Geometry & Topology 16 (2012) 2037–2065

Rational algebraic K–theory of topological K–theory

CHRISTIAN AUSONI
JOHN ROGNES

We show that after rationalization there is a homotopy fiber sequence

$$BBU_{\otimes} \to K(ku) \to K(\mathbb{Z})\,.$$

We interpret this as a correspondence between the virtual 2–vector bundles over a space X and their associated anomaly bundles over the free loop space $\mathcal{L}X$. We also rationally compute $K(KU)$ by using the localization sequence, and $K(MU)$ by a method that applies to all connective S–algebras.

55N15; 18F25, 19Lxx

1 Introduction

We are interested in the algebraic K–theory $K(ku)$ of the connective complex K–theory spectrum ku. By the calculations of the authors [4, Theorem 0.4] and the first author [2, Theorem 1.1], the "mod p and v_1" homotopy of $K(ku)$ is purely v_2–periodic, a distinctive homotopy theoretic property it shares with the spectra representing elliptic cohomology in the sense of Landweber–Ravenel–Stong [22] and topological modular forms in the sense of Hopkins et al. [18, Section 4]. The theory of 2–vector bundles from Baas–Dundas–Rognes [6] and Baas–Dundas–Richter–Rognes [5] therefore exhibits $K(ku)$ as a geometrically defined form of elliptic cohomology. In Section 6 we outline how a 2–vector bundle with connecting data, similar to a connection in a vector bundle, is thought to specify a $(1+1)$–dimensional conformal field theory. Since these 2–vector bundles are also effective cycles for the form of elliptic cohomology theory represented by $K(ku)$, we have some justification for referring to them as elliptic objects, as proposed by Segal [32].

As illustrated by the authors' calculations referred to above, the arithmetic and homotopy-theoretic information captured by algebraic K–theory becomes more accessible after the introduction of suitable finite coefficients. However, for the extraction of $\mathbb{C}$–valued numerical invariants from a conformal field theory, only the rational homotopy type of $K(ku)$ will matter. We are grateful to Ib Madsen and Dennis Sullivan for insisting that for such geometric applications, we should first want to compute $K(ku)$

Published: 3 September 2012 DOI: 10.2140/gt.2012.16.2037

rationally. To this end we can offer the following theorem. There is a unit inclusion map

$$w\colon BBU_{\otimes} \to BGL_1(ku) \to BGL_{\infty}(ku)^{+} \to K(ku)\,.$$

Let $\pi\colon K(ku) \to K(\mathbb{Z})$ be induced by the zeroth Postnikov section $ku \to H\mathbb{Z}$. The composite $\pi \circ w$ is the constant map to the base-point of the 1–component of $K(\mathbb{Z})$.

Theorem 1.1 (a) *After rationalization,*

$$BBU_{\otimes} \xrightarrow{w} K(ku) \xrightarrow{\pi} K(\mathbb{Z})$$

is a split homotopy fiber sequence. A rational splitting of π is given in Corollary 2.4(a) as the map $K(S) \to K(ku)$ induced by the unit map $S \to ku$.

(b) *The Poincaré series of $K(ku)$ is*

$$1 + \frac{t^3}{1-t^2} + \frac{t^5}{1-t^4} = 1 + \frac{t^3 + 2t^5}{1-t^4}\,.$$

(c) *There is a rational determinant map*

$$\det_{\mathbb{Q}}\colon BGL_{\infty}(ku)^{+} \to BGL_1(ku)_{\mathbb{Q}}$$

that, in its relative form for $ku \to H\mathbb{Z}$, rationally splits w. In other words, the composite of $w\colon BBU_{\otimes} \to K(\pi) \simeq BGL_{\infty}(\pi)^{+}$ and the determinant map $BGL_{\infty}(\pi)^{+} \to BGL_1(\pi)_{\mathbb{Q}}$ is a rational equivalence, where $K(\pi)$ denotes the homotopy fiber of $\pi\colon K(ku) \to K(\mathbb{Z})$, and similarly for the other functors.

By the Poincaré series of a space X of finite type, we mean the formal power series $\sum_{n\ge 0} r_n t^n$ in $\mathbb{Z}[\![t]\!]$, where r_n is the rank of $\pi_n(X)$. Theorem 1.1 is proved by assembling Theorem 3.3(a) and Theorem 5.5(a). The other parts of those theorems prove similar results for $K(ko)$ and $K(\ell)$, where $\ell = BP\langle 1\rangle$.

The splitting of $K(ku)_{\mathbb{Q}}$ shows that for a (virtual) 2–vector bundle over X, represented by a map $\mathcal{E}\colon X \to K(ku)$, the rational information splits into two pieces. The less interesting piece is the decategorified information carried by the dimension bundle $\dim(\mathcal{E}) = \pi \circ \mathcal{E}\colon X \to K(\mathbb{Z})$. The more interesting piece is the determinant bundle

$$|\mathcal{E}| = \det_{\mathbb{Q}} \circ \mathcal{E}\colon X \to (BBU_{\otimes})_{\mathbb{Q}}\,.$$

To specify $|\mathcal{E}|$ is equivalent to specifying a rational virtual vector bundle $\mathcal{H}\colon \mathcal{L}X \to (BU_{\otimes})_{\mathbb{Q}}$ over the free loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$, subject to a coherence condition relating the composition of free loops, when defined, to the tensor product of virtual vector spaces. See diagram (6-1). We call $\mathcal{H}$ the “anomaly bundle”.

The conclusion is that for rational purposes the essential information in a 2–vector bundle $\mathcal{E}$ over X is encoded in its anomaly bundle $\mathcal{H}$ over $\mathcal{L}X$ (subject to the indicated coherence condition, which we think of as implicit, and together with the dimension bundle $\dim(\mathcal{E})$ over X, which we tend to ignore). In physical language, the fiber of the anomaly bundle at a free loop $\gamma\colon S^1 \to X$ plays the role of the state space of γ viewed as a closed string in X. The advantage of 2–vector bundles over their homotopy-theoretic alternatives, such as representing maps to classifying spaces or bundles of ku–modules, is that they are geometrically modeled in terms of vector bundles, rather than virtual vector bundles. This seems to become an essential virtue when one wants to treat differential-geometric structures like connections on these bundles.

We also compute the rational algebraic K–theory $K(KU)$ of the periodic complex K–theory spectrum KU. To this end we evaluate the (rationalized) transfer map π_* in the localization sequence

$$K(\mathbb{Z}) \xrightarrow{\pi_*} K(ku) \xrightarrow{\rho} K(KU)$$

predicted by the second author and established by Blumberg–Mandell [7].

Theorem 1.2 (a) *There is a homotopy fiber sequence of infinite loop spaces*

$$K(ku) \xrightarrow{\rho} K(KU) \xrightarrow{\partial} BK(\mathbb{Z})$$

where ρ is induced by the connective cover map $ku \to KU$, and $BK(\mathbb{Z})$ denotes the first connected delooping of $K(\mathbb{Z})$. This sequence splits rationally.

(b) *The Poincaré series of $K(KU)$ is*

$$(1+t) + \frac{t^3 + 2t^5 + t^6}{1 - t^4}\,.$$

See Theorem 3.6 for our proof.

As stated, Theorems 1.1 and 1.2 only concern the algebraic K–theory of topological K–theory, but we develop our proofs in the greater generality of arbitrary connective S–algebras. In Section 2 we observe how the calculation by Goodwillie [15] of the relative rational algebraic K–theory for a 1–connected map $R \to \pi_0 R$ of simplicial rings (which generalizes earlier calculations by Hsiang–Staffeldt [20] for simplicial group rings), also applies to determine the relative rational algebraic K–theory for a 1–connected map $A \to H\pi_0 A$ of connective S–algebras. The answer is given in terms of negative cyclic homology; see Theorem 2.1 and Corollary 2.2.

When $\pi_0 A$ is close to $\mathbb{Z}$, and $A \to H\pi_0 A$ is a "rational de Rham equivalence", we get a very simple expression for the relative rational algebraic K–theory as the image of Connes' B–operator on Hochschild homology; see Proposition 2.3 and Corollary 2.4. These hypotheses apply to a number of interesting examples of connective S–algebras, including the K–theory spectra ku, ko and ℓ, and the bordism spectra MU, MSO and MSp. We work these examples out in Theorem 3.3 and Theorem 4.2, respectively.

In Section 5 we consider the unit inclusion map $w\colon BGL_1(A) \to K(A)$. For commutative A, the rationalization $A_{\mathbb{Q}}$ is equivalent as a commutative $H\mathbb{Q}$–algebra to the Eilenberg–Mac Lane spectrum HR of a commutative simplicial $\mathbb{Q}$–algebra R, so we can use the determinant $GL_n(R) \to GL_1(R)$ to define a rational determinant map

$$\det_{\mathbb{Q}}\colon BGL_\infty(A)^+ \to BGL_1(A)_{\mathbb{Q}} .$$

We show in Proposition 5.4 that the composite $\det_{\mathbb{Q}} \circ w$ is the rationalization map, and apply this in Theorem 5.5 to show that w induces a rational equivalence from $BBU_\otimes$ to the homotopy fiber of $\pi\colon K(ku) \to K(\mathbb{Z})$, and similarly for ko and ℓ. This last step is a counting argument; it does not apply for MU or the other bordism spectra.

Acknowledgements In an earlier version of this paper, we emphasized a trace map to $THH(ku)$ over the determinant map to $(BBU_\otimes)_{\mathbb{Q}}$, in order to detect the image of w in $K(ku)$. We are grateful to Bjørn Dundas for reminding us of the existence of determinants for commutative simplicial rings, which is half of the basis for the existence of the map $\det_{\mathbb{Q}}$ defined in Lemma 5.3. We are also grateful to Mike Mandell and Brooke Shipley for help with some of the references concerning commutative simplicial $\mathbb{Q}$–algebras given in Section 2.1.

2 Rational algebraic K–theory of connective S–algebras

2.1 S–algebras

We work in one of the modern symmetric monoidal categories of spectra constructed by Elmendorf–Kriz–Mandell–May [14], Hovey–Shipley–Smith [19] or Mandell–May–Schwede–Shipley [24], which we shall refer to as S–modules. The monoids (resp. commutative monoids) in these categories are called S–algebras (resp. commutative S–algebras), and are equivalent to the A_∞ ring spectra (resp. E_∞ ring spectra) considered since the 1970's. The Eilenberg–Mac Lane functor $H\colon R \mapsto HR$ maps the category of simplicial rings (resp. commutative simplicial rings) to the category of S–algebras (resp. commutative S–algebras).

