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Definable triangulations with regularity conditions

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We prove that every definable in an o-minimal structure set has a definable triangulation which is locally Lipschitz and weakly bi-Lipschitz on the natural simplicial stratification of a simplicial complex. We also distinguish a class $\mathcal{T}$ of regularity conditions and give a universal construction of a definable triangulation with a $\mathcal{T}$ condition of a definable set. This class includes the Whitney (B) and the Verdier conditions.

14P05, 14P10, 32B20, 32B25

Introduction

It has been known for more than 40 years – since papers of Whitney [22] and Łojasiewicz [14] – that analytic and semianalytic subsets of euclidean spaces admit stratifications with Whitney regularity conditions, a result later generalized to subanalytic (see Hironaka [9] and Łojasiewicz–Stasica–Wachta [16]) and finally to subsets definable in any o-minimal structures on $\mathbb{R}$ (see Ta Lê Loi [12; 11]). Since Łojasiewicz [13], it has also been known that semianalytic and subsequently subanalytic (see Hardt [8], Hironaka [10] and Łojasiewicz [15]) and definable in o-minimal structures (see van den Dries [20]) sets are triangulable.

A challenging problem stated by Łojasiewicz and Thom was to combine the both results, that is, to construct a triangulation of semi(sub)analytic sets which is a stratification with Whitney conditions. A main difficulty was that the construction of Whitney stratifications was by downward induction on dimension in contrast to the triangulation which goes by upward induction on dimension. It was not clear how to overcome this divergence.

A first positive solution to the problem was given by Masahiro Shiota [18]. In his eight-page article concerning semialgebraic case he proposed a solution based on a technique of controlled tube systems developed in his book [17]. However, his proof is difficult to understand.

In the present article we give a direct constructive solution to the problem based on the theory of weakly Lipschitz mappings (see Czapla [4]) and on Guillaume Valette's

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description of Lipschitz structure of definable sets [19]. Our solution is general in the sense that it concerns an arbitrary o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (or even on any real closed field) and, moreover, in the sense that we describe a class $\mathcal{T}$ of regularity conditions including the Whitney and the Verdier conditions, such that for any condition $\mathcal{Q}$ from $\mathcal{T}$ a definable triangulation with $\mathcal{Q}$ condition is possible.

The paper is organized as follows. In Section 1 we recall the notion of a weakly Lipschitz mapping and enlist its important properties. Section 2 is devoted to regular projection theorem of Guillaume Valette (Theorem 2.8) and its more detailed version (Theorem 2.10). They are used in Section 3 to prove the existence of definable locally Lipschitz, weakly bi-Lipschitz triangulation (Theorem 3.12) which in some sense (Theorem 4.11) preserves regularity conditions. In Section 4 we introduce a class of triangulable regularity conditions and prove the main result of the paper – the existence of a definable, locally Lipschitz triangulation with a triangulable condition (Theorem 4.15). In Section 5 we show that the Whitney (B) condition and the Verdier condition are triangulable.

A natural setting for our results is the theory of o-minimal structures (or more generally geometric categories), as presented in papers by van den Dries [20] and van den Dries–Miller [21]. In the whole paper the adjective *definable* (that is, definable subset, definable mapping) will refer to any fixed o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (or more generally on a real closed field).

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1 Weakly Lipschitz mappings

In this section we recall the notion of a weakly Lipschitz mapping and list its important properties.

We denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^n$. In the whole paper C^q denotes the class of smoothness of a mapping, so $q \in \{1, 2, \dots, \infty, \omega\}$. Now we remind briefly the notion of a C^q stratification.

Definition 1.1 Let A be a subset of $\mathbb{R}^n$. A C^q *stratification* of the set A is a (locally) finite partition $\mathfrak{X}_A$ of A into connected C^q submanifolds of $\mathbb{R}^n$ (called *strata*) such that for each $\Gamma \in \mathfrak{X}_A$ the set $(\overline{\Gamma} \setminus \Gamma) \cap A$ is a union of some strata from $\mathfrak{X}_A$ of dimension $< \dim \Gamma$.

We say that the stratification $\mathfrak{X}_A$ is *compatible with a family of sets* $B_i \subset A$, $i \in I$, if every set B_i is a union of some strata of $\mathfrak{X}_A$.

Actually, we will be interested only in finite stratifications.

Definition 1.2 Let A be a subset of $\mathbb{R}^n$ and let $\mathfrak{X}_A$ be a finite C^q stratification of the set A. Consider a mapping $f\colon A \to \mathbb{R}^m$. We say that f is *weakly Lipschitz of class C^q with respect to the stratification* $\mathfrak{X}_A$, if for each stratum $\Gamma \in \mathfrak{X}_A$ the restriction $f|_\Gamma$ is of class C^q and for any point $a \in \Gamma$ there exists a neighbourhood U_a of a in $\mathbb{R}^n$ (or equivalently in A) such that the mapping

$$\psi\colon (\Gamma \cap U_a) \times ((A \setminus \Gamma) \cap U_a) \ni (x, y) \mapsto \frac{|f(x) - f(y)|}{|x - y|} \in \mathbb{R}$$

is bounded.

Remark 1.3 For another equivalent description see Czapla [4, Definition 2.1].

Remark 1.4 If $f\colon A \to \mathbb{R}^m$ is weakly Lipschitz on a stratification $\mathfrak{X}_A$ of the set A, then f is continuous on A.

The weak lipschitzianity is a generalization of the Lipschitz condition. Indeed, we have the following proposition.

Proposition 1.5 *Let $f\colon A \to \mathbb{R}^m$ be a locally Lipschitz mapping. Assume that the set A admits a C^q stratification $\mathfrak{X}_A$ such that for all strata $\Gamma \in \mathfrak{X}_A$ the map $f|_\Gamma$ is of class C^q. Then f is weakly Lipschitz of class C^q on the stratification $\mathfrak{X}_A$.*

Since every definable subset admits a C^q stratification on strata on which a given definable mapping is C^q (see van den Dries [20] and van den Dries–Miller [21]), we have the following corollary.

Corollary 1.6 *Let $f\colon A \to \mathbb{R}^m$ be a definable locally Lipschitz mapping. There exists a definable C^q stratification $\mathfrak{X}_A$ of the set A, such that f is weakly Lipschitz of class C^q on $\mathfrak{X}_A$.*

Remark 1.7 Weakly Lipschitz mappings may not be locally Lipschitz (see Czapla [4, Examples 2.6 and 2.7]).

Proofs of the following propositions are straightforward.

Proposition 1.8 *Let $A \subset \mathbb{R}^n$, $\mathfrak{X}_A$ be a C^q stratification of the set A and $f\colon A \to \mathbb{R}^n$ be weakly Lipschitz of class C^q on $\mathfrak{X}_A$. Let $B \subset A$. Then for any C^q stratification $\mathfrak{X}_B$ of the set B, compatible with $\mathfrak{X}_A$, the mapping $f|_B$ is weakly Lipschitz of class C^q on the stratification $\mathfrak{X}_B$.*

Proposition 1.9 *Let Λ be a C^q submanifold of $\mathbb{R}^n$. Let $\{\Gamma_i\}_{i\in I}$ be a C^q stratification of $\overline{\Lambda} \setminus \Lambda$ with strata Γ_i of dimension $< \dim \Lambda$. Let $f\colon \overline{\Lambda} \to \mathbb{R}^m$ be a continuous mapping. Then f is weakly Lipschitz of class C^q on $\{\Lambda\} \cup \{\Gamma_i\}_{i\in I}$ if and only if for any $i \in I$ the mapping f is weakly Lipschitz of class C^q on $\{\Lambda, \Gamma_i\}$.*

Proposition 1.10 *Let $f\colon A \to \mathbb{R}^p$ be a weakly Lipschitz C^q mapping on a C^q stratification $\mathfrak{X}_A$ of a set $A \subset \mathbb{R}^n$ and let $g\colon B \to \mathbb{R}^r$ be a weakly Lipschitz C^q mapping on a C^q stratification $\mathfrak{X}_B$ of a set $B \subset \mathbb{R}^p$. Assume that the image under f of each stratum from $\mathfrak{X}_A$ is contained in some stratum from $\mathfrak{X}_B$ (in particular, $f(A) \subset B$). Then $g \circ f\colon A \to \mathbb{R}^r$ is a weakly Lipschitz C^q mapping on $\mathfrak{X}_A$.*

Proposition 1.11 *Let $f\colon A \to \mathbb{R}^p$ be a weakly Lipschitz C^q mapping on a C^q stratification $\mathfrak{X}_A$ of a set $A \subset \mathbb{R}^n$ and let $g\colon B \to \mathbb{R}^r$ be a weakly Lipschitz C^q mapping on a C^q stratification $\mathfrak{X}_B$ of a set $B \subset \mathbb{R}^m$. Then $f \times g\colon A \times B \to \mathbb{R}^p \times \mathbb{R}^r$ is weakly Lipschitz of class C^q on a C^q stratification $\mathfrak{X}_{A\times B} = \{\Lambda' \times \Lambda'' : \Lambda' \in \mathfrak{X}_A, \Lambda'' \in \mathfrak{X}_B\}$.*

Definition 1.12 Let $A \subset \mathbb{R}^n$. For a homeomorphic embedding $f\colon A \to \mathbb{R}^m$ and a C^q stratification $\mathfrak{X}_A$ of A such that for any $\Gamma \in \mathfrak{X}_A$ the mapping $f|_\Gamma$ is a C^q embedding, we have a natural C^q stratification of the image $f(A)$

$$f\mathfrak{X}_A = \{f(\Gamma) : \Gamma \in \mathfrak{X}_A\}.$$

This leads to the following definition of a weakly bi-Lipschitz homeomorphism:

Definition 1.13 Let $A \subset \mathbb{R}^n$ be a set, $f\colon A \to \mathbb{R}^m$ be a homeomorphic embedding. Let $\mathfrak{X}_A$ be a C^q stratification ($q \geq 1$) of A such that for each $\Gamma \in \mathfrak{X}_A$ the mapping $f|_\Gamma$ is a C^q embedding.

We say that the mapping f is *weakly bi-Lipschitz of class C^q on the stratification $\mathfrak{X}_A$*, if f is weakly Lipschitz of class C^q on $\mathfrak{X}_A$ and its inverse $f^{-1}\colon f(A) \to A \subset \mathbb{R}^n$ is weakly Lipschitz of class C^q on the stratification $f\mathfrak{X}_A$.

In order to check if the inverse mapping is weakly Lipschitz on $f\mathfrak{X}_A$, we can use the following proposition (see Czapla [4, Proposition 2.14]).

