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Lipschitz retraction and distortion for subgroups of $\mathsf{Out}(F_n)$

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Given a free factor A of the rank n free group F_n, we characterize when the subgroup of $\mathsf{Out}(F_n)$ that stabilizes the conjugacy class of A is distorted in $\mathsf{Out}(F_n)$. We also prove that the image of the natural embedding of $\mathsf{Aut}(F_{n-1})$ in $\mathsf{Aut}(F_n)$ is nondistorted, that the stabilizer in $\mathsf{Out}(F_n)$ of the conjugacy class of any free splitting of F_n is nondistorted and we characterize when the stabilizer of the conjugacy class of an arbitrary free factor system of F_n is distorted. In all proofs of nondistortion, we prove the stronger statement that the subgroup in question is a Lipschitz retract. As applications we determine Dehn functions and automaticity for $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$.

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Given a finitely generated group G, a finitely generated subgroup H is *undistorted* in G if the inclusion map $H \hookrightarrow G$ is a quasi-isometric embedding with respect to word metrics. Nondistortion may be verified by constructing a Lipschitz retraction $G \mapsto H$ (see Lemma 11), and this is particularly useful because it has extra consequences: if G is finitely presented, and if there is a Lipschitz retraction $G \to H$, then H is finitely presented and the Dehn function of G is dominated below by the Dehn function of H (see Proposition 12).

In studying the large scale geometry of the mapping class group $\mathcal{MCG}(S)$ of any closed oriented surface S, the Lipschitz projection methods introduced by Masur and Minsky [12, Section 2] are an important tool. They proved that there is a Lipschitz projection from (a geometric complex for) $\mathcal{MCG}(S)$ to the curve complex of S. In the course of investigating the results in this paper, we discovered that the same techniques from [12] easily generalize to product a Lipschitz retraction from $\mathcal{MCG}(S)$ onto the subgroup that stabilizes the isotopy class of any essential curve or subsurface of S, and so such subgroups are undistorted.

As a step in understanding the large scale geometry of the outer automorphism group $\mathsf{Out}(F_n)$ of a rank n free group F_n, one might consider the following. The action of the automorphism group $\mathsf{Aut}(F_n)$ on subgroups of F_n induces an action of $\mathsf{Out}(F_n)$

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on conjugacy classes of subgroups of F_n. Given a free factor $A < F_n$, the stabilizer $\mathsf{Stab}[A]$ of its conjugacy class is a finitely generated subgroup of $\mathsf{Out}(F_n)$. Regarding a free factor as an analogue of an essential subsurface, one might ask whether $\mathsf{Stab}[A]$ is an undistorted subgroup of $\mathsf{Out}(F_n)$. The answer is surprising:

Theorem 1 *Let $A < F_n$ be a proper, nontrivial free factor.*

(1) *If $\operatorname{rank}(A) = n-1$ then $\mathsf{Stab}[A]$ is a Lipschitz retract of $\mathsf{Out}(F_n)$, and therefore is undistorted in $\mathsf{Out}(F_n)$.*

(2) *If $\operatorname{rank}(A) \le n-2$ then $\mathsf{Stab}[A]$ is distorted in $\mathsf{Out}(F_n)$.*

Upon discovering this theorem, we searched for and found wider classes of Lipschitz retract subgroups and distorted subgroups, and we applied Lipschitz retracts to settle questions about Dehn functions and automaticity of $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$, all of which is explained in the results to follow.

More Lipschitz retracts, and applications

Theorem 2 *There is a Lipschitz retraction from $\mathsf{Out}(F_n)$ to a subgroup isomorphic to $\mathsf{Aut}(F_{n-1})$, which is therefore undistorted.*

The proof is given in Section 3.4, as an application of Theorem 7.

Around 2002 Jose Burillo asked the authors whether the natural $\mathsf{Aut}(F_{n-1})$ subgroup of $\mathsf{Aut}(F_n)$ is undistorted. This subgroup is defined as all automorphisms of $F_n = \langle a_1, \dots, a_{n-1}, a_n \rangle$ which preserve the free factor $\langle a_1, \dots, a_{n-1} \rangle$ and the element a_n. The methods of Theorem 1(1) apply to answer this question affirmatively, although the proof is even easier in this case, for which reason we give this proof first, in Section 2.

Theorem 3 *There is a Lipschitz retraction from $\mathsf{Aut}(F_n)$ to the natural subgroup isomorphic to $\mathsf{Aut}(F_{n-1})$, which is therefore undistorted.*

Combining Theorem 2, Theorem 3 and an evident induction using that a composition of Lipschitz retractions is a Lipschitz retraction, we obtain:

Corollary 4 *If $n \ge 4$ there are Lipschitz retractions from each of the groups $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$ to subgroups isomorphic to $\mathsf{Aut}(F_3)$.*

In [9], Hatcher and Vogtmann found exponential upper bounds for Dehn functions of $\mathsf{Aut}(F_n)$ and $\mathsf{Out}(F_n)$ when $n \geq 3$. In [3], Bridson and Vogtmann found exponential lower bounds when $n = 3$, thereby completely determining the growth types of those Dehn functions to be exponential. Lower bounds in rank ≥ 4 were left unresolved.

Combining Corollary 4 with the lower bounds in rank 3 from [3], and with the fact that the Dehn function of a group is dominated below by the Dehn function of any Lipschitz retract (see Proposition 12), we obtain the desired exponential lower bounds for $n \geq 4$, thus completely determining the growth types of the Dehn functions of $\mathsf{Aut}(F_n)$ and $\mathsf{Out}(F_n)$:

Corollary 5 *For all $n \geq 4$, the Dehn functions of the groups $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$ have exponential lower bounds, and therefore they have exponential growth type.*

See also Bridson and Vogtmann [4] for another, later proof of Corollary 5.

In [3], Bridson and Vogtmann use their exponential lower bound for Dehn functions to prove that $\mathsf{Aut}(F_3)$ and $\mathsf{Out}(F_3)$ are not automatic, because the Dehn function of every automatic group has a quadratic upper bound; see Epstein, Cannon, Holt, Levy, Paterson and Thurston [6]. Bridson and Vogtmann also proved that $\mathsf{Aut}(F_n)$ and $\mathsf{Out}(F_n)$ are not biautomatic for all $n \geq 4$, but automaticity was left unresolved in those cases. Using Corollary 5 we can now resolve this issue:

Corollary 6 *For all $n \geq 4$, the groups $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$ are not automatic.*

Remark The proof in [3] that $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$ are not biautomatic is inductive, based on exploiting theorems about centralizers of finite sets in biautomatic groups due to Gersten and Short [7]. The method of Lipschitz retraction to subgroups may prove effective in other contexts similar to the present one for inductive verification of lower bounds on isoperimetric functions and failure of automaticity.

Stabilizers of free splittings

The two special cases of Lipschitz retracts of $\mathsf{Out}(F_n)$ mentioned so far — of Theorem 1, case (1) and Theorem 2 — are each best expressed in the language of trees. This leads to a rich collection of undistorted subgroups of $\mathsf{Out}(F_n)$, as expressed in our next theorem.

Define a *free splitting* of F_n to be a minimal action of F_n on a nontrivial, simplicial tree T with trivial edge stabilizers. As is traditional in this subject, the notation T will incorporate both the tree and the action of F_n; we shall denote the action of

$g \in F_n$ on T by $x \mapsto g \cdot x$. Two free splittings T, T' are *conjugate* if there exists a homeomorphism $f\colon T \mapsto T'$ such that $f(g \cdot x) = g \cdot f(x)$. The group $\mathsf{Out}(F_n)$ acts on the set of conjugacy classes of free splittings of F_n, and the stabilizer of the conjugacy class of a free splitting T is denoted $\mathsf{Stab}[T]$. A free splitting may also be understood using Bass–Serre theory, being the Bass–Serre tree of its quotient graph of groups.

Theorem 7 *Given a free splitting T of F_n, there is a Lipschitz retraction from $\mathsf{Out}(F_n)$ to its subgroup $\mathsf{Stab}[T]$, which is therefore undistorted in $\mathsf{Out}(F_n)$.*

Section 3 contains the proof of Theorem 7. Section 3.1 contains a sketch of the proof, outlining the major ideas, for a special case that is relevant to Theorem 1(1), namely a free splitting with one vertex orbit and one edge orbit. Section 3.3 contains the detailed proof of the general case.

The application of Theorem 7 to proving Theorem 2 is found in Section 3.4. The application to proving Theorem 1(1) is found in Section 4.1, the idea being that for every corank 1 free factor there exists a free splitting with a single edge orbit such that the stabilizers of their conjugacy classes are the same subgroup of $\mathsf{Out}(F_n)$. See also the paragraph just after the statement of Theorem 8 below, which discusses a broader setting for this application of Theorem 7, in terms of "coedge 1 free splittings" and "coindex 1 free factor systems".

Stabilizers of free factor systems

We extend Theorem 1 as follows. A nonempty set $\mathcal{F} = \{[A_1], \dots, [A_K]\}$ of conjugacy classes of nontrivial free factors of F_n is called a *free factor system* if it is represented by subgroups $A_1, \dots, A_K$ such that there exists a free factorization of the form $F_n = A_1 * \cdots * A_K * B$, where B may be trivial. The action of $\mathsf{Out}(F_n)$ on conjugacy classes of free factors extends to an action on free factor systems. The *coindex* of $\mathcal{F} = \{[A_1], \dots, [A_K]\}$ is the number

$$\operatorname{coindex}(\mathcal{F}) = |\chi(F_n)| - \sum_{k=1}^{K} |\chi(A_k)| = (n-1) - \sum_{k=1}^{K} (\operatorname{rank}(A_k) - 1).$$

Topologically this is the number of edges that must be attached to the vertices of the union of K connected graphs of ranks equal to $\operatorname{rank}(A_1), \dots, \operatorname{rank}(A_K)$, respectively, in order to make a connected graph of rank n. It is also the minimum, among all graphs of rank n having a subgraph with K components of ranks equal to $\operatorname{rank}(A_1), \dots, \operatorname{rank}(A_K)$, of the number of edges not in the subgraph; this minimum is achieved if and only if every edge not in the subgraph has each endpoint in the subgraph.

Theorem 8 *Let $\mathcal{F}$ be a free factor system of F_n.*

(1) *If* $\mathrm{coindex}(\mathcal{F}) = 1$ *then* $\mathsf{Stab}(\mathcal{F})$ *is a Lipschitz retract of* $\mathsf{Out}(F_n)$*, and therefore is undistorted in* $\mathsf{Out}(F_n)$*.*

(2) *If* $\mathrm{coindex}(\mathcal{F}) \geq 2$ *then* $\mathsf{Stab}(\mathcal{F})$ *is distorted in* $\mathsf{Out}(F_n)$*.*

In Section 4.1 we prove Theorem 8(1), by showing that for any free factor system $\mathcal{F}$ of coindex 1 there is a free splitting T with one edge orbit such that $\mathsf{Stab}(\mathcal{F}) = \mathsf{Stab}[T]$.

In Section 4.3 we prove Theorem 8(2), by describing an explicit sequence $\phi_k \in \mathsf{Stab}(\mathcal{F})$ whose word length in $\mathsf{Out}(F_n)$ grows linearly but whose word length in $\mathsf{Stab}(\mathcal{F})$ grows exponentially. The phenomenon which allows this to happen is that for any marked graph G having a core subgraph $K \subset G$ representing the free factor system $\mathcal{F}$, and any oriented edge $E \subset G \setminus K$ having terminal endpoint in K, the edge E may be "Dehn twisted" around any circuit u in K whatsoever — as long as u is based at the terminal point of E — by a map on G taking E to Eu and restricting to the identity on K. By careful choice of a sequence of twist circuits u_k one defines the sequence ϕ_k with the desired properties.

Contents

Section 1 contains preliminary results about quasi-isometric geometry, and about $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$, the spine of outer space denoted $\mathcal{K}_n$, and the spine of autre espace denoted $\mathbb{A}_n$. Corollary 10 sets the stage for proofs of coarse Lipschitz retraction or distortion of finitely generated subgroups of any finitely generated group, by showing that these are equivalent to coarse Lipschitz retraction or distortion of corresponding subcomplexes, via the Milnor–Švarc Lemma.

Proofs regarding subgroups of $\mathsf{Out}(F_n)$ and $\mathsf{Aut}(F_n)$ that are coarse Lipschitz retracts are contained in Sections 2, 3 and 4.1. Proofs regarding distorted subgroups are contained in Section 4.3, with supporting material in Section 4.2. The distortion proofs and the nondistortion proofs are almost entirely independent, and one can choose to read one or the other first, with overlapping material mostly isolated in Section 1. In particular, readers interested in distortion may, after reviewing relevant material in Section 1, skip straight to Section 4.2 or even to Section 4.3.

Section 2 contains the proof of Theorem 3, which is given before the proof of Theorem 7 because of its more elementary nature.

Section 3 contains the proof of Theorem 7, regarding $\mathsf{Stab}[T] < \mathsf{Out}(F_n)$. Section 3.2 describes a subcomplex $\mathcal{K}_n^T \subset \mathcal{K}_n$ corresponding to $\mathsf{Stab}[T]$, and contains the proof of Lemma 18 which includes a description of the algebraic structure of (a finite index subgroup of) $\mathsf{Stab}[T]$, one part of which was first proved by Levitt in [11]. The heart of the proof of Theorem 7 is in Section 3.3 which describes a coarse Lipschitz retraction $\mathcal{K}_n \mapsto \mathcal{K}_n^T$.

Section 4 contains the proof of Theorem 8, regarding $\mathsf{Stab}(\mathcal{F}) < \mathsf{Out}(F_n)$. Section 4.1 handles the case where $\text{coindex}(\mathcal{F}) = 1$, by an application of Theorem 7. Section 4.2 describes a subcomplex $\mathcal{K}_n^{\mathcal{F}} \subset \mathcal{K}_n$ corresponding to $\mathsf{Stab}(\mathcal{F})$. Section 4.3 handles the case where $\text{coindex}(\mathcal{F}) \geq 2$ by proving that the subcomplex $\mathcal{K}_n^{\mathcal{F}}$ is distorted in $\mathcal{K}_n$.

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1 Preliminaries

1.1 Quasi-isometries and Lipschitz retracts

Given two metric spaces X, Y, a function $f\colon X \to Y$, and constants $K \geq 1$, $C \geq 0$, we say that f is K, C *coarse Lipschitz* if $d_Y(f(a), f(b)) \leq K d_X(a,b) + C$ for all $a, b \in X$, we say that f is a K, C *quasi-isometric embedding* if it is K, C coarse Lipschitz and $\frac{1}{K} d_X(a,b) - C \leq d_Y(f(a), f(b))$ for all $a, b \in X$, and we say that f is a K, C *quasi-isometry* if it is a K, C quasi-isometric embedding and for each $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) \leq C$. For each of these terms we often drop the reference to the constants K, C. A composition of quasi-isometries is a quasi-isometry, and any quasi-isometry $f\colon X \to Y$ has a *coarse inverse*, a map $\bar f\colon Y \to X$ such that each of the maps $\bar f \circ f$ and $f \circ \bar f$ move points a uniformly bounded distance; the quasi-isometry constants of $\bar f$ and the distance bounds for $\bar f \circ f$ and $f \circ \bar f$ depend only on the quasi-isometry constants of f.

A metric space X is *proper* if every closed ball is compact, and X is *geodesic* if for any $a, b \in X$ there is a rectifiable path from a to b whose length equals $d_X(a,b)$. For example (see Section 1.3), every simplicial complex supports a geodesic metric in which each simplex is isometric to a Euclidean simplex of side length 1; if the complex is locally finite then this metric is proper. Any action $G \curvearrowright X$ on a simplicial complex which respects simplicial coordinates is an isometry.

Lemma 9 (Milnor [13] and Švarc [15]) *For any group G and any proper, geodesic metric space X, if there exists a properly discontinuous, cocompact, isometric action $G \curvearrowright X$ then G is finitely generated. Furthermore, for any such action and any point $x \in X$, the orbit map $g \mapsto g \cdot x$ is a quasi-isometry $\mathcal{O}\colon G \to X$, where G is equipped with the word metric of any finite generating set.*

Given a geodesic metric space X, a subspace $Y \subset X$ is said to be *rectifiable* if any two points $x, y \in Y$ are endpoints of some path in Y which is rectifiable in X and whose length is minimal amongst all rectifiable paths in Y with endpoints x, y. It follows that minimal path length defines a geodesic metric on Y. Examples include any connected subcomplex of any connected simplicial complex, using a natural simplicial metric where each simplex is isometric to a Euclidean simple of side length 1. We say that Y is a *(coarse) Lipschitz retract* of X if there exists a *(coarse) Lipschitz retraction* $f\colon X \mapsto Y$, meaning a (coarse) Lipschitz function that restricts to the identity on Y.

Corollary 10 *If G is a finitely generated group acting properly discontinuously and cocompactly by isometries on a connected locally finite simplicial complex X, if $H < G$ is a subgroup, and if $Y \subset X$ is a nonempty connected subcomplex which is H–invariant and H–cocompact, then:*

(1) *H is finitely generated.*

(2) *H is undistorted in G if and only if the inclusion $Y \hookrightarrow X$ is a quasi-isometric embedding.*

(3) *The following are equivalent:*
   (a) *H is a Lipschitz retract of G.*
   (b) *The 0–skeleton of Y is a Lipschitz retract of the 0–skeleton of X.*
   (c) *The 1–skeleton of Y is a Lipschitz retract of the 1–skeleton of X.*
   (d) *Y is a coarse Lipschitz retract of X.*

Proof The actions of G on X and H on Y satisfy the hypotheses of the Milnor–Švarc Lemma so (1) follows. Also, choosing a base vertex $x \in Y$, it follows that the orbit map $g \mapsto g \cdot x$ defines quasi-isometries $\mathcal{O}_G\colon G \mapsto X$ and $\mathcal{O}_H\colon H \mapsto Y$. Denoting inclusions as $i_Y\colon Y \hookrightarrow X$ and $i_H\colon H \hookrightarrow G$ we have $\mathcal{O}_G \circ i_H = i_Y \circ \mathcal{O}_H$. It follows that i_Y is a quasi-isometric embedding if and only if i_H is, proving (2). Choose $C \geq 0$ so that each point of X is within distance C of image$(\mathcal{O}_G)$ and each point of Y is within distance C of image$(\mathcal{O}_H)$.

To prove (a)$\Longrightarrow$(b), suppose $\pi_H\colon G \to H$ is a Lipschitz retraction. Define π_Y from the 0–skeleton of X to the 0–skeleton of Y as follows: given a vertex $v \in X$, choose $g_v \in G$ so that $\mathcal{O}_G(g_v) = g_v \cdot x$ is within distance C of v, and define $\pi_Y(v) = \mathcal{O}_H \circ \pi_H(g_v)$. Chasing through diagrams, π_Y is easily seen to be Lipschitz, although it might not be a retraction. But if $v \in Y$ then we can choose g_v to be in H and so $\pi_Y(v) = \mathcal{O}_H(\pi_H(g_v)) = \mathcal{O}_H(g_v) = g_v \cdot x$ is within distance C of v and so, by moving $\pi_Y(v)$ a uniformly bounded distance for all vertices v of Y, we obtain a Lipschitz retraction.