Schwede proved in [31, Theorem 4.5] that H is part of a Quillen equivalence from the category of simplicial rings to the category of connective $H\mathbb{Z}$–algebras. There is a similar equivalence between the category of commutative simplicial $\mathbb{Q}$–algebras and the category of connective commutative $H\mathbb{Q}$–algebras.

One form of the latter equivalence appears in Kriz–May [21, II.1.3]. In a little more detail, the category of connective commutative $H\mathbb{Q}$–algebras is "connective Quillen equivalent" (see Mandell–May–Schwede–Shipley [24, page 445]) to the category of connective E_∞ $H\mathbb{Q}$–ring spectra (see Elmendorf–Kriz–Mandell–May [14, II.4]), and connective E_∞ $H\mathbb{Q}$–ring spectra are the E_∞ objects in connective $H\mathbb{Q}$–modules, which are Quillen equivalent to E_∞ simplicial $\mathbb{Q}$–algebras (see Schwede [31, Theorem 4.4]). The monads defining E_∞ algebras and commutative algebras in simplicial $\mathbb{Q}$–modules are weakly equivalent, since for every $j \geq 0$ the group homology of Σ_j with coefficients in any $\mathbb{Q}$–module is concentrated in degree zero. Hence E_∞ simplicial $\mathbb{Q}$–algebras are Quillen equivalent to commutative simplicial $\mathbb{Q}$–algebras (see Mandell [23, Theorem 6.7]).

The homotopy categories of commutative simplicial rings and connective commutative $H\mathbb{Z}$–algebras are not equivalent.

2.2 Linearization

Let A be a connective S–algebra. We write $\pi = \pi_A\colon A \to H\pi_0 A$ for its zeroth Postnikov section, and define the linearization map $\lambda = \lambda_A\colon A \to H\mathbb{Z} \wedge A$ to be $\pi \wedge \mathrm{id}\colon A \cong S \wedge A \to H\mathbb{Z} \wedge A$. It is a π_0–isomorphism and a rational equivalence of connective S–algebras. For each (simplicial or topological) monoid G let $S[G] = \Sigma^\infty G_+$ be its unreduced suspension spectrum. For $A = S[G]$, the linearization map $\lambda\colon S[G] \to H\mathbb{Z} \wedge S[G] \cong H\mathbb{Z}[G]$ agrees with the map considered by Waldhausen [34, page 43].

In general, $H\mathbb{Z} \wedge A$ is a connective $H\mathbb{Z}$–algebra, so by the first Quillen equivalence above there is a naturally associated simplicial ring R with $H\mathbb{Z} \wedge A \simeq HR$. For connective commutative A, the rationalization $A_\mathbb{Q} = H\mathbb{Q} \wedge A$ is a connective commutative $H\mathbb{Q}$–algebra, so by the second Quillen equivalence above there is a naturally associated commutative simplicial $\mathbb{Q}$–algebra R with $A_\mathbb{Q} \simeq HR$.

2.3 Algebraic K–theory

For a general S–algebra A, the algebraic K–theory space $K(A)$ can be defined as $\Omega|hS_\bullet\mathcal{C}_A|$, where $\mathcal{C}_A$ is the category of finite cell A–module spectra and their retracts, $S_\bullet$ denotes Waldhausen's $S_\bullet$–construction [36, Section 1.3], and $|h(-)|$ indicates the nerve of the subcategory of weak equivalences. By iterating the $S_\bullet$–construction, we may also view $K(A)$ as a spectrum. For connective S–algebras, $K(A)$ can alternatively be defined in terms of Quillen's plus-construction as $K_0(\pi_0 A) \times BGL_\infty(A)^+$, and then the two definitions are equivalent, see Elmendorff–Kriz–Mandell–May [14, VI.7.1]. We write $K(R)$ for $K(HR)$, and similarly for other functors defined on S–algebras.

Any map $A \to A'$ of connective S–algebras that is a π_0–isomorphism and a rational equivalence induces a rational equivalence $K(A) \to K(A')$, see Waldhausen [34, Proposition 2.2]. The proof goes by observing that $BGL_n(A) \to BGL_n(A')$ is a rational equivalence for each n. In particular, for R with $H\mathbb{Z} \wedge A \simeq HR$ there is a natural rational equivalence $\lambda\colon K(A) \to K(H\mathbb{Z} \wedge A) \simeq K(R)$. For $X \simeq BG$, Waldhausen writes $A(X)$ for $K(S[G])$, and $\lambda\colon A(X) \to K(\mathbb{Z}[G])$ is a rational equivalence. In this case, $A(X)$ can also be defined as the algebraic K–theory of a category $\mathcal{R}_f(X)$ of suitably finite retractive spaces over X, see Waldhausen [36, Section 2.1].

2.4 Cyclic homology

There is a natural trace map tr: $K(A) \to THH(A)$ to the topological Hochschild homology of A, see Bökstedt–Hsiang–Madsen [8, Section 3]. The target is a cyclic object in the sense of Connes, hence carries a natural S^1–action. There exists a model for the trace map that factors through the fixed points of this circle action, see Dundas [13], hence it also factors through the homotopy fixed points $THH(A)^{hS^1}$. We get a natural commutative triangle

$$\begin{tikzcd}
K(A) \arrow[r, "\alpha"] \arrow[dr, "\mathrm{tr}"'] & THH(A)^{hS^1} \arrow[d, "F"] \\
& THH(A)
\end{tikzcd}$$

where the Frobenius map F forgets about S^1–homotopy invariance. For any simplicial ring R there are natural equivalences $THH(HR_{\mathbb{Q}}) \simeq HH(R \otimes \mathbb{Q})$ (Hochschild homology space) and $THH(HR_{\mathbb{Q}})^{hS^1} \simeq HC^-(R \otimes \mathbb{Q})$ (negative cyclic homology space). See, for example, Elmendorff–Kriz–Mandell–May [14, IX.1.7] and Cohen–Jones [12, Lemma 1.3(3)]. With these identifications, the triangle above realizes the commutative diagram of Goodwillie [15, II.3.1]. In [15, II.3.4], Goodwillie proved the following:

Theorem 2.1 *Let $f\colon R \to R'$ be a map of simplicial rings, with $\pi_0 R \to \pi_0 R'$ a surjection with nilpotent kernel. Then*

$$\begin{tikzcd}
K(R)_{\mathbb{Q}} \arrow[r, "\alpha"] \arrow[d, "f"'] & HC^-(R \otimes \mathbb{Q}) \arrow[d, "f"] \\
K(R')_{\mathbb{Q}} \arrow[r, "\alpha"] & HC^-(R' \otimes \mathbb{Q})
\end{tikzcd}$$

is homotopy Cartesian, that is, the map of vertical homotopy fibers

$$\alpha\colon K(f)_{\mathbb{Q}} \to HC^-(f \otimes \mathbb{Q})$$

is an equivalence.

Here we write $K(f)$ for the homotopy fiber of $K(R) \to K(R')$, so that there is a long exact sequence

$$\cdots \to K_{*+1}(R') \to K_*(f) \to K_*(R) \to K_*(R') \to \cdots ,$$

and similarly for other functors from (S–)algebras to spaces. (Goodwillie writes $K(f)$ for a delooping of our $K(f)$, but we need to emphasize fibers over cofibers.) We write $K(R)_{\mathbb{Q}}$ for the rationalization of $K(R)$, and similarly for other spaces and S–algebras.

Corollary 2.2 *Let $g\colon A \to A'$ be a map of connective S–algebras, with $\pi_0 A \to \pi_0 A'$ a surjection with nilpotent kernel. Then*

$$\begin{tikzcd}
K(A)_{\mathbb{Q}} \arrow[r, "\alpha"] \arrow[d, "g"'] & THH(A_{\mathbb{Q}})^{hS^1} \arrow[d, "g"] \\
K(A')_{\mathbb{Q}} \arrow[r, "\alpha"] & THH(A'_{\mathbb{Q}})^{hS^1}
\end{tikzcd}$$

is homotopy Cartesian, that is, the map of vertical homotopy fibers

$$\alpha\colon K(g)_{\mathbb{Q}} \to THH(g_{\mathbb{Q}})^{hS^1}$$

is an equivalence.

Proof Given $g\colon A \to A'$ we find $f\colon R \to R'$ with $H\mathbb{Z} \wedge A \simeq HR$ and $H\mathbb{Z} \wedge A' \simeq HR'$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda"] \arrow[d, "g"'] & HR \arrow[d, "Hf"] \\
A' \arrow[r, "\lambda"] & HR'
\end{tikzcd}$$

homotopy commute. Then $\lambda\colon K(A) \to K(R)$ is a rational equivalence and $A_{\mathbb{Q}} \simeq H(R \otimes \mathbb{Q})$, so the square in the corollary is equivalent to the square in Goodwillie's theorem. □

2.5 De Rham homology

The (spectrum level) circle action on $THH(A)$ induces a suspension operator

$$d\colon THH_*(A) \to THH_{*+1}(A),$$

analogous to Connes' operator $B\colon HH_*(R) \to HH_{*+1}(R)$. When $A_{\mathbb{Q}} \simeq H(R \otimes \mathbb{Q})$, these operators are compatible under the isomorphism $THH_*(A_{\mathbb{Q}}) \cong HH_*(R \otimes \mathbb{Q})$. In general $dd = d\eta$ is not zero (see Hesselholt [16, (1.4.4)]), where η is the stable Hopf map, but in the algebraic case $BB = 0$, so one can define the de Rham homology

$$H^{dR}_*(R) = \ker(B)/\operatorname{im}(B)$$

of a simplicial ring R as the homology of $HH_*(R)$ with respect to the B–operator.

For a map $g\colon A \to A'$ of S–algebras, the homotopy fiber $THH(g)$ of $THH(A) \to THH(A')$ inherits a circle action and associated suspension operator. Similarly, for a map $f\colon R \to R'$ of simplicial rings there is a relative B–operator acting on the term $HH_*(f)$ in the long exact sequence

$$\dots \to HH_{*+1}(R') \to HH_*(f) \to HH_*(R) \to HH_*(R') \to \dots ,$$

and we define $H^{dR}_*(f)$ to be the homology of $HH_*(f)$ with respect to this B–operator. We say that $f\colon R \to R'$ is a de Rham equivalence if $H^{dR}_*(f) = 0$, and that f is a rational de Rham equivalence if $H^{dR}_*(f \otimes \mathbb{Q}) = 0$. If we assume that $HH_*(R) \to HH_*(R')$ is surjective in each degree, then there is a long exact sequence

$$\dots \to H^{dR}_{*-1}(R') \to H^{dR}_*(f) \to H^{dR}_*(R) \to H^{dR}_*(R') \to \dots ,$$

in which case f is a de Rham equivalence if and only if $f_*\colon H^{dR}_*(R) \to H^{dR}_*(R')$ is an isomorphism in every degree.

