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Poincaré invariants are Seiberg–Witten invariants

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We prove a conjecture of Dürr, Kabanov and Okonek that provides an algebro-geometric theory of Seiberg–Witten invariants for all smooth projective surfaces. Our main technique is the cosection localization principle (Kiem and Li [8]) of virtual cycles.

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1 Introduction

Recently there has been a renewed interest in Donaldson invariants and Seiberg–Witten invariants due to the influx of virtual intersection theory. See Mochizuki [15], Göttsche, Nakajima and Yoshioka [6], and [4], for instance. The purpose of this paper is to prove a conjecture (Theorem 1.1 below) of Dürr, Kabanov and Okonek in [4], which provides a natural algebro-geometric theory of Seiberg–Witten invariants. Our main technique is the cosection localization principle in Kiem and Li [8] that effectively localizes the virtual cycle when there is a cosection of the obstruction sheaf.

In the mid-1980s, Donaldson defined his famous invariants as intersection numbers on the Uhlenbeck compactification of the space of anti-self-dual (ASD) connections on a fixed hermitian vector bundle of rank 2 on a compact oriented 4–manifold X [2]. Because of the difficulty in calculating Donaldson invariants, an algebro-geometric theory of Donaldson invariants was highly anticipated from the beginning. Donaldson proved that when the 4–manifold X is an algebraic surface over $\mathbb{C}$, there is a diffeomorphism between the space of irreducible ASD connections and an open subset of the moduli space of Gieseker semistable sheaves of rank 2 and given Chern classes. In 1991, J Li [12] and Morgan [16] extended Donaldson's diffeomorphism to a continuous map from the Gieseker moduli space of semistable sheaves to the Uhlenbeck compactification and proved that Dondalson invariants are intersection numbers on the Gieseker moduli space. In fact, J Li furthermore proved that the Uhlenbeck compactification admits a scheme structure and the map from the Gieseker moduli space to the Uhlenbeck compactification is an algebraic morphism. In 1993, Kronheimer and Mrowka proved the celebrated structure theorem that expresses all the Donaldson invariants in terms of a

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finite number of classes $K_1, \dots, K_l \in H^2(X, \mathbb{Z})$ and rational numbers $\alpha_1, \dots, \alpha_l \in \mathbb{Q}$ if X is of simple type [11]. The condition of being a simple type roughly means that the point insertions do not provide new information on X. The mystery of the simple type condition, the basic classes $K_1, \dots, K_l$ and the rational numbers $\alpha_1, \dots, \alpha_l$ was elucidated by the advent of Seiberg–Witten theory in 1994.

A Spin^c–structure on a 4–manifold X refers to a pair of rank 2 hermitian vector bundles $E^{\pm}$ such that $\det E_+ \cong \det E_- =: L$. Taking the first Chern class of L provides us with a bijection from the collection of all Spin^c–structures on X to $H^2(X, \mathbb{Z})$. Seiberg and Witten stated a pair of equations on a pair (A, φ) where A is a connection on L and φ is a section of E_+. The collection of all solutions of Seiberg–Witten equations forms a compact topological space and Seiberg–Witten invariants are defined as intersection numbers on the solution space. In 1994, Witten in [19] conjectured that every Kähler surface X with a nontrivial holomorphic 2–form $\theta \in H^0(K_X)$ is of simple type and that for any Kähler surface X of simple type:

(1) The basic classes $K_1, \dots, K_l$ of Kronheimer and Mrowka satisfy
$$K_i \cdot (K_i - k_X) = 0 \quad \forall i \quad \text{where } k_X = c_1(T_X^*).$$
(2) The Seiberg–Witten invariants $\mathrm{SW}(\gamma)$ are zero if $\gamma \cdot (\gamma - k_X) \neq 0$.

(3) The rational numbers α_i in the structure theorem of Kronheimer and Mrowka are the Seiberg–Witten invariants $\mathrm{SW}(K_i)$ up to a constant, which depends only on $b_1(X), b_2^{\pm}(X)$.

Furthermore, Witten showed by physical means that the calculation of Seiberg–Witten invariants may be localized to a neighborhood of a canonical divisor when $p_g(X) > 0$. (See Witten [19, page 12], and Donaldson [3, page 54].)

When X is a Kähler surface with $b_1(X) = 0$, it was observed by Witten [19, page 18] that the solution space of Seiberg–Witten equations with fixed $\gamma = c_1(L) \in H^2(X, \mathbb{Z})$ is a projective space $\mathbb{P} H^0(X, L)$ and a theorem of Friedman and Morgan [5, Theorem 3.1] shows that the Seiberg–Witten invariants in this case are the integrals of cohomology classes multiplied by the Euler class of a certain vector bundle. Hence in the special case of $b_1(X) = 0$, we have an algebro-geometric theory of Seiberg–Witten invariants. Using this, T Mochizuki in [15] proved a formula that expresses the Donaldson invariants in terms of the Seiberg–Witten invariants of surfaces with $b_1(X) = 0$. Subsequently in [6], Göttsche, Nakajima and Yoshioka proved that Mochizuki's formula implies Witten's conjecture for algebraic surfaces with $b_1(X) = 0$. However this beautiful story could not be generalized to the case where $b_1(X) > 0$ because we still lack in an algebro-geometric definition of Seiberg–Witten invariants. Moreover, the proofs of Mochizuki and Göttsche, Nakajima and Yoshioka do not seem to explain the localization behavior of Seiberg–Witten invariants to a canonical divisor.

