On some convex cocompact groups
in real hyperbolic space

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We generalize to a wider class of hyperbolic groups a construction by Misha Kapovich yielding convex cocompact representations into real hyperbolic space.

1 Introduction

In this paper we study discrete cocompact isometry groups of CAT(-1) polygonal complexes, and try to represent them faithfully as convex cocompact groups of $\mathbb{H}^p$ (for some large integer p). The case of a (large) polygon of finite groups was first handled by Misha Kapovich in [19]. In fact the argument of Kapovich generalizes to a much wider class of groups. We now present more precisely our results.

1.1 Background on Coxeter groups and polygonal complexes

Recall that a group $\Gamma < \mathrm{Isom}(\mathbb{H}^p)$ is *convex-cocompact* provided Γ acts properly on $\mathbb{H}^p$ and is cocompact on the convex hull of its limit set. Equivalently, any orbital map $\Gamma \to \mathbb{H}^p$, sending γ to γx, is a quasi-isometric embedding. More generally for any geodesic metric space Y we say that a representation $\rho: \Gamma \to \mathrm{Isom}(Y)$ is *convex-cocompact* whenever there exists a closed convex subspace $Z \subset Y$ which is invariant under Γ and the Γ–action on Z is proper and cocompact. (Recall that a subspace $Z \subset Y$ is *convex* if any geodesic segment with endpoints inside Z is entirely contained inside Z.) When the metric space Y is *hyperbolic in the sense of Gromov* and Γ is convex cocompact in Y, it follows that Γ is *word-hyperbolic* (see Gromov [13] or Ghys and de la Harpe [12] for references on hyperbolic groups). In particular when a group is convex cocompact in $\mathbb{H}^p$ then it is a word-hyperbolic group. The converse problem is then:

What kind of word-hyperbolic groups are convex cocompact in $\mathbb{H}^p$?

The above question is extremely general and we will focus on a very particular class of groups. Any convex cocompact group of $\mathbb{H}^p$ inherits the Haagerup property of $\text{Isom}(\mathbb{H}^p)$ (see Faraut and Harzallah [10]), so we must investigate the class of word-hyperbolic groups with this property.

Recall first that Coxeter groups correspond to presentations of the form

$$W = \langle s_1, \ldots, s_r \mid (s_i)^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Precisely the data here are: a graph $\mathcal{G}$ on the set $\{1, \ldots, r\}$, and for each edge $\{i, j\}$ of $\mathcal{G}$ a natural number $m_{ij} \geq 2$. When $\{i, j\}$ is not an edge and $i \neq j$ there is no relation involving s_i, s_j, by convention we set $m_{ij} = \infty$. When $i = j$ we set $m_{ij} = 1$. The pair $(W, S = \{s_1, \ldots, s_r\})$ is called a *Coxeter system*.

Every Coxeter group has the Haagerup property (see Bozejko, Januszkiewicz and Spatzier [5]). Many Coxeter groups are word-hyperbolic, but very few of these can be discrete cocompact in $\mathbb{H}^p$, since their visual boundary is almost never a sphere. To be more precise a Coxeter group (W, S) is always a discrete cocompact group of automorphism of its Davis complex $\Sigma(W, S)$ (see Davis [8] for the construction of $\Sigma(W, S)$ as a combinatorial object). Moussong proved that $\Sigma(W, S)$ always admits a CAT(0) metric and proved that when W does not contain "obvious" free abelian groups of rank ≥ 2, then $\Sigma(W, S)$ has a CAT(-1) metric, and thus W is word-hyperbolic. See Moussong's thesis [20] for the geometrization of $\Sigma(W, S)$; we refer to Bridson and Haefliger [6] for the general facts on CAT(0) and CAT(-1) metric spaces, which generalize Hadamard manifolds. For example the word-hyperbolicity of W follows from the assumption

$$\text{for all } i, j \text{ with } i \neq j, \ m_{ij} \geq 4.$$

A natural question is then:

Are word-hyperbolic Coxeter groups convex cocompact in some $\mathbb{H}^p$?

This question appears for example in Kapovich's survey [18], where Kapovich notices that some hyperbolic Coxeter groups cannot be realized as convex cocompact *reflection* subgroups, a fact discovered by Felikson and Tumarkin [11]. In his thesis [20] and in [21], Moussong had already given examples of hyperbolic Coxeter groups that are not discrete cocompact in hyperbolic space $\mathbb{H}^p$.

In this paper we obtain the following.

Theorem 1.1 Let $\mathcal{G}$ denote any finite graph of girth ≥ 4. Let $m \geq 4$ be any integer. Let $W(\mathcal{G}, m)$ be the Coxeter group with one generating involution s_v for each vertex v of $\mathcal{G}$, and one relation $(s_v s_w)^m = 1$ for each edge $\{v, w\}$ of $\mathcal{G}$.