Proposition 2.3 *If $f\colon R \to R'$ is a de Rham equivalence, then there is an exact sequence*

$$0 \to HC^-_*(f) \xrightarrow{F} HH_*(f) \xrightarrow{B} HH_{*+1}(f)$$

that identifies $HC^-_*(f)$ with $\ker(B) \subset HH_*(f)$.

Proof By analogy with the homotopy fixed point spectral sequence for $THH(g)^{hS^1}$, there is a second quadrant homological spectral sequence

$$E^2_{**} = \mathbb{Q}[t] \otimes HH_*(f) \Longrightarrow HC^-_*(f)$$

with $t \in E^2_{-2,0}$ and $d^2(t^i \cdot x) = t^{i+1} \cdot B(x)$ for all $x \in HH_*(f)$, $i \geq 0$. See, for example, the second author's paper [28, Section 3.3]. So E^3_{**} is the sum of $\ker(B) \subset HH_*(f)$ in the zeroth column and a copy of $H^{dR}_*(f)$ in each even, negative column. By assumption the latter groups are all zero, so the spectral sequence collapses to the zeroth column at the E^3–term. The Frobenius F is the edge homomorphism for this spectral sequence, and the assertion follows. □

Corollary 2.4 *Let A be a connective S–algebra such that $\pi_0 A$ is any localization of the integers, and let R be a simplicial $\mathbb{Q}$–algebra with $A_{\mathbb{Q}} \simeq HR$.*

(a) *The homotopy fiber sequence*

$$K(\pi_A) \to K(A) \xrightarrow{\pi_A} K(\pi_0 A)$$

is rationally split, where $\pi_A \colon A \to H\pi_0 A$ is the zeroth Postnikov section.

(b) *There are equivalences*

$$K(\pi_A)_{\mathbb{Q}} \underset{\simeq}{\overset{\alpha}{\to}} THH(\pi_{A\mathbb{Q}})^{hS^1} \simeq HC^-(\pi_R)\,,$$

where $\pi_R \colon R \to \pi_0 R = \mathbb{Q}$ is the zeroth Postnikov section.

Suppose furthermore that $H^{dR}_*(R) \cong \mathbb{Q}$ is trivial in positive degrees.

(c) *The map π_R is a de Rham equivalence, and the Frobenius map identifies $HC^-_*(\pi_R)$ with the positive-degree part of*

$$\ker(B) \subset HH_*(R) \cong THH_*(A) \otimes \mathbb{Q}\,.$$

That part is also equal to $\mathrm{im}(B) \subset THH_*(A) \otimes \mathbb{Q}$.

(d) *The trace map* tr: $K(A) \to THH(A)$ *induces the composite identification of $K_*(\pi_A) \otimes \mathbb{Q}$ with the positive-degree part of* $\ker(B) \subset THH_*(A) \otimes \mathbb{Q}$.

Proof (a) Write $\pi_0 A = \mathbb{Z}_{(P)}$ for some (possibly empty) set of primes P. The unit map $i \colon S \to A$ factors through $S_{(P)}$, and the composite map $S_{(P)} \to A \to H\pi_0 A$ is a π_0–isomorphism and a rational equivalence. Hence the composite

$$K(S_{(P)}) \to K(A) \xrightarrow{\pi_A} K(\pi_0 A)$$

is a rational equivalence.

(b) The map $\pi_A\colon A \to H\pi_0 A$ induces the identity on π_0, so α is an equivalence by Corollary 2.2. We recalled the second identification in Section 2.4. It is clear that $\pi_0 R = \pi_0 A_{\mathbb{Q}} = \pi_0 A \otimes \mathbb{Q} = \mathbb{Q}$.

(c) Since $HH_*(\mathbb{Q}) = \mathbb{Q}$ is trivial in positive degrees, the map $HH_*(R) \to HH_*(\mathbb{Q})$ is surjective in each degree, so π_R is a de Rham equivalence if (and only if) $H^{dR}_*(R) \cong H^{dR}_*(\mathbb{Q}) = \mathbb{Q}$ is trivial in all positive degrees. The homotopy fiber sequence

$$HH(\pi_R) \to HH(R) \to HH(\mathbb{Q})$$

identifies $HH_*(\pi_R)$ with the positive-degree part of $HH_*(R)$, so $\ker(B) \subset HH_*(\pi_R)$ is the positive-degree part of $\ker(B) \subset HH_*(R)$. The identification $\operatorname{im}(B) = \ker(B)$ in positive degrees is of course equivalent to the vanishing of $H^{dR}_*(R)$ in positive degrees.

(d) The trace map factors as $\mathrm{tr} = F \circ \alpha$. □

3 Examples from topological K–theory

3.1 Connective K–theory spectra

Let ku be the connective complex K–theory spectrum, ko the connective real K–theory spectrum, and $\ell = BP\langle 1\rangle$ the Adams summand of $ku_{(p)}$, for p an odd prime. These are all commutative S–algebras. We write $\Omega^\infty ku = BU \times \mathbb{Z}$, $\Omega^\infty ko = BO \times \mathbb{Z}$ and $\Omega^\infty \ell = W \times \mathbb{Z}_{(p)}$ for the underlying infinite loop spaces (see May [25, V.3-4]). The homotopy units form infinite loop spaces, namely $GL_1(ku) = BU_\otimes \times \{\pm 1\}$, $GL_1(ko) = BO_\otimes \times \{\pm 1\}$ and $GL_1(\ell) = W_\otimes \times \mathbb{Z}^\times_{(p)}$. The homotopy algebras are $\pi_* ku = \mathbb{Z}[u]$ with $|u| = 2$, $\pi_* ko = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2{-}4\beta)$ with $|\eta| = 1$, $|\alpha| = 4$, $|\beta| = 8$, and $\pi_*\ell = \mathbb{Z}_{(p)}[v_1]$ with $|v_1| = 2p-2$. The complexification map $ko \to ku$ takes η to 0, α to $2u^2$ and β to u^4. The inclusion $\ell \to ku_{(p)}$ takes v_1 to u^{p-1}.

Proposition 3.1 (a) *There are π_0–isomorphisms and rational equivalences*

$$\kappa\colon S[\Omega S^3] \to S[K(\mathbb{Z},2)] \to ku$$

of S–algebras, so $ku_{\mathbb{Q}} \simeq H\mathbb{Q}[\Omega S^3]$ as homotopy commutative $H\mathbb{Q}$–algebras, where $\mathbb{Q}[\Omega S^3]$ is a simplicial $\mathbb{Q}$–algebra, and $ku_{\mathbb{Q}} \simeq H\mathbb{Q}[K(\mathbb{Z},2)]$ as commutative $H\mathbb{Q}$–algebras, where $\mathbb{Q}[K(\mathbb{Z},2)]$ is a commutative simplicial $\mathbb{Q}$–algebra.

(b) *There are π_0–isomorphisms and rational equivalences $\bar\alpha\colon S[\Omega S^5] \to ko$ and $\bar v_1\colon S_{(p)}[\Omega S^{2p-1}] \to \ell$, so $ko_{\mathbb{Q}} \simeq H\mathbb{Q}[\Omega S^5]$ and $\ell_{\mathbb{Q}} \simeq H\mathbb{Q}[\Omega S^{2p-1}]$ as $H\mathbb{Q}$–algebras, where $\mathbb{Q}[\Omega S^5]$ and $\mathbb{Q}[\Omega S^{2p-1}]$ are simplicial $\mathbb{Q}$–algebras.*

(c) *In particular, there is a rational equivalence $\kappa\colon A(S^3) \to K(ku)$ of S–algebras, a rational equivalence $A(K(\mathbb{Z},3)) \to K(ku)$ of commutative S–algebras, and a rational equivalence $\bar{\alpha}\colon A(S^5) \to K(ko)$ of S–modules.*

Proof (a) Let $BS^3 \to K(\mathbb{Z},4)$ represent a generator of $H^4(BS^3)$. It induces a double loop map $\Omega S^3 \to K(\mathbb{Z},2)$, such that the composite $S^2 \to \Omega S^3 \to K(\mathbb{Z},2)$ represents a generator of $\pi_2 K(\mathbb{Z},2)$. The inclusions $K(\mathbb{Z},2) \simeq BU(1) \to BU_\otimes \to GL_1(ku)$ are infinite loop maps, and the generator of $\pi_2 K(\mathbb{Z},2)$ maps to a generator of $\pi_2 GL_1(ku)$. By adjunction we have an E_2 ring spectrum map $S[\Omega S^3] \to S[K(\mathbb{Z},2)]$ and an E_∞ ring spectrum map $S[K(\mathbb{Z},2)] \to ku$, with composite the E_2 ring spectrum map $\kappa\colon S[\Omega S^3] \to ku$.

These are rational equivalences, because $\pi_* S[\Omega S^3] \otimes \mathbb{Q} \cong H_*(\Omega S^3;\mathbb{Q}) \cong \mathbb{Q}[x]$, $H_*(K(\mathbb{Z},2);\mathbb{Q}) \cong \mathbb{Q}[b]$ and $\pi_* ku \otimes \mathbb{Q} = \mathbb{Q}[u]$, with κ mapping x via b to u. We may take the Kan loop group of S^3 (a simplicial group, see for example Waldhausen [37]) as our model for ΩS^3, and rigidify κ to a map of S–algebras. Following Brun–Fiedorowicz–Vogt [10, Theorem C], there remains an $E_1 = A_\infty$ operad action on these S–algebras and κ, which in particular implies that $\kappa_{\mathbb{Q}}\colon H\mathbb{Q}[\Omega S^3] \to ku_{\mathbb{Q}}$ is a map of homotopy commutative $H\mathbb{Q}$–algebras.

(b) For the real case, let $S^4 \to BO_\otimes \subset GL_1(ko)$ represent a generator of $\pi_4 GL_1(ko)$. By the loop structure on the target, it extends to a loop map $\Omega S^5 \to GL_1(ko)$, with left adjoint an A_∞ ring spectrum map $\bar{\alpha}\colon S[\Omega S^5] \to ko$. It is a rational equivalence, because $\pi_* S[\Omega S^5] \otimes \mathbb{Q} \cong H_*(\Omega S^5;\mathbb{Q}) \cong \mathbb{Q}[y]$ and $\pi_* ko \otimes \mathbb{Q} = \mathbb{Q}[\alpha]$, with $\bar{\alpha}$ mapping y to α. We interpret ΩS^5 as the Kan loop group, and form the simplicial $\mathbb{Q}$–algebra $\mathbb{Q}[\Omega S^5]$ as its rational group ring. The Adams summand case is entirely similar, starting with a map $S^{2p-2} \to W_\otimes \subset GL_1(\ell)$.