In 2007, Dürr, Kabanov and Okonek proved in [4] that if X is a smooth projective surface and $\gamma \in H_2(X, \mathbb{Z})$, the Hilbert scheme Hilb_X^γ of divisors D on X whose homology classes are γ admits a perfect obstruction theory and thus we obtain a virtual fundamental class $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}$ by Li and Tian [13], and Behrend and Fantechi [1]. By integrating cohomology classes over $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}$, they defined new invariants of X called the *Poincaré invariants* and conjectured that the Poincaré invariants coincide with the Seiberg–Witten invariants for algebraic surfaces. See Section 3 for more details. Our main result in this paper is that the following conjecture of Dürr, Kabanov and Okonek in [4] is true.

Theorem 1.1 *The Poincaré invariants are the Seiberg–Witten invariants for all smooth projective surfaces.*

This theorem gives us a completely algebro-geometric definition of Seiberg–Witten invariants for all smooth projective surfaces and can be thought of as a natural generalization of [5, Theorem 3.1]. Since Hilb_X^γ parametrizes embedded curves, the Poincaré invariants may be viewed as "algebro-geometric Gromov invariants" and Theorem 1.1 may be considered as an algebraic version of Taubes' Theorem [18]. In fact, combining Taubes' theorem $\mathrm{Gr} = \mathrm{SW}$ and Theorem 1.1, we obtain the following.

Corollary 1.2 *The Poincaré invariants are the Gromov invariants for all smooth projective surfaces.*

So we now have the equivalence of the three invariants:

$$\begin{tikzcd}
\text{Poincare} \arrow[rr, Leftrightarrow, "\text{Theorem 1.1}"] \arrow[dr, Leftrightarrow, "\text{Corollary 1.2}"'] & & \text{Seiberg–Witten} \arrow[dl, Leftrightarrow, "\text{Taubes}"] \\
& \text{Gromov} &
\end{tikzcd}$$

Perhaps one may be able to give a direct proof of Corollary 1.2 by using the machinery of comparing algebraic and symplectic virtual fundamental classes. (See eg [14; 17].) But it looks very difficult with many technical issues to be handled. A direct proof of Corollary 1.2, combined with Taubes' theorem, should give us an alternative proof of Theorem 1.1, but that seems like a gigantic detour through hard analysis, compared to our concise purely algebraic proof below.

The authors of [4] proved deformation invariance, a blow-up formula and wall crossing formulas for the Poincaré invariants and reduced the proof of Theorem 1.1 to the following [4, page 286].

Theorem 1.3 *Let X be a minimal surface of general type. If $p_g(X) > 0$,*

$$\deg[\mathrm{Hilb}_X^{k_X}]^{\mathrm{vir}} = (-1)^{\chi(\mathcal{O}_X)}$$

where k_X is the homology class of a canonical divisor.

The main technique for our proof of Theorem 1.1 and Theorem 1.3 is the cosection localization principle [8], which tells us that if there is a cosection

$$\sigma\colon \mathcal{O}b_M \longrightarrow \mathcal{O}_M$$

of the obstruction sheaf $\mathcal{O}b_M = h^1(E^\vee)$ of a perfect obstruction theory $\phi\colon E \to \mathbb{L}_M$ over a Deligne–Mumford stack M, then the virtual fundamental class of (M, ϕ) localizes to the zero locus of σ.

We apply the principle to $M = \mathrm{Hilb}_X^\gamma$. Let $\theta \in H^0(X, K_X)$ be a nonzero holomorphic 2–form on X whose vanishing locus is denoted by Z. For $D \in \mathrm{Hilb}_X^\gamma$, the obstruction space at D by [4] is $H^1(\mathcal{O}_D(D))$. The connecting homomorphism $H^1(\mathcal{O}_D(D)) \to H^2(\mathcal{O}_X)$ from the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0$$

gives us a homomorphism

$$\mathrm{Ob}_{\mathrm{Hilb}_X^\gamma, D} = H^1(\mathcal{O}_D(D)) \longrightarrow H^2(\mathcal{O}_X) \xrightarrow{\theta} H^2(\mathcal{O}_X(Z)) \cong H^2(K_X) = \mathbb{C}.$$

By relativizing, we obtain a cosection

$$\sigma\colon \mathcal{O}b_{\mathrm{Hilb}_X^\gamma} \longrightarrow \mathcal{O}_{\mathrm{Hilb}_X^\gamma}$$

whose vanishing locus is the closed subscheme $\mathrm{Hilb}_X^\gamma(Z) \subset \mathrm{Hilb}_X^\gamma$ of curves D contained in Z. Therefore the virtual fundamental class is localized to the locus of effective divisors contained in Z and the calculation of the Poincaré invariants takes place within the canonical divisor Z, exactly as Witten told us about localization of Seiberg–Witten invariants mentioned above.