The implication (b)$\Longrightarrow$(c) follows by extending any Lipschitz retract of 0–skeleta continuously so as to map each edge to an edge path of bounded length. To prove the implication (c)$\Longrightarrow$(d), extend across each higher-dimensional simplex σ by the identity on the interior of σ if $\sigma \subset Y$, and otherwise extend so that the image of the interior of σ is contained in the image of the 1–skeleton of σ; continuity is ignored in this extension, hence one obtains only a *coarse* Lipschitz retract.

To prove (d)$\Longrightarrow$(a), suppose $\pi_Y\colon X \to Y$ is a coarse Lipschitz retract. For each $g \in G$, choose $h = \pi_H(g) \in H$ so that $\pi_Y(\mathcal{O}_G(g))$ is within distance C of $\mathcal{O}_H(h)$. Chasing through diagrams π_H is again easily seen to be Lipschitz. It might not be a retraction, but if $g \in H$ then $\mathcal{O}_G(g) = \mathcal{O}_H(g) \in Y$ is equal to $\pi_Y(\mathcal{O}_H(g))$ and is within C of $\mathcal{O}_H(h)$, and so $d_H(g,h)$ is uniformly bounded; by moving $h = \pi_H(g)$ a uniformly bounded distance to g we therefore obtain a Lipschitz retraction. $\square$

Remark If one were not satisfied with the statement of item (d), and if X and Y were to satisfy higher connectivity properties, then one could pursue the issue of a continuous Lipschitz retraction up through higher-dimensional skeleta. This is useful for investigating higher isoperimetric functions.

Lipschitz retracts give a handy method for verifying nondistortion:

Lemma 11 *If X is a geodesic metric space and $Y \subset X$ is a rectifiable subspace, and if Y is a coarse Lipschitz retract of X, then the inclusion $Y \hookrightarrow X$ is a quasi-isometric embedding.*

Proof By definition of a rectifiable subspace, $d_X(x,y) \le d_Y(x,y)$ for all $x,y \in Y$. For the other direction, if $f\colon X \to Y$ is a (K,C)–coarse Lipschitz retract then for all $x,y \in Y$ we have $d_Y(x,y) = d_Y(f(x),f(y)) \le K d_X(x,y) + C$. $\square$

The Dehn function of a finitely presented group Consider a finitely presented group $G = \langle g_i \mid R_j \rangle$, $i = 1,\dots,I$, $j = 1,\dots,J$. Let C be the presentation CW–complex, with one vertex, one oriented edge e_i labeled by each generator g_i, and one 2–cell σ_j whose boundary is attached along the closed path given by each defining relator R_j, inducing an isomorphism $G \approx \pi_1(C)$. Given a *relator* in G — a word w in the generators that represents the identity element — the path in C labeled by w is null homotopic. A *Dehn diagram* for a relator w is a continuous function $d\colon D \to C$ defined on a closed disc D such that, with respect to some CW–structure on D, each vertex of D maps to the vertex of C, each edge of D maps to an edge path of C, the restriction $d|\partial D$ is a parameterization of the edge path w, and the restriction of d to each 2–cell σ of D satisfies one of two possibilities: $d(\sigma)$ is contained in the 1–skeleton of C in which case we say that σ *collapses*; or there is a relator R_j such that the restriction of d to the boundary of σ traces out the path R_j and the restriction of d to the interior of σ is a homeomorphism to the interior of σ_j. The number of 2–cells of the latter type is denoted $\mathrm{Area}(d)$. Denote by $\mathrm{Area}(w)$ the minimum of $\mathrm{Area}(d)$ over all Dehn diagrams d of w. The *Dehn function* of the presentation is the function $f\colon \mathbb{N} \to \mathbb{N}$, where $f(m)$ equals the maximum of $\mathrm{Area}(w)$ amongst all words of length $\le m$. If $f'(m)$ is the Dehn function of any other finite presentation of G then there exist constants $A, B, C > 0$ such that

$$f'(m) \le Af(Bm) + Cm \quad \text{and} \quad f(m) \le Af'(Bm) + Cm \quad \text{for any } m \in \mathbb{N}.$$

More generally, given two functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we say that f is *dominated above* by g or that g is *dominated below* by f if there exist constants $A, B, C > 0$ such that $f(m) \le Ag(Bm) + Cm$. In particular, for a finitely presented group G, either all of its Dehn functions are dominated below by an exponential function, or none of them are.

Proposition 12 *Given a finitely presented group G and a finitely generated subgroup H, if H is a Lipschitz retract of G then H is finitely presented and the Dehn function of G is dominated below by the Dehn function of H.*

Proof The proof is based on familiar techniques. Choose finite generating sets for G and H so that the generators of H are a subset of the generators of G, and hence there is an inclusion of Cayley graphs $\Gamma_H \subset \Gamma_G$ which satisfies the hypotheses of Corollary 10. Applying that corollary we obtain a K–Lipschitz retraction $\pi\colon \Gamma_G \to \Gamma_H$. Choose a finite set of defining relators for G, and let ℓ be an upper bound for their lengths. By attaching a 2–cell to each closed edge path of Γ_H of length $\le K\ell$, the result is a simply connected 2–complex on which $\pi_1(H)$ acts properly and cocompactly, from which finite presentation of H follows, and from which one can derive the desired inequality of Dehn functions.

In more detail, for any word w in the generators of H that defines the trivial element, there is a minimal area Dehn diagram d for w defined using the relators of G. Let D be the domain of d and let $\widehat{D}$ denote the union of the 1–skeleton of D and its collapsing 2–cells. We may regard the restriction $d \mid \widehat{D}$ as mapping into the Cayley graph of G. The composition $\pi \circ d \mid \widehat{D}$ maps into the Cayley graph of H, and the image of the boundary of any noncollapsing 2–cell σ of D is a closed edge path in the Cayley graph of H of length at most $K\ell$, which is the boundary of one of the 2–cells attached to Γ_H. We may therefore extend $\pi \circ (d \mid \widehat{D})$ across these 2–cells to obtain a Dehn diagram for w in the relators of H, proving that H is finitely presented. Furthermore, the area of this new Dehn diagram is equal to the area of d itself, proving that the Dehn function of G is dominated below by the Dehn function of H. □

Remark The same result is true for higher-dimensional isoperimetric functions of groups with higher finiteness properties, using a similar proof, where Corollary 10 is extended to (noncoarse) Lipschitz retracts between higher-dimensional skeleta as remarked upon following the proof of that corollary.

1.2 Free groups

The notation $F = \langle a_1, \ldots, a_k \rangle$ will mean that F is the free group with free basis $\{a_1, \ldots, a_k\}$. Associated to F and its given free basis is a rose R_F, a CW 1–complex with one vertex x and with k oriented edges called *petals* labeled $a_1, \ldots, a_k$, such that F is identified with $\pi_1(R_F, x)$ and each free basis element a_i is identified with the path homotopy class of the petal with the label a_i. The number k is the *rank* of F.

For the notation of this paper, $F_n = \langle a_1, \ldots, a_n \rangle$ and $R_n = R_{F_n}$ will denote a fixed rank n free group with free basis and its associated rose. However, for the rest of Section 1, as well as in various places around the paper, we will couch some of our definitions and statements in terms of a general finite rank free group denoted F, in order that these definitions can apply as well to subgroups of F_n.

The action of the automorphism group $\mathsf{Aut}(F)$ on elements $g \in F$ and subgroups of $\mathsf{Aut}(F)$ induces an action of the outer automorphism group $\mathsf{Out}(F) = \mathsf{Aut}(F)/\mathsf{Inn}(F)$ on conjugacy classes of elements and subgroups. We generally use square brackets $[\,\cdot\,]$ to denote conjugacy classes, or more generally any variety of equivalence relation on which $\mathsf{Out}(F)$ acts.

A *free factorization* of F, written $F = A_1 * \cdots * A_K$, is a set of subgroups $\{A_1, \ldots, A_K\}$ with pairwise trivial intersections having the property that each nontrivial $g \in F$ can be written uniquely as a product of nontrivial elements $g = a_1 \cdots a_m$ such that each a_i is an element of one of $A_1, \ldots, A_K$. A *free factor system* of F is a nonempty set of nontrivial subgroup conjugacy classes of the form $\mathcal{F} = \{[A_1], \ldots, [A_K]\}$ such that there exists a free factorization of the form $F = A_1 * \cdots * A_K * B$, where B may be trivial; $\mathcal{F}$ is *improper* if $K = 1$ and $A_1 = F$, otherwise $\mathcal{F}$ is *proper*. The individual conjugacy classes $[A_1], \ldots, [A_K] \in \mathcal{F}$ are called the *components* of $\mathcal{F}$, and $\mathcal{F}$ is *connected* if it has one component. We use literal set theory notation $\bigcup \mathcal{F}$ to denote the union of the elements of $\mathcal{F}$, that is, for any subgroup $A < F$, we have $A \in \bigcup \mathcal{F}$ if and only if $[A] \in \mathcal{F}$.

We define a partial order on free factor systems: $\mathcal{F} \sqsubset \mathcal{F}'$ if and only if for each $A \in \bigcup \mathcal{F}$ there exists $A' \in \bigcup \mathcal{F}'$ such that $A < A'$. Recalling the coindex defined in the introduction, we have the following elementary fact:

Lemma 13 *For any free factor systems* $\mathcal{F}, \mathcal{F}'$ *in* F, *if* $\mathcal{F} \sqsubset \mathcal{F}'$ *then* $\operatorname{coindex}(\mathcal{F}') \le \operatorname{coindex}(\mathcal{F})$, *with equality if and only if* $\mathcal{F} = \mathcal{F}'$. □

Given an isomorphism $\rho\colon F \to F'$ of finite rank free groups, there is an induced map denoted ρ_*, well defined up to pre- and postcomposition by inner automorphisms, from conjugacy classes in F to conjugacy classes in F', and in particular from free factor systems in F to free factor systems in F'. We also use the notation ρ_* for similar purpose when ρ is a homotopy equivalence between finite graphs.

The action of $\mathsf{Out}(F)$ on conjugacy classes of subgroups induces an action on free factor systems. The *stabilizer* of a free factor system $\mathcal{F}$ is the subgroup

$$\mathsf{Stab}(\mathcal{F}) = \{\phi \in \mathsf{Out}(F) \mid \phi(\mathcal{F}) = \mathcal{F}\}.$$

When $\mathcal{F} = \{[A]\}$ is connected we write $\mathsf{Stab}[A] = \mathsf{Stab}(\mathcal{F})$.

1.3 Graphs

In this paper a *simplicial complex* X will be the geodesic metric space associated with an abstract simplicial complex (see for example [14, Chapter 3, Section 1]), in which every simplex comes equipped with coordinates making it isometric to a Euclidean simplex of side length 1; these metrics on simplices extend uniquely to a geodesic metric on X which is proper if and only if X is locally finite. The underlying CW–complex structure also induces a *weak topology* on X which equals the metric topology if and only if the complex is locally finite.

A *graph* is a 1–dimensional CW–complex. A *simplicial tree* is a simplicial complex which is a contractible graph. A *simplicial forest* is a graph whose components are simplicial trees. On a simplicial tree we may impose a geodesic metric making each

1–cell isometric to a closed interval in $\mathbf{R}$ of any positive length, resulting in a *simplicial* $\mathbf{R}$*–tree*; an $\mathbf{R}$*–graph* is similarly defined.

A map between graphs that takes each 0–cell to a 0–cell and each 1–cell either to a 0–cell or homeomorphically to a 1–cell is called a *cellular map* or a *simplicial map* depending on the context. A surjective cellular map is called a *forest collapse* if the inverse image of any 0–cell is a tree. A bijective cellular map is called an *isomorphism*, and a self-isomorphism of a graph is called an *automorphism*.

Any graph or tree not homeomorphic to a circle and having no isolated ends has a unique *natural* cell structure with no 0–cells of valence 2; phrases such as "natural vertex" or "natural edge" will refer to this natural structure. Every CW decomposition is a refinement of this natural cell structure. Every homeomorphism of the weak topology between graphs equipped with natural cell structures is an isomorphism of those structures.

1.4 Free splittings

A *splitting* of a group G is a simplicial action of G on a nontrivial simplicial tree T such that the action is minimal, meaning no proper subtree of T is invariant under the action, and for each edge $e \subset T$ and $g \in G$, if $g(e) = e$ then g restricts to the identity on e. Formally a splitting is given by a homomorphism $\chi\colon G \to \mathsf{Aut}(T)$ with values in the group of simplicial automorphisms of T. We will denote this by $\chi\colon G \curvearrowright T$, or suppressing the action just $G \curvearrowright T$, or suppressing the group just T. We often use notation like $(g, x) \mapsto g \cdot x \in T$ to refer to the image of the action of $g \in G$ on $x \in T$; formally $g \cdot x = \chi(g)(x)$.

In this paper, the groups which act are all free of finite rank, and all splittings will be "free splittings" meaning that all edges have trivial edge stabilizers, and we shall immediately specialize our definitions to that context. Also, although we often work in the context of fixed rank n free group F_n and its finite rank subgroups, in this section we couch all definitions in terms of an arbitrary finite rank free group, in order to accommodate free splittings of subgroups of F_n.

A *free splitting* of a finite rank free group F is a splitting $F \curvearrowright T$ such that the stabilizer subgroup of every edge $e \subset T$ is trivial, meaning that if $g \in F$ and $g(e) = e$ then g is the identity element of F. Associated to a free splitting $F \curvearrowright T$ is a free factor system denoted $\mathcal{F}(T)$, consisting of the conjugacy classes of all nontrivial subgroups of the form $\mathsf{Stab}(v) = \{g \in F \mid g \cdot v = v\}$. A free splitting $F \curvearrowright T$ is *proper* if $\mathcal{F}(T) = \varnothing$, i.e., the stabilizer subgroup of every vertex is trivial; equivalently, the action is free and properly discontinuous.

A *conjugacy* from a free splitting $\chi\colon F \curvearrowright T$ to a free splitting $\chi'\colon F \curvearrowright T'$ is a homeomorphism $f\colon T \to T'$ such that f is *equivariant*, meaning that $f \circ \chi(g) = \chi'(g) \circ f$ for all $g \in F$. We say that T and T' are *conjugate* if there exists a conjugacy $f\colon T \to T'$. Conjugacy is an equivalence relation amongst free splittings, and we use the notation $[T]$ to refer to the equivalence class. Note that any conjugacy f is an isomorphism of natural simplicial structures, although we do not require f to be an isomorphism of whatever subdivided simplicial structures T and T' may have.

The outer automorphism group $\mathsf{Out}(F)$ has a natural right action on the set of conjugacy classes of free splittings: the action of $\phi \in \mathsf{Out}(F)$ on the class of a free splitting $\chi\colon F \curvearrowright T$ is the class of the free splitting $\chi \cdot \phi\colon F \curvearrowright T$ which is well-defined up to conjugacy by the formula $\chi \cdot \phi(g) = \chi(\Phi(g))$ for any choice of an automorphism $\Phi \in \mathsf{Aut}(F)$ representing ϕ. We also denote this action by $[T]\cdot\phi$ or just $T \cdot \phi$. Notice that, suppressing the role of χ, the image of the action $(g,x) \mapsto g \cdot x$ can be denoted as the action $(g,x) \mapsto \Phi(g) \cdot x$. The *stabilizer* of the class of a free splitting $\chi\colon F \curvearrowright T$ is the group

$$\mathsf{Stab}[T] = \{\phi \in \mathsf{Out}(F) \mid T \cdot \phi = T\}.$$

A *semiconjugacy* from a free splitting $\chi\colon F \curvearrowright T$ to a free splitting $\chi'\colon F \curvearrowright T'$ is a continuous equivariant map $f\colon T \to T'$ that takes vertices to vertices and restricts on each edge of T to an embedding or a constant map. Since χ and χ' are minimal, any semiconjugacy between them is surjective.

1.5 The spines of outer space et les autres espaces

In this section, given a finite rank free group F, we review the spine of outer space denoted $\mathcal{K}_F$ on which $\mathsf{Out}(F)$ acts, and the spine of autre espace denoted $\mathbb{A}_F$ on which $\mathsf{Aut}(F)$ acts. Vertices of $\mathcal{K}_F$ are marked graphs, and vertices of $\mathbb{A}_F$ are marked graphs with a base point, modulo an appropriate equivalence in each case. For details and further references for this material, see [16]. We also consider a variation on $\mathbb{A}_F$, whose vertices are marked graphs with multiple base points.

Marked graphs Choose a basis for F with associated rose R_F, inducing an identification $\pi_1(F) = \pi_1(R_F)$. An *F–marked graph* is a pair (G,ρ), where G is a finite *core graph* meaning that no vertex has valence 1, equipped with a homotopy equivalence $\rho\colon R_F \to G$ called the *marking* (the usual definition requires no vertices of valence ≤ 2, but we will allow such vertices and simply use the natural cell structure as needed). Two marked graphs (G,ρ) and (G',ρ') are *equivalent* if there exists a homeomorphism $h\colon G \to G'$ such that $h\rho, \rho'\colon R_F \to G'$ are homotopic. We often suppress the marking ρ from the notation, using phrases like "a marked graph G", just as we often suppress the action in the notation for a free splitting; for instance, the equivalence class of (G,ρ) is formally denoted $[G,\rho]$ or just $[G]$ when ρ is understood. The right action of the group $\mathsf{Out}(F)$ on equivalence classes of marked graphs is defined by $[G,\rho]\cdot\phi = [G,\rho\circ\Phi]$, where $\Phi\colon R_F \to R_F$ is a homotopy equivalence inducing $\phi \in \mathsf{Out}(F) = \mathsf{Out}(\pi_1 R_F)$.

The universal covering construction defines a bijection between the set of conjugacy classes of proper free splittings of F and the equivalence classes of F–marked graphs, which associates to each F–marked graph (G,ρ) a proper, minimal action $F \curvearrowright \widetilde G$ on the universal covering space which is well-defined up to precomposition by an inner automorphism of F; the inverse of this bijection associates to each proper free splitting $F \curvearrowright T$ the quotient core graph $G = T/F$ with a marking $\rho\colon R_F \to G$ that is well-defined up to homotopy. This bijection clearly respects the actions of $\mathsf{Out}(F)$ on conjugacy classes of proper free splittings and on conjugacy classes of F–marked graphs.

Given two F–marked graphs (G,ρ), (G',ρ') and a homotopy equivalence $h\colon G \to G'$, we say that h *preserves marking* if $h\circ\rho$ is homotopic to ρ'.

Given an F–marked graph (G,ρ), the marking $\rho\colon R_F \to G$ induces a well-defined bijection ρ_* between conjugacy classes in $F = \pi_1(R_F)$ and conjugacy classes in $\pi_1(G)$, and between free factor systems in F and free factor systems in $\pi_1(G)$. Every proper, connected, noncontractible subgraph $H \subset G$ determines a well-defined connected free factor system in $\pi_1(G)$ denoted $[\pi_1 H]$, and one in F_n denoted $[H]$ which is characterized by the equation $\rho_*[H] = [\pi_1 H]$. More generally, for every proper subgraph $H \subset G$ with noncontractible components $H_1,\dots,H_k$ there is an associated free factor system in F_n denoted $[H] = \{[H_1],\dots,[H_k]\}$.