Then $W(\mathcal{G}, m)$ is convex-cocompact in some $\mathbb{H}^p$.

(We insist that the CAT(-1) Davis complex $\Sigma(W(\mathcal{G}, m))$ is *two-dimensional*.)

Warning The representation we get is not *by reflections*, in the sense that the generators s_v do not act by reflections on $\mathbb{H}^p$. We do not know whether a (discrete, convex cocompact) reflection-representation exists.

To obtain Theorem 1.1 we first consider a very particular class of two-dimensional Coxeter groups:

Lemma 1.2 *For $m \geq 4$ let $W(p, m)$ denote the Coxeter group with p generating involutions $s_1, \ldots, s_p$, and relations $(s_i s_j)^m = 1$ (for any pair i, j with $i \neq j$).*

Then $W(p, m)$ is convex-cocompact in $\mathbb{H}^{p-1}$. Moreover the convex-cocompact representation $W(p, m) \to \mathrm{Isom}(\mathbb{H}^{p-1})$ extends to a faithful convex-cocompact representation $\mathrm{Aut}(\Sigma(W(p, m))) \to \mathrm{Isom}(\mathbb{H}^{p-1})$.

(The group $\mathrm{Aut}(\Sigma(W(p, m)))$ above is simply the semi-direct product of $W(p, m)$ with the symmetric group of its p generators.)

Our argument consists in simply checking that the Tits form of $W(p, m)$ has signature $(p - 1, 1)$. Note that the Tits–Witt representation sends the generators to reflections.

We are in fact interested in groups that have a priori nothing to do with Coxeter groups: discrete cocompact groups of two-dimensional objects.

For us a *polygonal complex* is a cell complex X of dimension two such that

(i)   the 1–skeleton X^1 is a metric graph where each edge has length 1,

(ii) each 2–cell is a regular euclidean convex polygon (thus called a *polygon of X*),

(iii) the attaching map of a polygon P is a local isometry $\partial P \to X^1$, and

(iv) the link of a 0–cell is a combinatorial graph: it has no loops (this follows from (ii)) and no multiple edges.

The complex is *even-gonal* provided each polygon has an even number of edges. The complex is *k–gonal* provided each polygon has k edges.

For any polygonal complex we consider the following combinatorial quantities:

(i) $n(X)$ denotes the minimum of the number of sides of a polygon of X.

(ii) $\mu(X)$ denotes the minimum of the girth of the links of vertices of X.

It is easily seen that if $n(X) \geq 6$ (respectively, 7) or $\mu(X) \geq 6$ (respectively, 7) then X admits a locally CAT(0) (respectively, CAT(-1)) metric. It is not seldom that discrete cocompact groups of simply-connected polygonal complexes with $n(X) = 3$ have property (T) of Kazhdan, which is strongly opposite to Haagerup property, and thus these kinds of groups cannot be convex cocompact in $\mathbb{H}^p$. We thus concentrate on the case when Γ is a discrete cocompact group of automorphisms of a simply-connected even-gonal complex X with $n(X) \geq 6$. In this case it is well-known that X has natural "hyperplanes" and thus Γ has the Haagerup property.

The last notion we need was introduced by Haglund and Wise in [16], to which we refer for details. In the present paper we say a group Γ is *cubically special* provided there is a right-angled Coxeter group (W, S) (a Coxeter group all of whose finite m_{ij} are equal to 2), and a convex subcomplex $X \subset \Sigma(W, S)$ such that $\Gamma \subset W$, and the CAT(0) cube complex X is invariant and cocompact under Γ. It happens that being cubically special is related to strong subgroup separability properties, where we recall a subgroup is called *separable* if it is an intersection of finite index subgroups. Combining [16, Theorem 7.3 and Theorem 8.13] we obtain, in the context of polygonal complexes, the following.

Theorem 1.3 Let Γ be a discrete cocompact automorphism group of an even-gonal complex X with either $n(X) \geq 8$, or $n(X) \geq 6$ and $\mu(X) \geq 4$. Then the hyperbolic group Γ is virtually cubically special if and only if every quasi-convex subgroup of Γ is separable.

What we call here "cubically special" was simply named "special" (or C–special) in [16]. The reason is that in the present paper we consider CAT(0) complexes built of even polygons that are more general than squares, and we define special actions on these kinds of complexes. Having a special action on an even polygonal complex results in a convex cocompact representation in some related Coxeter group. This group is a 2–dimensional Coxeter group but it is no longer right-angled in general. We have been able to represent this kind of 2–dimensional Coxeter groups into Isom($\mathbb{H}^p$), whereas we couldn't find interesting representations for 2–dimensional right-angled Coxeter groups.

