $\Psi \circ \Theta^{-1}$ is a smooth map.

For $B = (v_1, \dots, v_n) \in \mathrm{GL}_n(\mathbb{R})$, we put

$$L_B^\Phi = D(\Theta \circ \Phi)_{(I,B)},$$
$$L_B^\Psi = D(\Psi \circ \Theta^{-1})_{(Q_{v_1}, \dots, Q_{v_n})}.$$

We identify the tangent spaces of $\mathcal{M}_0$ and $\mathcal{S}^2(\mathbb{R}^n)^n$ of each point with $M_n(\mathbb{R}^n)^2$ and $\mathcal{S}^2(\mathbb{R}^n)^n$, respectively. Then, (2) and (4) imply that

$$L_B^\Phi(A', B') = (A' \circ Q_{v_i} - Q_{v_i} \circ (A', I) - Q_{v_i} \circ (I, A') + Q_{\omega_i})_{1 \le i \le n},$$
$$L_B^\Psi(q_1, \dots, q_n) = ([q_i, Q_{v_j}] - [q_j, Q_{v_i}])_{1 \le i < j \le n}$$

for any $(A', B') \in M_n(\mathbb{R})^2$ with $B' = (\omega_1, \dots, \omega_n)$ and any $(q_1, \dots, q_n) \in \mathcal{S}^2(\mathbb{R}^n)^n$. The following proposition can be shown by a formal computation and we postpone the proof until Section 2.3.

Proposition 2.8 $\mathrm{Ker}\, L_B^\Psi = \mathrm{Im}\, L_B^\Phi$.

Theorem 2.1 follows from Corollary 2.5, Theorem 2.6, and the proposition since H is a homeomorphism between $\mathcal{M}_1$ and $\mathcal{S}^2(\mathbb{R}^n)^n$.

2.2 From local to global

In this subsection, we prove the main theorem. For a discrete group Γ and a Γ–action ρ on a manifold M, we say a point $p \in M$ is *a global fixed point* if $\rho^\gamma(p) = p$ for any $\gamma \in \Gamma$. Note the point ∞ is the unique global fixed point of ρ_B for any $B \in \mathrm{GL}_n(\mathbb{R})$.

In this subsection, we assume that $n \ge 2$ since the case $n = 1$ was already shown by Burslem and Wilkinson. First, we show that any local conjugacy to the standard $\Gamma_{n,k}$–action extends to a global one.

Proposition 2.9 *Suppose that an action $\rho \in \mathrm{Hom}(\Gamma_{n,k}, \mathrm{Diff}(M))$ admits a global fixed point p_∞ and there exists a smooth coordinate ϕ of S^n at p_∞ and $B \in \mathrm{GL}_n(\mathbb{R})$ such that $\phi(p_\infty) = 0$ and $\phi \circ \rho^\gamma \circ \phi^{-1} = P_B^\gamma$ as elements of $\mathcal{D}(\mathbb{R}^n, 0)$ for any $\gamma \in \Gamma_{n,k}$. Then, ρ is smoothly conjugate to ρ_B.*

Proof Recall $\bar{\phi} \colon S^n \to \mathbb{R}^n$ is the local coordinate at ∞ given by $\bar{\phi}(x) = (1/\|x\|^2) \cdot x$ and the local action P_B is defined by $P_B^\gamma = \bar{\phi} \circ \rho_B^\gamma \circ \bar{\phi}$. We put $U_r = S^n \setminus [-r, r]^n$ for $r > 0$ and $\Lambda_b = \{b_1^{\pm 1}, \dots, b_n^{\pm 1}\}$. By assumption, there exists $R > 0$ and a neighborhood U' of p_∞ such that

$$\phi \circ \rho^\gamma \circ \phi^{-1} = \bar{\phi} \circ \rho_B^\gamma \circ \bar{\phi}^{-1}$$

on $\bar{\phi}(U_R)$ for any $\gamma \in \{a^{\pm 1}\} \cup \Lambda_b$. Since $\rho_B^{b_1^m}(x)$ converges to ∞ as n goes to infinity for any $x \in S^n$, we can take $m_x \geq 0$ such that $\rho_B^{b_1^{m_x}}(x)$ is contained in U_R. Define a map $h \colon S^n \to S^n$ by

$$h(x) = \rho^{b_1^{-m_x}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^{m_x}}(x).$$

First, we see $h(x)$ does not depend on the choice of m_x. Suppose $\rho_B^{b_1^m}(x)$ is contained in U_R. Since ρ_B^γ is a translation for any $\gamma \in \Lambda_b$ and $S^n \setminus U_R = [-R, R]^n$ is a convex subset of $\mathbb{R}^n$, there exists a sequence $(\gamma_j)_{1 \leq j \leq l}$ of elements of Λ_b so that $b_1^m = \gamma_l \cdots \gamma_1 b_1^{m_x}$ and $\rho_B^{\gamma_j \cdots \gamma_1 b_1^{m_x}}(x)$ is contained in U_R for any $j = 1, \dots, l$.[1] Then,

$$\rho^{\gamma_{j+1}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{\gamma_j \cdots \gamma_1 b_1^{m_x}}(x) = (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{\gamma_{j+1} \gamma_j \cdots \gamma_1 b_1^{m_x}}(x).$$

This implies that

$$\begin{aligned}
\rho^{b_1^{-m}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^m}(x) &= \rho^{b_1^{-m}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{\gamma_l \cdots \gamma_1} \circ \rho_B^{b_1^{m_x}}(x) \\
&= \rho^{b_1^{-m}} \circ \rho^{\gamma_l \cdots \gamma_1} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^{m_x}}(x) \\
&= \rho^{b_1^{-m_x}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^{m_x}}(x) \\
&= h(x).
\end{aligned}$$

Therefore, $h(x)$ does not depend on the choice of m_x.

