Asymptotically cylindrical Calabi–Yau 3–folds from weak Fano 3–folds

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We prove the existence of asymptotically cylindrical (ACyl) Calabi–Yau 3–folds starting with (almost) any deformation family of smooth *weak Fano* 3–folds. This allow us to exhibit hundreds of thousands of new ACyl Calabi–Yau 3–folds; previously only a few hundred ACyl Calabi–Yau 3–folds were known. We pay particular attention to a subclass of weak Fano 3–folds that we call *semi-Fano* 3–folds. Semi-Fano 3–folds satisfy stronger cohomology vanishing theorems and enjoy certain topological properties not satisfied by general weak Fano 3–folds, but are far more numerous than genuine Fano 3–folds. Also, unlike Fanos they often contain $\mathbb{P}^1$s with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, giving rise to compact rigid holomorphic curves in the associated ACyl Calabi–Yau 3–folds.

We introduce some general methods to compute the basic topological invariants of ACyl Calabi–Yau 3–folds constructed from semi-Fano 3–folds, and study a small number of representative examples in detail. Similar methods allow the computation of the topology in many other examples.

All the features of the ACyl Calabi–Yau 3–folds studied here find application in [17] where we construct many new compact G_2–manifolds using Kovalev's twisted connected sum construction. ACyl Calabi–Yau 3–folds constructed from semi-Fano 3–folds are particularly well-adapted for this purpose.

1 Introduction

Compact Calabi–Yau manifolds have been studied intensively ever since Yau's resolution of the Calabi conjecture [101] allowed algebraic geometers to produce them in abundance. Nevertheless, some fundamental questions about compact Calabi–Yau manifolds even in dimension three remain open. For example, are there finitely many or infinitely many topological types of nonsingular Calabi–Yau 3–folds?

There has also been important work on complete noncompact Kähler Ricci-flat (KRF) metrics by many authors: Calabi, Yau, Eguchi–Hansen, Gibbons–Hawking, Hitchin, Kronheimer, Anderson–Kronheimer–LeBrun, Atiyah–Hitchin, Tian–Yau, Joyce, Nakajima, Biquard and Carron to name only a small selection. Nevertheless, compared to the compact nonsingular case, current understanding of noncompact KRF metrics is much less complete and demands further study; several open questions in this area go back as far as Yau's 1978 ICM address.

The simplest classes of noncompact KRF metrics are:

(a) those of maximal volume growth, that is, Euclidean volume growth;

(b) those of minimal volume growth, that is, linear volume growth.

The maximal volume growth case – especially the class of so-called ALE metrics – has already attracted considerable attention, for example, Kronheimer's classification results for ALE hyper-Kähler 4–manifolds [60] and Joyce's higher-dimensional existence results [44, Section 8]; part of the reason for the focus on the ALE case has been the intimate link to the theory of (noncollapsed) metric degenerations of compact Einstein manifolds with bounded diameter. Another obvious model for noncollapsed metric degenerations of compact Einstein manifolds is provided by the development of long "almost cylindrical necks". For this reason it is important to understand asymptotically cylindrical (ACyl) Einstein metrics. The simplest class of such ACyl Einstein metrics are the asymptotically cylindrical Calabi–Yau metrics studied in the present paper; see also Haskins–Hein–Nordström [33].

ACyl Calabi–Yau 3–*folds* play a distinguished role because they can also be used as building blocks in Kovalev's twisted connected sum construction of compact manifolds with holonomy G_2; see Kovalev [57], Kovalev–Lee [58] and the more recent developments in Corti–Haskins–Nordström–Pacini [17]. The twisted connected sum construction – first developed in [57] – constituted a major advance in the understanding of compact G_2–manifolds; along with Joyce's original orbifold resolution construction [44, Sections 11 and 12] it remains one of only two methods available to produce compact G_2–manifolds.

Given a pair of ACyl Calabi–Yau 3–folds $V_\pm$ the twisted connected sum construction gives a way to combine the pair of noncompact ACyl 7–manifolds $\mathbb{S}^1 \times V_\pm$ – both of which have holonomy $SU(3) \subset G_2$ – to construct a compact 7–manifold with holonomy the full group G_2. The twisted connected sum construction is possible only when a certain compatibility between the cylindrical ends of $V_\pm$ can be arranged; studying this "matching" problem for pairs of ACyl Calabi–Yau 3–folds is therefore very important for our applications to G_2–geometry in [17].

While we know the existence of huge numbers of deformation classes of compact Calabi–Yau 3–folds, until the present paper only a couple of hundred families of ACyl Calabi–Yau 3–folds were known. In the present paper we prove that it is possible to construct deformation families of ACyl Calabi–Yau 3–folds from (almost) any deformation family of smooth *weak Fano* 3–folds. As a consequence we prove that there are at least several hundred thousand deformation classes of ACyl Calabi–Yau 3–folds.

A *Fano* 3–fold Y is a smooth projective variety for which $-K_Y$ is ample or positive: complex projective space $\mathbb{P}^3$, smooth quadrics, cubics and quartics in $\mathbb{P}^4$ being the simplest examples. Fano 3–folds have been important objects in algebraic geometry since Fano's work in the 1930s and are still very much an active research area in contemporary algebraic geometry. A *weak Fano* 3–fold[1] is a smooth projective 3–fold for which $-K_Y$ is big and nef (but not ample). Differential geometers are encouraged to think of a line bundle being big and nef as the algebraic–geometric formulation of admitting a hermitian metric whose curvature is sufficiently semi-positive. All weak Fano 3–folds can be obtained by choosing suitable resolutions of mildly singular Fano 3–folds.

A number of properties of Fano manifolds generalise without too much difficulty to weak Fanos; we replace applications of the Kodaira vanishing theorem with its generalisation the Kawamata–Viehweg vanishing theorem. Kovalev [57] used Fano 3–folds to construct ACyl Calabi–Yau 3–folds with ends asymptotic to $\mathbb{C}^* \times S$, where S is a smooth K3 surface and suggested that other constructions of suitable ACyl Calabi–Yau 3–folds might be possible [57, page 148]; we prove that starting only with a *weak Fano* 3–fold (satisfying one further very mild restriction which is also needed even in the Fano case) we can still construct ACyl Calabi–Yau 3–folds with ends asymptotic to $\mathbb{C}^* \times S$. However, in order to solve the "matching" problem for pairs of ACyl Calabi–Yau 3–folds constructed from weak Fano 3–folds it turns out to be important to distinguish the subclass of *semi-Fano* 3–folds, that is, weak Fano 3–folds whose anticanonical morphism is a semi-small map.[2]

There are two principal advantages in generalising from Fano to weak Fano or semi-Fano 3–folds. It is well-known that there are exactly 105 deformation families of smooth Fano 3–folds (see Iskovskih [36; 37], Mori–Mukai [67; 68; 69], Mukai–Umemura [72] and Takeuchi [94]): in the paper, we will refer to this result as the

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[1] Some authors call this an *almost Fano* 3–fold.

[2] There seems to be no established terminology for this particular subclass of weak Fano 3–folds, so the term *semi-Fano* is our invention; it is intended to suggest that a *semi*-Fano 3–fold has *semi*-small anticanonical morphism. Warning: Chan et al [12] used the term semi-Fano manifold to mean something even weaker than weak Fano, that is, a complex manifold for which $-K_Y$ is nef (but not necessarily big).

"Iskovskih–Mori–Mukai classification". On the other hand, there are at least hundreds of thousands of deformation families of smooth weak Fano or semi-Fano 3–folds and their topology is less restrictive than for Fano 3–folds; unlike the Fano case there is at present no classification theory for weak Fano or semi-Fano 3–folds except under very special geometric assumptions. Thus generalising from Fano to weak Fano or semi-Fano 3–folds allows us to construct a significantly larger number of ACyl Calabi–Yau 3–folds.

For applications to the twisted connected sum construction of compact G_2–manifolds the following feature is also important; whereas on any Fano 3–fold the anticanonical class satisfies $-K_Y \cdot C > 0$ for any complex curve C, weak Fano 3–folds can contain special complex curves C for which $K_Y \cdot C = 0$ (the weakening of $-K_Y$ being positive to sufficiently semi-positive is crucial here). Moreover, in many cases C is a smooth rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (where $\mathcal{O}(d)$ denotes $\mathcal{O}_{\mathbb{P}^1}(d)$). In particular, C is rigid, that is, it has no infinitesimal (holomorphic) deformations. These special K–trivial curves C in weak Fanos allow us to construct compact rigid curves in the associated (noncompact) ACyl Calabi–Yau 3–folds. The fact that we can construct compact holomorphic curves in our ACyl Calabi–Yau 3–folds and that these curves have no infinitesimal deformations will be key to our construction of rigid associative 3–folds in compact G_2–manifolds [17].

We also discuss the following topics in some detail (keeping in mind applications of ACyl Calabi–Yau 3–folds to the twisted connected sum construction of G_2–manifolds):

  (i)   The topology of ACyl Calabi–Yau 3–folds; see Section 5.

 (ii)   Which hyper-Kähler K3 surfaces can appear as the ACyl limits of our ACyl Calabi–Yau 3–folds; see Section 6.

(iii)   Some representative ACyl Calabi–Yau 3–folds obtained from semi-Fano 3–folds, including computations of the topology of these examples and the number of rigid holomorphic curves they contain; see Section 7.

(iv)   Some general methods available for constructing (and in some cases classifying) weak Fano and semi-Fano 3–folds and some indication how the methods used in (iii) can be deployed in this more general context; see Section 8.

We now describe the structure of the rest of the paper.

Section 2 introduces (exponentially) ACyl Calabi–Yau manifolds and explains how to construct ACyl Calabi–Yau structures on certain types of quasiprojective manifold; see Theorem 2.6. Underpinning Theorem 2.6 is an analytic existence theorem for ACyl Calabi–Yau manifolds recently proven by Haskins–Hein–Nordström [33, Theorem D];

this result is related to previous work of Tian–Yau [95] and Kovalev [57]. Building on the previous work of Tian–Yau, Kovalev claimed to prove the existence of exponentially asymptotically cylindrical Calabi–Yau manifolds, improving substantially the asymptotics previously established by Tian–Yau. Unfortunately Kovalev's proof of the improved asymptotics contains an error (see the discussion following the statement of Theorem 2.6 and also [33] for further details). Other errors in Kovalev [57] occur in the construction of hyper-Kähler rotations (especially Lemma 6.47 which is used in the proof of the main Theorem 6.44) while several other points are unclear. For this reason, in both this paper and in [17] we chose not to rely on arguments from [57], and to give proofs or alternative references for the main results we need. To this end [33] gives a short self-contained proof of the existence of exponentially asymptotically cylindrical Calabi–Yau metrics that also bypasses the difficult existence theory of Tian–Yau [95].

A significant fraction of this paper then concerns trying to find a large number of quasiprojective 3–folds satisfying the hypotheses of Theorem 2.6. In Proposition 4.24 we show that if we can find a closed Kähler 3–fold Y with an anticanonical pencil that has some smooth member and whose base locus is a smooth curve, then blowing up that curve gives a 3–fold satisfying the hypotheses of Theorem 2.6, and hence an ACyl Calabi–Yau 3–fold. In turn, almost any weak Fano 3–fold satisfies the hypotheses of Proposition 4.24. To prove this and to show the relative abundance of weak Fano 3–folds requires a certain amount of algebro-geometric background; this background is developed in Sections 3 and 4.

Section 3 contains some material from algebraic geometry needed for our discussion of weak Fano 3–folds. We have included this algebro-geometric material in an attempt to make the paper accessible to a wide readership. The first part of the section deals with various notions of weak positivity for line bundles on projective manifolds and related vanishing theorems; these vanishing theorems generalise the classical Kodaira vanishing theorem (and its extension due to Akizuki–Nakano) for ample line bundles. The key results from this section are the Kawamata–Viehweg vanishing theorem for big and nef line bundles and the Sommese–Esnault–Viehweg vanishing result for l–ample line bundles. Also important for us is the Lefschetz theorem for semi-small morphisms; this is a special case of Goresky–MacPherson's vast generalization of the classical Lefschetz hyperplane theorem allowing a weaker positivity assumption on the line bundle than ampleness.

The second part of Section 3 contains material on mildly singular 3–folds and their crepant and small resolutions. We are interested in Gorenstein terminal and canonical 3–fold singularities; the anticanonical model of a smooth weak Fano 3–fold is a Fano 3–fold with Gorenstein canonical singularities; see Remark 4.10. The simplest terminal 3–fold singularity, the ordinary double point (ODP for short), or ordinary node, plays

a particularly important role throughout the paper. Conversely, given a mildly singular Fano 3–fold we can often construct smooth weak Fano 3–folds by finding appropriate resolutions. In the terminal singularities case any crepant resolution is a so-called small resolution, that is, the exceptional set contains no divisors. The existence of a small resolution of a singular variety X forces it to be non-$\mathbb{Q}$–factorial, that is, there are Weil divisors on X no multiple of which are Cartier. We explain the intimate link between small birational morphisms with target X and such Weil divisors on X. An important role is played by the defect of a Gorenstein canonical 3–fold X; the defect quantifies the failure of X to be $\mathbb{Q}$–factorial. We also recall some basic properties of flops in dimension three; for many weak Fano 3–folds we can use flops to produce many non-isomorphic weak Fano 3–folds from a single weak Fano 3–fold. The final part of the section recalls some basic terminology and facts from Mori theory for 3–folds; this is used only in Section 8 in our discussion of the classification scheme for weak Fano 3–folds with Picard rank $\rho = 2$.

Section 4 defines weak Fano 3–folds and recalls a number of their basic properties. Foremost among these properties is Theorem 4.7 (due to Reid and Paoletti): a general anticanonical divisor in a nonsingular weak Fano 3–fold is a nonsingular K3 surface; this is the fundamental property that allows us to construct ACyl Calabi–Yau 3–folds out of weak Fano 3–folds. Propositions 4.24 and 4.25 show how one can obtain quasiprojective 3–folds on which we can construct ACyl Calabi–Yau structures by blowing up suitable curves in suitable Kähler 3–folds; the earlier material shows that suitable 3–folds include almost any weak Fano 3–fold. These results are central to the paper.

As mentioned above, we also introduce an important subclass of weak Fano 3–folds which we call *semi-Fano* 3–folds: the anticanonical morphism of a semi-Fano 3–fold is a *semi-small* birational morphism, that is, it contracts no divisor to a point. Although weak Fano 3–folds suffice to construct ACyl Calabi–Yau 3–folds, for applications to the construction of compact G_2–manifolds using the twisted connected sum construction, we will often need to restrict to ACyl Calabi–Yau 3–folds obtained from semi-Fano 3–folds. The basic advantage is the stronger cohomology vanishing theorems available for semi-Fano 3–folds.

Section 5 is concerned with computing the topology of ACyl Calabi–Yau 3–folds and in particular the topology of the ACyl Calabi–Yau 3–folds we construct out of semi-Fano 3–folds. We compute the full integral cohomology groups of our ACyl Calabi–Yau 3–folds and note in particular that the only potential source of torsion comes from H^3 of the semi-Fano. We do not know any semi-Fano 3–folds for which H^3 has torsion but we have no general proof of its absence. We also establish simple-connectedness of our ACyl Calabi–Yau 3–folds and study the second Chern class c_2, particularly

properties related to its divisibility. These results on the primary topological invariants of ACyl Calabi–Yau 3–folds play an important role in [17]; there they are used to identify for the first time the diffeomorphism type of many compact G_2–manifolds.

Section 6 studies anticanonical divisors in semi-Fano 3–folds in detail. By Theorem 4.7 any general anticanonical divisor in a weak Fano 3–fold is a smooth K3 surface. A natural geometric question about ACyl Calabi–Yau 3–folds constructed from a weak Fano 3–fold is the following: which K3 surfaces can appear as asymptotic limits of our ACyl Calabi–Yau 3–folds as we vary both the weak Fano 3–fold in its deformation class and the chosen smooth anticanonical divisor? Addressing this question turns out to be crucial to the construction of so-called hyper-Kähler rotations between pairs of ACyl Calabi–Yau 3–folds and therefore to the construction of compact G_2–manifolds via the twisted connected sum construction.

To answer this question we need to develop some appropriate moduli/deformation theory. On the K3 side this requires recalling basic facts about lattice polarised K3 surfaces and versions of the Torelli theorem in this setting. We also need to extend Beauville's results [6] about the moduli stack parameterising pairs (Y, S), where Y belongs to a given deformation family of smooth Fano 3–folds and $S \in |-K_Y|$ is a smooth K3 section. The key observation – see Theorem 6.6 – is that the appropriate moduli stack is still smooth when Y is a semi-Fano 3–fold; here we use the stronger cohomology vanishing theorems available for semi-Fano 3–folds. The immediate payoff is Theorem 6.8 which gives us a good understanding of which K3 surfaces appear as smooth anticanonical divisors in a deformation class of semi-Fano 3–folds. It is likely that most of these facts hold, with appropriate modification, for more general weak Fano 3–folds but we do not pursue this here; however see for instance the recent paper by Sano [88].

Section 7 constructs a handful of ACyl Calabi–Yau 3–folds from a carefully chosen selection of Fano and semi-Fano 3–folds and computes the topology of these ACyl Calabi–Yau 3–folds in detail using the results from Section 5. In this section we only construct a very small number of typical examples making no attempt to be systematic. Similar methods can be used to produce many more ACyl Calabi–Yau 3–folds and to compute their topology.

Section 8 gives many further examples of semi-Fano 3–folds from which one can construct many more ACyl Calabi–Yau 3–folds. Our basic aim is to back up our assertion that there are *many* more weak Fano or semi-Fano 3–folds than Fano 3–folds. Unlike smooth Fano 3–folds, smooth weak Fano 3–folds are far from being classified and even in the longer-term such a classification may in practice be out of reach. Various classes of weak Fano 3–folds with special geometric or topological properties are much

closer to being classified. We consider in some detail several such special classes: (a) weak Fano 3–folds with Picard rank $\rho = 2$, (b) toric weak Fano 3–folds and (c) weak Fano 3–folds obtained by small resolutions of nodal cubics.

Thanks to recent work of various authors – including Arap–Cutrone–Marshburn [2], Blanc–Lamy [8], Cutrone–Marshburn [18], Jahnke–Peternell–Radloff [40; 41], Kaloghiros [45] and Takeuchi [93] – class (a) is known to consist of over 150 distinct deformation classes of semi-Fano 3–folds; many of these can be obtained by blowing up an appropriate smooth irreducible curve in an appropriate smooth rank one Fano 3–fold. This makes it relatively straightforward to determine many of the basic topological properties of such weak Fano 3–folds.

Class (b) gives rise to hundreds of thousands of distinct deformation classes of semi-Fano 3–folds (discussed in a forthcoming paper by Coates, Haskins, Kasprzyk and Nordström [15]). Toric semi-Fano 3–folds can be understood completely in terms of the geometry of so-called reflexive polytopes of dimension three; such reflexive polytopes were completely classified by Kreuzer–Skarke [59] and there are over four thousand such reflexive polytopes. Moreover, the topology of toric semi-Fano 3–folds is relatively simple and easily computed in terms of the reflexive polytope. This makes toric semi-Fano 3–folds a very convenient class for producing large numbers of ACyl Calabi–Yau 3–folds and computing their topology.

Class (c) all consist of so-called weak del Pezzo[3] 3–folds, that is, weak Fano 3–folds for which $-K_Y \in H^2(Y; \mathbb{Z})$ is divisible by 2. There are very few smooth del Pezzo 3–folds, of which smooth cubics in $\mathbb{P}^4$ form one deformation family. Degenerating a smooth cubic 3–fold to a cubic 3–fold with only ordinary nodes and seeking projective small resolutions of these singular del Pezzo 3–folds yields a method to produce numerous weak del Pezzo 3–folds – all of the same anticanonical degree but with increasing Picard rank – from a single deformation family of smooth del Pezzo 3–folds. This particular family of examples – studied in detail by Finkelnberg [24], Finkelnberg–Werner [26] and Werner [99] – illustrates a general principle that a single deformation family of smooth Fano 3–folds can spawn many different deformation families of smooth weak Fano 3–folds; this helps to explain why weak Fano 3–folds can be expected to be so numerous.

Acknowledgements

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[3]Some authors use almost del Pezzo.

2 Asymptotically cylindrical Calabi–Yau 3–folds

By a *Calabi–Yau manifold* we mean a Kähler manifold (M^{2n}, I, g, ω) with a parallel complex n–form Ω. Then the Riemannian holonomy of (M, g) is contained in $\mathrm{SU}(n)$. (At this stage we do not insist that $\mathrm{Hol}(g) = \mathrm{SU}(n)$; however, this will be the case for all the noncompact Calabi–Yau 3–folds constructed later in this paper.) We further impose a normalisation condition that

$$(2\text{-}1) \qquad \frac{\omega^n}{n!} = i^{n^2} 2^{-n} \Omega \wedge \overline{\Omega}$$

(equivalently Ω has constant norm 2^n). The complex structure and metric can be recovered from the pair (ω, Ω), and we refer to this as a *Calabi–Yau structure*. Ω is holomorphic, so the canonical bundle of (M, I) is trivial. The well-known relation between the curvature of the canonical bundle and the Ricci curvature of a Kähler metric implies that ω is Ricci-flat.

This relation implies also that if (M, I, ω) is a Ricci-flat Kähler manifold, then the restricted holonomy (that is, the group generated by parallel transport around contractible closed curves in M, or equivalently the identity component of $\mathrm{Hol}(M)$) is contained in $\mathrm{SU}(n)$, but if M is not simply connected then there need not be any global holomorphic section of K_M. In other words, the canonical bundle need not be trivial, though the real first Chern class $c_1(M) \in H^2(M; \mathbb{R})$ must vanish. Conversely, Yau's proof of the Calabi conjecture [101] shows that any compact Kähler manifold M with $c_1(M) = 0 \in H^2(M; \mathbb{R})$ admits Ricci-flat Kähler metrics. More precisely, every Kähler class on M contains a unique Kähler Ricci-flat metric.

We now turn our attention to a special type of non-compact complete manifold called asymptotically cylindrical.

Definition 2.2 We say that V_∞^{2n} is a *Calabi–Yau (half)cylinder* if $V_\infty \cong \mathbb{R}^+ \times X^{2n-1}$ is equipped with an $\mathbb{R}^+$–translation invariant Calabi–Yau structure $(I_\infty, g_\infty, \omega_\infty, \Omega_\infty)$, such that g_∞ is a product metric $dt^2 + g_X^2$ and X is a smooth closed manifold called the *cross-section* of V_∞.

The only Calabi–Yau cylinders that will play any significant role in this paper have cross-section $X = \mathbb{S}^1 \times S$ for a Calabi–Yau $(n-1)$–fold $(S^{2n-2}, I_S, g_S, \omega_S, \Omega_S)$, and $V_\infty := \mathbb{R}^+ \times \mathbb{S}^1 \times S$ (biholomorphic to $\Delta^* \times S$ where $\Delta^* \subset \mathbb{C}$ denotes the unit disc in $\mathbb{C}$ with the origin removed) has product structure

$$(2\text{-}3) \qquad \begin{aligned} I_\infty &:= I_\mathbb{C} + I_S, & g_\infty &:= dt^2 + d\vartheta^2 + g_S, \\ \omega_\infty &:= dt \wedge d\vartheta + \omega_S, & \Omega_\infty &:= (d\vartheta - i\,dt) \wedge \Omega_S, \end{aligned}$$

where t and ϑ denote the standard variables on $\mathbb{R}^+$ and $\mathbb{S}^1$. (The choice of phase for the $d\vartheta - i\,dt$ factor makes no material difference, but helps some equations in [17] take a more pleasant form.)

Definition 2.4 Let $(V, g, I, \omega, \Omega)$ be a complete Calabi–Yau manifold. We say that V is an *asymptotically cylindrical* (or *ACyl* for short) Calabi–Yau manifold if there exist (i) a compact set $K \subset V$, (ii) a Calabi–Yau cylinder V_∞ and (iii) a diffeomorphism $\eta: V_\infty \to V \setminus K$ such that for all $k \geq 0$, for some $\lambda > 0$ and as $t \to \infty$,

$$\eta^* \omega - \omega_\infty = d\varrho \quad \text{for some } \varrho \text{ such that } |\nabla^k \varrho| = O(e^{-\lambda t}),$$

$$\eta^* \Omega - \Omega_\infty = d\varsigma \quad \text{for some } \varsigma \text{ such that } |\nabla^k \varsigma| = O(e^{-\lambda t}),$$

where ∇ and $|\cdot|$ are defined using the metric g_∞ on V_∞. We will refer to V_∞ as the *asymptotic end* of V.

Remark 2.5 Our definition demands that $\eta^* \omega$ be cohomologous to ω_∞ on the end of V. However, as long as $|\eta^* \omega - \omega_\infty| \to 0$, this is automatic. The main point of the definition is thus to impose the existence of specific ϱ and ς with the stated rate of decay.

Since the complex structures on both $\mathbb{R}^+ \times \mathbb{S}^1 \times S$ and V are determined by the corresponding complex volume forms, similar estimates also hold for $|\nabla^k(\eta^* I - I_\infty)|$. The same is true for the metrics.

For the examples of this paper, we will be concerned with the case of complex dimension $n = 3$. Let us remark briefly on the relation between the holonomy of an ACyl Calabi–Yau manifold V and its topology in this case.

- Hol(V) is exactly SU(3) if and only if $\pi_1(V)$ is finite.

- If Hol(V) = SU(3) and the asymptotic end is a product $\mathbb{R}^+ \times \mathbb{S}^1 \times S$, then S is a projective K3 surface.

- If the asymptotic end is a product $\mathbb{R}^+ \times \mathbb{S}^1 \times S$ and S is a K3 surface, then Hol(V) = SU(3) unless V is a quotient of $\mathbb{R} \times \mathbb{S}^1 \times S$ by an involution; up to deformation there is a unique V of the latter kind.

For the proofs of these claims, and more general considerations of holonomy of ACyl Calabi–Yau manifolds, see Haskins–Hein–Nordström [33, Section 2].

We now want to review a method for constructing ACyl Calabi–Yau manifolds. It is based on the following ACyl version of the Calabi–Yau theorem. Note that if S is a smooth anticanonical divisor in a closed Kähler manifold Z, then the canonical bundle K_S is trivial, so each Kähler class on S contains a Ricci-flat metric by Yau's proof of the Calabi conjecture.

Theorem 2.6 *Let Z be a closed Kähler manifold with a morphism $f\colon Z \to \mathbb{P}^1$, with a smooth connected reduced fibre $S \in |-K_Z|$, and let $V = Z \setminus S$. If Ω_S is a non-vanishing holomorphic $(n-1)$–form on S, ω_S a Ricci-flat Kähler metric on S satisfying the normalisation condition* (2-1), *and $[\omega_S] \in H^{1,1}(S)$ is the restriction of a Kähler class on Z, then there is an ACyl Calabi–Yau structure (ω, Ω) on V whose asymptotic limit has the complex product form* (2-3).

Closely related statements were made first by Tian–Yau in [95, Theorem 5.2] and later by Kovalev in [57, Theorem 2.4]. Tian–Yau establish the existence of a Calabi–Yau structure on V by solving a complex Monge–Ampère equation, but not that this structure is asymptotically cylindrical in the sense defined in Definition 2.4, that is, they do not prove that the metric they construct decays exponentially to the complex product form (2-3). The exponential decay is crucial for the gluing argument used to construct compact G_2–manifolds from a pair of ACyl Calabi–Yau 3–folds via the twisted connected sum construction. Kovalev used Tian–Yau's work as a starting point and then attempted to prove the exponential decay as a separate step. Unfortunately, Kovalev's exponential decay argument [57, page 132] relies on an estimate established by Tian–Yau in their work on complete Kähler-Ricci-flat metrics with maximal volume growth [96, page 52], but the estimate from [96] crucially relies on a Euclidean type Sobolev inequality that definitely fails for any volume growth rate less than the maximal one. Thus until very recently no complete proof of the existence of ACyl Calabi–Yau manifolds existed in the literature. Haskins–Hein–Nordström [33] recently filled this gap by giving a short, direct self-contained proof of an ACyl version of the

Calabi conjecture [33, Theorem 4.1]; this proof avoids appealing to the more general (but technically more formidable and less precise) existence theory of Tian–Yau [95]. Proving the existence of ACyl Calabi–Yau metrics is relatively straightforward given the ACyl Calabi conjecture: see [33, Theorem D]. Below we explain how to deduce Theorem 2.6 from [33, Theorem D].

Proof By assumption there is a meromorphic n–form Ω on Z with a simple pole along S. Its residue is a non-vanishing holomorphic $(n-1)$–form on S. Since this is unique up to multiplication by a complex scalar, we can choose Ω so that its residue is Ω_S.

The restriction of Ω to V is a holomorphic volume form. Together with the exponential map $\mathbb{R}^+ \times \mathbb{S}^1 \cong \Delta^*$, a smooth local trivialisation $\Delta \times S \hookrightarrow Z$ for f yields a smooth map $\eta \colon \mathbb{R}^+ \times \mathbb{S}^1 \times S \to V$ that is a diffeomorphism onto the complement of a compact subset, and $\eta^*\Omega$ has the asymptotic behaviour required in Definition 2.4.

Now [33, Theorem D] shows that, in the restriction to V of any Kähler class on Z, there is a unique Ricci-flat Kähler metric ω such that (2-1) holds (implying that (ω, Ω) is a Calabi–Yau structure), and that ω is ACyl with respect to η. The asymptotic limit of ω has the form $\mu\, dt \wedge d\vartheta + \omega_S$, where ω_S is necessarily the unique Ricci-flat Kähler metric in the restriction of the Kähler class from Z to S. Because (ω_S, Ω_S) satisfies the normalisation condition for Calabi–Yau structures we must have $\mu = 1$. Thus the asymptotic limit of (ω, Ω) is precisely the product (2-3).   □

Remark Haskins–Hein–Nordström [33] also shows that the above construction is reversible in the following sense: If, as assumed in (2-3), V is an ACyl Calabi–Yau manifold whose cross-section X splits as a Riemannian product $\mathbb{S}^1 \times S$ for some smooth compact Calabi–Yau $(n-1)$–fold S, then if V is simply connected one can prove that there is a smooth closed Kähler (in fact projective) manifold Z with an anticanonical fibration over $\mathbb{P}^1$ such that applying Theorem 2.6 recovers V. It is not always the case that the asymptotic end of an ACyl Calabi–Yau manifold splits in this way; see [33, Example 1.5] for such a manifold. Provided that a simply connected ACyl Calabi–Yau V has irreducible holonomy and $\dim_{\mathbb{C}} V > 2$, then one can prove that a projective compactification Z still exists even when the asymptotic end is not such a Calabi–Yau product; in this case Z may have orbifold singularities, but V can still be recovered from Z by a generalisation of Theorem 2.6; see [33, Theorems B and C].

Kovalev [57] applies Theorem 2.6 to certain blow-ups Z of Fano 3–folds. Since there are 105 deformation classes of smooth Fano 3–folds this yields a similar number of

deformation classes of ACyl Calabi–Yau 3–folds. (Some Fano 3–folds Y can be blown up in several different ways to give different admissible Z, see, for example, Examples 7.8 and 7.9. This has not studied systematically, so it is difficult to be more precise with the enumeration here.) Kovalev–Lee [58] have also applied Theorem 2.6 to 3–folds Z of a different kind, obtained from K3 surfaces with non-symplectic involution. There are 75 deformation classes of K3 surfaces with non-symplectic involution to which their result applies; this gives another 75 deformation families of ACyl Calabi–Yau 3–folds. Together these existing constructions yield at most a few hundred ACyl Calabi–Yau 3–folds.

In Section 4 (for example, see Proposition 4.24 and the paragraph preceding it) we show that the same procedure used by Kovalev in the case of Fano 3–folds can be applied to the much larger class of weak Fano 3–folds: see Definition 4.1. Since, as we will explain in detail later, there are hundreds of thousands of deformation classes of weak Fano 3–folds this expands the number of known ACyl Calabi–Yau 3–folds from a few hundred to at least several hundred thousand. The topology of these ACyl manifolds is discussed in Section 5. In particular we find that they are simply connected, so their holonomy is exactly $SU(3)$.

3 Algebro-geometric preliminaries

We review briefly some definitions and results from algebraic geometry needed for our later discussion of weak Fano 3–folds; although these notions are well known to algebraic geometers they seem to be unfamiliar to many differential geometers interested in manifolds with special or exceptional holonomy. The reader should feel free to proceed to the section on weak Fano 3–folds, returning to this section as needed.