It turns out that in the context of hyperbolic complexes and groups the two notions of special actions are virtually equivalent, although one of them looks like it is a strict generalization of the other.

1.2 Statements of the results

Our main result relates uniform lattices of even-gonal complexes and two-dimensional word-hyperbolic Coxeter groups:

Theorem 1.4 *Let Γ be a uniform lattice of a simply-connected even-gonal complex X with $n(X) \geq 6$ and $\mu(X) \geq 4$. Assume Γ is virtually cubically special.*

There is a Coxeter group (W, S), with $n(X) \leq 2m_{ij}$ for $i \neq j$, and a convex-cocompact faithful representation $\rho\colon \Gamma \to \mathrm{Aut}(\Sigma(W, S))$ whose image is virtually contained in W.

All m_{ij} can be chosen to be finite, and we still get a convex-cocompact representation provided $n(X) \geq 8$.

If furthermore all polygons of X have the same number of sides, say $2m$, then the target Coxeter group W can be choosen to equal $W(p, m)$ (for some large number p).

Combining the above theorem with Lemma 1.2 we obtain the following.

Theorem 1.5 *Let X be a simply-connected $2m$–gonal complex with $m \geq 4$ and $\mu(X) \geq 4$.*

If Γ is a virtually cubically special uniform lattice of X then there is a faithful convex-cocompact representation $\rho\colon \Gamma \to \mathrm{Isom}(\mathbb{H}^p)$.

We note that Theorem 1.1 is an immediate application of the above theorem, since the Davis complex of $W(\mathcal{G}, m)$ satisfies the geometric assumptions, and word-hyperbolic Coxeter groups are virtually cubically special by Haglund–Wise [17].

We also (essentially) recover Kapovich's Theorem on even-gons of finite groups (see Kapovich [19, Theorem 1.1]). Indeed the universal cover of a $2m$–gon of finite groups is a simply-connected $2m$–gonal complex X with bipartite vertex links. It follows that $\mu(X) \geq 4$ and also that Γ is virtually cubically special, according to a deep result of Dani Wise (see Wise [26; 27] and the translation in the cubically special language in Haglund–Wise [16]). So our Theorem 1.5 applies (provided $m \geq 4$). Moreover it can be checked that Kapovich's representation $\Gamma \to \mathrm{Isom}(\mathbb{H}^{p-1})$ actually factors as the composite of our $\Gamma \to \mathrm{Aut}(\Sigma(W(p, m)))$ and our $\mathrm{Aut}(\Sigma(W(p, m))) \to \mathrm{Isom}(\mathbb{H}^{p-1})$. (In fact our construction still yields Kapovich's representation when $m = 3$, but in this case the methods developed in this paper do not allow to reprove that the representation is convex cocompact. We observe this convex cocompactness result was established in Kapovich's paper using an argument that is more elaborate than the one used for the generic case $m \geq 4$.)

We emphasize that both Theorems 1.4 and 1.5 provide a representation of the *whole* group Γ, even when Γ has torsion: it is not a virtual representation.

We want also to insist that when a word-hyperbolic group is virtually cubically special then by definition it is related to some right-angled Coxeter group. But in general this

Coxeter group is neither hyperbolic nor is it two-dimensional, so Theorem 1.4 requires a bit more of work.

Dani Wise has proved that when a CAT(0) polygonal complex X satisfies $n(X) \geq 6$ and has bipartite vertex-links then any uniform lattice of X is virtually cubically special (see Wise [26; 27] and Haglund–Wise [16]). This applies in particular to two-dimensional Tits-buildings (with $n(X) \geq 8$). Note the girth of bipartite graphs is at least four. Thus using Theorem 1.5 we obtain:

Corollary 1.6 *Let X be a simply-connected $2m$–gonal complex with $m \geq 4$. Assume X is a Tits building (that is: all vertex-links are isomorphic to a given generalized v–gon, $v \geq 2$), or more generally all vertex-links of X are bipartite graphs.*

Then any uniform lattice of X has a faithful convex-cocompact representation $\rho \colon \Gamma \to \mathrm{Isom}(\mathbb{H}^p)$.

We note that for $p = 2m \geq 8$ the above applies in particular when $X = I_{pq}$, the right-angled hyperbolic building all of whose chambers are regular right-angled p–gons and all of whose edges are contained in q polygons. In that case Marc Bourdon had already been constructing a convex cocompact representation for a certain graph products of finite groups Γ_{pq} acting geometrically on I_{pq} (see Bourdon [4]). The new thing we get here is that the pair $(I_{(2m)q}, \Gamma_{(2m)q})$ embeds equivariantly into $(\Sigma(W(m,r)), W(m,r))$ for some r (which can be computed more or less explicitly in this case).