We record the following elementary lemma for easy reference. Given an F–marked graph (G,ρ) and a natural subforest $E \subset G$, there is a quotient map $\pi\colon G \to G/E$ that collapses each component to a point. Each component of E clearly has ≥ 3 incident edges which are not in E, implying that every vertex of G/E has valence ≥ 3, and so G_i is indeed a core graph. The map π is a homotopy equivalence, and so by precomposing π with the marking on G we may regard G/E as a marked graph. We refer to $\pi\colon G \to G/E$ as a *forest collapse of marked graphs*.

Lemma 14 *Given a forest collapse $\pi\colon G \to G/E = G'$ of marked graphs, let $\widetilde E \subset \widetilde G$ be the full preimage of E in the universal cover of G and let $\widetilde\pi\colon \widetilde G \to \widetilde G'$ be the F–equivariant lift of π that collapses each component of $\widetilde E$ to a point.*

(1) *$\widetilde\pi$ induces a bijection between the set of edges of $\widetilde G - \widetilde E$ and the set of edges of $\widetilde G'$ and similarly for π_F.*

(2) *For any edge path in* $\widetilde{G}$ *— a finite arc, a ray, or a line — its* $\tilde{\pi}$*–image in* $\widetilde{G}'$ *is the edge path obtained by erasing the edges in* $\widetilde{E}$ *and identifying the remaining edges via the bijection of item (1) and similarly for edge paths in* G*.*

(3) *For each finitely generated subgroup* $B < G$*, the* $\tilde{\pi}$ *image of the* B*–minimal subtree of* $\widetilde{G}$ *is the* B*–minimal subtree of* $\widetilde{G}'$*.*

Proof Items (1) and (2) are obvious. Item (3) follows from items (1) and (2) and the fact that the minimal B–subtree is characterized as the union of all axes in T for the action of all nontrivial elements of B. □

The spine of outer space $\mathcal{K}_F$ This is an ordered simplicial complex whose set of 0–simplices corresponds bijectively to F–marked graphs up to equivalence, which using universal covering maps corresponds bijectively to proper, free splittings of F up to conjugacy. For each 0–simplex $\mathcal{K}_F$ represented by a marked graph G, for each $k \ge 1$, and for each properly increasing sequence of natural subforests $\varnothing = E_0 \subset E_1 \subset \cdots \subset E_k$ of G, there is an ordered k–simplex denoted $\Sigma(G; E_1 \subset \cdots \subset E_k)$ whose i^{th} vertex is represented by the marked graph $G_i = G_0/E_i$ obtained from $G = G_0$ by a collapsing the forest E_i. In particular, for any forest collapse $G = G_0 \mapsto G_1 = G/E$ we have an oriented 1–simplex $\Sigma(G; E)$ with initial vertex G_0 and terminal vertex G_1.

Lemma 15 (Facts about the spine)

(1) [5] $\mathcal{K}_F$ *is contractible. In particular, its* 1*–skeleton is connected.*

(2) [5] *The action of* $\mathsf{Out}(F)$ *on equivalence classes of marked graphs induces a properly discontinuous, cocompact, simplicial action* $\mathsf{Out}(F) \curvearrowright \mathcal{K}_F$*.*

(3) *For any vertex* G *of* $\mathcal{K}_F$ *the orbit map* $\mathsf{Out}(F) \mapsto \mathcal{K}_F$ *taking* G *to* $G \cdot \phi$ *is a quasi-isometry from the word metric on* $\mathsf{Out}(F)$ *to the simplicial metric on the* 1*–skeleton of* $\mathcal{K}_F$*.* □

Remarks on the proof Item (3) follows from item (1), item (2), local finiteness, and the Milnor–Švarc Lemma.

For later purposes we remind the reader of the proofs of the various finiteness properties of the action $\mathsf{Out}(F) \curvearrowright \mathcal{K}_F$. Cocompactness means that there are only finitely many orbits of simplices, for which it suffices to show that there are only finitely many vertex orbits. Indeed there is one orbit for each simplicial isomorphism class of core graphs whose rank equals $\operatorname{rank}(F)$ equipped with the natural simplicial structure, and there are only finitely many such classes because the number of natural edges is bounded above by $3\operatorname{rank}(F) - 3$. The subgroup of $\mathsf{Out}(F)$ that stabilizes each simplex is finite;

for 0–simplices this follows because the stabilizer in $\mathsf{Out}(F)$ equals the group of simplicial isomorphisms of the underlying core graph. Local finiteness follows because each marked graph G has only finitely many natural subgraphs. From all of this it follows that the action of $\mathsf{Out}(F)$ on $\mathcal{K}_F$ is properly discontinuous.

The spine of autre espace $\mathbb{A}_F$ An *F–marked pointed graph* consists of a triple (G, p, ρ), where G is a finite core graph, $p \in G$, and $\rho\colon (R_F, p_F) \to (G, p)$ is a pointed homotopy equivalence; when $\operatorname{rank}(F) \geq 2$ the point $p_F \in R_F$ is the unique natural vertex, whereas when $\operatorname{rank}(F) = 1$ the point $p_F \in R_F$ is chosen arbitrarily. We adopt the convention that G is equipped with the *relatively natural cell structure*, obtained from the natural cell structure by subdividing at the point p. Throughout Sections 1.5 and 2 we refer to this convention with phrases like "relatively natural vertex", "relative natural subforest", etc. Two F–marked pointed graphs (G, p, ρ), (G', p', ρ') are *pointed equivalent* if there exists a homeomorphism $h\colon (G, p) \to (G', p')$ such that $h\rho$ and $\rho'\colon (R_F, p_F) \to (G', p')$ are homotopic relative to p_F. One can also consider *proper, pointed free splittings* of F, that is, free splittings $F \curvearrowright T$ equipped with a base point $P \in T$, modulo the equivalence relation of *base point preserving conjugacy*.

The spine of autre espace is an ordered simplicial complex $\mathbb{A}_F$ whose 0–simplices correspond bijectively to F–marked pointed graphs up to pointed equivalence, which by universal covering space theory correspond bijectively to proper, pointed free splittings of F up to base point preserving conjugacy. For each 0–simplex represented by a pointed marked graph (G, p, ρ), for each $k \geq 1$, and for each properly increasing sequence of *relatively* natural subforests $\emptyset \neq E_1 \subset \cdots \subset E_k$ of G, there is an ordered k–simplex whose i^{th} vertex is represented by the core graph G/E_i, with base point and marking obtained by pushing forward p and ρ via the forest collapse $f_i\colon G \to G/E_i$, yielding $p_i = f_i(p) \in G/E_i$ and $\rho_i = f_i \circ \rho\colon (R_F, p_F) \to (G/E_i, p_i)$. In particular, a 1–simplex of the ordered simplicial complex $\mathbb{A}_F$ with initial 0–simplex represented by (G, p, ρ) and terminal 0–simplex represented by (G', p', ρ') is represented by a forest collapse $h\colon G \to G'$ taking p to p' such that $h\rho$ and $\rho'\colon (R_F, p_F) \to (G', p')$ are homotopic rel p_F.

The action of the group $\mathsf{Aut}(F)$ on $\mathbb{A}_F$ is induced by the action on the vertex set which is defined by precomposing a marking $\rho\colon (R_F, p_F) \to (G, p)$ with a homotopy equivalence $(R_F, p_F) \mapsto (R_F, p_F)$ realizing a given automorphism. The complex $\mathbb{A}_F$ is connected, and the action of $\mathsf{Aut}(F)$ is properly discontinuous and cocompact, with one orbit for each relatively natural isomorphism class of pairs (G, p), where G is a core graph of rank equal to $\operatorname{rank}(F)$ and $p \in G$. Applying the Milnor–Švarc Lemma, the orbit map $\mathsf{Aut}(F_n) \mapsto \mathbb{A}_n$ is a quasi-isometry.

For later purposes we give a proof of connectivity of $\mathbb{A}_F$ by reducing it to connectivity of $\mathcal{K}_F$. There is a simplicial map $\mathbb{A}_F \mapsto \mathcal{K}_F$ which "forgets the base point", defined on the 0–skeleton by taking the equivalence class of the pointed marked graph (G, p, ρ) to the equivalence class of the unpointed marked graph (G, ρ). The preimage in $\mathbb{A}_F$ of the vertex of $\mathcal{K}_F$ represented by (G, ρ) is a connected 1–complex identified with the first barycentric subdivision of the natural cell structure on the universal covering space $\widetilde{G}$: choosing the basepoint of $\widetilde{G}$ to be any natural vertex determines a 0–simplex of the preimage; choosing the basepoint to lie in the interior of any natural edge of $\widetilde{G}$ also determines a 0–simplex; and allowing the basepoint to move from the interior of a natural edge to either of its natural endpoints determines a 1–simplex. To prove that $\mathbb{A}_F$ is connected, it therefore suffices to show that each 1–simplex of $\mathcal{K}_F$ can be lifted via the forgetful map to a 1–simplex of $\mathbb{A}_F$. Consider a 1–simplex in $\mathcal{K}_F$ represented by a forest collapse of marked graphs $h\colon G \mapsto G'$, with markings $\rho\colon R_F \to G$ and $\rho'\colon R_F \to G'$. By homotoping ρ and ρ' we may assume that $p = \rho(p_F) \in G$ is a natural vertex and that $\rho' = h \circ \rho$ and so $p' = h(p) \in G'$ is a natural vertex; and with these assumptions it follows that the pointed marked graphs (G, p, ρ), (G', p', ρ') represent endpoints of a 1–simplex in $\mathbb{A}_F$ which is a lift of the given 1–simplex of $\mathcal{K}_F$.

The spine of M–pointed autre espace $\mathbb{A}_F^M$ Fix an integer $M \geq 1$. For use in Section 3.2 we need a variation of the autre espace $\mathbb{A}_F$ with its action by $\mathsf{Aut}(F)$, which allows for an M–tuple of base points in a marked graph. The group that will act, denoted $\mathsf{Aut}^M(F)$, is the subgroup of the M–fold direct sum $\mathsf{Aut}(F) \oplus \cdots \oplus \mathsf{Aut}(F)$ consisting of all M–tuples $(\Phi_1, \ldots, \Phi_M)$ whose images in $\mathsf{Out}(F)$ are all equal.

Define an *M–pointed F–marked graph* to be a tuple

$$(G; p_1, \ldots, p_M; \rho_1, \ldots, \rho_M) \quad \text{or, in shorthand,} \quad (G; p_m; \rho_m),$$

where G is a finite core graph, and for $m = 1, \ldots, M$ we have points $p_m \in G$ and pointed homotopy equivalences $\rho_m\colon (R_F, p_F) \to (G, p_m)$ such that the unpointed homotopy equivalences $\rho_1, \ldots, \rho_M\colon R_F \to G$ are all in the same free homotopy class. The *relatively natural* cell structure on G is the one whose vertex set is the union of the natural vertices of G and the set $\{p_1, \ldots, p_M\}$. Given another M–pointed F–marked graph $(G'; p'_m; \rho'_m)$, we say that the two are *pointed equivalent* if there is a homeomorphism $h\colon (G; p_1, \ldots, p_M) \to (G'; p'_1, \ldots, p'_M)$ such that for each m the pointed homotopy equivalences $h \circ \rho_m$, $\rho'_m\colon (R_F, p_F) \to (G', p'_m)$ are homotopic relative to p_F.

Alternatively, define a *proper, M–pointed free splitting* of F to be a free splitting $F \curvearrowright T$ equipped with an M–tuple of points $P_1, \ldots, P_M \in T$, modulo the equivalence relation of *M–tuple preserving conjugacy*. Again the universal covering map produces a bijection between equivalence classes of M–pointed F–marked graphs and M–tuple preserving conjugacy classes of proper, M–pointed free splittings. We shall describe this bijection explicitly. Consider an M–pointed F–marked graph $(G; p_m; \rho_m)$. In the universal cover $\widetilde{G} = T$ choose a lift P_1 of p_1, which by universal covering space theory determines an action $F \curvearrowright T$ which is a proper free splitting. Also, let $\widetilde{R}_F$ be the universal covering space of R_F, choose a lift P_F of p_F, determining an action $F \curvearrowright \widetilde{R}_F$. Let $\widetilde{\rho}_1 \colon (\widetilde{R}_F, P_F) \to (T, P_1)$ be the unique F–equivariant lift of $\rho_1 \colon (R_F, p_F) \to (G, p_1)$. For each $m = 2, \ldots, M$, any free homotopy between ρ_1 and $\rho_m \colon R_F \to G$ lifts uniquely to an equivariant homotopy between $\widetilde{\rho}_1$ and a certain lift $\widetilde{\rho}_m \colon \widetilde{R}_F \to T$ of ρ_m. Let $P_m = \widetilde{\rho}_m(P_F)$. The M–tuple $(P_1, \ldots, P_M)$ in T together with the action $F \curvearrowright T$ is a proper M–pointed free splitting, and this construction defines the desired bijection.

Define an ordered simplicial complex $\mathbb{A}_F^M$ whose 0–simplices are M–pointed F–marked graphs up to pointed equivalence, or alternatively M–pointed free splittings up to M–tuple preserving conjugacy. An ordered k–simplex of $\mathbb{A}_F^M$ is defined very much as for $\mathbb{A}_F$, starting with a choice of a 0–simplex represented by $(G; p_m; \rho_m)$ and a choice of a strictly increasing sequence of relatively natural subforests $\emptyset \neq E_1 \subset \cdots \subset E_k \subset G$; the i^{th} vertex has underlying graph G/E_i, with points and markings obtained by pushing forward $p_1, \ldots, p_M$ and $\rho_1, \ldots, \rho_M$. In particular there is an edge from the vertex represented by $(G; p_m; \rho_m)$ to the vertex represented by $(G'; p'_m; \rho'_m)$ if and only if there is a relatively natural forest collapse $h \colon G \mapsto G'$ such that for each $i = 1, \ldots, m$ we have $h(p_m) = p'_m$ and the homotopy equivalences $\rho'_m, h \circ \rho_m \colon (R_F, p_F) \to (G', p'_m)$ are homotopic rel p_F.

Connectivity of $\mathbb{A}_F^M$ is proved by induction on M, as follows. The map which forgets the last point p_m and marking ρ_M is a simplicial map $\mathbb{A}_F^M \mapsto \mathbb{A}_F^{M-1}$. As in the proof of connectivity of $\mathbb{A}_F$, the preimage in $\mathbb{A}_F^M$ of the vertex of $\mathbb{A}_F^{M-1}$ represented by $(G; p_1, \ldots, p_{M-1}; \rho_1, \ldots, \rho_{M-1})$ is simplicially isomorphic to the universal covering space of G equipped with the lift of the first barycentric subdivision of the relatively natural cell structure of $(G, p_1, \ldots, p_{M-1})$. Each 1–simplex of $\mathbb{A}_F^{M-1}$ lifts via the forgetful map to a 1–simplex of $\mathbb{A}_F^M$. The induction step follows immediately.

There is a natural group action $\mathsf{Aut}^M(F) \curvearrowright \mathbb{A}_F^M$, where the action of $(\Phi_1, \ldots, \Phi_M)$ on a 0–simplex represented by $(G; p_m; \rho_m)$ is the 0–simplex represented by $(G; p_m; \rho'_m)$, where the marking $\rho'_m \colon (R_F, p_F) \mapsto (G, p_m)$ is obtained by precomposing the marking $\rho_m \colon (R_F, p_F) \mapsto (G, p_m)$ with a homotopy equivalence of the pair (R_F, p_F) that represents the automorphism $\Phi_m \in \mathsf{Aut}(F)$. Since all of the automorphisms Φ_m are in the same outer automorphism classes, and since all of the marking ρ'_m are freely homotopic, it follows that all of the composed markings ρ'_m are still freely homotopic, and so $(G, p_m; \rho'_m)$ does indeed represent a 0–simplex of $\mathbb{A}_F^M$. The proof that this action is properly discontinuous and cocompact with finite cell stabilizers is a simple generalization of the proof for the action $\mathsf{Aut}(F) \curvearrowright \mathcal{A}_F$.

2 The $\mathsf{Aut}(F_{n-1})$ subgroup of $\mathsf{Aut}(F_n)$

In this section we prove Theorem 3: the naturally embedded subgroup

$$\mathsf{Aut}(F_{n-1}) < \mathsf{Aut}(F_n)$$

is a Lipschitz retract of $\mathsf{Aut}(F_n)$. Fixing the free basis $F_n = \langle a_1, \dots, a_{n-1}, a_n \rangle$, and identifying $F_{n-1} = \langle a_1, \dots, a_{n-1} \rangle < F_n$, this subgroup consists of all $\phi \in \mathsf{Aut}(F_n)$ that preserve the subgroup $\langle a_1, \dots, a_{n-1} \rangle$ and fix the element a_n. We assume that the edges of the rank n rose R_n correspond to $a_1, \dots, a_n$ and that the rank $(n-1)$ rose R_{n-1} is identified with the subrose of R_n corresponding to $a_1, \dots, a_{n-1}$.

The $\mathsf{Aut}(F_{n-1})$ equivariant embedding $\mathbb{A}_{n-1} \overset{j}{\hookrightarrow} \mathbb{A}_n$ The embedding j is defined as follows. Start with an F_{n-1}–marked pointed graph (H, p, ρ_H) representing a point in the spine $\mathbb{A}_{n-1}$, then attach to p a loop of length 1 to form a marked graph (G, p, ρ_G), where $\rho_G | R_{n-1}$ equals ρ_H, and ρ_G takes the a_n loop of R_n around the newly attached loop of G.

The image $j(\mathbb{A}_{n-1}) \subset \mathbb{A}_n$ is invariant under the action of the subgroup $\mathsf{Aut}(F_{n-1}) < \mathsf{Aut}(F_n)$. Since $\mathsf{Aut}(F_n)$ and $\mathsf{Aut}(F_{n-1})$ act properly discontinuously and cocompactly on $\mathbb{A}_n$ and $j(\mathbb{A}_{n-1})$ respectively, the hypotheses of Corollary 10(b) apply. We conclude that in order to prove that $\mathsf{Aut}(F_{n-1})$ is a Lipschitz retract of $\mathsf{Aut}(F_n)$ it suffices to prove that the 0–skeleton of $j(\mathbb{A}_{n-1})$ is a Lipschitz retract of the 0–skeleton of $\mathbb{A}_n$.

Retracting the 0–skeleton of $\mathbb{A}_n$ to the 0–skeleton of $j(\mathbb{A}_{n-1})$ Given an F_n–marked pointed graph (G, p, ρ) representing an arbitrary 0–simplex of $\mathbb{A}_n$ we will construct an F_{n-1}–marked pointed graph $r(G, p, \rho) = (K, q, \rho_K)$ representing a 0–simplex of $\mathbb{A}_{n-1}$ such that the restriction of rj to the 0–skeleton of $\mathbb{A}_{n-1}$ is the identity. The map $R = jr$ is then the desired retraction.

Let $(\widetilde{G}, \widetilde{p})$ be the universal cover with base point $\widetilde{p}$ that projects to p and with the action of F_n determined by ρ and the choice of $\widetilde{p}$. Let $S \subset \widetilde{G}$ be the minimal F_{n-1}–subtree and let $\widetilde{q} \in S$ be the nearest point to $\widetilde{p}$ in S. The quotient space (K, q) of $(S, \widetilde{q})$ by the action of F_{n-1} is an F_{n-1}–marked pointed graph. The marking ρ_K maps the edge in R_{n-1} corresponding to a_j to the projected image of the path in S from $\widetilde{q}$ to $\widetilde{q} \cdot a_j$. Define $r(G, p, \rho) = (K, q, \rho_K)$.