When $\gamma = k_X$, we will see that the vanishing locus of σ, as a scheme, consists of exactly one reduced point Z. Hence the virtual cycle of $\mathrm{Hilb}_X^{k_X}$ is localized to a neighborhood of the point Z. By using the results of Green and Lazarsfeld [7] on deforming cohomology groups of line bundles, we will find local defining equations near canonical divisors and show that there is a canonical divisor Z that is a smooth point of $\mathrm{Hilb}_X^{k_X}$ such that $\dim T_Z \mathrm{Hilb}_X^{k_X}$ has the same parity as $\chi(\mathcal{O}_X)$. By [8, Example 2.4], this implies that the (localized) virtual cycle of $\mathrm{Hilb}_X^{k_X}$ is $(-1)^{\chi(\mathcal{O}_X)}[Z]$ whose degree is precisely $(-1)^{\chi(\mathcal{O}_X)}$. This proves Theorem 1.3.

By Theorem 1.1, Mochizuki's formula in [15, Chapter 7] expresses the Donaldson invariants in terms of the Seiberg–Witten invariants. Therefore one may be able to generalize the arguments of [6] to answer the following interesting question.

Question 1.4 *Does Mochizuki's formula imply Witten's conjecture for all smooth projective surfaces X with $p_g(X) > 0$?*

We hope to get back to this question in the future.

Give a sheaf $\mathscr{F}$ over a scheme $\mathcal{Z}$, a cosection of $\mathscr{F}$ means a homomorphism of sheaves $\eta\colon \mathscr{F} \to \mathcal{O}_{\mathcal{Z}}$. The vanishing locus (also called zero locus), denoted by $\mathrm{zero}(\eta)$, is the maximal subscheme $T \subset \mathcal{Z}$ such that $\eta|_T\colon \mathscr{F}|_T \to \mathcal{O}_T$ vanishes. Thus $\mathrm{zero}(\eta)$ is a closed subscheme of $\mathcal{Z}$.

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2 Localization of virtual cycles by cosections

In this section we collect necessary materials on the cosection localization principle from [8].

Definition 2.1 *Let M be a Deligne–Mumford stack over $\mathbb{C}$. Let $\mathbb{L}_M$ denote the cotangent complex of M. A* perfect obstruction theory *on M is a morphism $\phi\colon E \to \mathbb{L}_M$ in the derived category $D^b(M)$ of bounded complex of coherent sheaves on M such that*

(1) *E is locally isomorphic to a two-term complex of locally free sheaves concentrated at $[-1, 0]$.*
(2) *$h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism.*

The obstruction sheaf *of (M, ϕ) is defined as $\mathcal{O}b_M = h^1(E^\vee)$ where $E^\vee$ denotes the dual of E. A* cosection *of the obstruction sheaf $\mathcal{O}b_M$ is a homomorphism $\mathcal{O}b_M \to \mathcal{O}_M$.*

By the construction in [1; 13], a perfect obstruction theory ϕ on M gives rise to a *virtual fundamental class* $[M]^{\mathrm{vir}}$ and many well-known invariants (such as Gromov–Witten and Donaldson–Thomas invariants) are defined as intersection numbers on the virtual fundamental classes of suitable moduli spaces. The cosection localization principle of [8] is a powerful technique of calculating these virtual intersection numbers.

Theorem 2.2 [8, Theorem 1.1] *Suppose there is a surjective cosection $\sigma\colon \mathcal{O}b_M|_U \to \mathcal{O}_U$ over an open $U \subset M$. Let $M(\sigma) = M - U$. Then the virtual fundamental class localizes to $M(\sigma)$ in the sense that there exists a localized virtual fundamental class*

$$[M]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(M(\sigma)),$$

which enjoys the usual properties of the virtual fundamental classes and such that

$$\imath_*[M]^{\mathrm{vir}}_{\mathrm{loc}} = [M]^{\mathrm{vir}} \in A_*(M) \quad \text{where } \imath\colon M(\sigma) \hookrightarrow M.$$

See [8; 9; 10] for direct applications of Theorem 2.2 to Gromov–Witten invariants of surfaces. From the construction of $[M]^{\mathrm{vir}}_{\mathrm{loc}}$ in [8], the following excision property follows immediately.

Proposition 2.3 *Let W be an open neighborhood of $M(\sigma)$ in M. Then we have*

$$[W]^{\mathrm{vir}}_{\mathrm{loc}} = [M]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(M(\sigma)).$$

The following special case will be useful.

Example 2.4 [8, Example 2.4] *Let M be an n–dimensional smooth scheme and E be a vector bundle of rank n on M. The zero map $0\colon \mathbb{T}_M \to E$ is a perfect obstruction theory with obstruction sheaf E. Let $\sigma\colon E \to \mathcal{O}_M$ be a cosection such that the scheme $\mathrm{zero}(\sigma)$ is a simple point p in M. Then $[M]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^n[p]$.*

3 Poincaré invariants

In this section, we recall the definition of Poincaré invariants from [4] as virtual intersection numbers on the Hilbert scheme Hilb^{γ}_X of divisors on a smooth projective surface X with $p_g(X) > 0$. For any nonzero $\theta \in H^0(X, K_X)$, we construct a cosection $\sigma_\theta\colon \mathcal{O}b_{\mathrm{Hilb}^{\gamma}_X} \to \mathcal{O}_{\mathrm{Hilb}^{\gamma}_X}$ of the obstruction sheaf and show that the vanishing locus of σ is the locus of divisors D contained in the zero locus Z of θ.