The representations in both Theorems 1.4 and 1.5 have a geometric companion.

In Theorem 1.4 there is a combinatorial map $X \to \Sigma(W, S)$ (sending polygons to polygons) which is equivariant under our $\Gamma \to \mathrm{Aut}(\Sigma(W, S))$. This map is an isometric embedding for the combinatorial distances on the 1–skeleta, but not an isometric embedding for the usual CAT(0) metrics. In Lemma 1.2 the representation $W(p, m) \to \mathrm{Isom}(\mathbb{H}^{p-1})$ also has a geometric companion: there is an equivariant map $\Sigma(W(p, m)) \to \mathbb{H}^{p-1}$ such that the image of each polygon of $\Sigma(W(p, m))$ is an isometric copy of some regular $2m$–gon of $\mathbb{H}^2$.

Thus in Theorem 1.5 the representation $\Gamma < \mathrm{Aut}(X) \to \mathrm{Isom}(\mathbb{H}^p)$ is *polygonal* in the sense that there is an equivariant locally injective map $X \to \mathbb{H}^p$ sending each polygon of X to a *regular planar polygon* of $\mathbb{H}^p$. A priori one could search for more general convex-cocompact representations, by requiring for example that there exists an equivariant locally injective map $X^1 \to \mathbb{H}^p$ that maps two "parallel" edges of X^1 to two segments bisected by the same hyperplane of $\mathbb{H}^p$.

1.3 Using "virtually Haken"

Shortly after this paper was submitted Ian Agol proved that any uniform lattice of a Gromov-hyperbolic CAT(0) cube complex is in fact virtually cubically special (see Agol, Groves and Manning [1]). Since the square subdivision of the polygonal complexes we consider are hyperbolic CAT(0) square complexes we can combine Agol's Theorem together with Theorem 1.5:

Theorem 1.7 *Let X be a simply-connected $2m$–gonal complex with $m \geq 4$ and $\mu(X) \geq 4$.*

Let Γ be a uniform lattice of X. Then Γ has a faithful convex-cocompact representation $\rho \colon \Gamma \to \mathrm{Isom}(\mathbb{H}^p)$.

1.4 Concluding remarks and questions

The results in this paper show that the Coxeter groups $W(p, m)$ contain as quasi-convex subgroups a lot of groups acting geometrically on negatively curved polygonal complexes. Similarly the polygonal complexes $\Sigma(W(p, m))$ contain (equivariantly) a lot of homogeneous $2m$–gonal complexes. This is a remarkable universality property of the so-called *Gromov polyhedra* $\Sigma(W(p, m))$ (see Haglund [14]).

We obtain a class of word-hyperbolic Coxeter groups generalizing $W(p, m)$ by demanding that all m_{st} be finite but allowing them to vary. Easy signature computations show that when the variation of the m_{st} is "small" with respect to the number of generators then the Tits form is still of signature $(p-1, 1)$. We thus propose the following.

Conjecture 1 *If a Coxeter group (W, S) has $4 \leq m_{st} < +\infty$ for every $s \neq t$, then W is convex-cocompact in $\mathbb{H}^p$.*

However we warn the reader that one cannot hope the above representations would be by reflections. We give a family of counterexamples in Corollary 4.25, where the coefficients m_{st} take two distinct values.

Conjecture 1, together with Theorem 1.4 and Agol's Theorem, would imply the following conjecture.

Conjecture 2 *Any uniform lattice of a simply-connected even-gonal complex X, with $n(X) \geq 8$ and $\mu(X) \geq 4$ is convex-cocompact in $\mathbb{H}^p$.*

In fact Conjecture 1 and the quotient Lemma 5.13 could be used to deduce the following conjecture.

Conjecture 3   *If a Coxeter group (W, S) has $4 \leq m_{st} \leq +\infty$ for every $s \neq t$, then W is convex-cocompact in $\mathbb{H}^p$.*

We would like to get rid of the assumption $\mu(X) \geq 4$ in Conjecture 2 above. However it is not always possible to map polygonally a CAT(-1) $2m$–gonal complex X with $\mu(X) = 3$ to a Davis complex $\Sigma(W(p, m))$. Indeed X may have non-trivial *holonomy* in the following sense (see Haglund [14]). There is a $2m$–gon P in X and a sequence $P_1, P_2, \ldots, P_{2m}$ of polygons such that:

(i) $P_1 \cap P, \ldots, P_{2m} \cap P$ are the consecutive edges of ∂P.

(ii) $P_1 \cap P_2$ is an edge, $P_2 \cap P_3$ is an edge, $\ldots$, $P_{2m-1} \cap P_{2m}$ is an edge (these edges are adjacent to P).