Convention 3.1 We always assume our varieties to be complex projective varieties and morphisms to be projective unless specifically stated otherwise.

Line bundles, weak positivity and vanishing theorems

We will need generalisations of the Kodaira–Akizuki–Nakano vanishing theorem and the Lefschetz theorem for sections of ample line bundles; the generalisations we need replace the ampleness/positivity of the line bundle with some condition of sufficient semi-positivity of the line bundle. Depending on what semi-positivity assumption we make on L we recover more or less of the cohomology vanishing results implied by the Kodaira–Akizuki–Nakano vanishing theorem. We refer the reader to Lazarsfeld's book [62] for a comprehensive treatment of positivity for line bundles.

Definition 3.2 Let L be a line bundle on a projective algebraic variety Y; we say that:

  (i) L is *very ample* if the sections in $H^0(Y, L)$ define an embedding into projective space.

 (ii) L is *ample* if for some integer $m > 0$ $L^{\otimes m}$ is very ample.

(iii) L is *semi-ample*, or *eventually free*, if for some integer $m > 0$ the sections in $H^0(Y, L^{\otimes m})$ define a morphism to projective space; equivalently, the linear system $|L^{\otimes m}|$ is base point free.

 (iv) L is *nef* if for every compact algebraic curve $C \subset Y$, $\deg L_{|C} = c_1(L) \cap C \geq 0$.

  (v) L is *big* if for some integer $m > 0$ the sections in $H^0(Y, L^{\otimes m})$ define a rational map to projective space which is birational on its image.

By replacing ample in the definition of a Fano manifold with the weaker condition big and nef we will obtain the definition of a weak Fano manifold; see Definition 4.1.

See also Definition 3.6 for the notion of an l–*ample* line bundle; this is intermediate between semi-ample and ample.

Remark It is well known that, if L is nef, then L is big if and only if

$$L^{\dim Y} := \int_Y c_1(L)^{\dim Y} > 0.$$

Suppose that L is a semi-ample line bundle on a normal projective variety Y. We denote by $M(Y, L)$ the sub-semigroup $M(Y, L) = \{m \in \mathbb{N} \mid L^{\otimes m}$ is base point free$\}$. We write e for the "exponent" of $M(Y, L)$, that is, the largest natural number dividing every element of $M(Y, L)$; in particular $L^{\otimes ke}$ is free for $k \gg 0$. Given $m \in M(Y, L)$, write $X_m = \varphi_m(Y)$ for the image of the morphism $\varphi_m = \varphi_{L^{\otimes m}} \colon Y \to \mathbb{P} H^0(Y, L^{\otimes m})^\vee$.

The following is a well-known result of Zariski; see Lazarsfeld [62, Theorem 2.1.27].

Theorem 3.3 (Semi-ample fibrations) *Let L be a semi-ample bundle on a normal projective variety Y. Then there is an algebraic fibre space $\varphi \colon Y \to X$ having the property that for any sufficiently large integer $k \in M(Y, L)$:*

$$X_k = X \quad \text{and} \quad \varphi_k = \varphi.$$

Furthermore there is an ample line bundle A on X such that $\varphi^* A = L^{\otimes e}$, where e is the exponent of $M(Y, L)$.

In other words, for $m \gg 0$ the mappings φ_m stabilise to define a fibre space structure on Y (essentially characterised by the fact that $L^{\otimes e}$ is trivial on the fibres).

Remark 3.4 A corollary of the previous theorem is the following fact: if L is a semi-ample line bundle then L is finitely generated, that is, $R(Y, L) := \bigoplus_{m \geq 0} H^0(Y, mL)$ is a finitely generated $\mathbb{C}$–algebra; see [62, 2.1.30].

If L is ample (or positive) we have the famous cohomology vanishing theorem due to Kodaira [52] and extended by Akizuki–Nakano [1]. If L is sufficiently semi-positive then we can also obtain similar cohomology vanishing theorems as we now describe.

We begin with the Kawamata–Viehweg vanishing theorem; this requires the weakest positivity assumption:

Theorem 3.5 (Kawamata–Viehweg vanishing) *Let L be a nef and big line bundle on a non-singular projective variety Y. Then $H^i(Y, K_Y \otimes L) = (0)$ for $i > 0$. Equivalently, by Serre duality, $H^i(Y, L^\vee) = (0)$ for $0 \leq i < \dim Y$.*

Remark We have stated a simplified form of the vanishing theorem of Kawamata and Viehweg that suffices for our purpose. The general statement – for example, see Kollár–Mori [56, Theorem 2.64] – and the proof of even the simplified form, require the use of fractional divisors.

In general the Akizuki–Nakano generalisation of Kodaira vanishing fails for big and nef line bundles; see Lazarsfeld [62, Example 4.3.4] for a big and nef line bundle L on Y, the one point blowup of $\mathbb{P}^3$, for which $H^1(Y, \Omega_Y^1 \otimes L^\vee) \neq 0$. However, we do have the following generalisation of the Akizuki–Nakano vanishing theorem, due to Sommese and improved by Esnault–Viehweg [23, 6.6].

Definition 3.6 A semi-ample line bundle L on a non-singular projective variety Y is l–*ample* for some integer $l \geq 0$ if the maximum dimension of any fibre of the semi-ample fibration $\varphi \colon Y \to X$ is $\leq l$.

An ample line bundle is 0–ample. For l–ample line bundles we get Akizuki–Nakano-type vanishing results but for a restricted range of cohomology groups that depends on l.

Theorem 3.7 (Sommese–Esnault–Viehweg vanishing) *Let L be an l–ample line bundle on a non-singular projective variety Y with semi-ample fibration $\varphi \colon Y \to X$. Then*

$$H^p(Y, \Omega_Y^q \otimes L^\vee) = 0, \quad \text{for } p + q < \min \{\dim X, \dim Y - l + 1\}.$$

In particular, if L is also big then $\dim Y = \dim X$ and so if $l \geq 1$ vanishing holds when $p + q < \dim Y - l + 1$.

For ample line bundles L we have the Lefschetz hyperplane theorem that relates the topology of sections of L to the topology of Y. For general big and nef line bundles the Lefschetz hyperplane theorem is false. However, there is a good generalisation of the Lefschetz hyperplane theorem to the case of a line bundle that defines a semi-small morphism, due – in its strongest and most general form – to Goresky and MacPherson.

We begin with the following, which we take from Goresky–MacPherson [32, page 151].

Definition 3.8 Let $f\colon Y \to X$ be a projective morphism of projective varieties (not necessarily of the same dimension) and for any non-negative integer k write

$$X^k = \{x \in X \mid \dim f^{-1}(x) = k\}.$$

We say that f is *semi-small* if

$$\dim Y - \dim X^k \geq 2k \quad \text{for every } k \geq 0.$$

Equivalently, f is semi-small if and only if there is no irreducible subvariety $E \subset Y$ such that $2 \dim E - \dim f(E) > \dim Y$.

Remark 3.9 If L is a semi-ample line bundle on a non-singular projective 3–fold Y and the semi-ample fibration $\varphi\colon Y \to X$ is birational then L is semi-small if and only if L is 1–ample.

The following Lefschetz theorem for semi-small morphisms is a more-or-less immediate consequence of Goresky–MacPherson's "relative Lefschetz hyperplane theorem with large fibers" [32, Theorem 1.1, page 150].

Proposition 3.10 *Let Y be a non-singular projective variety of complex dimension $n = \dim_{\mathbb{C}} Y$, $f\colon Y \to \mathbb{P}^N$ a semi-small morphism, and $S \in |f^{\star}\mathcal{O}_{\mathbb{P}^N}(1)|$ a non-singular member. Then, the restriction map*

$$H^m(Y; \mathbb{Z}) \to H^m(S; \mathbb{Z})$$

is an isomorphism for $m < n - 1$ and is primitive injective for $m = n - 1$.

Proof All statements follow from the fact that $H_m(Y, S; \mathbb{Z}) = (0)$ for $m \leq n - 1$. This fact is an immediate consequence of [32, Theorem 1.1, page 150]. Here we are applying the statement with their (X, H) being our (Y, S). The assumptions are

satisfied because the morphism $Y \to X$ is semi-small; see [32, Remark (2), page 151]. Note that by [32, Remark (1), page 151], we are allowed to replace H_δ with H. In summary the conclusion is that the usual statement of the Lefschetz theorem holds in our case for the pair (Y, S).

We discuss in some further detail the statement of primitivity of the inclusion. Consider the long exact sequence of cohomology of the pair (Y, S):

$$\cdots \longrightarrow H^{n-1}(Y;\mathbb{Z}) \xrightarrow{\rho} H^{n-1}(S;\mathbb{Z}) \xrightarrow{\delta} H^n(Y, S;\mathbb{Z}) \longrightarrow H^n(Y;\mathbb{Z}) \longrightarrow \cdots .$$

Notice that $\mathrm{Im}(\rho)$ is primitive iff $\mathrm{Coker}(\rho)$ is torsion-free, which is equivalent to $\mathrm{Im}(\delta)$ being torsion-free. It is thus enough to show that $H^n(Y, S)$ is torsion-free. By the universal coefficient theorem, the torsion of this group is isomorphic to the torsion of $H_{n-1}(Y, S)$, which is trivial by what we said. $\qquad\square$

Remark We will use Proposition 3.10 in the proof of Proposition 5.7(iii) (see also Lemma 6.4) to show that anticanonical sections of a semi-Fano 3–fold Y are Pic Y–polarised K3 surfaces.

Weak Fano 3–folds via resolutions of singularities

We will see shortly that every smooth weak Fano 3–fold Y – one of the main objects of interest in this paper – can be obtained as a special type of resolution of a mildly singular Fano 3–fold; see Remark 4.10. For this reason even though we are interested in constructing smooth weak Fano 3–folds we will need to deal with certain mildly singular 3–folds. This forces us to address several issues that arise only on singular varieties, for example, the fact that on a singular complex variety not every Weil divisor need be Cartier plays an important role in this paper.

Moreover, while resolutions of singularities exist very generally, the special sort of resolutions required to produce smooth weak Fanos from singular Fanos impose severe restrictions on the type of singularities we should consider. This leads us to consider in detail Gorenstein canonical and terminal singularities and special types of resolution of such singularities: so-called crepant and small resolutions. The existence of crepant and small resolutions is a delicate issue in general, as we will try to explain, but it is central to the construction of smooth weak Fano 3–folds from singular Fano 3–folds.

Divisors on singular varieties

We begin with some generalities about divisors on singular varieties; this issue comes up because we are forced to work with singular varieties.

We denote by Cl X, the *class group* of Weil divisors on X modulo linear equivalence, and by Pic X the Picard group of Cartier divisors on X modulo linear equivalence. A variety is *factorial* if every Weil divisor is Cartier or $\mathbb{Q}$–*factorial* if some integer multiple of every Weil divisor is Cartier. Being $\mathbb{Q}$–factorial is a local property in the Zariski topology of X, not the analytic topology. On any normal complex variety we can define the *canonical divisor* K_X (by extension from the regular part using the normality assumption) as a Weil divisor (unique up to linear equivalence). In general K_X is not a Cartier divisor; we say that X is *Gorenstein* (respectively $\mathbb{Q}$–Gorenstein) if K_X is Cartier (respectively there exists some $j \in \mathbb{N}$ so that jK_X is Cartier) and X is Cohen–Macaulay.

Convention 3.11 In the rest of the paper we assume all our varieties to be normal and Gorenstein, but many of the varieties we encounter will be neither factorial nor $\mathbb{Q}$–factorial.

Small projective birational morphisms and resolution of singularities

Let X and Y be normal complex algebraic varieties both of dimension n. Given a projective birational morphism $f\colon Y \to X$, define the f–exceptional set $E := \mathrm{Ex}(f)$ to be the closed subset where f is not a local isomorphism. f is surjective, $E = f^{-1}(f(E))$ and $\mathrm{codim}_X f(E) \geq 2$.

Definition 3.12 We call a projective birational morphism $f\colon Y \to X$ *small* if the exceptional set $E = \mathrm{Ex}(f)$ is of (complex) codimension at least 2.

Small projective birational morphisms and particularly projective small resolutions (that is, when Y is non-singular; see Definition 3.19) play important roles in this paper. Example 3.22 gives the simplest – and for this paper the most important – example of a small resolution.

Remark 3.13 If X is $\mathbb{Q}$–factorial every irreducible component of E has codimension 1 [19, Section 1.40]; in particular, if X is $\mathbb{Q}$–factorial then a projective birational morphism $f\colon Y \to X$ is never small. In other words, if there exists *any* projective birational morphism $f\colon Y \to X$ which is small (but not an isomorphism) then X cannot be $\mathbb{Q}$–factorial; this forces us to deal with singular varieties that are not $\mathbb{Q}$–factorial.

By Remark 3.13 if we are interested in small projective birational morphisms $f\colon Y \to X$ then X is forced to be non $\mathbb{Q}$–factorial. We now want to explain in detail the intimate link between non-$\mathbb{Q}$–Cartier divisors $D' \in \mathrm{Cl}(X)$ and small projective birational morphisms to X.

The following elementary lemma makes this connection precise: see, for example, Kawamata [49, Lemma 3.1] or Kollár's survey [53, Proposition 6.1.2].

Lemma 3.14 Let $f\colon Y \to X$ be a small projective birational morphism which is not an isomorphism and let D be an f–ample (see Remark 3.15) Cartier divisor on Y. Then the following hold:

(i) mf_*D is not Cartier if $m > 0$;

(ii) $f_*\mathcal{O}_Y(mD) = \mathcal{O}_X(mf_*D)$ for $m \geq 0$, and

(iii) $R(X, f_*D) := \bigoplus_{m \geq 0} \mathcal{O}_X(mf_*D)$ is a finitely generated $\mathcal{O}_X$–algebra and Y is recovered from $R(X, f_*D)$ by taking $\operatorname{Proj} R(X, f_*D)$.

Conversely, let D' be a Weil divisor on X which is not $\mathbb{Q}$–Cartier and for which

$$R(X, D') := \bigoplus_{m \geq 0} \mathcal{O}_X(mD')$$

is a finitely generated $\mathcal{O}_X$–algebra. Then $Y := \operatorname{Proj} R(X, D')$ is a normal projective variety, the projection map $f\colon Y \to X$ is a small projective birational morphism and $f_*^{-1}D'$ is f–ample (and $\mathbb{Q}$–Cartier).

Remark 3.15 Given a projective morphism $f\colon Y \to X$ of varieties there is a general notion of f–ampleness or ampleness of a divisor relative to the morphism f; see, for example, Lazarsfeld [62, Section 1.7]. Rather than give the general definition we recall the relative version of the Nakai criterion: a divisor D is f–*ample* if and only if $D^{\dim V} \cdot V > 0$ for every irreducible subvariety $V \subset Y$ of positive dimension which maps to a point under f.

Remark Suppose that $-D' = Z \geq 0$ is effective; then $\mathcal{O}_X(D') = I_Z \subset \mathcal{O}_X$ is the ideal sheaf of $Z \subset X$, and then the mth symbolic power of I_Z is $I_Z^{(m)} \cong \mathcal{O}(mD')$. (For this reason the algebra $R(X, D')$ is called the symbolic power algebra of D'.) If, in addition, we assume that the sheaf of algebras $\bigoplus_{m \geq 0} \mathcal{O}_X(mD')$ is generated by $\mathcal{O}_X(D') = I_Z \subset \mathcal{O}_X$, that is, generated in degree 1, then $Y = Bl_{I_Z} X$ is the blow up of the ideal of Z. In other words, in this case we can get Y by blowing up the non-$\mathbb{Q}$–Cartier divisor $Z \subset X$. In fact, this will be the case in all the examples considered in this paper; see Section 7.

To motivate the construction, recall first the universal property of blowups. The blowup of X in a subvariety (or, more generally, closed subscheme) $S \subset X$ with ideal $I_S \subset \mathcal{O}_X$ is a morphism $\pi\colon X' \to X$ such that the ideal $\pi^{-1}(I_S) \cdot \mathcal{O}_{X'} \subset \mathcal{O}_{X'}$ is a Cartier divisor

on X' and such that for any morphism $f\colon Y \to X$ with $f^{-1}(I_S)\cdot\mathcal{O}_Y$ a Cartier divisor on Y there exists a unique morphism $\rho\colon Y \to X'$ such that $f = \pi \circ \rho$. Informally: the blowup of $S \subset X$ is the "smallest" morphism to X that turns S into a Cartier divisor. In particular blowing up a Cartier divisor $D \subset X$ can only induce an isomorphism of X. However, if X is not $\mathbb{Q}$–factorial then blowing up a Weil divisor Z in X that is not $\mathbb{Q}$–Cartier must induce a non-trivial birational morphism to X – because it converts the Weil divisor Z into a Cartier divisor in the blowup. Since Z is of codimension 1 in X we expect that this birational morphism will not alter X too much; in fact, when the symbolic algebra of I_Z is generated by I_Z, Lemma 3.14 states that the induced birational morphism is *small* in the sense of Definition 3.12.

Remark To obtain a small projective birational morphism from a non-$\mathbb{Q}$–Cartier divisor $D' \in \mathrm{Cl}(X)$ as above we need to know that the symbolic power algebra $R(X, D')$ is a finitely generated $\mathcal{O}_X$–algebra. This is not true in general. However, for 3–folds with mild singularities Kawamata has shown that this is always true; we discuss this in more detail below in Theorem 3.35, after we have introduced appropriate classes of mildly singular 3–folds.

Remark 3.16 In general, blowing up different non-$\mathbb{Q}$–Cartier divisors $D' \in \mathrm{Cl}(X)$ as in Lemma 3.14 may give rise to the same small projective birational morphism $f\colon Y \to X$.

In this paper given some mildly singular variety X we will be particularly interested in constructing (special kinds of) projective birational morphisms $f\colon Y \to X$ where Y is *non-singular*.

Definition 3.17 A *resolution of singularities* or *desingularisation* of X is a projective birational morphism $f\colon Y \to X$ where Y is non-singular.

The so-called *ramification formula* for a resolution of singularities compares K_Y to the pullback f^*K_X and states that there exist unique integers $a_i \in \mathbb{Z}$ (recall that we always assume K_X Cartier) so that

$$(3\text{-}18) \qquad K_Y - f^*K_X = \sum_i a_i E_i,$$

where E_i are the *exceptional divisors* of f, that is, the irreducible components of the exceptional set E of codimension 1. The divisor $\sum_i a_i E_i$ is sometimes called the *discrepancy* of f. We say:

Definition 3.19 A resolution $f\colon Y \to X$ is *crepant* if $K_Y = f^* K_X$, that is, if all the coefficients $a_i \in \mathbb{Z}$ in (3-18) vanish.

A *small resolution* $f\colon Y \to X$ is a resolution of singularities in which the projective birational morphism f is small in the sense of Definition 3.12.

Remark

(i) If $f\colon Y \to X$ is a small resolution then the exceptional set E contains no divisors and hence $K_Y = f^* K_X$; so any small resolution is crepant.

(ii) In general a crepant resolution need not be small, however; see Remark 3.21(ii).

(iii) A resolution of singularities always exists (at least in characteristic 0) by Hironaka, whereas crepant or small resolutions exist only in very special circumstances. We will see below that the existence of a crepant or small resolution of X imposes strong constraints on the singularities X may have (for example, see Remark 3.21). Moreover, even when X satisfies these constraints determining whether a given (mildly) singular variety X admits a projective crepant or small resolution can be very delicate.

Terminal and canonical 3–fold Gorenstein singularities

One of the standard ways to define various classes of singularities is by assumptions on the coefficients a_i that appear in the ramification formula above. In this spirit we say:

Definition 3.20 A normal Gorenstein variety X has *terminal* (respectively *canonical*) singularities if for a given resolution of singularities $f\colon Y \to X$ all the coefficients $a_i \in \mathbb{Z}$ in (3-18) are positive (respectively non-negative).

One can show that this definition does not depend on the resolution f we chose. Numerous other equivalent definitions of terminal and canonical singularities can be found in Reid [86], which we recommend for an introduction to 3–fold terminal or canonical singularities.

Remark 3.21

(i) If X admits a crepant resolution f then (3-18) holds with all coefficients $a_i = 0$ and hence X has canonical singularities. If X admits a small resolution f then since the exceptional set contains no divisors, f vacuously satisfies the condition in Definition 3.20 and so X has terminal singularities.

(ii) If X has terminal singularities then it follows immediately from (3-18) and the definitions that any crepant resolution must be small. In other words, if X has terminal singularities then a resolution of X is crepant if and only if it is small.

(iii) If X is $\mathbb{Q}$–factorial with terminal singularities then X admits no crepant (necessarily small by the previous remark) resolutions by Remark 3.13.

(iv) In dimension two terminal points are non-singular. Any canonical singularity of a normal surface is (locally analytically) equivalent to a Du Val singularity, that is, to a hypersurface singularity in $\mathbb{C}^3$ of type A_n, D_n ($n \geq 4$), E_6, E_7 or E_8 (see Reid [84, Table 0.2] for a list of defining polynomials); Du Val singularities are the same as rational double points.

(v) A terminal normal 3–fold (respectively k–fold with $k \geq 3$) has only isolated singularities (respectively at worst codimension 3 singularities); canonical 3–folds have in general 1–dimensional singular loci.

(vi) One can prove that the notions of terminal and canonical singularities are (algebraically or analytically) local (see Matsuki [63, 4.1.2(iii)]); see also Proposition 3.26 for a concrete local characterisation of Gorenstein terminal 3–fold singularities.

The simplest example of a 3–fold Gorenstein terminal singularity is the ordinary double point or ordinary node; this singularity will play a crucial role in the paper.

Example 3.22 (The 3–fold ordinary double point)   Define a hypersurface $X \subset \mathbb{C}^4$ by
$$X := \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_1 z_2 = z_3 z_4 \}.$$

X is the affine cone over the quadric $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. X is non-singular away from the origin 0 where it has an isolated singular point, called the *ordinary double point* (ODP for short) or *ordinary node*. Blowing up the origin yields a non-singular variety $\widetilde{X}$ and a resolution of singularities $\pi \colon \widetilde{X} \to X$ whose exceptional set E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, but π is not a crepant resolution. $\mathbb{P}^1 \times \mathbb{P}^1$ has two rulings and we can contract the fibres of either ruling; this yields two other resolutions $\pi_\pm \colon \widehat{X}^\pm \to X$ whose exceptional set $E^\pm$ is isomorphic to $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. These two small resolutions of X can be realised concretely as complete intersections in $\mathbb{C}^4 \times \mathbb{P}^1$ as follows

(3-23a) $\widehat{X}^+ = \{(z, [x_1, x_2]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid z_1 x_2 = z_4 x_1, z_3 x_2 = z_2 x_1 \},$

(3-23b) $\widehat{X}^- = \{(z, [y_1, y_2]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid z_1 y_2 = z_3 y_1, z_4 y_2 = z_2 y_1 \},$

where $\pi^\pm \colon \widehat{X}^\pm \to X$ are the restrictions of the obvious projection $\mathbb{C}^4 \times \mathbb{P}^1 \to \mathbb{C}^4$. Since X admits small resolutions the origin is a (Gorenstein) terminal singularity.

Remark Affine cones over other del Pezzo surfaces give rise to canonical (non-terminal) 3–fold singularities. For example, take a non-singular del Pezzo surface S in $\mathbb{P}^8$ isomorphic to the 1–point blowup of $\mathbb{P}^2$. Then the vertex of the affine cone over S in $\mathbb{C}^9$ is an isolated canonical 3–fold singularity. This example illustrates that, unlike the ordinary double point (and Gorenstein terminal 3–fold singularities more generally see below), isolated Gorenstein canonical 3–fold singularities need not be of hypersurface type.

Definition 3.24 We call a (normal Gorenstein) terminal projective 3–fold X a *nodal 3–fold* if each of its singular points $P \in X$ is (locally analytically) equivalent to the ordinary double point of Example 3.22.

The Gorenstein terminal 3–fold singularities were classified by Reid [84]. To state Reid's classification we recall the notion of a cDV singularity.

Definition 3.25 A 3–fold singularity $P \in X$ is *cDV – compound Du Val –* if a general local analytic surface section $P \in S \subset X$ has Du Val singularities. Equivalently, $P \in X$ is cDV if it is (locally analytically) equivalent to a hypersurface singularity given by

$$f + tg = 0,$$

where $f \in \mathbb{C}[x, y, z]$ defines a Du Val singularity and $g \in \mathbb{C}[x, y, z, t]$ is arbitrary. Note that a general cDV singularity need not be isolated.

A cDV singularity $P \in X$ is said to be of type cA_n, cD_n, cE_n according to the Du Val singularity type of a sufficiently general surface section S through P.

Proposition 3.26 *Every cDV singularity of a Gorenstein 3–fold is canonical and the Gorenstein terminal 3–fold singularities are precisely the isolated cDV singularities; in particular Gorenstein terminal 3–fold singularities are all hypersurface double point singularities.*

Remark 3.27 Let $(X, 0)$ (or just X) be (the analytic germ of) an isolated Gorenstein 3–fold singularity, and $\pi \colon Y \to X$ a small resolution. Write the exceptional set as $E = \pi^{-1}(0) = C = \bigcup_i C_i$, where each C_i is irreducible. It is known that the C_i are non-singular rational curves meeting transversally and the normal bundle of C_i in Y is of type $(-1, -1)$ or $(0, -2)$ or $(1, -3)$: see Laufer [61, Theorem 4.1]. By the assumption that X admits a small resolution X has terminal singularities and therefore by Proposition 3.26 has isolated cDV singularities. The converse question of which isolated cDV singularities admit small resolutions is addressed by Katz–Morrison [48], Kawamata [50] and Pinkham [81]. For example, the cA_2 singularity defined by

$\{x^2 + y^2 + z^2 + t^{2n+1} = 0\}$ is factorial and hence admits no small resolutions, but this is not the case for the cA_2 singularity $\{x^2 + y^2 + z^2 + t^{2n} = 0\}$; see Friedman [27, Remark 1.7].

3–fold flops

Flops will play an important role later in the paper; they allow one to produce new (possibly many) weak Fano 3–folds from an existing one. This is a phenomenon which does not occur for smooth Fano 3–folds. We give only the basic definitions and state some of the foundational results about 3–fold flops; we refer the reader to Kollár's survey article [53] or to the book by Kollár–Mori [56] for much more comprehensive treatments.

Definition 3.28 Let Y be a normal variety. A *flopping contraction* is a projective birational morphism $f: Y \to X$ to a normal variety X that is small and such that K_Y is numerically f–trivial, that is, $K_Y \cdot C = 0$ for any curve C contracted by f.

Let $f: Y \to X$ be a flopping contraction, and D be a Cartier divisor on Y that is f–antiample, that is, $-D$ is f–ample (recall Remark 3.15). A projective birational small morphism $f^+: Y^+ \to X$ is called the D–*flop* of f if D^+, the proper transform of D on Y^+, is f^+–ample.

Remark 3.29 The D–flop of f is unique if it exists.

Theorem 3.30 (Kollár–Mori [56, Theorems 6.14–15]) *Let D be a Cartier divisor on a projective threefold Y with terminal singularities. Then D–flops exist and preserve the analytic singularity type of Y. In particular, if Y is non-singular then so is any flop Y^+.*

The 3–folds Y and Y^+ always have many similarities, for example, see [53, Theorem 3.2.2], but typically are not isomorphic or even diffeomorphic varieties.

Remark 3.31 If Y is non-singular and has Picard rank $\rho(Y) = 2$, then the D–flop of Y does not depend on the choice of divisor D.

Remark 3.32 We explain very briefly how flops occur in the context of weak Fano 3–folds; we will return to this point after we have developed the basic properties of weak Fano 3–folds in Section 4.

For many weak Fano 3–folds Y the anticanonical morphism $\varphi_{AC}: Y \to X$ (see Definition 4.9) will be a flopping contraction in the sense of Definition 3.28. By

choosing any φ_{AC}–antiample divisor D on Y we can perform the D–flop of φ_{AC}. This yields another weak Fano 3–fold Y^+ and another projective small birational morphism $\varphi^+: Y^+ \to X$; X is again the anticanonical model of Y^+ and Y^+ is also smooth by Theorem 3.30. In general Y and Y^+ are not isomorphic because the ring structure on cohomology is usually changed. Thus Y and Y^+ are usually different projective small resolutions of the same singular variety X; X itself will turn out to be a mildly singular but non-$\mathbb{Q}$–factorial Fano variety; see Remark 4.10.

In general Y^+ depends on the choice of the φ_{AC}–antiample divisor D. By Remark 3.31 the D–flop of Y does not depend on D when $\rho(Y) = 2$; that is, in this case there will be a unique flop of the rank two weak Fano 3–fold Y. However when $\rho(Y) \geq 3$ then Y may admit many different flops depending on the choice of D and all of them are smooth weak Fano 3–folds sharing many properties of Y. Remark 8.13(iii) exhibits a smooth weak Fano 3–fold with $\rho(Y) = 10$ which admits over 80 non-isomorphic projective small resolutions.

Defect, small $\mathbb{Q}$–factorialisations and small resolutions

Recall from Remark 3.13 that if a singular variety X admits a small birational morphism $f: Y \to X$ then X cannot be $\mathbb{Q}$–factorial. We now introduce a non-negative integer $\sigma(X)$, the defect of X, which quantifies the failure of a singular 3–fold X to be $\mathbb{Q}$–factorial; the defect also measures the failure of Poincaré duality on the singular variety X.

Definition 3.33 The *defect* of a Gorenstein canonical projective 3–fold is

$$\sigma(X) = \operatorname{rk} \operatorname{Cl}(X)/\operatorname{Pic}(X) = \operatorname{rk} H_4(X) - \operatorname{rk} H^2(X),$$

where $\operatorname{Cl}(X)$ denotes the class group of Weil divisors of X and $\operatorname{Pic}(X)$ the Picard group of X.

Remark 3.34

(i) By definition the defect of X is zero if and only if X is $\mathbb{Q}$–factorial. Hence any Gorenstein terminal 3–fold that admits a crepant (and hence small) resolution must have positive defect.

(ii) In fact the divisor class group $\operatorname{Cl}(X)$ of a terminal Gorenstein 3–fold X is torsion-free – see Kawamata [49, Lemma 5.1] who attributes the proof to Reid and Ue – so that X is $\mathbb{Q}$–factorial if and only if it is factorial. In particular, the defect $\sigma(X)$ of a terminal Gorenstein 3–fold is zero if and only if X is factorial, that is, every Weil divisor is Cartier.

If X admits a projective small resolution $f\colon Y \to X$ then by the previous remark X must have defect $\sigma(X) > 0$. We can therefore attempt to use the blowup construction from Lemma 3.14 to construct some small projective birational morphism to X by choosing a non-$\mathbb{Q}$–Cartier divisor $D' \in \mathrm{Cl}(X)$ and considering $\mathrm{Proj}\, R(X, D')$. However, we must verify that D' satisfies the condition that $R(X, D')$ is a finitely generated $\mathcal{O}_X$–algebra. This is not true in general; however for 3–folds Kawamata [49, Theorem 6.1] proved the following result.

Theorem 3.35 *Let X be a Gorenstein canonical 3–fold and $D' \in \mathrm{Cl}(X)$. Then $R(X, D')$ is finitely generated.*

Kawamata's proof uses the classification of Gorenstein terminal 3–fold singularities described earlier in a fundamental way. An easy corollary of Theorem 3.35, also due to Kawamata [49, Corollary 4.5] is the following.

Corollary 3.36 *For any projective 3–fold X with canonical (respectively terminal) singularities there exists a small projective birational morphism $f\colon Y \to X$ such that Y is $\mathbb{Q}$–factorial with at most canonical (respectively terminal) singularities. The morphism $f\colon Y \to X$ is said to be a (small) $\mathbb{Q}$–factorialisation of X.*

Proof The proof is simple given Theorem 3.35. Set $X = X_0$ and choose an arbitrary non-$\mathbb{Q}$–Cartier divisor $D_0 \in \mathrm{Cl}(X_0)$. Then by Theorem 3.35 and Lemma 3.14 $X_1 := \mathrm{Proj}\, R(X_0, D_0)$ is a normal projective variety and the natural projection to X gives a small projective birational morphism $f_1\colon X_1 \to X_0$. Clearly $\sigma(X_1) < \sigma(X_0) < \infty$. If X_1 fails to be $\mathbb{Q}$–factorial we repeat the process setting $X_2 = \mathrm{Proj}\, R(X_1, D_1)$ etc. This process terminates after at most $\sigma(X_0)$ steps and yields $Y = X_k$ a $\mathbb{Q}$–factorial variety with canonical (respectively terminal) singularities and a small projective birational morphism $f\colon Y \to X$. $\qquad\square$

Remark If Y is any $\mathbb{Q}$–factorialisation of X then the defect $\sigma(X)$ can also be calculated as
$$\sigma(X) = \mathrm{rk}\, \mathrm{Pic}(Y) - \mathrm{rk}\, \mathrm{Pic}(X).$$

Remark The existence of small $\mathbb{Q}$–factorialisations is now known in all dimensions as a consequence of the work of Birkar–Cascini–Hacon–McKernan [7]. One chooses an initial resolution of singularities and then runs an appropriate well-directed relative minimal model program which contracts the exceptional divisors of the resolution and whose output is the desired small $\mathbb{Q}$–factorialisation. However, Kawamata's result for canonical 3–folds suffices for our purposes.