Given a 0–simplex w of $\mathbb{A}_{n-1}$, letting $j(w) = (G, p, \rho_G)$ as in the definition of j, there is a connected core subgraph $H \subset G$ that contains p so that $\rho | R_{n-1}\colon R_{n-1} \to H$

is a homotopy equivalence and such $(H, p, \rho|R_{n-1})$ represents w. In this case, the minimal F_{n-1}–subtree $S \subset \widetilde{G}$ is a component of the full preimage of H and $\widetilde{p} = \widetilde{q} \in S$. Thus $r(G, p, \rho) = (H, p, \rho|R_{n-1})$ and $rj(w) = w$.

The Lipschitz constant of R is 1 If w and w' are 0–simplices of $\mathbb{A}_{n-1}$ that bound an edge in $\mathbb{A}_n$ then $j(w)$ and $j(w')$ bound an edge in $\mathbb{A}_n$. Thus j has Lipschitz constant 1. We will prove that r has Lipschitz constant 1 and hence that $R = jr$ has Lipschitz constant 1. It suffices to show that if (G, p, ρ) and (G', p', ρ') are the endpoints of an edge in $\mathbb{A}_n$ then

$$r(G, p, \rho) = (K, q, \rho_K) \quad \text{and} \quad r(G', p', \rho') = (K', q', \rho_{K'})$$

are either equal or are the endpoints of an edge in $\mathbb{A}_{n-1}$. In other words, if (G', p', ρ') is obtained from (G, p, ρ) by collapsing each component of a nontrivial relatively natural forest $F \subset G$ to a point then $(K', q', \rho_{K'})$ is obtained from (K, q, ρ_K) by collapsing each component of a (possibly trivial) relatively natural forest F' to a point.

Let $\widetilde{F} \subset \widetilde{G}$ be the full preimage of F in the universal cover $\widetilde{G}$ of G and, as above, let S be the minimal F_{n-1} subtree of $\widetilde{G}$. Lift the quotient map $\pi_F\colon G \to G'$ to the pointed universal covers, obtaining a map $\pi_{\widetilde{F}}\colon (\widetilde{G}, \widetilde{p}) \to (\widetilde{G}', \widetilde{p}')$ that collapses each component of $\widetilde{F}$ to a point. Lemma 14 implies that $S' = \pi_{\widetilde{F}}(S)$ is the minimal F_{n-1} subtree of $\widetilde{G}'$ and that the $\pi_{\widetilde{F}}$–image of the arc connecting $\widetilde{p}$ to $\widetilde{q}$ is an arc in $\widetilde{G}'$ connecting $\widetilde{p}'$ to $\widetilde{q}' := \pi_{\widetilde{F}}(\widetilde{q})$ that intersects S' only in $\widetilde{q}'$, proving that $\widetilde{q}' \in S'$ is the nearest point to $\widetilde{p}'$ in S'. Moreover, the arc in S' from $\widetilde{q}'$ to $\widetilde{q}' \cdot a_j$ is the $\pi_{\widetilde{F}}$–image of the arc in S from $\widetilde{q}$ to $\widetilde{q} \cdot a_j$. Let $\widetilde{F}'' = \widetilde{F} \cap S$ and let $F'' \subset K$ be the projected image of $\widetilde{F}''$. Then $(K', q', \rho_{K'})$ is obtained from (K, q, ρ_K) by collapsing each component of F'' to a point. However, the components of F'' need not be unions of relatively natural edges of K. We therefore define the relatively natural forest F' by including a relatively natural edge in F' if and only if it is entirely contained in F''. It is easy to check that replacing F'' by F' does not change the 0–simplex of $\mathbb{A}_{n-1}$ obtained by collapsing components to points. If F' is empty then $r(G, p, \rho) = r(G', p', \rho')$. Otherwise, collapsing the components of F' to points defines an edge in $\mathbb{A}_{n-1}$ with endpoints $r(G, p, \rho)$ and $r(G', p', \rho')$.

3 Stabilizers of free splittings in $\mathsf{Out}(F_n)$

In this section we prove Theorem 7: for any free splitting $F_n \curvearrowright T$, the subgroup $\mathsf{Stab}[T]$ is a Lipschitz retract of $\mathsf{Out}(F_n)$.

Throughout this section we drop the subscript n, using $\mathcal{K}$ to denote the spine of rank n outer space. By applying Corollary 10, it is sufficient to find a coarse Lipschitz retraction from the spine of outer space $\mathcal{K}$ to a nonempty, connected subcomplex which is invariant by and cocompact under the action of $\mathsf{Stab}[T]$. In Section 3.2 we construct such a subcomplex $\mathcal{K}^T \subset \mathcal{K}$; the construction and verification of the properties needed to apply Corollary 10 will be obtained by showing that $\mathcal{K}^T$ is simplicially isomorphic to a certain Cartesian product of multipointed autre espaces. In Section 3.3 we construct a coarse Lipschitz retraction $\mathcal{K} \mapsto \mathcal{K}^T$. As in the proof of Theorem 3, there are two steps in defining the retraction. The first uses minimal subtrees to define marked subgraphs realizing a given free factor system. In the context of $\mathbb{A}_n$, the free factor system is just $[F_{n-1}]$, in our current context it is $\mathcal{F}(T)$, and these minimal subtrees are found in precisely the same manner in these two different contexts. In the second step, the marked subgraphs are extended to marked graphs of full rank. This is essentially trivial in the context of $\mathbb{A}_n$ because the marked subgraph is naturally pointed and the full marked graph is obtained by attaching a marked loop to the basepoint. In our current context, this extension step is considerably more subtle; we refer to this as the "coedge attaching problem" in what follows. See the paragraph "Defining the map R" and the remark at the end of Section 3.3.

3.1 An informal sketch for a special case

Here is a sketch of the proof of Theorem 7 in the special case of a free splitting $F_n \curvearrowright T$ with one vertex orbit and one edge orbit, so that with respect to the free basis $F_{n-1} = \langle a_1, \dots, a_n \rangle$ there exists a vertex v of T whose stabilizer is the free factor $A = \langle a_1, \dots, a_n \rangle \approx F_{n-1}$. This also serves as a sketch of the proof of Theorem 1(1), together with an understanding that $\mathsf{Stab}[A] = \mathsf{Stab}[T]$, which is proved in Section 4.

In this special case, the subcomplex $\mathcal{K}^T \subset \mathcal{K}$ has as vertices all free splittings $F_n \curvearrowright S$ such that the minimal A–invariant subtree $S^A \subset S$ is disjoint from all of its translates by elements of $F_n - A$, and S has a unique edge orbit not contained in S^A. The group $\mathsf{Stab}[T]$ acts properly and cocompactly on the subcomplex $\mathcal{K}^T$; for details see Lemma 16.

By Corollary 10, existence of a Lipschitz retract $\mathsf{Out}(F_n) \mapsto \mathsf{Stab}[T]$ is equivalent to existence of a Lipschitz retract $\mathcal{K} \mapsto \mathcal{K}^T$ defined on the 0–skeleton.

Given a vertex $S' \in \mathcal{K}$, our task is therefore to construct a vertex of $\mathcal{K}^T$ denoted S. The given S' certainly has a well-defined minimal A–invariant subtree S'^A. However, the translates of S'^A might not be pairwise disjoint. This problem is easily remedied: pull all of these translates apart, forcing them to be pairwise disjoint. One obtains a forest $\widehat{S}$ on which F_n acts properly and cocompactly: as g varies over a collection of left coset representatives for A, the component $\widehat{S}^{gAg^{-1}}$ of $\widehat{S}$ stabilized by the conjugate subgroup gAg^{-1} may be regarded as a separate copy of the gAg^{-1}–minimal subtree of S'.

The tricky part of the proof is to attach an orbit of edges, which we refer to as "coedges", so as to make the forest $\widehat{S}$ into the desired tree S: picking $g \in F_n$ so that we have a free factorization $F_n = A * \langle g \rangle$, we must attach a coedge E with initial endpoint on $\widehat{S}^A$ and terminal endpoint on $\widehat{S}^{gAg^{-1}}$, and then we must extend this to an equivariant attachment of coedges. Furthermore, we must choose the initial and terminal endpoints of E in a manner which is quasi-isometrically natural, in order that the map $\mathcal{K} \mapsto \mathcal{K}^T$ taking the free splitting S' to the free splitting S be a Lipschitz retract as desired.

Solving this coedge attachment problem was for us the key step in proving Theorem 7. Our solution for finding the initial attaching point for E requires the choice of a parameter consisting of a point ξ in the Gromov boundary of F_n not lying in the boundary of the free factor A. For each 0–simplex S' of $\mathcal{K}$, one locates the unique point of S'^A that is closest to ξ in the Gromov compactification of S', and one uses the corresponding point of $\widehat{S}^A$ as the initial attaching point of E. Another similar choice is needed to determine the terminal attaching point of E on $\widehat{S}^{gAg^{-1}}$.

At the end of Section 3.3 we give a *Remark* that some seemingly natural schemes for attaching coedges fail to satisfy the Lipschitz requirement.

3.2 Free splittings, their stabilizers and their subcomplexes

Given a free splitting $F_n \curvearrowright T$ we shall construct a subcomplex of $\mathcal{K}$ which is a geometric model for $\mathsf{Stab}[T]$ as follows:

Lemma 16 *There exists a connected flag subcomplex $\mathcal{K}^T \subset \mathcal{K}$ preserved by $\mathsf{Stab}[T]$ such that the action $\mathsf{Stab}[T] \curvearrowright \mathcal{K}^T$ is properly discontinuous and cocompact. It follows that $\mathsf{Stab}[T]$ is finitely generated, and that for any 0–simplex $[S]$ of $\mathcal{K}^T$ the orbit map $\mathsf{Stab}[T] \mapsto \mathcal{K}^T$ taking ϕ to $[S] \cdot \phi$ is a quasi-isometry.*

The second sentence follows from the first by the Milnor–Švarc Lemma.

After the initial description of $\mathcal{K}^T$ in Definition 17, and after establishing some further notation used in this section and the next, we shall prove the remaining clauses of Lemma 16 by reducing them to a more precise and technical description of the combinatorial structure of $\mathcal{K}^T$ and the algebraic structure of (a finite index subgroup of) $\mathsf{Stab}[T]$. That description will be given in Lemma 18.

For the rest of the section we fix the free splitting $F \curvearrowright T$ and an F–invariant natural simplicial structure on T. Let $\mathcal{V}^{\mathrm{nt}} = \mathcal{V}^{\mathrm{nt}}(T)$ be the vertices of T with nontrivial stabilizer. For any $\phi \in \mathsf{Stab}[T]$ and any representative $\Phi \in \mathsf{Aut}(F_n)$ we denote by $h_\Phi \colon T \to T$ the simplicial homeomorphism uniquely characterized by the property that $h_\Phi \circ \Phi(g) = g \circ h_\Phi \colon T \to T$ for all $g \in F_n$.

Definition 17 (Definition of $\mathcal{K}^T$) Given a proper free splitting $F_n \curvearrowright S$, a semiconjugacy $f \colon S \to T$ is *tight* if the following hold:

(1) For any $V \in \mathcal{V}^{\mathrm{nt}}(T)$ the subgraph $S^V := f^{-1}(V)$ is the $\mathsf{Stab}(V)$–minimal subtree of S.

(2) The map f is injective over the set $T - \mathcal{V}^{\mathrm{nt}}(T)$.

The 0–simplices $\mathcal{K}_0^T$ are those proper free splittings $F_n \curvearrowright S$ for which there exists a tight semiconjugacy $f \colon S \to T$. Define $\mathcal{K}^T \subset \mathcal{K}$ to be the flag subcomplex with 0–skeleton $\mathcal{K}_0^T$.

$\mathsf{Stab}[T]$ invariance of $\mathcal{K}^T$ follows from invariance of $\mathcal{K}_0^T$: given $\phi \in \mathsf{Stab}[T]$ and a 0–simplex $[S] \in \mathcal{K}_0^T$ with tight semiconjugacy $f \colon S \to T$, for any representative $\Phi \in \mathsf{Aut}(F)$ it follows that $h_\Phi \circ f \colon S \to T$ is a tight semiconjugacy for $[S] \cdot \phi$ which is therefore in $\mathcal{K}_0^T$.

Notational conventions for T and $\mathcal{K}^T$ Let L be the number of orbits of the action $F_n \curvearrowright \mathcal{V}^{\mathrm{nt}}$ on the vertices of T with nontrivial stabilizer. Let $X = T/F_n$ be the quotient graph of groups, with simplicial structure inherited from the natural structure on T. Let $X^{\mathrm{nt}} = \{v_1, \ldots, v_L\} \subset X$ be image of $\mathcal{V}^{\mathrm{nt}}$, in other words, the vertices of X that are labeled by nontrivial subgroups. For each $\ell = 1, \ldots, L$, let M_ℓ denote the valence of v_ℓ in X^{nt}, and let $\eta_{\ell 1}, \ldots, \eta_{\ell M_\ell}$ denote an enumeration of the oriented edges of X^{nt} with initial vertex v_ℓ. Choose a lift $V_\ell \in \mathcal{V}^{\mathrm{nt}}$ of each v_ℓ, and let $A_\ell = \mathsf{Stab}(V_\ell)$, so $\mathcal{F}(T) = \{[A_1], \ldots, [A_L]\}$. The action of A_ℓ on oriented edges of T with initial vertex V_ℓ has M_ℓ orbits; choose orbit representatives denoted $\widetilde\eta_{\ell 1}, \ldots, \widetilde\eta_{\ell M_\ell}$, with $\widetilde\eta_{\ell m}$ mapping to $\eta_{\ell m}$.

With notation fixed as in the previous paragraph, henceforth we will let ℓ take values in $1, \ldots, L$ and, when the value of ℓ is fixed, we will let m take values in $1, \ldots, M_\ell$.

Define the *coedge forest* of T to be the F_n–forest $\widehat T$ obtained from T by removing the vertex subset $\mathcal{V}^{\mathrm{nt}}$ and taking the completion: for each ℓ, m and each $g \in F_n$, after removing the initial vertex $g \cdot V_\ell$ of the oriented edge $g \cdot \widetilde\eta_{\ell m} \subset T$, we add a valence 1 vertex denoted $\widehat V_{\ell m g} \in \widehat T$ in its place. There is a natural equivariant simplicial quotient map $\widehat T \to T$ that is infinite-to-one over each $g \cdot V_\ell$ and is otherwise injective, taking each $\widehat V_{\ell m g}$ to $g \cdot V_\ell$.

Given a tight semiconjugacy $f\colon S \to T$, denoting the map to the quotient marked graph as $q\colon S \to S/F_n = G$, and recalling from Definition 17 the minimal subtree S^{V_ℓ} for the restricted action of $\mathsf{Stab}(V_\ell)$, the subgraphs $H_\ell = q(S^{V_\ell})$ are pairwise disjoint, connected, core subgraphs representing the free factor system $\mathcal{F}(T) = \{[A_1], \dots, [A_L]\}$. Letting $[S] = [G]$ be the initial 0–simplex of some k–simplex $\Sigma(G; E_1, \dots, E_k) \subset \mathcal{K}$, we have $\Sigma(G; E_1, \dots, E_k) \subset \mathcal{K}^T$ if and only if $E_i \subset H_1 \cup \cdots \cup H_L$ for each $i = 1, \dots, k$, if and only if $E_k \subset H_1 \cup \cdots \cup H_L$.

Explicit description of $\mathsf{Stab}[T]$ For any $\phi \in \mathsf{Stab}[T]$ and any choice of $\Phi \in \mathsf{Aut}(F_n)$ representing ϕ, the simplicial homeomorphism $h_\Phi\colon T \to T$ descends to a corresponding simplicial homeomorphism of the quotient graph of groups X which is well-defined independent of the choice of Φ. This defines a simplicial action $\mathsf{Stab}[T] \curvearrowright X$ whose kernel is a finite index subgroup denoted $\mathsf{Stab}^f[T]$.

The formula for $\mathsf{Stab}^f[T]$ in (1) of the following lemma is a special case of a more general and more detailed formula previously given by Levitt in [11, Proposition 4.2].

Lemma 18 *For each free splitting $F \curvearrowright T$ with notation as above, there exists a group isomorphism*

$$\mathsf{Stab}^f[T] \approx \mathsf{Aut}^{M_1}(A_1) \oplus \cdots \oplus \mathsf{Aut}^{M_L}(A_L) \tag{1}$$

and a simplicial homeomorphism

$$\mathcal{I}\colon \mathcal{K}^T \approx \mathbb{A}^{M_1}_{A_1} \times \cdots \times \mathbb{A}^{M_L}_{A_L} \tag{2}$$

such that the actions $\mathsf{Stab}^f[T] \curvearrowright \mathcal{K}^T$ and $\mathsf{Aut}^{M_\ell}(A_\ell) \curvearrowright \mathbb{A}^{M_\ell}_{A_\ell}$ agree, for each ℓ.

Lemma 16 is an immediate consequence of Lemma 18 and the fact, proved in Section 1.5, that $\mathsf{Aut}^{M_\ell}(A_\ell) \curvearrowright \mathbb{A}^{M_\ell}_{A_\ell}$ is a properly discontinuous, cocompact action on a connected complex.

We also note the following special case of Lemma 18 which will be useful below for proving Theorem 2.

Corollary 19 *For any free splitting $F \curvearrowright T$ with quotient graph of groups $X = T/G$ and for any vertex $V \in X$, if V is labeled with a nontrivial free factor A, if every other vertex of X is labeled with the trivial group, and if V has valence 1 in T, then $\mathsf{Stab}[T]$ has a finite index subgroup isomorphic to $\mathsf{Aut}(A)$.* □

Proof of Lemma 18 We shall construct the simplicial homeomorphism $\mathcal{I}$ only on 1–skeleta, which suffices for the applications; the full description of $\mathcal{I}$ is left to the reader.

The homeomorphism $\mathcal{I}$ on 0–skeleta Given $[S] \in \mathcal{K}_0^T$ with tight semiconjugacy $f\colon S \to T$, for each ℓ, m let $\widehat{\eta}_{\ell m}$ be the unique natural oriented edge of S such that $f(\widehat{\eta}_{\ell m}) = \widetilde{\eta}_{\ell m}$. Let $Q_{\ell m} \in S^{V_\ell}$ be the initial vertex of $\widehat{\eta}_{\ell m}$; note that if $g \in F_n$, the point $g \cdot Q_{\ell m} \in S^{g \cdot V_\ell}$ is the initial vertex of $g \cdot \widehat{\eta}_{\ell m}$. Taking the action $A_\ell \curvearrowright S^{V_\ell}$ together with the M_ℓ tuple of points $Q_{\ell 1}, \ldots, Q_{\ell M_\ell} \in S^{V_\ell}$, we obtain an M–pointed proper A_ℓ splitting representing a 0–simplex $\mathcal{I}_\ell[S] \in \mathbb{A}_{A_\ell}^{M_\ell}$. Define the 0–simplex $\mathcal{I}[S] = (\mathcal{I}_1[S], \ldots, \mathcal{I}_L[S]) \in \mathbb{A}_{A_1}^{M_1} \times \cdots \times \mathbb{A}_{A_L}^{M_L}$.