(iii) $P_{2m} \cap P_1$ is a vertex (of P).

Question   If Γ is a virtually special uniform lattice of a *holonomy free* $2m$–gonal complex X with $m \geq 4$, does the pair (X, Γ) embed equivariantly in some pair $(\Sigma(W(p, m)), W(p, m))$?

A positive answer applied to certain of the counterexamples in Corollary 4.25 would imply the following.

Conjecture 4   *There are word-hyperbolic Coxeter groups (W, S) which are convex-cocompact in some $\mathbb{H}^p$, but which do not have faithful reflection-representations in any $\mathbb{H}^r$.*

1.5   Organization of the paper

In Section 2 our goal is to describe and study the hyperplanes and ramified hyperplanes in even-gonal complexes. We conclude this section by giving some more definitions on Coxeter groups.

In Section 3 we give local small-cancellation type conditions on a polygonal map $f \colon X \to Y$ that force $f \colon X^1 \to Y^1$ to be a global isometric embedding (Proposition 3.2).

In Section 4 we present more or less classical facts about real hyperbolic reflection groups. In particular we give necessary and sufficient conditions on a Poincaré polyhedron under which the associated reflection group is convex cocompact (Theorem 4.7). We then recall the definition of the Witt–Tits quadratic form. Applying Theorem 4.7 we then explain when the Witt–Tits representation is convex-cocompact in real hyperbolic space (under the assumption that the Witt–Tits quadratic form has hyperbolic signature; see Proposition 4.15). As an application we deduce Lemma 1.2 (see Corollary 4.22)

In [Section 5](#) we define special actions on polygonal complexes. We show that if a uniform lattice Γ of an even-gonal complex X with $\mu(X) \geq 4$ acts specially then there is a naturally associated 2–dimensional Coxeter system (W, S) and an isometric embedding of the pair (X, Γ) inside $(\Sigma(W, S), W)$. We check that if Γ is virtually cubically special then it also has a finite index subgroup which acts specially on the polygonal complex X (see [Corollary 5.4](#)), and in fact we can even get arbitrarily good lower bounds for the embedding radius of ramified hyperplanes. Combining this result with [Lemma 1.2](#) and [Proposition 3.2](#) we conclude this section with a proof of [Theorem 1.4](#).

2 Geometry of even-gonal complexes

(In this section all polygonal complexes are locally compact.)

2.1 Non-positive curvature conditions

Definition 2.1 (Piecewise euclidean metric) Let X be a polygonal complex.

By definition the 2–cells of X have a euclidean metric (and each edge of a polygon of X has unit length). We may thus consider on X the induced length metric, which we call the *piecewise euclidean metric* on X.

Let X, Y be polygonal complexes. A map $f \colon X \to Y$ is *combinatorial* provided it sends isometrically an edge (respectively, a polygon) of X onto an edge (respectively, a polygon) of Y.

Definition 2.2 (Non-positive and negative curvature conditions) We say a polygonal complex is *non-positively curved* provided its piecewise euclidean metric is locally $\mathrm{CAT}(0)$.

We say a polygonal complex is *negatively curved* provided it is non-positively curved, and moreover the piecewise euclidean metric has no embedded euclidean open disk through vertices.

Let X be a polygonal complex. For each vertex v of X and each k–gon P through v we assign the length $\alpha(v, P) = \pi(1 - \frac{2}{k})$ to the edge of $\mathrm{link}(v, X)$ corresponding to

P. This turns $\text{link}(v, X)$ to a metric graph and we denote by $\mu_{\text{eucl}}(v, X)$ its systole. Specifically $\mu_{\text{eucl}}(v, X)$ is the infimum of all sums $\alpha(v, P_1) + \cdots + \alpha(v, P_\ell)$, where $P_1, \ldots, P_\ell$ is a locally injective cycle of polygons around v. We set $\mu_{\text{eucl}}(X) = \inf_{v \in X^0} \mu_{\text{eucl}}(v, X)$.

We quote the following well-known result relating curvature and systole:

Lemma 2.3 (Bridson–Haefliger [6, Lemma 5.6]) *Let X be a polygonal complex. Then X is non-positively curved (negatively curved) if and only if $\mu_{\text{eucl}}(X) \geq 2\pi$ ($> 2\pi$).*

If X is non-positively curved and simply connected it is a CAT(0) space (see Bridson–Haefliger [6]) and thus we say X is a CAT(0) *polygonal complex*.

Definition 2.4 (Local largeness) We say X is *large at vertices* if the minimum of the girths of vertex-links is at least four: $\mu(X) \geq 4$.