(c) By [10, Theorem C] and naturality there is an A_∞ operad action on the induced map of spectra $A(S^3) = K(S[\Omega S^3]) \to K(ku)$ (rather than of spaces), which we can rigidify to a map of S–algebras. The S–algebra multiplication $A(S^3) \wedge A(S^3) \to A(S^3)$ is induced by the group multiplication $S^3 \times S^3 \to S^3$. □

Lemma 3.2 (a) *For any integer $n \geq 1$ the simplicial $\mathbb{Q}$–algebra $R = \mathbb{Q}[\Omega S^{2n+1}]$ has Hochschild homology*

$$HH_*(R) \cong \mathbb{Q}[x] \otimes E(dx)$$

with $|x| = 2n$, where Connes' B–operator satisfies $B(x) = dx$. Here $E(-)$ denotes the exterior algebra.

(b) *The de Rham homology $H_*^{dR}(R) \cong \mathbb{Q}$ is concentrated in degree zero, so $\pi_R\colon R \to \mathbb{Q}$ is a de Rham equivalence.*

(c) *The positive-degree part of* $\ker(B) \subset HH_*(R)$ *is*

$$\mathbb{Q}[x]\{dx\} = \mathbb{Q}\{dx, x\,dx, x^2\,dx, \dots\}\,.$$

Proof (a) The Hochschild filtration on the bisimplicial $\mathbb{Q}$–algebra $HH(R)$ yields a spectral sequence

$$E^2_{**} = HH_*(\pi_*(R)) \Longrightarrow HH_*(R)\,, \tag{3-1}$$

and $\pi_*(R) = \mathbb{Q}[x]$ with $|x| = 2n$. The Hochschild homology of this graded commutative ring is $\mathbb{Q}[x] \otimes E(dx)$, where $dx \in E^2_{1,2n}$ is the image of x under Connes' B–operator. The spectral sequence collapses at that stage, for bidegree reasons.

(b) and (c) The B–operator is a derivation, hence takes x^m to $mx^{m-1}\,dx$ for all $m \geq 0$. It follows easily that the de Rham homology is trivial in positive degrees, and that $\ker B$ is as indicated. □

By combining Corollary 2.4, Proposition 3.1 and Lemma 3.2, we obtain the following result.

Theorem 3.3 (a) *There is a rationally split homotopy fiber sequence*

$$K(\pi_{ku}) \to K(ku) \xrightarrow{\pi} K(\mathbb{Z})$$

and the trace map tr: $K(ku) \to THH(ku)$ *identifies*

$$K_*(\pi_{ku}) \otimes \mathbb{Q} \cong \mathbb{Q}[u]\{du\}$$

with its image in $THH_*(ku) \otimes \mathbb{Q} \cong \mathbb{Q}[u] \otimes E(du)$. *Here* $|u| = 2$ *and* $|du| = 3$, *so* $K(\pi_{ku})$ *has Poincaré series* $t^3/(1-t^2)$.

(b) *Similarly, there are rationally split homotopy fiber sequences*

$$K(\pi_{ko}) \to K(ko) \xrightarrow{\pi} K(\mathbb{Z})$$
$$K(\pi_{\ell}) \to K(\ell) \xrightarrow{\pi} K(\mathbb{Z}_{(p)})$$

and the trace maps identify

$$K_*(\pi_{ko}) \otimes \mathbb{Q} \cong \mathbb{Q}[\alpha]\{d\alpha\}$$
$$K_*(\pi_{\ell}) \otimes \mathbb{Q} \cong \mathbb{Q}[v_1]\{dv_1\}$$

with their images in $THH_*(ko) \otimes \mathbb{Q} \cong \mathbb{Q}[\alpha] \otimes E(d\alpha)$ *and* $THH_*(\ell) \otimes \mathbb{Q} \cong \mathbb{Q}[v_1] \otimes E(dv_1)$, *respectively. Hence* $K(\pi_{ko})$ *has Poincaré series* $t^5/(1-t^4)$, *whereas* $K(\pi_{\ell})$ *has Poincaré series* $t^{2p-1}/(1-t^{2p-2})$.

Remark 3.4 The Poincaré series of $K(\mathbb{Z})$ is $1+t^5/(1-t^4)$ by Borel's calculation [9]. Hence the (common) Poincaré series of $K(ku)$ and $A(S^3)$ is

$$1+t^3/(1-t^2)+t^5/(1-t^4) = 1+(t^3+2t^5)/(1-t^4)\,,$$

whereas the Poincaré series of $K(ko)$ and $A(S^5)$ is $1+2t^5/(1-t^4)$. More generally, we recover the Poincaré series $1+t^5/(1-t^4)+t^{2n+1}/(1-t^{2n})$ of $A(S^{2n+1})$ for $n \geq 1$, from Hsiang–Staffeldt [20, Corollary 1.2]. The group $K_1(\mathbb{Z}_{(p)})$ is not finitely generated, so we do not discuss the Poincaré series of $K(\ell)$.

3.2 Periodic K–theory spectra

Let KU be the periodic complex K–theory spectrum, KO the periodic real K–theory spectrum, and $L = E(1)$ the Adams summand of $KU_{(p)}$, for p an odd prime. We have maps of commutative S–algebras

$$H\mathbb{Z} \xleftarrow{\pi} ku \xrightarrow{\rho} KU$$

with associated maps of "brave new" affine schemes

$$\mathrm{Spec}(\mathbb{Z}) \xrightarrow{\pi} \mathrm{Spec}(ku) \xleftarrow{\rho} \mathrm{Spec}(KU) \tag{3-2}$$

in the sense of Toën–Vezzosi [33, Section 2]. Let $i\colon S \to S[\Omega S^3]$ and $c\colon S[\Omega S^3] \to S$ be induced by the inclusion map $* \to S^3$ and the collapse map $S^3 \to *$, respectively. We have a map of horizontal cofiber sequences

$$\begin{tikzcd}
\Sigma^2 S[\Omega S^3] \arrow[r, "x"] \arrow[d, "\Sigma^2\kappa"] & S[\Omega S^3] \arrow[r, "c"] \arrow[d, "\kappa"] & S \arrow[d, "\lambda"] \\
\Sigma^2 ku \arrow[r, "u"] & ku \arrow[r, "\pi"] & H\mathbb{Z}
\end{tikzcd} \tag{3-3}$$

where the top row exhibits S as a two-cell $S[\Omega S^3]$–module, and the bottom row exhibits $H\mathbb{Z}$ as a two-cell ku–module. (In each case, the two cells are in dimension zero and three.) There are algebraic K–theory transfer maps $c_*\colon A(*) \to A(S^3)$ and $\pi_*\colon K(\mathbb{Z}) \to K(ku)$ (with a lower star, in accordance with the variance conventions from algebraic geometry and (3-2)), that are induced by the functors that view finite cell S–modules as finite cell $S[\Omega S^3]$–modules, and finite cell $H\mathbb{Z}$–modules as finite cell ku–modules, respectively. In terms of retractive spaces, c_* is induced by the exact functor $\mathcal{R}_f(*) \to \mathcal{R}_f(S^3)$ that takes a pointed space $X \rightleftarrows *$ to the retractive space $X \times S^3 \rightleftarrows S^3$. The transfer maps are compatible, by (3-3), so we have a commutative diagram with vertical rational equivalences

$$\begin{tikzcd}
A(*) \arrow[r, "c_*"] \arrow[d, "\lambda"] & A(S^3) \arrow[d, "\kappa"] & \\
K(\mathbb{Z}) \arrow[r, "\pi_*"] & K(ku) \arrow[r, "\rho"] & K(KU) .
\end{tikzcd}$$

The bottom row is a homotopy fiber sequence by the localization theorem of Blumberg–Mandell [7].

Lemma 3.5 *The transfer map $c_*\colon A(*) \to A(S^3)$ is null-homotopic, as a map of $A(*)$–module spectra. The transfer map $\pi_*\colon K(\mathbb{Z}) \to K(ku)$ is rationally null-homotopic, again as a map of $A(*)$–module spectra.*

Proof The projection formula, also known as Frobenius reciprocity, asserts that c_* is an $A(S^3)$–module map, where

$$c\colon A(S^3) \to A(*)$$

makes $A(*)$ an $A(S^3)$–module. Restricting the module structures along $i\colon A(*) \to A(S^3)$, we see that c_* is a map of $A(*)$–module spectra, and the source is a free $A(*)$–module of rank one. Hence it suffices to show that c_* takes a generator of $\pi_0 A(*)$, represented say by $S^0 \rightleftarrows *$, to zero in $\pi_0 A(S^3) \cong \mathbb{Z}$. But c_* maps that generator to the class of $S^0 \times S^3 \rightleftarrows S^3$, which corresponds to its relative Euler characteristic $\chi(S^3) = 0$. The conclusion for π_* follows from that for c_*, via the rational equivalences λ and κ. □

Note the utility of the comparison with A–theory at this point, since we do not have an S–algebra map $K(\mathbb{Z}) \to K(ku)$ that is analogous to $i\colon A(*) \to A(S^3)$.

Theorem 3.6 *There are rationally split homotopy fiber sequences*

$$K(ku) \xrightarrow{\rho} K(KU) \xrightarrow{\partial} BK(\mathbb{Z})$$

$$K(\ell) \xrightarrow{\rho} K(L) \xrightarrow{\partial} BK(\mathbb{Z}_{(p)})$$

of infinite loop spaces. Hence the Poincaré series of $K(KU)$ is

$$(1+t) + (t^3 + 2t^5 + t^6)/(1-t^4) .$$

Proof The claims for KU follow by combining Theorem 3.3(a) and Lemma 3.5. The proof of the claim for L is completely similar, using that $H\mathbb{Z}_{(p)}$ is a two-cell ℓ–module, with cells in dimension zero and $(2p-1)$. By Blumberg–Mandell [7] there is a homotopy fiber sequence $K(\mathbb{Z}_{(p)}) \to K(\ell) \to K(L)$. □

Remark 3.7 We do not know how to relate $K(ko)$ with $K(KO)$, so we do not have a rational calculation of $K(KO)$. However, $KO \to KU$ is a $\mathbb{Z}/2$–Galois extension of commutative S–algebras, in the sense of the second author [29, Section 4.1], so it is plausible that $K(KO) \to K(KU)^{h\mathbb{Z}/2}$ is close to an equivalence. Here $\mathbb{Z}/2$ acts on KU by complex conjugation, and

$$\pi_*(K(KU)^{h\mathbb{Z}/2}) \otimes \mathbb{Q} \cong [K_*(KU) \otimes \mathbb{Q}]^{\mathbb{Z}/2} .$$

The conjugation action on ku fixes $K(\mathbb{Z})$, and acts on $K_*(\pi_{ku}) \otimes \mathbb{Q} \cong \mathbb{Q}[u]\{du\}$ by sign on u and du, hence fixes $\mathbb{Q}[u^2]\{udu\} \cong \mathbb{Q}[\alpha]\{d\alpha\} \cong K_*(\pi_{ko}) \otimes \mathbb{Q}$. So $K(ko) \to K(ku)^{h\mathbb{Z}/2}$ is a rational equivalence. The conjugation action also fixes $BK(\mathbb{Z})$ after rationalization, so the Poincaré series of $K(KU)^{h\mathbb{Z}/2}$ is $(1+t) + (2t^5 + t^6)/(1-t^4)$.