Remark 3.37

(i) Small $\mathbb{Q}$–factorialisations are not unique but one can prove that any two differ by a sequence of finitely many flops (see Kollár [53, 6.38]).

(ii) Suppose that X has only terminal singularities; if one small $\mathbb{Q}$–factorialisation of X is singular, then because terminal flops preserve singularities (recall Theorem 3.30) all small $\mathbb{Q}$–factorialisations are singular, and then X has no crepant resolutions.

A natural question is whether there are only finitely many distinct small $\mathbb{Q}$–factorialisations of a given Gorenstein terminal 3–fold X. This follows from the following more general finiteness result of Kawamata–Matsuki [51, Main Theorem].

Theorem 3.38 *Let X be a projective 3–fold with canonical singularities. Then there exist only finitely many projective birational crepant morphisms $f\colon Y \to X$ such that Y is a 3–fold with only canonical singularities.*

Remark Generalising Definition 3.19, we say a projective birational morphism $f\colon Y \to X$ is crepant if $K_Y = f^* K_X$.

In particular as an immediate corollary of Theorem 3.38 there are only finitely many different small $\mathbb{Q}$–factorialisations of a given terminal 3–fold X.

We summarise our discussion above. Given any terminal 3–fold X we can always find small $\mathbb{Q}$–factorialisations Y of X, but there are only finitely many of them and any two of them are related by a sequence of flops. For a general X, all $\mathbb{Q}$–factorialisations of X will still be singular; in this case X admits no projective small resolutions. In other words, for a general terminal 3–fold X it is quite rare that X admits a projective small resolution. For many purposes in algebraic geometry the existence of a small $\mathbb{Q}$–factorialisation of X often suffices; however for our later purposes in constructing smooth weak Fano 3–folds as projective small resolutions of terminal Fano 3–folds, it is crucial that the terminal Fano 3–fold admit a smooth small $\mathbb{Q}$–factorialisation, that is, a projective small resolution.

It can therefore be very subtle to determine whether a 3–fold X with Gorenstein terminal (respectively canonical) singularities admits a projective small (respectively crepant) resolution. Even if we suppose X has only terminal and therefore isolated cDV singularities and that locally (the analytic germ of) each singularity admits a small resolution then there are global reasons why X may admit no projective small resolutions. This occurs even in the simplest case where X is nodal, that is, has only ordinary double points.

Projective small resolutions of nodal 3–folds

We now consider in more detail the projective small resolution problem for the special case of *nodal* 3–folds: recall Definition 3.24; projective small resolutions of nodal Fano 3–folds will give rise to smooth weak Fano 3–folds containing special rigid holomorphic curves. These special rigid curves will play a crucial role in [17] because they give rise to rigid associative 3–folds in twisted connected sum G_2–manifolds.

As we now explain, it is not a problem to find small resolutions of X if we are prepared to leave the projective world and work in the *complex analytic category*; the difficulty is to find *projective* (or Kähler) small resolutions of X. Suppose the 3–fold X has k ordinary nodes $P_1, \ldots, P_k$ as its only singular points. Let $\widetilde{X}$ denote the blowup of X in all its singular points; $\pi \colon \widetilde{X} \to X$ is a non-singular projective 3–fold with k exceptional divisors $E_1, \ldots, E_k$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There are two natural projections $\pi_i^{\pm} \colon E_i \to \mathbb{P}^1$, (rulings of $\mathbb{P}^1 \times \mathbb{P}^1$) corresponding to a choice of $\mathbb{P}^1$ factor. For each exceptional divisor we make a choice of one of these two rulings; by Nakano [73; 28] the fibres of every $\pi_i^{\pm}$ can be blown down to yield a non-singular Moishezon 3–fold $\widehat{X}$, that is, a compact complex 3–fold with three algebraically independent meromorphic functions. Thus we obtain 2^k Moishezon small resolutions $\widehat{X}$ of the nodal 3–fold X in which each singular point P_i has been replaced by a non-singular rational curve C_i with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In general some of these 2^k small resolutions may be isomorphic. This happens when the nodal 3–fold X admits automorphisms permuting the nodes; such an automorphism will lift to an action on the set of all small resolutions of X and thereby give rise to isomorphic small resolutions.

Since all the small resolutions are Moishezon, a small resolution $\widehat{X}$ of X is projective if and only if it is Kähler. (Recall that Moishezon [65] proved that a Moishezon manifold is projective if and only if it is Kähler.) A natural but delicate question is therefore: given a nodal projective 3–fold X with k nodes how many of its 2^k Moishezon small resolutions are projective (Kähler)?

In general, even though our initial nodal 3–fold X is projective *none* of its 2^k small resolutions need be projective. In fact, from our previous results we have the following: all 2^k Moishezon small resolutions of X are non-projective if and only if any (and therefore all) projective small $\mathbb{Q}$–factorialisation of X is singular. Thus answering the question above about how many of the small resolutions are projective is rather subtle and it is equivalent to the following two questions

(i) Does X admit a *smooth* projective small $\mathbb{Q}$–factorialisation?

(ii) If so, how many different projective small $\mathbb{Q}$–factorialisations does X admit?

The existence of projective small resolutions of nodal projective 3–folds has been considered by various authors. To illustrate some of the issues in concrete cases – of interest later in this paper – we consider nodal cubics $X \subset \mathbb{P}^4$ with a small number of nodes; for a systematic study of projective small resolutions of nodal cubics in $\mathbb{P}^4$ see Finkelnberg [24], Finkelnberg–Werner [26] and Werner [99].

Small resolutions of nodal cubics and weak Fano 3–folds Finkelnberg–Werner [26] proved that if a nodal cubic $X \subset \mathbb{P}^4$ has fewer than 4 nodes then X itself is already $\mathbb{Q}$–factorial irrespective of the position of its nodes; therefore X admits no projective small resolutions. However, they showed that whether a nodal cubic $X \subset \mathbb{P}^4$ with 4 nodes is $\mathbb{Q}$–factorial or not depends on the position of the 4 nodes; X is $\mathbb{Q}$–factorial if and only if the 4 nodes are not contained in some projective plane $\Pi^2 \subset \mathbb{P}^4$. If the 4 nodes do lie in some plane then this special surface Π gives us a Weil divisor D' on X which is not $\mathbb{Q}$–Cartier and one projective small resolution Y is then obtained by blowing up this plane, that is, by taking $Y = \operatorname{Proj} R(X, D')$ as in Lemma 3.14.

In fact, since the nodal cubic X is a mildly singular (Gorenstein terminal) Fano 3–fold it will turn out that the projective small resolution Y is a smooth weak Fano 3–fold containing 4 rigid rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$: one from each of the 4 nodes in X. We will return to nodal cubics in $\mathbb{P}^4$ in Section 8 where we will explain how to obtain numerous smooth weak Fano 3–folds from such nodal cubics, generalising this example.

This example demonstrates clearly that the existence of projective small resolutions is a global question which depends on the location of singularities and not just the local analytic singularity type or number of singularities. It also illustrates that if X does not contain some relatively special surfaces (in this case the projective plane Π) then we have no candidate Weil non-$\mathbb{Q}$–Cartier divisors D' which we can "blowup" to obtain a nontrivial small birational morphism as in Lemma 3.14.

The number of small projective resolutions We highlight another aspect of the subtlety of the projectivity of small resolutions. A cubic X with 4 nodes containing a plane Π as above has defect $\sigma(X) = 1$ (see Finkelnberg–Werner [26, pages 190–191]) and hence the projective small resolution $\varphi \colon Y \to X$ obtained by blowing up the plane Π has Picard rank $\rho(Y) = 2$.[4] Therefore, by Remark 3.31, $\varphi \colon Y \to X$ has a unique flop, $\varphi^+ \colon Y^+ \to X$. Hence, by Remark 3.37, Y^+ is the only other projective small resolution of X (moreover by [26, page 191] these two projective small resolutions are not isomorphic). In other words, only 2 of the $2^4 = 16$ Moishezon small resolutions of X are projective.

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[4] See also Example 7.3 where we prove similar statements for a *quartic* 3–fold that contains a plane.

More generally, we will see that nodal Fano 3–folds arising as the anticanonical models of non-singular weak Fano 3–folds of Picard rank 2 have exactly *two* projective small resolutions (again because of the uniqueness of flops when the Picard rank $\rho = 2$). However, in Section 8 we will see that such rank 2 weak Fano 3–folds can have up to 46 nodes in their anticanonical models and therefore admit up to 2^{46} Moishezon small resolutions!

Small and crepant resolutions of toric Fano 3–folds From the point of view of understanding small (respectively crepant) projective resolutions one very nice class of Gorenstein terminal (respectively canonical) 3–folds are the *toric* Gorenstein Fano 3–folds. Some particularly pleasant features of the toric Gorenstein Fano world are:

(i) All singularities of toric terminal Gorenstein Fano 3–folds are ordinary double points.

(ii) Toric terminal (respectively canonical) Gorenstein Fano 3–folds are completely classified.

(iii) Every toric terminal Gorenstein Fano 3–fold has at least one projective small resolution, and moreover one can enumerate all possible projective small resolutions combinatorially.

We will give a more detailed description of the class of toric Gorenstein terminal (and more generally canonical) Fano 3–folds and small (respectively crepant) resolutions thereof later in Section 8; this will show that there is a very plentiful supply of toric weak Fano 3–folds.

Rudiments of Mori theory

We recall some basic terminology from Mori theory; see Debarre's book [19] for a more detailed introduction and Kollár–Mori's book [56] for a complete treatment. Mori theory will be needed only in Section 8 when we discuss the classification scheme for weak Fano 3–folds with Picard rank $\rho = 2$ and so the rest of this section may be safely ignored until then.

Definition 3.39 A 1–cycle on a projective variety Y is a formal linear combination of irreducible, reduced curves $C = \sum_i a_i C_i$. C is *effective* if $a_i \geq 0$ for every i. Two 1–cycles C and C' are *numerically equivalent* if they have the same intersection number with every Cartier divisor; we write $C \sim C'$. 1–cycles with real coefficients modulo numerical equivalence form a real vector space denoted $N_1(Y)$; the class of a 1–cycle C is denoted $[C]$.

Inside $N_1(Y)$ sits the (convex) cone of curves $NE(Y)$, the set of classes of effective 1–cycles.

Definition 3.40 The *cone of curves $NE(Y)$* is defined by

$$NE(Y) := \left\{ \sum a_i[C_i] \ \Big| \ C_i \subset X, 0 \leq a_i \in \mathbb{R} \right\} \subset N_1(Y),$$

where C_i are irreducible curves on Y. $\overline{NE}(Y)$ is defined to the closure of $NE(Y)$ in $N_1(Y)$.

Let X and Y be projective varieties. Define the *relative cone of a morphism $\pi\colon Y \to X$* as the convex subcone $NE(\pi) \subset NE(Y)$ generated by the classes of curves contracted by π. Since X is projective, an irreducible curve C is contracted by π if and only if $\pi_*[C] = 0$; in other words, being contracted is a numerical property. $NE(\pi)$ has the additional property that it is extremal.

Definition 3.41 Let V be a convex cone in $\mathbb{R}^n$. A subcone $W \subset V$ is *extremal* if it is closed and convex and if any two elements of V whose sum lies in W are both in W. Geometrically, this means that the cone V lies on one side of some hyperplane containing the extremal subcone W. An extremal cone of dimension 1 is called an *extremal ray*.

Lemma 3.42 (Debarre [19, Propositions 1.14 and 1.43–1.45]) *Let $\pi\colon Y \to X$ be a morphism of projective varieties.*

 (i) *The subcone $NE(\pi) \subset NE(Y)$ is a closed convex subcone which is extremal.*

 (ii) *If additionally we assume $\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X$ then the morphism π is determined by $NE(\pi)$ up to isomorphism.*

Lemma 3.42 says that a morphism determines an extremal subcone of $NE(Y)$ which, under the additional condition given in (ii), characterises that morphism. This motivates the following:

Definition 3.43 Let Y be a projective variety and $F \subset \overline{NE}(Y)$ an extremal face. A morphism $\mathrm{cont}_F\colon Y \to X$ to a projective variety X is called the *contraction of F* if

 (i) $\mathrm{cont}_F(C) = pt$, for an irreducible curve C if and only if $[C] \in F$; and

 (ii) $(\mathrm{cont}_F)_*\mathcal{O}_Y = \mathcal{O}_X$.

In general not every extremal face can be contracted. A central point in Mori theory is to find conditions guaranteeing the existence of cont_F. The main result in this direction is the following deep theorem often called the Contraction Theorem.

Theorem 3.44 (Contraction Theorem) *Let Y be a projective variety with at worst canonical singularities and let $F \subset \overline{NE}(Y)$ be an extremal face on which K_Y is negative; then the contraction cont_F (as in Definition 3.43) exists.*

Remark Kollár-Mori [56, Theorem 3.7.3] in fact proves a more general version of the Contraction Theorem (for klt pairs).

From now on we focus on the case of contractions associated with extremal rays. The following result says that a contraction associated with an extremal ray comes in three basic flavours; see [56, Proposition 2.5].

Proposition 3.45 *Let Y be a normal projective variety that is $\mathbb{Q}$–factorial. Let $\text{cont}_R \colon Y \to X$ be the contraction of an extremal ray $R \subset \overline{NE}(Y)$. Then one of the following holds:*

  (i) *(fibre type contraction)* $\dim Y > \dim X$;

  (ii) *(divisorial contraction)* f *is birational and* $\text{Ex}(f)$ *is an irreducible divisor;*

  (iii) *(small contraction)* f *is birational and* $\text{Ex}(f)$ *has codimension* ≥ 2.

Mori [66] gave a description of all contractions of extremal rays on a *non-singular* projective 3–fold.

Theorem 3.46 *Let Y be a non-singular projective 3–fold and $\text{cont}_R \colon Y \to X$ be the contraction of a K_Y–negative extremal ray $R \subset \overline{NE}(Y)$. The following is a complete list of possibilities for cont_R:*

  E *(exceptional)* $\dim X = 3$, cont_R *is birational and there are five types of local behaviour near the contracted surface.*

    E1 cont_R *is the inverse of the blowup of a non-singular curve in the non-singular threefold X.*

    E2 cont_R *is the inverse of the blowup of a non-singular point of the non-singular threefold X.*

    E3 cont_R *is the inverse of the blowup of an ordinary double point of X.*

    E4 cont_R *is the inverse of the blowup of an isolated cDV point of X which is locally analytically given by the equation $x^2 + y^2 + z^2 + w^3 = 0$.*

E5 cont$_R$ contracts a non-singular $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$ to a point on X which is locally analytically the quotient of $\mathbb{C}^3$ by the involution $(x, y, z) \mapsto -(x, y, z)$.

C *(conic bundle)* dim $X = 2$, cont$_R$ is a fibration whose fibres are plane conics.

D *(del Pezzo)* dim $X = 1$ and the general fibre of cont$_R$ is a del Pezzo surface of degree $d \neq 7$.

F *(Fano)* dim $X = 0$, $-K_X$ is ample and hence X is a Fano variety.

Remark 3.47 Note that for non-singular projective threefolds there are no small extremal contractions: type (iii) in Proposition 3.45.

4 Weak Fano 3–folds: basic theory

4.1 Weak Fano 3–folds and semi-Fano 3–folds

In this section, we review the definition and elementary properties of weak Fano 3–folds. We postpone any in-depth discussion of examples of weak Fano 3–folds until Sections 7 and 8 giving only two of the simplest weak Fano 3–folds as Examples 4.15 and 4.16.

Definition 4.1 A *weak Fano* 3–*fold* is a non-singular projective complex 3–fold Y such that the anticanonical sheaf $-K_Y$ is a nef and big line bundle (recall Definition 3.2 for the definitions of big and nef). The *index* of a weak Fano 3–fold Y is the integer $r = \operatorname{div} c_1(Y)$, that is, the greatest divisor of $c_1(Y) \in H^2(Y; \mathbb{Z})$.

Remark 4.2 The index $r(Y)$ of a weak Fano 3–fold belongs to $\{1, 2, 3, 4\}$. The only weak Fano 3–fold with index 4 is $\mathbb{P}^3$ (which of course is Fano). Weak Fano 3–folds with index 3 are classified; besides the quadric in $\mathbb{P}^4$ (which is Fano) there are only two further weak Fano 3–folds of index 3, namely Examples 4.15 and 4.16; see Casagrande–Jahnke–Radloff [11, Proposition 3.3] and Shin [89, Theorem 3.9]. Weak Fano 3–folds with index 2 are called *weak del Pezzo* (or sometimes *almost del Pezzo*) 3–folds. There are relatively few weak del Pezzo 3–folds; see Jahnke–Peternell [39] for a partial classification. However, note that a single deformation class of smooth del Pezzo 3–folds may give rise to a fairly large number of different deformation classes of weak del Pezzo 3–folds; see the discussion of nodal cubics in Section 8 for a more concrete demonstration of this phenomenon. Nevertheless, the vast majority of weak Fano 3–folds have index 1.

We construct a handful of examples of weak Fano 3–folds in Section 7 and discuss partial classification results and show the existence of many more examples in Section 8; the crucial point is that there are *many* more deformation families of weak Fano 3–folds than Fano 3–folds, though still only finitely many; see Theorem 4.13.

In the next several paragraphs, we summarise a few standard facts on weak Fano 3–folds which play an important role in this paper. These are properties of Fano 3–folds which extend without much difficulty to the case when the anticanonical bundle is only nef and big. Most of these follow by applying Kawamata–Viehweg vanishing (recall Theorem 3.5) wherever we would have used Kodaira vanishing in the Fano case. For an introduction to some of the basic properties of weak Fano 3–folds, see Reid [85, Section 4].

Corollary 4.3 *For any weak Fano 3–fold Y we have:*

(i) *All Hodge numbers $h^{i,0} = h^{0,i} = 0$ for $i > 0$.*

(ii) *The natural homomorphism $\mathrm{Pic}\, Y \to H^2(Y; \mathbb{Z})$ is an isomorphism.*

(iii) *$K_Y \cdot c_2(Y) = -24$, where $c_2(Y) \in H^4(Y; \mathbb{Z})$ denotes the second Chern class of Y.*

(iv) *The dimension of the space of holomorphic sections of $-K_Y$ is given by*

$$(4\text{-}4) \qquad h^0(Y, -K_Y) = g + 2, \quad \text{where } -K_Y^3 = 2g - 2.$$

Proof Part (i) follows immediately from Kawamata–Viehweg vanishing (Theorem 3.5) and Hodge theory. Part (ii) follows from the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ and $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$. For part (iii) recall that Riemann–Roch in the case of a line bundle L on a non-singular 3–fold Y gives

$$(4\text{-}5) \quad \chi(Y, L) := \sum_{i=0}^{3} (-1)^i h^i(L) = \tfrac{1}{6}L^3 - \tfrac{1}{4}L^2 K_Y + \tfrac{1}{12}L(K_Y^2 + c_2) - \tfrac{1}{24}K_Y \cdot c_2(Y).$$

Using part (i) and setting $L = 0$ we obtain $K_Y \cdot c_2(Y) = -24$. For part (iv) we now apply Kawamata–Viehweg vanishing with $L = -2K_Y$ to yield

$$h^0(Y, -K_Y) = \chi(Y, -K_Y) = -\tfrac{1}{2}K_Y^3 + 3 = g + 2. \qquad \square$$

Definition 4.6 The invariant g in (4-4) is called the *genus* of Y; the even integer $(-K_Y)^3 = 2g - 2$ is called the *anticanonical degree* of Y.

The following facts about anticanonical sections of weak Fano 3–folds are well known to algebraic geometers; they are central to the current paper.

It follows from the vanishing results in Corollary 4.3 together with adjunction that if a member $S \in |-K_Y|$ is non-singular, then it is a K3 surface. For smooth Fano 3–folds the existence of a non-singular member $S \in |-K_Y|$ is due to Shokurov [91]. For a weak Fano 3–fold Y with at worst canonical Gorenstein singularities Reid [85, Theorem 0.5] proved that a general $S \in |-K_Y|$ is an irreducible K3 with at worst rational double point singularities. Using Reid's result Paoletti [79, Proposition 2.1] deduced the following:

Theorem 4.7 *If Y is a non-singular weak Fano 3–fold then a general anticanonical member $S \in |-K_Y|$ is a non-singular K3 surface.*

To define the anticanonical morphism and the anticanonical model associated with any weak Fano 3–fold we need the following:

Theorem 4.8 *If Y is a weak Fano 3–fold, then $-K_Y$ is semi-ample.*

Proof The anticanonical divisor of Y is big and nef and hence by the Basepoint-free Theorem (apply Reid [85, Theorem 0.0] with $D = -K_Y$ and $a = 1$) the linear system $|-nK_Y|$ is basepoint-free for n sufficiently large. □

Since $-K_Y$ is semi-ample, by Remark 3.4 $-K_Y$ is finitely generated and the birational morphism $\varphi \colon Y \to \operatorname{Proj} R(Y, -K_Y)$ coincides with the algebraic fibre space $\varphi \colon Y \to X$ given by Theorem 3.3.

Definition 4.9 If Y is a weak Fano 3–fold, we call the finitely generated ring

$$R(Y, -K_Y) = \bigoplus_{n \geq 0} H^0(Y; -nK_Y)$$

the *anticanonical ring* of Y, and we call the birational morphism $\varphi \colon Y \to X = \operatorname{Proj} R(Y, -K_Y)$ attached to $|-K_Y|$ the *anticanonical morphism* and X the *anticanonical model* of Y.

We will sometimes abbreviate anticanonical as AC and therefore refer to the AC morphism or AC model of a weak Fano 3–fold Y.

Remark 4.10

(i) It is clear that Y is a resolution of singularities of the anticanonical model X. It is more-or-less a tautology that

$$K_Y = \varphi^* K_X \quad \text{and} \quad R(Y, -K_Y) = R(X, -K_X).$$

In particular, Y is a *crepant* resolution of X. It follows immediately that X has Gorenstein canonical singularities and that $-K_X$ is an ample line bundle; thus X is in its own right a singular Fano variety with at worst Gorenstein canonical singularities.

(ii) Conversely, given a Fano 3–fold X with Gorenstein canonical singularities one can ask whether X admits a non-singular projective crepant resolution Y; any such Y will be a non-singular weak Fano 3–fold with X as its anticanonical model.

(iii) If $|-K_Y|$ has two non-singular members S_1, S_2 intersecting *transversally* then the intersection $S_1 \cap S_2$ is a canonically polarized curve C (that is, $-K_{Y|C} = K_C$) of genus g.

(iv) In all examples we consider, the anticanonical ring $R(Y, -K_Y)$ is generated in degree 1; equivalently, $-K_X$ is very ample. In this case non-singular members S_1, S_2 always exist. The few Gorenstein canonical Fano 3–folds X for which $-K_X$ fails to be very ample are classified by Jahnke–Radloff [42, Theorem 1.1].

Now we define a subclass of weak Fano 3–folds that will play an important role throughout the rest of the paper and in our paper [17]. First recall the definitions of small and semi-small projective birational morphisms from Definition 3.8.

Definition 4.11 Let Y be a weak Fano 3–fold and $\varphi\colon Y \to X$ its anticanonical morphism. If φ is *semi-small*, we call Y a *semi-Fano* 3–fold.

Remark 4.12

(i) The anticanonical morphism $\varphi\colon Y \to X$ of a semi-Fano 3–fold may contract divisors to curves, or curves to points, but not divisors to points.

(ii) If the anticanonical morphism $\varphi\colon Y \to X$ of a semi-Fano 3–fold is *small* and not just semi-small then it contracts only a finite number of curves to points. X is then a Fano 3–fold with Gorenstein terminal and therefore isolated cDV singularities (recall Definition 3.25); the curves $C \subset Y$ contracted by φ give rise to the isolated cDV points in X. In this case φ is a flopping contraction in the sense of Definition 3.28. Hence if D is any φ–antiample (recall Remark 3.15) Cartier divisor on Y by Theorem 3.30 we may perform the D–flop of φ. This yields another semi-Fano 3–fold Y^+ whose anticanonical model $\varphi^+\colon Y^+ \to X$ is another small projective birational morphism, and where D^+ is φ^+–ample. Thus each semi-Fano 3–fold Y with small anticanonical morphism gives rise to at least one other semi-Fano 3–fold with small anticanonical morphism and the same anticanonical model X. (If $\rho(Y/X) > 1$, there can be several other semi-Fano 3–folds with the same anticanonical model. For all Cartier divisors D on Y, there is a sequence of flops $Y \dashrightarrow Y'$, with anticanonical model $\varphi'\colon Y' \to X$, such that D' is φ'–nef.)

(iii) In our construction of twisted connected sum G_2–manifolds in [17] we will be particularly interested in semi-Fano 3–folds Y with *nodal* anticanonical model X, that is, the only singular points of X are ordinary double points. In this special case of (ii) the curves contracted by φ are finitely many 'rigid' rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. These curves are the exceptional curves over the nodes of the anticanonical model X. As in the previous part we can flop $\varphi\colon Y \to X$ to obtain finitely many other semi-Fano 3–folds with the same nodal anticanonical model X. Each of these rigid rational curves C in Y will give rise to a compact rigid holomorphic curve in any ACyl Calabi–Yau 3–fold V constructed from Y using Proposition 4.24. These compact rigid holomorphic curves in our ACyl Calabi–Yau 3–folds will in turn be the source of compact rigid associative 3–folds in the twisted connected sum G_2–manifolds we construct in [17] out of pairs of ACyl Calabi–Yau 3–folds.

For smooth Fano 3–folds we know there are precisely 105 deformation families. For weak Fano 3–folds we still have:

Theorem 4.13 *There are only finitely many deformation families of smooth weak Fano 3–folds.*

Proof The anticanonical model X of a weak Fano 3–fold is a Gorenstein canonical Fano 3–fold. Gorenstein canonical Fano 3–folds form a bounded family; see Kollár–Miyaoka–Mori–Takagi [55, Corollary 1.3] (which proves the same holds for all canonical $\mathbb{Q}$–Fano 3–folds). Applying Theorem 3.38 we see that each deformation family

of Gorenstein canonical Fano 3–folds gives rise to only finitely many deformation families of weak Fano 3–folds.   □

We are not aware of another reference for Theorem 4.13 but it was surely known to various experts.

Remark 4.14

(i) The previous theorem does not yield any estimate on the number of deformation families of weak Fano 3–folds. For this we would need an improvement of Theorem 3.38 that gives a quantitative bound on the number of projective crepant resolutions of a given Gorenstein canonical Fano 3–fold.

(ii) Even if we restrict to the toric world we will see in Section 8 that there are over 4000 deformation families of toric Gorenstein canonical Fano 3–folds X. Each such X has at least one and often a large number of projective crepant (toric) resolutions and therefore gives rise to potentially many deformation families of toric weak Fano 3–folds with the same AC model X. Moreover, almost 900 of these toric Gorenstein canonical Fano 3–folds X give rise to semi-Fano 3–folds in the sense of Definition 4.11. So even toric semi-Fano 3–folds are very plentiful.

To make the discussion more concrete we give two of the simplest examples of semi-Fano 3–folds of different flavours; we will give many more examples of weak Fanos in Sections 7 and 8. Examples 4.15 and 4.16 are the only two weak Fano 3–folds of index three (recall Remark 4.2) besides the quadric Q^3 which of course is Fano.

Example 4.15 Let $X \subset \mathbb{P}^4$ be the projective cone over a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$; X is a Gorenstein terminal Fano 3–fold with Picard rank 1, defect 1, anticanonical degree 54, index 3 and 1 ODP at the apex of the cone. X has two small resolutions Y and Y^+ both of which are projective and isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$ (where $\mathcal{O}(d)$ denotes $\mathcal{O}_{\mathbb{P}^1}(d)$) which is a $\mathbb{P}^2$–bundle over $\mathbb{P}^1$. The anticanonical morphism $\varphi\colon Y \to X$ contracts the unique section C_0 with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Y is the unique non-singular toric weak Fano 3–fold with nodal anticanonical model and Picard rank $\rho = 2$; see Remark 8.13.

Next we give a simple example of a semi-Fano 3–fold Y with $\rho = 2$ which is semi-small but not small, that is, for which the anticanonical morphism contracts a divisor to a curve; as in the previous example Y is a $\mathbb{P}^2$–bundle over $\mathbb{P}^1$ and Y is toric.

Example 4.16 $Y = \mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O})$ (where as above $\mathcal{O}(d)$ denotes $\mathcal{O}_{\mathbb{P}^1}(d)$) is a non-singular rank 2 toric weak Fano 3–fold. As in the previous example Y is a weak Fano 3–fold of index 3 and anticanonical degree 54. However, in this case one can verify that the anticanonical morphism $\varphi \colon Y \to X$ contracts the divisor $D = \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$ to a curve along which X has A_1 singularities.

Remark If Y is a weak Fano 3–fold with index three then its anticanonical model X is a Gorenstein Fano 3–fold of index three with at worst canonical singularities. Hence by Shin [89, Theorem 3.9] X is isomorphic to some hyperquadric in $\mathbb{P}^4$.

Smoothing terminal Fano 3–folds and semi-Fano 3–folds

A very useful result which yields some modest control over terminal Gorenstein Fano 3–folds and hence non-singular weak Fano 3–folds with *small* anticanonical morphism is Namikawa's smoothing theorem for terminal Fano 3–folds; see Namikawa [74, Theorems 11 and 13] and also Namikawa–Steenbrink [75, Lemma 3.4].

Theorem 4.17 *Let X be a Fano 3–fold X with Gorenstein terminal singularities.*

   (i) *X is smoothable by a flat deformation, and hence is a degeneration of a non-singular Fano 3–fold X_t from the Iskovskih–Mori–Mukai classification. In particular, the anticanonical degrees, the Picard ranks and the Fano indices of X and X_t are equal.*

   (ii) *Suppose that a non-singular Fano 3–fold X_t degenerates to X by a flat deformation. Then we have*

$$(4\text{-}18) \qquad e(X) + \sum_{p \in Sing(X)} \mu(X, p) \leq 21 - \tfrac{1}{2}\chi(X_t) = h^{2,1}(X_t) + 20 - \rho(X_t),$$

where $\chi(X_t)$ is the topological Euler characteristic of X_t, $e(X)$ is the number of ordinary double points of X and μ is the non-negative integer invariant of an isolated rational singularity defined in [74, Section 2] (μ vanishes for an ODP and is positive for other Gorenstein terminal singularities).

   (iii) *In the case considered in (ii) we have*

$$H^i(X, \mathbb{Z}) \cong H^i(X_t, \mathbb{Z}) \qquad \text{for } i \neq 3, 4,$$

and the defect of X satisfies

$$\sigma(X) = b^3(X) - b^3(X_t) + \sum_{P \in Sing(X)} m_P,$$

where m_P denotes the Milnor number of the isolated hypersurface singularity $P \in X$. In particular, the defect of a nodal Fano 3–fold X with e nodes and Fano smoothing X_t satisfies

$$(4\text{-}19) \qquad\qquad \sigma(X) = b^3(X) - b^3(X_t) + e.$$

Remark Namikawa proves a slightly more general smoothing result than we have stated. His result generalises earlier work of Friedman [27, Corollary 4.2]. The anticanonical degree of a Gorenstein canonical Fano 3–fold can be as large as 72; in particular, since the maximal anticanonical degree of a non-singular Fano 3–fold is 64 (attained only by $\mathbb{P}^3$), there can be no general smoothing result for canonical Fano 3–folds analogous to Theorem 4.17. A more fundamental reason is that smoothings of canonical singularities are much more subtle than smoothings of terminal singularities.