We say X is *large-gonal* if the smallest number of sides of a polygon is at least six: $n(X) \geq 6$.

We say X is *locally large* if it is large at vertices and moreover the smallest number of sides of a polygon is at least four: $\mu(X) \geq 4$ and $n(X) \geq 4$.

Using Lemma 2.3 we have the following relations between local largeness conditions and curvature conditions:

Lemma 2.5 *Let X be a polygonal complex.*

  (i) *Assume X is large-gonal. Then X is non-positively curved .*

  (ii) *Assume X is large at vertices. Then X is non-positively curved if and only if X is locally large. If furthermore X is even-gonal, then X is negatively curved if and only if X is large-gonal.*

Since we do not really use the CAT(0) metric in this article the following could be considered to be a definition:

Lemma 2.6 (Combinatorial characterization of local isometries) *Let X, Y be polygonal complexes with $n(Y) \geq 4$. A combinatorial map $f\colon X \to Y$ is a local isometry if and only if f is locally injective and moreover $f(\text{link}(v, X))$ is a full subgraph of $\text{link}(f(v), Y)$.*

Proof The statement is local, so it suffices to prove it for CW–complexes X, Y, where X is a subcomplex of Y, and Y has a single vertex v and is obtained as a union of euclidean planar convex angular sectors, glued along rays at v. By the assumption "$n(Y) \geq 4$" the angle at the vertex of these sectors are $\geq \frac{\pi}{2}$. Using that the sectors are non-acute and the so-called Alexandrov's Lemma (see Bridson–Haefliger [6, Lemma 2.6]) we then make the following key observation:

Let p, p' be two points on distinct rays r, r' at v, with $p \neq v$ and $p' \neq v$. If $r \cup r'$ does not bound an angular sector of Y, then the geodesic between p, p' is the union of two segments contained inside $r \cup r'$.

If $\mathrm{link}(v, X)$ is not a full subgraph of $\mathrm{link}(v, Y)$ then there are two rays r, r' of X at v whose images in Y are boundary rays of some sector S of Y, but $r \cup r'$ does not bound a sector of X. Then for any two points $p \in r - \{v\}$ and $p' \in r' - \{v\}$ there is a shorter path in S between p and p', and so X is not isometrically embedded.

Assume conversely X is not isometrically embedded in Y. So there are points p, p' in X with $d_Y(p, p') < d_X(p, p')$. Note first $p \neq v$ and $p' \neq v$. A geodesic γ of Y between p and p' is a finite concatenation of geodesic segments of the sectors. So we may assume p, p' belong to rays r, r' at v. Applying once more the observation above we deduce that there must exist an angular sector in Y bounding $r \cup r'$. □

So the advantage when working with triangle-free polygonal complexes is that usual CAT(0) notions have a combinatorial characterization.

A subcomplex $Y \subset X$ is *locally convex* if the inclusion map $Y \to X$ is a local isometry. It is a standard fact that a local isometry with CAT(0) target is in fact a global isometry from its CAT(0) source onto its image (see Bridson–Haefliger [6, Proposition 4.14]). So when X is a CAT(0) polygonal complex any locally convex subcomplex $Y \subset X$ is in fact a geodesically convex subspace, and thus we say Y is a *convex subcomplex*.

2.2 Square subdivision

In this paper we will employ the expression *square complex* instead of the heavier 4–*gonal complex*.

Let P denote a euclidean convex polygon with n cyclically ordered vertices $v_1, \ldots, v_n$ (and the usual convention that $v_{n+1} = v_1$). Add an additional vertex v_0 in the interior of P, subdivide each edge $[v_i v_{i+1}]$ of ∂P by adding a vertex w_i in its interior, and then add the edges $[v_0 w_1], \ldots, [v_0 w_n]$. The resulting square complex is the *square subdivision of* P and is denoted by $P_\square$. So $P_\square$ consists of n unit squares.

Let X denote any polygonal complex. We may perform the square subdivision of each polygon of X individually, and then glue the subdivided polygons along the (subdivided) edges coming from the 1–skeleton of X, thus producing a square complex which we call the *square subdivision of X* and which we denote by $X_\square$. Note a vertex v of X is still a vertex of $X_\square$, and moreover we have $\mathrm{link}(v, X) = \mathrm{link}(v, X_\square)$.

We then observe that the square subdivision $X_\square$ is non-positively curved if and only if X is locally large ($n(X) \geq 4$ and $\mu(X) \geq 4$). We leave the verification of the following easy result to the reader:

Lemma 2.7 *Let $f\colon X \to Y$ be a combinatorial map of polygonal complexes. Then there is a naturally induced combinatorial map $f_\square\colon X_\square \to Y_\square$. If moreover $n(Y) \geq 4$ then $f_\square$ is a local isometry if and only if f is a local isometry.*

2.3 Straight and ramified hyperplanes of even-gonal complexes

Hyperplanes in even-gonal complexes have been used since a long time; see for example Haglund [14]. To define his representations Kapovich introduced what we call below *ramified hyperplanes* (see Kapovich [19]).