Remark 3.8 We expect that c_* and π_* are essential (not null-homotopic) as maps of $A(S^3)$–module spectra and $K(ku)$–module spectra, respectively. In other words, we expect that $K(ku) \to K(KU) \to \Sigma K(\mathbb{Z})$ is a non-split extension of $K(ku)$–module spectra. This expectation is to some extent justified by the fact that the cofiber $THH(ku|KU)$ of the THH–transfer map $\pi_*\colon THH(\mathbb{Z}) \to THH(ku)$ sits in a non-split extension $THH(ku) \to THH(ku|KU) \to \Sigma THH(\mathbb{Z})$ of $THH(ku)$–module spectra. See the first author's paper [1, Section 10.4], or Hesselholt–Madsen [17, Lemma 2.3.3] for a similar result in an algebraic case.

4 Examples from smooth bordism

4.1 Oriented bordism spectra

Let MU be the complex bordism spectrum, MSO the real oriented bordism spectrum, and MSp the symplectic bordism spectrum. These are all connective commutative S–algebras, given by the Thom spectra associated to infinite loop maps from BU, BSO and BSp to $BSF = BSL_1(S)$, respectively. We recall that

$$H_*(BU) \cong \mathbb{Z}[b_k \mid k \geq 1]$$

with $|b_k| = 2k$, while $H_*(BSO; \mathbb{Z}[1/2]) \cong H_*(BSp; \mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2][q_k \mid k \geq 1]$ with $|q_k| = 4k$.

The Thom equivalence $\theta\colon MU \wedge MU \to MU \wedge S[BU]$ induces an equivalence $H\mathbb{Z} \wedge MU \simeq H\mathbb{Z} \wedge S[BU] = H\mathbb{Z}[BU]$. Combined with the Hurewicz map $\pi\colon S \to H\mathbb{Z}$ we obtain a chain of maps of commutative S–algebras

$$MU \to H\mathbb{Z} \wedge MU \simeq H\mathbb{Z}[BU] \leftarrow S[BU] ,$$

that are π_0–isomorphisms and rational equivalences. There are similar chains $MSO \to H\mathbb{Z} \wedge MSO \simeq H\mathbb{Z}[BSO] \leftarrow S[BSO]$ and $MSp \to H\mathbb{Z} \wedge MSp \simeq H\mathbb{Z}[BSp] \leftarrow S[BSp]$, and all induce rational equivalences

$$\begin{aligned} K(MU) &\to K(\mathbb{Z}[BU]) \leftarrow A(BBU) \\ K(MSO) &\to K(\mathbb{Z}[BSO]) \leftarrow A(BBSO) \\ K(MSp) &\to K(\mathbb{Z}[BSp]) \leftarrow A(BBSp) \end{aligned}$$

of commutative S–algebras. Here we view $BU \simeq \Omega BBU$ as the Kan loop group of BBU, $\mathbb{Z}[BU]$ is the associated simplicial ring, and similarly for BSO and BSp.

Lemma 4.1 (a) *The simplicial* $\mathbb{Q}$*–algebra* $R = \mathbb{Q}[BU]$ *with* $\pi_* R = H_*(BU;\mathbb{Q}) = \mathbb{Q}[b_k \mid k \geq 1]$ *has Hochschild homology*

$$HH_*(R) \cong \mathbb{Q}[b_k \mid k \geq 1] \otimes E(db_k \mid k \geq 1)\,,$$

with Poincaré series

$$h(t) = \prod_{k \geq 1} \frac{1+t^{2k+1}}{1-t^{2k}}\,,$$

and Connes' operator acts by $B(b_k) = db_k$.

(b) *The de Rham homology* $H_*^{dR}(R) \cong \mathbb{Q}$ *is concentrated in degree zero, so* $\pi_R \colon R \to \mathbb{Q}$ *is a de Rham equivalence.*

(c) *The Poincaré series of* $\ker(B) \subset HH_*(R)$ *is*

$$k(t) = \frac{1+th(t)}{1+t}\,.$$

(d) *The simplicial* $\mathbb{Q}$*–algebra* $R_{so} = \mathbb{Q}[BSO] \simeq \mathbb{Q}[BSp]$, *with* $\pi_* R_{so} = \mathbb{Q}[q_k \mid k \geq 1]$, *has Hochschild homology*

$$HH_*(R_{so}) \cong \mathbb{Q}[q_k \mid k \geq 1] \otimes E(dq_k \mid k \geq 1)\,.$$

Its Poincaré series is $h_{so}(t) = \prod_{k\geq 1}(1+t^{4k+1})/(1-t^{4k})$. *The map* $R_{so} \to \mathbb{Q}$ *is a de Rham equivalence, and* $\ker(B) \subset HH_*(R_{so})$ *has Poincaré series* $k_{so}(t) = (1+th_{so}(t))/(1+t)$.

Proof (a) In this case the spectral sequence (3-1) has $E^2_{**} = HH_*(\mathbb{Q}[b_k \mid k \geq 1]) \cong \mathbb{Q}[b_k \mid k \geq 1] \otimes E(db_k \mid k \geq 1)$. The algebra generators are in filtrations 0 and 1, so $E^2 = E^\infty$. This term is free as a graded commutative $\mathbb{Q}$–algebra, so $HH_*(R)$ is isomorphic to the E^∞–term.

(b) The homology of $\mathbb{Q}[b_k] \otimes E(db_k)$ with respect to B is just $\mathbb{Q}$, for each $k \geq 1$, so by the Künneth theorem the de Rham homology of $HH_*(R)$ is also just $\mathbb{Q}$.

(c) Write H_n for $HH_n(R)$ and K_n for $\ker(B\colon H_n \to H_{n+1})$. Let $h_n = \dim_{\mathbb{Q}} H_n$, so $h(t) = \sum_{n\geq 0} h_n t^n$, and $k_n = \dim_{\mathbb{Q}} K_n$. In view of the exact sequence

$$0 \to \mathbb{Q} \to H_0 \xrightarrow{B} H_1 \xrightarrow{B} \dots \xrightarrow{B} H_{n-1} \to K_n \to 0$$

we find that $1-(-1)^n k_n = h_0 - h_1 + \dots + (-1)^{n-1} h_{n-1}$, so

$$\sum_{n\geq 0} t^n k_n - \sum_{n\geq 0} (-t)^n = th(t) - t^2 h(t) + \dots + (-t)^{m+1} h(t) + \dots .$$

It follows that the Poincaré series $k(t) = \sum_{n\geq 0} k_n t^n$ for $\ker(B)$ satisfies

$$k(t) - 1/(1+t) = th(t)/(1+t).$$

(d) The only change from the complex to the oriented real and symplectic cases is in the grading of the algebra generators, which (as long as they remain in even degrees) plays no role for the proofs. □

By combining Corollary 2.4 and Lemma 4.1, we obtain the following result.

Theorem 4.2 (a) *There is a rationally split homotopy fiber sequence*

$$K(\pi_{MU}) \to K(MU) \xrightarrow{\pi} K(\mathbb{Z})$$

and the trace map tr: $K(MU) \to THH(MU)$ *identifies* $K_*(\pi_{MU}) \otimes \mathbb{Q}$ *with the positive-degree part of* $\ker(B)$ *in*

$$THH_*(MU) \otimes \mathbb{Q} \cong \mathbb{Q}[b_k \mid k \geq 1] \otimes E(db_k \mid k \geq 1),$$

where $|b_k| = 2k$ *and* $B(b_k) = db_k$*. Hence* $K(\pi_{MU})$ *has Poincaré series*

$$k(t) - 1 = \frac{th(t) - t}{1+t}.$$

(b) *There are rationally split homotopy fiber sequences*

$$K(\pi_{MSO}) \to K(MSO) \xrightarrow{\pi} K(\mathbb{Z})$$

$$K(\pi_{MSp}) \to K(MSp) \xrightarrow{\pi} K(\mathbb{Z})$$

and the trace maps identify both $K_*(\pi_{MSO}) \otimes \mathbb{Q}$ *and* $K_*(\pi_{MSp}) \otimes \mathbb{Q}$ *with the positive-degree part of* $\ker(B)$ *in*

$$THH_*(MSO) \otimes \mathbb{Q} \cong THH_*(MSp) \otimes \mathbb{Q} \cong \mathbb{Q}[q_k \mid k \geq 1] \otimes E(dq_k \mid k \geq 1),$$

where $|q_k| = 4k$ *and* $B(q_k) = dq_k$*. Hence* $K(\pi_{MSO})$ *and* $K(\pi_{MSp})$ *both have Poincaré series* $k_{so}(t) - 1 = (th_{so}(t) - t)/(1+t)$*.*

Remark 4.3 Adding the Poincaré series of $K(\mathbb{Z})$, as in Remark 3.4, we find that the Poincaré series of $K(MU)$ and $A(BBU)$ is

$$\frac{t^5}{1-t^4} + \frac{1+th(t)}{1+t},$$

whereas the Poincaré series of $K(MSO)$, $A(BBSO)$, $K(MSp)$ and $A(BBSp)$ is $t^5/(1-t^4) + (1+th_{so}(t))/(1+t)$.

5 Units, determinants and traces

5.1 Units

For each connective S–algebra A there is a natural map of spaces

$$w\colon BGL_1(A) \to K(A)$$

that factors as the infinite stabilization map $BGL_1(A) \to BGL_\infty(A)$, composed with the inclusion $BGL_\infty(A) \to BGL_\infty(A)^+$ into Quillen's plus construction, and followed by the inclusion of $BGL_\infty(A)^+ \cong \{1\} \times BGL_\infty(A)^+$ into $K_0(\pi_0 A) \times BGL_\infty(A)^+ = K(A)$.