For the inverse of $\mathcal{I}$, a 0–simplex of $\mathbb{A}_{A_1}^{M_1} \times \cdots \times \mathbb{A}_{A_L}^{M_L}$ is determined by the following data for each ℓ: a proper, free splitting $A_\ell \curvearrowright S_\ell$, and an M_ℓ–tuple of points $Q_{\ell 1}, \ldots, Q_{\ell M_\ell} \in S_\ell$. From this data we construct a 0 simplex $[S] \in \mathcal{K}^T$ as follows. Defining $S^{V_\ell} = S_\ell$, the L–tuple of actions $A_\ell \curvearrowright S^{V_\ell}$ extends uniquely (up to conjugacy of F_n actions on graphs) to an ambient action $F_n \curvearrowright \bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$ on a forest having one component S^V for each $V \in \mathcal{V}^{\mathrm{nt}}$ so that S^V is stabilized by $\mathsf{Stab}[V]$ and so that the action of $\mathcal{A}_\ell = \mathsf{Stab}[V_\ell]$ on $S^{V_\ell} = S_\ell$ is the action originally given; a formal description of this action, which we leave to the reader, comes from the standard representation theory formula for inducting a subgroup representation up to a representation of the ambient group.

Define S be the quotient graph of the disjoint union of the coedge forest $\widehat{T}$ and the forest $\bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$ obtained by equivariantly identifying each valence 1 vertex $\widehat{V}_{\ell m g} \in \widehat{T}$ with the point $g \cdot Q_{\ell m} \in S^{g \cdot V_\ell}$. The actions of F_n on $\widehat{T}$ and $\bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$ clearly combine to give an action $F_n \curvearrowright S$. The equivariant maps $\widehat{T} \mapsto T$, $\bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V \mapsto \mathcal{V}^{\mathrm{nt}} \subset T$ clearly induce an equivariant map $f\colon S \mapsto T$. The graph S is a tree because T is a tree and the inverse image of each point of T is a subtree of S. Minimality of the action $F \curvearrowright S$ follows from minimality of the actions $A_\ell \curvearrowright S_\ell$ and $F_n \curvearrowright T$. The map $S \mapsto T$ is a tight semiconjugacy by construction.

Clearly these two constructions are inverse functions of each other, and so $\mathcal{I}$ is indeed a bijection of 0–skeleta in (2).

The homeomorphism $\mathcal{I}$ on 1–skeleta Consider a proper, free splitting $F \curvearrowright S$ and quotient marked graph $q\colon S \to S/F = G$ representing a 0–simplex $[S] = [G] \in \mathcal{K}^T$. It suffices to prove that $\mathcal{I}$ restricts to a bijection between the set of terminal 0–simplices of those 1–simplices in $\mathcal{K}^T$ with initial point $[G]$, and the set of terminal 0–simplices of those 1–simplices in $\mathbb{A}_{A_1}^{M_1} \times \cdots \times \mathbb{A}_{A_L}^{M_L}$ with initial point $\mathcal{I}[G]$. The image 0–simplex $\mathcal{I}[G] = (\mathcal{I}_1[G], \ldots, \mathcal{I}_L[G])$ is as described above, where each $\mathcal{I}_\ell[G]$ is represented by an M_ℓ–pointed A_ℓ–marked graph with underlying graph $H_\ell = q(S^{V_\ell})$ and M_ℓ–tuple of points $q(Q_{\ell m}) \in H_\ell$; the M_ℓ–tuple of pointed markings is understood. The 1–simplices of $\mathcal{K}^T$ with initial vertex $[G]$ have the form $\Sigma(G; E)$, where $E \subset H_1 \cup \cdots \cup H_L$ is a nonempty natural subforest of G. Each intersection $E_\ell = E \cap H_\ell$ determines an ordered simplex of $\mathbb{A}^{M_\ell}_{A_\ell}$ having initial 0–simplex $\mathcal{I}_\ell[S]$, of dimension 0 or 1 depending on whether E_ℓ is empty or nonempty, at least one of which has dimension 1 because at least one of E_ℓ is nonempty; taking these together we obtain a 1–simplex $\mathcal{I}(\Sigma(G;E)) \in \mathbb{A}^{M_1}_{A_1} \times \cdots \times \mathbb{A}^{M_L}_{A_L}$ with initial point $\mathcal{I}[G]$. Clearly an arbitrary such L–tuple $(E_1,\ldots,E_L)$ can occur, since we can choose any subforest $E_\ell \subset H_\ell$, empty or not, as long as at least one is nonempty, and then we take $E = E_1 \cup \cdots \cup E_L$. We therefore obtain a bijection of 1–simplices with initial point $[G]$ and 1–simplices with initial point $\mathcal{I}[G]$, and from the construction the terminal points of these 1–simplices clearly correspond under $\mathcal{I}$, as required.

The action isomorphism for $\mathsf{Stab}^f[T]$ For each $\phi \in \mathsf{Stab}^f[T]$, and each ℓ, all representatives $\Phi \in \mathsf{Aut}(F_n)$ of ϕ have the property that h_Φ preserves the F–orbit of each V_ℓ, and so amongst these representatives there exists one that satisfies $h_\Phi(V_\ell) = V_\ell$; any such Φ preserves $A_\ell = \mathsf{Stab}(V_\ell)$ and therefore gives an outer automorphism of A_ℓ which is well-defined independent of the choice of Φ, thereby producing a homomorphism $\theta_\ell\colon \mathsf{Stab}^f[T] \to \mathsf{Out}(A_\ell)$. For each ℓ, m, all representatives $\Phi \in \mathsf{Aut}(F_n)$ of ϕ for which $h_\Phi(V_\ell) = V_\ell$ have the property that h_Φ preserves the orbit of the oriented edge $\widetilde\eta_{\ell m}$ under the action of $\mathsf{Stab}(V_\ell)$, and so amongst these representatives there exists one which satisfies the further restriction that $h_\Phi(\widetilde\eta_{\ell m}) = \widetilde\eta_{\ell m}$. This gives a homomorphism $\Theta^m_\ell\colon \mathsf{Stab}^f[T] \to \mathsf{Aut}(A_\ell)$ well-defined independent of the restricted choice of Φ. By construction each Θ^m_ℓ is a lift of θ_ℓ, and so letting m vary we obtain a homomorphism

$$\Theta_\ell = (\Theta^m_1,\ldots,\Theta^{M_\ell}_\ell)\colon \mathsf{Stab}^f[T] \longrightarrow \mathsf{Aut}^{M_\ell}(A_\ell).$$

Letting ℓ vary we obtain a homomorphism

$$\Theta = (\Theta_1,\ldots,\Theta_L)\colon \mathsf{Stab}^f[T] \longrightarrow \mathsf{Aut}^{M_1}(A_1) \oplus \cdots \oplus \mathsf{Aut}^{M_L}(A_L).$$

By construction, the two actions of $\mathsf{Stab}^f[T]$ of $\mathbb{A}^{M_1}_{A_1} \times \cdots \times \mathbb{A}^{M_L}_{A_L}$ coincide; one coming from the homomorphism Θ, and the other from the action $\mathsf{Stab}^f[T] \curvearrowright \mathcal{K}^T$ followed by the simplicial isomorphism

$$\mathcal{K}^T \xrightarrow{\mathcal{I}} \mathbb{A}^{M_1}_{A_1} \times \cdots \times \mathbb{A}^{M_L}_{A_L}.$$

From injectivity of the action $\mathsf{Stab}^f[T] \curvearrowright \mathcal{K}^T$ it follows that Θ is injective.

For surjectivity of Θ we switch to the marked graph point of view. Pick a marked graph G representing a vertex of $\mathcal{K}^T$, with subgraphs $H_1,\ldots,H_L$ representing $[A_1],\ldots,[A_L]$. By collapsing a maximal forest of each H_ℓ to a point we may assume that H_ℓ is a rose whose base vertex q_ℓ equals its frontier in G. By projecting a tight semiconjugacy $\widetilde G \mapsto T$ we obtain a quotient map $G \mapsto X$ which collapses each H_ℓ

to v_ℓ and is otherwise injective. For ℓ, m let $e_{\ell m}$ be the oriented edge of G whose image in X is $\eta_{\ell m}$, and so $e_{\ell m}$ has initial vertex q_ℓ. Let $\epsilon_{\ell m}$ be the initial quarter of the oriented 1–simplex $e_{\ell m}$, and so as ℓ, m vary the oriented arcs $\epsilon_{\ell m}$ are pairwise disjoint except possibly at their initial points. Picking a base point for G and paths to q_ℓ for each ℓ, and rechoosing A_ℓ in its conjugacy class if necessary, we determine an isomorphism $A_\ell \approx \pi_1(H_\ell, q_\ell)$. An arbitrary $\Psi \in \mathsf{Aut}^{M_1}(A_1) \oplus \cdots \oplus \mathsf{Aut}^{M_L}(A_L)$ is determined by choosing $\Psi_{\ell m} \in \mathsf{Aut}(A_\ell)$ for each ℓ, m, subject to the requirement that for fixed ℓ all of the $\Psi_{\ell m}$ represent the same element of $\mathsf{Out}(A_\ell)$. This data is realized by a homotopy equivalence $f \colon G \to G$ as follows. First, f is the identity on $G \setminus ((\bigcup_\ell H_\ell) \cup (\bigcup_{\ell m} \epsilon_{\ell m}))$. Next, for each ℓ, f is the identity on $\epsilon_{\ell 1}$ and $f \mid H_\ell \colon (H_\ell, q_\ell) \to (H_\ell, q_\ell)$ is chosen to represent $\Psi_{\ell 1}$. Finally, for each ℓ and each $m = 2, \ldots, M_\ell$, the automorphisms $\Psi_{\ell 1}$ and $\Psi_{\ell m}$ differ by an inner automorphism of A_ℓ which is represented by a closed path $\gamma_{\ell m}$ in H_ℓ based at q_ℓ, and $f(\epsilon_{\ell m}) = \gamma_{\ell m}\epsilon_{\ell m}$. The element $\theta \in \mathsf{Out}(F)$ defined by the homotopy equivalence $f \colon G \to G$ is clearly in $\mathsf{Stab}^f[T]$, and by construction $\Theta(\theta) = \Psi$.

This completes the proof of Lemma 18. □

3.3 Retracting $\mathcal{K}$ to $\mathcal{K}^T$

To prove Theorem 7, by combining Corollary 10, Lemma 11 and Lemma 16 it suffices to prove:

Theorem 20 *For any free splitting $F_n \curvearrowright T$, there exists a coarse Lipschitz retraction of 0–skeleta $R \colon \mathcal{K}_0 \mapsto \mathcal{K}_0^T$.*

Throughout this section we will freely use the "Notational conventions" established in Section 3.2.

Suppose that we are given the following data:

(D1) An action $F_n \curvearrowright \widehat{S} = \bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$ on a forest $\widehat{S}$ having one component S^V for each $V \in \mathcal{V}^{\mathrm{nt}}$, so that S^V is stabilized by $\mathsf{Stab}[V]$ and the action $\mathsf{Stab}[V] \curvearrowright S^V$ is properly discontinuous and minimal.

(D2) A point $Q_{\ell m} \in S^{V_\ell}$, for each $\ell = 1, \ldots, L$ and each $m = 1, \ldots, M_\ell$.

From (D1) and (D2) we construct a 0–simplex $[S] \in \mathcal{K}_0^T$ using the exact method used in the proof of Lemma 16 for verifying surjectivity of the map $\mathcal{I}$ on 0–skeleta. As a graph, S is the disjoint union of the coedge forest $\widehat{T}$ and the forest $\widehat{S} = \bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$, modulo the identification of each valence 1 vertex $\widehat{V}_{\ell m g} \in \widehat{T}$ with the point $g \cdot Q_{\ell m} \in S^{g \cdot V_\ell}$. The actions of F_n on $\widehat{T}$ and on $\bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$ are clearly consistent with these identifications, and so they combine to induce a simplicial action $F_n \curvearrowright S$. There is an equivariant simplicial map $f\colon S \to T$ induced by the map that takes S^V to $V \in \mathcal{V}^{\text{nt}} \subset T$ and by the natural quotient map $\widehat{T} \to T$. Since the preimage of each point of T is a point or a tree in S, the graph S is a tree. The action $F_n \curvearrowright S$ is minimal because of minimality of the actions $\mathsf{Stab}[V] \curvearrowright S^V$ and $F_n \curvearrowright T$. By construction the action $F_n \curvearrowright S$ is proper, and $f\colon S \to T$ is a tight semiconjugacy, and so we have the desired 0–simplex $[S]$.

Defining the map $R\colon \mathcal{K}_0 \to \mathcal{K}_0^T$ Given an arbitrary 0–simplex $[S'] \in \mathcal{K}$ represented by a proper, free splitting $F_n \curvearrowright S'$, for each $V \in \mathcal{V}^{\text{nt}}$ let $S^V \subset S'$ be the minimal subtree for the restricted action $\mathsf{Stab}(V) \curvearrowright S'$. These subtrees S^V need not be pairwise disjoint in S'. Pull them apart by forming their disjoint union over $\mathcal{V}^{\text{nt}}$, resulting in a forest $\widehat{S} = \coprod_{V \in \mathcal{V}^{\text{nt}}} S^V$ equipped with an action of F_n that evidently satisfies data condition (D1); formally one may identify this disjoint union with the set of ordered pairs $(x, V) \in S' \times \mathcal{V}^{\text{nt}}$ such that $x \in S^V$.

The central idea of the proof is a method for describing the (D2) data of points $Q_{\ell m} \in S^{V_\ell}$ that is needed in order to complete with the definition of $R[S']$. Choose once and for all a base 0–simplex $[S_0] \in \mathcal{K}_0^T$, represented by a proper free splitting $F_n \curvearrowright S_0$ with tight semiconjugacy $S_0 \mapsto T$. Each of the oriented edges $\widetilde{\eta}_{\ell m}$ in T has a unique preimage in S_0 that we denote $e^0_{\ell m}$; the initial endpoint of $e^0_{\ell m}$ is in $S_0^{V_\ell}$. Choose once and for all, for each ℓ and m, a ray $\gamma^0_{\ell m}$ in S_0 with initial oriented edge $e^0_{\ell m}$. Let $\xi_{\ell m} \in \partial F_n$ be the ideal endpoint of $\gamma^0_{\ell m}$. Note that $\xi_{\ell m} \notin \partial A_\ell$ because $e^0_{\ell m}$ points out of $S_0^{V_\ell}$. To specify the data (D2) for $R[S']$, for each ℓ, m let $Q'_{\ell m}$ be the point in S^{V_ℓ} closest to $\xi_{\ell m}$; that is, let $\gamma_{\ell m}(S')$ be the unique ray in S' with ideal endpoint $\xi_{\ell m}$ whose intersection with S^{V_ℓ} equals its base point, and let $Q'_{\ell m}$ be the base point of $\gamma_{\ell m}(S')$. In the disjoint union $\widehat{S} = \coprod_{V \in \mathcal{V}^{\text{nt}}} S^V$, let $Q_{\ell m} \in S^{V_\ell}$ be the point corresponding to $Q'_{\ell m}$ (formally $Q_{\ell m}$ is the ordered pair $(Q'_{\ell m}, V_\ell)$). This completes the data specification (D2), and we may now complete the definition of $R[S']$ as above.

Example Fix a basis $F_n = \langle a_1, \dots, a_{n-1}, a_n\rangle$ and identify $F_{n-1} = \langle a_1, \dots, a_{n-1}\rangle < F_n$ as usual. Let R_n be the rank n rose with oriented edges $e_1, \dots, e_n$ corresponding to $a_1, \dots, a_n$, and identify $R_{n-1} = e_1 \cup \dots \cup e_{n-1} \subset R_n$. Equip the universal cover $\widetilde{R}_n$ with an F_n action using a lift $P \in \widetilde{R}_n$ of the unique vertex $p \in R_n$ as a basepoint. Let $\widetilde{R}_{n-1}$ be the component of the full preimage of R_{n-1} that contains P. Let T be the free splitting obtained from $\widetilde{R}_n$ by collapsing to a point each translate of $\widetilde{R}_{n-1}$; let V be the point to which $\widetilde{R}_{n-1}$ itself collapses. The collapse map $\widetilde{R}_n \to T$ is a tight semiconjugacy, and we choose the base simplex $[S_0]$ of $\mathcal{K}_0^T$ to be represented by the proper free splitting $F_n \curvearrowright \widetilde{R}_n$. The quotient graph of groups $X = T/F_n$ is obtained from R_n by collapsing R_{n-1} to a single point v, and having a single edge identified with e_n. In X there are two oriented edges: η_1 which is e_n with positive orientation, and η_2 with opposite orientation. The action of a_n on T has an axis which passes through V; let $\tilde\eta_1$, $\tilde\eta_2$ be the lifts of η_1, η_2 on that axis with initial vertex V. The action of a_n on $\widetilde{R}_n$ has an axis which touches $\widetilde{R}_{n-1}$ uniquely at P, and this axis is a union of two rays γ_1^0, γ_2^0 intersecting at P, so that the initial oriented edge e_i^0 of γ_i^0 maps to $\tilde\eta_i$ under the collapse map $\widetilde{R}_n \to T$. With these choices, the ideal endpoints ξ_1, ξ_2 of γ_1^0, γ_2^0 are the attracting and repelling fixed points, respectively, for the action of a_n on ∂F_n. For any proper free splitting $F_n \curvearrowright S'$, letting $S^V \subset S'$ denote the minimal subtree for the action of F_{n-1} as usual, and letting $L^{a_n} \subset S'$ denote the oriented axis of a_n, if $L \cap S^V$ is empty or a single point then $Q_1' = Q_2'$ is the point of S^V closest to L, and otherwise Q_1', Q_2' are the terminal and initial points, respectively, of the oriented arc $L \cap S^V$.

The map R is a retraction We prove that R fixes each 0–simplex of $\mathcal{K}_0^T$, using induction on distance from $[S_0]$ in the 1–skeleton of $\mathcal{K}^T$, starting with $R[S_0] = [S_0]$ which is clear by construction. Consider an oriented 1–simplex of the form $\Sigma(G_1, E) \subset \mathcal{K}^T$, with initial 0–simplex $[S_1] = [G_1]$ and terminal 0–simplex $[S_2] = [G_2]$, where $G_2 = G_1/E$ and $E \subset G_1$ is a nontrivial natural subforest. The induction step reduces to proving that $R[S_1] = [S_1]$ if and only if $R[S_2] = [S_2]$; the "only if" direction is used when $[S_1]$ is closest to $[S_0]$, the "if" direction when $[S_2]$ is closest.

Let $\widetilde{E} \subset \widetilde{G}_1 = S_1$ be the total lift of E, and so $S_2 = \widetilde{G}_2$ is obtained from S_1 by collapsing to a point each component of the forest $\widetilde{E}$. Letting $g\colon S_1 \to S_2$ be the collapse map, and letting $f_i\colon S_i \to T$ be tight semiconjugacies, we note that $f_2 \circ g = f_1$. It follows that each edge of $\widetilde{E}$ is collapsed by f_1, which implies that $\widetilde{E} \subset \coprod_{V \in \mathcal{V}^{\mathrm{nt}}} S_1^V$.