3.1 Perfect obstruction theory on Hilbert scheme

In this subsection, we recall the perfect obstruction theory on the Hilbert scheme $\mathrm{Hilb}^{\gamma}_{X}$ of divisors on X and the Poincaré invariants from [4].

Let $\mathcal{X} \to S$ be a flat projective morphism of relative dimension 2 and $\gamma \in H_2(\mathcal{X}, \mathbb{Z})$. Let $\mathrm{Hilb}^{\gamma}_{\mathcal{X}/S}$ be the relative Hilbert scheme parametrizing Cartier divisors D of fibers of $\mathcal{X} \to S$ with $[D] = \gamma \in H_2(\mathcal{X}, \mathbb{Z})$. Let

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[dr] & \mathrm{Hilb}^{\gamma}_{\mathcal{X}/S} \times_S \mathcal{X} \arrow[r] \arrow[d, "\pi"] & \mathcal{X} \arrow[d] \\ & \mathrm{Hilb}^{\gamma}_{\mathcal{X}/S} \arrow[r] & S \end{tikzcd}$$

be the universal family. Let $\mathcal{H} := \mathrm{Hilb}^{\gamma}_{\mathcal{X}/S}$ for simplicity. The triple $(\mathcal{D}, \mathcal{H} \times_S \mathcal{X}, \mathcal{X})$ induces a distinguished triangle

$$\mathbb{T}_{\mathcal{D}/(\mathcal{H}\times_S\mathcal{X})} \to \mathbb{T}_{\mathcal{D}/\mathcal{X}} \to \mathbb{T}_{(\mathcal{H}\times_S\mathcal{X})/\mathcal{X}} \xrightarrow{\delta} \mathbb{T}_{\mathcal{D}/\mathcal{H}\times_S\mathcal{X}}[1]$$

of the tangent complexes, which are by definition the duals of the cotangent complexes. Using $\mathbb{T}_{(\mathcal{H}\times_S\mathcal{X})/\mathcal{X}} \cong \pi^*\mathbb{T}_{\mathcal{H}/S}$ and $\mathbb{T}_{\mathcal{D}/\mathcal{H}\times_S\mathcal{X}}[1] \cong \mathcal{O}_{\mathcal{D}}(\mathcal{D})$ (because $\mathcal{D} \subset \mathcal{H} \times_S \mathcal{X}$ is a divisor), the morphism δ induces

$$\mathbb{T}_{\mathcal{H}/S} \longrightarrow R\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D}).$$

Taking its dual, one obtains

$$\phi\colon (R\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D}))^{\vee} \longrightarrow \mathbb{L}_{\mathcal{H}/S}. \tag{3-1}$$

The following is proved in [4, Theorem 1.7] and [4, Theorem 1.11].

Theorem 3.1 *ϕ is a relative perfect obstruction theory for $\mathcal{H} = \mathrm{Hilb}^{\gamma}_{\mathcal{X}/S} \to S$ in the sense of [1].*

When $S = \operatorname{Spec}\mathbb{C}$ and $\iota\colon \mathcal{D} \hookrightarrow \mathrm{Hilb}^{\gamma}_{X} \times X$ is a Cartier divisor (eg when X is smooth), we have a perfect obstruction theory on $\mathrm{Hilb}^{\gamma}_{X}$ whose obstruction sheaf is

$$\mathcal{O}b_{\mathrm{Hilb}^{\gamma}_{X}} = R^1\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D})$$

and by [1; 13] we obtain a virtual fundamental class $[\mathrm{Hilb}^{\gamma}_{X}]^{\mathrm{vir}}$ if X is projective. We will see below if there is a nonzero section $\theta \in H^0(K_X)$ for smooth X, there is a cosection

$$\sigma_\theta\colon \mathcal{O}b_{\mathrm{Hilb}^{\gamma}_{X}} \longrightarrow \mathcal{O}_{\mathrm{Hilb}^{\gamma}_{X}},$$

which enables us to define the localized virtual fundamental class $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ supported on the zero locus Z of θ.

Let X be a smooth projective surface. Then it is easy to find that the virtual dimension of Hilb_X^γ is precisely $\gamma\cdot(\gamma-k_X)$ where $k_X=c_1(K_X)$. The Poincaré invariants for X are now defined as intersection numbers on $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}$ but the precise definition is not necessary in this paper. See [4, Section 0] for the precise definition.

It was conjectured in [4] that the Poincaré invariants for X coincide with the Seiberg–Witten invariants. Furthermore, the authors of [4, page 286] proved that the conjecture follows if

$$\deg[\mathrm{Hilb}_X^{k_X}]^{\mathrm{vir}}=(-1)^{\chi(\mathcal{O}_X)}$$

for minimal surfaces of general type with $p_g>0$ (Theorem 1.3).