Remark 4.20

(i) Since we have only finitely many topological types of non-singular Fano 3–fold, $\chi(X_t)$ is bounded over all non-singular Fano 3–folds X_t and hence we get a bound on the maximum number of singular points (in particular ODPs) that any terminal Gorenstein Fano 3–fold X can have. Consulting the Iskovskih–Mori–Mukai classification we find that

$$10 \le 21 - \tfrac{1}{2}\chi(X_t) = h^{2,1}(X_t) + 20 - \rho(X_t) \le 71.$$

We can also consult the classification to compute $\chi(X_t)$ in any given case.

(ii) The term $h^{2,1}(X_t)$ on the RHS of (4-18) varies between 0 and 52. Only 11 Fano 3–folds have $h^{2,1} \ge 5$ and all such examples have relatively small anticanonical degree, for example, smooth quartics have $h^{2,1} = 30$ and anticanonical degree 4. On the other hand, $\rho(X_t)$ varies only between 1 and 10, and exceeds 5 only when the Fano 3–fold is the product of $\mathbb{P}^1$ with a del Pezzo surface. So the main contribution to the variation in the bound on the RHS of (4-18) comes from the variation of $h^{2,1}$. In particular only terminal Fano 3–folds which smooth to Fano 3–folds with large $h^{2,1}$ (which from the classification have small anticanonical degree) can have a large number of nodes. In [17] we construct compact G_2–manifolds from a pair of ACyl Calabi–Yau 3–folds via the twisted connected sum construction. When both ACyl Calabi–Yau 3–folds arise from blowing up a generic AC pencil on a semi-Fano 3–fold with nodal AC model X we can produce one rigid associative 3–fold with topology $S^1 \times \mathbb{P}^1$ for each node of X. Therefore bounds on the number of nodes of nodal Fano 3–folds imply bounds on the number of rigid associative 3–folds we can exhibit in our G_2–manifolds.

(iii) The maximum for $21 - \frac{1}{2}\chi(X_t)$ of 71 is achieved only for sextic double solids. The next highest value is 49 which is achieved only for quartics in $\mathbb{P}^4$; by de Jong–Shepherd-Barron–Van de Ven [43] a nodal quartic has at most 45 nodes (see Example 7.7 for such a quartic), so the bound from (4-18) is not sharp (but not so far from sharp either).

(iv) If the singularity at $P \in X$ is given by $\{f(x, y, z, t) = 0\}$ in local analytic coordinates (recall Definition 3.25), for f a polynomial with an isolated critical point at the origin, then the Milnor fibre is $\{f(x, y, z, t) = 1\}$. It is homotopic to a bouquet of 3–spheres and hence its cohomology is supported in degrees 0 and 3; the Milnor number m_P is equal to the number of spheres in the bouquet. In particular, $m_P = 1$ if and only if P is an ODP.

Let Y be a non-singular semi-Fano 3–fold with nodal anticanonical model X. We can compute the third Betti number of Y in terms of the defect σ of X, the number of nodes e and the third Betti number b of a Fano smoothing X_t of X as follows.

Lemma 4.21 *Let Y be a non-singular semi-Fano 3–fold with nodal anticanonical model X, and containing e exceptional $(-1, -1)$ curves. Let σ be the defect of X and $b = b^3(X_t)$ the third Betti number of a Fano smoothing X_t of the nodal Fano 3–fold X. Then*

$$(4\text{-}22) \qquad\qquad b^3(Y) = b - 2e + 2\sigma.$$

Proof We will compare Y to X via the small resolution $\varphi\colon Y \to X$ and also $X = X_0$ to a 1–parameter Fano smoothing X_t; the existence of the Fano smoothing of X follows from Theorem 4.17.

In the computation of the cohomology of the small resolution, and elsewhere in this paper, we work in the derived category of sheaves with (Zariski) constructible cohomology. If X is an algebraic variety then $\mathbb{Z}_X$ denotes the sheaf with constant fibre $\mathbb{Z}$, that is, if $U \subset X$ is open and connected, then $\mathbb{Z}_X(U) = \mathbb{Z}$. The sheaf cohomology groups of $\mathbb{Z}_X$ – calculated by taking an injective resolution $\mathbb{Z}_X \to \mathcal{I}^\bullet$ – are isomorphic to the singular cohomology groups of X (with integer coefficients). If $\varphi\colon Y \to X$ is a morphism then $R\varphi_\star \mathbb{Z}_Y$ denotes the derived direct image: it is a complex of sheaves on X with the property that $H^m(X, R\varphi_\star \mathbb{Z}_Y) = H^m(Y; \mathbb{Z})$.

We use the (nonsplit) exact triangle:

$$\mathbb{Z}_X \longrightarrow R\varphi_\star \mathbb{Z}_Y \longrightarrow \bigoplus_{i=1}^{e} \mathbb{Z}_{P_i}[-2] \xrightarrow{+1},$$

where $P_i \in X$ are the e nodes, and $\mathbb{Z}_{P_i}$ the skyscraper sheaf at P_i. This gives rise to the long exact sequence:

$$(0) \longrightarrow H^2(X) \longrightarrow H^2(Y) \longrightarrow \mathbb{Z}^e \longrightarrow H^3(X) \longrightarrow H^3(Y) \longrightarrow (0)$$

and $H^4(X) \simeq H^4(Y)$. The exact sequence shows that

$$(4\text{-}23) \qquad b^3(Y) = b^3(X) - e + b^2(Y) - b^2(X) = b^3(X) - e + \sigma.$$

From (4-19) we have $b^3(X) = b - e + \sigma$. (4-22) follows immediately by substituting for $b^3(X)$ into (4-23). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We will use Lemma 4.21 repeatedly to compute b^3 of the weak Fano 3–folds that arise in Section 7.

4.2 ACyl Calabi–Yau 3–folds from weak Fano 3–folds

We now explain how one can obtain a compact 3–fold Z, to which Theorem 2.6 can be applied to construct ACyl Calabi–Yau manifolds, from almost any weak Fano 3–fold Y. Recall that Z needs to fibre over $\mathbb{P}^1$ with fibres in the anticanonical linear system, and some smooth fibres. Since by Theorem 4.7 a generic anticanonical divisor on a weak Fano 3–fold Y is a smooth K3 surface, it is natural to consider the 3–fold Z obtained by resolving the indeterminacies of a pencil in $|-K_Y|$. We will mostly consider pencils with smooth base loci, so that we can perform the resolution by blowing it up. As explained in Remark 4.10(iv), this is the case for a generic anticanonical pencil on almost any weak Fano; we assume this from now on.

Assumption The linear system $|-K_Y|$ of the weak Fano 3–fold Y contains two non-singular members S_0, S_∞ intersecting transversally.

Under this additional (mild) assumption on the weak Fano Y we can apply the following Proposition to obtain a compact projective 3–fold Z that satisfies the hypotheses of the ACyl Calabi–Yau Theorem 2.6.

Proposition 4.24 *Let Y be a closed Kähler (respectively projective) 3–fold, and suppose that $|S_0, S_\infty| \subset |-K_Y|$ is a pencil with smooth (reduced) base locus C, and that $S \in |S_0, S_\infty|$ is a smooth divisor. Then the blow-up $\pi\colon Z \to Y$ at C is a closed Kähler (respectively projective) 3–fold with a fibration $f\colon Z \to \mathbb{P}^1$ with anticanonical fibres. The proper transform of S in Z is isomorphic to S, and the image in $H^{1,1}(S)$ of the Kähler cone of Z contains the image of the Kähler cone of Y.*

Proof The proper transform of each element of $|S_0, S_\infty|$ is an anticanonical divisor on Z, and together they form a base-point-free pencil in $|-K_Z|$, thus defining the required fibration. If $[\omega_0] \in H^{1,1}(Y)$ is a Kähler class, then there is $\lambda_0 > 0$ such that $\pi^*[\omega] - \lambda[E]$ is a Kähler class on Z for any Kähler class $[\omega]$ in a neighbourhood U of $[\omega_0]$ and $\lambda \leq \lambda_0$ (where E is the exceptional divisor). The map from the Kähler cone of Y to the image of $H^{1,1}(Y)$ in $H^{1,1}(S)$ is open, so for sufficiently small $\lambda > 0$ there is some $[\omega] \in U$ such that $[\omega]_S = [\omega_0]_{|S} + \lambda[C]$. Thus $[\omega_0]_{|S}$ lies in the image of the Kähler cone of Z. $\qquad\square$

We will also consider some pencils where the base locus is reducible, but each component C_i is smooth and of multiplicity one. Blowing up C_1 gives a new smooth Kähler 3–fold Z_1 with an anticanonical pencil whose base locus is the proper transform of the remaining components. We can thus obtain a suitable 3–fold Z by blowing up the components in sequence; compare with the discussion preceding Example 2.7 in Kovalev–Lee [58]. When the C_i meet transversely, this is equivalent to blowing up the base locus and then making a projective small resolution of the ordinary double points resulting from the double points of the base locus.

Proposition 4.25 *Let $Y = Z_0$ be a closed Kähler 3–fold, and suppose that $|S_0, S_\infty| \subset |-K_Y|$ is a pencil with base locus $C_1 \cup \cdots \cup C_k$ so that C_i is smooth (and reduced) and suppose $S \in |S_0, S_\infty|$ is a smooth divisor. Let Z_i be the blow-up of Z_{i-1} at the proper transform of C_i. Then $Z = Z_k$ satisfies the conclusions of Proposition 4.24.*

Proof The proof is a straightforward variation on the proof of Proposition 4.24. The base locus of the pencil $|S_0, S_\infty|$ is resolved by blowing up all of the curves C_i, in any order. Thus there is a fibration $f: Z \to \mathbb{P}^1$. $\qquad\square$

5 Topology

As explained in Theorem 2.6, we can obtain an ACyl Calabi–Yau manifold $V = Z \setminus S$ from a compact Kähler manifold Z fibred over $\mathbb{P}^1$ by a pencil of (generically smooth) anticanonical divisors where S is the smooth anticanonical divisor given by the fibre at ∞. In this section, we collect some tools to compute basic topological invariants of V and Z when the complex dimension is 3. The choice of topological invariants of V and Z we compute is motivated in part by applications to the twisted connected sum construction of compact G_2–manifolds. To compute the integral cohomology of the resulting 7–manifolds in [17] and in many cases also the diffeomorphism type we need sufficient topological information about the topology of the building blocks used.

All homology and cohomology groups in this section are over $\mathbb{Z}$ unless explicitly stated otherwise.

5.1 Cohomology of the ACyl manifolds

We begin by discussing how the topology of the ACyl manifold $V = Z \setminus S$ is related to the compact manifolds Z and S. For convenience, we include some topological assumptions in our definition of such building blocks (see Definition 5.1), and restrict to the case of complex dimension 3. Proposition 5.7 provides a large class of building blocks (Z, f, S) satisfying the conditions imposed in Definition 5.1.

Definition 5.1 A *building block* is a non-singular projective 3–fold Z together with a projective morphism $f: Z \to \mathbb{P}^1$ satisfying the following four assumptions:

(i) The anticanonical class $-K_Z \in H^2(Z)$ is primitive.

(ii) $S = f^\star(\infty)$ is a non-singular K3 surface and $S \sim -K_Z$.

The fibration structure implies that S has trivial normal bundle in Z so $c_1(Z)^2 \sim S \cdot S = 0$. We denote by $j: V = Z \setminus S \hookrightarrow Z$ the open embedding of the complement and we still denote by $f: V \to \mathbb{C}$ the restricted morphism. Since the normal bundle of S in Z is trivial, there is an inclusion $S \hookrightarrow V$ well-defined up to homotopy, and the restriction map $H^m(Z) \to H^m(S)$ factors through $H^m(V)$, in the sense that it coincides with the composition

$$H^m(Z) \to H^m(V) \to H^m(S).$$

We write $H = H^2(V)$ and assume to have identified S with a "standard" K3 and $H^2(S)$ with the K3 lattice L, and set

- $\rho: H \to L$ the natural restriction map,
- $K = \ker(\rho)$, and
- $N = \rho(H) \subset L$.

(iii) The inclusion $N \hookrightarrow L$ is primitive, that is, L/N is torsion-free.

(iv) The group $H^3(Z)$ is torsion-free. This implies that $H^4(Z)$ is also torsion-free.

Remark In the case that, as in Proposition 4.24, the building block Z is obtained by blowing up the smooth (reduced) base locus C of an anticanonical pencil on a Kähler 3–fold Y, then it will follow from Lemma 5.6 that $H^3(Z)$ is torsion-free if and only if $H^3(Y)$ is. The possibility of torsion in $H^3(Y)$ is discussed in Remark 5.8 when Y is weak Fano.

Lemma 5.2 If (Z, S) is a building block then $\pi_1(Z) = \pi_1(V) = (0)$.

Remark Together with assumption (iv), this implies that $H^*(Z)$ and $H_*(Z)$ are torsion-free.

Proof The critical values of the morphism f are discrete in $\mathbb{P}^1$. The statement will follow from the van Kampen theorem once we show that $\pi_1(V_\Delta) = (0)$, where $V_\Delta = f^{-1}(\Delta)$ for any disc $\Delta \subset \mathbb{P}^1$ containing at most 1 critical value x. To this end, write $\Delta^\times = \Delta \setminus \{x\}$ and $V_\Delta^\times = f^{-1}(\Delta^\times)$. Since $V_\Delta^\times$ is a C^∞ fiber bundle over $\Delta^\times$ with fibre a K3 surface, which is simply connected, we see from the long exact sequence of homotopy groups in a fibration that $\pi_1(V_\Delta^\times) = \pi_1(\Delta^\times) = \mathbb{Z}$. Now let $f^*(x) = \sum m_i F_i$ be the fibre at x, where $F_i \subset V$ are the irreducible components and m_i their multiplicities. Condition (i) in the definition of a building block implies $\gcd(m_i) = 1$. It is well known and easy to see that the natural homomorphism $j_\star: \pi_1(V_\Delta^\times) \to \pi_1(V_\Delta)$, induced by the inclusion $j: V_\Delta^\times \hookrightarrow V_\Delta$, is surjective. Examining the image of a loop that loops once around the generic point of F_i, we see that $m_i j_\star(1) = 0$ in $\pi_1(V_\Delta)$. Since, as we noted, $\gcd(m_i) = 1$, we conclude that $j_\star(1) = 0$, that is, $\pi_1(V_\Delta) = 0$ as was to be shown. $\square$

We regard N as a lattice with the quadratic form inherited from L via the primitive inclusion $N \subset L$. In examples N is almost never unimodular, thus the natural inclusion $N \hookrightarrow N^*$ is not an isomorphism. We write

$$T = \{l \in L \mid \langle l, n \rangle = 0 \ \forall n \in N\}.$$

T stands for "transcendental"; in examples, N and T are the Picard and transcendental lattices of a lattice polarized K3 surface. Notice that, unless N is unimodular, we cannot write $L = N \oplus T$. However, since by (iii) N is primitive and L is unimodular there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow L \longrightarrow N^* \longrightarrow 0,$$

that is, $L/T \simeq N^*$.

Lemma 5.3 *Let (Z, f, S) be a building block and $V := Z \setminus S$. Then:*

(i) $H^1(V) = (0)$.

(ii) *The class $[S] \in H^2(Z)$ fits in a split exact sequence*

$$(0) \longrightarrow \mathbb{Z} \xrightarrow{[S]} H^2(Z) \longrightarrow H^2(V) \longrightarrow (0),$$

hence $H^2(Z) \simeq \mathbb{Z}[S] \oplus H^2(V)$ and the restriction $H^2(Z) \to L$ maps onto N.

(iii) *There is a split exact sequence*

$$(0) \longrightarrow H^3(Z) \longrightarrow H^3(V) \longrightarrow T \longrightarrow (0),$$

hence $H^3(V) \simeq H^3(Z) \oplus T$.

(iv) *There is a split exact sequence*

$$(0) \longrightarrow N^* \longrightarrow H^4(Z) \longrightarrow H^4(V) \longrightarrow (0),$$

hence $H^4(Z) \simeq H^4(V) \oplus N^*$.

(v) $H^5(V) = (0)$.

In particular, $H^*(V)$ is torsion-free.

Proof We use the triangle

$$\mathbb{Z}_S[-2] \longrightarrow \mathbb{Z}_Z \longrightarrow Rj_\star \mathbb{Z}_V \xrightarrow{+1} .$$

The associated long exact sequence is isomorphic via Poincaré duality to the long exact sequence for homology of the pair (Z, S). It starts out as

$$(0) \longrightarrow H^1(Z) \longrightarrow H^1(V) \longrightarrow H^0(S) \hookrightarrow H^2(Z) \longrightarrow \cdots .$$

We already know from [Lemma 5.2](#) that the first two terms vanish (i). The exact sequence continues with

$$(0) \longrightarrow H^0(S) = \mathbb{Z}[S] \longrightarrow H^2(Z) \longrightarrow H^2(V) \longrightarrow (0) = H^1(S).$$

The sequence splits because we assumed that $[S] = -K_Z$ is primitive (ii). The long exact sequence continues with:

$$(0) \longrightarrow H^3(Z) \longrightarrow H^3(V) \longrightarrow L \longrightarrow H^4(Z) \longrightarrow H^4(V) \longrightarrow (0).$$

The Poincaré dual of $L \to H^4(Z)$ is $H_2(S) \to H_2(Z)$. This dualizes to $H^2(Z) \to H^2(S)$, which has image N. Identifying $L \simeq L^*$, the image of the dual map coincides with the orthogonal complement of the kernel of $L \to H^4(Z)$; in other words, the kernel is T. This implies exactness of (iii), and exactness of (iv) follows since $N^* \cong L/T$. The sequence (iii) is split exact because T, being a subgroup of L, is torsion-free. The inclusion $N^* \to H^4(Z)$ is primitive because the dual is surjective, so (iv) splits too. Finally, (v) is immediate from the last piece of the long exact sequence. □

Remark Apart from the conclusion that $H^*(V)$ is torsion-free, the proof did not use the condition that $H^3(Z)$ is torsion-free.

As a corollary of the proof we obtain the following description.

Corollary 5.4 Let (Z, f, S) be a building block and $V := Z \setminus S$. Since the normal bundle of S in Z is trivial, we get a natural inclusion of $S \times \mathbb{S}^1 \subset V$. Denote by $\mathbf{a}^0 \in H^0(\mathbb{S}^1)$, $\mathbf{a}^1 \in H^1(\mathbb{S}^1)$ the standard generators. The natural restriction homomorphisms

$$\beta^m \colon H^m(V) \longrightarrow H^m(S \times \mathbb{S}^1) = \mathbf{a}^0 H^m(S) \oplus \mathbf{a}^1 H^{m-1}(S)$$

are computed as follows:

  (i) $\beta^1 = 0$.
  (ii) $\beta^2 \colon H^2(V) \to H^2(S \times \mathbb{S}^1) = \mathbf{a}^0 H^2(S)$ is the homomorphism $\rho \colon H \to L$.
  (iii) $\beta^3 \colon H^3(V) \to H^3(S \times \mathbb{S}^1) = \mathbf{a}^1 H^2(S)$ is the composition $H^3(V) \twoheadrightarrow T \subset L$.
  (iv) The natural restriction homomorphism $H^4(Z) \to H^4(S) = \mathbb{Z}$ is surjective and factors through $\beta^4 \colon H^4(V) \to H^4(S \times \mathbb{S}^1) = \mathbf{a}^0 H^4(S) = \mathbb{Z}$ and, writing $K = \ker[H \to N]$ as in [Definition 5.1](#), there is a split exact sequence:

$$(0) \longrightarrow K^* \longrightarrow H^4(V) \xrightarrow{\ \beta^4\ } H^4(S) \longrightarrow (0).$$

Proof Part (i) is trivial. Part (ii) uses only that $\mathrm{pr}_1 \circ \beta^m \colon H^m(V) \to H^m(S)$ is the natural map specified in [Definition 5.1](#).

For (iii), we use that the homomorphism $H^m(V) \to H^{m-1}(S)$ in the long exact sequence in the proof of [Lemma 5.3](#) is the "Griffiths tube map", that is, it is the composition:

$$H^m(V) \xrightarrow{\ \beta^m\ } H^m(S \times \mathbb{S}^1) \xrightarrow{\ \mathrm{pr}_2\ } \mathbf{a}^1 H^{m-1}(S).$$

To see this, note that the boundary map $H^m(S \times \mathbb{S}^1) \to H^{m+1}(S \times \Delta,\ S \times \mathbb{S}^1) \cong H^{m-1}(S)$ is equivalent to pr_2 (where $S \times \Delta$ is a tubular neighbourhood of S). The restriction map $H^m(Z \setminus S) \to H^m(S \times \Delta^\times)$ is equivalent to β^m while $H^{m+1}(Z, Z \setminus S) \cong H^{m+1}(S \times \Delta,\ S \times \Delta^\times)$ by excision. Therefore $\beta^m \circ \mathrm{pr}_2$ is the boundary map in the long exact sequence for cohomology of the pair $(Z,\ Z \setminus S)$, which is Poincaré dual to the long exact sequence for homology of $(Z,\ S)$.

The content of (iv) is to show that β^4 is surjective and to determine its kernel. β^4 fits into the long exact sequence for cohomology of V relative to its "boundary" $S \times \mathbb{S}^1$, and surjectivity follows from $H^5(V, S \times \mathbb{S}^1) \cong H_1(V) = 0$. The cup product gives a perfect pairing between the free parts of $\ker \beta^m$ and $\ker \beta^{6-m}$ for any m, so in particular $\ker \beta^4 \cong (\ker \beta^2)^* = K^*$. $\qquad\square$

Remark 5.5 We can also compute the homology groups of V as follows:

  (i) $H_1(V) = 0$.

(ii) $0 \longrightarrow H_2(V) \longrightarrow H_2(Z) \longrightarrow \mathbb{Z} \longrightarrow 0$ split exact.

(iii) $0 \longrightarrow T^* \longrightarrow H_3(V) \longrightarrow H_3(Z) \longrightarrow 0$ split exact.

(iv) $H_4(V) \cong K$.

(v) $H_5(V) = 0$.

5.2   Building blocks from semi-Fano 3–folds

We now study the topology of the closed 3–folds Z produced in Proposition 4.24 by blowing up the base locus of a generic anticanonical pencil on a weak Fano 3–fold Y.

We will use the following simple lemma in numerous places in the rest of the paper.

Lemma 5.6   Let $\pi\colon (E \subset Z) \to (C \subset Y)$ be the blow up of a non-singular curve in a non-singular 3–fold Y. Then $H^m(Z) \simeq H^m(Y) \oplus H^{m-2}(C)$.

Proof   The decomposition theorem holds over $\mathbb{Z}$:

$$R\pi_\star \mathbb{Z}_Z \simeq \mathbb{Z}_Y \oplus \mathbb{Z}_C[-2];$$

hence $H^m(Z) \simeq H^m(Y) \oplus H^{m-2}(C)$.   □

The following result proves that we can always obtain a building block (in the sense of Definition 5.1) by blowing up the base locus of a generic AC pencil (provided that this is smooth) on a semi-Fano 3–fold with torsion-free H^3; see also Remark 5.8 for comments on the torsion-free assumption. We use the same notation as in Definition 5.1.

Proposition 5.7   Let Y be a weak Fano 3–fold, C the smooth base locus of a generic pencil in $|-K_Y|$, Z the blow-up of Y at C, and $f\colon Z \to \mathbb{P}^1$ the fibration induced by the pencil.

(i)   The anticanonical class $-K_Z \in H^2(Z)$ is primitive.

(ii)   Some fibre $S = f^*(\infty)$ is a non-singular K3 surface and $S \sim -K_Z$.

(iii)   The restriction maps from $H^2(Y)$, $H^2(Z)$ and $H^2(V)$ to $H^2(S) = L$ have the same image N. If Y is semi-Fano then $K = 0$ (that is, $H^2(V) \to H^2(S)$ is injective) and the inclusion $N \hookrightarrow L$ is primitive.

(iv)   The group $H^3(Z)$ is torsion-free if and only if $H^3(Y)$ is.

Furthermore, $\pi_1(Z) = (0)$.

Proof (i) and (ii) follow from the well-known formula $-K_Z = \pi^*(-K_Y) - E$ and Theorem 4.7. (iv) follows from Lemma 5.6. (i) and (ii) are the only hypotheses used in the proof of Lemma 5.3(ii), which entails that $H^2(V)$ and $H^2(Z)$ have the same image in L. The image of the class in $H^2(Z)$ of the exceptional divisor is $[C] = -K_{Y|S} \in L$, so $H^2(Y)$ has the same image too.

To complete the proof of (iii) we need the following fact: if Y is a semi-Fano 3–fold and $-K_Y \sim S \subset Y$ is a non-singular K3 surface then $H^2(Y) \to H^2(S)$ is a primitive inclusion. This follows from Proposition 3.10.

It was proved in Lemma 5.2 that (i) and (ii) imply $\pi_1(Z) = (0)$. This can also be deduced from some standard facts about weak Fano 3–folds. Recall that an algebraic variety Y is *rationally connected* if given any two general points $y_1, y_2 \in Y$, there exists a morphism $f \colon \mathbb{P}^1 \to Y$ such that y_1 and y_2 are both in the image of f. Campana [10, Theorem 3.5] proved that rationally connected varieties are simply connected and Kollár–Miyaoka–Mori [54, Corollary 3.11] established that any smooth weak Fano 3–fold is rationally connected. Simple-connectivity of Z now follows from the fact that $\pi_1(Z) \cong \pi_1(Y)$ for the blow-up of Y in a smooth curve. $\square$

Remark 5.8

(i) The torsion subgroup $T_2 \subset H^3(Y)$ is a birational invariant of a non-singular projective variety Y of any dimension n. In particular, $T_2 = 0$ if Y is rational, that is, Y is birational to $\mathbb{P}^n$. Rationality of Fano 3–folds (including those with mild singularities) is somewhat subtle and still an area of current research activity.

(ii) It follows from the classification of non-singular Fano 3–folds that there is no torsion in $H^3(Y)$ for any non-singular Fano 3–fold; we are not aware of any conceptual proof of this fact.

(iii) There is a well-known example due to Artin–Mumford of a singular Fano 3–fold with torsion in H^3 [4]. The Artin–Mumford example is a nodal quartic double solid with 10 nodes. Torsion in nodal double solids has been studied more systematically by Endraß [22]. Nodal quartic double solids can have any number of nodes e between 1 and 16; Endraß showed that for nodal double quartics non-zero torsion T_2 can occur only when $e = 10$ [22, Theorem 3.6] (as in the Artin–Mumford example). Very recently, a nodal double sextic solid with 35 nodes and non-zero torsion was constructed (see Iliev–Katzarkov–Przyjalkowski [34, Proposition 3.1]). However, since these examples do not admit any projective small resolution they do not give rise to weak Fano 3–folds with torsion in H^3.

(iv) We can prove that $H^3(Y)$ is torsion-free for many semi-Fano 3–folds, in which case by Proposition 5.7 Y gives rise to a building block in the sense of Definition 5.1. In fact we do not currently know of any weak Fano 3–fold Y with torsion in $H^3(Y)$.

For a Z obtained by a sequential blow-up as in Proposition 4.25, we replace part (iii) of Proposition 5.7 by a more limited claim.

Proposition 5.9 *Let Y be a weak Fano 3–fold, C the base locus of a pencil in $|-K_Y|$ such that each irreducible component $C_1, \ldots, C_k \subseteq C$ is smooth, and Z the blow-up of Y at the C_i in sequence. Then Z satisfies the conclusions (i), (ii) and (iv) of Proposition 5.7, and $\pi_1(Z) = (0)$.*

$H^2(Z)$ and $H^2(V)$ have the same image N in L. Let K_0 and N_0 be the kernel and image of $H^2(Y) \to L$. Then $\mathrm{rk}\, N/N_0 + \mathrm{rk}\, K - \mathrm{rk}\, K_0 = k - 1$.

Proof Parts (i), (ii), and (iv) are proved by repeated application of the proof of Proposition 5.7. Like there, it then follows that $\pi_1(Z) = (0)$ and that $H^2(Z)$ and $H^2(V)$ have equal image. The final claim is simply an application of rank-nullity, using $b^2(V) = b^2(Z) - 1 = b^2(Y) + k - 1$. $\square$

Examples 7.8, 7.9 and 7.11 consider in detail building blocks obtained from nongeneric AC pencils on Fano or semi-Fano 3–folds. In the terminology of the previous proposition $N = N_0$ in Examples 7.8 and 7.11. There the anticanonical pencil is spanned by a generic K3 S and a sum of smooth divisors D_i intersecting S transversely: then the image of the exceptional divisor in $H^2(Z)$ over $C_i = D_i \cap S$ is $[C_i] \in H^2(S)$, which is already the image of $[D_i] \in H^2(Y)$. On the other hand, if *all* the anticanonical divisors in the pencil are non-generic then $H^2(Z)$ can have bigger image than $H^2(Y)$. If, like in Example 7.9, we take a pencil of anticanonical divisors containing a special curve C_1, then $[C_1] \in H^2(S)$ will be contained in the image of $H^2(Z)$, but not in the image of $H^2(Y)$.

5.3 Chern classes

As in the previous subsection let $f : Z \to \mathbb{P}^1$ be a building block in the sense of Definition 5.1, S be a smooth fibre of the morphism f and $V = Z \setminus S$. Let us first consider briefly the characteristic class $c_2(V) \in H^4(V)$. When $K = 0$ (which by Proposition 5.7 always holds when Z is a building block obtained from a generic AC pencil on a semi-Fano 3–fold with torsion-free H^3), Corollary 5.4(iv) implies

that $H^4(V) \cong H^4(S) \cong \mathbb{Z}$, and $c_2(V)$ is completely determined by the fact that the restriction of $c_2(V)$ to S is $c_2(S) \cong \chi(S) = 24$.

More generally, $c_2(V)$ is the restriction of $c_2(Z)$. By Lemma 5.3(iv) the restriction map $H^4(Z) \to H^4(V)$ has non-trivial kernel N^*, so $c_2(Z)$ contains strictly more information than $c_2(V)$. In the twisted connected sum construction of compact G_2–manifolds in [17], it turns out that in order to determine the characteristic class $p_1(M)$ for the resulting smooth 7–manifold one needs to understand $c_2(Z)$ of the building blocks. In order to apply classification results for smooth 2–connected 7–manifolds there we are mainly concerned with determining the greatest divisors of these classes, see [17, Corollary 4.32]. We begin by observing:

Lemma 5.10 $c_2(Z) \in H^4(Z)$ *is even for any building block* Z.

Proof Recall from Definition 5.1 following assumption (ii), that $c_1(Z)^2 = 0$ for any building block Z. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and the induced "mod 2" maps $H^i(Z; \mathbb{Z}) \to H^i(Z; \mathbb{Z}/2\mathbb{Z})$. To prove the lemma it suffices to prove that, for any complex 3–fold, $c_2(Z) \bmod 2 = c_1(Z)^2 \bmod 2$. The proof requires several facts about characteristic classes for which we refer to the book by Milnor and Stasheff [64].

Let $w_i(Z) \in H^i(Z; \mathbb{Z}/2\mathbb{Z})$ denote the Stiefel–Whitney classes of Z. According to [64, Theorem 11.14], the classes $w_i(Z)$ can be written in terms of Steenrod squares of the Wu classes $v_i(Z)$ in terms of the equation $w_k(Z) = \sum_{i+j=k} Sq^i(v_j(Z))$. As in [64, page 171], $w_{2i}(Z) = c_i(Z) \bmod 2$ and $w_{2i+1}(Z) = 0$. Using the basic properties of Steenrod squares, compare with [64, pages 90ff], it follows that $v_1(Z) = 0$ and $v_2(Z) = w_2(Z)$. Since $w_2(Z) = c_1(Z) \bmod 2$, the image of this class under the Bockstein operator $\delta : H^2(Z; \mathbb{Z}/2\mathbb{Z}) \to H^3(Z; \mathbb{Z})$ vanishes. On the other hand it is known, compare with Steenrod–Epstein [92, page 2], that Sq^1 is the Bockstein operator $H^2(Z; \mathbb{Z}/2\mathbb{Z}) \to H^3(Z; \mathbb{Z}/2\mathbb{Z})$ defined by the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Relating the above two sequences of coefficients via the obvious "mod" maps proves that $Sq^1(w_2(Z)) = \delta(w_2(Z)) \bmod 2 = 0$. It follows that $v_3(Z) = 0$. Also $v_4(Z) = 0$ because Wu classes in degree greater than half the dimension of the manifold vanish, compare with [64, page 132]. Hence $w_4(Z) = Sq^2(v_2(Z)) = w_2(Z)^2$. $\square$

The remainder of the section is devoted to providing tools to compute $c_2(Z)$ for the examples of building blocks in this paper.