Remark 5.1 This w is an E_∞ map with respect to the multiplicative E_∞ structure on $K(A)$ that is induced by the smash product over A. However, we shall only work with the additive grouplike E_∞ structure on $K(A)$, which comes from viewing $K(A)$ as the underlying infinite loop space of the K–theory spectrum. So when we refer to infinite loop structures below, we are thinking of the additive ones.

We write $BSL_1(A) = BGL_1(\pi_A)$ for the homotopy fiber of the map $BGL_1(A) \to BGL_1(\pi_0 A)$ induced by $\pi_A\colon A \to H\pi_0 A$. In the resulting diagram

$$BSL_1(A) \xrightarrow{w} K(A) \xrightarrow{\pi} K(\pi_0 A)$$

the composite map has a preferred null-homotopy (to the base point of the 1–component of $K(\pi_0 A)$). The diagram is a rational homotopy fiber sequence if and only if $w\colon BSL_1(A) \to K(\pi_A)$ is a rational equivalence. Note that the natural inclusion $\{1\} \times BGL_\infty(A)^+ \to K(A)$ induces a homotopy equivalence

$$BGL_\infty(\pi_A)^+ \simeq K(\pi_A),$$

since $K_0(A) \cong K_0(\pi_0 A)$.

5.2 Determinants

Suppose furthermore that A is commutative as an S–algebra. One attempt at proving that w is injective could be to construct a map det: $K(A) \to BGL_1(A)$ with the property that $\det \circ w \simeq \mathrm{id}$. However, no such determinant map exists in general, as the following adaption of an argument of Waldhausen [35, Corollary 3.7] shows.

Example 5.2 When $A = S$, the map $\lambda \circ w\colon BF = BGL_1(S) \to A(*) \to K(\mathbb{Z})$ factors through $BGL_1(\mathbb{Z}) \simeq K(\mathbb{Z}/2, 1)$, and $\pi_2(\lambda)\colon \pi_2 A(*) \to K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ is an isomorphism, so $\pi_2(w)\colon \pi_2 BF \to \pi_2 A(*)$ is the zero map. But $\pi_2 BF \cong \pi_1(S) \cong \mathbb{Z}/2$ is not zero, so $\pi_2(w)$ is not injective. In particular, w is not split injective up to homotopy.

However, it is possible to construct a rationalized determinant map. Recall from Section 2.1 that $A_{\mathbb{Q}}$ is equivalent to HR for some naturally determined commutative simplicial $\mathbb{Q}$–algebra R.

Lemma 5.3 *Let R be a commutative simplicial ring. There is a natural infinite loop map*

$$\det\colon BGL_\infty(R)^+ \to BGL_1(R)$$

that agrees with the usual determinant map for discrete commutative rings, such that the composite $\det \circ w$ with $w\colon BGL_1(R) \to BGL_\infty(R)^+$ equals the identity.

Proof The usual matrix determinant det: $M_n(R) \to R$ induces a simplicial group homomorphism $GL_n(R) \to GL_1(R)$ and a pointed map $BGL_n(R) \to BGL_1(R)$ for each $n \geq 0$. These stabilize to a map $BGL_\infty(R) \to BGL_1(R)$, which extends to an infinite loop map

$$\det\colon BGL_\infty(R)^+ \to BGL_1(R)\,,$$

unique up to homotopy, by the multiplicative infinite loop structure on the target and the universal property of Quillen's plus construction. To make the construction natural, we fix a choice of extension in the initial case $R = \mathbb{Z}$, and define $\det_R$ for general R as the dashed pushout map in the following diagram:

$$\begin{tikzcd}
BGL_\infty(\mathbb{Z}) \arrow[r] \arrow[d] & BGL_\infty(\mathbb{Z})^+ \arrow[d] \arrow[dr, "\det_{\mathbb{Z}}"] & \\
BGL_\infty(R) \arrow[r] \arrow[drr] & BGL_\infty(R)^+ \arrow[dr, dashed, "\det_R"] & BGL_1(\mathbb{Z}) \arrow[d] \\
 & & BGL_1(R)
\end{tikzcd}$$

(Recall that Quillen's plus construction is made functorial by demanding that the left hand square is a pushout.) □

Proposition 5.4 *Let A be a connective commutative S–algebra. There is a natural infinite loop map*

$$\det_{\mathbb{Q}}\colon BGL_\infty(A)^+ \to BGL_1(A)_{\mathbb{Q}}$$

that agrees with the rationalized determinant map for a commutative ring R when $A = HR$, such that the composite

$$BGL_1(A) \xrightarrow{w} BGL_\infty(A)^+ \xrightarrow{\det_{\mathbb{Q}}} BGL_1(A)_{\mathbb{Q}}$$

is homotopic to the rationalization map.

Proof We define $\det_{\mathbb{Q}}$ as the dashed pullback map in the following diagram

$$\begin{tikzcd}
BGL_\infty(A)^+ \arrow[r] \arrow[d] \arrow[dr, dashed, "\det_{\mathbb{Q}}"] & BGL_\infty(A_{\mathbb{Q}})^+ \arrow[dr, "\det'_R"] & \\
BGL_\infty(\pi_0 A)^+ \arrow[dr, "(\det_{\pi_0 A})_{\mathbb{Q}}"'] & BGL_1(A)_{\mathbb{Q}} \arrow[r] \arrow[d] & BGL_1(A_{\mathbb{Q}}) \arrow[d] \\
 & BGL_1(\pi_0 A)_{\mathbb{Q}} \arrow[r] & BGL_1(\pi_0 A_{\mathbb{Q}})
\end{tikzcd}$$

where the vertical maps are induced by the Postnikov section $\pi\colon A \to H\pi_0 A$, and the horizontal maps are induced by the rationalization $q\colon A \to A_{\mathbb{Q}}$. The right hand square is a homotopy pullback, since $SL_1(A)_{\mathbb{Q}} \simeq SL_1(A_{\mathbb{Q}})$.

To define the map $\det'_R$, we take R to be a commutative simplicial $\mathbb{Q}$–algebra such that $A_{\mathbb{Q}} \simeq HR$ as commutative $H\mathbb{Q}$–algebras. A natural choice can be made for R, as discussed in Section 2.1, such that the identification $\pi_0 A_{\mathbb{Q}} \cong \pi_0 R$ is the identity. Then $\det'_R$ is the composite map

$$BGL_\infty(A_{\mathbb{Q}})^+ \simeq BGL_\infty(R)^+ \xrightarrow{\det_R} BGL_1(R) \simeq BGL_1(A_{\mathbb{Q}})\,,$$

with $\det_R$ from Lemma 5.3. It strictly covers the map $\det_{\pi_0 A_{\mathbb{Q}}}$, so the outer hexagon commutes strictly. This defines the desired map $\det_{\mathbb{Q}}$.

To compare $\det_{\mathbb{Q}} \circ w$ and $q\colon BGL_1(A) \to BGL_1(A)_{\mathbb{Q}}$, note that both maps have the same composite to $BGL_1(\pi_0 A)_{\mathbb{Q}}$, they have homotopic composites to $BGL_1(A_{\mathbb{Q}})$, and all composites (and homotopies) to $BGL_1(\pi_0 A_{\mathbb{Q}})$ are equal. Hence the maps to the homotopy pullback are homotopic, too. □

Theorem 5.5 (a) *The relative unit map*

$$BBU_{\otimes} = BGL_1(\pi_{ku}) \xrightarrow{w} BGL_{\infty}(\pi_{ku})^+ \simeq K(\pi_{ku})$$

is a rational equivalence, with rational homotopy inverse given by the relative rational determinant map

$$\det_{\mathbb{Q}} \colon BGL_{\infty}(\pi_{ku})^+ \to BGL_1(\pi_{ku})_{\mathbb{Q}} = (BBU_{\otimes})_{\mathbb{Q}} .$$

(b) *The relative unit maps*

$$\begin{aligned} BBO_{\otimes} &= BGL_1(\pi_{ko}) \to K(\pi_{ko}) \\ BW_{\otimes} &= BGL_1(\pi_{\ell}) \to K(\pi_{\ell}) \end{aligned}$$

are rational equivalences (with rational homotopy inverse $\det_{\mathbb{Q}}$ *in each case).*

Proof (a) By Proposition 5.4, the composite

$$BBU_{\otimes} \xrightarrow{w} K(\pi_{ku}) \xrightarrow{\det_{\mathbb{Q}}} (BBU_{\otimes})_{\mathbb{Q}}$$

is a rational equivalence, so w is rationally injective. Here $\pi_* BBU_{\otimes} \cong \pi_{*-1} BU_{\otimes}$ has Poincaré series $t^3/(1-t^2)$, just like $K(\pi_{ku})$ by Theorem 3.3(a). Thus w is a rational equivalence.

(b) The same proof works for ko and ℓ, using that $BBO_{\otimes}$ and $BW_{\otimes}$ have Poincaré series $t^5/(1-t^4)$ and $t^{2p-1}/(1-t^{2p-2})$, respectively. □

Remark 5.6 The analogous map $w \colon BSL_1(MU) \to K(\pi_{MU})$ is rationally injective, but not a rational equivalence. For the Poincaré series of the source is

$$t(p(t)-1) = t^3 + 2t^5 + 3t^7 + 5t^9 + \dots ,$$

where $p(t) = \prod_{k \geq 1} 1/(1-t^{2k})$, and the Poincaré series of the target is

$$(th(t)-t)/(1+t) = t^3 + 2t^5 + 3t^7 + t^8 + 5t^9 + \dots ,$$

by Theorem 4.2(a). These first differ in degree 8, since $\pi_8 BSL_1(MU) \cong \pi_7 MU$ is trivial, but $K_8(MU)$ and $K_8(\pi_{MU})$ have rank one. A generator of the latter group maps to $db_1 \cdot db_2$ in $\ker(B) \subset THH_*(MU) \otimes \mathbb{Q}$.

In the same way, $w \colon BSL_1(MSO) \to K(\pi_{MSO})$ and its symplectic variant are rationally injective, but not rational equivalences.

5.3 Traces

Our original strategy for proving that $w\colon BGL_1(A) \to K(A)$ is rationally injective for $A = ku$ was to use the trace map $\mathrm{tr}\colon K(A) \to THH(A)$, in place of the rational determinant map. By Schlichtkrull [30, Section 4], there is a natural commutative diagram

$$\begin{tikzcd}
BGL_1(A) \arrow[r, "w"] \arrow[d] & K(A) \arrow[d] \arrow[dr, "\mathrm{tr}"] & \\
B^{cy}GL_1(A) \arrow[r] & K^{cy}(A) \arrow[r] & THH(A) \\
GL_1(A) \arrow[u] \arrow[rr] & & \Omega^\infty A \arrow[u]
\end{tikzcd}$$

where B^{cy} and K^{cy} denote the cyclic bar construction and cyclic K–theory, respectively. The middle row is the geometric realization of two cyclic maps, hence consists of circle-equivariant spaces and maps.