3.2 Cosection of the obstruction sheaf

Suppose $p_g(X)>0$ and fix a nonzero holomorphic 2–form $\theta\in H^0(X,K_X)$ on X whose vanishing locus is denoted by Z so that $\mathcal{O}_X(Z)\cong K_X$.

For $D\in\mathrm{Hilb}_X^\gamma$, the obstruction space at D with respect to the perfect obstruction theory in [4] is $H^1(\mathcal{O}_D(D))$. From the short exact sequence

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_X(D)\longrightarrow\mathcal{O}_D(D)\longrightarrow 0 \tag{3-2}$$

we obtain a connecting homomorphism $H^1(\mathcal{O}_D(D))\to H^2(\mathcal{O}_X)$. Upon composing with the multiplication by θ, we obtain a homomorphism

$$\sigma_\theta\colon \mathrm{Ob}_{\mathrm{Hilb}_X^\gamma,D}=H^1(\mathcal{O}_D(D))\longrightarrow H^2(\mathcal{O}_X)\xrightarrow{\theta}H^2(\mathcal{O}_X(Z))=H^2(K_X)\cong\mathbb{C}. \tag{3-3}$$

This construction can be lifted to a cosection $\sigma\colon \mathcal{O}b_{\mathrm{Hilb}_X^\gamma}\to\mathcal{O}_{\mathrm{Hilb}_X^\gamma}$. Let

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[dr] & \mathrm{Hilb}_X^\gamma\times X \arrow[d, "\pi"] \\ & \mathrm{Hilb}_X^\gamma \end{tikzcd}$$

be the universal family and let

$$0\longrightarrow\mathcal{O}_{\mathrm{Hilb}_X^\gamma\times X}\longrightarrow\mathcal{O}_{\mathrm{Hilb}_X^\gamma\times X}(\mathcal{D})\longrightarrow\mathcal{O}_{\mathcal{D}}(\mathcal{D})\longrightarrow 0 \tag{3-4}$$

be the short exact sequence. By (3-1) the obstruction sheaf of Hilb_X^γ is

$$\mathcal{O}b_{\mathrm{Hilb}_X^\gamma}=R^1\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D}).$$

From (3-4), we obtain a homomorphism

$$R^1\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D}) \longrightarrow R^2\pi_*\mathcal{O}_{\mathrm{Hilb}^{\gamma}_X \times X}$$

and by composing it with the multiplication by the pullback $p_X^*\theta$ of θ via the projection $p_X\colon \mathrm{Hilb}^{\gamma}_X \times X \to X$ we obtain the cosection:

$$\text{(3-5)}\quad \sigma_\theta\colon \mathcal{O}b_{\mathrm{Hilb}^{\gamma}_X} \cong R^1\pi_*\mathcal{O}_{\mathcal{D}}(\mathcal{D}) \longrightarrow R^2\pi_*\mathcal{O}_{\mathrm{Hilb}^{\gamma}_X \times X} \xrightarrow{p_X^*\theta} R^2\pi_* p_X^* K_X = \mathcal{O}_{\mathrm{Hilb}^{\gamma}_X}$$

Lemma 3.2 *The vanishing locus* $\mathrm{zero}(\sigma_\theta)$ *is the closed subscheme*

$$\mathrm{Hilb}^{\gamma}_X(Z) := \{D \in \mathrm{Hilb}^{\gamma}_X \mid D \subset Z\}.$$

Proof Let $\varphi\colon T \to \mathrm{Hilb}^{\gamma}_X$ be any morphism and $\mathcal{D}_T = \mathcal{D} \times_{\mathrm{Hilb}^{\gamma}_X} T \subset T \times X \xrightarrow{\rho} T$ be the pullback of the universal family. Let $p\colon T \times X \to X$ be the projection. Fix a nonzero section $s \in H^0(\mathcal{O}_{T\times X}(\mathcal{D}_T))$ whose vanishing locus is $\mathcal{D}_T$. The exact sequence $0 \to \mathcal{O}_{T\times X} \xrightarrow{s} \mathcal{O}_{T\times X}(\mathcal{D}_T) \to \mathcal{O}_{\mathcal{D}_T}(\mathcal{D}_T) \to 0$ induces an exact sequence

$$R^1\rho_*(\mathcal{O}_{\mathcal{D}_T}(\mathcal{D}_T)) \longrightarrow R^2\rho_*(\mathcal{O}_{T\times X}) \xrightarrow{s} R^2\rho_*(\mathcal{O}_{T\times X}(\mathcal{D}_T)) \longrightarrow R^2\rho_*(\mathcal{O}_{\mathcal{D}_T}(\mathcal{D}_T)) = 0.$$

Then

$$\varphi^*(\sigma_\theta)\colon R^1\rho_*\mathcal{O}_{\mathcal{D}_T}(\mathcal{D}_T) \xrightarrow{p^*\theta} R^2\rho_* p^* K_X = \mathcal{O}_T$$

vanishes if and only if the second arrow factors as

$$p^*\theta\colon R^2\rho_*(\mathcal{O}_{T\times X}) \xrightarrow{s} R^2\rho_*(\mathcal{O}_{T\times X}(\mathcal{D}_T)) \xrightarrow{\mathfrak{f}} R^2\rho_* p^*(\mathcal{O}_X(Z)) = R^2\rho_* p^* K_X$$

for some $\mathfrak{f}$. By taking the duals, we find that this is equivalent to saying that

$$p^*\theta\colon \rho_*\mathcal{O}_{T\times X} \to \rho_* p^* K_X$$

factors as

$$\rho_*\mathcal{O}_{T\times X} \xrightarrow{\mathfrak{f}^\vee} \rho_* p^* K_X(-\mathcal{D}_T) \xrightarrow{s} \rho_* p^* K_X.$$