Proposition 1.14 *Let $A \subset \mathbb{R}^n$, $f\colon A \to \mathbb{R}^m$ be a homeomorphic embedding. Let $\mathfrak{X}_A$ be a C^q stratification of A and assume that for each stratum $\Gamma \in \mathfrak{X}_A$ the mapping $f|_\Gamma$ is a C^q embedding. Then the mapping $f^{-1}\colon f(A) \to A$ is weakly Lipschitz of class C^q on the stratification $f\mathfrak{X}_A$, if and only if for any strata $\Gamma, \Lambda \in \mathfrak{X}_A$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and for any point $c \in \Gamma$, if $\{a_\nu\}_{\nu\in\mathbb{N}}$, $\{b_\nu\}_{\nu\in\mathbb{N}}$ are arbitrary sequences such that $a_\nu \in \Gamma, b_\nu \in \Lambda$ for $\nu \in \mathbb{N}$, then*

$$a_\nu, b_\nu \to c \quad (\nu \to +\infty) \quad \Longrightarrow \quad \liminf_{\nu\to+\infty} \frac{|f(a_\nu) - f(b_\nu)|}{|a_\nu - b_\nu|} > 0.$$

2 Regular projection theorem of Guillaume Valette

Our important tool will be a detailed version of the Regular Projection Theorem of Guillaume Valette [19, Proposition 3.13]. In order to present it here we recall some definitions and notations after Valette [19].

Definition 2.1 Let A be a definable subset of $\mathbb{R}^n$ and let $\lambda \in \mathbb{R}^n$ be a unit vector (that is, $|\lambda| = 1$ or $\lambda \in \mathbb{S}^{n-1}$). We say that λ is *a regular direction for A* if there is a constant $\alpha > 0$ such that for any C^1 regular point a of A and any unit vector v tangent to A at a we have $|v - \lambda| \geq \alpha$. Then we say that the orthogonal projection in the direction λ is a *regular projection for A*.

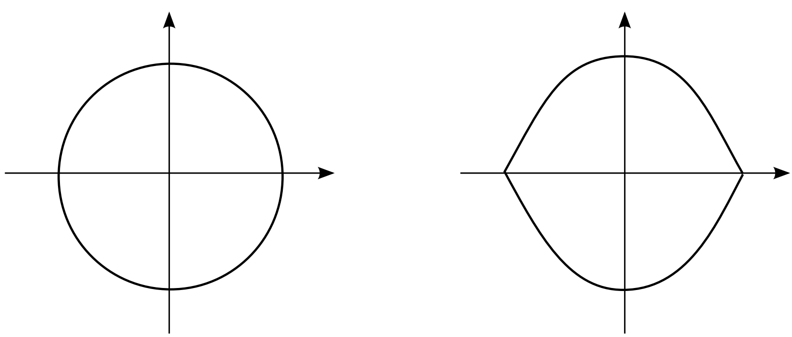

Figure 1: The circle on the left admits no regular projections, while the lens on the right does

Definition 2.2 For any linear hyperplane N in $\mathbb{R}^n$ let $\pi_N\colon \mathbb{R}^n \to N$ denote the orthogonal projection onto N. When $\lambda \in \mathbb{S}^{n-1}$, we will denote by N_λ the linear hyperplane orthogonal to λ and by π_λ the projection π_{N_λ}. For each $q \in \mathbb{R}^n$ let q_λ be the coordinate of q along λ. Then $q = \pi_\lambda(q) + q_\lambda \cdot \lambda$.

Definition 2.3 Let $\lambda \in \mathbb{S}^{n-1}$, A', A be subsets of $\mathbb{R}^n$ and $A' \subset N_\lambda$. Let $\xi\colon A' \to \mathbb{R}$ be a function. We say that A is *the graph of a function ξ for λ* if

$$A = \{q \in \mathbb{R}^n : \pi_\lambda(q) \in A' \text{ and } q_\lambda = \xi(\pi_\lambda(q))\}.$$

Definition 2.4 Let $H \subset \mathbb{R}^n$ be a graph of a function $\xi\colon N_\lambda \to \mathbb{R}$ for $\lambda \in \mathbb{S}^n$. A *subgraph of H for λ* is defined as the subset

$$E(H, \lambda) := \{q \in \mathbb{R}^n : q_\lambda \le \xi(\pi_\lambda(q))\}.$$

Definition 2.5 A *regular family of hypersurfaces of* $\mathbb{R}^n$ is a finite family $H = \{(H_k; \lambda_k)\}_{k=1,\dots,b}$ ($b \in \mathbb{N}$) of definable subsets H_k of $\mathbb{R}^n$ together with elements λ_k of $\mathbb{S}^n$ such that for each $k < b$,

(i) H_k and H_{k+1} are respectively graphs for λ_k of two global Lipschitz functions ξ_k, $\xi_k'\colon N_{\lambda_k} \to \mathbb{R}$ and in the coordinates $N_{\lambda_k} \oplus \mathbb{R}\lambda_k$ we have $\xi_k \le \xi_k'$ and H_b is also a graph for λ_b of a global Lipschitz function $\xi_b\colon N_b \to \mathbb{R}$; and

(ii) $E(H_{k+1}; \lambda_k) = E(H_{k+1}; \lambda_{k+1})$.

Definition 2.6 Let A be a subset of $\mathbb{R}^n$. We say that a regular family of hypersurfaces $H = \{(H_k, \lambda_k)\}_{k=1,\dots,b}$ is *compatible with* A, if $A \subset \bigcup_{k=1}^b H_k$.

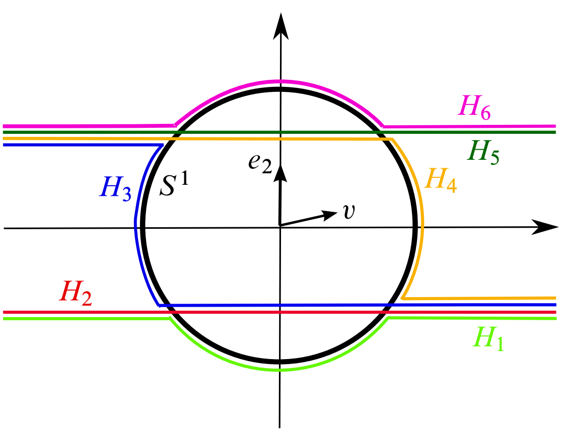

Figure 2: A regular family of hypersurfaces $\{(H_k, \lambda_k)\}_{k=1,\dots,6}$, where $\lambda_1 = \lambda_5 = \lambda_6 = e_2$ and $\lambda_2 = \lambda_3 = \lambda_4 = v$, compatible with the circle S^1.

Theorem 2.7 (Valette [19, Proposition 3.10]) *For every definable set $A \subset \mathbb{R}^n$ of empty interior there exists a regular family of hypersurfaces of $\mathbb{R}^n$ compatible with A.*

Theorem 2.7 is the main tool in the Valette's proof of his Regular Projection Theorem which reads as follows.

Theorem 2.8 (Valette [19, Proposition 3.13]) *Let $A \subset \mathbb{R}^n$ be a definable subset of empty interior. Then there exists a definable bi-Lipschitz homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\widetilde{h}(A)$ has a regular projection.*

Convention 2.9 When $\Lambda \subset \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$, we will sometimes identify the restricted function $f|_\Lambda$ with its graph

$$f|_\Lambda = \operatorname{graph} f|_\Lambda = \{(x,y) : x \in \Lambda, y = f(x)\}.$$

Moreover, if $g\colon \mathbb{R}^n \to \mathbb{R}$ is another function, then we set

$$(f,g)|_\Lambda = \{(x,y) : x \in \Lambda, f(x) < y < g(x)\}.$$

Here we allow the cases $f \equiv -\infty$ or $g \equiv +\infty$.

This is a detailed version of Theorem 2.8 that we will use in Section 3 to prove one of the main results.

Theorem 2.10 *Let C be a definable, nowhere dense subset of $\mathbb{R}^n$. Let $D_1, \dots, D_s$ be definable subsets of C. There exists a definable, bi-Lipschitz homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that the vector e_n is a regular direction for $\widetilde{h}(C)$ and there exists a definable C^q stratification $\mathcal{C}'$ of $\mathbb{R}^{n-1}$ and definable, Lipschitz functions $\eta_k\colon \mathbb{R}^{n-1} \to \mathbb{R}$ $(k = 1, \dots, b)$ such that $\eta_1 \le \eta_2 \le \dots \le \eta_b$, $\widetilde{h}(C) \subset \bigcup_{k=1}^b \eta_k$ and*

(a) *for any $\Gamma \in \mathcal{C}'$ and for any $k = 1, \dots, b$ the restriction $\eta_k|_\Gamma$ is of class C^q;*

(b) *for any $\Gamma \in \mathcal{C}'$ and $k = 1, \dots, b-1$, $\eta_k = \eta_{k+1}$ on Γ or $\eta_k < \eta_{k+1}$ on Γ;*

(c) *the family*

$$\begin{aligned}\mathcal{C} = &\big\{(\eta_k, \eta_{k+1})|_\Gamma : \Gamma \in \mathcal{C}', \eta_k|_\Gamma < \eta_{k+1}|_\Gamma, k = 1, \dots, b-1\big\}\\ &\cup \big\{(-\infty, \eta_1)|_\Gamma : \Gamma \in \mathcal{C}'\big\} \cup \big\{(\eta_b, +\infty)|_\Gamma : \Gamma \in \mathcal{C}'\big\}\\ &\cup \big\{\eta_k|_\Gamma : \Gamma \in \mathcal{C}', k = 1, \dots, b\big\}\end{aligned}$$

is a definable C^q stratification of $\mathbb{R}^n$ compatible with $\widetilde{h}(C)$ and $\widetilde{h}(D_j)$ for $j = 1, \dots, s$; and

(d) *for any $\Lambda \in \mathcal{C}$ the restriction $\widetilde{h}^{-1}|_\Lambda$ is a definable C^q embedding.*

Proof First we recall the construction of the homeomorphism $\widetilde{h}$ following Valette [19, Proposition 3.13].