When $A = ku$, the resulting B–operator on $H_*(B^{cy}GL_1(A);\mathbb{Q})$ takes primitive classes in the image from $H_*(BU_\otimes;\mathbb{Q}) \subset H_*(GL_1(A);\mathbb{Q})$ to primitive classes generating the image from $H_*(BGL_1(A);\mathbb{Q}) \cong H_*(BBU_\otimes;\mathbb{Q})$, so by a diagram chase we can determine the images of the latter primitive classes in $H_*(THH(A);\mathbb{Q})$. By an appeal to the Milnor–Moore theorem [26, Appendix], this suffices to prove that $\mathrm{tr}\circ w$ is rationally injective in this case.

In comparison with the rational determinant approach taken above, this trace method involves more complicated calculations. For commutative S–algebras, it is therefore less attractive. However, for non-commutative S–algebras, the trace method may still be useful, since no (rational) determinant map is likely to exist. We have therefore sketched the idea here, with a view to future applications.

6 Two-vector bundles and elliptic objects

The following informal discussion elaborates on the second author's work with Baas and Dundas in [6]. It is intended to explain some of our interest in Theorem 1.1.

6.1 Two-vector bundles

A 2–vector bundle $\mathcal{E}$ of rank n over a base space X is represented by a map $X \to |BGL_n(\mathcal{V})|$, where $\mathcal{V}$ is the symmetric bimonoidal category of finite dimensional complex vector spaces. A virtual 2–vector bundle $\mathcal{E}$ over X is represented by a map $X \to K(\mathcal{V})$, where $K(\mathcal{V})$ is the algebraic K–theory of the 2–category of finitely generated free $\mathcal{V}$–modules; see [6, Theorem 4.10]. By Baas–Dundas–Richter–Rognes [5, Theorem 1.1], spectrification induces a weak equivalence $\mathrm{Spt}\colon K(\mathcal{V}) \to K(ku)$, so the 2–vector bundles over X are geometric 0–cycles for the cohomology theory $K(ku)^*(X)$.

6.2 Anomaly bundles

The preferred rational splitting of $\pi\colon K(ku) \to K(\mathbb{Z})$ defines an infinite loop map

$$\det\nolimits_{\mathbb{Q}}\colon K(ku) \to (BBU_{\otimes})_{\mathbb{Q}} ,$$

which extends the rationalization map over $w\colon BBU_{\otimes} \to K(ku)$ and agrees with the relative rational determinant on $K(\pi_{ku})$. (We proved with Dundas in [3, Corollary 2.3] that there does not exist an integral determinant map $BGL_{\infty}(ku)^+ \to BBU_{\otimes}$.) We define the rational determinant bundle $|\mathcal{E}| = \det(\mathcal{E})$ of a virtual 2–vector bundle represented by a map $\mathcal{E}\colon X \to K(\mathcal{V}) \simeq K(ku)$, as the composite map

$$|\mathcal{E}|\colon X \xrightarrow{\mathcal{E}} K(ku) \xrightarrow{\det_{\mathbb{Q}}} (BBU_{\otimes})_{\mathbb{Q}} .$$

We define the rational anomaly bundle $\mathcal{H} \to \mathcal{L}X$ of $\mathcal{E}$ as the composite map

$$\mathcal{H}\colon \mathcal{L}X \xrightarrow{\mathcal{L}|\mathcal{E}|} \mathcal{L}(BBU_{\otimes})_{\mathbb{Q}} \xrightarrow{r_{\mathbb{Q}}} (BU_{\otimes})_{\mathbb{Q}} ,$$

where $r\colon \mathcal{L}BBU_{\otimes} \to BU_{\otimes}$ is the retraction defined as the infinite loop cofiber of the constant loops map $BBU_{\otimes} \to \mathcal{L}BBU_{\otimes}$. Up to rationalization, $\mathcal{H}$ is a virtual vector bundle of virtual dimension $+1$, that is, a virtual line bundle. Furthermore, the anomaly bundle relates the composition $\star$ of free loops, when defined, to the tensor product of virtual vector spaces: the square

(6-1)
$$\begin{tikzcd}
\mathcal{L}X \times_X \mathcal{L}X \arrow[r, "{(\mathcal{H},\mathcal{H})}"] \arrow[d, "\star"'] & (BU_{\otimes})_{\mathbb{Q}} \times (BU_{\otimes})_{\mathbb{Q}} \arrow[d, "\otimes"] \\
\mathcal{L}X \arrow[r, "\mathcal{H}"] & (BU_{\otimes})_{\mathbb{Q}}
\end{tikzcd}$$

commutes up to coherent isomorphism.

6.3 Gerbes

A 2–vector bundle of rank 1 over X is the same as a $\mathbb{C}^*$–gerbe $\mathcal{G}$, which is represented by a map $\mathcal{G}\colon X \to BBU(1)$. When viewed as a virtual 2–vector bundle, via $BBU(1) \to$

$BBU_{\otimes} \to K(ku)$, the associated anomaly bundle is the complex line bundle over $\mathcal{L}X$ that is represented by the composite

$$\mathcal{L}X \xrightarrow{\mathcal{L}\mathcal{G}} \mathcal{L}BBU(1) \xrightarrow{r} BU(1)\,.$$

This is precisely the anomaly line bundle for $\mathcal{G}$, as described in Brylinski's book [11, Section 6.2]. Note that the rational anomaly bundles of virtual 2–vector bundles represent general elements in

$$1 + \widetilde{K}^0(\mathcal{L}X) \otimes \mathbb{Q} \subset K^0(\mathcal{L}X) \otimes \mathbb{Q}\,,$$

whereas the anomaly line bundles of gerbes only represent elements in $H^2(\mathcal{L}X)$.

6.4 State spaces and action functionals

In physical language, we think of a free loop $\gamma\colon S^1 \to X$ as a closed string in a spacetime X. For a 2–vector bundle $\mathcal{E} \to X$, we think of the fiber H_γ (a virtual vector space) at γ of the anomaly bundle $\mathcal{H} \to \mathcal{L}X$ as the state space of that string. Then the state space of a composite of two strings (or a disjoint union of two strings) is the tensor product of the individual state spaces, as is usual in quantum mechanics. Similarly, the state space of an empty set of strings is $\mathbb{C}$. In the special case of anomaly line bundles for gerbes the resulting state spaces are complex lines, while in our generality they are virtual vector spaces. These are much closer to the Hilbert spaces usually considered in more analytical approaches to this subject.

There is evidence that a two-part differential-geometric structure (∇_1, ∇_2) on $\mathcal{E}$ over X (somewhat like a connection for a vector bundle, but providing parallel transport both for objects and for morphisms in the 2–vector bundle) provides $\mathcal{H} \to \mathcal{L}X$ with a connection, and more generally an action functional

$$S(\Sigma)\colon H_{\bar{\gamma}_1} \otimes \cdots \otimes H_{\bar{\gamma}_p} \to H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_q}\,,$$

where $\Sigma\colon F \to X$ is a compact Riemann surface over X, with p incoming and q outgoing boundary circles. The time development of the physical system is then given by the Euler–Lagrange equations of the action functional.

In a little more detail, the idea is that the primary form of parallel transport in $(\mathcal{E}, \nabla_1)$ around γ provides an endo-functor $\tilde{\gamma}$ of the fiber category $\mathcal{E}_x \cong \mathcal{V}^n$ over a chosen point x of γ. More precisely, parallel transport only provides a zig-zag of functors connecting $\mathcal{E}_x$ to itself, but the determinant in $(BU_{\otimes})_{\mathbb{Q}}$ is still well-defined. This "holonomy" is then the fiber $H_\gamma = \det(\tilde{\gamma})$ at γ of the anomaly bundle $\mathcal{H}$. For a moving string, say on the Riemann surface F, the secondary form of parallel transport ∇_2 specifies how the holonomy changes with the string, and this defines the connection ∇

on $\mathcal{H} \to \mathcal{L}X$. In the gerbe case, this theory has been worked out by Brylinski in [11, Section 5.3], where ∇_1 is called "connective structure" and ∇_2 is called "curving".

For a closed surface F, $S(\Sigma)\colon \mathbb{C} \to \mathbb{C}$ is multiplication by a complex number, which would only depend on the rational type of $\mathcal{E}$. Optimistically, this association can produce a conformal invariant of F over X, which in the case of genus 1 surfaces would lead to an elliptic modular form. Less naively, additional structure derived from a string structure on X should account for the weight of the modular form. With such structure, a 2–vector bundle $\mathcal{E}$ with connective structure ∇ would qualify as a Segal elliptic object over X.

6.5 Open strings

In the presence of D–branes in the space-time X, we can extend the anomaly bundle to also cover open strings with end points restricted to lie on these D–branes; see Moore [27, Section 3.4]. In this terminology, the (rational) determinant bundle $|\mathcal{E}| \to X$ plays the role of the B–field.

By a (rational) *D–brane* $(\mathcal{W}, E)$ in X we will mean a subspace $\mathcal{W} \subset X$ together with a trivialization E of the restriction of the (rational) determinant bundle $|\mathcal{E}|$ to $\mathcal{W}$. In terms of representing maps, E is a null-homotopy of the composite map

$$\mathcal{W} \subset X \xrightarrow{\mathcal{E}} K(ku) \xrightarrow{\det_{\mathbb{Q}}} (BBU_{\otimes})_{\mathbb{Q}} .$$

In similar terminology, we may refer to the determinant bundle $|\mathcal{E}| \to X$ as the (rational) *B–field*.

When the B–field $|\mathcal{E}|$ is rationally trivial, then a second choice of trivialization E amounts to a choice of null-homotopy of the trivial map $\mathcal{W} \to (BBU_{\otimes})_{\mathbb{Q}}$, or equivalently to a map $E\colon \mathcal{W} \to (BU_{\otimes})_{\mathbb{Q}}$. In other words, E is a virtual vector bundle over $\mathcal{W}$ of virtual dimension $+1$, up to rationalization. In this case, the K–theory class of $E \to \mathcal{W}$ in $1 + \widetilde{K}^0(\mathcal{W}) \otimes \mathbb{Q}$ is the "charge" of the D–brane $(\mathcal{W}, E)$. This conforms with the (early) view on D–branes as coming equipped with a charge $[E]$ in topological K–theory.