Definition 2.8 (Hyperplanes of an even-gon) Let P be an polygon with $2m$ edges, and let $e_1, e_2, \ldots, e_{2m}$ be a cyclic enumeration of the edges of P. As usual we consider the indices modulo $2m$ so that $e_{2m+1} = e_1$.

Let e be an edge of P: the *radial segment of P at e* is the straight euclidean segment joining the center of P to the midpoint of e.

We say e_i and e_{i+m} are *parallel in P*. So P has m parallelism classes of edges. A *straight hyperplane of P* (or simply a *hyperplane of P*) is the union of radial segments at e_i and e_{i+m}. Thus P has m hyperplanes.

Any two edges $e_i, e_{i+2\ell}$ whose indices differ by an even number are said to be *even-parallel in P*. Since P is an even-gon e_1 and e_2 are not even-parallel, and P has exactly two even-parallelism classes of edges. A *ramified hyperplane of P* is the union of radial segments at $e_i, e_{i+2}, \ldots, e_{i+2k}, \ldots$. Thus P has two ramified hyperplanes.

We will use the expression *straight hyperplanes* to insist on the difference with ramified hyperplanes. When we use the word hyperplane alone, it always means *straight hyperplane*. Note that when P is a square ramified hyperplanes are straight.

Definition 2.9 (Hyperplanes of an even-gonal complex) Let X be an even-gonal complex. The disjoint union of the hyperplanes of polygons of X naturally maps to X:

$$\bigsqcup_{\substack{h \text{ hyperplane of } P \\ P \text{ polygon of } X}} h \to X$$

We then identify two points $p, q \in \bigsqcup_{h,P} h$ if they have the same image in X and moreover $p = q$ is the midpoint of an edge of X. The resulting quotient graph $\mathcal{H}(X)$ is the *space of hyperplanes of X*. Its connected components are the *immersed hyperplanes of X*. Since radial segments have a length each immersed hyperplane is a metric graph.

Definition 2.10 (Hyperplanes neighborhoods) Let X be an even-gonal complex. The disjoint union of the polygons of X marked by their hyperplanes naturally maps to X:

$$\bigsqcup_{\substack{P \text{ polygon of } X \\ h \text{ hyperplane of } P}} P \times \{h\} \to X$$

and we identify two edges a in $P \times \{h\}$ and a' in $P' \times \{h'\}$ if a, a' map to the same edge in X and moreover the midpoint of $a = a'$ is an extremity of both h and h'. We denote by $N(\mathcal{H}(X))$ the resulting quotient polygonal complex. There is a natural map $i\colon \mathcal{H}(X) \to N(\mathcal{H}(X))$, and it is compatible with both maps $\mathcal{H}(X) \to X$ and $N(\mathcal{H}(X)) \to X$.

Lemma 2.11 *The family of orthogonal projections $P \times \{h\} \to h$ induces a map $r\colon N(\mathcal{H}(X)) \to \mathcal{H}(X)$ such that $r \circ i = \mathrm{id}$. In particular i is injective and it induces a 1–1 identification between the connected components of $\mathcal{H}(X)$ and the connected components of $N(\mathcal{H}(X))$.*

For every immersed hyperplane H of X we will thus denote by $N(H)$ the connected component of $N(\mathcal{H}(X))$ containing H, and we call $N(H)$ the *polygonal neighborhood of H*.

Lemma 2.12

(i) *Both maps $i\colon \mathcal{H}(X) \to N(\mathcal{H}(X))$ and $\mathcal{H}(X) \to X$ are local isometries.*

(ii) *Assume the even-gonal complex X is large at vertices ($\mu(X) \geq 4$), then $N(\mathcal{H}(X)) \to X$ is a local isometry.*

Sketch proof The retraction map $r\colon N(\mathcal{H}(X)) \to \mathcal{H}(X)$ is clearly 1–Lipschitz, thus $i\colon \mathcal{H}(X) \to N(\mathcal{H}(X))$ is a local isometry. Since $N(\mathcal{H}(X)) \to X$ is a local isometry in the neighborhood of every point of $\mathcal{H}(X)$, it thus follows by composition that $\mathcal{H}(X) \to X$ is a local isometry as well.