Let $s' \in H^0(\rho_* p^* K_X(-\mathcal{D}_T))$ be the image of the section 1 by $\mathfrak{f}^\vee$. Then $p^*\theta = ss'$ and thus $\mathcal{D}_T = \mathrm{zero}(s) \subset \mathrm{zero}(p^*\theta) = T \times Z$, ie, φ factors through $\mathrm{Hilb}^{\gamma}_X(Z)$. Therefore φ factors through $\mathrm{zero}(\sigma_\theta)$ if and only if it factors through $\mathrm{Hilb}^{\gamma}_X(Z)$. This proves the lemma. □

By the cosection localization principle in [8] (see Section 2), we obtain the following.

Proposition 3.3 *There exists a localized virtual fundamental class*

$$[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(\mathrm{Hilb}_X^\gamma(Z))$$

such that $\imath_*[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(\mathrm{Hilb}_X^\gamma)$ *is the ordinary virtual fundamental class* $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}$ *in Section 3.1. Furthermore, if* W *is an open neighborhood of* Z *in* Hilb_X^γ*, we have* $[W]^{\mathrm{vir}}_{\mathrm{loc}} = [\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}}$.

Therefore the calculation of the Poincaré invariants takes place near a canonical divisor Z. This is consistent with Witten's claim about localization of Seiberg–Witten invariants to a canonical divisor [19; 3].

Suppose $\gamma = k_X := c_1(K_X) \in H^2(X,\mathbb{Z})$. Then the virtual dimension $\gamma\cdot(\gamma - k_X)$ of Hilb_X^γ is 0. If $\varphi\colon T \to \mathrm{Hilb}_X^{k_X}(Z)$ is a morphism, we obtain a diagram:

$$\begin{tikzcd} \mathcal{D}_T \arrow[r, hook] \arrow[dr] & Z\times T \arrow[r, hook] \arrow[d] & X\times T \arrow[dl] \\ & T & \end{tikzcd}$$

Since each $D \in \mathrm{Hilb}_X^{k_X}$ and Z have the same Hilbert polynomial, $\mathcal{D}_T = Z \times T$ and hence the morphism $T \to \mathrm{Hilb}_X^{k_X}(Z)$ factors through the reduced point $\{Z\}$. Hence the scheme $\mathrm{Hilb}_X^{k_X}(Z)$ is simply the reduced point $\{Z\}$.

Corollary 3.4 *When* $\gamma = k_X$*, the zero locus of the cosection* σ_θ *is the single reduced point* $\{Z\}$*. Hence the localized virtual fundamental class* $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ *of* Hilb_X^γ *is supported at* $\{Z\}$ *and the Poincaré invariant of* $\mathrm{Hilb}_X^{k_X}$ *is the degree of* $[\mathrm{Hilb}_X^\gamma]^{\mathrm{vir}}_{\mathrm{loc}}$.

4 A proof of Theorem 1.1

In this section we will prove Theorem 1.3 and thus complete a proof of Theorem 1.1. Let X be a minimal projective surface of general type with $p_g > 0$. We will find local defining equations of $\mathrm{Hilb}_X^{k_X}$ near canonical divisors and show that there is a canonical divisor Z representing a smooth point of the Hilbert scheme whose dimension at Z has the same parity as $\chi(\mathcal{O}_X)$. Then Theorem 1.3 will follow directly from Example 2.4.

Let $M = \mathrm{Hilb}_X^{k_X}$ denote the Hilbert scheme of divisors D with $c_1(\mathcal{O}_X(D)) = k_X = c_1(K_X) \in H^2(X,\mathbb{Z})$. Let $P = \mathrm{Pic}^{k_X}(X)$ (resp. $P^0 = \mathrm{Pic}^0(X)$) denote the Picard variety of line bundles L on X with $c_1(L) = k_X$ (resp. $c_1(L) = 0$). Since a Cartier divisor defines a line bundle, we have a natural morphism

$$\tau\colon M \to P$$

whose fiber over an $L \in P$ is the complete linear system $\mathbb{P} H^0(X, L)$. It is easy to see that M is the fine moduli space of pairs (L, s) where $L \in P$ and $s \in H^0(X, L) - \{0\}$ where two such pairs (L, s) and (L', s') are isomorphic if there is an isomorphism $L \xrightarrow{\simeq} L'$ that sends s to s'.