For a general B–field $|\mathcal{E}|$, the possible trivializations E of its restriction to $\mathcal{W}$ instead form a torsor under the group $1 + \widetilde{K}^0(\mathcal{W}) \otimes \mathbb{Q}$. For two such trivializations E and E' differ by a loop of maps $\mathcal{W} \to (BBU_{\otimes})_{\mathbb{Q}}$, or equivalently a map $E' - E\colon \mathcal{W} \to (BU_{\otimes})_{\mathbb{Q}}$. So $[E' - E]$ is a topological K–theory class measuring the charge difference between the two D–branes $(\mathcal{W}, E)$ and $(\mathcal{W}, E')$.

Given two D–branes $(\mathcal{W}_0, E_0)$ and $(\mathcal{W}_1, E_1)$ in $(X, \mathcal{E})$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{W}_0 \arrow[r] \arrow[d, "E_0"'] & X \arrow[d, "|\mathcal{E}|"] & \mathcal{W}_1 \arrow[l] \arrow[d, "E_1"] \\
P(BBU_{\otimes})_{\mathbb{Q}} \arrow[r, "\pi"] & (BBU_{\otimes})_{\mathbb{Q}} & P(BBU_{\otimes})_{\mathbb{Q}} \arrow[l, "\pi"']
\end{tikzcd}$$

where $\pi\colon PY \to Y$ denotes the path space fibration covering a based space Y. An open string in X, constrained to $\mathcal{W}_0$ and $\mathcal{W}_1$ at its ends, is a map $\gamma\colon I \to X$ with $\gamma(0) \in \mathcal{W}_0$ and $\gamma(1) \in \mathcal{W}_1$. In other words, it is an element in the homotopy pullback of the top row in the diagram above. Let $\Omega(X, \mathcal{W}_0, \mathcal{W}_1)$ denote the space of such open strings. The homotopy pullback of the lower row is $\Omega(BBU_{\otimes})_{\mathbb{Q}} \simeq (BU_{\otimes})_{\mathbb{Q}}$. Hence the 2–vector bundle $\mathcal{E}$ and the two D–branes specify a map of homotopy pullbacks

$$\mathcal{H}\colon \Omega(X, \mathcal{W}_0, \mathcal{W}_1) \to (BU_{\otimes})_{\mathbb{Q}}$$

that we call the (rational, virtual) anomaly bundle of this space of open strings. Again, we think of each fiber H_γ at $\gamma\colon (I, 0, 1) \to (X, \mathcal{W}_0, \mathcal{W}_1)$ as the state space of that open string.

In the presence of a suitable connection (∇_1, ∇_2) on $\mathcal{E} \to X$, parallel transport in $(\mathcal{E}, \nabla_1)$ along γ induces a (zig-zag) functor $\tilde{\gamma}$ from $\mathcal{E}_x$ to $\mathcal{E}_y$, with determinant $\det(\tilde{\gamma})$ from the fiber of $|\mathcal{E}|$ at $x = \gamma(0)$ to the fiber at $y = \gamma(1)$. The trivializations of these two fibers provided by the D–brane data E_0 and E_1, respectively, then agree up to a correction term, which is the fiber H_γ in the anomaly bundle:

$$\det(\tilde{\gamma})(E_{0,x}) \cong H_\gamma \otimes E_{1,y}$$

Again, the secondary part of the connection may induce a connection on $\mathcal{H}$ over $\Omega(X, \mathcal{W}_0, \mathcal{W}_1)$, and more generally an action functional $S(\Sigma)$, where now $\Sigma\colon F \to X$ and the incoming and the outgoing parts of F are unions of circles and closed intervals. For example, an open string might split off a closed string. One advantage of the above perspective is that the state spaces of open and closed strings arise in a compatible fashion, as the holonomy of parallel transport in the 2–vector bundle $\mathcal{E}$, and this makes the construction of $S(\Sigma)$ feasible. The gerbe case is discussed by Brylinski in [11, Section 6.6].

References

[1] **C Ausoni**, *Topological Hochschild homology of connective complex K–theory*, Amer. J. Math. 127 (2005) 1261–1313 MR2183525

[2] **C Ausoni**, *On the algebraic K–theory of the complex K–theory spectrum*, Invent. Math. 180 (2010) 611–668 MR2609252

[3] **C Ausoni**, **B I Dundas**, **J Rognes**, *Divisibility of the Dirac magnetic monopole as a two-vector bundle over the three-sphere*, Doc. Math. 13 (2008) 795–801 MR2466184

[4] **C Ausoni**, **J Rognes**, *Algebraic K–theory of topological K–theory*, Acta Math. 188 (2002) 1–39 MR1947457

[5] **N A Baas**, **B I Dundas**, **B Richter**, **J Rognes**, *Stable bundles over rig categories*, J. Topol. 4 (2011) 623–640 MR2832571

[6] **N A Baas**, **B I Dundas**, **J Rognes**, *Two-vector bundles and forms of elliptic cohomology*, from: "Topology, geometry and quantum field theory", London Math. Soc. Lecture Note Ser. 308, Cambridge Univ. Press (2004) 18–45 MR2079370

[7] **A J Blumberg**, **M A Mandell**, *The localization sequence for the algebraic K–theory of topological K–theory*, Acta Math. 200 (2008) 155–179 MR2413133

[8] **M Bökstedt**, **W C Hsiang**, **I Madsen**, *The cyclotomic trace and algebraic K–theory of spaces*, Invent. Math. 111 (1993) 465–539 MR1202133

[9] **A Borel**, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. 7 (1974) 235–272 MR0387496

[10] **M Brun**, **Z Fiedorowicz**, **R M Vogt**, *On the multiplicative structure of topological Hochschild homology*, Algebr. Geom. Topol. 7 (2007) 1633–1650 MR2366174

[11] **J-L Brylinski**, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics 107, Birkhäuser, Boston, MA (1993) MR1197353

[12] **R L Cohen**, **J D S Jones**, *Algebraic K–theory of spaces and the Novikov conjecture*, Topology 29 (1990) 317–344 MR1066175

[13] **B I Dundas**, *The cyclotomic trace for S–algebras*, J. London Math. Soc. 70 (2004) 659–677 MR2096869

[14] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, American Mathematical Society (1997) MR1417719 With an appendix by M Cole

[15] **T G Goodwillie**, *Relative algebraic K–theory and cyclic homology*, Ann. of Math. 124 (1986) 347–402 MR855300

[16] **L Hesselholt**, *On the p–typical curves in Quillen's K–theory*, Acta Math. 177 (1996) 1–53 MR1417085

[17] **L Hesselholt**, **I Madsen**, *On the K–theory of local fields*, Ann. of Math. 158 (2003) 1–113 MR1998478

[18] **M J Hopkins**, *Algebraic topology and modular forms*, from: "Proceedings of the International Congress of Mathematicians, Vol I (Beijing, 2002)", Higher Ed. Press, Beijing (2002) 291–317 MR1989190

[19] **M Hovey**, **B Shipley**, **J Smith**, *Symmetric spectra*, J. Amer. Math. Soc. 13 (2000) 149–208 MR1695653

[20] **W C Hsiang**, **R E Staffeldt**, *A model for computing rational algebraic K–theory of simply connected spaces*, Invent. Math. 68 (1982) 227–239 MR666160

[21] **I Kříž**, **J P May**, *Operads, algebras, modules and motives*, Astérisque 233 (1995) MR1361938

[22] **P S Landweber**, **D C Ravenel**, **R E Stong**, *Periodic cohomology theories defined by elliptic curves*, from: "The Čech centennial (Boston, MA, 1993)", Contemp. Math. 181, Amer. Math. Soc., Providence, RI (1995) 317–337 MR1320998

[23] **M A Mandell**, *Topological André–Quillen cohomology and E_∞ André–Quillen cohomology*, Adv. Math. 177 (2003) 227–279 MR1990939

[24] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. 82 (2001) 441–512 MR1806878

[25] **J P May**, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics 577, Springer, Berlin (1977) MR0494077 With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave

[26] **J W Milnor**, **J C Moore**, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965) 211–264 MR0174052

[27] **G Moore**, *K–theory from a physical perspective*, from: "Topology, geometry and quantum field theory", London Math. Soc. Lecture Note Ser. 308, Cambridge Univ. Press (2004) 194–234 MR2079376

[28] **J Rognes**, *Trace maps from the algebraic K–theory of the integers (after Marcel Bökstedt)*, J. Pure Appl. Algebra 125 (1998) 277–286 MR1600028

[29] **J Rognes**, *Galois extensions of structured ring spectra. Stably dualizable groups*, Mem. Amer. Math. Soc. 192 (2008) viii+137 MR2387923

[30] **C Schlichtkrull**, *Units of ring spectra and their traces in algebraic K–theory*, Geom. Topol. 8 (2004) 645–673 MR2057776

[31] **S Schwede**, *Stable homotopical algebra and Γ–spaces*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 329–356 MR1670249

[32] **G Segal**, *Elliptic cohomology (after Landweber–Stong, Ochanine, Witten, and others)*, from: "Séminaire Bourbaki 1987/88", Astérisque 161–162 (1988) 187–201 MR992209

[33] **B Toën**, **G Vezzosi**, *Brave new algebraic geometry and global derived moduli spaces of ring spectra*, from: "Elliptic cohomology", London Math. Soc. Lecture Note Ser. 342, Cambridge Univ. Press (2007) 325–359 MR2330521

[34] **F Waldhausen**, *Algebraic K–theory of topological spaces I*, from: "Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1", Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, R.I. (1978) 35–60 MR520492

[35] **F Waldhausen**, *Algebraic K–theory of spaces, a manifold approach*, from: "Current trends in algebraic topology, Part 1 (London, Ont., 1981)", CMS Conf. Proc. 2, Amer. Math. Soc., Providence, R.I. (1982) 141–184 MR686115

[36] **F Waldhausen**, *Algebraic K–theory of spaces*, from: "Algebraic and geometric topology (New Brunswick, N.J., 1983)", Lecture Notes in Math. 1126, Springer, Berlin (1985) 318–419 MR802796

[37] **F Waldhausen**, *On the construction of the Kan loop group*, Doc. Math. 1 (1996) 121–126 MR1386050

LAGA, Université Paris 13
Sorbonne Paris Cité, F-93430, Villetaneuse, France

Department of Mathematics, University of Oslo
Box 1053 Blindern, 0316 Oslo, Norway

ausoni@math.univ-paris13.fr, rognes@math.uio.no

http://www.math.univ-paris13.fr/~ausoni/,
http://folk.uio.no/rognes/home.html

Proposed: John W Morgan
Seconded: Haynes Miller, Peter Teichner

Received: 10 June 2009
Accepted: 4 April 2012