By Theorem 2.7 we find a regular family of hypersurfaces $H = \{(H_k, \lambda_k)\}_{k=1,\dots,b}$ compatible with A. We will be defining a bi-Lipschitz homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that for any $k = 1, 2, \dots, b$

$$(\mathfrak{F}0)\quad \begin{array}{ll} \text{(i)} & \widetilde{h}(H_k)=F_k \text{ where } F_k \text{ is a graph of a Lipschitz map } \eta_k\colon \mathbb{R}^{n-1}\to\mathbb{R} \text{ for } e_n, \\ \text{(ii)} & \widetilde{h}(E(H_k,\lambda_k)) = E(F_k, e_n). \end{array}$$

The definition of $\widetilde{h}$ will be inductive extending it to the consecutive levels of the following tower:

$$\begin{aligned} E(H_1,\lambda_1) \subset E(H_2,\lambda_1) &= E(H_2,\lambda_2) \subset \cdots \\ &= E(H_k,\lambda_k) \subset E(H_{k+1},\lambda_k) \\ &= E(H_{k+1},\lambda_{k+1}) \subset \cdots \\ &= E(H_b,\lambda_b) \subset \mathbb{R}^n. \end{aligned}$$

Let $k = 1$ and let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism such that $\phi(N_{\lambda_1}) = \mathbb{R}^{n-1}\times\{0\}$ and $\phi(\lambda_1) = e_n$. Then we put

$$(\mathfrak{F}1)\qquad \widetilde{h}|_{E(H_1,\lambda_1)} := \phi|_{E(H_1,\lambda_1)}.$$

Let $k = 1, \dots, b$ and $q \in E(H_{k+1},\lambda_{k+1}) \setminus E(H_k,\lambda_k)$. Then

$$q = u + \xi_k(u)\cdot\lambda_k + \tau\cdot(\xi_k'(u) - \xi_k(u))\cdot\lambda_k,$$

with a unique point $u \in N_{\lambda_k}$ and $\tau \in (0, 1]$. Hence

$$(\mathfrak{F}2)\qquad \widetilde{h}(q) := \widetilde{h}(u + \xi_k(u)\cdot\lambda_k) + \tau\cdot(\xi_k'(u) - \xi_k(u))\cdot e_n.$$

If $q \in \mathbb{R}^n \setminus E(H_b,\lambda_b)$, then $q = u + \xi_b(u)\cdot\lambda_b + \tau\cdot\lambda_b$, with a unique point $u \in N_{\lambda_b}$ and $\tau \in (0, +\infty)$. Then we define

$$(\mathfrak{F}2')\qquad \widetilde{h}(q) := \widetilde{h}(u + \xi_b(u)\cdot\lambda_b) + \tau\cdot e_n.$$

The following inductive formula follows easily from $(\mathfrak{F}2)$, where we put $\tau = 1$:

$$(\mathfrak{F}3)\qquad \eta_{k+1}(z) := \eta_k(z) + (\xi_k' - \xi_k)\circ\pi_{\lambda_k}\circ\widetilde{h}^{-1}(z + \eta_k(z)\cdot e_n)$$

for any $z \in \mathbb{R}^{n-1}$ and $k = 1, \dots, b-1$.

The following remark gives another useful representation of the homeomorphism $\widetilde{h}$.

Remark 2.11 There exist unique, definable and bi-Lipschitz homeomorphisms $\psi_{0k}\colon N_{\lambda_k} \to \mathbb{R}^{n-1}$ for $k = 1, \dots, b$, such that

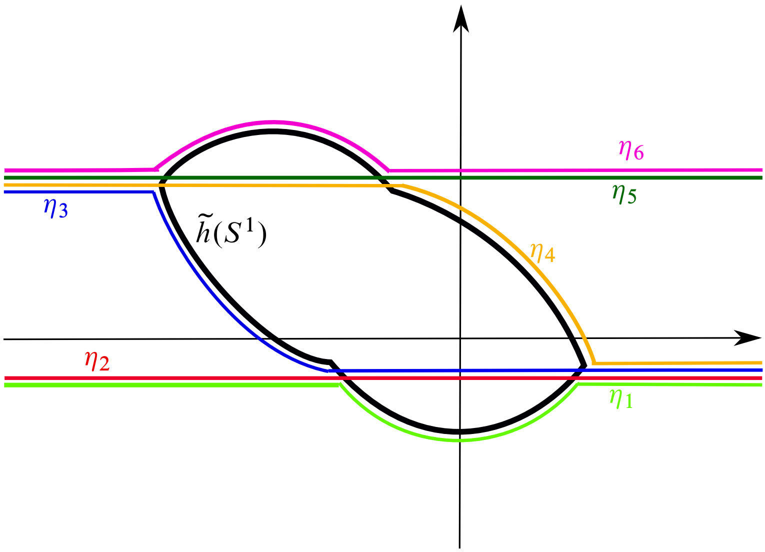

Figure 3: The circle S^1 from Figure 2 is mapped by $\widetilde{h}$ onto a set $\widetilde{h}(S^2)$, which has a regular direction e_2

(a) for all $u \in N_{\lambda_k}$ where $k = 1, \dots, b$,

$$\widetilde{h}(u + \xi_k(u) \cdot \lambda_k) = \psi_{0k}(u) + \eta_k(\psi_{0k}(u)) \cdot e_n;$$

(b) for all $u \in N_{\lambda_1}$ and $\tau \in (-\infty, 0]$,

$$\widetilde{h}(u + \xi_1(u) \cdot \lambda_1 + \tau \cdot \lambda_1) = \psi_{01}(u) + \eta_1(\psi_{01}(u)) \cdot e_n + \tau \cdot e_n;$$

(c) for all $u \in N_{\lambda_k}$ and $\tau \in (0, 1]$ where $k = 1, \dots, b-1$,

$$\widetilde{h}(u + \xi_k(u) \cdot \lambda_k + \tau \cdot (\xi'_k(u) - \xi_k(u)) \cdot \lambda_k) = \\ \psi_{0k}(u) + \eta_k(\psi_{0k}(u)) \cdot e_n + \tau\, (\eta_{k+1}(\psi_{0k}(u)) - \eta_k(\psi_{0k}(u))) \cdot e_n;$$

(d) for all $u \in N_{\lambda_b}$ and $\tau \in (0, +\infty)$,

$$\widetilde{h}(u + \xi_b(u) \cdot \lambda_b + \tau \cdot \lambda_b) = \psi_{0b}(u) + \eta_b(\psi_{0b}(u)) \cdot e_n + \tau \cdot e_n.$$

Proof of Remark 2.11 (a) Let $k = 1, \dots, b$. By $(\mathfrak{F}0)$(i) for any $u \in N_{\lambda_k}$

$$\widetilde{h}(u + \xi_k(u) \cdot \lambda_k) = z + \eta_k(z) \cdot e_n$$

with some $z \in \mathbb{R}^{n-1}$. As we have $\pi_{\mathbb{R}^{n-1}}|_{F_k} = (\mathrm{id}_{\mathbb{R}^{n-1}}, \eta_k)^{-1}$ then

$$z = (\mathrm{id}_{\mathbb{R}^{n-1}}, \eta_k)^{-1} \circ \widetilde{h}|_{H_k} \circ (\mathrm{id}_{N_{\lambda_k}}, \xi_k)(u),$$

so we obtain for $\psi_{0k}\colon N_\lambda \to \mathbb{R}^{n-1}$ the following formula

$$\psi_{0k} = (\mathrm{id}_{\mathbb{R}^{n-1}}, \eta_k)^{-1} \circ \widetilde{h}|_{H_k} \circ (\mathrm{id}_{N_{\lambda_k}}, \xi_k), \tag{$\mathfrak{F}4$}$$

and (a) is satisfied.

(b) It follows from ($\mathfrak{F}1$) that for any $z \in \mathbb{R}^{n-1}$

$$\eta_1(z) = \xi_1 \circ \phi^{-1}(z).$$

Then by ($\mathfrak{F}1$) and ($\mathfrak{F}4$) we have $\psi_{01} = \phi|_{N_{\lambda_1}}$ and for any $u \in N_{\lambda_1}$, $\tau \in (-\infty, 0]$,

$$\begin{aligned}\widetilde{h}(u + \xi_1(u)\cdot\lambda_1 + \tau\cdot\lambda_1) &= \phi(u + \xi_1(u)\cdot\lambda_1 + \tau\cdot\lambda_1)\\ &= \phi(u) + \xi_1(u)\cdot e_n + \tau\cdot e_n\\ &= \psi_{01}(u) + \xi_1 \circ \psi_{01}^{-1} \circ \psi_{01}(u)\cdot e_n + \tau\cdot e_n\\ &= \psi_{01}(u) + \eta_1(\psi_{01}(u))\cdot e_n + \tau\cdot e_n.\end{aligned}$$

(c) By ($\mathfrak{F}3$) and ($\mathfrak{F}4$) for any $k = 1, \dots, b-1$, $z \in \mathbb{R}^{n-1}$:

$$\eta_{k+1}(z) = \eta_k(z) + (\xi_k' \circ \psi_{0k}^{-1}(z) - \xi_k \circ \psi_{0k}^{-1}(z)). \tag{$\mathfrak{F}3'$}$$

Then for $k = 1, \dots, b-1$ and $u \in N_{\lambda_k}$, $\tau \in (0, 1]$ we have

$$\begin{aligned}&\widetilde{h}(u + \xi_k(u)\cdot\lambda_k + \tau\cdot(\xi_k'(u) - \xi_k(u))\cdot\lambda_k)\\ &\overset{(\mathfrak{F}2)}{=} \widetilde{h}(u + \xi_k(u)\cdot\lambda_k) + \tau\cdot(\xi_k'(u) - \xi_k(u))\cdot e_n\\ &\overset{\text{(a)}}{=} \psi_{0k}(u) + \eta_k(\psi_{0k}(u))\cdot e_n + \tau\cdot\big(\xi_k'(u) - \xi_k(u)\big)\cdot e_n\\ &\overset{(\mathfrak{F}3')}{=} \psi_{0k}(u) + \eta_k(\psi_{0k}(u))\cdot e_n + \tau\cdot(\eta_{k+1}(\psi_{0k}(u)) - \eta_k(\psi_{0k}(u)))\cdot e_n.\end{aligned}$$

(d) Let $u \in N_{\lambda_b}$, $\tau \in (0, +\infty)$. Then

$$\begin{aligned}\widetilde{h}(u + \xi_b(u)\cdot\lambda_b + \tau\cdot\lambda_b) &\overset{(\mathfrak{F}2')}{=} \widetilde{h}(u + \xi_b(u)\cdot\lambda_b) + \tau\cdot e_n\\ &\overset{\text{(a)}}{=} \psi_{0b}(u) + \eta_b(\psi_{0b}(u))\cdot e_n + \tau\cdot e_n.\end{aligned}$$

Corollary 2.12

(a) *for all* $k = 1, \dots, b$ *and* $z \in \mathbb{R}^{n-1}$,

$$\widetilde{h}^{-1}(z + \eta_k(z)\cdot e_n) = \psi_{0k}^{-1}(z) + \xi_k \circ \psi_{0k}^{-1}(z)\cdot\lambda_k;$$

(b) *for all* $z \in \mathbb{R}^{n-1}$ *and* $\tau \in (-\infty, 0]$,

$$\widetilde{h}^{-1}(z + \eta_1(z)\cdot e_n + \tau\cdot e_n) = \psi_{01}^{-1}(z) + \xi_1 \circ \psi_{01}^{-1}(z)\cdot\lambda_1 + \tau\cdot\lambda_1;$$

(c) *for all* $k = 1, \dots, b-1$, $z \in \mathbb{R}^{n-1}$ *and* $\tau \in (0,1]$,

$$\widetilde{h}^{-1}(z + \eta_k(z)\cdot e_n + \tau\cdot(\eta_{k+1}(z) - \eta_k(z))\cdot e_n) = \psi_{0k}^{-1}(z) + \xi_k \circ \psi_{0k}^{-1}(z)\cdot \lambda_k + \tau\cdot(\eta_{k+1}(z) - \eta_k(z))\cdot \lambda_k;$$

(d) *for all* $z \in \mathbb{R}^{n-1}$ *and* $\tau \in (0,+\infty)$,

$$\widetilde{h}^{-1}(z + \eta_b(z)\cdot e_n + \tau\cdot e_n) = \psi_{0b}^{-1}(z) + \xi_b \circ \psi_{0b}^{-1}(z)\cdot \lambda_b + \tau\cdot \lambda_b.$$

Now we finish the proof of Theorem 2.10. In view of Corollary 2.12 it is enough to take a definable C^q stratification of $\mathbb{R}^{n-1}$ compatible with the sets: $\{\eta_k = \eta_{k+1}\}$, $\pi_{\mathbb{R}^{n-1}}(\widetilde{h}(C) \cap \eta_k)$, $\pi_{\mathbb{R}^{n-1}}(\widetilde{h}(D_j) \cap \eta_k)$ and such that for each $\Gamma \in \mathcal{C}'$ the mappings $\eta_k|_\Gamma$, $\xi_k \circ \psi_{0k}^{-1}|_\Gamma$ are of class C^q and $\psi_{0k}^{-1}|_\Gamma$ is a C^q embedding. □

Remark 2.13 There is a misprint in the paper of Valette [19] in the proof of Proposition 3.13. It is written there that for $q \in \mathbb{R}^{n-1}$:

$$\eta_{k+1}(q) = \eta_k \circ \pi_{e_n}(q) + (\xi'_k - \xi_k) \circ \pi_{\lambda_k} \circ \widetilde{h}^{-1}(q; \eta_k \circ \pi_{e_n}(q)).$$

As $\pi_{e_n}(q) = q$, the formula reads simply as $(\mathfrak{F}3)$.

