msp

Geometry & Topology 17 (2013) 39–71

Noncoherence of arithmetic hyperbolic lattices

MICHAEL KAPOVICH

We prove that all arithmetic lattices in $O(n,1)$, $n \geq 4$, $n \neq 7$, are noncoherent. We also establish noncoherence of uniform arithmetic lattices of the simplest type in $SU(n,1)$, $n \geq 2$, and of uniform lattices in $SU(2,1)$ which have infinite abelianization.

11F06; 20F67

1 Introduction

Recall that a group Γ is called *coherent* if every finitely generated subgroup of Γ is finitely presented. This paper is motivated by the following.

Conjecture 1.1 *Let G be a semisimple Lie group (without compact factors) which is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. Then every lattice in G is noncoherent.*

In the case of lattices in $O(n,1)$, this conjecture is due to Dani Wise. Conjecture 1.1 is true for all lattices containing the direct product of two nonabelian free groups since the latter are incoherent. Therefore, it holds, for instance, for $\mathrm{SL}(n,\mathbb{Z})$, $n \geq 4$. The case $n = 3$ is unknown (this problem is due to Serre; see Wall's list of problems [44]).

Conjecture 1.1 is out of reach for nonarithmetic lattices in $O(n,1)$ and $SU(n,1)$, since we do not understand the structure of such lattices. However, all known constructions of nonarithmetic lattices lead to noncoherent groups: See the author, Potyagailo and Vinberg [28] for the case of Gromov–Piatetsky–Shapiro construction; the same argument proves noncoherence of nonarithmetic reflection lattices (see eg Vinberg [42]) and nonarithmetic lattices obtained via Agol's construction [1]. In the case of lattices in $PU(n,1)$, all known nonarithmetic groups are commensurable to the ones obtained via the construction of Deligne and Mostow [12]. Such lattices contain fundamental groups of complex-hyperbolic surfaces which fiber over hyperbolic Riemann surfaces. Noncoherence of such groups is proved by the author in [25]; see also Section 8.

In this paper we will discuss the case of arithmetic subgroups of rank 1 Lie groups. Conjecture 1.1 was proved in [28] for nonuniform arithmetic lattices in $O(n,1)$, $n \geq 6$

Published: 14 February 2013 DOI: 10.2140/gt.2013.17.39

(namely, it was proved that the noncoherent examples of the author and Potyagailo from [27] embed in such lattices). The proof of Conjecture 1.1 in the case of all arithmetic lattices of the *simplest type* appears as a combination of [28] and Agol [3]. In particular, it covers the case of all nonuniform arithmetic lattices ($n \geq 4$) and all arithmetic lattices in $O(n,1)$ for n even, since they are of the simplest type. For odd $n \neq 3, 7$, there are also arithmetic lattices in $O(n,1)$ of "quaternionic origin" (see Section 6 for the detailed definition), while for $n = 7$ there is one more family of arithmetic groups associated with octonions. One of the keys to the proof of noncoherence above is the virtual fibration theorem for hyperbolic 3–manifolds. The main result of the present paper (Theorem 1.3) depends heavily on the existence of such fibrations (conjectured by Thurston) proved recently by Ian Agol, with the help of the results of Dani Wise [49].

Theorem 1.2 (Agol [2]) *Every hyperbolic* 3*–manifold* M *of finite volume admits a virtual fibration, ie,* M *has a finite cover which fibers over the circle. Moreover,* M *admits infinitely many nonisotopic virtual fibrations.*

Our main results are the following.

Theorem 1.3 *Conjecture 1.1 holds for all arithmetic lattices in* $O(n,1)$ *of quaternionic type.*

The proof of the above theorem occupies most of the paper. In Section 8 we will also provide some partial corroboration to Conjecture 1.1 for arithmetic subgroups of $SU(n,1)$.

Theorem 1.4 *Conjecture 1.1 holds for*

(1) *all uniform arithmetic lattices of the simplest type (also called* type 1 arithmetic lattices*) in* $SU(n,1)$*;*

(2) *all uniform lattices (arithmetic or not) in* $SU(2,1)$ *with (virtually) positive first Betti number.*

Theorem 1.5 *Conjecture 1.1 holds for all lattices in the isometry groups of the quaternionic–hyperbolic spaces* $\mathbf{H}\mathbb{H}^n$ *and the octonionic–hyperbolic plane* $\mathbf{O}\mathbb{H}^2$*. These lattices are noncoherent since they contain arithmetic lattices coming from* $O(4,1)$ *or* $O(8,1)$*.*

Historical Remarks The fact that the group $F_2 \times F_2$ is noncoherent was known for a very long time (at least since Grunewald's paper [17]). Moreover, it was proved by Baumslag and Roseblade [5] that "most" finitely generated subgroups of $F_2 \times F_2$ are not finitely presented. It therefore follows that many higher rank lattices (eg, $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 4$) are noncoherent. It was proved by Scott [39] that finitely generated 3–manifold groups are all finitely presented. In particular, lattices in $\mathrm{SL}(2, \mathbb{C})$ are coherent. The first examples of incoherent geometrically finite groups in $SO(4, 1)$ were constructed by the author and Potyagailo [27; 36; 37]. These examples were generalized by Bowditch and Mess [9] who constructed incoherent uniform arithmetic lattices in $SO(4, 1)$. (For instance, the reflection group in the faces of right-angled 120–cell in $\mathbb{H}^4$ is one of such lattices.) Their examples, of course, embed in all other rank 1 Lie groups (except for $SO(n, 1)$, $SU(n, 1)$, $n = 1, 2, 3$). Noncoherent arithmetic lattices in $SU(2, 1)$ (and, hence, $SU(3, 1)$) were constructed in [25]. All these constructions were ultimately based on either existence of hyperbolic 3–manifolds fibering over the circle (in the case of discrete subgroups in $SO(n, 1)$) or existence of complex-hyperbolic surfaces which admit singular holomorphic fibrations over hyperbolic complex curves. A totally new source of noncoherent geometrically finite groups comes from the recent work of Wise [48]: He proved that fundamental groups of many (in some sense, most) polygons of finite groups are noncoherent. On the other hand, according to the author [26], the fundamental group of every even-sided (with at least 6 sides) hyperbolic polygons of finite groups embeds as a discrete convex-cocompact subgroup in some $O(n, 1)$.

Acknowledgments During the work on this paper I was supported by the NSF grants DMS-05-54349 and DMS-09-05802. I am grateful to Dani Wise for sharing with me the results [49], to Jonathan Hillman for providing reference to Theorem 3.1, to Ben McReynolds and Andrei Rapinchuk for help with classification of arithmetic subgroups of $\mathrm{Isom}(\mathbf{H}\mathbb{H}^n)$ and $\mathrm{Isom}(\mathbf{O}\mathbb{H}^2)$ and to John Millson for providing me with references. I am also grateful to Leonid Potyagailo and Ernest Vinberg as this paper has originated in our joint work [28] on noncoherence of nonuniform arithmetic lattices. Finally, I am grateful to the referee for useful remarks.

2 Relative separability for collections of subgroups

Recall that a subgroup H in a group G is called *separable* if for every $g \in G \setminus H$ there exists a finite index subgroup $G' \subset G$ containing H but not g. For instance, if $H = \{1\}$ then separability of H amounts to residual finiteness of G. A group G is called LERF if every finitely generated subgroup of G is separable. According to Scott [40], LERF property is stable under group commensurability. Examples of LERF groups include Free groups (Hall [20]), surface groups (Scott [40]), certain hyperbolic 3–manifold groups (Gitik [14], Wise [47]), groups commensurable to right-angled hyperbolic Coxeter groups (Scott [40], Haglund [18]), nonuniform arithmetical lattices of small dimension (Agol, Long and Reid [4] and Kapovich, Potyagailo, Vinberg [28]), and other classes of groups (Haglund and Świątkowski [19], Wise [46]). Note that in a number of these results one has to replace finite generation of subgroups with geometric finiteness/quasiconvexity. On the other hand, there are 3–dimensional graph-manifolds whose fundamental groups are not LERF; see Long and Niblo [32].

In this paper we will be using a *relative version* of subgroup separability, which deals with finite collections of subgroups of G. It is both stronger than subgroup separability (since it deals with collections of subgroups) and weaker than subgroup separability (since it does not require as much as separability in the case of a single subgroup). Actually, we will need this concept only in the case of pairs of subgroups, but we included the more general discussion for the sake of completeness.

Let G be a group, $H_1, H_2 \subset G$ be subgroups. We say that a double coset $H_1 g H_2$ is *trivial* if it equals the double coset $H_1 \cdot 1 \cdot H_2 = H_1 \cdot H_2$. In other words, $g \in H_1 \cdot H_2$. Given a finitely generated group G, we let Γ_G denote its Cayley graph (here we are abusing the notation by suppressing the choice of a finite generating set which will be irrelevant for our purposes). Recall also that a *geometric action* of a group G on a metric space X is an isometric properly discontinuous cocompact action.

Definition 2.1 Let G be a group, $\mathcal{H} = \{H_1, H_2, \dots, H_M\}$ be a collection of subgroups of G. We will say that $\mathcal{H}$ is *relatively separable* in G if for every finite collection of nontrivial double cosets $H_i g_k H_j$, $i, j \in \{1, \dots, M\}$, $k = 1, \dots, K$, there exists a finite index subgroup $G' \subset G$ which is disjoint from the above double cosets.

Remark 2.2 Again, the combination of the results by Ian Agol [2] and Dani Wise [49], shows that if M is a compact 3–dimensional hyperbolic manifold, then every finite collection of quasiconvex subgroups in $\pi_1(M)$ is relatively separable.

In the case when G, H_i, $i = 1, \dots, M$, are finitely generated, separability can be reformulated as follows.

Given a number R, there exists a finite index subgroup $G' \subset G$ so that for each $g \in G'$ either $g \in H_i \cdot H_j$ for some i, j (and, hence, $d(\Gamma_{H_i}, g\Gamma_{H_j}) = d(\Gamma_{H_i}, \Gamma_{H_j})$) or

$$d(g\Gamma_{H_i}, \Gamma_{H_j}) \geq R$$

for all $i, j \in \{1, \dots, M\}$.

Equivalently, in the above property one can replace Γ_G with a space X on which G acts geometrically and Γ_{H_i}'s with H_i–invariant subsets $X_i \subset X$ with compact quotients X_i/H_i, $i = 1, \dots, M$.

In what follows we will use the following notation.

Notation 2.3 Given a metric space X, a subset $Y \subset X$ and a real number $R \geq 0$, let $B_R(Y) := \{x \in X : d(x, Y) \leq R\}$, ie, the R–neighborhood of Y. For instance for $x \in X$, $B_R(x)$ is the closed R–ball in X centered at x. We let $\mathrm{proj}_Y \colon X \to Y$ denote the nearest-point projection of X to Y.

Below are several useful examples illustrating relative separability.

Example 2.4 Suppose that $M = 1$. Then $\mathcal{H}$ is relatively separable provided that $H = H_1$ is separable in G.

Proof Suppose that H is separable in G. Given R, there are only finitely many distinct nontrivial double cosets

$$Hg_k H, \quad k = 1, \dots K,$$

so that

$$d(g\Gamma_H, \Gamma_H) < R$$

for $g \in Hg_k H$. Let $G' \subset G$ be a subgroup containing H but not $g_1, \dots, g_K$. Then G' is disjoint from

$$\bigcup_{k=1}^{K} Hg_k H,$$

and the claim follows. $\square$

Although the converse to the above example is, probably, false, relative separability suffices for typical applications of subgroup separability. For instance, suppose that G is the fundamental group of a 3–manifold M and $\mathcal{H} = \{H\}$ is a relatively separable surface subgroup of G. Then a finite cover of M contains an incompressible surface (whose fundamental group is a finite index subgroup in H).