For each $i = 1, 2$ and each ℓ, m, consider the unique oriented edge $e^i_{\ell m} \subset S_i$ for which $f_i(e^i_{\ell m}) = \tilde\eta_{\ell m}$, and consider also the ray $\gamma^i_{\ell m} \subset S_i$ with ideal endpoint $\xi_{\ell m}$ that intersects $S_i^{V_\ell}$ solely in its base point. Note that the trees $S_i^{V_\ell}$ are already pairwise disjoint in S_i as required for the (D1) data, and that the tree S_i can then be reconstructed using the (D2) data where $Q^i_{\ell m}$ is the initial point of $e^i_{\ell m}$. This shows that the equation $R[S_i] = S_i$ is equivalent to the statement that $e^i_{\ell m}$ is the initial edge of the ray $\gamma^i_{\ell m}$. It therefore remains to prove that $e^1_{\ell m}$ is the initial oriented edge of $\gamma^1_{\ell m}$ if and only if $e^2_{\ell m}$ is the initial oriented edge of $\gamma^2_{\ell m}$.

Let $e^i \subset \gamma^i_{\ell m}$ be the initial oriented edges. By Lemma 14, the image $g(\gamma^1_{\ell m}) \subset S_2$ is a ray. Since $g\colon S_1 \to S_2$ is an equivariant quasi-isometry, the ideal endpoint of $g(\gamma^1_{\ell m})$ is still $\xi_{\ell m}$. Since $e^1 \not\subset \bigcup_{V \in \mathcal{V}^{\mathrm{nt}}} S^V$, we have $e^1 \not\subset \widetilde{E}$, in other words e^1 is not collapsed by g. It follows that $g(e^1)$ is the initial oriented edge of $g(\gamma^1_{\ell m})$.

Also, since $g(S_1^{V_\ell}) = S_2^{V_\ell}$, it follows that the initial endpoint of $g(e^1)$ is the unique point in which $g(\gamma_{\ell m}^1)$ intersects $S_2^{V_\ell}$, and so $g(\gamma_{\ell m}^1) = \gamma_{\ell m}^2$ and $g(e^1) = e^2$. Since $f_2 \circ g = f_1$, it follows that $f_1(e^1) = \widetilde{\eta}_{\ell m}$ if and only if $f_2(e^2) = \widetilde{\eta}_{\ell m}$. Since f_1, f_2 are injective over each edge of T, it follows that $e^1 = e_{\ell m}^1$ if and only if $e^2 = e_{\ell m}^2$.

The Lipschitz constant is 1 Consider 0–simplices $[S_i'] = [G_i'] \in \mathcal{K}_0$ $(i = 1, 2)$ represented by proper free splittings $F \curvearrowright S_i'$ with quotient graphs of groups G_i', and suppose that $[S_1']$, $[S_2']$ are endpoints of a 1–simplex in $\mathcal{K}$. To show that the map R has Lipschitz constant 1 it suffices to show that $[S_1] = R[S_1']$, $[S_2] = R[S_2']$ are either equal or are endpoints of a 1–simplex in $\mathcal{K}^T$.

Up to reordering we have $G_2' = G_1'/E'$ for some nontrivial natural subforest $E' \subset G_1'$. Letting $\widetilde{E}' \subset \widetilde{G}_1' = S_1'$ be the full preimage, the quotient map $G_1' \mapsto G_2'$ lifts to an equivariant simplicial map

$$\pi'\colon S_1' \longrightarrow S_2'$$

that collapses each component of $\widetilde{E}'$ to a point. In particular, π' induces the identity map on ∂F_n. Letting $S_1^V \subset S_1'$, $S_2^V \subset S_2'$ denote the $\mathsf{Stab}(V)$ minimal subtrees for each $V \in \mathcal{V}^{\mathrm{nt}}$, Lemma 14 implies that $\pi'(S_1^V) = S_2^V$. Define $\widetilde{E}^V = \widetilde{E}' \cap S_1^V$, and note that the restricted map

$$\pi^V = (\pi' | S_1^V)\colon S_1^V \longrightarrow S_2^V$$

collapses each component of $\widetilde{E}^V$ to a point. After forming the disjoint union of the trees S_i^V over $\mathcal{V}^{\mathrm{nt}}$ as formally described earlier, and forming the disjoint union of the subforests $\widetilde{E}^V$ over $\mathcal{V}^{\mathrm{nt}}$ — formally described as those pairs (x, V), where $x \in \widetilde{E}^V$, $V \in \mathcal{V}^{\mathrm{nt}}$ — the collection of maps π^V induces an F–equivariant map of forests

$$\Pi\colon \widehat{S}_1 = \coprod_{V \in \mathcal{V}^{\mathrm{nt}}} S_1^V \longrightarrow \coprod_{V \in \mathcal{V}^{\mathrm{nt}}} S_2^V = \widehat{S}_2$$

that collapses to a point each component of the forest $\widetilde{E} = \coprod_{V \in \mathcal{V}^{\mathrm{nt}}} \widetilde{E}^V$.

We next extend the collapse map Π over the attachments of the coedge forest $\widehat{T}$. For each $i = 1, 2$ and each ℓ, m, let $\gamma_{\ell m}^i \subset S_i'$ be the unique ray with ideal endpoint $\xi_{\ell m}$ intersecting $S_i^{V_\ell}$ solely in its base points $Q_{\ell m}'^{i}$. By Lemma 14 it also follows that $\pi'(\gamma_{\ell m}^1)$ is a ray in S_2'. Its ideal endpoint is still $\xi_{\ell m}$. Let $\alpha \subset \gamma_{\ell m}^1$ be the largest initial subsegment of $\gamma_{\ell m}^1$ such that $\pi'(\alpha)$ is a point. Then it follows that $\pi'(Q_{\ell m}'^1) = \pi'(\alpha)$ is the initial point of the ray $\pi'(\gamma_{\ell m}^1)$, that this initial point is contained in $S_2^{V_\ell}$, and that this initial point is the sole point of intersection of the ray $\pi'(\gamma_{\ell m}^1)$ with $S_2^{V_\ell}$; it follows that

$$\pi'(\gamma_{\ell m}^1) = \gamma_{\ell m}^2 \quad \text{and} \quad \pi'(Q_{\ell m}'^1) = Q_{\ell m}'^2.$$

After pulling things apart, it follows that $\Pi(Q^1_{\ell m}) = Q^2_{\ell m}$. Since, in forming S_i, the valence 1 vertex $V_{\ell m} \in \widehat{T}$ is attached to $Q^i_{\ell m} \in \coprod S^V_i$, and since the attachments are extended equivariantly, it follows that Π extends to an equivariant map $\Pi\colon S_1 \to S_2$.

By construction, Π is the equivariant quotient map that collapses to a point each component of the forest $\widetilde{E} \subset \coprod S^V_1 \subset S_1$, and so Π descends to a map of quotient marked graphs

$$G_1 = S_1/F_n \xrightarrow{\pi} S_2/F_n = G_2$$

that collapses to a point each component of $E = \widetilde{E}/B_n \subset G_1$. If E contains no natural edge then the map π is homotopic to a homeomorphism and $[S_1] = [S_2]$. Otherwise, letting $\widehat{E} \subset E$ be the union of natural edges in E, the map $\pi\colon G_1 \to G_2$ is homotopic to a map which collapses each component of $\widehat{E}$ to a point, and so there is a 1–simplex $\Sigma(G_1, \widehat{E})$ with initial vertex $[S_1]$ and terminal vertex $[S_2]$. This completes the proof of Theorem 20.

Remark Our definition of $R\colon \mathcal{K}_0 \to \mathcal{K}^T_0$ depends on a fairly arbitrary choice of points $\xi_{\ell m} \in \partial F_n$. It is tempting to make more canonical choices using one of various other approaches, based on graph theoretic or metric considerations, or based on surgery arguments using Hatcher's sphere complex [8]. Indeed, we tried several such approaches but none of them gave Lipschitz retractions. As an illustration, we return to the example that follows the definition of R, using a construction that arises naturally from the point of view of the sphere complex. If $[S'] \in \mathcal{K}^T$ then $S^V \cap a_n \cdot S^V = \varnothing$ from which it follows that Q'_1 is the point of S^V closest to $a_n \cdot S^V$ and Q'_2 is the point of S^V closest to $a_n^{-1} \cdot S^V$. Otherwise, $S^V \cap a_n S^V$ is a finite tree τ (which has a natural description in sphere complex language), and we tried choosing Q'_1 to be the centroid of τ, and similarly Q'_2 to be the centroid of the tree

$$a_n^{-1}(\tau) = S^V \cap a_n^{-1} \cdot S^V.$$

This approach defines a retraction of $\mathcal{K}$ to $\mathcal{K}^T$ that one can show is not Lipschitz. We also considered approaches which somehow pick out a certain valence 1 vertex of τ (corresponding to picking out an "innermost disc" in sphere complex language), but no such approach that we tried yields a Lipschitz retraction.

3.4 Proof of Theorem 2

Consider a graph of groups X having the shape of a "sewing needle" with two vertices V, W and two edges A, B, the edge A having one end on V and one on W, the edge B having both ends on W, the vertex V labeled by the group F_{n-1}, the vertex W labeled by the trivial group, and both edges labeled by the trivial group. By Bass–Serre theory, the Serre fundamental group of X is isomorphic to F_n, we obtain an action $F_n \curvearrowright T$ on the Bass–Serre tree T of X with quotient graph of groups X, and this action is clearly a free splitting of F_n. By Corollary 19 the group $\mathsf{Stab}[T]$ has a finite index subgroup isomorphic to $\mathsf{Aut}(F_{n-1})$ (the automorphism group of F_{n-1} group that labels the vertex V). By Theorem 7 there is a Lipschitz retraction from $\mathsf{Out}(F_n)$ to $\mathsf{Stab}[T]$. Clearly there is a Lipschitz retraction from $\mathsf{Stab}[T]$ to its finite index $\mathsf{Aut}(F_{n-1})$ subgroup. Composing these gives a Lipschitz retract from $\mathsf{Out}(F_n)$ to an $\mathsf{Aut}(F_{n-1})$ subgroup. □

4 Stabilizers of free factor systems in $\mathsf{Out}(F_n)$

In this section we prove Theorem 8, regarding coarse Lipschitz retraction versus distortion for the subgroup $\mathsf{Stab}(\mathcal{F}) < \mathsf{Out}(F_n)$ stabilizing a free factor system $\mathcal{F}$ of F_n.

In Section 4.1 we prove Theorem 8 item (1); this is the *only* place in Section 4 where we will appeal to the results of either of Sections 2 or 3. In Section 4.2 we study a subcomplex $\mathcal{K}_n^{\mathcal{F}} \subset \mathcal{K}_n$ corresponding to free factor system $\mathcal{F}$, to which Corollary 10 can be applied. In Section 4.3 we prove distortion of $\mathsf{Stab}(\mathcal{F})$ in $\mathsf{Out}(F_n)$ when coindex$(\mathcal{F}) \ge 2$, by proving distortion of $\mathcal{K}_n^{\mathcal{F}}$ in $\mathcal{K}_n$.

4.1 Proof of Theorem 8 item (1)

Suppose that $\mathcal{F}$ is a free factor system of coindex 1 in F_n: either $\mathcal{F} = \{[A]\}$, where $A < F_n$ is a free factor of rank $n-1$; or $\mathcal{F} = \{[A_1],[A_2]\}$, where $F_n = A_1 * A_2$. Choose a marked graph G in which $\mathcal{F}$ is realized by a core subgraph H such that each component of H is a rose and its complement $G \setminus H$ is a single oriented edge E. Define T to be the free splitting obtained from the universal cover $\widetilde{G}$ of G by collapsing each component of the full preimage $\widetilde{H}$ of H to a point. In light of Theorem 7, which says that $\mathsf{Stab}[T]$ is a Lipschitz retract of $\mathsf{Out}(F_n)$ and is therefore undistorted, it suffices to show that $\mathsf{Stab}(\mathcal{F}) = \mathsf{Stab}[T]$.

The inclusion $\mathsf{Stab}[T] \subset \mathsf{Stab}(\mathcal{F})$ follows from $\mathcal{F}(T) = \mathcal{F}$, which is immediate from the definition of T, together with $\mathsf{Out}(F_n)$–equivariance of the function $T \mapsto \mathcal{F}(T)$. For the reverse inclusion, suppose that $\phi(\mathcal{F}) = \mathcal{F}$. Choose a homotopy equivalence $f\colon G \to G$ that represents ϕ, preserves H, and restricts to an immersion on E. By [2, Corollary 3.2.2] there are (possibly trivial) paths $u, v \in H$ such that either $f(E) = uEv$ or $f(E) = uE^{-1}v$. In either case, f lifts to $\widetilde{f} : \widetilde{G} \to \widetilde{G}$ that preserves $\widetilde{H}$, inducing a bijection on the set of components of $\widetilde{H}$, and such that the image of any edge in the complement of $\widetilde{H}$ crosses exactly one edge in the complement of $\widetilde{H}$. Thus $\widetilde{f}$ induces a conjugacy between the given action $F_n \curvearrowright T$ and the precomposition of that action by an automorphism $\Phi \in \mathsf{Aut}(F_n)$ that represents ϕ, and it follows that $\phi \in \mathsf{Stab}[T]$.

4.2 Free factor systems, their stabilizers and their subcomplexes

Fix a proper free factor system $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$ of F_n and define a flag subcomplex $\mathcal{K}_n^{\mathcal{F}}$ of $\mathcal{K}_n$ by requiring that a 0–simplex $[G, \rho] \in \mathcal{K}_n$ is in $\mathcal{K}_n^{\mathcal{F}}$ if and only if G has a core subgraph H satisfying $\mathcal{F} = [H]$.

Lemma 21 *$\mathcal{K}_n^{\mathcal{F}}$ is a connected subcomplex of $\mathcal{K}_n$, preserved by $\mathsf{Stab}(\mathcal{F})$, and with finitely many orbits of cells under the action of $\mathsf{Stab}(\mathcal{F})$. It follows that $\mathsf{Stab}(\mathcal{F})$ is finitely generated, and that for any 0–simplex $[G, \rho]$ of $\mathcal{K}_n^{\mathcal{F}}$ the orbit map $\mathsf{Stab}(\mathcal{F}) \mapsto \mathcal{K}_n^{\mathcal{F}}$ taking ϕ to $[G, \rho]\phi$ is a quasi-isometry.*

Proof The second sentence follows from the first, by applying the Milnor–Švarc Lemma.

To prove that $\mathsf{Stab}(\mathcal{F})$ leaves $\mathcal{K}_n^{\mathcal{F}}$ invariant, it suffices to prove that its 0–simplices are invariant. Consider a 0–simplex $[G, \rho]$ in $\mathcal{K}_n^{\mathcal{F}}$ with subgraph $H \subset G$ satisfying $[H] = \mathcal{F}$; to be more explicit, under the isomorphism $\rho_*\colon F_n = \pi_1(R_n) \to \pi_1(G)$ we have $\rho_*(\mathcal{F}) = [H]$. Consider $\phi \in \mathsf{Stab}(\mathcal{F})$ represented by a self-homotopy equivalence $\Phi\colon R_n \to R_n$, so $\Phi_*(\mathcal{F}) = \mathcal{F}$ and $(G, \rho)\phi = (G, \rho\Phi)$. We have $(\rho\Phi)_*(\mathcal{F}) = \rho_*\Phi_*(\mathcal{F}) = \rho_*(\mathcal{F}) = [H]$. It follows that the subgraph H still represents the free factor system $\mathcal{F}$ with respect to the marking $\rho\Phi$ on the graph G, proving that $[G, \rho]\phi \in \mathcal{K}_n^{\mathcal{F}}$.

To prove that $\mathcal{K}_n^{\mathcal{F}}$ has only finitely many $\mathsf{Stab}(\mathcal{F})$ orbits of cells, again it suffices to prove this for 0–cells. Let $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$. Consider the set of triples (G, ρ, H) such that $[G, \rho]$ is a 0–simplex of $\mathcal{K}_n$ and $H \subset G$ is a subgraph with k noncontractible components, having ranks equal to $\operatorname{rank}(A_1), \ldots, \operatorname{rank}(A_k)$, respectively. Define an equivalence relation on this set of triples, where (G, ρ, H) is equivalent to (G', ρ', H') if there exists an isometry $h\colon G \to G'$ such that $h(H) = H'$ and $h \circ \rho$ is homotopic to ρ'. The group $\mathsf{Out}(F_n)$ acts on such equivalence classes just as it does on marked graphs themselves, and there are only finitely many orbits of this action, by the same proof that $\mathcal{K}_n$ has only finitely many $\mathsf{Out}(F_n)$ 0–simplex orbits. It therefore suffices to prove that if (G, ρ) and (G', ρ') represent 0–simplices of $\mathcal{K}_n^{\mathcal{F}}$, if $H \subset G$, $H' \subset G'$ are subgraphs with $[H] = [H'] = \mathcal{F}$, and if the triples (G, ρ, H), (G', ρ', H') are in the same $\mathsf{Out}(F_n)$ orbit, then they are in the same $\mathsf{Stab}(\mathcal{F})$ orbit. Choose $\phi \in \mathsf{Out}(F_n)$ so that $(G, \rho, H)\phi$ is equivalent to (G', ρ', H'), so there exists an isometry $h\colon G \to G'$ and there exists a homotopy equivalence $\Phi\colon R_n \to R_n$ representing ϕ such that $h(H) = H'$ and $h \circ \rho$ is homotopic to $\rho' \circ \Phi$. We have

$$\phi_*\mathcal{F} = \phi_*\rho_*^{-1}[\pi_1 H] = (\rho')_*^{-1} h_*[\pi_1 H] = (\rho')_*^{-1}[\pi_1 H'] = \mathcal{F}$$

and so $\phi \in \mathsf{Stab}(\mathcal{F})$.

We prove connectivity of the subcomplex $\mathcal{K}_n^{\mathcal{F}}$ using a Stallings fold argument. Choose a base vertex $[G'] \in \mathcal{K}_n^{\mathcal{F}}$ so that each component of the core subgraph $H' \subset G'$ realizing $\mathcal{F}$ is a rose and so that all vertices of G' are contained in H' (we are suppressing F_n–markings from the notation for marked graphs). Given an arbitrary vertex $[G] \in \mathcal{K}_n^{\mathcal{F}}$ we must find an edge path in $\mathcal{K}_n^{\mathcal{F}}$ connecting $[G']$ to $[G]$. By successively collapsing to a point each edge of $G \setminus H$ having exactly one endpoint in H, and collapsing to a point each edge in H with distinct endpoints, we may assume that each component of H is a rose with one vertex and that all vertices of G are contained in H. By moving each component of H through the connected autre espace of the corresponding component of $\mathcal{F}$, we may assume that the components of H and H' corresponding to the same component $[A_i] \in \mathcal{F}$ are equivalent as pointed marked A_i graphs. It follows that there exists a homotopy equivalence $h\colon (G, H) \to (G', H')$ that preserves F_n–markings, takes vertices to vertices, and restricts to a homeomorphism $H \mapsto H'$. Furthermore, the restriction of h to each edge e of $G \setminus H$ is homotopically nontrivial rel endpoints, and so by homotoping $h|e$ rel endpoints we may assume that $h|e$ is an immersion.