3 Definable, locally Lipschitz, weakly bi-Lipschitz triangulation

In this section, for any definable subset, we construct a triangulation that is locally Lipschitz and weakly bi-Lipschitz on the natural simplicial stratification of a simplicial complex. We work in an o-minimal structure on the ordered field $\mathbb{R}$, which admits C^q Cell Decomposition Theorem. For the convenience of the reader we start with recalling standard definitions concerning triangulations.

Definition 3.1 Let V be an affine subspace of $\mathbb{R}^n$, Γ be a C^q submanifold of V. Fix a point $c \in \mathbb{R}^n \setminus V$. The *cone with the vertex* c *and the basis* Γ is the following C^q submanifold

$$c * \Gamma := \{(1-t)\cdot c + t\cdot x : x \in \Gamma, t \in (0,1)\}.$$

Definition 3.2 Let $k \in \mathbb{N}$, $k \le n$. A *k–dimensional simplex* in $\mathbb{R}^n$ is the set

$$[y_0, \dots, y_k] = \left\{ \sum_{i=0}^{k} \beta_i \cdot y_i : \beta_i > 0, i = 0, \dots, k, \sum_{i=0}^{k} \beta_i = 1 \right\},$$

where $y_0, \dots, y_k$ are affinely independent in $\mathbb{R}^n$ and are called the *vertices*.

Remark 3.3 Observe that a simplex $\triangle = [y_0, \dots, y_k]$ is an open subset in the affine subspace L spanned by the points $y_0, \dots, y_k$. Then

$$\overline{\triangle} = \left\{ \sum_{i=0}^{k} \beta_i \cdot y_i : \beta_i \geq 0, i = 0, \dots, k, \sum_{i=0}^{k} \beta_i = 1 \right\}$$

is the closure of $\triangle$ and $\partial\triangle = \overline{\triangle} \setminus \triangle$ is the boundary of $\triangle$ in L.

Definition 3.4 Let $l \in \mathbb{N}$, $k \in \mathbb{N}$, $l \leq k$. An *l–dimensional face* of a simplex $\triangle = [y_0, \dots, y_k]$ is any of the following simplexes $\triangle'$:

$$\triangle' = [y_{\nu_0}, \dots, y_{\nu_l}]$$

where $0 \leq \nu_0 < \dots < \nu_l \leq k$.

Definition 3.5 If $\triangle = [y_0, \dots, y_k]$ is a k–dimensional simplex in $\mathbb{R}^n$, then a *barycentre* of $\triangle$ is the point

$$0_\triangle = \sum_{i=0}^{k} \frac{1}{k+1} \cdot y_i.$$

Definition 3.6 A *simplicial complex* in $\mathbb{R}^n$ is a finite family K of simplexes in $\mathbb{R}^n$, which satisfies the following conditions:

(1) If $S_1, S_2 \in K$, $S_1 \neq S_2$, then $S_1 \cap S_2 = \varnothing$.

(2) If $S \in K$ and S' is a face of S, then $S' \in K$.

A *polyhedron of a simplicial complex* K is the set

$$|K| = \bigcup K.$$

Observe that $|K|$ is a definable compact subset of $\mathbb{R}^n$ of dimension

$$\dim K = \max\{\dim \triangle : \triangle \in K\}.$$

Definition 3.7 For $l \leq n$, the *l–dimensional skeleton* of a simplicial complex K is the following simplicial complex

$$K^{(l)} = \{S \in K : \dim S \leq l\}.$$

Definition 3.8 Let K be a simplicial complex in $\mathbb{R}^n$. Then we define another simplicial complex called *barycentric subdivision* K^* of K by induction on $\dim K$:

(i) If $\dim K = 0$, put $K^* = K$.

(ii) Assume that $\dim K = d > 0$. By the induction hypothesis we have $\big(K^{(d-1)}\big)^*$ which is a stratification of $K^{(d-1)}$. Then we put

$$K^* = (K^{(d-1)})^* \cup \{0_\triangle * \triangle' : \triangle' \subset \partial\triangle, \triangle' \in (K^{(d-1)})^*, \triangle \in K, \dim\triangle = d\} \\ \cup \{0_\triangle : \triangle \in K, \dim\triangle = d\}.$$

Remark 3.9 Replacing in Definition 3.6 the notion of a simplex by the notion of an *open convex polyhedron* (that is, nonempty, bounded intersection of finite number of open affine half-spaces with an affine subspace) we obtain the notion of *a polyhedral complex*. In a similar way we introduce the notion of *a barycentric subdivision of a polyhedral complex*, where in the place of the barycentre of every polyhedron we can use any fixed point of it. Note that such a barycentric subdivision of a polyhedral complex is a simplicial complex.

Now we will prove a lemma crucial for the proof of the existence of a definable, locally Lipschitz, weakly bi-Lipschitz triangulation.

Lemma 3.10 *Let $T = [\tilde{y}_0, \dots, \tilde{y}_k]$ be a simplex in $\mathbb{R}^n$. Let*

$$K_T = \{[\tilde{y}_{\nu_0}, \dots, \tilde{y}_{\nu_l}] : 0 \le \nu_0 < \cdots < \nu_l \le k, l \in \{0, \dots, k\}\}$$

be the simplicial complex of all faces of T, so $|K_T| = \overline{T}$. Let $f\colon |\overline{T}| \to \mathbb{R}$, $g\colon |\overline{T}| \to \mathbb{R}$ be definable and Lipschitz functions such that for each $\triangle \in K_T$

(i) *$f = g$ on $\overline{\triangle}$ or there is a vertex w of $\triangle$ such that $f(w) < g(w)$*

(ii) *$f|_\triangle$, $g|_\triangle$ are of class C^q.*

Let $\psi_f\colon |\overline{T}| \to \mathbb{R}$, $\psi_g\colon |\overline{T}| \to \mathbb{R}$ be the functions defined by the formulae

$$\psi_f\left(\sum_{i=0}^{k} \beta_i \tilde{y}_i\right) = \sum_{i=0}^{k} \beta_i \cdot f(\tilde{y}_i), \qquad \psi_g\left(\sum_{i=0}^{k} \beta_i \tilde{y}_i\right) = \sum_{i=0}^{k} \beta_i \cdot g(\tilde{y}_i)$$

for any $y = \sum_{i=0}^k \beta_i \tilde{y}_i \in \overline{T}$, where $\sum_{i=0}^k \beta_i = 1$, $\beta_i \ge 0$. Consider the polyhedral complex

$$K = \{\psi_f|_\triangle : \triangle \in K_T\} \cup \{\psi_g|_\triangle : \triangle \in K_T\} \cup \{(\psi_f, \psi_g)|_\triangle : \triangle \in K_T, \psi_f|_\triangle < \psi_g|_\triangle\}$$

and put $|K| = \bigcup K$.

Then there exists a unique definable homeomorphism

$$H\colon |K| \to \{(y, z) \in \overline{T} \times \mathbb{R} : f(y) \le z \le g(y)\}$$

such that for any $y \in \overline{T}$ we have

$$H(y, \psi_f(y)) = (y, f(y)), \quad H(y, \psi_g(y)) = (y, g(y))$$

and H is an affine isomorphism of the line segment $[(y, \psi_f(y)), (y, \psi_g(y))]$ onto the line segment $[(y, f(y)), (y, g(y))]$. Moreover,

(a) *H is a Lipschitz mapping*

(b) *H^{-1} is locally Lipschitz on $\{(y,z) \in \overline{T} \times \mathbb{R} : f(y) \le z \le g(y), f(y) < g(y)\}$*

(c) *H is weakly bi-Lipschitz of class C^q on K.*

Proof It is clear that, for each $\triangle \in K_T$,

$$(\mathfrak{F}) \quad H(y,z) = \begin{cases} (y, f(y)) & \text{if } (y,z) \in \psi_f|_\triangle \\ \left(y, \frac{z-\psi_f(y)}{\psi_g(y)-\psi_f(y)} \cdot g(y) + \frac{\psi_g(y)-z}{\psi_g(y)-\psi_f(y)} \cdot f(y)\right) & \text{if } (y,z) \in (\psi_f, \psi_g)|_\triangle \\ (y, g(y)) & \text{if } (y,z) \in \psi_g|_\triangle . \end{cases}$$

Notice that H is a well-defined bijection due to $i)$ which implies that, for each $\triangle \in K_T$, $f < g$ on $\triangle$ if and only if $\psi_f < \psi_g$ on $\triangle$ and $f \equiv g$ on $\triangle$ if and only if $\psi_f \equiv \psi_g$ on $\triangle$. Now we will prove (a), (b) and (c), leaving the more standard part to the reader.