Example 2.5 Let G be an arithmetic lattice of the simplest type in $O(n, 1)$ and $H_1, \dots, H_M \subset G$ be the stabilizers of distinct "rational" hyperplanes $L_1, \dots, L_M$ in $\mathbb{H}^n$, ie, L_i/H_i has finite volume, $i = 1, \dots, M$. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G; see [28].

Example 2.6 As a special case of the second example, suppose that G is a surface group and $H_1, \dots, H_M$ are cyclic subgroups. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G.

Recall that finitely generated subgroups of surface groups are separable. Therefore, one can generalize the last example as follows.

Proposition 2.7 *Suppose that G is a word-hyperbolic group which is separable with respect to its quasiconvex subgroups. Let $H_1, \dots, H_M$ be residually finite quasiconvex subgroups with finite pairwise intersections. Then $\mathcal{H} = \{H_1, \dots, H_M\}$ is relatively separable in G.*

Proof Let $H \subset G$ be a subgroup generated by sufficiently deep finite index torsion-free subgroups $H_i' \subset H_i$ $(i = 1, \dots, M)$. Then, according to Gitik [15], H is isomorphic to the free product $H_1' * H_2' * \dots * H_M'$ and is quasiconvex. Let X denote the Cayley graph of G and X_i, $i = 1, \dots, M$ the Cayley graphs of $H_1, \dots, H_M$. Since the groups H_i are quasiconvex and have finite intersections, for every $R < \infty$, there exists r so that for $i \neq j$, the projection of $B_R(X_i)$ to X_j is contained in $B_r(1)$.

Let Y_j denote the preimage in X of $B_r(1)$ under the nearest-point projection $X \to X_j$. Thus, $B_R(X_j) \subset Y_i$, $i \neq j$. Moreover, if we choose r large enough then

$$Y_j^c \cap Y_i^c = \varnothing, \quad \forall i \neq j,$$

where Y_j^c denotes the complement of Y_j in X.

Since the group H_j is residually finite, there exists a finite index subgroup $H_j' \subset H_j$ so that Y_j is a *subfundamental domain* for the action $H_j' \curvearrowright X$, ie,

$$h(Y_j) \cap Y_j = \varnothing, \quad \forall h \in H_j' \setminus \{1\}.$$

Therefore, one can apply the ping-pong arguments to the collection of subfundamental domains $Y_1, \dots, Y_M$ as follows.

Every nontrivial element h is the product

$$h_{i_1} \circ h_{i_2} \circ \dots \circ h_{i_m},$$

where $h_{i_k} \in H_{i_k} \setminus \{1\}$ and $i_k \neq i_{k+1}$ for each $k = 1, \dots, m-1$. Then, arguing inductively on m, we see that for each j,

$$h(Y_j) \subset Y_l^c,$$

where $l = i_1$.

We now claim that $d(X_i, h(X_j)) \geq R$ provided that $h \notin H_i \cdot H_j$. We write down h in the normal form as above with $l = i_1$.

Case 1 Suppose first that $i \neq l$. Then $B_R(X_i) \subset Y_l$. On the other hand, by taking any $m \neq j$, we get

$$h(X_j) \subset h(Y_m) \subset Y_l^c.$$

Thus $B_R(X_i) \subset X_l$ has empty intersection with $h(X_j) \subset Y_l^c$ and the claim follows.

Case 2 Suppose now that $i = l = i_1$. Then $s = i_2 \neq i$; set

$$g := h_{i_2} \circ \cdots \circ h_{i_m}.$$

Then

$$d(X_i, h(X_j)) = d(X_i, h_i g(X_j)) = d(X_i, g(X_j)).$$

Now, by appealing to Case 1, we get

$$d(X_i, h(X_j)) = d(X_i, g(X_j)) \geq R.$$

We hence conclude that $\mathcal{H}$ is relatively separable in the subgroup generated by $H_1, \dots, H_M$.

There are only finitely many nontrivial double coset classes $H_i g_k H_j$, $k = 1, \dots, K$, in $H_i \backslash G / H_j$ so that for the elements $g \in H_i g_k H_j \subset G$, we have

$$d(X_i, g(X_j)) < R.$$

Note that $g_k \notin H$ unless $g_k \in H_i \cdot H_j$ (in which case the corresponding double coset would be trivial). Since H is quasiconvex in G and does not contain g_k, $k = 1, \dots, K$, by the subgroup separability of G, there exists a finite index subgroup $G' \subset G$ containing H, so that $g_1, \dots, g_K \notin G'$. Therefore, G' has empty intersection with each of the double cosets $H_i g_k H_j$, $k = 1, \dots, K$ (for all $i, j \in \{1, \dots, M\}$). It therefore follows that for every $g \in G' \setminus H_i \cdot H_j$,

$$d(X_i, g(X_j)) \geq R.$$

Hence $\mathcal{H}$ is relatively separable in G. □

3 Normal subgroups of Poincaré duality groups

Recall that a group Γ is called an n–dimensional *Poincaré Duality group (over $\mathbb{Z}$)*, abbreviated *PD*(n) group, if there exists $z \in H_n(\Gamma, D)$, so that

$$\cap z \colon H^i(\Gamma, M) \to H_{n-i}(\Gamma, \overline{M})$$

is an isomorphism for $i = 0, \dots, n$ and every $\mathbb{Z}\Gamma$–module M. Here $\overline{M} = D \otimes M$, where $D \cong H^n(\Gamma, \mathbb{Z}\Gamma)$ is the *dualizing module*. For instance, if X is a closed n–manifold so that $X = K(\Gamma, 1)$, then Γ is a *PD*(n) group. The converse holds for $n = 2$, while for $n \geq 3$ the converse is an important open problem (for groups which admit finite $K(\Gamma, 1)$).

Recall also that a group Γ is called FP_r (over $\mathbb{Z}$) if there exists a partial resolution

$$P_r \to P_{r-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of finitely generated projective $\mathbb{Z}\Gamma$–modules. For instance, Γ is FP_1 if and only if it is finitely generated, while every finitely presented group is FP_2 (the converse is false; see Bestvina and Brady [6]). We refer the reader to Bieri [7] for a comprehensive discussion of $PD(n)$ and FP_r groups. We will need the following theorem in the case $n = 4$, $r = 2$.

Theorem 3.1 (Hillman [23, Theorem 1.19]; see also Hillman and Kochloukova [24]) *Let π be a $PD(n)$–group with an FP_r normal subgroup K such that $G = \pi/K$ is a $PD(n-r)$–group and $2r \geq n-1$. Then K is a $PD(r)$–group.*

4 Bisectors

Given $p \neq q \in \mathbb{H}^n$ the *bisector* $\mathrm{Bis}(p,q)$ is the set of points in $\mathbb{H}^n$ equidistant from p and q. Define the closed half-spaces $\mathrm{Bis}(p,q)^{\pm}$ bounded by $\mathrm{Bis}(p,q)$ by requiring $\mathrm{Bis}(p,q)^{+}$ to contain q and $\mathrm{Bis}(p,q)^{-}$ to contain p.

In this section we consider configurations $\mathcal{C}$ of 3–dimensional subspaces H_1, H_2, H_3 in $\mathbb{H}^n$ and geodesics $\gamma_1, \ldots, \gamma_4$, so that

(1) $\gamma_i \subset H_i$, $i = 1, 2, 3$, $\gamma_4 \subset H_3$;

(2) $H_1 \cap H_2 = \gamma_2$, $H_2 \cap H_3 = \gamma_3$ and the angles at these intersections are greater than or equal to $\alpha > 0$;

(3) γ_i, γ_{i+1} $(i = 1, 2, 3)$ are at least distance $R_0 > 0$ apart.

The following is elementary.

Lemma 4.1 *Under the above assumptions, for every $r > 0$, there exists $R_* = R_*(\alpha, r)$ so that*

$$d(\gamma_2, \gamma_3) \geq R_* \Rightarrow d(H_1, H_3) \geq r.$$

Therefore, from now on, we will also fix a number $r > 0$ and consider only configurations $\mathcal{C}$ which satisfy

(4) H_1, H_3 are at least distance r apart.

We will call the triple (α, r, R_0) the *parameters* of $\mathcal{C}$.

Let $p_i, q_i \in \gamma_i$ denote the points closest to $\gamma_{i-1}, \gamma_{i+1}$ respectively. We let m_i denote the midpoint of $\overline{q_i p_{i+1}}$, $i = 1, 3$. Let n_i denote the midpoint of $\overline{p_i q_i}$, $i = 2, 3$. Our assumptions imply that $d(q_i, p_{i+1}) \geq R_0$, $i = 1, \ldots, 3$.

We would like to find conditions on $\mathcal{C}$ that ensure that some of the bisectors $\mathrm{Bis}(p_i, q_i)$, $\mathrm{Bis}(q_j, p_{j+1})$ are disjoint. Clearly, if all the mutual distances between points p_i, q_i (lying on the same geodesic γ_i) and q_j, p_{j+1} (lying on the same subspace H_j) are sufficiently large relative to the parameters (α, r, R_0), then all the bisectors will be disjoint. It could happen, however, that some of the above distances are relatively small, and still, if we have enough relatively large distances between the points, then, say, the bisectors $\mathrm{Bis}(q_1, p_2)$, $\mathrm{Bis}(q_3, p_4)$ are some definite distance apart. The goal of this section is to establish various conditions which ensure separation of bisectors (and more); see Figure 1.

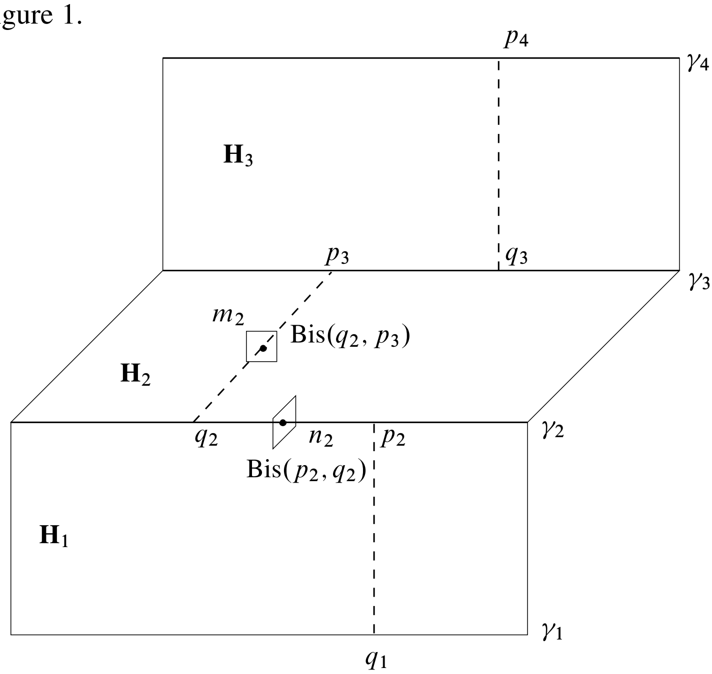

Figure 1: Configuration of planes and lines

Lemma 4.2 *There exist $R_1 = R_1(\alpha, r, R_0) > R_0$, $\rho_1 = \rho_1(\alpha, r, R_0) > 0$, so that if $d(q_2, p_2) \geq R_1$ then for every configuration $\mathcal{C}$ with parameters (α, r, R_0) we have*

(1) *the bisectors $\mathrm{Bis}(p_2, q_2)$ and $\mathrm{Bis}(q_2, p_3)$ are positive distance away from each other;*

(2) *$\gamma_1 \subset \mathrm{Bis}(p_2, q_2)^-$ and $H_3 \subset \mathrm{Bis}(p_2, q_2)^+$;*

(3) *$d(\gamma_1, \mathrm{Bis}(p_2, q_2)) \geq \rho_1$ and $d(H_3, \mathrm{Bis}(p_2, q_2)) \geq \rho_1$.*

Proof Observe that as $d = d(q_2, p_2)$ goes to infinity, the bisector $\mathrm{Bis}(p_2, q_2)$ converges to an ideal point of the geodesic γ_2. Since both γ_2, $\mathrm{Bis}(q_2, p_3)$ are orthogonal to $\overline{q_2 p_3}$ and $d(p_3, q_2) \geq R_0 > 0$, it follows that $\mathrm{Bis}(p_2, q_2)$, $\mathrm{Bis}(q_2, p_3)$ are disjoint for large d. Similarly, since γ_2 is within positive distance from H_3, it follows that $H_3 \subset \mathrm{Bis}(p_2, q_2)^+$ for large d. Similarly, $\gamma_1 \subset \mathrm{Bis}(p_2, q_2)^-$ for large d (since $d(q_1, p_2) \geq R_0 > 0$). Alternatively, one computes R_1, ρ_1 directly using hyperbolic trigonometry. □

Note that the same conclusion occurs if $d(q_3, p_3) \geq R_1$, we just have to interchange the indices $1 \leftrightarrow 4$, $2 \leftrightarrow 4$.