Now factor h as a composition of Stallings folds

$$G = G_0 \xrightarrow{h_1} G_1 \longrightarrow \cdots \longrightarrow G_{k-1} \xrightarrow{h_k} G_k = G'$$

and recall the following properties: each G_i is a marked graph and each h_i preserves markings; for each $i = 0, \dots, k-1$, denoting $g_i = h_k \circ \cdots \circ h_{i+1}\colon G_i \to G'$, there exist oriented edges $e_{i1}, e_{i2} \subset G_i$ with the same initial vertex, and there exist initial segments $\eta_{i1} \subset g_{i1}$ such that $g_i(\eta_{i1})$ and $g_i(\eta_{i2})$ are equal as edge paths in G', the initial segments η_{i1}, η_{i2} are maximal with respect to this property, and the map h_i identifies the segments η_{i1}, η_{i2} respecting g_i–images. Since h restricts to a homeomorphism from H to H' it follows that in each G_i there is a subgraph H_i such that h_i restricts to a homeomorphism from H_{i-1} to H_i. It follows that each H_i is a core subgraph representing $\mathcal{F}$ and so $[G_i] \in \mathcal{K}_n^{\mathcal{F}}$ for each i.

Proceeding by induction on k, we may assume that $[G_1]$, $[G']$ are connected by an edge path in $\mathcal{K}_n^{\mathcal{F}}$, and it remains to find an edge path connecting $[G_0]$ and $[G_1]$. The fold map $h_1\colon G_0 \to G_1$ may be factored as

$$G_0 \xrightarrow{q'} G' \xrightarrow{q''} G_1,$$

where q' identifies initial segments of η_{i1} and η_{i2} which are proper in e_{i1}, e_{i2}, respectively, and q'' identifies the remaining segments of η_{i1}, η_{i2}. Since h_1 restricts to a homeomorphism from H_0 to H_1 it follows that q' restricts to a homeomorphism from H_0 to H' and q'' restricts to a homeomorphism from H' to H_1, and again it follows that H' is a core subgraph representing $\mathcal{F}$ and so $[G'] \in \mathcal{K}_n^{\mathcal{F}}$. There is a forest collapse $G' \to G_0$ which collapses to a point the natural segment $q'(\eta_{i1}) = q'(\eta_{i2}) \subset G'$, and so $[G], [G']$ are endpoints of an edge of $\mathcal{K}_n^{\mathcal{F}}$. There is also a forest collapse $G' \to G_1$ which collapses to a point the union of the two segments $q'(e_{i1} - \eta_{i1})$ and $q'(e_{i2} - \eta_{i2})$ (one or both of which may be proper segments of natural edges), and it follows that $[G']$, $[G_1]$ are either equal or connected by an edge. □

4.3 Stabilizers of free factor systems of coindex ≥ 2 in $\mathsf{Out}(F_n)$

In this section we prove the remaining half of Theorem 8: if $\mathcal{F}$ is a free factor system of F_n with coindex$(\mathcal{F}) \geq 2$ then $\mathsf{Stab}(\mathcal{F})$ is distorted in $\mathsf{Out}(F_n)$. To do this we construct a sequence $\phi_k \in \mathsf{Stab}(\mathcal{F})$ whose word length in $\mathsf{Out}(F_n)$ has a linear upper bound in terms of k but whose word length in $\mathsf{Stab}(\mathcal{F})$ has an exponential lower bound.

A sketch of the proof in the special case of a one component free factor system In the special case considered in Theorem 1(2), one has a free factor system $\mathcal{F} = \{[A]\}$ whose single component is the conjugacy class of a proper, nontrivial free factor $A < F_n$ of rank $r \leq n-2$. The sequence $\phi_k \in \mathsf{Stab}(\mathcal{F}) = \mathsf{Stab}[A]$ is described as follows. Recall the rose R_n with n oriented edges labeled $a_1, \ldots, a_n$, which allows us to identify $F_n \approx \langle a_1, \ldots, a_n \rangle \approx \pi_1(R_n)$. By conjugating A with the appropriate element of $\mathsf{Out}(F_n)$, we may assume that $A = \langle a_1, \ldots, a_r \rangle$ with $1 \leq r \leq n-2$. Let $\theta \in \mathsf{Out}(F_n)$ be defined by a homotopy equivalence $\Theta\colon R_n \to R_n$ which is the identity on $a_{r+2} \cup \cdots \cup a_n$, and whose restriction to the subgraph $a_1 \cup \cdots \cup a_{r+1}$ is an irreducible train track map of exponential growth (for this special case we make no other assumptions on Θ, although in the general case below we pick a particular Θ). By the train track property, for each $k \geq 0$ the path $u_k = \Theta^k(a_1)$ has no cancellation and so may be regarded as a reduced word in the generators $a_1, \ldots, a_{r+1}$. Now define $\phi_k \in \mathsf{Stab}[A]$ by the automorphism

$$\begin{cases} a_i \mapsto a_i & \text{if } i < n, \\ a_n \mapsto a_n u_k. \end{cases}$$

From the expression $\phi_k = \theta^k \phi_0 \theta^{-k}$ one sees immediately that the word length of ϕ_k in $\mathsf{Out}(F_n)$ has a linear upper bound in k. The hard work is to prove that the word length of ϕ_k in $\mathsf{Stab}[A]$ has an exponential lower bound. The key properties of u_k are that it is a reduced word in the letters of the rank $r+1$ free factor $B = \langle a_1, \ldots, a_r, a_{r+1} \rangle$, and that the number of occurrences in the word u_k of the letter a_{r+1} grows exponentially in k.

The technique at the heart of the proof is counting occurrences of a_{r+1}. Fixing any conjugacy class γ in F_n that is not represented by an element of B, for any $\psi \in \mathsf{Stab}[A]$

we will describe a way to count occurrences of the letter a_{r+1} in the conjugacy class $\psi(\gamma)$, and we will show that this count defines a Lipschitz function on $\mathsf{Stab}[A]$ with respect to word metric; see Lemma 22. Applying this to the conjugacy class $\gamma = [a_n]$ and to any factorization of $\psi = \phi_k$ as a word in the generators of $\mathsf{Stab}[A]$, it will follow that the word length of ϕ_k in $\mathsf{Stab}[A]$ is bounded below by a constant multiple of the number of occurrences of a_{r+1} in u_k, which has an exponential lower bound.

The sequence ϕ_k in general We denote $\mathcal{F} = \{[A_1], \ldots, [A_I]\}$. As needed we will impose further restrictions on the free factors $A_1, \ldots, A_I$. On first reading the proof, the reader may wish to focus on the special case $I = 1$ considered just above. To start the construction we choose some marked graphs and subgraphs. At first we describe these choices in general terms. Later we will specify further details of these choices in three separate cases, depending on whether the number I of components of $\mathcal{F}$ is 1, 2 or ≥ 3.

Choose a "base" marked graph G_b representing a vertex of the subcomplex $\mathcal{K}_n^{\mathcal{F}}$ of $\mathcal{K}_n$, with core subgraph $H_b \subset G_b$ representing the free factor system $\mathcal{F}$. Also choose an edge E_b of $G_b \setminus H_b$. Make these choices so as satisfy the following requirements. First, the components $H_b = H_{b,1} \cup \cdots \cup H_{b,I}$ are roses, with rose vertex denoted $x_{b,i} \in H_{b,i}$, so that $[H_{b,i}] = [A_i]$. Next, each endpoint of each edge of $G_b \setminus H_b$ equals some $x_{b,i}$; it follows that $G_b \setminus H_b$ consists of exactly coindex$(\mathcal{F})$ edges. Finally, if E_b is a loop then its base point is $x_{b,1}$, otherwise its two endpoints are $x_{b,1}, x_{b,2}$.

Define a subgraph $J_b \subset G_b$ as follows: if E_b is a loop then $J_b = H_{b,1}$; otherwise $J_b = H_{b,1} \cup H_{b,2}$. Let $K_b = J_b \cup E_b$. Let $L_b = H_b \cup E_b$, and so K_b is the component of L_b containing E_b. Denote the connected free factor system $\mathcal{B} = [K_b] = \{[B]\}$. From the hypothesis that coindex$(\mathcal{F}) \geq 2$ it follows that $B < F_n$ is a proper free factor of rank ≥ 2. The components of J_b determine a free factor system $\mathcal{A} \subset \mathcal{F}$ such that $\mathcal{A} \sqsubset \mathcal{B}$ and $\mathcal{A}$ has coindex 1 in $\mathcal{B}$. If E_b is a loop then we may choose A_1 so that $\mathcal{A} = \{[A_1]\}$ and $A_1 < B$ is a corank 1 free factor of B; otherwise we may choose A_1, A_2 so that $\mathcal{A} = \{[A_1], [A_2]\}$ and so that we have a free factorization $B = A_1 * A_2$.

Define a marked graph G_c (which *need not* represent a vertex of $\mathcal{K}_n^{\mathcal{F}}$) as follows: when E_b is a loop then $G_c = G_b$; and when E_b is not a loop then G_c is obtained from G_b by collapsing E_b. In either case the quotient map $G_b \mapsto G_c$ is a homotopy equivalence preserving marking which restricts to homotopy equivalences from $L_b, K_b \subset G_b$ to core subgraphs $L_c, K_c \subset G_c$, respectively, and so K_c is a component of L_c and $\mathcal{B} = [K_c] = [K_b]$. Note that K_c is a rose of rank ≥ 2: when E_b is a loop then $K_c = K_b = H_{b,1} \cup E_b$; otherwise K_c is obtained from $K_b = H_{b,1} \cup E_b \cup H_{b,2}$ by collapsing E_b.

Orient the edges of the rose K_c and enumerate them as $e_1, \dots, e_m$, which we can take as a basis for the free group B. Let $\theta \in \mathsf{Out}(F_n)$ be represented by a homotopy equivalence $\Theta\colon G_c \to G_c$ which is the identity on $G_c \setminus K_c$ and which restricts to the following exponentially growing train track map on K_c:

$$\begin{aligned}\Theta(e_1) &= e_1 e_m, \\ \Theta(e_i) &= e_{i-1} \quad \text{if } 2 \le i \le m,\end{aligned}$$

which is easily seen to be a homotopy equivalence since $m \ge 2$. This map Θ is a train track map, meaning that for all $k \ge 0$ the restriction of Θ^k to each edge e_i is an edge path $\Theta^k(e_i)$ without cancellation; this is true because Θ is a positive map, respecting orientation of edges. Also, by an application of the Perron–Frobenius Theorem [10], the action of Θ on K_c is exponentially growing, meaning that there exist constants $C > 0$, $r > 1$ such that for all $i, j = 1, \dots, m$ and all sufficiently large $k \ge 1$ the number of occurrences of e_j in the edge path $\Theta^k(e_i)$ exceeds Cr^k; again this uses that $m \ge 2$. Also let $\overline{\Theta}\colon G_c \to G_c$ be any homotopy inverse for Θ which is the identity on $G_c \setminus K_c$.

Pick an oriented edge $\eta \subset G_c \setminus L_c$ whose terminal vertex is the rose vertex of K_c. We define the sequence of outer automorphisms $\phi_k \in \mathsf{Out}(F_n)$, $k \ge 0$, by picking representative homotopy equivalences $\Phi_k\colon G_c \to G_c$ which restrict to the identity on $G_c \setminus \eta$ and which satisfy

$$\Phi_k(\eta) = \eta u_k, \quad \text{where } u_k = \Theta^k(e_1).$$

Notice that if $\mathcal{F}$ has one component, whose rank r must therefore satisfy $1 \le r \le n-2$, then (replacing "e" by "a") by taking $F_n = \langle a_1, \dots, a_n \rangle$, $\mathcal{F} = \{[\langle a_1, \dots, a_r \rangle]\}$, $B = \langle a_1, \dots, a_{r+1} \rangle$, and $\eta = a_n$, this sequence $\phi_k \in \mathsf{Out}(F_n)$ is exactly the one described in the special case above.

Clearly $\Theta^k \Phi_0 = \Phi_k \Theta^k$, so $\phi_k = \theta^k \phi_0 \theta^{-k} \in \mathsf{Out}(F_n)$, an expression which patently demonstrates that the word length of ϕ_k in the group $\mathsf{Out}(F_n)$ has a linear upper bound in k. Also, since $\mathcal{F} \sqsubset [L_b] = [L_c] \sqsubset [G_c \setminus \eta]$ we have $\phi_k \in \mathsf{Stab}(\mathcal{F})$. For the same reason, ϕ_k also stabilizes the free factor systems $\mathcal{A}$ and $\mathcal{B}$, that is, $\phi_k(\mathcal{A}) = \mathcal{A}$ and $\phi_k(\mathcal{B}) = \mathcal{B}$, a fact which we will use below.

The exponential lower bound It remains to prove that the word length of ϕ_k in the group $\mathsf{Stab}(\mathcal{F})$ has an exponential lower bound.

We shall define a nonnegative integer-valued function $i_\gamma(G)$, where the parameter γ is a conjugacy class in F_n that is not represented by an element of the free factor B, and where the argument G is a marked graph representing a vertex of the subcomplex

$\mathcal{K}_n^{\mathcal{F}} \subset \mathcal{K}_n$, so that $i_\gamma(G)$ is a well-defined function of the vertex represented by G. The function $i_\gamma(G)$ depends implicitly on two other parameters as well, namely the free factor systems $\mathcal{A}$ and $\mathcal{B}$, and although we usually suppress this dependence, we shall write $i_\gamma(G) = i(G;\gamma,\mathcal{A},\mathcal{B})$ when we need to emphasize all three parameters. Our strategy for finding an exponential lower bound to the word length of ϕ_k will be to prove in Lemma 22 that for each fixed γ the function $i_\gamma(G)$ is a 2–Lipschitz function on the vertex set of $\mathcal{K}_n^{\mathcal{F}}$, and then to exhibit a value of the parameter γ such that $i_\gamma(G_b \cdot \phi_k)$ has an exponential lower bound in k.

We set the notation needed to define $i_\gamma(G)$. Let $H \subset G$ be the core subgraph representing $\mathcal{F}$, with components $H = H_1 \cup \cdots \cup H_I$ so that $[H_i] = [A_i]$. Let $J = \bigcup_j H_j$ be the union of the one or two components of H representing $\mathcal{A}$, so $j \in \{1\}$ if $\mathcal{A}$ has one component and $j \in \{1,2\}$ otherwise. In the universal cover $\widetilde{G}$ consider the minimal B–subtree $S \subset \widetilde{G}$, with quotient B–marked graph $K = S/B$. The composition $S \hookrightarrow \widetilde{G} \mapsto G$ factors as

$$S \mapsto S/B = K \xrightarrow{q} G,$$

where q is an immersion, and so K is the so-called "Stallings graph" of B. For each j consider the A_j–minimal subtree $S_j \subset S$, and note that the universal covering map $\widetilde{G} \mapsto G$ restricts to a universal covering map $S_j \to H_j$; passing to the quotient for each j, we obtain an embedding $J \hookrightarrow K$ whose composition with $q\colon K \to G$ equals the inclusion map $J \hookrightarrow G$. Since $\mathcal{A} = [J]$ has coindex 1 in $\mathcal{B} = [K]$ it follows that the complement $K \setminus J$ has one of the following forms: either $K \setminus J$ is a single natural edge denoted $E \subset K$ having both endpoints in J; or $K \setminus J$ is a *lollipop* consisting of a single natural loop edge disjoint from J denoted E, and another edge connecting the vertex of E with some vertex of J.

For any $g \in F_n$ representing γ we define a nonnegative integer $i(g,G)$ as follows. Let $\mathrm{Axis}(g) \subset \widetilde{G}$ be the axis of the action of g on $\widetilde{G}$. Since $g \notin B$, the intersection $\mathrm{Axis}(g) \cap S$ is a (possibly empty) finite path in S that projects to a (possibly empty) finite path μ in K whose endpoints need not be natural vertices of K. Define $i(g,G)$ to be the number of times that μ entirely crosses the natural edge E in either direction.

Now define

$$i_\gamma(G) = \sup_{g \in \gamma} i(g,G).$$

To see that this is finite, note that $i(bgb^{-1},G) = i(g,G)$ for all $b \in B$ because $\mathrm{Axis}(g) \cap S$ and $\mathrm{Axis}(bgb^{-1}) \cap S = b(\mathrm{Axis}(g)) \cap S = b(\mathrm{Axis}(g) \cap S)$ project to the same path in K. For any given compact subset $X \subset \widetilde{G}$ there exist only finitely many g representing γ such that $\mathrm{Axis}(g) \cap X$ is nontrivial. Choosing $X \subset S$ to be a compact fundamental domain for the action of B, if g represents γ and if $\mathrm{Axis}(g)$ has nontrivial intersection with S then there exists b such that $b(\mathrm{Axis}(g)) = \mathrm{Axis}(bgb^{-1})$ has nontrivial intersection with X. It follows that $i_\gamma(G)$ is the maximum value of $i_g(G)$ as g varies over the finite subset of γ for which $\mathrm{Axis}(g) \cap X$ is nonempty. Since each $i_g(G)$ is finite by construction, the maximum value is also finite.

We note the following properties of $i_\gamma(G) = i(G; \gamma, \mathcal{A}, \mathcal{B})$:

Conjugacy invariance $i(G; \gamma, \mathcal{A}, \mathcal{B})$ depends only on the equivalence class of the marked graph G and it depends only on $\mathcal{A}$ and $\mathcal{B}$ not on the choice of representative subgroups A_j, B.

Equivariance $i(G \cdot \psi; \gamma, \mathcal{A}, \mathcal{B}) = i(G; \psi(\gamma), \psi(\mathcal{A}), \psi(\mathcal{B}))$ for each $\psi \in \mathsf{Stab}(\mathcal{F})$.

Note that for the quantity $i(G; \gamma, \mathcal{A}, \mathcal{B})$ to make sense it is necessary that $\mathcal{A}$ be represented by some subgraph of G, which is guaranteed by the requirement that G represents a vertex of $\mathcal{K}_n^{\mathcal{F}}$. That same requirement also guarantees that the two sides of the equivariance condition are defined for any $\psi \in \mathsf{Stab}(\mathcal{F})$ (as long as γ is not represented by an element of B).

We also note an important special case in which $i_\gamma(G)$ can be computed "by inspection", namely, when the immersion $g\colon K \to G$ is an embedding. In this case $g(E)$ is either a based embedded loop or an embedded arc intersecting J at most at its endpoints (in the lollipop case the intersection is empty). For any conjugacy class γ represented by a circuit σ in G that is not contained in K, the number $i_\gamma(G)$ may be computed as follows: enumerate the maximal subpaths of σ contained in K, count the number of occurrences of $g(E)$ in each such subpath, and take the maximum count over all such subpaths of σ.