To prove (a) first observe that using the following Lipschitz automorphism

$$\chi \colon \overline{T} \times \mathbb{R} \ni (y,z) \mapsto (y, z - f(y)) \in \overline{T} \times \mathbb{R}$$

we can assume without any loss of generality that $f \equiv \psi_f \equiv 0$. Clearly, we can also assume that $g > 0$ on T. Put $S = (0, \psi_g)|_T$ and $H(y,z) = (y, H_2(y,z))$. Since S is a convex set and $H|_{\overline{S}}$ is continuous and $H|_S$ is of class C^q, to prove that H is Lipschitz it suffices[1] to check that all first-order partial derivatives of H are bounded on S. Since

$$\frac{\partial H_2}{\partial y_i}(y,z) = \frac{z}{\psi_g(y)} \cdot \frac{\partial g(y)}{\partial y_i} - \frac{z}{\psi_g(y)} \cdot \frac{g(y)}{\psi_g(y)} \cdot \frac{\partial \psi_g(y)}{\partial y_i}$$

$$\text{and} \quad \frac{\partial H_2}{\partial z}(y,z) = \frac{g(y)}{\psi_g(y)},$$

it is enough to check that $\frac{g}{\psi_g}$ is bounded on T. This is clear if $\psi_g(\tilde{y}_j) = g(\tilde{y}_j) > 0$ for all j, so assume that $\{\tilde{y}_0, \dots, \tilde{y}_l\} = \{\tilde{y}_j : g(\tilde{y}_j) = 0\}$, where $0 \le l < k$.

For any $y = \sum_{\nu=0}^k \beta_\nu \tilde{y}_\nu \in T$, where $\beta_\nu > 0$ and $\sum_{\nu=0}^k \beta_\nu = 1$ take

$$x = \frac{\beta_0}{\sum_{\nu=0}^l \beta_\nu} \cdot \tilde{y}_0 + \cdots + \frac{\beta_l}{\sum_{\nu=0}^l \beta_\nu} \cdot \tilde{y}_l .$$

[1] By the Mean Value Theorem, see Dieudonné [5, Theorem 8.5.2].

Then $g(x) = 0$ by (i) and

$$x - y = \beta_{l+1}(x - \tilde{y}_{l+1}) + \cdots + \beta_k(x - \tilde{y}_k).$$

Consequently,

$$\begin{aligned}\left|\frac{g(y)}{\psi_g(y)}\right| &= \left|\frac{g(y) - g(x)}{\psi_g(y)}\right| \\ &\leq \frac{L_g \cdot |x - y|}{\psi_g(y)} \\ &\leq L_g \cdot \frac{\sum_{\nu=l+1}^{k} \beta_\nu \cdot |x - \tilde{y}_\nu|}{\sum_{\nu=l+1}^{k} \beta_\nu \cdot g(\tilde{y}_\nu)} \\ &\leq \frac{L_g \cdot \operatorname{diam} T \cdot (k - l)}{\min\{g(\tilde{y}_\nu) : \nu > l\}}\end{aligned}$$

This completes the proof of (a).

To prove (b) take any point $(c, d) \in \overline{T} \times \mathbb{R}$ such that $f(c) \leq d < g(c)$ (for the case $f(c) < d \leq g(c)$ the argument will be similar). Again, using the Lipschitz automorphism χ we can assume without loss of generality that $f \equiv \psi_f \equiv 0$. Put

$$\Pi = \{(y, z) \in T \times \mathbb{R} : 0 < z < g(y)\}.$$

Then $(c, d) \in \overline{\Pi}$ and it is clear that (c, d) admits arbitrarily small neighbourhoods U in $\mathbb{R}^n$ such that $U \cap \Pi$ is convex. Hence now it is enough to notice that all first-order partial derivatives of

$$H^{-1}(y, z) = \left(y, \frac{\psi_g(y)}{g(y)} \cdot z\right)$$

are bounded on some $U \cap \Pi$. This completes the proof of (b).

To prove (c) take any pair Γ, $\Lambda \in K$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$. In view of Proposition 1.9 (a) and (b), it suffices to check that H^{-1} is weakly Lipschitz on $\{H(\Lambda), H(\Gamma)\}$ for the case when $\Lambda = (\psi_f, \psi_g)|_T$ and $\Gamma = \psi_f|_\triangle = \psi_g|_\triangle$, where $\triangle \subset \overline{T} \setminus T$. To use Proposition 1.14 take any $c \in \Gamma$ and two sequences $a_\nu \in \Gamma$, $b_\nu \in \Lambda$ $(\nu \in \mathbb{N})$ such that $a_\nu, b_\nu \to c$. Put $a_\nu = (a'_\nu, a_{\nu n})$, $b_\nu = (b'_\nu, b_{\nu n})$, where $a'_\nu, b'_\nu \in \mathbb{R}^{n-1}$, $a_{\nu n}, b_{\nu n} \in \mathbb{R}$. Then $a_\nu = (a'_\nu, f(a'_\nu)) = (a'_\nu, g(a'_\nu))$ and $b_\nu = (b'_\nu, \lambda_\nu f(b'_\nu) + \mu_\nu g(b'_\nu))$, where $\lambda_\nu > 0$, $\mu_\nu > 0$ and $\lambda_\nu + \mu_\nu = 1$, hence

$$\liminf_{\nu \to +\infty} \frac{|H(a_\nu) - H(b_\nu)|}{|a_\nu - b_\nu|} \geq \liminf_{\nu \to +\infty} \frac{|a'_\nu - b'_\nu|}{|a_\nu - b_\nu|} \geq \frac{1}{1 + L_f + L_g},$$

where L_f and L_g are Lipschitz constants for f and g respectively. □

We now remind now briefly a definition of a definable C^q triangulation.

Definition 3.11 Let A be a compact definable set in $\mathbb{R}^n$. A *definable C^q triangulation* of the set A is a pair (K, f) where K is a simplicial complex in some space $\mathbb{R}^m$ and $f\colon |K| \to A$ a definable homeomorphism such that for each $\triangle \in K$ the set $f(\triangle)$ is a definable C^q submanifold of $\mathbb{R}^n$ and $f|_\triangle$ is a definable C^q diffeomorphism onto $f(\triangle)$. If A is not compact, then a *definable C^q triangulation* of A is a pair (K', f), where K' is a subfamily of a simplicial complex K and $f\colon |K'| \to A$ is a definable homeomorphism of $|K'| = \bigcup K'$ onto A such that for each $\triangle \in K'$ the set $f(\triangle)$ is a definable C^q submanifold of $\mathbb{R}^n$ and $f|_\triangle\colon \triangle \to f(\triangle)$ is a definable C^q diffeomorphism.

Let $A_1, \dots, A_r$ be definable subsets of A. We say that a triangulation (K, f) is *compatible with the sets* $A_1, \dots, A_r$, if the stratification $\{f(\triangle) : \triangle \in K\}$ is compatible with $A_1, \dots, A_r$.

We may now prove one of the main theorems in this paper.

Theorem 3.12 (Definable, locally Lipschitz, weakly bi-Lipschitz triangulation) *Let A be a definable subset of $\mathbb{R}^n$, let $A_1, \dots, A_r$ be definable subsets of A, $r \in \mathbb{N}$. Then there exists a definable C^q triangulation (K, H) of the set A, compatible with $A_1, \dots, A_r$ and such that*

(a) *H is a locally Lipschitz mapping; and*

(b) *H is weakly bi-Lipschitz of class C^q on the natural simplicial stratification K of the set $|K|$.*[2]

Proof By the following definable C^ω diffeomorphism

$$\zeta\colon \mathbb{R}^n \ni x \mapsto \frac{x}{\sqrt{1+|x|^2}} \in \{u \in \mathbb{R}^n : |u| < 1\}$$

without any loss of generality we can assume that A is compact. Observe also that it suffices to find a definable C^q triangulation (K, H) of the set A, compatible with $A_1, \dots, A_r$ and such that for any $\triangle \in K$

(a) $H|_{\overline{\triangle}}$ is a Lipschitz mapping;[3] and

(b) $H|_{\overline{\triangle}}$ is weakly bi-Lipschitz of class C^q on the natural simplicial stratification $\mathfrak{X}_{\overline{\triangle}}$ of the set $\overline{\triangle}$.

[2] This theorem with only conclusion (a) was proved in a different way by Coste and Reguiat [3, Theorem 3].

The proof is proceeded by induction on the dimension n of the ambient space. Cases $n=0$, $n=1$ are trivial. Let $n \geq 2$ and the statement be true for $n-1$. Denote $D_0 = A \setminus \operatorname{int} A$, $D_i = A_i \setminus \operatorname{int} A_i$, $D_{i+r} = \overline{A}_i \setminus \operatorname{int} A_i$, $i = 1, 2, \dots, r$. Let $C = \bigcup_{i=0}^{2r} D_i$. Observe that C is nowhere dense in $\mathbb{R}^n$ and that any stratification of $\mathbb{R}^n$, compatible with all the sets $D_0, \dots, D_{2r}$ is also compatible with all the sets $A, A_1, \dots, A_r$.

Step 1 By Theorem 2.10 we can assume that there exists a definable C^q stratification $\mathcal{C}'$ of $\mathbb{R}^{n-1}$ and a finite family of definable Lipschitz functions $\eta_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$ ($k = 1, \dots, b$) such that $\eta_1 \leq \dots \leq \eta_b$, $C \subset \bigcup_{k=1}^{b} \eta_k$ and for any $\Gamma \in \mathcal{C}'$, $\eta_k|_\Gamma$ is of class C^q and either $\eta_k|_\Gamma \equiv \eta_{k+1}|_\Gamma$ or $\eta_k|_\Gamma < \eta_{k+1}|_\Gamma$ and the family

$$\begin{aligned}\mathcal{C} := \{(\eta_k, \eta_{k+1})|_\Gamma : \Gamma \in \mathcal{C}', \eta_k|_\Gamma < \eta_{k+1}|_\Gamma, k{=}1, \dots, b{-}1\} \\ \cup \{(\eta_b, +\infty)|_\Gamma : \Gamma \in \mathcal{C}'\} \cup \{(-\infty, \eta_1)|_\Gamma : \Gamma \in \mathcal{C}'\} \cup \{\eta_k|_\Gamma : \Gamma \in \mathcal{C}', k{=}1, \dots, b\}\end{aligned}$$

is a C^q stratification of $\mathbb{R}^n$, compatible with $A, A_1, \dots, A_r$. Now it suffices to construct a desired triangulation compatible with $\mathcal{C}$ for the set

$$\{(y, z) \in A' \times \mathbb{R} : \eta_1(y) \leq z \leq \eta_b(y)\},$$

where A' is the projection of A into $\mathbb{R}^{n-1}$.

Step 2 From now on we will follow the classical construction of triangulation (compare Hironaka [10] or Coste [2, Theorem 4.4]). By the induction hypothesis there exists a definable C^q triangulation (K', h) of the set A' compatible with the finite family $\{\Gamma \in \mathcal{C}' : \Gamma \subset A'\}$ and such that for every simplex $\triangle \in K'$

(a) $h|_{\overline{\triangle}}$ is a Lipschitz mapping;

(b) $h|_{\overline{\triangle}}$ is weakly bi-Lipschitz on the natural simplicial stratification $\mathfrak{X}_{\overline{\triangle}}$ of $\overline{\triangle}$, where $\mathfrak{X}_{\overline{\triangle}} = \{\triangle' \in K' : \triangle' \subset \partial\triangle\} \cup \{\triangle\}$.