For any given $x_0 \in S^n$, there is a choice of $(m_x)_{x \in S^n}$ which is constant on a small neighborhood of x_0. This implies that h is a locally diffeomorphic at x_0, and hence, h is a covering map. Since S^n is simply connected, h is diffeomorphism.

---

[1] We need $n \geq 2$ here.

It is easy to see $h \circ \rho_B^\gamma = \rho^\gamma \circ h$ for any $\gamma \in \Lambda_b$. For any given $x \in S^n$, there is $m \geq 1$ such that $\rho_B^{b_1^{km}}(x)$ is contained in U_R. Then,

$$
\begin{aligned}
h \circ \rho_B^a(x) &= \rho^{b_1^{-km}} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^{km}} \circ \rho_B^a(x) \\
&= \rho b_1^{-km} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^a \circ \rho_B^{b_1^m}(x) \\
&= \rho b_1^{-km} \circ \rho^a \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^m}(x) \\
&= \rho^a \circ \rho b_1^{-m} \circ (\phi^{-1} \circ \bar{\phi}) \circ \rho_B^{b_1^m}(x) \\
&= \rho^a \circ h(x).
\end{aligned}
$$

Therefore, h is a smooth conjugacy between ρ_B and ρ. □

Next, we give a criterion for the persistence of a global fixed point of a $\Gamma_{n,k}$–action.

Lemma 2.10 Let M be a manifold and ρ be an action in $\mathrm{Hom}(\Gamma_{n,k}, \mathrm{Diff}(M))$. Suppose that ρ_0 has a global fixed point p_0 such that $(D\rho_0^a)_{p_0} = k^{-1}I$ and $(D\rho_0^{b_i})_{p_0} = I$ for any $i = 1, \ldots, n$. Then, there exists a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma_{n,k}, \mathrm{Diff}(M))$ of ρ_0 and a continuous map $\hat{p} \colon \mathcal{U} \to M$ such that $\hat{p}(\rho_0) = p_0$ and $\hat{p}(\rho)$ is a global fixed point of ρ for any $\rho \in \mathcal{U}$.

Proof Take $k^{-1} < \lambda < 1$ and $\delta > 0$ so that $\lambda + k\delta < 1$. Fix an open neighborhood U of p_0 and a local coordinate $\phi \colon U \to \mathbb{R}^n$. There exist convex neighborhoods V and V_1 of $\phi(p_0)$ and a neighborhood $\mathcal{U}_0$ of ρ_0 which satisfy the following conditions for any $\rho \in \mathcal{U}_0$ and $i = 1, \ldots, n$.

- $\phi \circ \rho^{a^l b_i^m} \circ \phi^{-1}$ is well-defined on V for any $l = 0, 1$ and $m = 0, \ldots, k$
- $\phi \circ \rho^{b_i} \circ \phi^{-1}(V_1) \subset V$
- $\|D(\phi \circ \rho^a \circ \phi^{-1})_z\| < \lambda$ and $\|D(\phi \circ \rho^{b_i^m} \circ \phi^{-1})_z - I\| < \delta$ for any $z \in V$ and $m = 1, \ldots, k$

By the persistence of attracting fixed point, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of ρ_0 and a continuous map $\hat{p} \colon \mathcal{U} \to \phi^{-1}(V_1 \cap V)$ such that $\hat{p}(\rho_0) = p_0$ and $\hat{p}(\rho)$ is an attracting fixed point of ρ^a for any $\rho \in \mathcal{U}$. Since $\rho_0^{b_i}(p_0) = p_0$, by replacing $\mathcal{U}$ with a smaller neighborhood of ρ_0, we may assume $\rho^{b_i}(\hat{p}(\rho)) \in \phi^{-1}(V_1 \cap V)$ for any $\rho \in \mathcal{U}$ and $i = 1, \ldots, n$.

Fix $i = 1, \ldots, n$ and $\rho \in \mathcal{U}$. Put $z_* = \phi(\hat{p}(\rho))$, $F = \phi \circ \rho^a \circ \phi^{-1}$, and $G = \phi \circ \rho^{b_i} \circ \phi^{-1}$. We will show $G(z_*) = z_*$. Since z_* and $G(z_*)$ are contained in V,

$$
\|F \circ G(z_*) - F(z_*)\| \leq \lambda \|G(z_*) - z_*\|,
$$
$$
\|(G^{m+1}(z_*) - G(z_*)) - (G^m(z_*) - z_*)\| \leq \delta \|G(z_*) - z_*\|,
$$

for $m = 0, \ldots, k-1$. Since $F \circ G = G^k \circ F$ and $F(z_*) = z_*$, the former implies

$$\|G^k(z_*) - z_*\| \le \lambda \|G(z_*) - z_*\|.$$

Hence,

$$k \cdot \|z_* - G(z_*)\| \le \|G^k(z_*) - z_*\| + \sum_{m=0}^{k-1} \|G^{m+1}(z_*) - G^m(z_*) - G(z_*) + z_*\|$$

$$\le (\lambda + k\delta) \cdot \|G(z_*) - z_*\|.$$

Since $\lambda + k\delta < 1$, we have $G(z_*) = z_*$. Thus, $\hat{p}(\rho)$ is a global fixed point of ρ. $\quad\square$

Now, we prove the main theorem.