Proposition 5.11 *Let Y be a compact complex 3–fold, $S \subset Y$ a smooth anticanonical divisor and let $\pi\colon Z \to Y$ be obtained by blowing up, in sequence, smooth curves $C_1, \ldots, C_k \subset S$, such that $-K_{Y|S} = [C_1] + \cdots + [C_k]$. Then*

$$(5\text{-}12) \quad c_2(Z) = \pi^*(c_2(Y) + c_1(Y)^2) + K_Y^3[\mathbb{P}_1^1] + \sum_{i=1}^{k-1} K_{Z_i}^3([\mathbb{P}_{i+1}^1] - [\mathbb{P}_i^1]) \in H^4(Z),$$

where $\mathbb{P}_i^1$ is a fibre in the exceptional set over C_i and Z_i are the intermediate blow-ups if $k > 1$.

Proof Since $c_1(Z)^2 = 0$, the result follows from the following claim by induction. Let Y be a compact complex 3–fold, S a non-singular anticanonical divisor, and $\pi\colon Z \to Y$ the blow-up of Y at a non-singular curve $C \subset S$. Then

$$(5\text{-}13) \quad c_2(Z) + c_1(Z)^2 = \pi^*(c_2(Y) + c_1(Y)^2) + (-K_Z^3 + K_Y^3)[\mathbb{P}^1] \in H^4(Z),$$

where $[\mathbb{P}^1]$ is a fibre in the exceptional set E over C.

For $d := c_2(Z) + c_1(Z)^2 - \pi^*(c_2(Y) + c_1(Y)^2)$ is 0 away from E, so it is Poincaré dual to some class in $H_2(E)$. To show that d is a multiple of $[\mathbb{P}^1]$ it suffices to show that $d \cdot (E + \widetilde{S}) = 0$, where $\widetilde{S}$ is the proper transform of S. The line bundle corresponding to $E + \widetilde{S}$ is $\pi^*(-K_Y)$.

Because $\pi\colon Z \to Y$ is a blow-up in a smooth curve, $H^k(Y, L) \cong H^k(Z, \pi^*L)$ for any line bundle L on Y and any k.[5] In particular $\chi(Z, \pi^*L) = \chi(Y, L)$. Applying this and the Riemann–Roch formula (4-5) when L is trivial gives that $c_1(Z)c_2(Z) = c_1(Y)c_2(Y)$. Applying it to $L = -K_Y$ gives

$$\pi^*(-K_Y)(c_2(Z) + c_1(Z)^2) = -K_Y(c_2(Y) + c_1(Y)^2)$$

because all the other terms in the Riemann–Roch formula agree. The right-hand side equals $\pi^*(-K_Y)\pi^*(c_2(Y) + c_1(Y)^2)$, so $d \cdot \pi^*(-K_Y) = 0$ as required.

To pin down the coefficient of $[\mathbb{P}^1]$ we just evaluate on $-K_Z$. $\qquad\square$

Remark Fulton [30, Theorem 15.4] gives a general formula for the difference between the Chern class of a blow-up and the pull-back of the Chern class of the base. In addition, Fulton's Example 15.4.3 helpfully distills the formula for blow-ups in a smooth codimension two variety. In our notation, it states

$$c_2(Z) - \pi^*c_2(Y) = \pi^*[C] - (\pi^*c_1(Y))[E]$$

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[5]The projection formula states $R\pi_\star(\pi^\star L) = L \otimes R\pi_\star \mathcal{O}_Z$; here $R\pi_\star \mathcal{O}_Z = \mathcal{O}_Y$; so, finally $H^k(Z, \pi^\star(L)) = H^k(Y, R\pi_\star(\pi^\star L)) = H^k(Y, L)$.

for a single step blow-up. We can use this to prove (5-13); however the ad hoc proof in terms of Riemann–Roch makes it easier to identify the terms we want.

Recall that the index of a weak Fano Y is $r = \text{div}(c_1(Y))$, the greatest divisor of $c_1(Y)$.

Corollary 5.14 *Let Z be a building block obtained by blowing up the smooth base locus of an AC pencil on a semi-Fano 3–fold Y with torsion-free H^3 as in Proposition 5.7. Then*

$$2 \mid \text{div}(c_2(Z)) \mid \gcd\left(\tfrac{1}{r}(24 + K_Y^3), 24\right),$$

with equality on the right if $k = 1$ and Y has Picard rank 1. In particular, if $r = 1$ and K_Y^3 is not divisible by 3 or 4 then $\text{div}(c_2(Z)) = 2$.

Proof $c_2(Z)$ is even for any building block Z according to Lemma 5.10.

$$\text{div}\left(c_2(Y) + c_1(Y)^2\right) \mid \tfrac{1}{r}\left(c_2(Y)c_1(Y) + c_1(Y)^3\right) = \frac{24 + K_Y^3}{r}$$

since $\tfrac{1}{r}c_1(Y)$ is integral. If Y has Picard rank 1 then Y is Fano so $H^3(Y)$ is torsion-free. It follows that $\tfrac{1}{r}c_1(Y)$ spans $H^2(Y)$ and $H^4(Y) \times H^2(Y) \to \mathbb{Z}$ is a perfect pairing, leading to equality. Thus, using Proposition 5.11,

$$\text{div}(c_2(Z)) \mid \gcd\left(c_2(Y) + c_1(Y)^2, K_Y^3\right) \mid \gcd\left(\tfrac{1}{r}(24 + K_Y^3), K_Y^3\right)$$

with equality on the left if $k = 1$ and equality on the right if Y has Picard rank 1. $\square$

If the Picard rank of the weak Fano Y is not 1 and the corollary does not force $\text{div}\, c_2(Z) = 2$, then we can compute it by applying the lemma below to non-singular divisors that together with $-K_Y$ form a basis for $H_4(Y)$. In the examples considered in Section 7 there are obvious choices of such divisors.

Lemma 5.15 *Let D be a non-singular divisor in a non-singular complex 3–fold Y. Then*

$$(5\text{-}16) \quad (c_2(Y) + c_1(Y)^2)_{|D} = \int_D (c_2(D) - c_1(D)^2) + D(-D - 2K_Y)(-K_Y).$$

Proof $TY_{|D} = TD + D_{|D}$ implies that $c_2(Y)_{|D} = c_2(D) + c_1(D)D_{|D}$ and $c_1(D) = (-D - K_Y)_{|D}$. $\square$

Remark 5.17 In examples where Y is a small resolution of a singular 3–fold X, taking a different small resolution Y^+ (which is therefore related to Y by a finite sequence of flops) leaves the quadratic form on the Picard lattice unchanged, and hence also the last term of (5-16). However, the divisors are transformed birationally, which changes the first term of the RHS of (5-16). More precisely, observe that if D^+ is the proper transform in Y^+ of D, then

$$c_1(D)^2 - c_1(D^+)^2 = D(D + K_Y)^2 - D^+(D^+ + K_{Y^+})^2 = D^3 - (D^+)^3.$$

Since $c_2(D) + c_1(D)^2$ is a birational invariant, it follows that $c_2(Y)_{|D} - c_2(Y^+)_{|D^+} = -2(D^3 - (D^+)^3)$. Hence it is easy to understand the change in div $c_2(Z)$ under a flop if we know the difference of the intersection numbers $D^3 - (D^+)^3$.

In the case when Z is obtained by blowing up a reducible base locus of an AC pencil, changing the order of the components also corresponds to a flop, and according to (5-12) div $c_2(Z)$ may depend on $K_{Z_i}^3$ of the intermediate blow-ups.

In the fundamental case when the block Z is constructed from a weak Fano by a single blow-up, viewing $c_2(Z)$ in a different basis can make it easier to determine the greatest divisor of $c_2(Z)$ modulo certain subgroups of $H^4(Z)$; this is useful in certain applications of [17, Corollary 4.32] to compute characteristic classes of twisted connected sum G_2–manifolds. (5-12) expresses $c_2(Z)$ in terms of the decomposition $H^4(Z) = \pi^* H^4(Y) \oplus \mathbb{Z}[\mathbb{P}^1]$, but we could also use the exactness of the sequence

$$0 = H^3(S) \longrightarrow H^4_{cpt}(V) \longrightarrow H^4(Z) \longrightarrow H^4(S) \longrightarrow H^5_{cpt}(V) = 0$$

to write $H^4(Z) = H^4_{cpt}(V) \oplus \mathbb{Z}[\mathbb{P}^1]$. Composing $\pi^*\colon H^4(Y) \to H^4(Z)$ with the projection $H^4(Z) \to H^4_{cpt}(V)$ gives an isomorphism $g\colon H^4(Y) \to H^4_{cpt}(V)$. Now (5-12) implies $c_2(Z) = g(c_2(Y) + c_1(Y)^2) + 24[\mathbb{P}^1]$. (This gives another way to phrase the proof of Corollary 5.14.)

There are natural maps $i_Z\colon L \cong H_2(S) \to H_2(Z) \cong H^4(Z)$ and $i_Y\colon L \to H^4(Y)$. Because S has trivial self-intersection, the image of i_Z is contained in $H^4_{cpt}(V)$, and $g \circ i_Y = i_Z$. Hence:

Lemma 5.18 *For any subgroup $N' \subset L$,*

$$\operatorname{div}(c_2(Z) \bmod i_Z(N')) = \gcd(c_2(Y) + c_1(Y)^2 \bmod i_Y(N'),\ 24).$$

In turn, $\operatorname{div}(c_2(Y) + c_1(Y)^2 \bmod i_Y(N'))$ can be computed by using Lemma 5.15 to evaluate $c_2(Y) + c_1(Y)^2$ on elements $D \in H^2(Y)$ such that $i_Y^*(D)$ is perpendicular to N'.

6 Anticanonical divisors in semi-Fano 3–folds

Almost any non-singular weak Fano 3–fold Y – recall both the standing Assumption preceding Proposition 4.24 and Remark 4.10(iv) – can be blown up as in Proposition 4.24 to obtain a projective 3–fold Z with an anticanonical K3 divisor S, such that Theorem 2.6 produces ACyl Calabi–Yau structures on $V = Z \setminus S$. The asymptotic limit is the product of the cylinder $\mathbb{R}^+ \times \mathbb{S}^1$ and S as in (2-3). S is equipped with a non-vanishing holomorphic 2–form Ω_S and a Kähler form ω_S, and we can regard the pair (Ω_S, ω_S) as a hyper-Kähler structure. We now wish to understand better *which* hyper-Kähler structures on K3 occur as the asymptotic limits of ACyl Calabi–Yau 3–folds constructed this way from a given family of weak Fanos. In Proposition 6.9 we show that when Y is a semi-Fano 3–fold then the subset of asymptotic limit hyper-Kähler structures on K3 is "large" (as characterised by the de Rham cohomology classes of the 2–forms) in the space of adapted hyper-Kähler structures, that is, those satisfying the a priori necessary polarisation condition described below.

In view of Theorem 2.6, we are interested in which complex 2–forms on K3 are holomorphic with respect to some smooth embedding of K3 as an anticanonical divisor in an element Y of a family of weak Fano 3–folds, and which real 2–forms on K3 are restrictions of Kähler forms on Y. We are therefore led to study the deformation theory of pairs (Y, S) where Y is a weak Fano 3–fold and S is a non-singular anticanonical divisor.

If Y is a weak Fano 3–fold then by Theorem 4.7 a generic $S \in |-K_Y|$ is a smooth K3 surface. If moreover Y is semi-Fano then by Lemma 6.4 below, the natural restriction homomorphism $\mathrm{Pic}\, Y \to \mathrm{Pic}\, S$ is a primitive embedding. This implies that the K3 surfaces appearing as anticanonical divisors in a given (deformation class of) semi-Fano 3–fold are very special; we see only those K3 surfaces S that contain a primitive sublattice $\mathrm{Pic}\, Y \subset \mathrm{Pic}\, S$. Such K3 surfaces are called *lattice polarised K3 surfaces* and the moduli theory of lattice polarised K3 surfaces is well understood; we recall it below.

In order to understand when a given lattice polarised K3 surface S appears as a smooth anticanonical divisor in a given deformation class of semi-Fano 3–folds, we also need to construct a sufficiently well-behaved moduli space (stack) parameterising pairs consisting of a deformation class of semi-Fano 3–folds Y and the choice of a smooth anticanonical section $S \in |-K_Y|$; see Definition 6.5. The semi-Fano assumption on Y is used in our proof that the appropriate moduli stack parameterising such pairs is a smooth stack; see Theorem 6.6. The smoothness proof we give relies on the fact that semi-Fano 3–folds satisfy slightly better vanishing theorems (Theorem 3.7) than the standard Kawamata–Viehweg vanishing (Theorem 3.5). (However, see also the

Remark following Theorem 6.6 for the general weak Fano case.) Most importantly of all, we need to understand the forgetful map $(Y, S) \mapsto S$ from such pairs (Y, S) to the moduli (stack) of lattice polarised K3 surfaces; see Theorem 6.8 for the statement of such a result.

Theorem 6.8 is a crucial ingredient in arguments in our paper [17, Section 6] that allows us in many cases to solve the so-called "matching problem" for a pair of hyper-Kähler K3 surfaces and therefore construct many compact 7–manifolds with holonomy group G_2 using the twisted connected sum construction.

We now describe the relevant moduli theory first for lattice polarised K3 surfaces, secondly for pairs of semi-Fano 3–folds and smooth anticanonical sections and then study the natural map between these two moduli spaces (stacks).

6.1 Lattice polarised K3 surfaces and the Torelli theorem

We recall some standard facts about moduli of lattice polarised K3 surfaces. Our purpose is to fix notation and recall just the facts that we need, not to give an introduction to moduli of K3. The constructions here are described in greater detail for example in Dolgachev [21, Sections 1 and 3].

We denote by $L = 2E_8(-1) \perp 3U$ an abstract copy of the K3 lattice. Fix a triple (N, A, j) of a lattice N of signature $(1, \rho)$ ($\rho = 0$ is allowed), an element $A \in N$ with $A^2 = 2g - 2 > 0$, and a primitive lattice embedding $j\colon N \hookrightarrow L$. (In general, there may be several inequivalent such embeddings.)

Write $\Delta = \{\delta \in N \mid \delta^2 = -2\}$. As we will see shortly, to specify the moduli space of N–polarised K3 we need to choose a partition $\Delta = \Delta^+ \sqcup \Delta^-$ satisfying the properties:

  (i) $\Delta^- = \{-\delta \mid \delta \in \Delta^+\}$,

  (ii) if $\delta_1, \ldots, \delta_k \in \Delta^+$ and $\delta = \sum \lambda_i \delta_i \in \Delta$ with all $\lambda_i \geq 0$ then also $\delta \in \Delta^+$, and

  (iii) $A \geq 0$ on Δ^+.

In what follows, we always (implicitly) assume that such a choice has been made.

Remark 6.1 In general, it is not easy to make explicit the choice of a partition of $\Delta = \Delta^+ \sqcup \Delta^-$ as just discussed. In almost all cases of interest to us, it will be possible to verify that for all $\delta \in \Delta$, $A \cdot \delta \neq 0$. When this is the case, property (iii) specifies that $\Delta^+ = \{\delta \in \Delta \mid A \cdot \delta > 0\}$.

Let $V^+ \subset N_{\mathbb{R}}$ be the connected component of the cone $V = \{\xi \mid \xi^2 > 0\} \subset N_{\mathbb{R}}$ containing A, and write

$$C^+ = \{\xi \in V^+ \mid \xi \cdot \delta > 0 \text{ for all } \delta \in \Delta^+\}.$$

Definition 6.2 The stack $\mathfrak{K}^{N,A}$ of (N, A, j)–polarised K3 surfaces (we often just say N–polarised K3 surfaces) is the category whose objects are: families $f \colon \mathcal{S} \to B$ of (non-singular) K3 surfaces, together with an isometry

$$N \hookrightarrow \underline{\mathrm{Pic}}(\mathcal{S}/B) \subset \mathbb{L} = R^2 f_\star \mathbb{Z}_{\mathcal{S}}$$

(where $\underline{\mathrm{Pic}}(\mathcal{S}/B)$ is the relative Picard group functor[6]) such that:

   (i)   For every $b \in B$, the embedding $N \subset \mathbb{L}_b$ is equivalent to $j \colon N \subset L$.

   (ii) $C^+ \cap \mathrm{Amp}(\mathcal{S}/B) \neq \varnothing$.

Morphisms in the category are Cartesian diagrams.

It is well known that $\mathfrak{K}^{N,A}$ is a smooth and connected Deligne–Mumford stack with quasi-projective coarse moduli space that we denote by $\mathcal{K}^{N,A}$. (Our only reason for working with stacks, and not with spaces, is that, because of smoothness, we can use certain infinitesimal arguments below. The reader who wishes to do so can pretend that the stack is in fact a smooth space, even though this is not, strictly speaking, true.)

Next we summarise the construction of the coarse moduli space from Hodge theory.

- Denote by D the Griffiths domain of oriented positive real 2–dimensional vector subspaces $\Pi \subset N_{\mathbb{R}}^{\perp} \subset L_{\mathbb{R}}$. Recall that giving Π is equivalent to giving a polarised Hodge structure on L:

$$L \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

where $\Pi \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$ is the complex structure on Π where multiplication by $\sqrt{-1}$ is achieved by the (positive) rotation by 90 degrees. Giving the real 2–plane $\Pi \subset L \otimes \mathbb{R}$ is equivalent to giving the complex line $H^{2,0} \in L \otimes \mathbb{C}$. The *period point* of a K3 surface S is the plane $\Pi(S)$ corresponding to $H^{2,0} = H^{2,0}(S)$.

---

[6]Ie, the sheaf on B for the faithfully flat topology associated to the presheaf $(B'/B) \mapsto \mathrm{Pic}(\mathcal{S} \times_B B')$. This sheaf is representable by a group scheme over B that we also denote by $\underline{\mathrm{Pic}}(\mathcal{S}/B)$.

- The stack $\mathfrak{M}^{N,A}$ of *marked* (N, A, j)–polarised K3 surfaces is the category whose objects are: objects $f\colon \mathcal{S} \to B$ of $\mathfrak{K}^{N,A}$, together with an isometry (marking)

$$h\colon R^2 f_\star \mathbb{Z}_\mathcal{S} \to L \otimes \mathbb{Z}_B$$

such that: (i) for every $b \in B$, the composition $N \subset \mathbb{L}_b = H^2(S_b; \mathbb{Z}) \xrightarrow{h_b} L$ is $j\colon N \subset L$, and (ii) $C^+ \cap \mathrm{Amp}(\mathcal{S}/B) \neq \varnothing$. Morphisms in the category are Cartesian diagrams.

  If $\delta \in N^\perp \subset L$ is a class with $\delta^2 = -2$, we denote by $H_\delta \subset D$ the hypersurface consisting of $\Pi \subset \delta^\perp$. Taking the union over all such δ, we write

(6-3)
$$D^0 = D \setminus \bigcup H_\delta \subset D.$$

  The *period map* is the morphism $\Pi\colon \mathfrak{M}^{N,A} \to D^0$ that maps $f\colon \mathcal{S} \to B$ to the polarised variation of Hodge structure on $R^2 f_\star \mathbb{Z}_\mathcal{S}$, that is, it maps the surface S to its period point $\Pi(S)$. A variant of the Torelli theorem for K3 surfaces (see Dolgachev [21, Corollary 3.2]) states that Π is an isomorphism.

- It follows from the Torelli theorem just stated that

$$\mathfrak{K}^{N,A} = [D^0/\Gamma]$$

  as stacks, and $\mathcal{K}^{N,A} = D^0/\Gamma$ as spaces, where $\Gamma \subset O(L)$ is a discrete group acting properly and discontinuously on D^0; see Dolgachev [20; 21] for details. For our purposes, we only need to know that Γ is commensurable with the set of isometries of L which restrict to the identity on N. We do not need the precise description of Γ.

6.2 Semi-Fano 3–folds and their K3 sections

We now come to the key purpose of this section, which is to extend some of the notions and results of Beauville [6] to the case of semi-Fano 3–folds.

Lemma 6.4 *Let Y be a non-singular semi-Fano 3–fold, and let $S \in |-K_Y|$ be a non-singular surface, necessarily a K3 surface. Then, the natural restriction homomorphism* $\mathrm{Pic}\, Y \to \mathrm{Pic}\, S$ *is a primitive embedding.*

Proof This was already shown in the proof of Proposition 5.7. □

Remark By Theorem 4.13 we know that the Picard rank $\rho < c$ for all weak Fano 3–folds, but in general we have no estimate of c. Lemma 6.4 implies that $\rho(Y) :=$

rk Pic $Y \le 20$ for any non-singular semi-Fano 3–fold Y; Example 7.7 gives a semi-Fano 3–fold that has Picard rank $\rho = 16$ and a nodal AC model.

For a general non-singular weak Fano 3–fold this upper bound of 20 on the Picard rank is false; according to Remark 8.9 there exists a toric weak Fano 3–fold with $\rho(Y) = 35$. This is the maximal Picard rank that occurs for toric weak Fano 3–folds and is currently the largest Picard rank known for any non-singular weak Fano 3–fold.

If Y is a semi-Fano 3–fold, we regard Pic $Y \cong H^2(Y, \mathbb{Z})$ as a lattice by means of the quadratic form $(D_1, D_2) \mapsto -D_1 \cdot D_2 \cdot K_Y$; this lattice has the distinguished element $A = -K_Y$ with $A^2 = 2g - 2$. (Note that A is *not* a Kähler class on Y when Y is semi-Fano but not Fano.)

Definition 6.5 Fix now a lattice N of signature $(1, \rho)$, with a distinguished element A with $A^2 = 2g - 2$. We also fix an embedding $j \colon N \subset L$ in the K3 lattice. The stack $\mathfrak{F}^{N,A}$ is the category whose objects are families $f \colon (\mathcal{S}, \mathcal{Y}) \to B$, such that

   (i)   for every geometric point $b \in B$, the fibre Y_b is a non-singular semi-Fano 3–fold, and the fibre $S_b \subset Y_b$ is a non-singular K3 surface in the linear system $|-K_{Y_b}|$,

together with an isometry $N \cong \underline{\mathrm{Pic}}(\mathcal{Y}/B)$ sending A to $-K_{\mathcal{Y}}$, such that

   (ii)   for every geometric point $b \in B$, the composition $N \to \mathrm{Pic}(Y_b) \to H^2(S_b; \mathbb{Z})$ is equivalent to j.

Theorem 6.6 *The stack $\mathfrak{F}^{N,A}$ is a smooth algebraic stack.*

Remark 6.7 The stack $\mathfrak{F}^{N,A}$ is often not connected: in examples, the connected components can often be understood in terms of flops relating different (partial) resolutions of singular Fano 3–folds.

Proof In the Fano case, Beauville [6] shows that $\mathfrak{F}^{N,A}$ is a smooth algebraic stack. The proof in [6] works word for word once we establish that $H^2(Y, T_Y) = (0)$, which implies that the stack $\mathfrak{F}^{N,A}$ is smooth. But $H^2(Y, T_Y) = H^2(Y, \Omega_Y^2 \otimes (-K_Y))$ is Serre dual to $H^1(Y, \Omega_Y^1 \otimes K_Y)$, and this group vanishes for any semi-Fano 3–fold (but not for a general weak Fano) thanks to Theorem 3.7. $\square$

Remark $H^2(Y, T_Y)$ does not always vanish for a weak Fano Y (see [88, Example 2.7]) and therefore Beauville's proof of smoothness of $\mathfrak{F}^{N,A}$ does not work in the general weak Fano setting. However, this does not necessarily mean that the stack

$\mathfrak{F}^{N,A}$ fails to be smooth. In fact, using Paoletti's result (Theorem 4.7) that a generic anticanonical member $S \in |-K_Y|$ is a non-singular K3 surface, it follows from work of Ran [83, Corollary 3], using the so-called T^1–lifting method, that $\mathfrak{F}^{N,A}$ is still smooth for any (smooth) weak Fano 3–fold Y. (Very recently, Sano [88, Theorem 1.1] considered the extension of this result to weak Fano n–folds for $n > 3$ in which case it is no longer true that a general $S \in |-K_Y|$ need be smooth.)

The only reason that we use moduli stacks rather than spaces here is so we can use infinitesimal arguments in the proof of Theorem 6.8 below: the stack is smooth even when the space is not. As we already noted, for any semi-Fano 3–fold Y the restriction homomorphism $\operatorname{Pic} Y \to \operatorname{Pic} S \subset H^2(S; \mathbb{Z})$ is a primitive embedding. Thus, we view S as an (N, A)–polarised K3 surface. As above, let $\mathfrak{K}^{N,A}$ be the stack of (N, A)–polarised K3 surfaces. There is an obvious forgetful morphism

$$s^{N,A} \colon \mathfrak{F}^{N,A} \longrightarrow \mathfrak{K}^{N,A}.$$

The following is the key result of this section and lies at the core of the matching argument in [17, Section 6].

Theorem 6.8 *The morphism $s^{N,A} \colon \mathfrak{F}^{N,A} \to \mathfrak{K}^{N,A}$ is smooth and generically surjective. More precisely, let $\mathfrak{F} \subset \mathfrak{F}^{N,A}$ be any connected component, and denote by $s \colon \mathfrak{F} \to \mathfrak{K}^{N,A}$ the restriction of $s^{N,A}$ to $\mathfrak{F}$. Then s is smooth and generically surjective.*

Proof Beauville's proof in [6] works word for word. □

Remark As already remarked above, Lemma 6.4 can definitely fail for general weak Fano 3–folds. Hence, in the general weak Fano case it is not a priori clear what moduli space (stack) of lattice polarised K3 surfaces $\mathfrak{K}$ should appear as the target of the forgetful morphism s above. An appropriate modification of Theorem 6.8 may still hold in the general weak Fano case; we will not consider this issue further in this paper.

In order to show that the set of hyper-Kähler structures that appear in the limits of our ACyl Calabi–Yau manifolds is large in the sense we want, it remains to find a sufficient condition for a class in $L_{\mathbb{R}}$ to correspond to a restriction of a Kähler class from Y.

Proposition 6.9 *As in the previous theorem, let $\mathfrak{F} \subset \mathfrak{F}^{N,A}$ be any connected component of the moduli stack $\mathfrak{F}^{N,A}$ of semi-Fano 3–folds, and $s \colon \mathfrak{F} \to \mathfrak{K}^{N,A}$ the forgetful morphism. Recall that, according to the discussion in the previous section, $\mathfrak{K}^{N,A} = [D^0 / \Gamma]$, where $D^0 \subset D$ is an open subset of the appropriate Griffiths domain of oriented positive planes $\Pi \subset N_{\mathbb{R}}^{\perp}$. Then there exist:*

(1)   a subset $U_{\mathfrak{F}} \subseteq D^0$ with complement a locally finite union of complex analytic submanifolds of positive codimension, and

(2)   an open subcone $\mathrm{Amp}_{\mathfrak{F}} \subset N_{\mathbb{R}}$,

with the following property: Fix any pair (Π, k) of $\Pi \in U_{\mathfrak{F}}$ and $k \in \mathrm{Amp}_{\mathfrak{F}}$; denote by (S, h) the marked (N, A, j)–polarized K3 surface with period point $\Pi(S, h) = \Pi$ (this means, in particular, that $h\colon H^2(S; \mathbb{Z}) \to L$ is an isometry); then there is an embedding $S \subset Y$ in a semi-Fano 3–fold Y, and a Kähler class $[\omega] \in (\mathrm{Pic}\, Y) \otimes \mathbb{R}$ such that $h([\omega_{|S}]) = j(k) \in L_{\mathbb{R}}$.

Proof   By the following Lemma 6.10, there is a Zariski open subset $\mathfrak{F}^0 \subset \mathfrak{F}$ (the only open stratum of the Zariski locally closed stratification in the statement of that lemma) such that the cone $\mathrm{Amp}\, Y_b \subset N_{\mathbb{R}}$ is constant for $b \in \mathfrak{F}^0$. Let $\mathrm{Amp}_{\mathfrak{F}}$ denote this constant cone: it is an open cone, because ampleness is an open property. By Theorem 6.8 the restriction of s to $\mathfrak{F}^0$ is generically surjective. Therefore the image $s(\mathfrak{F}^0)$ contains a Zariski open subset $W \subset s(\mathfrak{F}^0) \subset \mathfrak{K}^{N,A}$ and, denoting by $p\colon D^0 \to [D^0 / \Gamma] = \mathfrak{K}^{N,A}$ the natural projection, we take $U_{\mathfrak{F}} = p^{-1}(W)$. Here (1) holds because Γ is a discrete group and the action on D^0 is properly discontinuous. We claim that the open $U_{\mathfrak{F}}$ and the cone $\mathrm{Amp}_{\mathfrak{F}}$ just defined satisfy the conclusion.

Indeed, choose a pair (Π, k) with $\Pi \in U_{\mathfrak{F}}$ and $k \in \mathrm{Amp}_{\mathfrak{F}}$. By construction, $p(\Pi) \in W \subset s(\mathfrak{F}^0)$: that is, $S = p(\Pi)$ is part of a pair $(S \subset Y) \in \mathfrak{F}^0$ where Y is a semi-Fano 3–fold and $k \in \mathrm{Amp}_{\mathfrak{F}} = \mathrm{Amp}\, Y$, so tautologically k corresponds to a Kähler class $[\omega]$ on Y under the identification $N = \mathrm{Pic}(Y)$ that is part of the data of the moduli problem. The statement that $h([\omega_{|S}]) = j(k)$ is a tautology. $\square$

The proof of Lemma 6.10 used above rests on Paoletti's study [79] of how the Kähler cone of a weak Fano (quasi-Fano in his terminology) 3–fold changes under deformation. It is well known that the Kähler cone of a non-singular Fano n–fold is locally constant under deformation (see Wiśniewski [100]). Paoletti's main result [79, Theorem 1.1] is a characterisation of how the Kähler cone of a weak Fano 3–fold can fail to be locally constant under deformation. We don't actually need the precise formulation of his result; we only need to know that, in an algebraic family, the cone is constant on a Zariski open subset. Also note [79, Corollary 1.2] stating that the Kähler cone is constant in a family of weak Fano 3–folds whose anticanonical morphism is small and in particular for any semi-Fano obtained as a (projective) small resolution of nodal Fano 3–fold; this is the case for almost all of the examples we consider in detail in this paper.

Lemma 6.10 *Let $f\colon \mathcal{Y} \to B$ be a flat algebraic family of semi-Fano 3–folds together with an isometry $N \cong \underline{\mathrm{Pic}}(\mathcal{Y}/B)$. There is a Zariski locally closed stratification $\coprod B_i = B$ of B such that for all i the ample cone $\mathrm{Amp}\, Y_b \subset N_\mathbb{R}$ is constant in $b \in B_i$.*

Proof The result follows easily from [79, Theorem 1.1]. Indeed consider the flat family $\mathcal{X} \to B$ of anticanonical models of the family $\mathcal{Y}$. For all $b \in B$, let $E_b \subset Y_b$ be the exceptional set of the birational morphism $Y_b \to X_b$, with its reduced scheme structure. Let $\coprod B_i \to B$ be a Zariski locally closed stratification of B such that for all i:

$$E_i = \bigcup_{b \in B_i} E_b \longrightarrow B_i$$

is a flat family. Now [79, Theorem 1.1] immediately implies that $\mathrm{Amp}\, Y_b \subset N_\mathbb{R}$ is constant in $b \in B_i$. Indeed, if for some $b_0 \in B_i$ E_{b_0} contains a surface $F_b \subset Y_b$ contracting to a curve $C_b \subset X_b$, then, by flatness, so does every $b \in B_i$. $\square$

Remark It is important to understand that $\mathrm{Amp}_{\mathfrak{F}}$ is not the whole Kähler cone of S, even generically. If Y is semi-Fano but not Fano, and the anticanonical morphism $Y \to X$ is small, then $-K_Y$ is not a Kähler class on Y but it is when restricted to a generic S.

There is, however, an issue even in the strict Fano case when rank ≥ 2. For example consider a tridegree $(2, 2, 2)$ hypersurface S in $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathrm{Amp}\, Y_{|S}$ is a fundamental domain for the action of $\mathrm{Aut}\, S$ (a free group on the three involutions) on $\mathrm{Amp}\, S$: see Oguiso [78].

Remark Different components $\mathfrak{F} \subset \mathfrak{F}^{N,A}$ have different $\mathrm{Amp}_{\mathfrak{F}}$, for example, in Example 7.3 where we consider a generic quartic containing a plane the $\mathrm{Amp}_{\mathfrak{F}}$ of the two different small resolutions are the two components of $\mathrm{Amp}\, S \setminus \langle A \rangle$.