Our next goal is to analyze what happens if $d(p_2, q_2) < R_1$. In this situation, we could have either $d(q_3, p_3) \geq R_1$ or $d(q_3, p_3) < R_1$. Note, however, that the separation properties of the configuration $\mathcal{C}$ and the associated bisectors only improve as we increase $d(p_3, q_3)$. Thus, we will only analyze the case when $d(p_2, q_2) < R_1$ and $d(q_3, p_3) < R_1$.

The next lemma is proved by the same arguments as Lemma 4.2 and we will omit the proof.

Lemma 4.3 *Assume that* $R_0 \leq d(p_2, q_2) < R_1$, $R_0 \leq d(p_3, q_3) < R_1$. *Then there exist* $R_2 = R_2(\alpha, r, R_0)$, $\rho_2 = \rho_2(\alpha, r, R_0) > 0$ *so that if*

$$d(q_1, p_2) \geq R_2, \quad d(q_3, p_4) \geq R_2,$$

then

(1) *the bisectors* $\mathrm{Bis}(q_1, p_2)$ *and* $\mathrm{Bis}(q_3, p_4)$ *are positive distance apart from each other;*

(2) $H_2 \cup H_3 \subset \mathrm{Bis}(q_1, p_2)^+$ *and* $H_2 \cup H_1 \subset \mathrm{Bis}(q_3, p_4)^-$*;*

(3) $d(H_2 \cup H_3, \mathrm{Bis}(q_1, p_2)) \geq \rho_2$, $d(H_2 \cup H_1, \mathrm{Bis}(q_3, p_4)) \geq \rho_2$.

We now set $\rho := \min(\rho_1, \rho_2)$. Thus, $\rho := \rho(\alpha, r, R_0)$.

We will say that for $i = 1$, $i = 4$, the subspace H_i is (α, r, R_0)– *large* (large relative to the triple (α, r, R_0)) if $d(q_i, p_{i+1}) \geq R_2$. Similarly, we will say that for $i = 2$, $i = 3$, the geodesic γ_i is (α, r, R_0)–*large* (large relative to the triple (α, r, R_0)). If H_i or γ_i is not large, we will call it *small*.

Thus, if one of γ_2, γ_3 is large, then $\mathrm{Bis}(q_2, p_2)$ separates γ_1 from γ_4; if both γ_2, γ_3 are small but both H_1, H_3 are large, then bisectors $\mathrm{Bis}(q_1, p_2)$, $\mathrm{Bis}(q_3, p_4)$ are disjoint and separate γ_1 from γ_4. Furthermore, in either one of these cases, separation between geodesics γ_1, γ_4 is at least ρ_2.

5 A combination theorem for quadrilaterals of groups

Combination theorems in theory of Kleinian groups provide a tool for proving that a subgroup of $O(n,1)$ generated by certain discrete subgroups is also discrete and, moreover, has prescribed algebraic structure. The earliest example of such theorem is "Schottky construction" (actually, due to Klein) producing free discrete subgroups of $O(n,1)$. This was generalized by Maskit in the form of Klein–Maskit combination theorems where one constructs amalgamated free products and HNN extensions acting properly discontinuously on $\mathbb{H}^n$. More generally, the same line of arguments applies to graphs of groups. Complexes of groups are higher-dimensional generalizations of graphs of groups. A combination theorem for polygons of finite groups was proved in [26]. The goal of this section is to prove a combination theorem for certain quadrilaterals of infinite groups.

Let G_1 be a discrete subgroup in $\mathrm{Isom}(L)$, $L \cong \mathbb{H}^3$. Pick two nonconjugate maximal cyclic subgroups $G_{\epsilon_1}, G_{\epsilon_2}$ in G_1, which generate a rank 2 free subgroup of G_1. For $i = 1,2$, let $\gamma_i = L(\epsilon_i) \subset L$ denote the invariant geodesics of G_{ϵ_i}.

We will assume the following.

Assumption 5.1 (0) The distance between the geodesics γ_1, γ_2 is $\mathbf{R}_0 > 0$.

(1) There exists $\mathbf{R}_1 > 0$ so that for each $\gamma = \gamma_i$ with $i = 1,2$ and geodesics $\beta_1, \beta_2 \in G_1 \cdot (\gamma_1 \cup \gamma_2)$ so that $d(\beta_j, \gamma) = \mathbf{R}_0$ ($j = 1,2$), it follows that
$$d(\mathrm{proj}_\gamma(\beta_1), \mathrm{proj}_\gamma(\beta_2)) \geq \mathbf{R}_1.$$

(2) There exists $\mathbf{R}_2 > \mathbf{R}_0$ such that for all distinct geodesics β, γ in the G_1–orbits of γ_1, γ_2, the distance $d(\beta, \gamma)$ is at least $\mathbf{R}_2$, unless there exists $g \in G_1$ which carries $\beta \cup \gamma$ to $\gamma_1 \cup \gamma_2$.

We will specify the choices of $\mathbf{R}_1$ and $\mathbf{R}_2$ later on.

We then define a quadrilateral $\mathcal{Q}$ of groups where the vertex groups $G_1, \dots, G_4$ are copies of G_1 and the edge groups $G_{\epsilon_1}, \dots, G_{\epsilon_4}$ are copies of $G_{\epsilon_1}, G_{\epsilon_2}$. In particular, $\mathcal{Q}$ has trivial face-group. We let Q be the quadrilateral underlying this quadrilateral of groups. We refer the reader to Bridson and Haefliger [10] for the precise definitions of complexes of groups and their fundamental groups, we note here only that for each vertex v of Q incident to an edge e, the structure of a quadrilateral of groups prescribes an embedding $G_e \hookrightarrow G_v$. The fundamental group $G = \pi_1(\mathcal{Q})$ of this quadrilateral of groups is the direct limit of the diagram of groups and homomorphisms given by $\mathcal{Q}$.

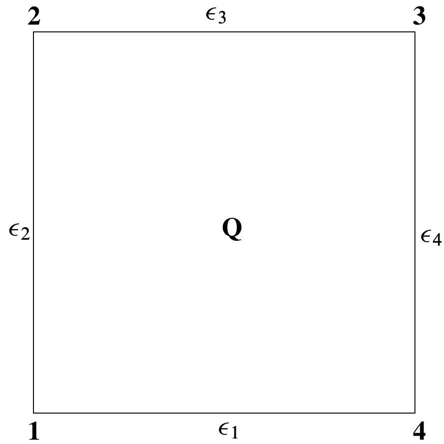

Figure 2: Square of groups

More specifically, we require that $\mathcal{Q}$ admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$–action, generated by two involutions σ_1, σ_2 acting on Q, so that

$$\sigma_1(1) = 2, \quad \sigma_2(1) = 4,$$
$$\sigma_2(G_1) = G_2, \quad \sigma_1(G_1) = G_4,$$

and σ_i fixes G_{ϵ_i}, $i = 1, 2$. The isomorphisms

$$G_1 \to G_i, \quad i = 2, 3, 4,$$

are induced by $\sigma_1, \sigma_1\sigma_2, \sigma_2$ respectively.

Let X denote the universal cover of the quadrilateral of groups $\mathcal{Q}$. Then X is a square complex where the links of vertices are bipartite graphs. We will identify Q with one of the squares in X. Since $G_{\epsilon_1} \cap G_{\epsilon_2} = \{1\}$, it follows that the links of X contain no bigons. Thus, X is a CAT(0) square complex (see [10]). Let X^1 denote 1–skeleton of the complex X; we metrize X^1 by declaring every edge to have unit length. We let dist_X denote the resulting distance function on X^1.

Since the edge groups $G_{\epsilon_i}, i = 1, 2$ in G_1 generate a free subgroup of G_1, the links of the vertices of X are trees. Define the group $\widetilde{G} = \langle G, \sigma_1, \sigma_2 \rangle$ generated by G, σ_1, σ_2. Then $\widetilde{G}$ is a finite extension of G. The group $\widetilde{G}$ acts on X with exactly two orbit types of edges.

Lemma 5.2 *The subgroups $\langle G_{\epsilon_1}, G_{\epsilon_3} \rangle$, $\langle G_{\epsilon_2}, G_{\epsilon_4} \rangle$ of G generated by the respective edge-groups, are free groups of rank 2.*

Proof It suffices to prove lemma for the subgroup $\langle G_{\epsilon_1}, G_{\epsilon_3} \rangle$. The proof is the same as the proof of Corollary 2.7 in [26]. Namely, let τ denote the a geodesic segment in Q orthogonal to the edges ϵ_1, ϵ_3 and disjoint from the vertices. Let $\widetilde{\tau}$ be orbit of τ under the group $\langle G_{\epsilon_1}, G_{\epsilon_3}\rangle$. By the same argument as in the proof of Lemma 2.6 in [26], the graph $\widetilde{\tau}$ is a simplicial tree. The group $\langle G_{\epsilon_1}, G_{\epsilon_3}\rangle$ acts on $\widetilde{\tau}$ with trivial edge stabilizers, so that the vertex stabilizers are conjugate to the groups $G_{\epsilon_1}, G_{\epsilon_3}$. Hence, $\langle G_{\epsilon_1}, G_{\epsilon_3}\rangle \cong G_{\epsilon_1} * G_{\epsilon_3}$. □

We next describe a construction of representations $\phi\colon G \to O(5,1)$ which are discrete and faithful provided that $\mathbf{R}_1, \mathbf{R}_2$ are sufficiently large. We identify $\mathbb{H}^3 = L$ with a 3–dimensional hyperbolic subspace $L \subset \mathbb{H}^5$. Let S_i be some 3–dimensional subspaces in $\mathbb{H}^5$ which intersect L along γ_i at the angles $\alpha_i \geq \alpha > 0$, $i = 1, 2$. We assume that $S_1 \cap S_2$ is a geodesic which is distance greater than or equal to $\mathbf{r}/2 > 0$ away from L and that S_1 is orthogonal to S_2. Let σ_i, $i = 1, 2$ denote commuting isometric involutions in $\mathbb{H}^5$ with the fixed-point sets S_i, $i = 1, 2$ respectively.