Proof of Main Theorem Take open neighborhoods $U \subset S^n$ of ∞ and $V \subset \mathbb{R}^n$ of 0 and a family $(\phi_p)_{p \in U}$ of diffeomorphisms from U to V such that $\phi_\infty = \bar{\phi}$, $\phi_p(p) = 0$ for any $p \in U$ and the map $(p, q) \mapsto \phi_p(q)$ is smooth. Fix $B \in \mathrm{GL}_n(\mathbb{R})$. The action ρ_B satisfies the assumption of Lemma 2.10. Hence, there exists a neighborhood $\mathcal{U}_1$ of ρ_B and a continuous map $\hat{p} \colon \mathcal{U}_1 \to U$ such that $\hat{p}(\rho)$ is a global fixed point of ρ for any $\rho \in \mathcal{U}_1$. We define a local action $P_\rho \in \mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$ by $P_\rho^\gamma = \phi_{\hat{p}(\rho)} \circ \rho^\gamma \circ \phi_{\hat{p}(\rho)}^{-1}$. Then, the map $\rho \mapsto P_\rho$ is a C^2_{loc}–continuous map from $\mathcal{U}_1$ to $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D}(\mathbb{R}^n, 0))$. By Theorem 2.1, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of ρ_B such that P_ρ is smoothly conjugate to $P_{B'}$ for some $B' = B'(\rho) \in \mathrm{GL}_n(\mathbb{R})$ for any $\rho \in \mathcal{U}$. By Proposition 2.9, ρ is smoothly conjugate to $\rho_{B'}$. $\quad\square$

2.3 Proof of Proposition 2.8

In this subsection, we give a proof of the following proposition, which we have not shown in Section 2.1.

Proposition 2.8 $\mathrm{Ker}\, L_B^\Psi = \mathrm{Im}\, L_B^\Phi$.

Our proof is formal and lengthy computation. It may be interesting to find a more geometric proof.

Fix $B = (v_1, \ldots, v_n) \in \mathrm{GL}_n(\mathbb{R})$. Recall the linear maps $L_B^\Phi \colon M_n(\mathbb{R}^n)^2 \to \mathcal{S}^2(\mathbb{R}^n)^n$ and $L_B^\Psi \colon \mathcal{S}^2(\mathbb{R}^n)^n \to \mathcal{S}^3(\mathbb{R}^n)^{n(n-1)/2}$ are given by

$$L_B^\Phi(A', B') = (A' \circ Q_{v_i} - Q_{v_i} \circ (A', I) - Q_{v_i} \circ (I, A') + Q_{\omega_i})_{1 \le i \le n},$$

$$L_B^\Psi(q_1, \ldots, q_n) = ([q_i, Q_{v_j}] - [q_j, Q_{v_i}])_{1 \le i < j \le n}$$

for any $(A', B') \in M_n(\mathbb{R})^2$ with $B' = (\omega_1, \ldots, \omega_n)$ and any $(q_1, \ldots, q_n) \in \mathcal{S}^2(\mathbb{R}^n)^n$, where

$$(5) \qquad Q_v(\xi, \eta) = \langle \xi, \eta \rangle \cdot v - \langle \xi, v \rangle \cdot \eta - \langle \eta, v \rangle \cdot \xi,$$

$$\begin{aligned}
[Q, Q'](\xi, \eta, \theta) = {} & \{Q(\xi, Q'(\eta, \theta)) + Q(\eta, Q'(\theta, \xi)) + Q(\theta, Q'(\xi, \eta))\} \\
& - \{Q'(\xi, Q(\eta, \theta)) + Q'(\eta, Q(\theta, \xi)) + Q'(\theta, Q(\xi, \eta))\}.
\end{aligned}$$

First, we reduce the problem to the case $B = I$.

Lemma 2.11 For $B, B' \in \mathrm{GL}_n(\mathbb{R})$, $\mathrm{Ker}\, L_B^\Psi = \mathrm{Im}\, L_B^\Phi$ if and only if $\mathrm{Ker}\, L_{B'}^\Psi = \mathrm{Im}\, L_{B'}^\Phi$.

Proof Put $B = (v_1, \ldots, v_n)$ and $B' = (w_1, \ldots, w_n)$. Take $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{R})$ such that $B' = BA$. Since the map $v \mapsto Q_v$ is linear, we have

$$(Q_{w_1}, \ldots, Q_{w_n}) = (Q_{v_1}, \ldots, Q_{v_n}) \cdot A.$$

It implies that $\mathrm{Im}\, L_{B'}^\Phi = \mathrm{Im}\, L_B^\Phi \cdot A$. For $(q_1, \ldots, q_n) \in \mathrm{Ker}\, L_B^\Psi$ we have

$$\left[\left(\sum_{k=1}^n a_{ki} q_k \right), Q_{w_j} \right] - \left[\left(\sum_{l=1}^n a_{lj} q_l \right), Q_{w_i} \right] = \sum_{k,l=1}^n a_{ki} a_{lj} ([q_k, Q_{v_l}] - [q_l, Q_{v_k}]) = 0.$$

Hence, $\mathrm{Ker}\, L_B^\Psi \cdot A$ is a subspace of $\mathrm{Ker}\, L_{B'}^\Psi$. Similarly, $\mathrm{Ker}\, L_{B'}^\Psi \cdot A^{-1}$ is a subspace of $\mathrm{Ker}\, L_B^\Psi$. Thus, $\mathrm{Ker}\, L_{B'}^\Psi = \mathrm{Ker}\, L_B^\Psi \cdot A$. $\square$

By the lemma, it is sufficient to show Proposition 2.8 for $B = I$. Put $I = (e_1, \ldots, e_n)$. It is easy to check the following properties of Q_v.