Example 6.11 The restriction $-K_{Y|S}$ is ample if and only if the anticanonical morphism $Y \to X$ is small. The following examples further illustrate the statements of Theorem 6.8 and Proposition 6.9:

(i) $Y = \mathbb{F}_2 \times \mathbb{P}^1$, where $\mathbb{F}_2$ is the Segre surface. The anticanonical morphism contracts the surface $E = e \times \mathbb{P}^1$ where $e \subset \mathbb{F}_2$ is the curve of self-intersection $e^2 = -2$. In particular, if $S \in |-K_Y|$ is non-singular then $-K_{Y|S}$ is not ample and it always contracts *two* curves of self-intersection -2. Consider the basis of $N = \mathrm{Pic}\, Y$ consisting of $D = \mathbb{F}_2 \times \{\mathrm{pt}\}$, E as above, and $F = f \times \mathbb{P}^1$ where

$f \subset \mathbb{F}_2$ is the class of a fibre. $-K_Y = 2D + 2E + 4F$, and the matrix of the intersection form is:

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -4 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Note that there are no -2 classes in N. In fact, Pic S always has rank 4: as Y deforms to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the surface $E \subset Y$ "evaporates," and S deforms to a rank 3 K3 surface.

(ii) Let X be a general quartic 3–fold containing a double line $\ell \subset \mathbb{P}^4$. It is easy to check that the proper transform Y of X in the blowing up of $\ell \subset \mathbb{P}^4$ is non-singular and the exceptional divisor $E \subset Y$ mapping to ℓ is a conic bundle surface with 6 singular fibres. In this case Y has rank 2 and $E^2 \cdot A = -2$, thus $E \in N$ is a -2 class. Moreover, it is clear that E survives all deformations of Y. (See Example 7.12 below.)

Remark Example 6.11(i) illustrates that the property of being Fano is unstable under deformation; see also Paoletti [79, Example 1.3] for another such example. Also a weak Fano 3–fold with small AC morphism may be deformation equivalent to a weak Fano 3–fold with AC morphism which is not small; see [79, Example 1.6].

7 Examples: building blocks

We construct a handful of building blocks and compute the topological invariants considered in Section 5 of these building blocks. The results are summarized in Tables 7.1 and 7.2. These building blocks and their topological invariants will be used in [17] to construct examples of compact G_2–manifolds and to determine their topology. In this section we make no attempt at being systematic: we only construct a very small number of typical examples.

In Section 8 we discuss many more general classes of semi-Fano 3–folds to illustrate the variety of examples available. Applying Proposition 5.7 to any of these examples yields a building block (Z, f, S) in the sense of Definition 5.1 and therefore by Theorem 2.6 an ACyl Calabi–Yau structure on the quasiprojective 3–fold $Z \setminus S$. Similar methods to those utilised in the present section would also allow the computation of the basic topological invariants of the corresponding building blocks and ACyl Calabi–Yau 3–folds.

To construct the examples in this section typically we start with a singular Fano 3–fold X with only ordinary double points; resolve this to a non-singular semi-Fano Y and

anticanonical morphism $Y \to X$; choose a K3 surface S and pencil in $|-K_Y|$, and resolve the indeterminacies to obtain $S \subset Z \to \mathbb{P}^1$ where S is the fibre of $\infty \in \mathbb{P}^1$. According to Proposition 5.7, Z is a "building block" in the sense of Definition 5.1, and therefore by Theorem 2.6, $V = Z \setminus S$ admits ACyl Calabi–Yau metrics.

We compute the following topological invariants of the building blocks: the degree $-K_Y^3$, the integral cohomology groups $H^2(Z)$ and $H^3(Z)$, the primitive sublattice $N \subset L$, the kernel K of $H^2(V) \to L$, the greatest divisor of $c_2(Z)$, and the number $e(Z)$ of $(-1,-1)$–curves. In examples involving small resolutions of 3–folds with ordinary double points, all these invariants are independent of the choice of small resolution, except possibly the greatest divisor of $c_2(Z)$ (recall Remark 5.17).

In the calculation of $H^m(Z)$ we use Lemma 5.6; to compute $b^3(Y)$ we use Lemma 4.21; to compute $c_2(Z)$ we use Proposition 5.11. In all cases, except in Example 7.9 which requires some extra work, it is immediate from Proposition 5.7 that the sublattice $N \subset L$ is primitive.

Example 7.1 A class of examples, already considered in Kovalev [57], is to take Y to be a Fano "of the first species", that is, a member of one of the 17 deformation families of smooth Fano 3–folds with Picard rank 1, and let Z be the building block arising from blowing up the (smooth) base locus of a generic transverse anticanonical pencil. Let r be the index of Y, and $d = (-\frac{1}{r}K_Y)^3$ the degree. Then by definition $-K_Y = rH$ for H a generator for $\operatorname{Pic} Y$, and $(-K_Y)H^2 = rd$. So the polarising lattice is $N = \langle rd \rangle$.

For these examples, Corollary 5.14 easily gives the greatest divisor of $c_2(Z)$. Consulting Iskovskih [37, Table 6.5] and Iskovskih–Prokhorov's book [38, Table 12.2] we summarise the values of $b^3(Z)$ and greatest divisor of $c_2(Z)$ in Table 7.1.

Example 7.2 We can also readily construct building blocks from the 36 rank 2 Fanos in the Mori–Mukai classification, but do not describe the examples in further detail here.

Examples 7.3 through 7.6 arise in a uniform way. We impose the condition that a quartic in $\mathbb{P}^4$ contain a special surface W: a projective plane Π, a quadric surface Q_2^2, a cubic scroll surface $\mathbb{F}$ and the complete intersection of two quadrics $F_{2,2}$ respectively. The generic such quartic X has only ODPs, the number e of which is determined by the special surface W imposed and all of which are contained in W. W gives us a Weil divisor on X which is not $\mathbb{Q}$–Cartier; blowing up $W \subset X$ as in Lemma 3.14 yields a smooth projective small resolution Y with anticanonical morphism $\varphi : Y \to X$. Y is a smooth semi-Fano 3–fold with Picard rank $\rho = 2$ (so the defect $\sigma(X)$ is 1) whose

Y	r	$-K_Y^3$	$b^3(Y)$	$b^3(Z)$	$\operatorname{div} c_2(Z)$
$\mathbb{P}^3$	4	4^3	0	66	2
$Q_2 \subset \mathbb{P}^4$	3	$3^3 \cdot 2$	0	56	2
$V_1 \to W_4$	2	2^3	42	52	8
$V_2 \to \mathbb{P}^3$	2	$2^3 \cdot 2$	20	38	4
$Q_3 \subset \mathbb{P}^4$	2	$2^3 \cdot 3$	10	36	24
$V_{2 \cdot 2} \subset \mathbb{P}^5$	2	$2^3 \cdot 4$	4	38	4
$V_5 \subset \mathbb{P}^6$	2	$2^3 \cdot 5$	0	42	8
$V_2 \to \mathbb{P}^3$	1	2	104	108	2
$Q_4 \subset \mathbb{P}^4$	1	4	60	66	4
$V_{2 \cdot 3} \subset \mathbb{P}^5$	1	6	40	48	6
$V_{2 \cdot 2 \cdot 2} \subset \mathbb{P}^6$	1	8	28	38	8
$V_{10} \subset \mathbb{P}^7$	1	10	20	32	2
$V_{12} \subset \mathbb{P}^8$	1	12	14	28	12
$V_{14} \subset \mathbb{P}^9$	1	14	10	26	2
$V_{16} \subset \mathbb{P}^{10}$	1	16	6	24	8
$V_{18} \subset \mathbb{P}^{11}$	1	18	4	24	6
$V_{22} \subset \mathbb{P}^{13}$	1	22	0	24	2

Table 7.1: Building blocks Z from Fanos Y with Picard rank 1

integral cohomology group $H^2(Y)$ is spanned by the anticanonical class $A = -K_Y$ and $\widetilde{W}$, the proper transform of the special surface $W \subset X$. Since the anticanonical morphism $\varphi \colon Y \to X$ is small by Theorem 3.30 we can flop φ to obtain another smooth weak Fano Y^+ with $\rho(Y^+) = 2$; by Remark 3.31 there is a unique such flop of φ. In general the flop Y^+ is not isomorphic to Y but shares the same topological invariants except possibly for $c_2(Y)$ which we compute.

Example 7.3 (Generic AC pencil on a small resolution of a generic quartic containing a plane) The following semi-Fano also appears in work of Cheltsov [13, Lemma 25], Jahnke–Peternell–Radloff [41, 3.15] and Takeuchi [93, 2.9.6 and 6.6.6]. Fix a 2–plane $\Pi \subset \mathbb{P}^4$ and let $\Pi \subset X \subset \mathbb{P}^4$ be a general quartic 3–fold containing Π. Choose homogeneous coordinates $x_0, \dots, x_4$ on $\mathbb{P}^4$ such that $\Pi = (x_0 = x_1 = 0)$; then $X = (f_4 = 0)$ is the zero locus of a homogeneous quartic in the ideal (x_0, x_1):

$$f_4 = x_0 a_3 + x_1 b_3,$$

where a_3, b_3 are degree 3 homogeneous in $x_0, \ldots, x_4$. To say that X is general is to say that the forms a_3, b_3 are general; thus, X has 9 ordinary double points $x_0 = x_1 = a_3 = b_3 = 0$ – see also the remark at the end of this example. Blowing up Π yields a non-singular semi-Fano 3–fold $Y \to X$ with $e = 9$ $(-1, -1)$–curves resolving the 9 ordinary double points of X on Π. We show that:

- $H^2(Y) = \mathbb{Z}^2$ with basis $\tilde{\Pi}$ (the proper transform of Π) and $A = -K_Y$, and quadratic form in this basis

$$\begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix};$$

- $H^3(Y) \simeq \mathbb{Z}^{44}$.

Below we discuss how to compute $H^2(Y)$ and $H^3(Y)$. The building block $f \colon Z \to \mathbb{P}^1$ is obtained by blowing up the base locus of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. The base locus is a non-singular curve C of genus 3 (naturally a plane quartic); hence, $H^3(Z) \simeq H^3(Y) \oplus H^1(C) \simeq \mathbb{Z}^{44} \oplus \mathbb{Z}^6$.

To calculate $H^2(Y)$ and $H^3(Y)$, we proceed as follows. First, Y is the proper transform of X in the blowup $G \to \mathbb{P}^4$ of the plane Π; this is the scroll with weight data

s_0	s_1	x_2	x_3	x_4	x
1	1	0	0	0	-1
0	0	1	1	1	1

with morphism to $\mathbb{P}^4$ given by $x_0 = s_0 x, x_1 = s_1 x$. Denoting by L the line bundle on G with sections s_0, s_1 and M the line bundle with sections $x_2, \ldots$, we see that the equation of $Y \subset G$ is:

$$s_0 a_3 + s_1 b_3,$$

that is $Y \in |L + 3M|$. Thus $Y \subset G$ is an ample divisor; it then follows easily from the Lefschetz theorems that the restriction $H^2(G) \to H^2(Y)$ is an isomorphism and $H^3(Y)$ is torsion-free. To see this consider the long exact cohomology sequence of the pair G, Y:

$$\cdots \longrightarrow H^m(G, Y) \longrightarrow H^m(G) \longrightarrow H^m(Y) \longrightarrow H^{m+1}(G, Y) \longrightarrow \cdots,$$

note that $H^m(G, Y) = H^m_{cpt}(G \setminus Y) = H_{8-m}(G \setminus Y)$, and recall that the Lefschetz homotopy dimension theorem states that $G \setminus Y$ has the homotopy type of a CW complex of real dimension 4. It follows that $H^m(G, Y) = (0)$ for $m < 4$ and that $H^4(G, Y)$ is torsion-free.

We calculate $b^3(Y)$ by applying Lemma 4.21. In the present case we have $e = 9$, $\sigma = b^2(Y) - b^2(X) = 2 - 1 = 1$ and $b = 60$ since the third Betti number of a smooth quartic in $\mathbb{P}^4$ is 60. Hence by (4-22) we have

$$b^3(Y) = 60 - 2 \times 9 + 2 \times 1 = 44.$$

It remains to compute $c_2(Z)$ for both Y and its (unique) flop Y^+; for this we first need to compute $c_2(Y)$ and then apply the blow-up formula Proposition 5.11 to compute $c_2(Z)$. $\tilde{\Pi}$ is the blow-up of Π at 9 points, so $c_2(\tilde{\Pi}) = 12$ and $c_1(\tilde{\Pi})^2 = 0$. The second term in (5-16) we can compute from the quadratic form on $H^2(Y)$:

$$(-\tilde{\Pi}^2 + 2\tilde{\Pi}(-K_Y))(-K_Y) = -(-2) + 2 = 4.$$

Hence Lemma 5.15 gives $(c_2(Y) + c_1(Y)^2)_{|\tilde{\Pi}} = 16$. Since $\chi(C) = -4$, it follows from Proposition 5.11 that $\operatorname{div} c_2(Z) = 4$. If we flop Y to the other small resolution Y^+ of X, then the proper transform of Π is isomorphic to Π, $(c_2(Y^+) + c_1(Y^+)^2)_{|\Pi} = -2$ and $\operatorname{div} c_2(Z^+) = 2$.

Remark We have the following elementary lemma, for example, see Finkelnberg [24, Proposition 1.1]: if Π is a plane contained in a hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 2$ then X is singular and Π contains at least one singular point of X. If X contains only finitely many singular points then it contains at most $(d-1)^2$ singular points. If X has only nodes there are exactly $(d-1)^2$ singular points on X.

Example 7.4 (Small resolution of a generic quartic containing a quadric surface) See also Cheltsov [13, Example 10]. Fix a quadric surface $Q = Q_2^2 \subset \mathbb{P}^4$ and let $Q \subset X \subset \mathbb{P}^4$ be a general quartic 3–fold containing Q. Choose homogeneous coordinates $x_0, \ldots, x_4$ on $\mathbb{P}^4$ such that $Q = (x_0 = x_1 x_2 + x_3 x_4 = 0)$; then $X = (f_4 = 0)$ is the zero locus of a homogeneous quartic in the ideal of Q:

$$f_4 = x_0 a_3 + (x_1 x_2 + x_3 x_4) b_2,$$

where a_3, b_2 are general homogeneous forms of degrees 3, 2 in $x_0, \ldots, x_4$. Thus, X has 12 ordinary double points $x_0 = x_1 x_2 + x_3 x_4 = a_3 = b_2 = 0$. Blowing up Q yields a non-singular semi-Fano 3–fold $Y \to X$ with $e = 12$ $(-1, -1)$–curves resolving the 12 ordinary double points of X on Q. We show that:

- $H^2(Y) = \mathbb{Z}^2$ with basis $\tilde{Q}, A$, and quadratic form in this basis

$$\begin{pmatrix} -2 & 2 \\ 2 & 4 \end{pmatrix};$$

- $H^3(Y) \simeq \mathbb{Z}^{38}$.

The building block $f\colon Z \to \mathbb{P}^1$ is obtained by blowing up the base locus of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. The base locus is a non-singular curve C of genus 3 (naturally a plane quartic); hence, $H^3(Z) \simeq H^3(Y) \oplus H^1(C) \simeq \mathbb{Z}^{44}$.

To calculate $H^2(Y)$ and $H^3(Y)$ we proceed as follows. First, Y is the proper transform of X in the blowup $G \to \mathbb{P}^4$ of the quadric $Q = (x_0 = x_1 x_2 + x_3 x_4 = 0)$. We realize G as the hypersurface with equation

$$s x_0 + t(x_1 x_2 + x_3 x_4) = 0$$

in the 5–dimensional toric scroll with weight data:

x_0	x_1	x_2	x_3	x_4	s	t
1	1	1	1	1	0	−1
0	0	0	0	0	1	1

We denote by L, respectively M, the line bundles on this scroll with global sections x_i, respectively $s, t x_i$. Thus, Y is given in G by the two simultaneous equations:

$$\begin{cases} s x_0 + t(x_1 x_2 + x_3 x_4) = 0, \\ s b_2 - t a_3 = 0. \end{cases}$$

Hence, $Y \subset G$ is the complete intersection of two ample hypersurfaces of type $L + M$ and $2L + M$; as in the previous example, it follows from Lefschetz that the restriction $H^2(G) \to H^2(Y)$ is an isomorphism and $H^3(Y)$ is torsion-free. We compute $b^3(Y)$ as in the previous example using Lemma 4.21. Since in this case we have $e = 12$, $\sigma = 1$ and $b = 60$, (4-22) yields $b^3(Y) = 60 - 24 + 2 = 38$.

$c_2(\tilde{Q}) - c_1(\tilde{Q})^2 = 20$, and Lemma 5.15 gives $(c_2(Y) + c_1(Y)^2)|_{\tilde{Q}} = 26$. Since $\chi(C) = -4$, Proposition 5.11 implies $\operatorname{div} c_2(Z) = 2$. Flopping does not change any invariants of Y, since it corresponds to blowing up a different quadric surface $(x_0 = b_2 = 0)$ contained in Y.

Example 7.5 (See also Entry 30 in Kaloghiros [45, Table 1]) Fix a cubic scroll surface $\mathbb{F} \subset \mathbb{P}^4$ and let $\mathbb{F} \subset X \subset \mathbb{P}^4$ be a general quartic 3–fold containing $\mathbb{F}$. One can choose homogeneous coordinates $x_0, \ldots, x_4$ on $\mathbb{P}^4$ such that $\mathbb{F}$ is the locus where the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

has rank < 2; then $X = (f_4 = 0)$ is the zero locus of a homogeneous quartic in the ideal of the 2×2 minors of M:

$$f_4 = \begin{pmatrix} x_1 x_4 - x_2 x_3 & -x_0 x_4 + x_2^2 & x_0 x_3 - x_1 x_2 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

where a_2, b_2, c_2 are general homogeneous forms of degrees 2 in $x_0, \ldots, x_4$. A straightforward calculation with the Porteous formula (see Arbarello–Cornalba–Griffiths–Harris [3, Chapter II, (4.2)]) shows that X has 17 ordinary double points; the singularities of $X \subset \mathbb{P}^4$ are the locus in $\mathbb{P}^4$ where the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

has rank 1. Blowing up $\mathbb{F}$ yields a non-singular semi-Fano 3–fold $Y \to X$ with $e = 17$ $(-1, -1)$–curves resolving the 17 ordinary double points of X on $\mathbb{F}$. We show that:

- $H^2(Y) = \mathbb{Z}^2$ with basis $\widetilde{\mathbb{F}}, A$, and quadratic form in this basis

$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix};$$

- $H^3(Y) \simeq \mathbb{Z}^{28}$.

The building block $f \colon Z \to \mathbb{P}^1$ is obtained by blowing up the base locus of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. The base locus is a non-singular curve C of genus 3 (naturally a plane quartic); hence, $H^3(Z) \simeq H^3(Y) \oplus H^1(C) \simeq \mathbb{Z}^{28} \oplus \mathbb{Z}^6$.

To calculate $H^2(Y)$ and $H^3(Y)$, the strategy, as usual, is to show that Y is a complete intersection of ample hypersurfaces in a non-singular toric variety. Indeed, the blow up G of $\mathbb{F} \subset \mathbb{P}^4$ is the complete intersection given by equations:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = 0 \quad \text{in } \mathbb{P}^4 \times \mathbb{P}^2_{y_0, y_1, y_2}$$

and Y is given in G by the equation:

$$a_2 y_0 + b_2 y_1 + c_2 y_2 = 0.$$

Thus, Y is the complete intersection of 3 ample hypersurfaces in $\mathbb{P}^4 \times \mathbb{P}^2$. Everything else from now on proceeds as in the previous examples. Since $e = 17$, $\sigma = 1$ and $b = 60$, (4-22) yields $b^3(Y) = 60 - 34 + 2 = 28$.

$\mathbb{F}$ is $\mathbb{P}^2$ blown up in 1 point, and $\widetilde{\mathbb{F}}$ is $\mathbb{F}$ blown up in 17 points. Lemma 5.15 gives $(c_2(Y) + c_1(Y)^2)_{|\widetilde{\mathbb{F}}} = 38$. Hence $\mathrm{div}\, c_2(Z) = 2$ by Proposition 5.11. In the other small resolution Y^+ of X, the proper transform of $\mathbb{F}$ is isomorphic to $\mathbb{F}$. There $(c_2(Y^+) + c_1(Y^+)^2)_{|\mathbb{F}} = 4$ and $\mathrm{div}\, c_2(Z^+) = 4$.

Example 7.6 (See also Cheltsov [13, Theorem 11, Lemma 21], Jahnke–Peternell –Radloff [41, 3.9.II.6.a], Kaloghiros [47, Example 3.9], and Takeuchi [93, 2.11.10].) Fix the complete intersection of two quadrics $F = F_{2,2} \subset \mathbb{P}^4$ (that is, a del Pezzo surface of degree 4) and let $F \subset X \subset \mathbb{P}^4$ be a general quartic 3–fold containing F. In homogeneous coordinates $x_0, \ldots, x_4$ on $\mathbb{P}^4$, $F = (p_2 = q_2 = 0)$ where p_2, q_2 are general homogeneous quadratic polynomials; then $X = (f_4 = 0)$ is the zero locus of a homogeneous quartic in the ideal of F:

$$f_4 = p_2 a_2 + q_2 b_2,$$

where a_2, b_2 are general homogeneous quadratic forms in $x_0, \ldots, x_4$. Thus, X has 16 ordinary double points $p_2 = q_2 = a_2 = b_2 = 0$. Blowing up F yields a non-singular semi-Fano 3–fold $Y \to X$ with $e = 16$ $(-1, -1)$–curves resolving the 16 ordinary double points of X on F. By using the methods described in the previous examples, it is easy enough to show that:

- $H^2(Y) = \mathbb{Z}^2$ with basis $\widetilde{F}, A$, and quadratic form in this basis

$$\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix};$$

- $H^3(Y) \simeq \mathbb{Z}^{30}$.

The building block $f \colon Z \to \mathbb{P}^1$ is obtained by blowing up the base locus of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. The base locus is a non-singular curve C of genus 3 (naturally a plane quartic); hence, $H^3(Z) \simeq H^3(Y) \oplus H^1(C) \simeq \mathbb{Z}^{30} \oplus \mathbb{Z}^6$.

Proposition 5.11 implies $\mathrm{div}\, c_2(Z) = 4$, using $(c_2(Y) + c_1(Y)^2)_{|\widetilde{F}} = 44$. Flopping does not change any invariants of Y, since it corresponds to blowing up another complete intersection of quadrics ($p_2 = q_2 = 0$) contained in X.

For the next example we again consider a weak Fano 3–fold Y whose AC model is a nodal quartic $X \subset \mathbb{P}^4$ but in this case with the maximal number of possible nodes (which is 45); such an X is unique up to projective equivalence. It is a classical fact that X admits projective small resolutions Y. Unlike the previous four examples in which $\rho(Y) = 2$ in this case we will show that $\rho(Y) = 16$ and hence X has defect

$\sigma = 15$, that is, X contains many Weil divisors that are not $\mathbb{Q}$–Cartier; by a result of Kaloghiros [46], 15 is also the maximal possible defect for any quartic in $\mathbb{P}^4$ with only terminal singularities. Because of the high Picard rank of Y the computation of the lattice structure on $H^2(Y)$ is considerably more involved in this case. As far as we know the number of distinct projective small resolutions of Y has not been computed in this case.

Example 7.7 The Burkhardt quartic 3–fold is the hypersurface

$$X = \left(x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\right) \subset \mathbb{P}^4.$$

It is well known (and one can verify by inspection), that: X contains 40 planes, has 45 ordinary nodes as singularities, defect $\sigma = 15$, and several projective small resolutions. (See Finkelnberg's thesis [25] for these and other facts on the Burkhardt quartic.) Below we take one such projective small resolution $Y \to X$, and make a building block $f: Z \to \mathbb{P}^1$ by blowing up the (non-singular) base curve of a general pencil $|S_0, S_\infty| \subset |-K_Y|$. In what follows, we establish the following facts about X, Y, and Z:

- Write $N = H_4(X)$ with the integral quadratic form

$$D_1, D_2 \mapsto q(D_1, D_2) = (-K_X) \cdot D_1 \cdot D_2.$$

  Then: N is a hyperbolic lattice of rank 16; N is 3–elementary; more precisely, the discriminant of N is $(\mathbb{Z}/3\mathbb{Z})^5$ (thus $\ell = 5$); and, finally:

$$N \cong E_6^*(-3) \perp E_8(-1) \perp U .$$

  (Here U is the rank 2 hyperbolic lattice, while E_6^* is the dual lattice of the lattice E_6. In other words, if B is the intersection matrix for E_6, then B^{-1} is the intersection matrix for E_6^*. In particular B^{-1} is not an integer matrix: it has entries in $\frac{1}{3}\mathbb{Z}$; however, $E_6^*(-3)$ is an integral lattice. Since E_6 has rank 6 and discriminant $\mathbb{Z}/3\mathbb{Z}$, it immediately follows that $E_6^*(-3)$ has discriminant $(\mathbb{Z}/3\mathbb{Z})^5$, which of course can also be checked by direct computation.)

- The embedding $N \subset L$ in the K3 lattice L is unique, and $N^\perp = T = A_2(-1) \perp 2U(3)$, where $A_2(-1)$ and $U(3)$ denote the rank 2 lattices with intersection forms

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

- All projective small resolutions $Y \to X$ have 45 $(-1,-1)$–curves, $H^2(Y) \cong N$, and $H^3(Y) = (0)$.

- Let $f: Z \to Y$ be the blow up of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. Then Z is a building block with $H^2(Z) = \mathbb{Z}^{17}$, $H^3(Z) = \mathbb{Z}^6$.

We now prove all of these claims. Todd [97] gives an explicit birational map $\mathbb{P}^3 \dashrightarrow X$. Resolving this map by explicit blow ups, Finkelnberg [25] constructs a small resolution $Y \to X$ and a basis of $H_4(Y)$ consisting of planes. His notation for this basis is:

$$V, \ E_1^k, \ E_2^k, \ E_3^k, \ E_1^l, \ E_2^l, \ E_3^l, \ E_1^m, \ E_2^m, \ E_3^m, \ F_1^1, \ F_2^1, \ F_1^2, \ F_2^2, \ F_1^3, \ F_2^3.$$

Finkelnberg also makes a list of the curves contracted by $Y \to X$; using this information, it is not difficult (though tedious) to write down the matrix of the intersection form on $H_4(Y)$ in this basis:

$$\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & -2 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & -2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & -2 & 1 & 0 & 0 \\
 & & & & & & & & & & & & & -2 & 0 & 0 \\
 & & & & & & & & & & & & & & -2 & 1 \\
 & & & & & & & & & & & & & & & -2
\end{pmatrix}$$

From this it is easy to compute (for example, by computer algebra) that the discriminant $A \cong (\mathbb{Z}/3\mathbb{Z})^5$. Recall that, for p prime, a lattice is said to be p–elementary if the discriminant is of the form $(\mathbb{Z}/p\mathbb{Z})^\ell$: we have just shown that N is 3–elementary with $\ell = 5$. Rudakov and Shafarevich [87, Section 1, Theorem] prove that an even, hyperbolic (meaning it has signature $(1, r-1)$), p–elementary for $p \neq 2$ prime, lattice of rank ≥ 3 is uniquely determined by its discriminant (that is, equivalently, the invariant ℓ). This implies that

$$N \cong E_6^*(-3) \perp E_8(-1) \perp U.$$

The proof of [87, Section 1, Theorem] goes through, with the appropriate small modifications, for lattices of any indefinite signature. This implies that the transcendental lattice

$$T = N^\perp \cong A_2(-1) \perp 2U(3),$$

as this is the unique lattice with signature $(2, 4)$ and discriminant $(\mathbb{Z}/3\mathbb{Z})^5$. By Dolgachev [20, Theorem 1.4.8], the fact that $\mathrm{rk}\, T + \ell(T) + 2 \leq \mathrm{rk}\, L$ implies that the primitive embedding $T \subset L$ is unique up to automorphisms, and so the same is true for the embedding $N \subset L$ (note that $\mathrm{rk}\, N + \ell(N) + 2 > \mathrm{rk}\, L$, so [20, Theorem 1.4.8] does not apply directly to N).

All other assertions are straightforward.

We can compute $\mathrm{div}\, c_2(Z)$ by evaluating it on the basis of planes. If the proper transform in Y of some plane remains a plane (that is, no points are blown up on the plane) then that forces $\mathrm{div}\, c_2(Z) = 2$ like in Example 7.3. This is easy to arrange. For instance choose a plane $\Pi \subset X$ and let $Y' = \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathcal{O}_X(n\Pi)\right) \to X$. Then the proper transform $\Pi' \subset Y$ is relatively ample and isomorphic to Π and therefore Y' is non-singular in a neighbourhood of Π'. Let next $Y \to Y'$ be a small resolution of Y': this does not alter Y' in a neighbourhood of Π', hence the proper transform of Π in Y is still isomorphic to $\mathbb{P}^2$.

The next pair of examples consider building blocks slightly different from those above. They are obtained by blowing up the base locus of a *nongeneric* AC pencil on the simplest smooth Fano 3–fold $Y = \mathbb{P}^3$.

Example 7.8 (Compare with Kovalev–Lee [58, Example 2.7]) Instead of a generic transverse pencil we consider the pencil $|S_0, S_\infty| \subset |\mathcal{O}(4)|$, where $S_0 = (x_0 x_1 x_2 x_3 = 0)$ is the sum of the four coordinate planes, and S_∞ is a non-singular quartic surface meeting all coordinate planes transversely. The base curve of the pencil is the union $C = \sum_{i=0}^3 \Gamma_i$ of the four non-singular curves $\Gamma_i = (x_i = 0) \cap S_\infty$. Let Z be obtained from $Y = \mathbb{P}^3$ by blowing up the four base curves one at a time; Z is a non-singular building block containing $e = 6 \times 4 = 24$ $(-1, -1)$–curves; the blow-up resolves the base locus of the pencil, which then defines a (projective) morphism $Z \to \mathbb{P}^1$. It is clear that $H^2(Z) \simeq H^2(\mathbb{P}^3) \oplus \bigoplus_{i=0}^3 H^0(\Gamma_i) \simeq \mathbb{Z}^5$ and, since each Γ_i is a curve of genus 3, $H^3(Z) \simeq \bigoplus_{i=0}^3 H^1(\Gamma_i) \simeq \mathbb{Z}^{24}$.

The image of $H^2(Z)$ in $H^2(S)$ equals the image of $H^2(\mathbb{P}^3)$, that is, it is generated by the hyperplane class. This is because the image of each exceptional divisor is just the hyperplane class, so they contribute only to the kernel K of $H^2(V) \to H^2(S)$, which is $\mathbb{Z}^3$. Since $c_2(\mathbb{P}^3) + c_1(\mathbb{P}^3)^2 = 22H^2$ while $-K_{\mathbb{P}^3}^3 = 64$, it follows from Proposition 5.11 that $\mathrm{div}\, c_2(Z) = 2$.

Example 7.9 We can also consider a pencil of anticanonical divisors in $Y = \mathbb{P}^3$ where *each* divisor is non-generic. Fix a pair of generic plane conics $C_1, C_2 \subset \mathbb{P}^3$. It is easy to see that a generic quartic surface S containing both C_1 and C_2 is non-singular. The curves C_1, C_2 and the hyperplane class generate a lattice $N \subset H^2(S)$ with intersection form represented by

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

We next argue that $N \subset H^2(S)$ is a primitive sublattice. Indeed consider the blow up Y' of $\mathbb{P}^3$ along $C_1 \sqcup C_2$. Since the union $C_1 \sqcup C_2$ is cut out scheme theoretically by quartics, it follows that the anticanonical linear system $|-K_{Y'}| = |I_{C_1 \sqcup C_2}(4)|$ is base point free on Y' and hence $-K_{Y'}$ is nef. A small calculation gives $-K_{Y'}^3 = 64 - 36 = 28$ so $-K_{Y'}$ is also big and Y' is a weak Fano 3–fold. It is clear from the construction that N is the image of $H^2(Y') \to H^2(S)$, hence, in particular, $H^2(Y') \to H^2(S)$ is injective (the matrix above is non-singular). This implies that Y' is a semi-Fano 3–fold (any contracted divisor would lie in the kernel). The lattice $N \subset H^2(S)$ is therefore primitive by Proposition 5.7.