Let $L_1 := L$, $L_2 := \sigma_2(L_1)$, $L_3 := \sigma_1\sigma_2(L_1)$, $L_4 := \sigma_1(L_1)$. Then

$$d(L_1, L_3) = d(L_2, L_4) \geq \mathbf{r}. \tag{1}$$

We have the (identity) discrete embedding $\phi_1\colon G_1 \to \mathrm{Isom}(L_1) \subset \mathrm{Isom}(\mathbb{H}^5)$. We will assume that $\phi_1(G_{\epsilon_i})$ stabilizes γ_i, $i = 1, 2$.

Given this data, we define a representation $\phi\colon G \to \mathrm{Isom}(\mathbb{H}^5)$ so that the symmetries σ_1, σ_2 of the quadrilateral of groups Q correspond to the involutions $\sigma_1, \sigma_2 \in O(5,1)$:

(1) $\phi_1 = \phi|G_1$;

(2) $\phi_2 = \rho|G_2 = \mathrm{Ad}_{\sigma_2} \circ \phi_1$, $\phi_4 = \phi|G_4 = \mathrm{Ad}_{\sigma_1} \circ \phi_1$;

(3) $\phi|G_3 = \mathrm{Ad}_{\sigma_1\sigma_2} \circ \phi_1$.

(Here Ad_σ is the inner automorphism of $O(5,1)$ induced by conjugation via $\sigma \in O(5,1)$.) Thus ϕ extends to a representation (also called ϕ) of the group $\widetilde{G} = \langle G, \sigma_1, \sigma_2\rangle$ generated by G, σ_1, σ_2.

Our main result is the following.

Theorem 5.3 *If in the above construction $\mathbf{R}_1, \mathbf{R}_2$ are sufficiently large (with fixed $(\alpha, \mathbf{r}, \mathbf{R}_0)$, then ϕ is discrete and faithful.*

Proof Our proof is analogous to the one in [26]. Every vertex x of X is associated with a 3–dimensional hyperbolic subspace $L(x) \subset \mathbb{H}^5$, namely, it is a subspace stabilized by the vertex group of x in G. If $x = g(x_1)$, $g \in \widetilde{G}$, where $x_1 \in Q \subset X$ is stabilized by G_1, then $L(x) = g(L_1)$.

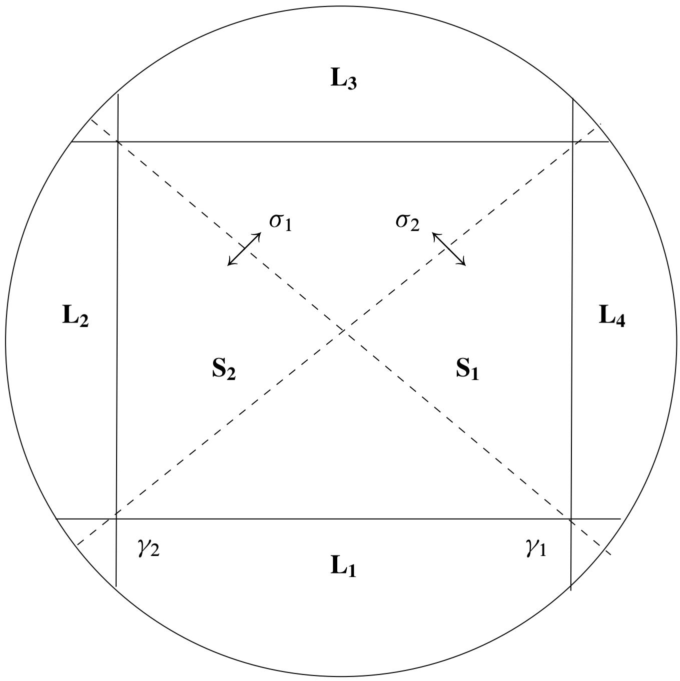

Figure 3: Constructing a representation ϕ: Projective model of $\mathbb{H}^5$

Similarly, every edge e of X is associated with a geodesic $L(e) \subset \mathbb{H}^5$ stabilized by G_e. Hence, if e connects vertices x, y then

$$L(e) = L(x) \cap L(y).$$

The following properties of the subspaces $L(x)$, $L(e)$ follow directly from the inequality (1) and Assumption 5.1.

If x, y are vertices of X which belong to a common 2–face and $\mathrm{dist}_X(x, y) = 2$, the subspaces $L(x), L(y)$ are distance $\mathbf{r}$ apart. If e, f are distinct edges incident to a common vertex x of X then

(a) either e, f belong to a common 2–face of X, in which case

$$d(L(e), L(f)) = \mathbf{R}_0;$$

(b) otherwise, $d(L(e), L(f)) \geq \mathbf{R}_2$.

In order to prove Theorem 5.3, it suffices to show that for sufficiently large $\mathbf{R}_1, \mathbf{R}_2$, there exists $\delta > 0$ so that for all distinct edges f, f' in X,

$$d(L(f), L(f')) \geq \delta. \tag{2}$$

Recall that for $k > 0$ a path $\mathfrak{q}$ in X^1 is a k–local geodesic if every subpath of length k in $\mathfrak{q}$ is a geodesic in X^1.

Consider an edge-path $\mathfrak{c}$ of length 4 in X^1, which is the concatenation of distinct edges $f_i = [y_{i-1}, y_i]$, $i = 1, \ldots, 4$, so that $\mathfrak{c}$ is a 3–local geodesic in X^1. Each path $\mathfrak{c} = (f_1, \ldots, f_4)$ corresponds to a configuration $\mathcal{C} = \mathcal{C}(\mathfrak{c})$ of three hyperbolic 3–dimensional subspaces $L(y_1), L(y_2), L(y_3)$, and four geodesics $L(f_1), \ldots, L(f_4)$ as in Section 4. The numbers $(\alpha, \mathbf{r}, \mathbf{R}_0)$ are the *parameters* of $\mathcal{C}$ in the sense of Section 4.

We now describe the choices of $\mathbf{R}_1$ and $\mathbf{R}_2$ that will be fixed from now on.

(1) We will assume that $\mathbf{R}_1$ is such that for each path $\mathfrak{c}$ as above, the corresponding configuration $\mathcal{C}$ of hyperbolic subspaces in $\mathbb{H}^5$ satisfies $\mathbf{R}_1 \geq R_1(\alpha, \mathbf{r}, \mathbf{R}_0)$ where the function R_1 is defined in Section 4.

(2) We will assume that $\mathbf{R}_2$ is chosen so that for each $\mathfrak{c}$, the corresponding configuration $\mathcal{C}$ of hyperbolic subspaces in $\mathbb{H}^5$ satisfies
$$\mathbf{R}_2 \geq R_2(\alpha, \mathbf{r}, \mathbf{R}_0).$$
We set $\boldsymbol{\rho} := \rho(\alpha, \mathbf{r}, \mathbf{R}_0)$, where ρ is the function defined in Section 4.

Our claim is that, with this choice of $\mathbf{R}_1$ and $\mathbf{R}_2$, for any pair of edges f, f' in X, one can take $\delta = \min(2\boldsymbol{\rho}, \mathbf{r}, \mathbf{R}_0)$ in (2); once this claim is established, the theorem would follow.

If edges f, f' share a vertex then $d(L(f), L(f')) \geq \mathbf{R}_0$ (see Assumption 5.1). If f, f' do not share a vertex and belong to a common 2–face then $d(L(f), L(f')) \geq \mathbf{r}$. We, therefore, consider the generic case when none of the above occurs.

Consider a geodesic edge-path in X^1 connecting f to f'; this path is a concatenation of the edges
$$e_2 \cup \cdots \cup e_{k-1}.$$
Then $k \geq 3$ and for $e_1 = f, e_k = f'$, the concatenation
$$\mathfrak{p} = e_1 \cup e_2 \cup \cdots \cup e_{k-1} \cup e_k$$
is a 3–local geodesic. We will use the notation $e_i = [x_i, x_{i+1}]$, $i = 1, \ldots, k$. Following Section 4, for each geodesic $L(e_i)$ we define points p_i (closest to $L(e_{i-1})$) and q_i (closest to $L(e_{i+1})$).

Definition 5.4 We will say that an edge e in $\mathfrak{p}$ is *large* if it is $(\alpha, \mathbf{r}, \mathbf{R}_0)$–large in the sense of Section 4, ie, $d(p_i, q_i) \geq \mathbf{R}_1$. Similarly, a vertex x_i in $\mathfrak{p}$ is *large* if the subspace $L(x_i)$ is $(\alpha, \mathbf{r}, \mathbf{R}_0)$–large in the sense of Section 4, ie, $d(q_i, p_{i+1}) \geq \mathbf{R}_2$. An edge or a vertex is called *small* if it is not large.

The next lemma immediately follows from Assumption 5.1.

Lemma 5.5 *For every edge $e_i = [x_i, x_{i+1}]$, at least one of e_i, x_i, x_{i+1} is large.*

Remark 5.6 Note that we can have a vertex x_i so that all three x_i, e_i, e_{i-1} are small.

We also define a sequence of bisectors in $\mathbb{H}^5$ corresponding to the path $\mathfrak{p}$ as follows.

For each large cell $c = e_i$, $c = x_i$ in $\mathfrak{p}$, we take the corresponding bisector defined as in Section 4, $\mathrm{Bis}(c) = \mathrm{Bis}(p_i, q_i)$, $\mathrm{Bis}(c) = \mathrm{Bis}(q_i, p_{i+1})$. Recall that we also have half-spaces $\mathrm{Bis}(c)^{\pm} \subset \mathbb{H}^n$ bounded by the bisectors $\mathrm{Bis}(c)$.

We define the natural total order $>$ on the cells in $\mathfrak{p}$ by requiring that $e_j > e_i$, $x_j > x_i$ if $j > i$ and that $e_i > x_i$.

Proposition 5.7 (1) *If $c > c'$ then $\mathrm{Bis}(c')^+ \subset \mathrm{Bis}(c)^+$.*

(2) *If $e_i \neq c$ then $d(L(e_i), \mathrm{Bis}(c)) \geq \rho > 0$. Moreover, if $e_i < c$ then $L(e_i) \subset \mathrm{Bis}(c)^-$, while if $e_i > c$ then $L(e_i) \subset \mathrm{Bis}(c)^+$.*

Proof For part (1), if c, c' are consecutive (with respect to the order $>$) large cells in $\mathfrak{p}$ then they belong to a common length 4 edge-path. Therefore, the assertion follows from Lemmas 4.2 and 4.3 in this case. The general case follows by induction since for three consecutive large cells $c_1 < c_2 < c_3$ in $\mathfrak{p}$, we have

$$\mathrm{Bis}(c_3)^+ \subset \mathrm{Bis}(c_2)^+ \subset \mathrm{Bis}(c_1)^+.$$

For part (2), if e_i, c are not separated (with respect to the order $<$) by any large cells, then they belong to a common length 4 edge-path and the assertion follows from Lemmas 4.2, 4.3 and the definition of ρ. In the general case, the assertion follows from the above and part (1). □

Corollary 5.8 $d(L(e_1), L(e_k)) \geq 2\rho > 0$.

Proof Since the (combinatorial) length of $\mathfrak{p}$ is at least 3, by Lemma 5.5 there exists a large cell c in $\mathfrak{p}$, since every edge e_i, $1 < i < k$, contains a large cell c. Then

$$e_1 < c < e_k,$$

and, by Proposition 5.7(2),

$$L(e_1) \subset \mathrm{Bis}(c)^-, \quad L(e_k) \subset \mathrm{Bis}(c)^+.$$

Moreover

$$\min(d(L(e_1), \mathrm{Bis}(c)), d(L(e_k), \mathrm{Bis}(c))) \geq \rho.$$

The corollary follows. □

This concludes the proof of Theorem 5.3. □

Remark 5.9 By using the arguments of the proof of Theorem 5.3, one can also show that $\phi(G)$ is convex-cocompact.