Without loss of generality, by taking the barycentric subdivision of K', we can assume (see Proposition 1.8) that (K', h) has still the above properties a) and b) and furthermore for any $\triangle \in K'$ and $k = 1, \dots, b-1$, either

$$\eta_k \circ h|_{\overline{\triangle}} = \eta_{k+1} \circ h|_{\overline{\triangle}}$$

[3]It suffices to apply the following lemmas about "glueing" Lipschitz mappings: Let $A_i \subset \mathbb{R}^p$, $i = 1, \dots, r$ and $B = \bigcup_{i=1}^{r} A_i$ be a quasi-convex[4]set, $f \colon B \to \mathbb{R}^m$ be such that for $i = 1, \dots, r$, $f|_{A_i}$ is a Lipschitz mapping. Then f is a Lipschitz mapping. Also if $G_1, \dots, G_r \subset \mathbb{R}^n$ are disjoint and compact, $f \colon \bigcup_{i=1}^{r} G_i \to \mathbb{R}^m$ is such that $f|_{G_i}$ is Lipschitz, then f is Lipschitz.

[4]A set $A \subset \mathbb{R}^n$ is *quasi-convex*, if there exists a constant $C > 0$ such that for $x, y \in A$ there exists a continuous arc $\lambda \colon [0,1] \to A$, piecewise C^1 such that $\lambda(0) = x$, $\lambda(1) = y$ and $|\lambda| \leq C \cdot |x - y|$. In particular the connected components of $|K|$, where K is a simplicial complex in $\mathbb{R}^n$, are quasi-convex.

or there exists a vertex w of $\triangle$ such that

$$\eta_k \circ h(w) < \eta_{k+1} \circ h(w).$$

Thus since now without any loss of generality we can assume[5] that $A' = |K'|$, $h = \mathrm{id}_{A'}$ and

(♯) $\eta_k|_{\overline{\triangle}} = \eta_{k+1}|_{\overline{\triangle}}$ or there is a vertex w of $\triangle$ such that $\eta_k(w) < \eta_{k+1}(w)$

Consider the following functions $\psi_k \colon |K'| \to \mathbb{R}$:

$$\psi_k\left(\sum_{i=0}^{l} \beta_i y_i\right) = \sum_{i=0}^{l} \beta_i \cdot \eta_k(y_i)$$

for any simplex $[y_0, \dots, y_l] \in K'$ and $\beta_i > 0$, $i = 0, \dots, l$ and $\sum_{i=0}^{l} \beta_i = 1$. Observe that every ψ_k is affine on the closure of each simplex $\triangle \in K'$ and by (♯)

$$\begin{aligned} \psi_k < \psi_{k+1} \text{ on } \triangle \quad &\Longleftrightarrow \quad \eta_k < \eta_{k+1} \text{ on } \triangle \\ \text{and} \quad \psi_k = \psi_{k+1} \text{ on } \overline{\triangle} \quad &\Longleftrightarrow \quad \eta_k = \eta_{k+1} \text{ on } \overline{\triangle}. \end{aligned}$$

Consider the polyhedral complex

$$K = \{\psi_k|_\triangle : \triangle \in K', k = 1, \dots, b\} \cup \\ \{(\psi_k, \psi_{k+1})|_\triangle : \triangle \in K', \psi_k|_\triangle < \psi_{k+1}|_\triangle, k = 1, \dots, b-1\}.$$

Put $|K| = \bigcup K$. Take now the mapping $H\colon |K| \to \{(y,z) \in A' \times \mathbb{R} : \eta_1(y) \le z \le \eta_b(y)\}$ such that $H(y, \psi_k(y)) = (y, \eta_k(y))$, for each affine isomorphism of the line segment $[(y, \psi_k(y)), (y, \psi_{k+1}(y))]$ onto the line segment $[(y, \eta_k(y)), (y, \eta_{k+1}(y))]$, for each $y \in A'$ and $k = 1, \dots, b-1$. By Lemma 3.10 H is a homeomorphism, $\{H(S) : S \in K\}$ is compatible with $\mathcal{C}$ and for any $S \in K$

(a) $H|_{\overline{S}}$ is a Lipschitz mapping; and

(b) $H|_{\overline{S}}$ is weakly bi-Lipschitz of class C^q on the natural polyhedral stratification $\mathfrak{X}_{\overline{S}}$ of the set $\overline{S}$, where $\mathfrak{X}_{\overline{S}} = \{S' \in K : S' \subset \overline{S}\}$.

Finally, passing to the barycentric subdivision K^* of K completes the construction (see Proposition 1.8). □

Remark 3.13 It follows that K can be always (a subfamily of) a simplicial complex in $\mathbb{R}^n$ for a set $A \subset \mathbb{R}^n$.

Remark 3.14 If A is compact, we can have $H\colon |K| \to A$ Lipschitz.

[5]Compare Propositions 1.10 and 1.11.

4 A class of triangulable regularity conditions

In this section we define a class $\mathcal{T}$ of regularity conditions and using the results from Czapla [4] we prove that every definable compact set in $\mathbb{R}^n$ has a definable C^q triangulation with a $\mathcal{T}$ condition. We remind briefly some notions from Czapla [4].

Let $\mathcal{Q}$ be a regularity condition of pairs (Λ, Γ) at a point $x \in \Gamma$, where $\Lambda, \Gamma \subset \mathbb{R}^n$ are C^q submanifolds such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$.

Definition 4.1 We say that $\mathcal{Q}$ is *local*, if for an open neighbourhood U of the point $x \in \Gamma$ the pair (Λ, Γ) satisfies the condition $\mathcal{Q}$ at x if and only if the pair $(\Lambda \cap U, \Gamma \cap U)$ satisfies the condition $\mathcal{Q}$ at the point x.

Since now we will consider only local conditions. We will use the following notation:

- $\mathcal{W}^{\mathcal{Q}}(\Lambda,\Gamma,x)$: The condition $\mathcal{Q}$ is satisfied for the pair (Λ,Γ) at the point $x \in \Gamma$.
- $\mathcal{W}^{\mathcal{Q}}(\Lambda,\Gamma)$: For any point $x \in \Gamma$ we have $\mathcal{W}^{\mathcal{Q}}(\Lambda,\Gamma,x)$.
- $\sim \mathcal{W}^{\mathcal{Q}}(\Lambda,\Gamma,x)$: The pair (Λ,Γ) does not satisfy $\mathcal{Q}$ at the point x.

In the natural way we may define a stratification with the condition $\mathcal{Q}$.

Definition 4.2 Let $\mathcal{Q}$ be a regularity condition, $A \subset \mathbb{R}^n$ be a set. A C^q *stratification with the condition* $\mathcal{Q}$ (or a $\mathcal{Q}$ *stratification of class* C^q) of the set A is a C^q stratification $\mathfrak{X}_A^{\mathcal{Q}}$ such that for any two strata Λ, $\Gamma \in \mathfrak{X}_A^{\mathcal{Q}}$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, we have $\mathcal{W}^{\mathcal{Q}}(\Lambda, \Gamma)$.

In the next part of this section we focus on describing common features of regularity conditions. Some of them are well-known features of Whitney's conditions but some seem to be new (the lifting property, the projection property, the conical property).

Definition 4.3 Let $\mathcal{Q}$ be a regularity condition. We say that $\mathcal{Q}$ is *definable*, if for any definable C^q submanifolds $\Gamma, \Lambda \subset \mathbb{R}^n$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the set

$$\{x \in \Gamma : \mathcal{W}^{\mathcal{Q}}(\Lambda, \Gamma, x)\}$$

is definable.

Definition 4.4 Let $\mathcal{Q}$ be a definable regularity condition. We say that $\mathcal{Q}$ is *generic*, if for any definable C^q submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the set

$$\{x \in \Gamma : \sim \mathcal{W}^{\mathcal{Q}}(\Lambda, \Gamma, x)\}$$

is nowhere dense in Γ.

Remark 4.5 If $\mathcal{Q}$ is a definable and generic condition, then for any two definable C^q submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$, such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and $\dim \Gamma = 0$, we have $\mathcal{W}^q(\Lambda, \Gamma)$.

Definition 4.6 Let $\mathcal{Q}$ be a regularity condition. We say that $\mathcal{Q}$ is C^q *invariant* (or *invariant under* C^q *diffeomorphisms*), if for any C^q submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and $\dim \Gamma < \dim \Lambda$, and for any point $x \in \Gamma$, if U is an open neighbourhood of x and $\phi\colon U \to \mathbb{R}^m$ is a C^q embedding, then

$$\mathcal{W}^{\mathcal{Q}}(\Lambda, \Gamma, x) \iff \mathcal{W}^{\mathcal{Q}}(\phi(\Lambda \cap U), \phi(\Gamma \cap U), \phi(x)).$$

Definition 4.7 Let $A \subset \mathbb{R}^n$ and let $f\colon A \to \mathbb{R}^m$ be a continuous mapping, $\mathfrak{X}_A$ be a C^q stratification of the set A such that $f|_\Gamma$ is of class C^q for all $\Gamma \in \mathfrak{X}_A$. Then by the *induced* C^q *stratification of graph f*, we will mean the following:

$$\mathfrak{X}_{\operatorname{graph} f}(\mathfrak{X}_A) = \{\operatorname{graph} f|_\Gamma : \Gamma \in \mathfrak{X}_A\}.$$

Definition 4.8 We say that a condition $\mathcal{Q}$ has *the projection property with respect to weakly Lipschitz mappings of class* C^q if for any C^q mapping $f\colon A \to \mathbb{R}^m$ weakly Lipschitz on a C^q stratification $\mathfrak{X}_A$ of a set $A \subset \mathbb{R}^n$, we have

$$\mathfrak{X}_{\operatorname{graph} f}(\mathfrak{X}_A) \text{ is a } \mathcal{Q} \text{ stratification} \quad \to \quad \mathfrak{X}_A \text{ is a } \mathcal{Q} \text{ stratification.}$$

Definition 4.9 We say that a condition $\mathcal{Q}$ has *the lifting property with respect to locally Lipschitz mappings of class* C^q if for any two C^q submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, and for any locally Lipschitz mapping $f\colon \Lambda \cup \Gamma \to \mathbb{R}^m$ such that the restrictions $f|_\Lambda$, $f|_\Gamma$ are of class C^q and for any C^q submanifolds $M, N \subset \mathbb{R}^n$ such that $M, N \subset \Lambda \cup \Gamma$, $N \subset \overline{M} \setminus M$ and $\{M, N\}$ is compatible with $\{\Lambda, \Gamma\}$, we have

$$\mathcal{W}^{\mathcal{Q}}(M, N), \mathcal{W}^{\mathcal{Q}}(\operatorname{graph} f|_\Lambda, \operatorname{graph} f|_\Gamma) \implies \mathcal{W}^{\mathcal{Q}}(\operatorname{graph} f|_M, \operatorname{graph} f|_N).$$

Definition 4.10 We say that a regularity condition $\mathcal{Q}$ is a $\mathcal{WL}$ *condition of class* C^q, if and only if it has all the following properties:

- definability,
- genericity,
- invariance under definable C^q diffeomorphisms,
- the projection property with respect to weakly Lipschitz mappings of class C^q,
- the lifting property with respect to locally Lipschitz mappings of class C^q.