Now take a generic pencil of quartic K3s containing both C_1 and C_2. The base locus consists of C_1, C_2 and a degree 12 curve C_3 (of genus 15) meeting each of C_1 and C_2 in 10 points. Let Z be obtained by blowing up the C_i in any order, and S the proper transform of a smooth element of the pencil. Then (Z, S) is a building block. $H^2(Z) \to L$ maps onto N, and $H^2(V) \to L$ is injective. Regardless of the order of the blow-ups, $\operatorname{div} c_2(Z) = 2$ like in the previous example.

By varying C_1, C_2 and the pencils, we get three different families of blocks, depending on whether we blow up C_3 first, second or last. By Theorem 6.8, a generic N–polarised K3 surface S can be embedded as an anticanonical divisor in a deformation of Y', and hence as a quartic K3 in $\mathbb{P}^3$ containing a pair of conics. It will therefore occur as the K3 fibre of a building block in each of the three families.

The next pair of examples arise from blowing up AC pencils on a projective small resolution Y of a very particular terminal Gorenstein *toric* Fano 3–fold X_{22}. X_{22} is a singular Fano 3–fold with Picard rank 1, AC degree 22 and 9 ODPs. Every Gorenstein toric Fano variety X has an associated combinatorial object called a reflexive polytope which determines X; see Chapters 1 and 2 in the thesis of Nill [76] for a review of basic definitions and facts in toric Fano geometry. See also Section 8 for a brief overview of basic properties of toric weak Fano 3–folds in general.

All such 3d reflexive polytopes and hence all Gorenstein toric Fano 3–folds were classified by Kreuzer–Skarke [59]. The terminal toric Fano 3–folds are precisely those reflexive polytopes whose facets are either standard triangles or standard parallelograms (see Lemma 8.10). Small resolutions of X are also toric and their projectivity can be seen in terms of the combinatorics of the associated reflexive polytope; in particular one can prove that any Gorenstein toric Fano 3–fold admits at least one projective small resolution (see Proposition 8.7). For the toric Fano 3–fold X_{22} chosen in Examples 7.10 and 7.11 one can prove that *all* $512 = 2^9$ possible small resolutions of X are projective (this follows from the fact, shown below, that the defect $\sigma(X) = 9$ is equal to the number of nodes of X); using computer algebra one can show that these 512 projective small resolutions consist of 84 distinct isomorphism classes of weak Fano 3–folds; see also Remark 8.13.

For any toric weak Fano 3–fold Y all odd cohomology groups vanish; in particular we never have to worry about the possibility of torsion in $H^3(Y)$ for toric weak Fano 3–folds. Toric semi-Fano 3–folds therefore give rise to a very large number of building blocks.

Example 7.10 Let X be the terminal Gorenstein toric Fano 3–fold with Fano polytope the reflexive polytope in $N = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T})$ with vertices

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 \end{pmatrix};$$

this is polytope 1942 in the Sage implementation of Kreuze–Skarke's database of 4319 reflexive polytopes in 3 dimensions. X has Picard rank $1 = \mathrm{rk}\, H^2(X)$ and $10 = \mathrm{rk}\, H_4(X)$. The polytope can be viewed in Sage (see also below). (Note, incidentally, that the polytope is self-polar: thus, there is no point in wasting your efforts trying to determine whether you are working in the fan picture or its dual: your conclusions will be correct in either case.) Direct inspection shows that X has a toric projective small semi-Fano resolution $Y \to X$ with $e = 9$ $(-1, -1)$–curves resolving the ordinary nodes of X. Note that the defect $\sigma = e = 9$. Below we show that $H^2(Y) = \mathbb{Z}^{10}$ and

$$N = E_8(-1) \perp \langle 8 \rangle \perp \langle -16 \rangle$$

and $H^3(Y) = (0)$ since Y is a toric variety. The building block $f: Z \to \mathbb{P}^1$ is obtained blowing up the base locus of a pencil $|S_0, S_\infty| \subset |-K_Y|$ where S_0, S_∞ are non-singular and meet transversely. Below we also denote by S a general member of the pencil $|S_0, S_\infty|$. The base locus is a non-singular curve C of genus 12; hence, $H^3(Z) \simeq H^3(Y) \oplus H^1(C) \simeq \mathbb{Z}^{24}$.

Since $\chi(C) = -22$, Proposition 5.11 implies that $\operatorname{div} c_2(Z) = 2$.

Now we calculate the lattice N. Inspecting the polytope – see also Figure 1 – and in particular the boundary surface, shows that, in $\operatorname{Pic}(Y)$,

$$S = -K_Y = \sum_{i=1}^{9} Q_i + \sum_{j=1}^{4} \Pi_j$$

is the union of 9 copies Q_i of $\mathbb{P}^1 \times \mathbb{P}^1$ and 4 copies Π_i of $\mathbb{P}^2$, and $-K_{Y|Q_i} \simeq \mathcal{O}(1,1)$ and $K_{Y|\Pi_j} \simeq \mathcal{O}(1)$ so the total degree of the surface is $2 \times 9 + 4 = 22$, as it should be. From this it follows that the curves $S \cap Q_i$ and $S \cap \Pi_j$ are all rational curves, hence they are all curves of self-intersection -2 on S (S is a K3). These curves meet in a configuration with a dual graph that looks like Figure 1. (The vertices of the graph correspond to -2–curves on S/components of the "boundary" surface of X/vertices of the polytope. Two vertices are connected by an edge if and only if the corresponding -2–curves intersect. The figure signifies that the vertex G is joined to the vertices E_1, E_3, E_5, E_7.)
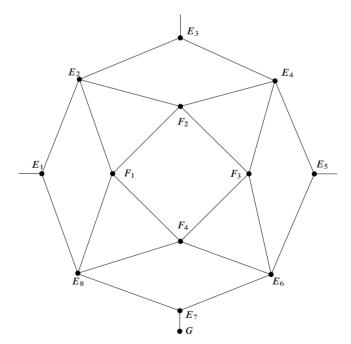

Figure 1: The dual graph

Note that the curves E_1, E_2, E_3, E_4, F_1, F_4, E_6, E_7 generate a sublattice of type $E_8(-1)$. Since $E_8(-1)$ is unimodular, it follows that $N = E_8(-1) \perp (E_8(-1)^{\perp})$,

where $E_8(-1)^\perp$ is a lattice of rank 2. Our next task is to compute $E_8(-1)^\perp$. Looking at elliptic fibrations on S we discover the following relations in $N = \mathrm{Pic}(S)$:

$$2G + E_1 + E_3 + E_5 + E_7 = F_1 + F_2 + F_3 + F_4$$
$$E_1 + E_2 + F_1 + E_8 = F_3 + E_4 + E_5 + E_6$$
$$F_1 + F_2 + E_2 = G + E_5 + E_6 + E_7$$

The first of these, for instance, is obtained from an elliptic fibration with fibres $2G + E_1 + E_3 + E_5 + E_7$ (a fibre of type $\tilde{D}_4$) and $F_1 + F_2 + F_3 + F_4$ (a fibre of type $\tilde{A}_3$). The other two relations are obtained similarly. To find a basis for $E_8(-1)^\perp$, we look at these relations modulo $E_8(-1)$:

$$
\begin{array}{ccccc}
E_5 & -F_2 & -F_3 & +2G & \equiv 0 \\
-E_5 +E_8 & & -F_3 & & \equiv 0 \quad \mathrm{mod}\ E_8(-1) \\
-E_5 & +F_2 & & -G & \equiv 0
\end{array}
$$

It is immediate from these relations that E_5, F_2 is a basis of $N \mod E_8(-1)$. It is easy to check that the vectors

$$
\begin{array}{llllllll}
E_5 & +8E_4 & +15E_3 & +22E_2 & +18F_1 & +14F_4 & +10E_6 & +5E_7 & +11E_1, \\
F_2 & +22E_4 & +43E_3 & +64E_2 & +52F_1 & +39F_4 & +26E_6 & +13E_7 & +32E_1
\end{array}
$$

are perpendicular to $E_8(-1)$ (for instance by computing 16 inner products); thus, by what has been said, they form a basis of $E_8(-1)^\perp$. In this basis the intersection matrix is computed to be

$$
\begin{pmatrix} 16 & 48 \\ 48 & 136 \end{pmatrix}.
$$

and a small change of coordinates then puts this in the form $\langle 8 \rangle \perp \langle -16 \rangle$.

Example 7.11 In this example, $Y \to X$ is the same as in the previous Example 7.10, but we construct the building block Z by blowing up a different pencil. Indeed, let us choose the more interesting pencil $|S_0, S_\infty| \subset |\mathcal{O}(4)|$, where $S_0 = \sum_{i=1}^{9} Q_i + \sum_{j=1}^{4} \Pi_j$ is the toric boundary surface of Y and S_∞ is a non-singular element of $|-K_Y|$ meeting all the components of S_0 transversely. The base curve of the pencil is the union $C = \sum_{i=1}^{9} \Gamma_i + \sum_{j=1}^{4} G_j$ of 13 non-singular rational curves. Let Z be obtained from Y by blowing up the 13 curves one at a time; Z is a non-singular building block containing $e = 9 + 24 = 33$ $(-1, -1)$–curves: 9 that were present in Y, and 24 from the intersection points of C (corresponding to edges in Figure 1). The blow-up resolves the base locus of the pencil, which then defines a (projective) morphism $Z \to \mathbb{P}^1$. It is clear that $H^2(Z) \simeq H^2(Y) \oplus \mathbb{Z}^{13} \simeq \mathbb{Z}^{23}$ and $H^3(Z) = (0)$.

From Proposition 5.11, $\operatorname{div} c_2(Z) \mid (-K_Y^3) = 22$. Since also $\operatorname{div} c_2(Z) \mid 24$, it must be 2.

Example 7.12 Now we give an example using a semi-Fano 3–fold whose anti-canonical morphism is not small, but contracts a divisor to a curve. Let $X \subset \mathbb{P}^4$ be defined by

$$\sum_{0 \le i \le j \le 2} X_i X_j Q_{ij},$$

where Q_{ij} are homogeneous quadrics. This is the general form of a quartic "containing a double line" $\ell = \{X_0 = X_1 = X_2 = 0\}$. For generic Q_{ij}, the sextic polynomial $\det(Q_{ij})$ on ℓ has simple zeros, and the blow-up Y of X at ℓ is smooth, compare with Conte–Murre [16, Lemma 1.15]. Then Y is a crepant resolution of X, and X is the anticanonical model of Y. In particular, Y is semi-Fano.

A generic hyperplane section S of Y is the resolution of a quartic K3 with a single node, so $N = \operatorname{Pic} S = \langle 4 \rangle \perp \langle -2 \rangle$.

To understand more about the topology of Y, consider it as the proper transform of X in G, the blow-up of $\mathbb{P}^4$ in ℓ. Thinking of G as the union of all planes containing ℓ identifies it with the total space of a $\mathbb{P}^2$–bundle over $\mathbb{P}^2$. To be precise, $\pi \colon G \to \mathbb{P}^2$ is the projectivisation of $V = 2\mathcal{O} \oplus \mathcal{O}(-1)$ on $\mathbb{P}^2$. Let T be the associated tautological bundle on G, and $F = \pi^* \mathcal{O}(1)$. The exceptional divisor of $G \to \mathbb{P}^4$ is $E = -T - F$, while the tautological bundle on $\mathbb{P}^4$ pulls back to T. Therefore Y is a section of $-4T - 2E = -2T + 2F$ (so Y is a conic bundle over $\mathbb{P}^2$). This is an ample class ($-T$ and F span the nef cone of G), so $H^3(Y)$ is torsion-free by the Lefschetz theorem, and we can apply Proposition 5.7 to get a building block Z.

To compute characteristic classes of Y, first note that as a complex vector bundle $TG = T_{\mathrm{vert}}G \oplus \pi^* T\mathbb{P}^2$, which is stably isomorphic to $(T^{-1} \otimes \pi^* V) \oplus \pi^*(3\mathcal{O}(1)) = 2T^{-1} \oplus T^{-1}F^{-1} \oplus 3F$. Therefore the total Chern class of Y is

$$\begin{aligned} c(Y) &= (1-T)^2(1-T-F)(1+F)^3 \frac{1}{1-2T+2F} \\ &= 1 - T + T^2 - 5FT + T^3 - 4FT^2 + 7F^2T, \end{aligned}$$

where the addition and multiplication are now in the cohomology ring, and we use that $F^3 = 0$ in $H^6(G; \mathbb{Z})$. By interpreting T^2 as the class of the section $\mathbb{P}(\mathcal{O}(-1))$ of $G = \mathbb{P}(2\mathcal{O} \oplus \mathcal{O}(-1))$, we see that $T^4 = -FT^3 = F^2T^2 = [G]$. Hence

$$\chi(Y) = \int_Y c_3(Y) = (T^3 - 4FT^2 + 7F^2T)(-2T + 2F) = -34,$$

What	$-K_Y^3$	$H^2(Z)$	N	K	$H^3(Z)$	div $c_2(Z)$	e
Example 7.3	4	$\mathbb{Z}^3$	$\begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}$	(0)	$\mathbb{Z}^{50}$	2, 4	9
Example 7.4	4	$\mathbb{Z}^3$	$\begin{pmatrix} -2 & 2 \\ 2 & 4 \end{pmatrix}$	(0)	$\mathbb{Z}^{44}$	2	12
Example 7.5	4	$\mathbb{Z}^3$	$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$	(0)	$\mathbb{Z}^{34}$	2, 4	17
Example 7.6	4	$\mathbb{Z}^3$	$\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$	(0)	$\mathbb{Z}^{36}$	4	16
Example 7.7	4	$\mathbb{Z}^{17}$	$E_6^*(-3) \perp E_8(-1) \perp U$	(0)	$\mathbb{Z}^6$	2	45
Example 7.8	64	$\mathbb{Z}^5$	$\langle 4 \rangle$	$\mathbb{Z}^3$	$\mathbb{Z}^{24}$	2	24
Example 7.9	64	$\mathbb{Z}^4$	$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & 4 \end{pmatrix}$	(0)	$\mathbb{Z}^{30}$	2	20
Example 7.10	22	$\mathbb{Z}^{11}$	$E_8(-1) \perp \langle 8 \rangle \perp \langle -16 \rangle$	(0)	$\mathbb{Z}^{24}$	2	9
Example 7.11	22	$\mathbb{Z}^{23}$	$E_8(-1) \perp \langle 8 \rangle \perp \langle -16 \rangle$	$\mathbb{Z}^{12}$	(0)	2	33
Example 7.12	4	$\mathbb{Z}^3$	$\langle 4 \rangle \perp \langle -2 \rangle$	(0)	$\mathbb{Z}^{46}$	2	12

Table 7.2: A small number of examples of building blocks

so $b^3(Y) = 40$. Similarly we find that $c_2(Y) + c_1(Y)^2 = 2T^2 - 5FT$ evaluates to -28 on T (as it should, since $T = K_Y$) and 18 on F. Hence $b^3(Z) = 46$, and div $c_2(Z) = 2$.

The exceptional set of $Y \to X$ is a conic bundle over ℓ with 6 degenerate fibres. Each degenerate fibre consists of two $\mathbb{P}^1$s intersecting in a single point. These 12 $\mathbb{P}^1$s have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

8 Weak Fano 3–folds: further examples and partial classification results

In this section we give some further examples of weak Fano 3–folds. Our aim is to back up our statement that there are *many* more non-singular weak Fano 3–folds than non-singular Fano 3–folds. We will not be too systematic since weak Fano 3–folds are far from being classified.

Any weak Fano Y for which $-K_Y$ is big and nef but not ample has

$$\rho(Y) = \mathrm{rk}\,\mathrm{Pic}(Y) \geq 2;$$

thus weak Fano 3–folds with $\rho = 2$ are the simplest class of weak Fano 3–folds that are not actually Fano 3–folds.

Examples 7.3 to 7.6 already gave a small number of semi-Fano 3–folds with Picard rank $\rho = 2$: all of anticanonical degree 4 obtained by a small resolution of a (sufficiently generic) quartic containing a special surface; Examples 4.15 and 4.16 are *toric* weak Fano 3–folds with $\rho = 2$. As we will discuss below there are many other weak Fano 3–folds that generalise both classes of examples: toric or $\rho = 2$.

For the purposes of the differential geometry of ACyl Calabi–Yau 3–folds, we are interested in the classification of weak Fano 3–folds up to *deformation*. In Example 6.11(i) we considered how the semi-Fano $\mathbb{F}_2 \times \mathbb{P}^1$ can deform to the rigid Fano $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$; these are different varieties from the algebraic point of view as one is Fano and the other is not, but the ACyl Calabi–Yau 3–folds we construct from them using Proposition 4.24 are deformation-equivalent. We will not discuss the problem of classifying weak Fano 3–folds up to deformation in depth. We use the deformation properties of extremal contractions to distinguish between many of the rank two examples we describe. For the toric examples, we determine whether they are rigid (as complex manifolds) in order to get a crude lower bound on the number of deformation classes.

Weak Fano 3–folds with Picard rank $\rho = 2$: classical examples

We now exhibit some further concrete examples of weak Fano 3–folds with Picard rank $\rho = 2$. These weak Fanos were studied initially because of their connection to so-called *elementary rational maps* between rank one Fano 3–folds; for example, see Iskovskih–Prokhorov's book [38, Section 4.1] to which we refer the reader for further details and references. All these rank two weak Fano 3–folds arise as blowups in points or in low degree curves in rank one Fano 3–folds.

Rank two semi-Fano 3–folds from smooth blowups of Fano 3–folds We have seen that one way to obtain smooth weak Fano 3–folds is to look for projective small (respectively crepant) resolutions of Gorenstein terminal (respectively canonical) Fano 3–folds, but often it is difficult to determine if a projective small (respectively crepant) resolution exists. Another potential way to obtain weak Fano 3–folds is to realise them as smooth blowups of other simpler 3–folds.

In this direction we have the following result of Fujino–Gongyo [29, Theorem 4.5] generalising the analogous result by Kollár–Mori [54, Corollary 2.9] in the Fano setting.

Theorem 8.1 Let $f\colon Y \to W$ be a smooth projective morphism between smooth projective varieties. If Y is weak Fano (respectively Fano) then W is also weak Fano (respectively Fano).

In particular if Y is weak Fano 3–fold with Picard rank $\rho = 2$ and $f\colon Y \to W$ is the inverse of the blowup of a smooth point or curve, then W is a weak Fano 3–fold with $\rho(W) = 1$; but since $\rho(W) = 1$ this forces W to be Fano not just weak Fano. In other words, to find rank two weak Fano 3–folds we ought to consider smooth blowups of smooth rank one Fano 3–folds; in fact we will see below – in our discussion of the classification scheme for rank two weak Fano 3–folds – that the majority of all rank two weak Fano 3–folds arise this way. The particular rank two weak Fano 3–folds that arise as blowups of smooth rank one Fano 3–folds in low degree curves C – that is, lines, conics, and rational normal cubics – have been known at least since the late 1980s and in some special cases since the late 1970s; see Iskovskih–Prokhorov [38, Sections 4.3–4.6] for further details and references.

We will use several times the following well-known result on the behaviour of the canonical class of a smooth threefold under blow-up of a smooth curve or a point.

Lemma 8.2 Let $C \subset W$ be a smooth curve of genus $g(C)$ in a smooth threefold W, let $\pi\colon Y \to W$ be the blowup of C and let E denote the exceptional divisor of π. Then

$$(-K_Y)^3 = (-K_W)^3 + 2K_W \cdot C - 2 + 2g(C);$$
$$(-K_Y)^2 \cdot E = -K_W \cdot C + 2 - 2g(C);$$
$$-K_Y \cdot E^2 = 2g(C) - 2;$$
$$E^3 = K_W \cdot C + 2 - 2g(C).$$

Let Y be the blowup of a smooth threefold W in a point. Then

$$(-K_Y)^3 = (-K_W)^3 - 8;$$
$$(-K_Y)^2 \cdot E = 4;$$
$$-K_Y \cdot E^2 = -2;$$
$$E^3 = 1.$$

Proof The result follows from the fact that $K_Y = \pi^* K_W + E$; see Blanc–Lamy [8, Lemma 2.4] for details. $\qquad\square$

Remark 8.3 If a weak Fano 3–fold Y arises as the blowup of a smooth curve in a smooth rank one Fano 3–fold W then by Lemma 5.6 $H^*(Y)$ is torsion-free since the cohomology of W is torsion-free. Therefore whenever Y is semi-Fano (recall Proposition 5.7(iv)) we can obtain building blocks Z satisfying Definition 5.1 from Y by blowing up the base locus of a generic AC pencil.

Lemma 8.2 allows us to compute the lattice structure on $\mathrm{Pic}(Y)$ and Lemma 5.6 the Betti numbers of Y from those of W and the genus of the curve $g(C)$. We can also understand $c_2(Y)$ and therefore $c_2(Z)$ for the associated building block Z by using the behaviour of c_2 under smooth blowups. Therefore we can obtain all the topological information we need about building blocks that arise this way with relatively little work.

Recall from the Iskovskih classification of smooth rank 1 Fano 3–folds that there are 17 families of examples: $\mathbb{P}^3$, the quadric $Q \subset \mathbb{P}^4$, the del Pezzo (that is, Fano with index 2) 3–folds $V_1, \ldots, V_5$ and 10 index one Fanos 3–folds V_{2g-2} with genus $g \in \{2, \ldots, 10, 12\}$. We shall concentrate on weak Fano 3–folds obtained by blowing up curves in index one rank one Fano 3–folds.

If W is a rank 1 Fano 3–fold of index 1 and genus g and $C \subset W$ is a smooth curve of degree $\deg C := -K_W \cdot C$ and genus $g(C)$ and $Y = Bl_C(W)$ then Lemma 8.2 specialises to yield

$$(8\text{-}4) \qquad -K_Y^3 = 2g' - 2, \quad \text{where } g' = g + g(C) - \deg C - 1.$$

In particular if C is a line, quadric or rational normal cubic then $g' = g - 2$, $g' = g - 3$ or $g' = g - 4$ respectively. If $E \subset Y$ denotes the exceptional divisor of the blowup then the Picard lattice of Y is generated by $-K_Y$ and E. The lattice structure induced on Y is also determined by the information in Lemma 8.2. In particular, if $Y = Bl_C(W)$ is the blowup of a line, conic or rational normal cubic then with respect to the basis E, $A = -K_Y$ of $\mathrm{Pic}(Y)$ the quadratic form $-K_Y \cdot D_1 \cdot D_2$ is

$$\begin{pmatrix} -2 & 3 \\ 3 & 2(g-3) \end{pmatrix}, \qquad \begin{pmatrix} -2 & 4 \\ 4 & 2(g-4) \end{pmatrix}, \qquad \begin{pmatrix} -2 & 5 \\ 5 & 2(g-5) \end{pmatrix},$$

respectively.

To ensure that Y is a rank 2 weak Fano 3–fold one needs to ensure that $-K_Y$ is big and nef. As soon as one shows that $-K_Y$ is nef then for bigness we need only show $-K_Y^3 > 0$ and this can be checked immediately from Equation (8-4). One also needs to ensure the existence of lines, conics and rational normal cubics on the appropriate rank one Fano 3–folds. To show that the AC morphism is small one also needs to know that it contracts only a finite number of curves.

Blowups of lines Iskovskih–Prokhorov [38, Proposition 4.3.1] shows that for every line C on an anticanonically embedded rank one Fano 3–fold W of genus $g \geq 5$ the blowup $Y = Bl_C(W)$ is a rank two semi-Fano of genus $g' = g - 2$ with small AC morphism; moreover the fibres of the AC morphism are all $\mathbb{P}^1$s and they can be understood in terms of the geometry of W, for example, the generic fibre type is any curve $F \subset Y$ whose proper transform in W intersects the chosen line $C \subset W$. In particular by blowing up any line on a rank one Fano 3–fold W of genus $g = 5$ we get a rank two semi-Fano 3–fold Y of genus $g' = 3$ with small AC morphism and quadratic form given in the basis E and $-K_Y$ by

$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}.$$

This is the same quadratic form that appeared in Example 7.5: the small resolution of a general quartic containing a cubic scroll surface. Indeed the rank two semi-Fano Y we have constructed is a projective small resolution of such a nodal quartic 3–fold; see also Entry 30 in Kaloghiros [45, Table 1]. We have similar rank two semi-Fano 3–folds of genus $4, 5, 6, 7, 8, 10$ by blowing up lines on rank one Fano 3–folds of genus $6, 7, 8, 9, 10, 12$ respectively.

Blowups of conics If $Y = Bl_C(W)$ is the blowup of any smooth conic C on an anticanonically embedded rank one Fano 3–fold of genus g then by Iskovskih–Prokhorov [38, 4.4.3] Y is a weak Fano 3–fold of genus $g' = g - 3$ for $g \geq 5$. Furthermore, if $g \geq 7$ then Y is a semi-Fano 3–fold with small AC morphism for any sufficiently generic conic in W and if $g \geq 9$ the same holds for all conics. If we take $g = 6$ then Y is a weak Fano 3–fold of genus 3, that is, its AC model is a terminal quartic 3–fold; see also Kaloghiros [45, Table 1, number 25]. Its quadratic form in the basis E, $-K_Y$ is

$$\begin{pmatrix} -2 & 4 \\ 4 & 4 \end{pmatrix}.$$

We have similar rank two semi-Fano 3–folds of genus $4, 5, 6, 7, 9$ by blowing up sufficiently generic conics on rank one Fano 3–folds of genus $7, 8, 9, 10, 12$ respectively.

Blowups of points If $Y = Bl_P(W)$ is the blowup of a point P not lying on a line in W (such points exist; see [38, 4.2.2]) on an anticanonically embedded rank one Fano 3–fold of genus $g \geq 6$, then Y is a rank two weak Fano 3–fold of genus $g' = g - 4$, moreover for a sufficiently general point P the AC morphism of Y is small [38, 4.5.1]. If we take $g = 7$ then Y is a weak Fano 3–fold of genus 3, that is, its AC model is a terminal quartic 3–fold; see also [45, Table 1, number 24]. Its quadratic form in the

basis E, $-K_Y$ is

$$\begin{pmatrix} -2 & 4 \\ 4 & 4 \end{pmatrix}$$

which is the same lattice which arose above by considering the blowup of a genus 6 Fano 3–fold in a sufficiently generic conic. This pair of rank two weak Fano 3–folds with the same Picard lattice structure are not deformation-equivalent; for instance because they have extremal contractions of different types, compare with Mori [66, Theorem 3.47]. We have similar rank two semi-Fano 3–folds of genus $4, 5, 6, 10$ by blowing up sufficiently generic conics on rank one Fano 3–folds of genus $8, 9, 10, 12$ respectively.

Blowups of rational normal cubics   A rank one Fano 3–fold V_{2g-2} of $g \geq 5$ which contains a line and a conic also contains a rational normal cubic [38, 4.6.1]. So we can also consider blowups along rational normal cubics. If $Y = Bl_C(W)$ is the blowup of any (respectively a sufficiently general) rational normal cubic on a rank one Fano 3–fold of genus $g \geq 7$ (respectively $g \geq 6$), then Y is a rank two weak Fano 3–fold of genus $g' = g - 4$ [38, 4.6.2]. If we take $g = 7$ then Y is a weak Fano 3–fold of genus 3, that is, its AC model is a Gorenstein at worst canonical quartic 3–fold. Its quadratic form in the basis E, $-K_Y$ is

$$\begin{pmatrix} -2 & 5 \\ 5 & 4 \end{pmatrix}.$$

The classification scheme for rank two weak Fano 3–folds

In the Mori–Mukai classification there are 36 families of non-singular Fano 3–folds with $\rho = 2$; see Iskovskih–Prokhorov [38, Table 12.3] for the list. The classification of non-singular weak Fano 3–folds with $\rho = 2$ was initiated recently by Jahnke–Peternell–Radloff [40; 41] with subsequent contributions by Takeuchi [93], Cutrone–Marshburn [18], Arap–Cutrone–Marshburn [2] and Blanc–Lamy [8]; see also related work by Kaloghiros [45; 46]. The classification is not yet complete, but already more than 200 families of rank two weak Fano 3–folds are known (with around 50 further cases still to be settled). Below we summarise the basic strategy of this classification scheme and some of the main results obtained; we refer the reader to the references above for further details.

Throughout the rest of this section Y will denote a non-singular weak Fano 3–fold of rank 2 and $\varphi\colon Y \to X$ its anticanonical morphism. In general the anticanonical model X of a rank two weak Fano 3–fold Y is a Gorenstein canonical Fano 3–fold with $\rho(X) = 1$ whose anticanonical degree is the same as that of Y. There are two main classes:

(i) the anticanonical morphism $\varphi \colon Y \to X$ is divisorial, that is, it contracts a divisor. In this case, X is a Gorenstein Fano 3–fold with canonical non-terminal singularities, $\rho(X) = 1$ and $\sigma(X) = 0$. (In the vast majority of cases we will see that φ is semi-small, so that Y is a semi-Fano 3–fold in the sense of Definition 4.11).

(ii) the anticanonical morphism $\varphi \colon Y \to X$ is small. X is a non-$\mathbb{Q}$–factorial Gorenstein Fano 3–fold with terminal singularities, $\rho(X) = 1$ and $\sigma(X) = 1$. (In many of these cases X has only ordinary double points).

Recall that in the classification of non-singular rank 2 Fano 3–folds a fundamental role is played by the two different Mori contractions that any such 3–fold admits. By Mori's classification of non-singular 3–fold extremal rays (Theorem 3.46) the possible contractions are completely understood and fall into three basic classes: type C (conic bundle type), D (del Pezzo fibre type) and E (exceptional/divisorial) type. For example if a Fano 3–fold with $\rho = 2$ admits an extremal contraction of type E1, that is, the inverse of the blowup of a non-singular curve in a smooth 3–fold W, then by Theorem 8.1 W itself must be a smooth Fano 3–fold with $\rho(W) = 1$. In general the existence of two extremal rays of known type together with the condition of being Fano put severe constraints on the 3–fold, enough to allow a complete classification.

For non-singular rank 2 *weak* Fano 3–folds there is only a single Mori contraction $\psi \colon Y \to W$. A substitute for the missing second extremal ray is provided by the AC morphism $\varphi \colon Y \to X$. When the anticanonical morphism contracts a divisor, an almost complete classification was given recently by Jahnke–Peternell–Radloff [40]. When the anticanonical morphism is small the analysis is more involved and the classification is not yet close to complete. Nevertheless, as we will describe below, many examples are known and there are classification results under additional assumptions.

Rank 2 weak Fanos with divisorial AC morphism In [40], Jahnke, Peternell and Radloff classify rank two weak Fano 3–folds of type (i) – where the AC morphism φ is a divisorial contraction – according to the type of the Mori contraction $\psi \colon Y \to W$; see [40, Tables A.2–A.5, pages 627–630]. There are at most 59 deformation families (the existence of two possible families A.2.7, A.2.8 remains to be shown) with (even) anticanonical degrees $-K_Y^3$ between 2 and 72; because of the length and complexity of the classification we do not reproduce it here. A key technical role is played by Mukai's classification [70] of all Gorenstein Fano threefolds with canonical singularities such that the anticanonical divisor does not admit a moving decomposition; see [40, 4.7].

One important fact to note from the classification is that rank two weak Fano 3–folds that are *not* semi-Fano are extremely rare; when the extremal ray is of type D or

E2–5 one can show that Y is always semi-Fano [40, 2.3 and 5.2] and there is a single exception out of 25 cases with an extremal ray of type E1 [40, Section 4]. Altogether only in four (A.3.1, 3.9, 3.12 and 4.25 in [40]) out of the 59 families does the AC morphism contract a divisor to a point. Hence we have 53(+2?) non-singular rank 2 semi-Fano 3–folds for which the anticanonical morphism $\varphi\colon Y \to X$ contracts a divisor D to a curve B. In all such cases $B \subset X$ is a non-singular curve of cDV singularities, Y is the blowup of X in the curve B and D is a conic bundle over B [40, 1.8].

Rank 2 weak Fanos with small AC morphism   As mentioned above, the classification of rank two semi-Fano 3–folds whose AC morphism φ is small is more involved and not yet complete, despite recent activity in this direction by several authors. In this case the anticanonical model X is a non-$\mathbb{Q}$–factorial Gorenstein Fano 3–fold with terminal singularities, which by Proposition 3.26 are isolated cDV singularities; in many cases X has only ordinary double points.