6 Arithmetic subgroups of $O(n,1)$

The goal of this section is to describe a "quaternionic" construction of arithmetic subgroups in $O(n,1)$. For $n \neq 3,7$, this construction covers all arithmetic subgroups. Our discussion follows Vinberg and Shvartsman [43], and we refer the reader to Li and Millson [30] for the detailed proofs.

We begin by reviewing quaternion algebras over number fields and "hermitian vector spaces" over such algebras.

Let K be a field, D be a central quaternion algebra over K. In other words, the algebra $D = D(a,b)$ has the basis $\{1, i, j, k\}$, subject to the relations

$$i^2 = a \in K, \quad j^2 = b \in K, \quad ij = -ji = k,$$

and so that 1 generates the center of D. For instance, for $a = b = -1$ and $K = \mathbb{R}$, we get the algebra of Hamilton's quaternions $\mathbf{H}$. Similarly, if $a = b = 1$ then D is naturally isomorphic to the algebra of 2×2 matrices $\mathrm{End}(\mathbb{R}^2)$. One uses the notation

$$\lambda = x1 + yi + zj + wk = x + yi + zj + wk$$

for the elements of D, with $x, y, z, w \in K$. An element of D is *imaginary* if $x = 0$. We will identify K with the center of D:

$$K = K \cdot 1 \subset D.$$

One defines the *conjugation* on D by

$$\lambda = x + yi + zj + wk \mapsto \bar{\lambda} = x - yi - zj - wk.$$

Then $\mathrm{Tr}(\lambda) = \lambda + \bar{\lambda}$, $N(\lambda) = \lambda\bar{\lambda}$ are the *trace* and the *norm* on D. Clearly, both the trace and the norm are elements of K. In the case when $D \cong \mathrm{End}(\mathbb{R}^2)$, the trace is twice the matrix trace and the norm is the matrix determinant. Suppose that K is a subfield of $\mathbb{R}$. We say that $\lambda \in D$ is *positive* (resp. *negative*) if it has positive (resp. negative) norm.

In what follows, we will assume that K is a totally real number field and D is a division algebra.

We will consider finite-dimensional "vector spaces" V over D, ie, finite-dimensional right D–modules where we use the notation

$$v\lambda = v \cdot \lambda \in V$$

for $v \in V$, $\lambda \in D$. Such a module is isomorphic to D^n for some $n < \infty$, where n is the dimension of V as a D–module. Given $v \in V$ define $[v]$ as the submodule in V generated by v.

A *skew-hermitian form* on V is a function $F(u,v) = \langle u, v\rangle \in D$, $u, v \in V$, so that

$$F(u_1+u_2, v) = F(u_1, v) + F(u_2, v),$$
$$F(u\lambda, v\mu) = \bar{\lambda} F(u,v)\mu, \quad F(u,v) = -\overline{F(v,u)}.$$

Similarly, F is hermitian if

$$F(u\lambda, v\mu) = \bar{\lambda} F(u,v)\mu, \quad F(u,v) = \overline{F(v,u)}.$$

The form F is *nondegenerate* if

$$F(v,u) = 0 \quad \forall u \in V \Rightarrow v = 0.$$

In coordinates, we have

$$F(x,y) = \sum_{l,m} \bar{x}_l a_{lm} y_m, \quad a_{lm} = -\overline{a_{ml}}.$$

In particular, the diagonal entries of the Gramm matrix of F are imaginary. A *null-vector* is a vector with $F(v,v) = 0$, equivalently, $F(v,v)$ is not imaginary. We say that a vector v is *regular* if it is not null. From now on, we fix F.

Define $U(V,F)$, the group of *unitary* automorphisms of V, ie, invertible endomorphisms which preserve F.

For a regular vector v, define the submodule $v^\perp$ in V by

$$v^\perp = \{u \in V \mid \langle u, v\rangle = 0\}.$$

We will see below that

$$V = [v] \oplus v^\perp,$$

and the restriction of the form F to $v^\perp$ is again nondegenerate.

Orthogonal projection Suppose $\langle v, v\rangle = a \neq 0$. Define

$$\mathrm{Proj}_v\colon V \to [v], \quad \mathrm{Proj}_v(u) = v \cdot a^{-1}\langle v, u\rangle.$$

Then $\mathrm{Proj}_v \in \mathrm{End}(V)$, $\mathrm{Proj}_v|_{[v]} = \mathrm{Id}$ and $\mathrm{Ker}(\mathrm{Proj}_v) = v^{\perp}$. In particular, for a vector $u \in V$,

$$u' = u - \mathrm{Proj}_v(u) \in v^{\perp}.$$

Hence, $u = u' + u''$, with $u'' = \mathrm{Proj}_v(u)$. It is now immediate that

$$V = [v] \oplus v^{\perp}.$$

Since F is nondegenerate, the restriction $F|_{v^{\perp}}$ is also nondegenerate.

The existence of projections allows us to define Gramm–Schmidt orthogonalization in V. In particular, V has an orthogonal basis in which F is diagonal.

Reflections Given a regular vector $v \in V$, define the *reflection* $\sigma_v \in \mathrm{End}(V)$ by

$$\sigma_v(u) := u - 2\,\mathrm{Proj}_v(u).$$

It is immediate that $\sigma_v|[v] = -\mathrm{Id}$ and $\sigma_v|v^{\perp} = \mathrm{Id}$. In particular, σ_v is an involution in $U(V, F)$.

Observe that reflections σ_u, σ_v commute if and only if either $[u] = [v]$ or $\langle u, v\rangle = 0$.

Proof of noncoherence of arithmetic lattices will use the following technical result.

Lemma 6.1 *Let V be 3–dimensional, p_1, p_2 be regular vectors which span a 2–dimensional submodule P in V so that the restriction of F to P is nondegenerate. Then there exist $u_1, u_2 \in V$ so that*

(1) $\langle p_m, u_m\rangle = 0$, $m = 1, 2$;

(2) $\langle u_1, u_2\rangle = 0$.

Proof Since $F|_P$ is nondegenerate, it follows from the Gramm–Schmidt orthogonalization that there exists $v \in V$ so that $P = v^{\perp}$ and $V = P \oplus [v]$. In particular, v is a regular vector.

Orthogonalization implies that there exist vectors $u_1', u_2' \in P$ orthogonal to p_1, p_2 respectively. We will find vectors u_1, u_2 in the form

$$u_m = u_m' + v \cdot \lambda_m, \quad \lambda_m \in D,\ m = 1, 2.$$

It is immediate that $\langle p_m, u_m \rangle = 0$, $m = 1, 2$. We have

$$\langle u_1, u_2 \rangle = \langle u'_1, u'_2 \rangle + \overline{\lambda}_1 \langle v, u'_2 \rangle + \langle u'_1, v \rangle \lambda_2 + \overline{\lambda}_1 \langle v, v \rangle \lambda_2.$$

Set $\alpha := \langle v, v \rangle$, $\nu_1 := \langle u'_1, v \rangle$, $\nu_2 := \langle v, u'_2 \rangle$, $\mu := \langle u'_1, u'_2 \rangle$. By scaling u'_1 if necessary, we get

$$\nu_1 + \alpha \neq 0.$$

We now set $\lambda_1 = 1$, $\lambda_2 = \lambda$ (the unknown). Then the equation $\langle u_1, u_2 \rangle = 0$ has the solution

$$\lambda = -(\mu + \nu_2)(\nu_1 + \alpha)^{-1} \in D. \qquad \square$$

Next, we now relate (V, F) to hyperbolic geometry. Regarding K as a subfield of $\mathbb{R}$, we define the completions $D_{\mathbb{R}}$ of D and $V_{\mathbb{R}}$ of V. We require D to be such that $D_{\mathbb{R}} \cong \mathrm{SL}(2, \mathbb{R})$, ie, to have zero divisors. Hence, at least one of the generators i, j, k of D is negative, ie, has negative norm. We assume that this is i; thus $i^2 > 0$. By interchanging j and k if necessary, we obtain that j is also negative. By scaling i, j by appropriate real numbers we get $i^2 = j^2 = 1$. By abusing the terminology, we retain the notation i, j for these real multiples of the original generators of D.

Since $i^2 = 1$, the right multiplication by i,

$$I(v) = v \cdot i, v \in V,$$

determines an involutive linear transformation of $V_{\mathbb{R}}$ (regarded as the real vector space). We obtain the eigenspace decomposition

$$V_{\mathbb{R}} = V_+ \oplus V_-,$$

where $I|V_\pm = \pm \,\mathrm{Id}$. (Note that $V_\pm \cap V = 0$.) Let $J\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ be given by the right multiplication by j: This is again a linear automorphism. For $v \in V_+$ we have

$$vji = -vij = -vj.$$

Therefore, J determines an isomorphism $V_+ \to V_-$.

Our next goal is to analyze the subspace V_+ (which has dimension $2n$). For $u, v \in V_+$ we have

$$c = \langle u, v \rangle = -i \langle u, v \rangle = \langle ui, v \rangle = \langle u, vi \rangle = \langle u, v \rangle i.$$

Hence,

$$-ic = c = ci.$$

Such c necessarily has the form $c = t(k - j)$, $t \in \mathbb{R}$. Set $\alpha := k - j$. Then $F|_{V_+}$ takes values in $\mathbb{R}\alpha$. In particular, $F|_{V_+} = \alpha\varphi$, where φ is a real bilinear form.

Define $\beta := 1 + i$. Then

$$\alpha i = \alpha, \quad i\beta = \beta i = \beta, \quad \alpha^2 = 0, \quad \beta\overline{\beta} = 0,$$
$$\alpha\beta = \overline{\beta}\alpha = 2\alpha, \quad k\beta = \alpha, \quad k\alpha = -\beta.$$

We now consider the case when V is 1–dimensional, ie, $V = D$. Then the form F is given by

$$F(x, y) = \overline{x}ay, \quad a \in D,\ a = -\overline{a}.$$

Therefore,

$$D_+ = \{x\beta + y\alpha \mid x, y \in \mathbb{R}\}.$$

We now compute the form φ so that $F|D_+ = \alpha\varphi$. Note that the group

$$\mathrm{SL}_1(D) = \{g \mid N(g) = g\overline{g} = 1\}$$

acts on the space of traceless matrices

$$D_0 = \{\lambda \in D_{\mathbb{R}} \mid \mathrm{Tr}(\lambda) = \lambda + \overline{\lambda} = 0\}$$

by

$$\mathrm{Ad}_g(\lambda) = \overline{g}\lambda g = g^{-1}\lambda g.$$

This action preserves the nondegenerate indefinite quadratic form $\lambda\overline{\lambda}$. Hence, this is a the orthogonal action on $\mathbb{R}^{2,1}$ which has three nonzero orbit types. The relevant ones are positive ($N(\lambda) > 0$) and negative ($N(\lambda) < 0$) vectors in D_0. They are represented by $\lambda = k$ (in which case $N(k) = 1$) and $\lambda = i$ (in which case $N(i) = -1$).

By changing the generator in $D_{\mathbb{R}}$, we replace a (in the definition of F) with $\overline{g}ag$. Hence, our analysis of the form φ reduces to two cases: $a = i, a = k$.

Case 1 $a = i$, $N(a) < 0$. Then for $v = a\beta + y\alpha \in D_+$, we have

$$\begin{aligned} F(v, v) &= \langle x\beta + y\alpha, x\beta + y\alpha\rangle = (x\overline{\beta} + y\overline{\alpha})i(x\beta + y\alpha) \\ &= (x\overline{\beta} - y\alpha)(x\beta - y\alpha) = -2xy\alpha. \end{aligned}$$

Hence, $\varphi(v, v) = -2xy$, an indefinite form.