The goal is to show that there is a smooth open subvariety in M whose dimension has the same parity as $\chi(\mathcal{O}_X)$. Here is the idea of our proof: One picks $s \in H^0(K_X)$ such that s: $H^1(\mathcal{O}_X) \to H^1(K_X)$ has maximal rank. Then by semicontinuity, there is an open neighborhood $\mathbb{P}U$ of $Z = \mathrm{zero}(s)$ in $\mathbb{P} H^0(X, K_X)$ where the rank of the homomorphism $H^1(\mathcal{O}_X) \to H^1(K_X)$ is constant so that the kernels form a vector bundle on U. By using [7], one can show that there is an analytic neighborhood of $\mathbb{P}U$ in the above vector bundle of kernels that is isomorphic to an analytic neighborhood of $\mathbb{P}U$ in M. This implies the smoothness of M at Z. The parity follows from the fact that

$$H^1(\mathcal{O}_X) \xrightarrow{s} H^1(K_X) \cong H^1(\mathcal{O}_X)^\vee$$

is skew-symmetric. The details are as follows.

Let T be a ball in $H^1(\mathcal{O}_X) \cong \mathbb{C}^q$. Consider the isomorphism $P^0 \to P$ defined by $L \mapsto L^{-1} \otimes K_X$ and the exponential map $H^1(\mathcal{O}_X) \to P^0$ that sends 0 to the trivial line bundle $\mathcal{O}_X$. Let κ: $T \to P$ be the composition of

$$T \xrightarrow{\subset} H^1(\mathcal{O}_X) \quad \text{with } H^1(\mathcal{O}_X) \to P^0 \to P.$$

Pulling back the universal family over $P \times X$ by κ, we obtain a family $\mathcal{L} \to T \times X$ of line bundles such that $\mathcal{L}|_{0 \times X} \cong K_X$. Let π, ρ denote the projections from $T \times X$ to T and X respectively.

Let $H^i(\mathcal{O}_X)_T = H^i(\mathcal{O}_X) \otimes \mathcal{O}_T$ and let $D_T^\bullet$ denote the complex

$$0 \longrightarrow H^0(\mathcal{O}_X)_T \longrightarrow H^1(\mathcal{O}_X)_T \longrightarrow H^2(\mathcal{O}_X)_T \longrightarrow 0 \tag{4-1}$$

where the differentials are $\lambda \mapsto \lambda \wedge t$ for $t \in T$. The following is a special case of [7, Theorem 3.2].

Lemma 4.1 *Under the above assumptions, we have isomorphism*

$$(R^i \pi_*(\mathcal{L}^{-1} \otimes \rho^* K_X))_0 \cong \mathcal{H}^i(D_T^\bullet)_0 \tag{4-2}$$

where the subscript 0 *means stalk at zero.*

In [7], the authors construct the relative Dolbeault complex $\mathcal{A}_T^{0,\bullet}$ on $T \times X$, which turns out to be a resolution of $\mathcal{L}^{-1} \otimes \rho^* K_X$ [7, Lemma 2.2], so that $R\pi_*(\mathcal{L}^{-1} \otimes \rho^* K_X)$ is quasi-isomorphic to $A_T^{0,\bullet} = \pi_* \mathcal{A}_T^{0,\bullet}$. Representing $H^i(\mathcal{O}_X)$ by harmonic forms, one has a natural morphism $D_T^\bullet \to A_T^{0,\bullet}$. Then Green and Lazarsfeld show that the morphism

$$D_T^\bullet \longrightarrow A_T^{0,\bullet} \cong R\pi_*(\mathcal{L}^{-1} \otimes \rho^* K_X) \tag{4-3}$$

is a quasi-isomorphism over the stalk at 0. Since the sheaves $R^2\pi_*(\mathcal{L}^{-1} \otimes \rho^* K_X)$ and $\mathcal{H}^2(D_T^\bullet)$ are coherent sheaves over T, we have isomorphisms of stalks

$$(\pi_*\mathcal{L})_0 \cong (R^2\pi_*(\mathcal{L}^{-1} \otimes \rho^* K_X))_0^\vee \cong \mathcal{H}^2(D_T^\bullet)_0^\vee \cong (\mathcal{H}^2(D_T^\bullet)^\vee)_0 \cong \mathcal{H}^0(D_T^{\bullet\vee})_0$$

where $D_T^{\bullet\vee}$ is the dual complex

$$0 \longrightarrow H^0(K_X)_T \xrightarrow{\wedge t} H^1(K_X)_T \xrightarrow{\wedge t} H^2(K_X)_T \longrightarrow 0. \tag{4-4}$$

Let $\iota_S \colon S \to T$ be a morphism of analytic schemes sending a closed point $s \in S$ to $0 \in T$ and let $\mathcal{L}_S$ be the pullback of $\mathcal{L}$ by $\iota_s \times \mathrm{id}_X$. Let $\pi_S \colon S \times X \to S$ be the projection. Then the pullback $\mathcal{A}_S^{0,\bullet}$ of $\mathcal{A}_T^{0,\bullet}$ to $S \times X$ is a resolution of the pullback of $\mathcal{L}^{-1} \otimes \rho^* K_X$ because $\mathcal{L}$ and $\mathcal{A}_T^{0,i}$ are all flat over T. By repeating the same argument, we obtain an isomorphism of stalks

$$(\pi_{S*}\mathcal{L}_S)_s \cong \mathcal{H}^0(D_S^{\bullet\vee})_s$$

where $D_S^{\bullet\vee}$ is (4-4) with T replaced by S. In particular, for each n, we have an isomorphism

$$\pi_{T_n*}\mathcal{L}_{T_n} \cong \mathcal{H}^0(D_T^{\bullet\vee}|_{T_n}) \tag{4-5}$$

where $T_n = \operatorname{Spec} \mathbb{C}[t_1, \dots, t_q]/(t_1, \dots, t_q)^n$ is the n^{th} infinitesimal neighborhood of 0 in T.