Lemma 2.12 For $v \in \mathbb{R}^n$ and mutually disjoint $i, j, k = 1, \ldots, n$,

$$\begin{aligned}
Q_{e_i}(e_i, v) = Q_{e_i}(v, e_i) &= -v, \\
Q_{e_i}(e_j, e_j) &= e_i, \\
Q_{e_i}(e_j, e_k) &= 0.
\end{aligned}$$

$\square$

Let W be the subspace of $\mathcal{S}^2(\mathbb{R}^n)^n$ consisting of $(q_1, \ldots, q_n)$ such that

$$(6) \qquad\qquad\qquad q_j(e_j, e_j) = 0,$$

$$(7) \qquad\qquad \langle e_i, q_j(e_i, e_i) \rangle + \langle e_j, q_i(e_j, e_j) \rangle = 0,$$

$$(8) \qquad\qquad\qquad \langle e_1, q_1(e_j, e_j) \rangle = 0,$$

for any $i, j = 1, \ldots, n$.

Lemma 2.13 If $\operatorname{Ker} L_I^{\Psi} \cap W = \{0\}$, then $\operatorname{Ker} L_I^{\Psi} = \operatorname{Im} L_I^{\Phi}$.

Proof We show $\mathcal{S}^2(\mathbb{R}^n)^n = W + \operatorname{Im} L_I^{\Phi}$. Then the assumption $\operatorname{Ker} L_I^{\Psi} \cap W = \{0\}$ implies $\operatorname{Ker} L_I^{\Psi} = \operatorname{Im} L_I^{\Phi}$ since $\operatorname{Im} L_I^{\Phi} \subset \operatorname{Ker} L_I^{\Psi}$.

For $A, B \in M_n(\mathbb{R})$, let $q_j^{A,B}$ be the j–th component of $L_I^{\Phi}(A, B)$. Fix $(q_1, \ldots, q_n)$ in $\mathcal{S}^2(\mathbb{R}^n)^n$ and we will find $A, B \in M_n(\mathbb{R})$ such that

$$(9) \qquad q_j^{A,B}(e_j, e_j) = q_j(e_j, e_j),$$

$$(10) \qquad \langle e_i, q_j^{A,B}(e_i, e_i) \rangle + \langle e_j, q_i^{A,B}(e_j, e_j) \rangle = \langle e_i, q_j(e_i, e_i) \rangle + \langle e_j, q_i(e_j, e_j) \rangle,$$

$$(11) \qquad \langle e_1, q_1^{A,B}(e_j, e_j) \rangle = \langle e_1, q_1(e_j, e_j) \rangle.$$

These equations imply that $(q_1, \ldots, q_n) - L_I^{\Phi}(A, B)$ is an element of W.

Take $A = (a_{ij})$, $B = (b_{ij}) \in M_n(\mathbb{R})$. A direct computation with [Lemma 2.12](#) implies

$$q^{A,B}{}_j(e_j, e_j) = A \circ Q_{e_j}(e_j, e_j) - 2Q_{e_j}(Ae_j, e_j) + Q_{Be_j}(e_j, e_j)$$

$$= Ae_j + \sum_{k=1}^{n} b_{kj} Q_{e_k}(e_j, e_j)$$

$$(12) \qquad = (a_{jj} - b_{jj})e_j + \sum_{k \neq j}(a_{kj} + b_{kj})e_k,$$

$$q_i^{A,B}(e_j, e_j) = A \circ Q_{e_i}(e_j, e_j) - 2Q_{e_i}(Ae_j, e_j) + Q_{Be_i}(e_j, e_j)$$

$$= Ae_i - 2\sum_{k=1}^{n} a_{kj} Q_{e_i}(e_k, e_j) + \sum_{k=1}^{n} b_{ki} Q_{e_k}(e_j, e_j)$$

$$= (a_{ii} - 2a_{jj} + b_{ii})e_i + (a_{ji} + 2a_{ij} - b_{ji})e_j + \sum_{k \neq i,j}(a_{ki} + b_{ki})e_k.$$

for any mutually distinct $i, j = 1, \ldots, n$. The latter equation implies that

$$(13) \qquad \langle e_i, q_j^{A,B}(e_i, e_i) \rangle + \langle e_j, q_i^{A,B}(e_j, e_j) \rangle = 3(a_{ij} + a_{ji}) - (b_{ij} + b_{ji})$$

for any mutually distinct $i, j = 1, \ldots, n$ and

$$(14) \qquad \langle e_1, q_1^{A,B}(e_j, e_j) \rangle = a_{11} - 2a_{jj} + b_{11}$$

for any $j = 2, \ldots, n$.

Put $s_{ij} = \langle e_i, q_j(e_j, e_j) \rangle$, $t_{ij} = \langle e_j, q_i(e_j, e_j) \rangle$ and $u_j = \langle e_1, q_1(e_j, e_j) \rangle$ for $i, j = 1, \ldots, n$. Notice that $s_{11} = t_{11} = u_1$. Put $a_{11} = s_{11}/2$, $b_{11} = -s_{11}/2$,

$$a_{jj} = -u_j/2, \quad b_{jj} = -s_{jj} - (u_j/2)$$

for $j = 2, \ldots, n$, and

$$a_{ij} = \frac{1}{4}(s_{ij} + t_{ij}), \quad b_{ij} = s_{ij} - a_{ij} = \frac{1}{4}(3s_{ij} - t_{ij}),$$

for any mutually distinct $i, j = 1, \ldots, n$. By (12), (13), and (14), $A = (a_{ij})$ and $B = (b_{ij})$ satisfy (9), (10), and (11). $\qquad\square$

Fix $(q_1, \ldots, q_n) \in \mathrm{Ker}\, L_I^{\Psi} \cap W$. By the lemma, the goal is to show $q_1 = \cdots = q_n = 0$.

Lemma 2.14 $\quad q_j(e_i, e_j) = q_j(e_j, e_i) = 0$ for any $i, j = 1, \ldots, n$.