Case 2 $a = k$, $N(a) > 0$. Then for $v = a\beta + y\alpha \in D_+$, we have

$$\begin{aligned} F(v, v) &= \langle x\beta + y\alpha, x\beta + y\alpha\rangle = (x\overline{\beta} + y\overline{\alpha})k(x\beta + y\alpha) \\ &= (x\overline{\beta} + y\overline{\alpha})(x\alpha - y\beta) = 2(x^2 + y^2)\alpha. \end{aligned}$$

Therefore, in this case the form φ is positive-definite.

To summarize, if $N(a)<0$ then φ is indefinite, while if $N(a)>0$ then the form φ is positive-definite; in both cases, φ is nondegenerate.

We now consider the general case. Without loss of generality, we may assume that F is diagonal, $\{v_1,\dots,v_n\}$ is an orthogonal basis in V. Clearly,

$$V_+=\bigoplus_{l=1}^{n}[v_1]_+,\quad V_-=\bigoplus_{l=1}^{n}[v_1]_-,$$

where

$$[v_1]_+=v_l\cdot D_+,\quad [v_1]_-=v_l\cdot D_-.$$

We say a vector $v\in V$ is *positive* (resp. *negative*) if $N(\langle v,v\rangle)>0$ (resp. $N(\langle v,v\rangle)<0$).

Let $n=p+q$. We will say that F has "signature" (p,q) if some (equivalently, every) orthogonal basis $\{v_1,\dots,v_n\}$ of V contains exactly q negative vectors. Thus we obtain the following.

Proposition 6.2 *The form φ is always nondegenerate. Then F has "signature" (p,q) if and only if φ has signature $(2p+q,q)$.*

We assume from now on F has the "signature" $(p,1)$, ie, φ has signature $(2p+1,1)$. Then (V,F) determines the hyperbolic $2p+1$–space $H(V)$ associated with the Lorentzian space (V_+,φ). A vector $v\in V$ determines a geodesic $H([v])\subset H(V)$ if and only if v is a negative vector. Similarly, $F|_{v^\perp}$ has hyperbolic signature if and only if v is a positive vector; thus $H(v^\perp)\subset H(V)$ is a hyperbolic hyperplane.

Definition 6.3 For a subspace $W_+\subset V_+$ we define W_+^- as the closed light-cone

$$\{w\in W_+\mid \varphi(w)\le 0\}. \tag{3}$$

We obtain an embedding $U(V,F)\hookrightarrow \mathrm{Isom}(H(V))$ given by the action of $U(V,F)$ on the Lorentzian space $(V_+,\varphi)\cong\mathbb{R}^{2p+1,1}$.

Lemma 6.4 *Suppose that u,v are positive vectors in V. Then the hyperbolic hyperplanes $H(u^\perp),H(v^\perp)$ are orthogonal if and only if u is orthogonal to v.*

Proof The lemma immediately follows from the fact that the following are equivalent:

(1) $H(u^\perp), H(v^\perp)$ are orthogonal or equal;

(2) σ_u commutes with σ_v;

(3) u and v are orthogonal or generate the same submodule in V. □

Given the hyperbolic space $H(V)$, we say that a hyperplane $L \subset H(V)$ is K–rational if $L = H(u^\perp)$ for some positive vector $u \in V$. A subspace in $H(V)$ is K–rational if it appears as intersection of K–rational hyperplanes. One observes (using orthogonalization) that a geodesic $\gamma \subset H(V)$ is K–rational if and only if $\gamma = H([v])$, where $v \in V$ is a negative vector.

We therefore obtain the following.

Corollary 6.5 *Suppose that γ_1, γ_2 are distinct K–rational geodesics in the hyperbolic 5–space $\mathbb{H}^5 = H(V)$, $\dim(V) = 3$. Then there exist K–rational hyperplanes H_1, H_2 containing γ_1, γ_2, so that H_1 is orthogonal to H_2.*

Proof Let $p_l \in V$ be such that $\gamma_l = H([p_1])$, $l = 1, 2$. Then, according to Lemma 6.1, there exist $u_1, u_2 \in V$ so that

$$[p_1] \subset u_l^\perp, \quad l = 1, 2, \quad \langle u_1, u_2 \rangle = 0.$$

Since $F|_{[p_1]}$ is indefinite, it follows that $F|u_l^\perp$ is indefinite as well ($l = 1, 2$). Hence, $u_l^\perp$ determines a K–rational hyperbolic hyperplane $H_l = H(u_l^\perp) \subset \mathbb{H}^3$. Clearly, $\gamma_l \subset H_l$, $l = 1, 2$. In view of the previous lemma, H_1 and H_2 are orthogonal provided that they actually intersect in $\mathbb{H}^5$.

Pick a generator w of $W = u_1^\perp \cap u_2^\perp$. Since u_1, u_2 are positive and u_1, u_2, w form an orthonormal basis in V, then w is negative. Therefore, $\gamma = H(W) = H([w])$ is a geodesic in $\mathbb{H}^5$. Hence, H_1, H_2 intersect along the geodesic γ at the right angle. □

Let $\sigma_l := \sigma_{u_l}, l = 1, 2$ denote the reflections in the subspaces $U_l = u_l^\perp$. Since σ_1, σ_2 commute, their product is the involution τ whose fixed-point set is the 1–dimensional subspace $W = U_1 \cap U_2$. We now assume that the geodesics γ_1, γ_2 are positive distance apart from each other. Set $\gamma := H(W)$ (a K–rational geodesic in $\mathbb{H}^5$) and set $L := H(P)$ (a 3–dimensional K–rational hyperbolic subspace in $\mathbb{H}^5$).

Lemma 6.6 *H and γ, L and $\tau(L)$ are positive distance apart from each other.*

Proof Observe that $U_l \neq P$, $l = 1, 2$ (since u_l is not a multiple of v). In particular, $U_l \cap P = [p_1]$. Hence,

$$U_{l+}^- \cap P_+^- = [p_1]_+^-.$$

Here and below we are using the notation from (3), eg $[p_1]_+^-$ denotes the closed negative light cone in the subspace $[p_1]_+$, etc.

If $w \in W_+^- \cap P_+^-$ then $w \in U_{l+}^- \cap P_+^- = [p_1]_+^-$, $l = 1, 2$. However, $[p_1]_+^- \cap [p_2]_+^- = 0$ since we assumed that γ_1, γ_2 are within positive distance from each other. Thus $w = 0$. In particular, L and γ are within positive distance from each other.

Let $\zeta \subset \mathbb{H}^5$ denote the geodesic segment with the endpoints in L, γ, which is orthogonal to both. Then $\zeta \cup \tau(\zeta)$ is a geodesic segment connecting L and $\tau(L)$ and orthogonal to both subspaces. Hence, L and $\tau(L)$ are positive distance apart from each other. $\square$

Suppose that F is a skew-hermitian form on V. Given an embedding $\tau\colon K \to \mathbb{R}$ we define the signature $\mathrm{sig}_\tau(F)$ with respect to the subfield $\tau(F) \subset \mathbb{R}$. Note that the notion of positivity and negativity in V (and, hence, the signature) depends on the embedding τ.

At last, we are ready to define the class of "quaternionic" arithmetic lattices $G \subset O(n, 1)$. Let K be totally real, D be a quaternion algebra over K and V be an $n+1$–dimensional module over D. We assume that F is a nondegenerate hermitian form on V satisfying the following:

(1) F has the signature $(n, 1)$;

(2) For every nontrivial embedding $\tau\colon K \to \mathbb{R}$, the signature $\mathrm{sig}_\tau(F)$ is $(n+1, 0)$ (or $(0, n+1)$).

Next, we need a notion of an "integer" automorphism of V. Let $O \subset D$ be an *order*, ie, a lattice in D regarded as a vector space over K. An example of such order is given by $A^4 \subset D$, where A is the ring of integers of K.

The order O also determines the lattice $O^{n+1} \subset V = D^{n+1}$. We let $\mathrm{GL}(V, O)$ denote the group of automorphisms of the D–module V which preserve the lattice O^{n+1}. If we regard automorphisms of V as "matrices" with coefficients in D, then the elements of $\mathrm{GL}(V, O)$ are "matrices" with coefficients in O which admit inverses with the same property.

We now fix an order $O \subset D$. Then every subgroup Γ of $U(V, F)$ commensurable to the intersection $U(V, F) \cap \mathrm{GL}(V, O)$ is *an arithmetic group of quaternionic type*. The embedding $U(V, F) \hookrightarrow O(n, 1)$ (induced by the identity embedding $K \hookrightarrow \mathbb{R}$) realizes Γ as a lattice in $O(n, 1)$.

Theorem 6.7 [30] *Except for $n = 3, 7$, every arithmetic subgroup of $O(n,1)$ appears as one of the groups Γ as above. In the case $n = 7$, there is an extra class of arithmetic groups associated with octaves rather than quaternions. For $n = 3$, there is yet another construction, also of quaternionic origin, which covers all arithmetic groups in this dimension; see Maclachlan and Reid [33] for the detailed description.*

7 Proof of noncoherence of arithmetic groups of quaternionic origin

Let $G_0 \subset O(3,1)$ be an arithmetic lattice. By Agol's solution of the virtual fibration conjecture, there exists a finite index (torsion-free) subgroup $G_0' \subset G_0$ so that the manifold $M^3 = \mathbb{H}^3/G_0'$ fibers over the circle. Let $F_0 \lhd G_0'$ denote the normal surface subgroup corresponding to the fundamental group of the surface fiber in this fibration. Pick two nonconjugate maximal cyclic subgroups G_{e_i}', $i = 1,2$ in G_0' so that $G_{e_i}' \cap F_0 = \{1\}$. In particular, subgroups generated by F_0 and G_{e_i}' have finite index in G_0.

Since we know that G_0 is LERF and the subgroups G_{e_i}', being cyclic, are quasiconvex, it follows from Proposition 2.7 that the pair $\{G_{e_1}', G_{e_2}'\}$ is relatively separable in G_0'.

We let γ_i, $i = 1,2$ denote the invariant geodesic of G_{e_i}'. Let $\mathbf{R}_0 := d(\gamma_1, \gamma_2)$. We will assume, as in the beginning of Theorem 5.3, that $\mathbf{R}_0$ is the distance between the projections of γ_1, γ_2 to the manifold M^3.

Since the pair of subgroups $\{G_{e_1}', G_{e_2}'\}$ is relatively separable in G_0', given a number $\mathbf{R}_1$, there exists a finite-index subgroup $G_0'' \subset G_0'$ so that the following holds.

For each $\gamma = \gamma_i$, $i = 1,2$ and geodesics $\beta_1, \beta_2 \in G_0'' \cdot (\gamma_1 \cup \gamma_2)$ so that $d(\beta_j, \gamma) = \mathbf{R}_0$ ($j = 1,2$), it follows that

$$d(\mathrm{proj}_\gamma(\beta_1), \mathrm{proj}_\gamma(\beta_2)) \geq \mathbf{R}_1.$$

Set $G_{e_i}'' := G_{e_1}' \cap G_0''$, $i = 1,2$. Without loss of generality, we may assume that G_{e_1}'', G_{e_2}'' generate a free subgroup G_0''. Using relative separability of $\{G_{e_1}', G_{e_2}'\}$ again, given a number $\mathbf{R}_2$, one can find a finite-index subgroup $G_0''' \subset G_0''$ so that the following holds.

For all distinct geodesics β, γ in the G_0'''–orbit of γ_1, γ_2, the distance $d(\beta, \gamma)$ is at least $\mathbf{R}_2$ unless there exists $g \in G_0'''$ which carries $\beta \cup \gamma$ to $\gamma_1 \cup \gamma_2$.