The class $\mathcal{WL}$ and properties of weakly Lipschitz mappings were widely discussed in Czapla [4]. The main theorem of Czapla [4] states that the $\mathcal{WL}$ conditions are invariant under weakly Lipschitz mappings in the following sense:

Theorem 4.11 (Invariance of the $\mathcal{WL}$ conditions under definable, locally Lipschitz, weakly bi-Lipschitz homeomorphisms) *Let $\mathcal{Q}$ be a regularity $\mathcal{WL}$ condition of class C^q, where $q \in \{1, 2, \dots, \infty, \omega\}$. Let $B \subset \mathbb{R}^n$ be a definable set and consider a definable homeomorphic embedding $f\colon B \to \mathbb{R}^m$, that is weakly bi-Lipschitz of class C^q on definable C^q stratification $\mathfrak{X}_B$. Assume additionally that for any two sumbanifolds Λ, $\Gamma \in \mathfrak{X}_B$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the mapping $f|_{\Lambda \cup \Gamma}$ is locally Lipschitz.*

Then there exists a definable C^q stratification $\mathfrak{X}'_B$ of the set B, compatible with $\mathfrak{X}_B$ and such that

$$\{\Gamma \in \mathfrak{X}_B : \dim \Gamma = \dim B\} = \{\Gamma' \in \mathfrak{X}'_B : \dim \Gamma' = \dim B\}$$

and the condition $\mathcal{Q}$ is invariant with respect to the pair $(f, \mathfrak{X}'_B)$ in the following sense

for any definable C^q submanifolds $M, N \subset B$ such that $N \subset \overline{M} \setminus M$ and $\{M, N\}$ are compatible with the stratification $\mathfrak{X}'_B$

$$\mathcal{W}^{\mathcal{Q}}(M, N) \quad \Longrightarrow \quad \mathcal{W}^{\mathcal{Q}}(f(M), f(N)).$$

Proof See Czapla [4, Theorem 3.15]. □

In order to prove the triangulation theorem with regularity conditions, we will have to narrow the class $\mathcal{WL}$ of regularity conditions by imposing an extra condition.

Definition 4.12 Let $\mathcal{Q}$ be a regularity condition. We say that $\mathcal{Q}$ has *the conical property of class* C^q, if for any $n \in \mathbb{N}$, any affine subspace $V \subset \mathbb{R}^n$ and any C^q submanifolds M, $N \subset V$ such that $N \subset \overline{M} \setminus M$ and for any point $c \in \mathbb{R}^n \setminus V$ we have

$$\mathcal{W}^{\mathcal{Q}}(M, N) \quad \Longrightarrow \quad \begin{cases} \text{(a)} & \mathcal{W}^{\mathcal{Q}}(c * M, M), \mathcal{W}^{\mathcal{Q}}(c * N, N) \\ \text{(b)} & \mathcal{W}^{\mathcal{Q}}(c * M, c * N) \\ \text{(c)} & W^{\mathcal{Q}}(c * M, N). \end{cases}$$

Remark 4.13 Observe that for any affine subspace $S \subset \mathbb{R}^n$ and for any point $c \in \mathbb{R}^n \setminus S$ the mapping

$$\varphi\colon S \times (0, +\infty) \ni (x, t) \mapsto (1-t) \cdot c + t \cdot x \in \mathbb{R}^n$$

is a C^ω embedding. Therefore, if a condition $\mathcal{Q}$ is invariant under definable C^q diffeomorphisms (for example, when $\mathcal{Q}$ is a $\mathcal{WL}$ condition of class C^q), then it has the conical property of class C^q if and only if for any C^q submanifolds M, $N \subset \mathbb{R}^n$ such that $N \subset \overline{M} \setminus M$ we have

$$\mathcal{W}^{\mathcal{Q}}(M,N) \quad \Longrightarrow \quad \begin{cases} \text{(a)} & \mathcal{W}^{\mathcal{Q}}(M\times(0,1), M\times\{1\}), \mathcal{W}^{\mathcal{Q}}(N\times(0,1), N\times\{1\}) \\ \text{(b)} & \mathcal{W}^{\mathcal{Q}}(M\times(0,1), N\times(0,1)) \\ \text{(c)} & W^{\mathcal{Q}}(M\times(0,1), N\times\{1\}). \end{cases}$$

Now we can describe a class of triangulable conditions.

Definition 4.14 A regularity condition $\mathcal{Q}$ is called a *triangulable C^q condition*, if it is a $\mathcal{WL}$ condition of class C^q and it has the conical property of class C^q. Let $\mathcal{T}$ denote the class of triangulable conditions.

In Section 5 we will prove that the Whitney (B) and the Verdier conditions belong to the class $\mathcal{T}$. Now we will prove a general theorem about definable C^q triangulation with a triangulable condition.

Theorem 4.15 *(Definable, locally Lipschitz triangulation with a triangulable condition) Let $\mathcal{Q}$ be a triangulable condition of class C^q, where $q \in \{1,2,\dots,\infty,\omega\}$. Let $A \subset \mathbb{R}^n$ be a definable set and $A_1,\dots,A_r$ be definable subsets of A. Then there exists a definable C^q triangulation (K,H) of A, such that the family $\{H(\triangle) : \triangle \in K\}$ is a definable C^q stratification with the condition $\mathcal{Q}$ of the set A compatible with $A_1,\dots,A_r$ and $H\colon |K| \to A$ is a locally Lipschitz mapping.*

Proof As in the proof of Theorem 3.12, without any loss of generality we can assume that A is compact. Then we will prove the conclusion with a Lipschitz mapping H. We proceed by induction on dimension of the set A. Let $\dim A = d$.

The case $d=0$ is trivial. In the case $d=1$ to get the desired triangulation it suffices to apply Theorem 3.12 (Remark 3.14). Let $d>1$ and the theorem be true for the sets of dimension $\le d-1$.

Step 1 By Theorem 3.12 (Remark 3.14) we find a definable C^q triangulation (K_1,h_1) of the set A, compatible with $A_1,\dots,A_r$, Lipschitz and weakly bi-Lipschitz of class C^q on the simplicial stratification K_1 of the polyhedron $|K_1|$.

Step 2 By Theorem 4.11, where we put $f=h_1, B=|K_1|, \mathfrak{X}_B = K_1$, we find a definable C^q stratification $\mathfrak{X}'_{|K_1|}$ of the polyhedron $|K_1|$ that is compatible with K_1 and such that the condition $\mathcal{Q}$ and the pair $(h_1, \mathfrak{X}'_{|K_1|})$ satisfy the conclusion of Theorem 4.11. In particular

$$\{\Gamma \in \mathfrak{X}'_{|K_1|} : \dim\Gamma = d\} = \{\triangle \in K_1 : \dim\triangle = d\}.$$

Observe also that the following family of definable C^q submanifolds

$$\mathfrak{X}'_{|K_1^{(d-1)}|} := \{\Gamma \in \mathfrak{X}'_{|K_1|} : \dim \Gamma \leq d-1\}$$

is a definable C^q stratification of the polyhedron $\left|K_1^{(d-1)}\right|$, compatible with its natural simplicial stratification

$$K_1^{(d-1)} = \{\triangle \in K_1 : \dim \triangle \leq d-1\}.$$

Step 3 Consider the polyhedron $\left|K_1^{(d-1)}\right|$ and its stratification $\mathfrak{X}'_{|K_1^{(d-1)}|}$. As we have $\dim \left|K_1^{(d-1)}\right| < d$, then by the induction hypothesis there exists a definable C^q triangulation (K_2, h_2) of the polyhedron $\left|K_1^{(d-1)}\right|$, such that the family

$$\{h_2(\triangle) : \triangle \in K_2\}$$

forms a definable C^q stratification with the condition $\mathcal{Q}$ of the set $\left|K_1^{(d-1)}\right|$, compatible with the stratification $\mathfrak{X}'_{|K_1^{(d-1)}|}$ and $h_2\colon \left|K_2\right| \to \left|K_1^{(d-1)}\right|$ is a Lipschitz mapping.

Step 4 We define[6] now a new simplicial complex K_3.

Let $\{a_1, \dots, a_\alpha\}$ be the set of all vertices of K_2 and let $\{\triangle_{\alpha+1}, \dots, \triangle_T\}$ be the set of all d–dimensional simplexes of the complex K_1. Denote by $\{0_{\triangle_{\alpha+1}}, \dots, 0_{\triangle_T}\}$ the set of the barycentres of the simplexes $\triangle_{\alpha+1}, \dots, \triangle_T$. First we define the set of vertices of K_3:

$$\mathrm{Vert}(K_3) = \{e_1, \dots, e_\alpha, e_{\alpha+1}, \dots, e_T\},$$

where e_j, $j = 1, \dots, T$ are the vertices in $\mathbb{R}^T$ of a standard $(T-1)$–dimensional simplex

$$\triangle^{T-1} = [e_1, \dots, e_\alpha, e_{\alpha+1}, \dots, e_T].$$

Now we define the simplicial complex K_3 as the subcomplex

$$\triangle \in K_3 \iff \begin{cases} \quad \triangle = [e_{\beta_1}, \dots, e_{\beta_s}] & \text{for some } [a_{\beta_1}, \dots, a_{\beta_s}] \in K_2 \\ \text{or } \triangle = [e_j, e_{\beta_1}, \dots, e_{\beta_s}] & \text{for some } [a_{\beta_1}, \dots, a_{\beta_s}] \in K_2 \text{ and } \triangle_j, \\ & \text{where } j \in \{\alpha+1, \dots, T\} \text{ is such that} \\ & h_2([a_{\beta_1}, \dots, a_{\beta_s}]) \subset \overline{\triangle}_j \\ \text{or } \triangle = [e_j] & \text{for some } j = \alpha+1, \dots, T. \end{cases}$$

of the natural simplicial complex

$$K_{\triangle^{T-1}} = \left\{[e_{\delta_1}, \dots, e_{\delta_s}] : \delta_1, \dots, \delta_s \in \{1, \dots, T\}, \delta_k \neq \delta_l \text{ for } k \neq l\right\}.$$

[6] Compare the construction in Dugundji–Granas [6, Theorem 4.2], also Engelking–Sieklucki [7, Example 9.5.3].

Observe that K_3 is a d–dimensional simplicial complex in $\mathbb{R}^T$ and there exists a $(d-1)$–dimensional subcomplex

$$L = \{[e_{\beta_1}, \dots, e_{\beta_s}] : [a_{\beta_1}, \dots, a_{\beta_s}] \in K_2\} \subset K_3$$

which is simplicially isomorphic to K_2 by the following piecewise linear homeomorphism $f\colon |L| \to |K_2|$

$$f\left(\sum_{j=1}^{s} \lambda_j \cdot e_{\beta_j}\right) = \sum_{j=1}^{s} \lambda_j \cdot a_{\beta_j},$$

for any $\sum_{j=1}^{s} \lambda_j \cdot e_{\beta_j} \in [e_{\beta_1}, \dots, e_{\beta_s}] \in L$.