Proof When $i = j$, it is just shown by (6) in the definition of W. Take mutually distinct $i, j = 1, \ldots, n$. Then,

$$\begin{aligned}
0 &= \frac{1}{3}([q_i, Q_{e_j}] - [q_j, Q_{e_i}])(e_j, e_j, e_j) \\
&= \{q_i(e_j, Q_{e_j}(e_j, e_j)) - Q_{e_j}(e_j, q_i(e_j, e_j))\} \\
&\qquad\qquad\qquad\qquad - \{q_j(e_j, Q_{e_i}(e_j, e_j)) - Q_{e_i}(e_j, q_j(e_j, e_j))\} \\
&= \{q_i(e_j, -e_j) + q_i(e_j, e_j)\} - \{q_j(e_j, e_i) - Q_{e_i}(e_j, 0)\} \\
&= -q_j(e_j, e_i).
\end{aligned}$$

Since q_j is symmetric, we also obtain that $q_j(e_i, e_j) = 0$. $\qquad\square$

Lemma 2.15 *For any* $i, j = 1, \ldots, n$,

$$(15) \qquad\qquad \langle e_i, q_i(e_j, e_j) \rangle + \langle e_j, q_j(e_i, e_i) \rangle = 0.$$

For any $i, j, k = 1, \ldots, n$ *with* $i \neq k$,

$$(16) \qquad\qquad \langle e_k, q_i(e_j, e_j) \rangle = 0.$$

Proof When $i = j$, Lemma 2.15 follows from the definition of W. Suppose $i \neq j$. Since $q_i(e_i, e_j) = q_j(e_i, e_j) = q_j(e_j, e_j) = 0$ by Lemma 2.14 and (6) in the definition of W, we have

$$\begin{aligned}
[q_i, Q_{e_j}](e_i, e_j, e_j) &= \{q_i(e_i, Q_{e_j}(e_j, e_j)) + 2q_i(e_j, Q_{e_j}(e_i, e_j))\} \\
&\qquad\qquad - \{Q_{e_j}(e_i, q_i(e_j, e_j)) + 2Q_{e_j}(e_j, q_i(e_i, e_j))\} \\
&= \{q_i(e_i, -e_j) + 2q_i(e_j, -e_i)\} \\
&\qquad - \{\langle e_i, q_i(e_j, e_j) \rangle \cdot e_j - \langle e_j, q_i(e_j, e_j) \rangle \cdot e_i + 2Q_{e_j}(e_j, 0)\} \\
&= \langle e_j, q_i(e_j, e_j) \rangle \cdot e_i - \langle e_i, q_i(e_j, e_j) \rangle \cdot e_j,
\end{aligned}$$

$$[q_j, Q_{e_i}](e_i, e_j, e_j) = \{q_j(e_i, Q_{e_i}(e_j, e_j)) + 2q_j(e_j, Q_{e_i}(e_i, e_j))\}$$
$$- \{Q_{e_i}(e_i, q_j(e_j, e_j)) + 2Q_{e_i}(e_j, q_j(e_i, e_j))\}$$
$$= \{q_j(e_i, e_i) + 2q_j(e_j, -e_j)\} - \{Q_{e_i}(e_i, 0) + 2Q_{e_i}(e_j, 0)\}$$
$$= q_j(e_i, e_i).$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_j}] = 0$,

$$q_j(e_i, e_i) = \langle e_j, q_i(e_j, e_j) \rangle \cdot e_i - \langle e_i, q_i(e_j, e_j) \rangle \cdot e_j.$$

By taking the inner product with e_k, we obtain $\langle q_i(e_j, e_j), e_k \rangle = 0$ for $k \neq i, j$. By taking the inner product with e_i and e_j, we also have

$$\langle e_i, q_j(e_i, e_i) \rangle - \langle e_j, q_i(e_j, e_j) \rangle = 0,$$
$$\langle e_j, q_j(e_i, e_i) \rangle + \langle e_i, q_i(e_j, e_j) \rangle = 0.$$

The latter is (15). Equation (16) follows from the former and (7) in the definition of W. □

Equations (8) and (15) imply that

$$(17) \qquad\qquad \langle e_1, q_1(e_j, e_j) \rangle = \langle e_j, q_j(e_1, e_1) \rangle = 0$$

for any $j = 1, \ldots, n$. Now, we prove Proposition 2.8 for $n = 2$.

Proposition 2.16 *If $n = 2$, then $\operatorname{Ker} L_I^{\Psi} = \operatorname{Im} L_I^{\Phi}$.*

Proof For $(q_1, q_2) \in \operatorname{Ker} L_I^{\Psi} \cap W$, $\langle e_i, q_j(e_k, e_l) \rangle = 0$ for any $i, j, k, l = 1, 2$ by Lemma 2.14, Lemma 2.15 and (17). Therefore, $q_1 = q_2 = 0$. Lemma 2.13 implies that $\operatorname{Ker} L_I^{\Psi} = \operatorname{Im} L_I^{\Phi}$. Proposition 2.8 for $n = 2$ follows from Lemma 2.11. □

We continue the proof for $n \geq 3$.