Therefore, Assumption 5.1 (from Theorem 5.3) is satisfied by the group G_0''' and its subgroups $G_{e_i} \cap G_0'''$, $i = 1,2$, with respect to the numbers $\mathbf{R}_1$ and $\mathbf{R}_2$. Assumption 5.1

continues to hold if we replace the lattice G_0''' and its cyclic subgroups $G_{e_i} \cap G_0'''$ by smaller finite index subgroups. Therefore, we do the following: Consider the homomorphism $\psi\colon G_0''' \to \mathbb{Z} = G_0'''/F_1$, $F_1 := (F_0 \cap G_0''')$. We have that the intersection $Z := \psi(G_{e_1} \cap G_0''') \cap \psi(G_{e_2} \cap G_0''')$ is an infinite cyclic group. Now, set

$$G_1 := \psi^{-1}(Z), \quad G_{\epsilon_i} = \langle t_i \rangle := G_{e_i} \cap \psi^{-1}(Z), \quad i = 1, 2.$$

Then $F_1 \lhd G_1$ is a normal surface subgroup with cyclic quotient and

$$G_1 = \langle F_1, G_{\epsilon_i} \rangle, \quad i = 1, 2.$$

Below we explain how to choose numbers $\mathbf{R}_1$ and $\mathbf{R}_2$.

We let S_i, $i = 1, 2$, be orthogonal 3–dimensional K–rational subspaces in $\mathbb{H}^5$ intersecting L_1 along the geodesics γ_1, γ_2; let $L_2 := \sigma_s(L_1)$, $L_4 := \sigma_1(L_1)$. Since S_1, S_2 are K–rational, the involutions σ_1, σ_2 belong to the commensurator of the lattice Γ. Since the group generated by σ_1, σ_2 is finite, without loss of generality we may assume that these involutions normalize Γ (otherwise, we first pass to a finite-index subgroup in Γ).

Then $r > 0$ is the distance $d(L_2, L_4)$. Let α_1, α_2 denote the angles $\angle(L_1, L_4)$, $\angle(L_1, L_2)$ and $\alpha := \min(\alpha_1, \alpha_2)$. In view of Lemma 4.1, we will use $R_0 \ge R_*$ so that for every $g \in G_1$,

$$d(g(L_i), L_j) \ge r, \quad i, j \in \{2, 4\}, \quad g(L_i) \ne L_j.$$

Lastly, we set $\mathbf{R}_1 := R_1(\alpha, R_0, r)$ and $\mathbf{R}_2 := R_2(\alpha, R_0, r)$, where R_1, R_2 are the functions defined in Section 4.

As in Section 5, we define a quadrilateral $\mathcal{Q}$ of groups with vertex groups isomorphic to G_1, so that these isomorphisms send $G_{\epsilon_1}, G_{\epsilon_4}$ to edge groups. Let $G := \pi_1(\mathcal{Q})$. Using the involutions σ_1, σ_2 as in Section 5, we construct a discrete and faithful representation $\phi\colon G \to O(n, 1)$. Since $G_1 \subset \Gamma$ and σ_1, σ_2 normalize Γ, the image of this representation is contained in Γ. (This provides yet another proof of discreteness of $\phi(G)$, however we still have to use Theorem 5.3 in order to conclude ϕ is faithful.)

In order to show incoherence of Γ it suffices to prove the following.

Lemma 7.1 *The group G is noncoherent.*

Proof Let $\epsilon_1, \dots, \epsilon_4$ denote the edges of Q and G_{ϵ_i}, $i = 1, \dots, 4$ denote the corresponding edge groups. Recall that $F_1 \lhd G_1$ is a normal surface subgroup. Let F_i

denote the normal surface subgroups of G_i, $i = 2, 3, 4$, which are the images of F_1 under the isomorphisms $G_1 \to G_i$. Then

$$F_i \cap G_{\epsilon_{i-1}} = F_i \cap G_{\epsilon_i} = \{1\};$$

here and in what follows i is taken mod 4. Let F denote the subgroup of G generated by $F_1, \dots, F_4$. Clearly, this group is finitely generated. We will show that F is not finitely presented by proving that $F \cong F_+ *_N F_-$, where $F_\pm$ are finitely generated (actually, finitely presented) and N is a free group of infinite rank.

We first describe G as an amalgamated free product: We cut the quadrilateral Q in half so that one half contains the vertices $1, 2$, while the other half contains the vertices $3, 4$. Accordingly, set

$$G_- := \langle G_1, G_2 \rangle \cong G_1 *_{G_{\epsilon_2}} G_2, \quad G_+ := \langle G_3, G_4 \rangle \cong G_3 *_{G_{\epsilon_4}} G_4,$$
$$E := \langle G_{\epsilon_1}, G_{\epsilon_3} \rangle \cong G_{\epsilon_1} * G_{\epsilon_3} \cong \mathbb{Z} * \mathbb{Z},$$

the latter follows from Lemma 5.2. Then

$$G \cong G_- *_E G_+.$$

Similarly, we set

$$F_+ := F \cap G_+ \cong F_1 * F_2, \quad F_- := F \cap G_- \cong F_3 * F_4.$$

Since G_1 is generated by t_2 and F_1, the group G_2 is generated by t_2 and F_2. Hence, F_- is normal in G_- and $G_-/F_- \cong \mathbb{Z}$. Moreover, F_- has trivial intersection with G_{ϵ_1} (since F_1 does). It is immediate that $N := F_- \cap E$ is an infinite index subgroup of E. Let us verify that N is nontrivial. Indeed, if $N = \{1\}$, then the group E projects injectively to $G_-/F_- \cong \mathbb{Z}$. This contradicts the fact that E is free of rank 2. Thus, N is nontrivial.

Since E is free of rank 2, it follows that N is a free group of infinite rank. Clearly, $N = F_+ \cap E$ and we obtain

$$N \cong F_+ *_N F_-.$$

Therefore, F is finitely generated and infinitely presented since (by considering the Mayer–Vietoris sequence associated with the amalgam $N \cong F_+ *_N F_-$) the homology group $H_2(F, \mathbb{Z})$ has infinite rank. Thus, G is noncoherent. □

8 Complex-hyperbolic and quaternionic lattices

It is an important open problem in theory of lattices in rank 1 Lie groups $O(n,1)$ and $SU(n,1)$ if a lattice has positive virtual first Betti number, ie, contains a finite-index subgroup with infinite abelianization. In this section we relate this problem to noncoherence in the case of $SU(n,1)$. It was proved by Kazhdan [29] (see also Wallach [45]) that arithmetic lattices of the *simplest type* (or, first type) in $SU(n,1)$ admit finite index (congruence) subgroups with infinite abelianization. Certain classes of nonarithmetic lattices in $SU(2,1)$ (the ones violating integrality condition for arithmetic groups) are proved to have positive virtual first Betti number by Yeung [50].

On the other hand Rogawski [38] proved that for arithmetic lattices Γ in $SU(n,1)$ of *second type* (associated with division algebras), every congruence subgroup $\Gamma' \subset \Gamma$ has finite abelianization. It is unknown if noncongruence subgroups in such lattices (if they exist at all!) can have infinite abelianization. Our noncoherence results say nothing about this class of lattices, although we find it very unlikely that they could be coherent.

Below is the description of arithmetic lattices of the simplest type in $SU(n,1)$ following McReynolds [34] and Stover [41]. Let K be a totally real number field; take a totally imaginary quadratic extension L/K and let $\mathcal{O}_L$ be the ring of integers of L. Let $\sigma_1, \sigma_2, \dots, \sigma_k \colon L \to \mathbb{C}$ be the embeddings. Next, take a hermitian quadratic form in $n+1$ variables

$$\varphi(z,\overline{z}) = \sum_{p,q=1}^{n+1} a_{pq} z_p \overline{z}_q$$

with coefficients in L. We require $\varphi^{\sigma_1}, \varphi^{\bar{\sigma}_1}$ to have signature $(n,1)$ and require the forms φ^{σ_j} to have signature $(n+1,0)$ for the rest of the embeddings σ_j. Let $SU(\varphi)$ denote the group of special unitary automorphisms of the form φ on L^{n+1}. The embedding σ_1 defines a homomorphism $SU(\varphi) \to SU(n,1)$ with relatively compact kernel. We will identify L with $\sigma_1(L)$, so $\sigma_1 = \mathrm{Id}$.

Definition 8.1 A subgroup Γ of $SU(n,1)$ is said to be an arithmetic lattice of the *simplest type* if it is commensurable to $SU(\varphi, \mathcal{O}_L) = SU(\varphi) \cap \mathrm{SL}(n+1, \mathcal{O}_L)$.

By diagonalizing the form φ, we see that L^{n+1} contains a 3-dimensional subspace L^3 so that the restriction of φ to L^3 is a form of the signature $(2,1)$. Therefore, an arithmetic lattice $\Gamma \subset SU(n,1)$ of the simplest type intersects $SU(\varphi|L^3)$ along a lattice Γ' of the simplest type (regarded as a subgroup of $SU(2,1)$). If Γ' is noncoherent, so is Γ.

Therefore, we restrict our discussion to the case of isometries of the complex-hyperbolic plane $\mathbb{CH}^2$.

Suppose that $\Gamma \subset SU(2,1)$ is a torsion-free uniform lattice with infinite abelianization. Therefore, $b_1(M) > 0$, where $M = \mathbb{CH}^2/\Gamma$, Since M is Kähler, its Betti numbers are even; therefore, there exists an epimorphism $\psi\colon \Gamma \to \mathbb{Z}^2$. There are two cases to consider.

Case 1: $\mathrm{Ker}(\psi)$ **is not finitely generated** Then, according to Delzant [13], there exists a holomorphic fibration $M \to R$ with connected fibers, where R is a hyperbolic Riemann surface–orbifold. It was proved in [25] that the kernel K of the homomorphism $\Gamma \to \pi_1(R)$ is finitely generated but not finitely presented. Hence, Γ is noncoherent in this case.

Remark 8.2 Jonathan Hillman [21] suggested an alternative proof that K is not finitely presented. Namely, if K is of type FP_2 (eg, is finitely presented) then it is a $PD(2)$–group (see Theorem 3.1) and, hence, a surface group. It was proved by Hillman in [22] that the holomorphic fibration $M \to R$ has no singular fibers. Such fibrations cannot exist due to a result of Liu [31]. Thus, K is not finitely presented.

Case 2: $F = \mathrm{Ker}(\psi)$ **is finitely generated** If Γ were coherent, F would be also finitely presented. It is proved by Jonathan Hillman (Theorem 3.1) that F has to be a surface group. We obtain the associated homomorphism $\eta\colon \mathbb{Z}^2 \to \mathrm{Out}(F)$ (the mapping class group of a surface). Since Γ contains no rank 2 abelian subgroups, η is injective. Rank 2 abelian subgroups of the mapping class group have to contain nontrivial reducible elements; see Birman, Lubotzky and McCarthy [8]. Let $\gamma \in \mathbb{Z}^2 \setminus \{1\}$ be such that $\eta(\gamma)$ is a reducible element of the mapping class group. Hence, $\eta(\gamma)$ fixes a conjugacy class of some $\alpha \in F \setminus \{1\}$. It follows that Γ contains $\mathbb{Z}^2$ (generated by a lift of γ to Γ and by α). This is a contradiction.

We thus obtain the following.

Theorem 8.3 *Suppose that $\Gamma \subset SU(2,1)$ is a cocompact arithmetic group with infinite abelianization. Then Γ is noncoherent.*

Corollary 8.4 *Suppose that $\Gamma \subset SU(n,1)$ is a cocompact arithmetic group of the simplest type, where $n \geq 2$. Then Γ is noncoherent.*

We now consider quaternionic-hyperbolic lattices. Recall that all lattices in $\mathbf{H}\mathbb{H}^n$, $n \geq 2$, are arithmetic according to Corlette [11] and Gromov and Schoen [16]. On the other hand, $\mathrm{Isom}(\mathbf{H}\mathbb{H}^1) \cong \mathrm{Isom}(\mathbb{H}^4)$ and, hence, this group contains nonarithmetic lattices as well.