The quantity $i_\gamma(G)$ plays the same role in our proof of an exponential lower bound as is played by the function denoted $\widetilde{\alpha}(\gamma)$ in Alibegović's paper [1], where an infinite cyclic subgroup $\mathsf{Out}(F_n)$ is proved to be undistorted by providing a linear lower bound to the word length of the powers of the generator. The quantity $\widetilde{\alpha}(\gamma)$ is a "count" of the maximum number of consecutive fundamental domains of other group elements that occur inside the axes of elements in the conjugacy class γ. [1, Lemma 2.4] says that this count barely changes under small moves, namely when $\widetilde{\alpha}(\gamma)$ is replaced by $\widetilde{\alpha}(g(\gamma))$ for a generator g of $\mathsf{Out}(F_n)$.

The following lemma shows that $i_\gamma(G)$ barely changes under small moves, namely as G moves along an edge in $\mathcal{K}_n^{\mathcal{F}}$.

Lemma 22 *For any nontrivial conjugacy class γ in F_n which is not represented by an element of B, the map $G \mapsto i_\gamma(G)$ is an integer-valued 2–Lipschitz function on the vertex set of the complex $\mathcal{K}_n^{\mathcal{F}}$.*

Proof Consider two marked graphs G, G' representing 0–simplices in $\mathcal{K}_n^{\mathcal{F}}$ that are endpoints of a 1–simplex. Given $g \in F_n$ representing the conjugacy class γ, in particular $g \notin B$, it suffices to prove that $|i(g,G) - i(g,G')| \le 2$.

The notation for defining $i_\gamma(G)$ and $i(g,G)$ was set above, and we adopt similar notation for $i_\gamma(G')$ and $i(g,G')$ as follows. The subgraph of G' realizing $\mathcal{A} = \{[A_j]\}_j$ is $J' = \bigcup_j H_j'$, with $j \in \{1\}$ when $\mathcal{A}$ has one component and $j \in \{1,2\}$ otherwise. The B–minimal subtree of $\widetilde{G}'$ is $S' \subset \widetilde{G}'$, with quotient $K' = G'/B$ and induced immersion $K' \mapsto G'$. The A_j–minimal subtree of S' is $S_j' \subset S'$, and passing to quotients we have an induced embedding $J' \hookrightarrow K'$ so that the composition $J' \hookrightarrow K' \mapsto G'$ is the usual embedding $J' \hookrightarrow G'$. There is a natural edge $E' \subset K'$ such that either $K' \setminus J' = E'$ or $K' \setminus J'$ is a lollipop consisting of the loop edge E' disjoint from J' and another edge connecting the base point of E' to a vertex in J'. The axis of the action of γ on $\widetilde{G}'$ is $\mathrm{Axis}'(\gamma)$, the intersection $S' \cap \mathrm{Axis}'(\gamma)$ is finite, and the image of this intersection is a finite path μ' in K'. The number $i(g,G')$ equals the number of times that μ' crosses E' in either direction.

Up to reversing the order of G, G' we may assume there is a quotient map $\pi\colon G \to G'$ that preserves the marking and that collapses to a point each component of a natural subforest $F \subset G$. Lift π to an F_n–equivariant quotient map $\widetilde{\pi}\colon \widetilde{G} \to \widetilde{G}'$ that collapses to a point each component of the full preimage $\widetilde{F} \subset \widetilde{G}$ of F. By Lemma 14, $\widetilde{\pi}(S) = S'$, and so there is an induced quotient map $\pi_K\colon K \mapsto K'$ which is a homotopy equivalence. This quotient map collapses to a point each component of the subforest $F_K \subset K$ which is the downstairs image in K of the B–equivariant subforest $\widetilde{F}_S = S \cap \widetilde{F}$ of S. Also by Lemma 14, $\widetilde{\pi}(S_j) = S_j'$ for each j, and so $\pi_K(J) = J'$ and $\pi_K(K \setminus J) = K' \setminus J'$.

In what follows one should keep in mind that F_K is a union of edges in the simplicial structure on K induced by the simplicial structure on $\widetilde{G}$, but not necessarily in the natural simplicial structure on K.

We claim that $E \not\subset F_K$ and that $\pi_K(E) = E'$. To prove this we use that $\pi_K\colon (K,J) \to (K',J')$ is a homotopy equivalence of pairs, and that $\pi_K(K \setminus J) = K' \setminus J'$. In the case that $K \setminus J$ equals E we have $E \not\subset F_K$, for otherwise $\pi_K\colon J \to J'$ would not be a homotopy equivalence; moreover, in this case the fact that $K \setminus J = E$ is a natural edge of K implies that $K' \setminus J'$ is a natural edge of K', which must therefore equal E'. In the case that $K \setminus J$ is a lollipop with loop edge E and with edge α connecting the base vertex of E to a vertex of J, we have $E \not\subset F_K$ for otherwise $\pi_K\colon K \to K'$ would not be a homotopy equivalence. In the subcase that $\alpha \not\subset F_K$ it follows that $K' \setminus J'$ is a lollipop with loop edge E' whose base point is connected to a vertex of J' by the edge $\alpha' = \pi_K(\alpha)$, and π_K takes E to E'. In the subcase that $\alpha \subset F_K$ it follows $K' \setminus J'$ is the natural edge E'. This proves the claim.

In all cases of the proof of the above claim, it follows furthermore that π_K restricts to a map $E \mapsto E'$ that collapses to a point each component of the subforest $F_E = E \cap F_K$.

By Lemma 14 we have $\widetilde{\pi}(\mathrm{Axis}(g)) = \mathrm{Axis}'(g)$, and so $\widetilde{\pi}(S \cap \mathrm{Axis}(g)) = S' \cap \mathrm{Axis}'(g)$. It follows that sequence of edges of the edge path μ' in K' is obtained from the sequence of edges of the edge path μ in K by erasing edges in F_K and then taking the π_K images of the remaining edges. From this it follows that different crossings of E by μ are mapped by π_K to different crossings of E' by μ'. Furthermore, the only way that there can be any additional crossings of E' by μ' beyond those just described is when μ' has an initial or terminal subpath which equals an initial or terminal subsegment of E' whose complement in E' is contained in F_K, and there can be at most 2 such subpaths of μ', one for each endpoint. It follows that $i(g, G) \le i(g, G') \le i(g, G) + 2$. $\square$

We complete the proof of distortion of $\mathsf{Stab}(\mathcal{F})$ by applying Lemma 22 to find an exponential lower bound in k to the word length of ϕ_k in the group $\mathsf{Stab}(\mathcal{F})$. We do this in three cases depending on the cardinality of $\mathcal{F}$. In each case we proceed as follows: first we specify additional details of the choices of G_b and E_b; we then specify the choice of the edge η; we next specify the choice of a conjugacy class denoted γ_0 that is not represented by an element of B; and we show that $i_{\gamma_0}(G_b \cdot \phi_k)$ grows exponentially in k.

Case 1: $\mathcal{F}$ has one component In this case we choose G_b to be a rose with subrose H_b. The edge E_b must be a loop. The subgraphs $H_b = J_b \subset K_b = L_b$ are all roses. Since E_b is a loop we have $G_c = G_b$, and each object with "b" subscript equals its corresponding object with "c" subscript; we shall use the "b" subscript henceforth in Case 1.

The chosen edge $\eta \subset G_b \setminus L_b$ must be a loop. Choose the conjugacy class γ_0 to be the one represented in G_b by the loop η. By inspection we have $i_{\gamma_0}(G_b) = 0$. Let $\gamma_k = \phi_k(\gamma_0)$, which is the conjugacy class in F_n represented by

$$\Phi_k(\eta) = \Theta^k \Phi\, \overline{\Theta}^k(\eta) = \Theta^k \Phi(\eta) = \Theta^k(\eta e_1) = \eta\Theta^k(e_1),$$

where the second equation is true because the homotopy inverse $\overline{\Theta}$ is the identity on $G_b \setminus K_b$. This is clearly a circuit in G_b, i.e., there is no cancellation, because $\Theta | K_b$ is a train track map. This circuit is not contained in K_b, and it has exactly one maximal path in K_b, namely $\Theta^k(e_1)$. By inspection it follows that $i_{\gamma_k}(G_b)$ is equal to the number of occurrences of E_b in the edge path $\Theta^k(e_1)$, and as noted above after the definition of Θ, this number has an exponential lower bound in k. It follows that the number

$$\begin{aligned}|i_{\gamma_0}(G_b \cdot \phi_k) - i_{\gamma_0}(G_b)| = i_{\gamma_0}(G_b \cdot \phi_k) &= i(G_b \cdot \phi_k; \gamma_0, \mathcal{A}, \mathcal{B}) \\ &= i(G_b; \phi_k(\gamma_0), \phi_k(\mathcal{A}), \phi_k(\mathcal{B})) \\ &= i(G_b; \gamma_k, \mathcal{A}, \mathcal{B}) = i_{\gamma_k}(G_b)\end{aligned}$$

has an exponential lower bound in k. Applying Lemma 22, the length of the shortest path in $\mathcal{K}_n^{\mathcal{F}}$ from G_b to $G_b\phi_k$ has an exponential lower bound in k. Applying Lemma 21 and Corollary 10, the word length of ϕ_k in the group $\mathsf{Stab}(\mathcal{F})$ has an exponential lower bound in k.

Case 2: $\mathcal{F}$ has two components If it so happens that $\mathrm{coindex}(\mathcal{F}) \geq 3$ then one can almost directly copy the proof of Case 1: choose G_b to be a union of two roses with an edge connecting their base points, so that the second rose equals $H_{b,2}$ and the first rose is the union of $H_{b,1}$ plus at least two other loops; choose one of the latter loops to be E_b and another to be η; choose γ_0 to be the conjugacy class represented by η.

But we can dispatch Case 2 in full generality, using only the hypothesis $\mathrm{coindex}(\mathcal{F}) \geq 2$, by a slightly different argument. Choose G_b to be the union of two roses with the edge E_b connecting their base points, so that the first rose equals $H_{b,1}$ and the second rose is the union of $H_{b,2}$ plus at least one other loop; we have $J_b = H_b = H_{b,1} \cup H_{b,2}$ and $L_b = K_b = H_b \cup E_b$. The quotient map $G_b \mapsto G_c$ collapses the edge E_b to a point. The marked graph G_c is a rose, the union of the proper subrose $L_c = K_c = H_{c,1} \cup H_{c,2}$ and at least one other loop. Choose $\eta = \eta_c \subset G_c \setminus L_c$ to be one of the latter loops. The general formulas given earlier now apply, for the homotopy equivalences $\Phi, \Theta\colon G_c \to G_c$, and for the sequence ϕ_k represented by the homotopy equivalences $\Phi_k = \Theta^k \Phi\, \overline{\Theta}^k\colon G_c \to G_c$.

Choose γ_0 to be the conjugacy class represented in G_c by the loop η_c. Note that γ_0 is also represented in G_b by a loop edge which we shall denote η' and which intersects K_b only in the base point of the rose $H_{b,2}$, and so by inspection we have $i_{\gamma_0}(G_b) = 0$. Let $\gamma_k = \phi_k(\gamma_0)$. Exactly as in Case 1, γ_k is represented in G_c by the circuit $\eta_c u_k$, where $u_k = \Theta^k(e_1)$ and e_1 is the first enumerated edge in K_c.

We claim that $i_{\gamma_k}(G_b)$ has an exponential lower bound in k. Once this claim is established, the exact same argument as in Case 1 shows that the word length of ϕ_k in $\mathsf{Stab}(\mathcal{F})$ has exponential lower bound.

Intuitively, to prove the claim we count occurrences of the edge E_b in the path u_k. But this does quite not make sense because E_b lives in G_b while u_k lives in G_c. Instead we proceed as follows. The path u_k lives in the subrose K_c which is the union of the two subroses $H_{c,1}$ and $H_{c,2}$ intersecting only at their common rose vertex. Note that there exists k_0 such that the edge path $\Theta^{k_0}(e_1)$ contains some edge of $H_{c,1}$ and some edge of $H_{c,2}$ and so by connectivity $\Theta^{k_0}(e_1)$ contains a 2–edge subpath ρ consisting of one edge of $H_{c,1}$ and one edge of $H_{c,2}$. Since $\Theta^{k-k_0}(e_1)$ contains an exponentially growing number of copies of e_1, it follows that $u_k = \Theta^k(e_1)$ contains an exponentially growing number of copies of the subpath ρ.

We may assume that the edges $e_1, \dots, e_m$ of K_c are listed so that $e_1, \dots, e_\ell$ are the edges of $H_{c,1}$ and $e_{\ell+1}, \dots, e_m$ are the edges of $H_{c,2}$, where $1 \le \ell < m$. We may then list the edges of $H_{b,1}$ and $H_{b,2}$ as $e'_1, \dots, e'_\ell$ and $e'_{\ell+1}, \dots, e'_m$, respectively, so that e'_i maps to e_i under the map $G_b \mapsto G_c$ that collapses E_b. There is a unique closed edge path u'_k in G_b with initial and terminal points at the rose vertex of $H_{b,2}$ such that the image of u'_k under the collapse map $G_b \mapsto G_c$ equals u_k: the path u'_k is obtained from u_k by adding prime symbols $'$ to edges in the edge path u_k, inserting a copy of E_b or $\overline{E}_b$ wherever u_k has a two-edge subpath which crosses between $H_{b,1}$ and $H_{b,2}$, inserting an initial copy of $\overline{E}_b$ if u_k starts with an edge of $H_{c,1}$, and inserting a terminal copy of E_b if u_k ends with an edge of $H_{c,1}$. From this description, and the fact that u_k contains exponentially many copies of ρ, it follows that u'_k at least as many copies of E_b. Furthermore, it follows that γ_k is represented in G_b by $\eta' u'_k$. It now follows by inspection that $i_{\gamma_k}(G_b)$ grows exponentially in k, proving the claim and completing Case 2.

Case 3: $\mathcal{F}$ has three or more components If it so happens that $\text{coindex}(\mathcal{F}) \ge 4$ then one can handle Case 3 by copying the method of Case 1, and if $\text{coindex}(\mathcal{F}) \ge 3$ then one can copy Case 2. But we can dispatch Case 3 in full generality by a slightly different argument.

Choose G_b so that there is an edge from the rose vertex x_{i-1} of $H_{b,i-1}$ to the rose vertex x_i of $H_{b,i}$ for $i = 2, \dots, I$, and with additional loop edges, if any, based at the rose vertex x_I of $H_{b,I}$. Choose E_b to be the unique edge connecting x_1 and x_2, and so $K_b = H_{b,1} \cup E_b \cup H_{b,2}$. Under quotient map $G_b \mapsto G_c$ that collapses E_b, we have a subrose $K_c = H_{c,1} \cup H_{c,2} \subset G_c$. Choose η to be the unique edge of G_c connecting the rose vertices of K_c and $H_{c,3}$, oriented to have terminal vertex on the rose K_c. The general formula now applies for defining the homotopy equivalences $\Phi, \Theta\colon K_c \to K_c$, and the homotopy inverse $\overline{\Theta}\colon K_c \to K_c$, and for defining the sequence ϕ_k represented by homotopy equivalences $\Phi_k = \Theta^k \Phi\, \overline{\Theta}^k\colon G_c \to G_c$.

The remaining issue is to choose an appropriate conjugacy class γ_0. In Cases 1 and 2 we chose γ_0 to be represented in G_c by the loop edge η, but in the present case η is not a loop. Instead we choose γ_0 to be the conjugacy class represented in G_c by a loop of the form $\tau = \rho\eta\sigma\overline{\eta}$. We choose ρ to be any loop in $H_{c,3}$ based at its rose vertex. We choose σ to be a loop in K_c based at its rose vertex, taking care that for each k we avoid cancellation in the loop

$$\Phi_k(\tau) = \Phi_k(\rho)\,\Phi_k(\eta)\,\Phi_k(\sigma)\,\Phi_k(\overline{\eta}) = \rho\eta u_k\sigma\overline{u}_k\overline{\eta}$$

which represents $\gamma_k = \phi^k(\gamma_0)$ in G_c. The only possibilities for cancellation are at the concatenation point of $u_k\sigma$ and at the concatenation point of $\sigma\overline{u}_k$. Here we take advantage of the fact that the original formula for Θ is a positive map on the rose K_c with respect to the chosen orientations of its edges $e_1,\dots,e_m$, and so u_k is a positive word in these edges for each k. To avoid cancellation it therefore suffices to choose $\sigma = e_1\overline{e}_2$; this is possible because K_c has ≥ 2 edges.

As in Case 2, the edge path u_k, and so also the loop $\Phi_k(\tau)$, contains exponentially many 2–edge paths consisting of one edge of $H_{c,1}$ and another edge of $H_{c,2}$. In the marked graph G_b, the conjugacy class γ_k is therefore represented by a loop that contains exponentially many copies of E_b and $\overline{E}_b$, and furthermore that loop contains a unique maximal subpath in K_b. From this it follows by inspection that $i_{\gamma_k}(G_b)$ grows exponentially in k. The proof now proceeds as in Case 1 to show that the word length of ϕ_k in $\mathsf{Stab}(\mathcal{F})$ grows exponentially.

References

[1] **E Alibegović**, *Translation lengths in* $\mathrm{Out}(F_n)$, Geom. Dedicata 92 (2002) 87–93 MR1934012

[2] **M Bestvina**, **M Feighn**, **M Handel**, *The Tits alternative for* $\mathrm{Out}(F_n)$*, I: Dynamics of exponentially-growing automorphisms*, Ann. of Math. 151 (2000) 517–623 MR1765705

[3] **M R Bridson**, **K Vogtmann**, *On the geometry of the automorphism group of a free group*, Bull. London Math. Soc. 27 (1995) 544–552 MR1348708

[4] **M R Bridson**, **K Vogtmann**, *The Dehn functions of* $\mathrm{Out}(F_n)$ *and* $\mathrm{Aut}(F_n)$, Ann. Inst. Fourier 62 (2012) 1811–1817

[5] **M Culler**, **K Vogtmann**, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986) 91–119 MR830040

[6] **D B A Epstein**, **J W Cannon**, **D F Holt**, **S V F Levy**, **M S Paterson**, **W P Thurston**, *Word processing in groups*, Jones and Bartlett, Boston, MA (1992) MR1161694

[7] **S M Gersten**, **H B Short**, *Rational subgroups of biautomatic groups*, Ann. of Math. 134 (1991) 125–158 MR1114609

[8] **A Hatcher**, *Homological stability for automorphism groups of free groups*, Comment. Math. Helv. 70 (1995) 39–62 MR1314940

[9] **A Hatcher**, **K Vogtmann**, *Isoperimetric inequalities for automorphism groups of free groups*, Pacific J. Math. 173 (1996) 425–441 MR1394399

[10] **T Hawkins**, *Continued fractions and the origins of the Perron–Frobenius theorem*, Arch. Hist. Exact Sci. 62 (2008) 655–717 MR2457065

[11] **G Levitt**, *Automorphisms of hyperbolic groups and graphs of groups*, Geom. Dedicata 114 (2005) 49–70 MR2174093

[12] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR1791145

[13] **J Milnor**, *A note on curvature and fundamental group*, J. Differential Geometry 2 (1968) 1–7 MR0232311

[14] **E H Spanier**, *Algebraic topology*, McGraw-Hill, New York (1966) MR0210112

[15] **A S Švarc**, *A volume invariant of coverings*, Dokl. Akad. Nauk SSSR 105 (1955) 32–34 MR0075634 In Russian

[16] **K Vogtmann**, *Automorphisms of free groups and outer space*, Geom. Dedicata 94 (2002) 1–31 MR1950871

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