Let $\widetilde{M}$ be the moduli functor that assigns to any $\iota_S \colon S \to T$ the set $\Gamma(S \times X, \mathcal{L}_S)$ where $\mathcal{L}_S$ is the pullback of $\mathcal{L}$ by $\iota_S \times \mathrm{id}_X$. By the relative Hilbert scheme construction for $\mathcal{L} \to T \times X \to T$, $\widetilde{M}$ is represented by a quasi-projective scheme over T with central fiber $\widetilde{M}|_0 = H^0(K_X)$. On the other hand, if we let $\widetilde{M}'$ be the moduli functor that assigns to any $\iota_S \colon S \to T$ the set $\Gamma(S, \mathcal{H}^0(D_S^{\bullet\vee}))$, then it is easy to see that $\widetilde{M}'$ is represented by the fiber product

$$\begin{tikzcd} \widetilde{M}' \arrow[r] \arrow[d] & H^0(K_X) \times T \arrow[d] \\ T \arrow[r] & H^1(K_X) \times T \end{tikzcd}$$

where the right vertical arrow is $(s,t) \mapsto (s \wedge t, t)$ and the bottom horizontal is $t \mapsto (0,t)$. In other words, $\widetilde{M}'$ is the isomorphic to

$$\{(s,t) \in H^0(K_X) \times T \mid s \wedge t = 0\} \subset H^0(K_X) \times H^1(\mathcal{O}_X).$$

By (4-5), we find that the formal completion of $\widetilde{M}$ along the central fiber $H^0(K_X)$ is isomorphic to the formal completion of $\widetilde{M}'$ along the central fiber $H^0(K_X)$. Note that, by the definition of $\widetilde{M}$, M is the projectivization of $\widetilde{M}$ over T, ie, $M|_T = M \times_P T = \mathbb{P}\widetilde{M}$.

Let us consider the map

$$\eta\colon H^0(K_X) \times H^1(\mathcal{O}_X) \longrightarrow H^1(K_X), \quad \eta(s,t) = s \wedge t.$$

By lower semicontinuity of rank, there is a $\mathbb{C}^*$–invariant open set U of $H^0(K_X) - \{0\}$ where the rank of the homomorphism

$$\eta_s\colon H^1(\mathcal{O}_X) \longrightarrow H^1(K_X), \quad \eta_s(t) = s \wedge t$$

is maximal for all $s \in U$. Hence, the kernel of η gives us a vector bundle over U whose fiber of $s \in U$ is $\ker(\eta_s)$. This implies that $\widetilde{M}'$ near $U \times \{0\}$ is smooth of complex dimension

$$\dim U + \dim H^1(\mathcal{O}_X) - r = p_g + q - r$$

where r is the rank of η_s for any $s \in U$. Therefore we conclude that M is smooth of complex dimension $p_g + q - r - 1$ near the open set $\mathbb{P}U$ of the central fiber $\mathbb{P}H^0(K_X)$.

By Serre duality, $H^1(K_X) \cong H^1(\mathcal{O}_X)^\vee$ and it is easy to check that the homomorphism

$$\eta_s\colon H^1(\mathcal{O}_X) \xrightarrow{s} H^1(K_X) \cong H^1(\mathcal{O}_X)^\vee$$

is skew-symmetric. Hence its rank r is even, so the parity of $p_g + q - r - 1$ equals that of $\chi(\mathcal{O}_X) = 1 - q + p_g$. In summary we proved the following.

Proposition 4.2 *Let X be a smooth minimal projective surface of general type with $p_g > 0$. Then there is a canonical divisor Z in X, which represents a smooth point in the Hilbert scheme $\mathrm{Hilb}_X^{k_X}$. Furthermore, the dimension of $\mathrm{Hilb}_X^{k_X}$ at Z has the same parity as $\chi(\mathcal{O}_X)$.*

By the cosection localization in Section 3, the virtual cycle of $\mathrm{Hilb}_X^{k_X}$ is localized to any open neighborhood of Z. Since Z is a smooth point, we can excise the singular part of $\mathrm{Hilb}_X^{k_X}$ and may assume that $\mathrm{Hilb}_X^{k_X}$ is a smooth variety whose dimension has the same parity as $\chi(\mathcal{O}_X)$. Since the virtual dimension is zero, the obstruction sheaf is a locally free sheaf E whose rank equals the dimension of $\mathrm{Hilb}_X^{k_X}$. Since

$M = \mathrm{Hilb}_X^{k_X}$ is smooth, 0: $\mathbb{T}_M \to E$ is a perfect obstruction theory. Now Theorem 1.3 follows immediately from Example 2.4 and Corollary 3.4. This completes our proof of Theorem 1.1.

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