Proposition 8.5 *Every arithmetic lattice in $\mathbf{H}\mathbb{H}^n$ is noncoherent.*

Proof According to Platonov and Rapinchuk [35], we have that all arithmetic lattices in $\mathrm{Isom}(\mathbf{H}\mathbb{H}^n) \cong \mathrm{Sp}(n,1)$ have the following form.

Let $K \subset \mathbb{R}$ be a totally real number field, D be a central quaternion algebra over K, V be an $n+1$–dimensional right D–module, F be a hermitian bilinear form on V (see Section 6). Choose a basis where F is diagonal:

$$F(x,y) = \sum_{m=1}^{n+1} \bar{x}_m a_m y_m,$$

$a_m = \bar{a}_m$. Then the signature of F is (p,q) if (after permuting the coordinates) $a_m > 0$, $m = 1, \dots, p$ and $a_m < 0$, $m = p+1, \dots, n+1 = p+q$. Let $U(V,F)$ be the group of unitary transformations of (V,F).

Given an embedding $\sigma\colon K \to \mathbb{R}$, we define a new form F^σ by applying σ to the coefficients of F. We now require F, D and K to be such that

(1) F has signature $(n,1)$ and F^σ is definite for all embeddings σ different from the identity;
(2) the completions of D with respect to all the embeddings $\sigma\colon K \to \mathbb{R}$ are isomorphic to Hamilton's quaternions $\mathbf{H}$ (ie, are division algebras).

In particular, the embedding $D \to \mathbf{H}$, induced by the identity embedding $K \hookrightarrow \mathbb{R}$, gives rise to a homomorphism $\eta\colon U(V,F) \to \mathrm{Sp}(n,1) = \mathrm{Isom}(\mathbf{H}\mathbb{H}^n)$.

Let O be an order in D and set $\Gamma_{V,O} := U(V,F) \cap \mathrm{SL}(V,O)$. Lastly, a group commensurable to $\eta(\Gamma_{V,O}) \subset \mathrm{Sp}(n,1)$ is called an *arithmetic* lattice in $\mathrm{Sp}(n,1)$. Note that the kernel of the homomorphism

$$\eta\colon \Gamma_{V,O} \to \mathrm{Sp}(n,1)$$

is finite. Hence, very arithmetic lattice in $\mathrm{Sp}(n,1)$ is abstractly commensurable to $\Gamma_{V,O}$ for some choice of K, D and O.

By restricting the form F to the 2–dimensional submodule W in V spanned by the first and last basis vectors, we obtain a hermitian form of signature $(1,1)$. Therefore, every arithmetic lattice Γ in $\mathrm{Sp}(n,1)$ will contain a subgroup commensurable to $\eta(\Gamma_{W,O})$. The latter is an arithmetic lattice in $\mathbb{H}^4$ and, hence, is noncoherent according to [28] and [3]. Thus, Γ is incoherent as well. □

The same argument applies to lattices Γ in the isometry group of the hyperbolic plane over Cayley octaves $\mathrm{Isom}(\mathbf{O}\mathbb{H}^2)$, as every such lattice is arithmetic and contains an arithmetic sublattice $\Gamma' \subset \mathrm{Isom}(\mathbf{O}\mathbb{H}^1) \cong \mathrm{Isom}(\mathbb{H}^8)$. (I owe this remark to Andrei Rapinchuk.) Since Γ' is noncoherent, so is Γ.

References

[1] **I Agol**, *Systoles of hyperbolic* 4*–manifolds* `arXiv:math.GT/0612290`

[2] **I Agol**, *The virtual Haken conjecture* `arXiv:1204.2810`

[3] **I Agol**, *Criteria for virtual fibering*, J. Topol. 1 (2008) 269–284 MR2399130

[4] **I Agol**, **D D Long**, **A W Reid**, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. 153 (2001) 599–621 MR1836283

[5] **G Baumslag**, **J E Roseblade**, *Subgroups of direct products of free groups*, J. London Math. Soc. 30 (1984) 44–52 MR760871

[6] **M Bestvina**, **N Brady**, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997) 445–470 MR1465330

[7] **R Bieri**, *Homological dimension of discrete groups*, 2nd edition, Queen Mary College Mathematical Notes, Queen Mary College Department of Pure Mathematics, London (1981) MR715779

[8] **J S Birman**, **A Lubotzky**, **J McCarthy**, *Abelian and solvable subgroups of the mapping class groups*, Duke Math. J. 50 (1983) 1107–1120 MR726319

[9] **B H Bowditch**, **G Mess**, *A* 4*–dimensional Kleinian group*, Trans. Amer. Math. Soc. 344 (1994) 391–405 MR1240944

[10] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer, Berlin (1999) MR1744486

[11] **K Corlette**, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. 135 (1992) 165–182 MR1147961

[12] **P Deligne**, **G D Mostow**, *Commensurabilities among lattices in* $PU(1, n)$, Annals of Mathematics Studies 132, Princeton Univ. Press (1993) MR1241644

[13] **T Delzant**, *L'invariant de Bieri–Neumann–Strebel des groupes fondamentaux des variétés kählériennes*, Math. Ann. 348 (2010) 119–125 MR2657436

[14] **R Gitik**, *Doubles of groups and hyperbolic LERF* 3*–manifolds*, Ann. of Math. 150 (1999) 775–806 MR1740992

[15] **R Gitik**, *Ping-pong on negatively curved groups*, J. Algebra 217 (1999) 65–72 MR1700476

[16] **M Gromov**, **R Schoen**, *Harmonic maps into singular spaces and* p*–adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. (1992) 165–246 MR1215595

[17] **F J Grunewald**, *On some groups which cannot be finitely presented*, J. London Math. Soc. 17 (1978) 427–436 MR500627

[18] **F Haglund**, *Finite index subgroups of graph products*, Geom. Dedicata 135 (2008) 167–209 MR2413337

[19] **F Haglund**, **J Świątkowski**, *Separating quasi-convex subgroups in* 7*–systolic groups*, Groups Geom. Dyn. 2 (2008) 223–244 MR2393180

[20] **M Hall, Jr**, *Coset representations in free groups*, Trans. Amer. Math. Soc. 67 (1949) 421–432 MR0032642

[21] **J A Hillman**, *Private communication* (1998)

[22] **J A Hillman**, *Complex surfaces which are fibre bundles*, Topology Appl. 100 (2000) 187–191 MR1733043

[23] **J A Hillman**, *Four-manifolds, geometries and knots*, Geometry & Topology Monographs 5 (2002) MR1943724

[24] **J A Hillman**, **D H Kochloukova**, *Finiteness conditions and* PD_r*–group covers of* PD_n*–complexes*, Math. Z. 256 (2007) 45–56 MR2282258

[25] **M Kapovich**, *On normal subgroups in the fundamental groups of complex surfaces* (1998) `arXiv:1204.2810`

[26] **M Kapovich**, *Representations of polygons of finite groups*, Geom. Topol. 9 (2005) 1915–1951 (electronic) MR2175160

[27] **M Kapovich**, **L Potyagailo**, *On the absence of Ahlfors' finiteness theorem for Kleinian groups in dimension three*, Topology Appl. 40 (1991) 83–91 MR1114093

[28] **M Kapovich**, **L Potyagailo**, **E Vinberg**, *Noncoherence of some lattices in* $\mathrm{Isom}(\mathbb{H}^n)$, from: "The Zieschang Gedenkschrift", (M Boileau, M Scharlemann, R Weidmann, editors), Geom. Topol. Monogr. 14, Geom. Topol. Publ., Coventry (2008) 335–351 MR2484708

[29] **D Kazhdan**, *Some applications of the Weil representation*, J. Analyse Mat. 32 (1977) 235–248 MR0492089

[30] **J-S Li**, **J J Millson**, *On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group*, Duke Math. J. 71 (1993) 365–401 MR1233441

[31] **K Liu**, *Geometric height inequalities*, Math. Res. Lett. 3 (1996) 693–702 MR1418581

[32] **D D Long**, **G A Niblo**, *Subgroup separability and* 3*–manifold groups*, Math. Z. 207 (1991) 209–215 MR1109662

[33] **C Maclachlan**, **A W Reid**, *The arithmetic of hyperbolic* 3*–manifolds*, Graduate Texts in Mathematics 219, Springer, New York (2003) MR1937957

[34] **D B McReynolds**, *Arithmetic lattices in* $\mathrm{SU}(n,1)$ (2007) Available at `http://www.ma.utexas.edu/dmcreyn/downloads.html`

[35] **V Platonov**, **A Rapinchuk**, *Algebraic groups and number theory*, Pure and Applied Mathematics 139, Academic Press, Boston, MA (1994) MR1278263

[36] **L Potyagaĭlo**, *The problem of finiteness for Kleinian groups in* 3*–space*, from: "Knots 90", (A Kawauchi, editor), de Gruyter, Berlin (1992) 619–623 MR1177449

[37] **L Potyagaĭlo**, *Finitely generated Kleinian groups in 3–space and 3–manifolds of infinite homotopy type*, Trans. Amer. Math. Soc. 344 (1994) 57–77 MR1250823

[38] **J D Rogawski**, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies 123, Princeton Univ. Press (1990) MR1081540

[39] **G P Scott**, *Compact submanifolds of 3–manifolds*, J. London Math. Soc. (2) 7 (1973) 246–250 MR0326737

[40] **G P Scott**, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) 17 (1978) 555–565 MR0494062

[41] **M Stover**, *Property (FA) and lattices in $SU(2,1)$*, Internat. J. Algebra Comput. 17 (2007) 1335–1347 MR2372600

[42] **È B Vinberg**, *Discrete groups generated by reflections in Lobačevskiĭ spaces*, Mat. Sb. 72 (114) (1967) 471–488; correction, ibid. 73 (115), 303 MR0207853

[43] **È B Vinberg**, **O V Shvartsman**, *Discrete groups of motions of spaces of constant curvature*, from: "Geometry, II", Encyclopaedia Math. Sci. 29, Springer, Berlin (1993) 139–248 MR1254933

[44] **C Wall**, *List of problems*, from: "Homological group theory", (C T C Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 369–394 MR564438

[45] **N R Wallach**, *Square integrable automorphic forms and cohomology of arithmetic quotients of* $\mathrm{SU}(p,q)$, Math. Ann. 266 (1984) 261–278 MR730169

[46] **D T Wise**, *Subgroup separability of graphs of free groups with cyclic edge groups*, Q. J. Math. 51 (2000) 107–129 MR1760573

[47] **D T Wise**, *Subgroup separability of the figure 8 knot group*, Topology 45 (2006) 421–463 MR2218750

[48] **D T Wise**, *Morse theory, random subgraphs, and incoherent groups*, Bull. Lond. Math. Soc. 43 (2011) 840–848 MR2854555

[49] **D T Wise**, *The structure of groups with a quasiconvex hierarchy* (2012)

[50] **S-K Yeung**, *Virtual first Betti number and integrality of compact complex two-ball quotients*, Int. Math. Res. Not. 2004 (2004) 1967–1988 MR2063566

Department of Mathematics, University of California
Davis, CA 95616, USA

kapovich@math.ucdavis.edu

http://www.math.ucdavis.edu/~kapovich

Proposed: Martin Bridson Received: 7 September 2011
Seconded: Walter Neumann, Leonid Polterovich Revised: 2 September 2012