Lemma 2.17 $q_i(e_j, e_k) = q_j(e_k, e_i) = q_k(e_i, e_j)$ *for mutually distinct $i, j, k = 1, \ldots, n$.*

Proof Since i, j, k are mutually distinct, Lemma 2.15 implies

$$\frac{1}{3} \cdot [q_i, Q_{e_j}](e_k, e_k, e_k) = q_i(e_k, Q_{e_j}(e_k, e_k)) - Q_{e_j}(e_k, q_i(e_k, e_k))$$
$$= q_i(e_k, e_j) - \{\langle e_k, q_i(e_k, e_k) \rangle \cdot e_j - \langle e_j, q_i(e_k, e_k) \rangle \cdot e_k\}$$
$$= q_i(e_k, e_j).$$

Similarly, we have $(1/3) \cdot [q_j, Q_{e_i}](e_k, e_k, e_k) = q_j(e_k, e_i)$. Hence,

$$q_i(e_k, e_j) - q_j(e_k, e_i) = \frac{1}{3} \cdot ([q_i, Q_{e_j}] - [q_j, Q_{e_i}])(e_k, e_k, e_k) = 0.$$

It implies $q_i(e_j, e_k) = q_i(e_k, e_j) = q_j(e_k, e_i)$. By permutations of indices (i, j, k), we obtain that $q_j(e_k, e_i) = q_k(e_i, e_j)$. □

Lemma 2.18 *For* $i, j, k = 1, \ldots, n$,

(18)
$$q_i(e_j, e_j) = 0,$$

(19)
$$\langle e_i, q_i(e_j, e_k) \rangle = \langle e_j, q_i(e_j, e_k) \rangle = \langle e_k, q_i(e_j, e_k) \rangle = 0.$$

Proof For mutually distinct $i, j, k = 1, \ldots, n$, we have

$$\begin{aligned}
[q_i, Q_{e_j}](e_j, e_k, e_k) &= \{q_i(e_j, Q_{e_j}(e_k, e_k)) + 2q_i(e_k, Q_{e_j}(e_j, e_k))\} \\
&\quad - \{Q_{e_j}(e_j, q_i(e_k, e_k)) + 2Q_{e_j}(e_k, q_i(e_j, e_k))\} \\
&= \{q_i(e_j, e_j) + 2q_i(e_k, -e_k)\} \\
&\quad - \{-q_i(e_k, e_k) + 2\left(\langle e_k, q_i(e_j, e_k) \rangle \cdot e_j - \langle e_j, q_i(e_j, e_k) \rangle \cdot e_k\right)\} \\
&= q_i(e_j, e_j) - q_i(e_k, e_k) \\
&\quad - 2\langle e_k, q_i(e_j, e_k) \rangle \cdot e_j + 2\langle e_j, q_i(e_j, e_k) \rangle \cdot e_k,
\end{aligned}$$

$$\begin{aligned}
[q_j, Q_{e_i}](e_j, e_k, e_k) &= \{q_j(e_j, Q_{e_i}(e_k, e_k)) + 2q_j(e_k, Q_{e_i}(e_j, e_k))\} \\
&\quad - \{Q_{e_i}(e_j, q_j(e_k, e_k)) + 2Q_{e_i}(e_k, q_j(e_j, e_k))\} \\
&= \{q_j(e_j, e_i) + 2q_j(e_k, 0)\} \\
&\quad - \{\langle e_j, q_j(e_k, e_k) \rangle \cdot e_i - \langle e_i, q_j(e_k, e_k) \rangle \cdot e_j + 2Q_{e_i}(e_k, 0)\} \\
&= -\langle e_j, q_j(e_k, e_k) \rangle \cdot e_i.
\end{aligned}$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$, we obtain that

$$\begin{aligned}
q_i(e_j, e_j) &- q_i(e_k, e_k) \\
&= -\langle q_j(e_k, e_k), e_j \rangle \cdot e_i + 2\langle q_i(e_j, e_k), e_k \rangle \cdot e_j - 2\langle q_i(e_j, e_k), e_j \rangle \cdot e_k.
\end{aligned}$$

By taking the inner product of the with e_i and e_j,

$$\langle e_i, q_i(e_j, e_j) \rangle - \langle e_i, q_i(e_k, e_k) \rangle = -\langle e_j, q_j(e_k, e_k) \rangle,$$

(20)
$$\langle e_j, q_i(e_j, e_j) \rangle - \langle e_j, q_i(e_k, e_k) \rangle = 2\langle e_k, q_i(e_j, e_k) \rangle.$$

The former equation for $i = 1$ implies $\langle e_j, q_j(e_k, e_k) \rangle = 0$ for any mutually distinct $j, k = 2, \ldots, n$. By (17), the same equation holds for $j = 1$ or $k = 1$. Combined with (16), we obtain (18).

Equations (16) and (20) imply $\langle e_k, q_i(e_j, e_k) \rangle = 0$. By permutations of indices (i, j, k) and Lemma 2.17, we obtain (19) for mutually distinct i, j, k. Equation (19) for other cases follows from Lemma 2.14 and (18). ☐

Proposition 2.8 for $n = 3$ follows from Lemma 2.18.

Proposition 2.19 *If $n = 3$, then* $\operatorname{Ker} L_I^\Psi = \operatorname{Im} L_I^\Phi$.

Proof For $(q_1, q_2, q_3) \in \operatorname{Ker} L_I^\Psi \cap W$, (19) in Lemma 2.18 implies $q_1 = q_2 = q_3 = 0$ if $n = 3$. By Lemma 2.13, we have $\operatorname{Ker} L_I^\Psi = \operatorname{Im} L_I^\Phi$. Proposition 2.8 for $n = 3$ follows from Lemma 2.11. ☐

The following lemma completes the proof for $n \geq 4$.

Lemma 2.20 $q_i(e_j, e_k) = 0$ *for any* $i, j, k = 1, \ldots, n$.