By Namikawa's smoothing result (Theorem 4.17) X admits a smoothing $\mathcal{X} \to \Delta \subset \mathbb{C}$ such that $\mathcal{X}_0 \simeq X$ and $\mathcal{X}_t$ for $t \neq 0$ is a non-singular Fano 3–fold (this is not always true in the case of Gorenstein canonical singularities); moreover, the Picard groups (over $\mathbb{Z}$) of X and the general $\mathcal{X}_t$ are isomorphic. Hence X and $\mathcal{X}_t$ have the same Fano index and $\mathcal{X}_t$ is a non-singular Fano 3–fold of Picard rank 1. The cases where X has index > 1 are relatively straightforward – see Jahnke–Peternell–Radloff [41, 2.12–3] – and the main case is when X has index 1. In this case the Iskovskih classification of rank 1 non-singular Fano 3–folds (see [38, Table 12.2] for a convenient list or see our Table 7.1 in Section 7) implies

$$2 \leq -K_Y^3 \leq 22 \quad \text{with} \quad -K_Y^3 \neq 20$$

and, in fact, all such possible anticanonical degrees actually occur.

The anticanonical morphism $\varphi\colon Y \to X$ is a flopping contraction (recall Definition 3.28) and thus it can be flopped (recall Theorem 3.30); that is, there is another non-singular rank 2 weak Fano 3–fold Y^+, whose (small) anticanonical morphism we denote $\varphi^+\colon Y^+ \to X$. Y^+ has the same anticanonical degree as Y. For any divisor D on Y let D^+ denote the strict transform of D under the flop χ; the map $D \to D^+$ induces an isomorphism between the Picard groups of Y and Y^+. Moreover, the lattice structures induced on the Picard lattices of Y and Y^+ are isomorphic, that is, for any divisors D_1 and D_2 on Y we have

(8-5) $$-K_Y \cdot D_1 \cdot D_2 = -K_{Y^+} \cdot D_1^+ \cdot D_2^+.$$

Y^+ also admits a (K_{Y^+}–negative) extremal contraction $\psi^+\colon Y^+ \to W^+$. Everything fits into the following diagram:

(8-6)
$$
\begin{array}{ccc}
Y & \overset{\chi}{\dashrightarrow} & Y^+ \\
\downarrow{\scriptstyle\psi} \quad \searrow{\scriptstyle\varphi} \quad \swarrow{\scriptstyle\varphi^+} & & \downarrow{\scriptstyle\psi^+} \\
W \qquad\quad X \qquad\quad & & W^+
\end{array}
$$

The classification programme has two steps: a *numerical classification* stage and the more delicate *geometric realisability* question. In the numerical classification stage one first writes down a system of Diophantine equations determined by the relations among various intersection numbers that any non-singular weak Fano 3–fold of rank 2 with small AC morphism would have to satisfy. The precise form of these Diophantine equations depends on the pair of Mori contractions ψ and ψ^+; as a result there are various subcases depending on the type of the pair of Mori contractions. See Cutrone–Marshburn [18, Section 2.1] – particularly equations (2.6) and (2.7) therein – for the Diophantine equations in the case where both Mori contractions ψ and ψ^+ are of type E1; in this latter case both Y and its flop Y^+ arise as the blowups of smooth curves C and C^+ in rank one Fano 3–folds W and W^+. So rank two weak Fanos with link type E1–E1 constitute a direct generalisation of the concrete rank two weak Fanos constructed in the previous subsection as blowups of low degree curves in non-singular index one rank one Fano 3–folds.

A solution of the numerical classification problem means a finite list of all possible solutions to these Diophantine equations. Each such solution is referred to as a *numerical link*. For some pairs of Mori contractions there are many numerical links while for others there are relatively few. However, not every numerical link is realisable by a weak Fano 3–fold. For each numerical link further (often more delicate) argument is required either to find a weak Fano realising that numerical link or to prove that no such weak Fano exists. This is the geometric realisability question.

Jahnke–Peternell–Radloff [41] give a complete list of numerical links in the case that at *most* one of the Mori contractions ψ or ψ^+ from (8-6) is of type E. Cutrone–Marshburn [18] completed the numerical classification when both Mori contractions are of type E. Takeuchi [93] considers the case where ψ is of type D and gives a complete classification including the geometric realisability question when this del Pezzo fibration has degree different from 6; this augments (but also overlaps considerably with) the geometric realisability studies contained in [41, Section 3]. Thus the classification of rank two weak Fano 3–folds where at least one Mori contraction is not of type E is now close to complete; the 6? entries in [41, Table 7.7] and the del Pezzo fibrations of degree 6 are still outstanding.

The classification of rank two weak Fano 3–folds where both Mori contractions are of type E is substantially less complete. Cutrone–Marshburn [18] gives a list of 111 numerical links of type E1–E1: meaning both Mori contractions are of type E1, that is, both Y and Y^+ arise as the blowup of smooth curves in rank one Fano 3–folds W and W^+. Of these 111 numerical links, they prove 11 to be geometrically realisable, 13 not to be geometrically realisable, and leave 87 numerical links unsettled; see [18, 5.1]. Recall from Remark 8.3 that if Y is a rank two weak Fano 3–fold with link type E1–E1 (or more generally E1–**) then $H^3 Y$ is torsion-free. Hence we can always construct building blocks in the sense of Definition 5.1 from any such weak Fano 3–fold.

More recently Blanc–Lamy [8] settled the geometric realisability question when the weak Fano Y arises as the blowup of a space curve in $\mathbb{P}^3$; this gives the existence of 13 further pairs (Y and its unique flop Y^+) of rank two semi-Fano 3–folds with small AC morphism. Very recently Arap–Cutrone–Marshburn [2] settled most of the geometric realisability questions in the cases when the weak Fano Y arises as the blowup of a curve in: a smooth quadric in $\mathbb{P}^4$, a pair of quadrics in $\mathbb{P}^5$ or a del Pezzo 3–fold of degree 5. These give another 8, 6 and 13 pairs of examples of rank two semi-Fano 3–folds with small AC morphism respectively.

So geometric realisability currently remains open for approximately 50 of the 111 numerical links of type E1–E1 listed in [18]. The situation for other numerical links of type E–E is far more heavily constrained with only relatively few numerical link types; for most of these numerical links the geometric realisability question is already solved [18, 5.2–5.7].

Rough enumeration of rank two weak Fano 3–folds with small AC morphism Let us give a rough enumeration of the number of deformation types generated by the rank two weak Fano 3–folds with small AC morphism currently known to exist.

If Y and Y' are smooth rank two weak Fano 3–folds belonging to the same deformation type then by Mori's deformation theory for extremal rays [66, Theorem 3.47] both Y and Y' admit extremal rays of the same type except possibly in the cases E3/E4 (the point being that E3 can degenerate to E4). In particular, if both Y and Y' have small AC morphisms and different numerical links then Y and Y' do not belong to the same deformation type.

Takeuchi [93, 2.2–2.13] gives a list of 33 families of del Pezzo fibred non-singular rank two Fano 3–folds with small AC morphism and shows that none of them are deformation equivalent [93, Theorem 2.15]. He also lists their anticanonical models X and their flops Y^+; in almost all cases the anticanonical model X has only ordinary double points and therefore both Y and Y^+ have nodal AC model. The number of

curves contracted by the anticanonical morphism φ varies between 1 and 46. In 19 cases Y^+ is not itself del Pezzo fibred and is therefore not deformation equivalent to any of the rank two weak Fanos in Takeuchi's list of 33. Hence we obtain 52 distinct families of rank 2 Fano 3–folds with small AC morphism from Takeuchi's work, almost all of which have nodal AC model. When Y is a del Pezzo fibration of degree 6 [41, A.2–A.4] provides 5 additional examples (plus their flops which are different) and leaves open a further 8 possibilities for del Pezzo fibrations of degree 6. Finally [41, Tables 7.5–7.7] yields $12 = 2 + 3 + 7$ cases (plus their flops) where ψ is a conic bundle (with the geometric realisability of 6 further numerical links left open).

In total this gives us 84 (that is, $52 + 2 \times 5 + 1 \times 2 + 2 \times 3 + 2 \times 7$) currently known deformation types generated by rank 2 Fano 3–folds with small AC morphism for which at least one of the Mori contractions ψ and ψ^+ is *not* birational, and the majority of these have nodal AC model. In addition we have 26 cases from [18] where both Mori contractions ψ and ψ^+ are birational and over 40 further examples of link type E1–E1 from Arap–Cutrone–Marshburn [2] and Blanc–Lamy [8].

To summarise: we have at least 150 deformation types arising from known families of rank two semi-Fano 3–folds for which the AC morphism is small (many of which have nodal AC model) in addition to the 36 deformation types of rank two genuine Fano 3–folds. (There are additional deformation types which arise from the known rank two semi-Fano 3–folds for which the AC morphism is only semi-small, but we must take some care enumerating these because these may belong to the 150+ deformation types above or may be deformation equivalent to a rank two genuine Fano 3–fold.)

This abundance of rank two semi-Fanos will allow us to construct a large number of new compact G_2–holonomy manifolds in [17]. If we use at least one building block built from one of the many semi-Fano 3–folds with nodal AC model, then we will be able to construct G_2–holonomy manifolds containing a variety of different numbers of rigid associative 3–folds.

Toric weak Fano 3–folds

In this section we give an overview of the results one can obtain for projective small (respectively crepant) resolutions of toric Gorenstein terminal (respectively canonical) Fano 3–folds. This will prove the existence of very many (hundreds of thousands of) non-singular toric weak Fano 3–folds. We will also see that the set of non-singular toric weak Fano 3–folds with nodal AC model is essentially disjoint from the many Picard rank 2 semi-Fanos discussed above. The building blocks in Examples 7.10 and 7.11 both use one very particular toric semi-Fano with nodal AC model.

Although by Batyrev's work [5] there are only 18 deformation classes of non-singular toric Fano 3–fold (see also the table in [38, Appendix 12.8]), there are many deformation classes of singular toric Fano 3–fold as soon as one allows even relatively mild singularities. In the following whenever we refer to a toric Fano 3–fold we shall mean a Gorenstein toric Fano 3–fold; these automatically have at worst canonical singularities [5, 2.2.5].

Toric Fano 3–folds correspond (uniquely up to isomorphism) to so-called *reflexive polytopes*, see for example, Nill's thesis [76, Chapters 1 and 2] for basic definitions in toric Fano geometry. Kreuzer–Skarke [59] developed an algorithm to classify reflexive polyhedra in arbitrary dimensions; as an application of this algorithm they showed that there are 4319 3–dimensional reflexive polytopes, including the 18 that correspond to non-singular toric Fano 3–folds.

A big advantage of Gorenstein toric Fano 3–folds compared to more general Gorenstein canonical Fano 3–folds (where often no projective crepant resolution exists, for example, any nodal quartic in $\mathbb{P}^4$ with fewer than 9 nodes) is that one can use toric geometry to prove that *any* toric Fano 3–fold admits a projective crepant resolution. Since every such crepant resolution is a non-singular toric weak Fano 3–fold this proves the existence of at least 4301 deformation families of toric weak Fano 3–fold. In fact there are many more such families because many singular toric Fano 3–folds admit numerous non-isomorphic projective crepant resolutions; moreover, all the projective crepant resolutions are toric and can be enumerated purely combinatorially (see below). The topology of toric weak Fano 3–folds is also relatively straightforward: as smooth toric varieties they have no cohomology in odd degree and their even cohomology is torsion-free. In particular we never have to worry about the condition $H^3(Y)$ being torsion-free. These features make toric weak Fano 3–folds a very rich class of examples which nonetheless can be studied relatively easily.

Proposition 8.7 *Any 3–dimensional Gorenstein toric Fano variety X admits at least one projective crepant resolution Y; Y is a non-singular toric weak Fano 3–fold whose anticanonical model is X.*

Remark 8.8

(i) This result is not true for higher-dimensional Gorenstein toric varieties X; what is true is that there is a projective birational morphism $f: X' \to X$, such that f is crepant and X' is toric with only $\mathbb{Q}$–factorial terminal singularities [5, Theorem 2.2.24]. Batyrev calls $f: X' \to X$ a *maximal projective crepant partial desingularisation* of X or *MPCP-desingularisation* for short. Proposition 8.7

is a special case of the existence of MPCP-desingularisations; since any 3–dimensional Gorenstein toric variety with $\mathbb{Q}$–factorial terminal singularities must in fact be non-singular, any 3–dimensional MPCP-desingularisation is non-singular.

(ii) Crepant resolutions of a toric variety X correspond to fans Δ' refining the original fan Δ defining X. The toric variety X' associated to the fan Δ' is in general not projective; when the toric variety associated to the fan Δ' is again projective, the fan Δ' is called a *coherent crepant refinement* of Δ.

(iii) Batyrev shows that any MPCP-desingularisation $f: X' \to X$ defines a "maximal projective triangulation" of the reflexive polytope P associated to X and conversely that any maximal projective triangulation of the reflexive polytope P determines a MPCP-desingularisation of X. Since Gelfand, Kapranov and Zelevinsky [31] already proved the existence of maximal projective triangulations (regular triangulations in their terminology) of any integral polyhedron P, the existence of MPCP-desingularisations (and hence projective crepant resolutions in the 3–dimensional case) then follows immediately.

(iv) One can use the correspondence between projective crepant resolutions of a toric Fano 3–fold and maximal projective triangulations of the corresponding reflexive polytope to enumerate *all* projective crepant resolutions of a given toric Gorenstein Fano 3–fold. Together with Tom Coates and Al Kasprzyk we have used TOPCOM [82] in combination with PALP and Sage to find *all* toric semi-Fano 3–folds up to isomorphism. A more detailed description of this computation, the full data and a systematic treatment of G_2–manifolds arising from them will appear elsewhere [15].

Many features of any crepant projective resolution of a toric Fano 3–fold can be read immediately from the associated reflexive polytope. For example we have the following:

Remark 8.9 The Picard rank ρ of *any* crepant resolution of a toric Fano 3–fold equals the number of lattice points (including the origin) of the corresponding reflexive polytope minus 4. Hence from Kreuzer–Skarke [59, Table 2] we have that for a non-singular toric weak Fano 3–fold Y, $\rho = b^2(Y)$ can attain any value between 2 and 35 except 32 and 33.

We can also recognise the various flavours of non-singular toric weak Fano Y from the geometry of the reflexive polytope associated with its (Gorenstein toric Fano) anticanonical model X.

For toric Fano 3–folds with small AC morphism we have:

Lemma 8.10 (Terminal toric Fano 3–folds)

(i) *A toric Fano 3–fold X is terminal if and only if all facets of its reflexive polytope are either standard triangles or standard parallelograms.*

(ii) *The only singularities of a terminal toric Fano 3–fold are ordinary double points and the number of ODPs of X is equal to the number of parallelograms in its reflexive polytope. In particular, every toric weak Fano 3–fold with small AC morphism has nodal AC model.*

(iii) *Every terminal toric Fano 3–fold X admits at least one small projective resolution Y; Y is a non-singular toric semi-Fano 3–fold. Conversely every non-singular toric semi-Fano 3–fold Y with nodal AC model arises as a small projective resolution of a terminal (nodal) toric Fano 3–fold X.*

Proof For (i) and (ii) see the thesis of Nill [76, 4.2.4 and 4.3.1–4.3.2]. (iii) is a special case of Proposition 8.7; see also the remark below. □

Remark 8.11 In the special case of a terminal toric (and therefore nodal) Fano 3–fold X any "crepant refinement" of the reflexive polytope of X as in Remark 8.8 arises as follows: for each parallelogram facet in the reflexive polytope pick one of its two diagonals and make a new polytope by adding the chosen diagonals as additional edges to the reflexive polytope. Clearly there are 2^e such refinements where e is the number of parallelograms (by Lemma 8.10(ii) parallelograms correspond to the nodes of X); each such refinement gives a (toric) small but not necessarily projective resolution of X. By Lemma 8.10(iii) at least one of these small resolutions is projective.

Corollary 8.12

(i) *There are precisely 82 singular toric Fano 3–folds with terminal singularities.*

(ii) *The Picard rank ρ of a terminal toric Fano 3–fold X can be 1, 2, 3 or 4.*

(iii) *The Picard rank ρ of a toric semi-Fano 3–fold with nodal AC model takes all values between 2 and 11.*

(iv) *The genus g of a toric semi-Fano 3–fold with nodal AC model takes all values in $\{11,\dots,25\}\cup\{28\}$.*

(v) *The defect σ of a terminal toric Fano 3–fold takes all values in $\{1,\dots,7\}\cup\{9\}$.*

(vi) *The number e of exceptional $(-1,-1)$ curves of a toric semi-Fano 3–fold with nodal AC model takes all values in $\{1,\dots,9\}\cup\{12\}$.*

(vii) *Every toric semi-Fano* 3*–fold* Y *with nodal AC model is rigid, that is,*

$$H^1(Y, \mathcal{T}_Y) = (0).$$

(viii) *There are precisely* 1009 *deformation types of toric semi-Fano* 3*–fold with nodal AC model.*

Proof   (i) follows either from Nill's thesis [76] or from the Kreuzer–Skarke classification of reflexive polytopes in three dimensions and the characterisation of the terminal ones from Lemma 8.10(i). (ii)–(vi) now follow from an examination of the 82 possible terminal reflexive polytopes, for example, see the list of terminal toric Fano 3–folds on the Graded Rings database [9]. (vii) follows immediately from Ilten's thesis [35, Corollary 4.2.6]. For (viii) we first enumerate all projective small resolutions of the 82 terminal reflexive polytopes. Next we identify projective small resolutions of a given terminal polytope which differ by a lattice automorphism. This yields the number of non-isomorphic projective small resolutions for each polytope. The details of these calculations will appear in [15]. The total number of non-isomorphic projective small resolutions turns out to be 1009: since by (vii) all these varieties are rigid the number of deformation types is also equal to 1009.   □

Remark 8.13   (Toric semi-Fano 3–folds with nodal AC model and near-extremal Picard rank)

(i)   Let us a consider toric weak Fano 3–fold Y with the minimal possible Picard rank $\rho = 2$ (we assume Y is not already Fano so $\rho \geq 2$). By Remark 8.9 the reflexive polytope corresponding to its AC model X has exactly 6 lattice points. Consulting the classification we find that among the 82 terminal reflexive polytopes there is precisely one such polytope. The corresponding terminal toric Fano 3–fold $X \subset \mathbb{P}^4$ is the projective cone over a non-singular quadric $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and has two (isomorphic) projective small resolutions as described in Example 4.15. Therefore up to isomorphism there is precisely one toric semi-Fano 3–fold with $\rho = 2$ and nodal AC model; as remarked previously it has index 3. In particular, only this toric semi-Fano 3–fold appears in our earlier count of over 150 semi-Fano 3–folds with $\rho = 2$ and small AC morphism.

(ii)   By Corollary 8.12(iii) any toric semi-Fano 3–fold Y with nodal AC model has Picard rank at most 11. By counting lattice points again the classification of terminal reflexive polytopes shows that if the Picard rank of Y equals 11 then its anticanonical model X is the unique terminal toric Fano 3–fold X_{20} of degree 20; X_{20} has Picard rank 2 and defect 9. (We know X_{20} could not have Picard rank 1 because there is no smooth rank 1 Fano of degree 20 which

could degenerate to X_{20}.) X_{20} corresponds to polytope 2355 in the Sage list of 3–dimensional reflexive polytopes; this polytope contains 12 parallelograms and hence X contains 12 nodes. Using TOPCOM to count regular triangulations of the polytope we find that this polytope admits 3608 (out of all $2^{12} = 4096$ possible small resolutions) projective small resolutions and that these lie in 125 distinct isomorphism classes. Apart from these 125 isomorphism classes of toric semi-Fano 3–folds with nodal AC model and Picard rank 11, all other toric semi-Fano 3–folds with nodal AC model have Picard rank between 2 and 10.

(iii) Similarly, any toric semi-Fano 3–fold with nodal AC model and Picard rank equal to 10 has anticanonical model the unique terminal toric Fano 3–fold X_{22} of degree 22; X_{22} has Picard rank 1 and defect 9. It corresponds to polytope 1942 in the Sage list of 3–dimensional reflexive polytopes; this polytope contains 9 parallelograms and hence X contains 9 nodes. Note that the number of nodes of X is equal to its defect. Using TOPCOM to count regular triangulations of the polytope we find that all 512 small resolutions of this polytope are projective and these consist of 84 distinct isomorphism classes. Building blocks constructed from this particular polytope were discussed in detail in Examples 7.10 and 7.11. We selected this particular polytope because it has maximal defect $\sigma = 9$ and because the polytope is self-dual.

Similarly we can recognise more general toric semi-Fano 3–folds from the geometry of the associated reflexive polytope:

Lemma 8.14 (Toric semi-Fano 3–folds)

(i) *A toric weak Fano 3–fold Y is semi-Fano if and only if the reflexive polytope corresponding to the (toric Fano) anticanonical model X of Y has no facets that contain lattice points strictly in their interior, that is, every lattice point on any facet lies on an edge of the facet. In this case by a slight abuse of terminology we will say that the reflexive polytope (or the singular toric Fano 3–fold X) is semi-small.*

(ii) *Every semi-small toric Fano 3–fold X admits at least one projective crepant resolution Y; Y is a non-singular toric semi-Fano 3–fold.*

Proof   (i) is obvious; (ii) is a special case of Proposition 8.7.   □

Corollary 8.15

(i) *There are 799 semi-small 3–dimensional reflexive polytopes (excluding the $100 = 18 + 82$ corresponding to non-singular or terminal toric Fanos). These are precisely the polytopes for which every facet contains no interior lattice points, but that contain at least one boundary lattice point that is not a vertex of the polytope. 435 of these polytopes contain at least one standard parallelogram.*

(ii) *The Picard rank ρ of a semi-small toric Fano 3–fold X can be 1, 2, 3 or 4.*

(iii) *The Picard rank ρ of a non-singular toric semi-Fano 3–fold Y can be any integer between 2 and 15.*

(iv) *The genus g of a toric semi-Fano 3–fold Y can be any integer between 7 and 29.*

(v) *The defect σ of a semi-small toric Fano 3–fold X is at least 1 and at most 13.*

(vi) *There are 526 130 isomorphism classes of non-singular toric semi-Fano 3–fold (including the $18 + 1009$ corresponding to smooth toric Fanos and toric Fanos with terminal AC model); 435 459 of these are rigid.*

Proof (i) follows from the Kreuzer–Skarke explicit list of all 4319 reflexive polytopes and the characterisation given in Lemma 8.14(i). (ii)–(v) follow from computations based on the explicit list of semi-small reflexive polytopes. The first part of (vi) follows by enumerating all projective crepant resolutions of the semi-small reflexive polytopes as described in the proof of Corollary 8.12(viii). It is not the case that every toric semi-Fano 3–fold is rigid; Example 6.11(i) exhibits a toric semi-Fano 3–fold that is not rigid. To determine which toric semi-Fano 3–folds are rigid first we compute $h^0 = \dim H^0(Y, \mathcal{T}_Y)$ by finding the number of Demazure roots of the associated fan, compare with Oda's book [77, Corollary 3.13]. To compute $h^1 = \dim H^1(Y, \mathcal{T}_Y)$ we use the fact that

$$h^1 - h^0 = -\chi(Y, \mathcal{T}_Y) = 19 - \rho - g + h^{2,1}(Y),$$

where ρ and g are the Picard rank and genus respectively of the semi-Fano Y, compare with Mukai [71, Section 4]. (In the toric case we always have $h^{2,1} = 0$). The detailed calculations will appear in [15]. □

In particular, there are at least 435 459 deformation types of toric semi-Fano 3–folds (including the 1027 corresponding to smooth toric Fanos and toric semi-Fanos with terminal AC model). More effort would be needed to determine how many deformation types are realised by the remaining non-rigid toric semi-small Fano 3–folds.

Terminal Fano 3–folds via degenerations of non-singular Fano 3–folds

Given any non-singular semi-Fano 3–fold Y with small AC morphism, we can associate a deformation class of non-singular Fano 3–folds as follows. By Remark 4.12(ii), the anticanonical model X of Y is a terminal Gorenstein Fano 3–fold which thanks to Namikawa's smoothing result (Theorem 4.17) is smoothable by a flat deformation to a family of non-singular Fano 3–folds X_t. The anticanonical degrees and indexes of Y, X and X_t are all the same and $\rho(X) = \rho(X_t)$ but $\rho(Y) = \rho(X) + \sigma$ where σ is the defect of X. For instance in the case where Y is a rank 2 semi-Fano 3–fold with small AC morphism (as considered earlier) this associates to Y one of the 17 deformation classes of non-singular rank 1 Fano 3–folds from the Iskovskih classification.

Semi-Fano 3–folds associated with degenerations of a cubic in $\mathbb{P}^4$ to a nodal cubic are completely understood; see below for a summary of these results. On the other hand weak Fano 3–folds associated with say degenerations of a quartic in $\mathbb{P}^4$ are still very far from understood in general; see below for some further discussion of nodal quartics and their small resolutions.

Weak del Pezzo 3–folds from nodal cubics   From our earlier remarks about the behaviour of the index and degree under smoothing and small resolution, we see immediately that because a smooth cubic has index 2 and degree 24, any semi-Fano 3–fold arising as the small resolution of a nodal cubic also has index 2 and degree 24. In particular, they are all weak del Pezzo 3–folds. Finkelnberg–Werner [26] understood how many nodes can occur on a degeneration of a smooth cubic, what defects occur and in each case how many of the small resolutions are projective. Their results demonstrate clearly how one single deformation class of smooth del Pezzo 3–folds can give rise to a much larger number of deformation classes of weak del Pezzo 3–folds.

Finkelnberg–Werner show that the number of nodes k can take any value up to 10 and the defect any value up to 5. Table 8.1 lists the possible number of nodes e, the defect σ, the number of projective small resolutions s, the number of planes P contained in the nodal cubic and $b^3(Y)$ denotes the third Betti number of any projective small resolution of the nodal cubic (if any exists); the latter is computed using (4-22) and the fact that $b^3 = 10$ for a non-singular cubic 3–fold.

Remark 8.16

(i)   A nodal cubic 3–fold $X \subset \mathbb{P}^4$ is nonrational if and only if it is smooth by Clemens–Griffiths [14, Theorem 13.12]. Any projective small resolution Y of a nodal cubic X therefore has no torsion in $H^3(Y)$ (recall Remark 5.8) and hence gives rise to a building block Z in the sense of Definition 5.1 via the construction of Proposition 5.7.

e	σ	s	P	$\rho(Y)$	$b^3(Y)$
0	0	0	0	1	10
1	0	0	0	–	–
2	0	0	0	–	–
3	0	0	0	–	–
4	0	0	0	–	–
	1	2	1	2	4
5	1	0	1	–	–
	1	0	1	–	–
6	1	2	0	2	0
	2	6	2	3	2
7	2	6	2	3	0
	2	0	3	–	–
8	3	24	5	4	0
9	4	102	9	5	0
10	5	332	15	6	0

Table 8.1: The possible nodal degenerations of cubic 3–folds; e denotes the number of ODPs, σ the defect, s the number of projective small resolutions and P the number of planes contained in the nodal cubic. $\rho(Y)$ and $b^3(Y)$ denote the Picard rank and third Betti number of any projective small resolution Y of the nodal cubic (when one exists).

(ii) All the examples in Table 8.1 with $\rho(Y) > 2$ have nodal AC model and are not already included in any of the classes described earlier in the paper; to see this we only need note the following: a non-singular cubic and hence any nodal degeneration has Picard rank $\rho = 1$ and anticanonical degree 24, whereas the classification of toric terminal Fano 3–folds shows none of the three degree 24 examples has Picard rank 1.

(iii) (4-18) applied to a degeneration of a non-singular cubic yields $e \leq 5+20-1=24$, whereas in fact we have $e \leq 10$. So in this case the bound from (4-18) is quite far from being sharp.

(iv) (4-22) and non-negativity of $b^3(Y)$ immediately implies $e - \sigma \leq 5$; Table 8.1 shows that there are 5 different possible combinations of e and σ realising $e - \sigma = 5$ (which forces $b^3(Y) = 0$).

(v) From Table 8.1 we see that $3 \leq e - \sigma \leq 5$ for any nodal cubic that admits a projective small resolution.

(vi) In Table 8.1 some (sometimes many) of the s small projective resolutions of a given nodal cubic may give rise to projective varieties that are abstractly isomorphic; if the nodal cubic X admits a nontrivial discrete group of automorphisms then this group acts on the set of all small resolutions and different small resolutions in the same orbit are isomorphic. For example, the unique (up to projective equivalence) nodal cubic with 10 nodes called the Segre cubic has automorphism group the symmetric group S_6. In [24] Finkelnberg showed that there are 13 different orbits of S_6 acting on the set of all $2^{10} = 1024$ small resolutions of the Segre cubic; 6 of these orbits consist of projective small resolutions while 7 contain only non-projective small resolutions. In particular, we obtain 6 different isomorphism classes of semi-Fano 3–fold with index 2 (weak del Pezzo 3–fold), degree 24, Picard rank 6 and nodal AC morphism. For other nodal cubics with close to the maximal number of nodes the number of non-isomorphic projective small resolutions does not seem to have been determined.

Semi-Fano 3–folds from nodal quartics   Examples 7.3 to 7.6 all give examples of defect 1 semi-Fano 3–folds arising from projective small resolutions of nodal quartics in $\mathbb{P}^4$. Example 7.7 is a defect 15 weak Fano 3–fold associated with a nodal quartic in $\mathbb{P}^4$ (with the maximal number of nodes $e = 45$; moreover, 15 is the maximal possible defect for a terminal quartic 3–fold; see below). There currently does not seem to be a good understanding of semi-Fano 3–folds associated with nodal quartics when the defect is not either 1 or close to the maximum 15. Even for the maximal defect $\sigma = 15$ it does not seem that the number of projective small resolutions of the Burkhardt quartic has been determined. (Recall it has at least one projective small resolution and exactly $2^{45} \simeq 3.5 \times 10^{13}$ Moishezon but not necessarily projective small resolutions).

The following statement summarises some of the main known results about nodal quartics and their defects and projective small resolutions.

Theorem 8.17   *Let X be a nodal quartic in $\mathbb{P}^4$, let e denote the number of nodes of X and $\sigma(X)$ its defect.*

(i) *If $e < 9$ then $\sigma = 0$ and hence X admits no projective small resolutions.*

(ii) If $e = 9$ then $\sigma = 0$ if and only if X contains no planes Π. In particular, a general quartic with $e = 9$ admits no projective small resolutions. If $e = 9$ and X contains a plane Π then $\sigma = 1$ and blowing up Π in X yields a projective small resolution as in Example 7.3.

(iii) If $e < 12$ and X contains no planes then $\sigma = 0$ and hence X admits no projective small resolutions.

(iv) If $e = 12$ then $\sigma = 0$ unless X contains a quadric surface. A sufficiently general quartic containing an irreducible quadric Q_2^2 has precisely 12 nodes all contained in Q_2^2 and has $\sigma = 1$. Blowing up Q_2^2 yields a projective small resolution as in Example 7.4.

(v) $e \leq 45$ with equality if and only if X is projectively equivalent to the Burkhardt quartic as in Example 7.7.

(vi) $\sigma \leq 15$ with equality if and only X is projectively equivalent to the Burkhardt quartic. Moreover, $\sigma \leq 10$ if X contains no planes.

Proof (i) and (ii) are proved in Cheltsov [13, Theorems 2 and 5]. (iii) and (iv) are proved in Shramov [90, Theorem 1.3]. (v): $e \leq 45$ was proved in Varchenko [98]; the case of equality was treated in de Jong–Shepherd-Barron–Van de Ven [43]. (vi) is proved in Kaloghiros [46, Theorem 1.1]; see the erratum for a correction to the original claim of Theorem 1.1.(ii). □

Remark 8.18 The special class of nodal *determinantal quartics* has been studied in some detail. A determinantal quartic is a hypersurface in $\mathbb{P}^4$ given as the zero-locus of the determinant of a 4×4 matrix of linear forms in $[z_0, \ldots, z_4]$. A determinantal quartic is never smooth but generically has only nodes; this makes determinantal quartics a good source of nodal quartics. A nodal determinantal quartic has $20 \leq e \leq 45$ and the generic one has $e = 20$. The Burkhardt quartic is determinantal. Every determinantal quartic is rational and hence any projective small resolution Y has no torsion in $H^3(Y)$. Pettersen's thesis [80] used a particular rationalisation to study nodal determinantal quartics. He gave classification results for nodal determinantal quartics with $e \geq 42$ and showed any such quartic admits at least one projective small resolution. Such resolutions are semi-Fano 3–folds with $H^3(Y)$ torsion-free and thus give rise to building blocks in the sense of Definition 5.1. Pettersen [80, Section 6.2] constructed determinantal quartics with $e = 40$ and $\sigma = 10$ which contain no plane; thus the defect bound from Theorem 8.17(vi) is sharp.

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