Then $(L, h_2 \circ f)$ is still a definable C^q triangulation of $\left|K_1^{(d-1)}\right|$ with the condition $\mathcal{Q}$, compatible with $\mathfrak{X}'_{|K_1^{(d-1)}|}$ and $h_2 \circ f$ is also a Lipschitz mapping.

Step 5 Now we extend the mapping $h_2 \circ f$ on the polyhedron $|K_3|$ in a conical way: $h_3\colon |K_3| \to |K_1|$, where

$$h_3(z) = \begin{cases} h_2 \circ f(z) & \text{for } z \in \triangle', \triangle' \in L, \\ (1-t) \cdot 0_{\triangle_j} + t \cdot h_2 \circ f(x) & \text{for } z \in \triangle, \triangle \in K_3 \setminus L, \triangle = [e_j, e_{\beta_1}, \dots, e_{\beta_s}], \\ & z = (1-t) \cdot e_j + t \cdot x, x \in [e_{\beta_1}, \dots, e_{\beta_s}], \\ & t \in (0,1), \\ 0_{\triangle_j} & \text{for } z = e_j, j \in \{\alpha+1, \dots, T\}. \end{cases}$$

It follows from the construction that h_3 has the following properties:

(i) $h_3|_\triangle$ is a definable C^q embedding for any $\triangle \in K_3$

(ii) $h_3((e_j, e_{\beta_1}, \dots, e_{\beta_s})) = 0_{\triangle_j} * h_2 \circ f((e_{\beta_1}, \dots, e_{\beta_s}))$, for any simplex $\triangle \in K_3 \setminus L$, $\triangle = [e_j, e_{\beta_1}, \dots, e_{\beta_s}]$

(iii) $\{h_3(\triangle) : \triangle \in L\}$ is a C^q stratification with the condition $\mathcal{Q}$ of $\left|K_1^{(d-1)}\right|$, compatible with the stratification $\mathfrak{X}'_{|K_1^{(d-1)}|}$

(iv) h_3 is a Lipschitz mapping.[7]

By the conical property and Remark 4.5 the family $\{h_3(\triangle) : \triangle \in K_3\}$ is a definable C^q stratification with the condition $\mathcal{Q}$ of the set $|K_1|$ compatible with the stratification $\mathfrak{X}'_{|K_1|}$. Now it is clear that the set

$$\{h_1 \circ h_3(\triangle) : \triangle \in K_3\}$$

[7] It suffices to observe that for any $\triangle \in K_3$ the first order derivatives of $h_3|_\triangle$ are bounded.

is a definable C^q stratification with the condition $\mathcal{Q}$ of the set A, compatible with $\{h_1(\triangle) : \triangle \in K_1\}$. It is also compatible with the sets $A_1, \dots, A_r$, because the family $\{h_1(\triangle) : \triangle \in K_1\}$ is compatible with $A_1, \dots, A_r$ (see Step 1). Hence, $(K_3, h_1 \circ h_3)$ is the desired definable C^q triangulation. □

5 The Whitney (B) and the Verdier conditions as triangulable conditions

In this section we will show, using the results from Czapla [4], that the Whitney (B) and the Verdier condition belong to the class $\mathcal{T}$. First remind basic definitions.

Definition 5.1 Let N, M be C^q submanifolds of $\mathbb{R}^n$ $(q \geq 1)$ such that $N \subset \overline{M} \setminus M$ and let $a \in N$. We say that the pair (M, N) satisfies *the Whitney (B) condition at the point a* if for any sequences $\{a_\nu\}_{\nu\in\mathbb{N}} \subset N$, $\{b_\nu\}_{\nu\in\mathbb{N}} \subset M$ both converging to a and such that the sequence of the secant lines $\{\mathbb{R}(a_\nu - b_\nu)\}_{\nu\in\mathbb{N}}$ converges to a line L in $\mathbb{P}_{n-1}$ and the sequence of the tangent spaces $\{T_{b_\nu} M\}_{\nu\in\mathbb{N}}$ converges to a subspace $T \subset \mathbb{R}^n$ in $\mathbb{G}_{\dim M,n}$, we have always $L \subset T$.

When a pair of C^q submanifolds (M, N) satisfies (respectively, does not satisfy) the Whitney (B) condition at a point $a \in N$, we write $\mathcal{W}^B(M, N, a)$ (respectively $\sim \mathcal{W}^B(M,N,a)$). If for any point $a \in N$ we have $\mathcal{W}^B(M,N,a)$, we write $\mathcal{W}^B(M,N)$.

Definition 5.2 Let $v \in \mathbb{S}^{n-1}$ and let W be a nonzero linear subspace of $\mathbb{R}^n$. We put

$$d(v, W) = \inf\{\sin(v, w) : w \in W \cap S^{n-1}\},$$

where $\sin(v, w)$ denotes the sine of the angle between the vectors v and w. We also put $d(u, W) = 1$, if $W = \{0\}$.

Definition 5.3 For any $P \in \mathbb{G}_{k,n}$ and $Q \in \mathbb{G}_{l,n}$, we put

$$d(P, Q) = \sup\{d(\lambda; Q) : \lambda \in P \cap S^{n-1}\},$$

when $k > 0$ and $d(P, Q) = 0$, when $k = 0$.

Now we list some elementary properties of the function d leaving the proofs to the reader.

Proposition 5.4 (a) *Consider the following metric on $\mathbb{P}_{n-1}$:*

$$\widetilde{d}(\mathbb{R}v, \mathbb{R}w) = \min\{|u - w|, |u + w|\} \text{ for } u, w \in S^{n-1}.$$

Then we have

$$\frac{1}{\sqrt{2}}\,\widetilde{d}(\mathbb{R}v,\mathbb{R}w) \le d(\mathbb{R}v,\mathbb{R}w) \le \widetilde{d}(\mathbb{R}v,\mathbb{R}w).$$

(b) *If V, W are linear subspaces of $\mathbb{R}^n$, then $d(V\times\mathbb{R}, W\times\mathbb{R}) = d(V,W)$.*

(c) *If $Q' \subset Q$, then $d(P,Q) \le d(P,Q')$.*

(d) *For any $k \in \mathbb{N}$, $k \le n$ the function d is a metric on $\mathbb{G}_{k,n}$.*

(e) *Let E be a linear subspace of $\mathbb{R}^n$ of dimension $k(<n)$, let $E^{\perp}$ denote the orthogonal complement of E and let $\mathcal{L}(E,E^{\perp})$ denote the space of linear mappings $f\colon E \to E^{\perp}$ with the norm $\|f\| = \sup\{|f(v)| : v \in E, |v| = 1\}$. Put $\widehat{f} := \{v + f(v) : v \in E\} \in \mathbb{G}_{k,n}$, for any $f \in \mathcal{E}, \mathcal{E}^{\perp}$. Then*

$$d(\widehat{f_1}, \widehat{f_2}) \le 2\|f_1 - f_2\|,$$

for any $f_1, f_2 \in \mathcal{L}(E,E^{\perp})$.

Definition 5.5 Let Λ, Γ be C^2 submanifolds of $\mathbb{R}^n$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$. We say that the pair (Λ,Γ) satisfies *the Verdier condition at a point* $x_0 \in \Gamma$ (notation: $\mathcal{W}^V(\Lambda,\Gamma,x_0)$), if there exists an open neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ and $C_{x_0} > 0$ such that

$$d(T_x\Gamma, T_y\Lambda) \le C_{x_0}|x-y|.$$

for all $x \in \Gamma \cap U_{x_0}$ and $y \in \Lambda \cap U_{x_0}$. If a pair of submanifolds (Λ,Γ) satisfies the Verdier condition at each point $x_0 \in \Gamma$, we will write $\mathcal{W}^V(\Lambda,\Gamma)$.

The following theorem was proved in Czapla [4], assuming that our o-minimal structure admits definable C^q cell decompositions, when $q = \infty$ or $q = \omega$.

Theorem 5.6 *The Whitney (B) condition is a $\mathcal{WL}$ condition of class C^q, $q \ge 1$. Also the Verdier condition is a $\mathcal{WL}$ condition of class C^q, $q \ge 2$.*

In order to show that the Whitney (B) condition and the Verdier condition belong to the class $\mathcal{T}$ we have to prove the following

Proposition 5.7 *The Whitney (B) condition has the conical property of class C^q, $q \ge 1$. The Verdier condition has the conical property of class C^q, $q \ge 2$.*

Proof We will prove the part concerning the Verdier condition, leaving the easier part concerning the Whitney (B) condition to the reader. Using Remark 4.13, assume that M and N are two C^q submanifolds, where $q \ge 2$, such that $N \subset \overline{M} \setminus M$ and $\mathcal{W}^V(M,N)$.

(a) Let $x'_0 \in M$. Since M is C^2, there is a constant $C > 0$ such that

$$d(T_{x'}M, T_{y'}M) \leq C|x' - y'|$$

for x', y' in a neighbourhood of x'_0 in M (use Proposition 5.4(e)). Then, by Proposition 5.4(c),

$$\begin{aligned} d(T_{(x',1)}(M \times \{1\}), T_{(y',y_{n+1})}(M \times (0,1))) &= d(T_{x'}M \times \{0\}, T_{y'}M \times \mathbb{R}) \\ &\leq d(T_{x'}M, T_{y'}M) \\ &\leq C|x' - y'| \\ &\leq C|(x',1) - (y', y_{n+1})|. \end{aligned}$$

(b) Let $x'_0 \in N$. Then, for $x' \in N$ and $y' \in M$ in a neighbourhood of x'_0,

$$\begin{aligned} d\big(T_{(x',x_{n+1})}(N \times (0,1)), &T_{(y',y_{n+1})}(M \times (0,1))\big) \\ &= d\big(T_{x'}N \times \mathbb{R}, T_{y'}M \times \mathbb{R}\big) \\ &= d\big(T_{x'}N, T_{y'}M\big) \\ &\leq C_{x'_0}|x' - y'| \qquad \text{(Definition 5.5)} \\ &\leq C_{x'_0}|(x', x_{n+1}) - (y', y_{n+1})|. \end{aligned}$$

(c) Let $x'_0 \in N$. Then, similarly as in (a) and (b),

$$d\left(T_{(x',1)}(N \times \{1\}), T_{(y',y_{n+1})}(M \times (0,1))\right) \leq d(T_{x'}N, T_{y'}M) \leq C_{x'_0}|x' - y'|. \quad \square$$

Corollary 5.8 *The Whitney (B) condition and the Verdier condition belong to the class $\mathcal{T}$.*

Remark 5.9 The (r) condition of Kuo does not belong to the class $\mathcal{T}$, since it has not the conical property (see Brodersen–Trotman [1]).

Open question Can the simplicial complex K in Theorem 4.15 be chosen in the space $\mathbb{R}^n$, where A lies?

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