Proof By Lemmas 2.14 and 2.18, it is enough to show $\langle e_i, q_j(e_k, e_l) \rangle = 0$ for mutually distinct $i, j, k, l = 1, \ldots, n$. Take mutually disjoint $i, j, k, l = 1, \ldots, n$. Then,

$$[q_i, Q_{e_j}](e_k, e_l, e_l)$$
$$= \{q_i(e_k, Q_{e_j}(e_l, e_l)) + 2q_i(e_l, Q_{e_j}(e_k, e_l))\}$$
$$\qquad\qquad\qquad - \{Q_{e_j}(e_k, q_i(e_l, e_l)) + 2Q_{e_j}(e_l, q_i(e_k, e_l))\}$$
$$= \{q_i(e_k, e_j) + 2q_i(e_l, 0)\} - \{Q_{e_j}(e_k, 0) + 2(\langle e_l, q_i(e_k, e_l) \rangle \cdot e_j - \langle e_j, q_i(e_k, e_l) \rangle \cdot e_l)\}$$
$$= q_i(e_j, e_k) - 2\langle e_j, q_i(e_k, e_l) \rangle \cdot e_l.$$

Similarly, we obtain that

$$[q_j, Q_{e_i}](e_k, e_l, e_l) = q_j(e_i, e_k) - 2\langle e_i, q_j(e_k, e_l) \rangle \cdot e_l.$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$,

$$q_i(e_j, e_k) - q_j(e_i, e_k) = \{\langle e_j, q_i(e_k, e_l) \rangle - \langle e_i, q_j(e_k, e_l) \rangle\} \cdot e_l.$$

By Lemma 2.17, $q_i(e_j, e_k) = q_j(e_k, e_l)$ and $q_i(e_k, e_l) = q_k(e_l, e_i)$. Hence, we have

$$\langle e_j, q_k(e_l, e_i) \rangle = \langle e_i, q_j(e_k, e_l) \rangle.$$

By take permutations of indices (i, j, k, l),

$$(21) \qquad \langle e_l, q_i(e_j, e_k) \rangle = \langle e_i, q_j(e_k, e_l) \rangle = \langle e_j, q_k(e_l, e_i) \rangle = \langle q_l(e_i, e_j), e_k \rangle.$$

On the other hand, we have

$$[q_i, Q_{e_j}](e_j, e_k, e_l)$$
$$= \{q_i(e_j, Q_{e_j}(e_k, e_l)) + q_i(e_k, Q_{e_j}(e_l, e_j)) + q_i(e_l, Q_{e_j}(e_j, e_k))\}$$
$$\qquad - \{Q_{e_j}(e_j, q_i(e_k, e_l)) + Q_{e_j}(e_k, q_i(e_l, e_j)) + Q_{e_j}(e_l, q_i(e_j, e_k))\}$$
$$= \{q_i(e_j, 0) + q_i(e_k, -e_l) + q_i(e_l, -e_k)\}$$
$$\qquad - \{-q_i(e_k, e_l) + \langle e_k, q_i(e_l, e_j)\rangle \cdot e_j + \langle e_l, q_i(e_j, e_k)\rangle \cdot e_j\}$$
$$= -q_i(e_k, e_l) - 2\langle e_j, q_i(e_k, e_l)\rangle \cdot e_j,$$
$$[q_j, Q_{e_i}](e_j, e_k, e_l)$$
$$= \{q_j(e_j, Q_{e_i}(e_k, e_l)) + q_j(e_k, Q_{e_i}(e_l, e_j)) + q_j(e_l, Q_{e_i}(e_j, e_k))\}$$
$$\qquad - \{Q_{e_i}(e_j, q_j(e_k, e_l)) + Q_{e_i}(e_k, q_j(e_l, e_j)) + Q_{e_i}(e_l, q_j(e_j, e_k))\}$$
$$= \{q_j(e_j, 0) + q_j(e_k, 0) + q_j(e_l, 0)\}$$
$$\qquad - \{-\langle e_i, q_j(e_k, e_l)\rangle \cdot e_j + Q_{e_i}(e_k, 0) + Q_{e_i}(e_l, 0)\}$$
$$= \langle e_j, q_i(e_k, e_l)\rangle \cdot e_j.$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$,

$$q_i(e_k, e_l) + 3 \cdot \langle e_j, q_i(e_k, e_l)\rangle \cdot e_j = 0.$$

By taking the inner product with e_j, we have $\langle e_j, q_i(e_k, e_l)\rangle = 0$. Hence,

$$\langle e_i, q_j(e_k, e_l)\rangle = 0$$

by permuting indices (i, j, k, l). $\qquad\square$

Now, we prove Proposition 2.8 for $n \geq 4$. The last lemma implies that $q_1 = \cdots = q_n = 0$ for any $(q_1, \ldots, q_n) \in \operatorname{Ker} L_I^\Psi \cap W$. By Lemma 2.13, we obtain that $\operatorname{Ker} L_I^\Psi = \operatorname{Im} L_I^\Phi$. Proposition 2.8 follows from Lemma 2.11.

3 Classification of the standard actions

In this section, we classify the standard $\Gamma_{n,k}$–actions up to smooth conjugacy. Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$.

Proposition 3.1 For $B, B' \in \operatorname{GL}_n(\mathbb{R})$, ρ_B and $\rho_{B'}$ are smoothly conjugate if and only if there exists $T \in O(n)$ and $c > 0$ such that $B' = (cT)B$.

Note that all standard $\Gamma_{n,k}$–actions are *topologically conjugate* to each other, ie there exists a *homeomorphism* h of S^n such that $\rho_{B'}^\gamma \circ h = h \circ \rho_B^\gamma$ for any $\gamma \in \Gamma_{n,k}$. In

fact, if $B' = AB$ for some $A \in \mathrm{GL}_n(\mathbb{R})$, then the linear map $x \mapsto Ax$ on $\mathbb{R}^n$ extends to a *homeomorphism* h_A on S^n. It is easy to check that $\rho_{B'}^\gamma \circ h_A = h_A \circ \rho_B^\gamma$ for any $\gamma = a, b_1, \ldots, b_n$. When $A = cT$ with $c > 0$ and $T \in O(n)$, then h_A is a *diffeomorphism*. Hence, ρ_B and $\rho_{B'}$ are smoothly conjugate in this case.

To prove the "only if" part of Proposition 3.1, we need a technical lemma. Recall that $Q_v \in \mathcal{S}^2(\mathbb{R}^n)$ is defined by

$$(22) \qquad Q_v(\xi, \eta) = \langle \xi, \eta \rangle \cdot v - \langle \xi, v \rangle \cdot \eta - \langle \eta, v \rangle \cdot \xi.$$

Lemma 3.2 *Suppose that $A \circ Q_v = Q_w \circ (A, A)$ for $v, w \in \mathbb{R}^n \setminus \{0\}$ and $A \in \mathrm{GL}_n(\mathbb{R})$. Then, $A = cT$ for some $c > 0$ and $T \in O(n)$.*

Proof By a direct computation, we have

$$Q_w(A\xi, A\xi) = \|A\xi\|^2 \cdot w - 2\langle A\xi, w \rangle \cdot A\xi,$$
$$A \circ Q_v(\xi, \xi) = \|\xi\|^2 \cdot Av - 2\langle \xi, v \rangle \cdot A\xi.$$

Hence,

$$(23) \qquad \|A\xi\|^2 \cdot w - \|\xi\|^2 \cdot Av = 2\left(\langle A\xi, w \rangle - \langle \xi, v \rangle\right) \cdot A\xi$$

for any $\xi \in \mathbb{R}^n$. Put $\lambda = \left(2\langle Av, w \rangle - \|v\|^2\right) / \|Av\|^2$. Then, the equation for $\xi = v$ implies $w = \lambda Av$. By substituting it to (23), we have

$$(\lambda \|A\xi\|^2 - \|\xi\|^2) Av = 2\left(\lambda \langle A\xi, Av \rangle - \langle \xi, v \rangle\right) \cdot A\xi.$$

Since A is invertible, it implies that $\|A\xi\| = \lambda^{-1}\|\xi\|$ for any $\xi \in \mathbb{R}^n \setminus \mathbb{R}v$. Since $\mathbb{R}^n \setminus \mathbb{R}v$ is a dense subset of $\mathbb{R}^n$, the same holds for any $\xi \in \mathbb{R}^n$. Hence, there exists $T \in O(n)$ such that $A = \lambda^{-1}T$. $\qquad \square$

Proof of Proposition 3.1 It is enough to show the "only if" part. Suppose ρ_B and $\rho_{B'}$ are smoothly conjugate. Take a diffeomorphism h of S^n such that $\rho_{B'}^\gamma \circ h = h \circ \rho_B^\gamma$ for any $\gamma \in \Gamma_{n,k}$. Since ∞ is the unique global fixed point of ρ_B and $\rho_{B'}$, the diffeomorphism h fixes ∞. Recall P_B and $P_{B'}$ are the local $\Gamma_{n,k}$–actions defined by $P_B^\gamma = \bar{\phi} \circ \rho_B^\gamma \circ \bar{\phi}^{-1}$ and $P_{B'}^\gamma = \bar{\phi} \circ \rho_{B'}^\gamma \circ \bar{\phi}^{-1}$, where $\bar{\phi}(x) = (1/\|x\|^2) \cdot x$. Put $H = \bar{\phi} \circ h \circ \bar{\phi}^{-1}$ and $A = D_0^{(1)}H$. Then, $P_{B'}^\gamma \circ H = H \circ P_B^\gamma$, and hence,

$$(24) \quad D_0^{(1)} P_{B'}^\gamma \circ D_0^{(2)} H + D_0^{(2)} P_{B'}^\gamma \circ (A, A)$$
$$= A \circ D_0^{(2)} P_B^\gamma + D_0^{(2)} H \circ (D_0^{(1)} P_B^\gamma, D_0^{(1)} P_B^\gamma).$$

Since $P_B^a(x) = P_{B'}^a(x) = k^{-1}x$, Equation (24) for $\gamma = a$ implies $k^{-1} D_0^{(2)} H = k^{-2} D_0^{(2)} H$. Therefore, $D_0^{(2)} H = 0$. Put $B = (v_1, \ldots, v_n)$ and $B' = (w_1, \ldots, w_n)$.

Since $D_0^{(2)} P_B^{b_i} = 2Q_{v_i}$ and $D_0^{(2)} P_{B'}^{b_i} = 2Q_{w_i}$, (24) for $\gamma = b_i$ implies

$$Q_{w_i} \circ (A, A) = A \circ Q_{v_i}$$

for any $i = 1, \ldots, n$. By Lemma 3.2, there exists $c > 0$ and $T \in O(n)$ such that $A = cT$. Since T preserves the inner product,

$$
\begin{aligned}
(cT) \circ Q_{v_i}(\xi, \eta) &= Q_{w_i}((cT)\xi, (cT)\eta) \\
&= c^2 \{ \langle T\xi, T\xi \rangle \cdot w_i - \langle T\xi, w_i \rangle \cdot T\eta - \langle T\eta, w_i \rangle \cdot T\xi \} \\
&= (cT) \circ \{ \langle \xi, \eta \rangle \cdot (cT^{-1})w_i - \langle \xi, (cT^{-1})w_i \rangle \cdot \eta - \langle \eta, (cT^{-1})w_i \rangle \cdot \xi \} \\
&= (cT) \circ Q_{cT^{-1}w_i}(\xi, \eta)
\end{aligned}
$$

for any $\xi, \eta \in \mathbb{R}^n$. It implies that $v_i = cT^{-1}w_i$ for any $i = 1, \ldots, n$. Therefore, $B' = (c^{-1}T)B$. $\qquad \square$

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