Assume now $\mu(X) \geq 4$ and let $\overline{v}$ be a vertex of $N(\mathcal{H}(X))$, and denote by v its image in X. Let P be a polygon of X containing v, and let h be a hyperplane of P such that $\overline{v}$ is the image of $v \times \{h\}$. We let a, a' denote the two edges of ∂P which are perpendicular to h at their midpoint. If $v \notin a \cup a'$, then in fact $\mathrm{link}(\overline{v}, N(\mathcal{H}(X)))$ identifies with $\mathrm{link}(v, P)$, and this edge is obviously a full subgraph of $\mathrm{link}(v, X)$. Else $v \in a \cup a'$, say $v \in a$. Consider the polygons $P_1 = P, \ldots, P_k$ of X which contain a. Then $\mathrm{link}(\overline{v}, N(\mathcal{H}(X)))$ identifies with $\bigcup_i \mathrm{link}(v, P_i) \simeq \mathrm{Star}(a, \mathrm{link}(v, X))$. Now any subgraph of diameter two in a graph of girth ≥ 4 has to be full. $\square$

Using the previous lemma and the standard incompressibility of local isometries in non-positive curvature we get:

Corollary 2.13 (Haglund–Paulin [15]) *Let X be a CAT(0) even-gonal complex. Then every immersed hyperplane embeds in X. Its image is a convex subtree of X that disconnects X into two connected components.*

If moreover X is large at vertices then the union of polygons of X meeting a given hyperplane is a convex subcomplex.

Another classical fact is that the hyperplanes of a CAT(0) polygonal complex explain the combinatorial distance on the 1–skeleton:

Theorem 2.14 *Let X be a CAT(0) even-gonal complex. The combinatorial distance between two vertices of the 1–skeleton X^1 equals the number of hyperplanes separating the vertices. Moreover an edge-path $(e_1, \ldots, e_\ell)$ of X^1 is a combinatorial geodesic between its endpoints if and only if the sequence of hyperplanes it crosses at each edge e_i has no repetition.*

We refer to Haglund–Paulin [15] for an argument. Sometimes using hyperplanes is easier, so the combinatorial distance on X^1 is more adapted than the CAT(0) distance on X. Note however that if there is an upper bound for the number of vertices in a polygon and on the number of edges containing a vertex in the CAT(0) polygonal complex X, then the inclusion $(X^1, d_{\mathrm{comb}}) \to (X, d_{\mathrm{eucl}})$ is a quasi-surjective quasi-isometry. The upper bounds clearly exist if X admits a cocompact group of automorphism.

We now define the (immersed) ramified hyperplanes of an even-gonal complex.

Definition 2.15 (Ramified hyperplanes and their neighborhoods)   Let X be a non-positively curved even-gonal complex. The disjoint union of the ramified hyperplanes of polygons of X naturally maps to X:

$$\bigsqcup_{\substack{h \text{ a ramified hyperplane of } P \\ P \text{ polygon of } X}} h \to X$$

and identify two points $p, q \in \bigsqcup_{h,P} h$ if they have the same image in X and moreover $p = q$ is the midpoint of an edge of X. The resulting quotient graph $\mathcal{H}_r(X)$ is the *space of ramified hyperplanes of X*. Its connected components are the *immersed ramified hyperplanes of X*.

Consider also:

$$\bigsqcup_{\substack{P \text{ polygon of } X \\ h \text{ a ramified hyperplane of } P}} P \times \{h\} \to X$$

and identify two edges a in $P \times \{h\}$ and a' in $P' \times \{h'\}$ if a, a' map to the same edge in X and moreover the midpoint of $a = a'$ belong to both h and h'. We denote by $N(\mathcal{H}_r(X))$ the resulting quotient polygonal complex. There is a natural map $i\colon \mathcal{H}_r(X) \to N(\mathcal{H}_r(X))$, and it is compatible with both maps $\mathcal{H}_r(X) \to X$ and $N(\mathcal{H}_r(X)) \to X$.

In order to study the immersed ramified hyperplanes of the non-positively curved even-gonal complex X, it is convenient to work with the square subdivision $X_\square$ of X. Indeed we note that a ramified hyperplane h of an even-gon P is a subcomplex of $P_\square$. This turns h to a metric graph whose edges have unit length. Note there is a retraction $P_\square \to h$ which projects orthogonally each square C of $P_\square$ onto its unit edge $C \cap h$.

We will consider the induced combinatorial maps

$$\mathcal{H}_r \to X_\square, \quad \mathcal{H}_r \to N_\square(\mathcal{H}_r(X)), \quad N_\square(\mathcal{H}_r(X)) \to X_\square,$$

where $N_\square(\mathcal{H}_r(X))$ denotes the square subdivision of $N(\mathcal{H}_r(X))$.

Lemma 2.16   *The family of retractions $P_\square \times \{h\} \to h$ induces a map $r\colon N_\square(\mathcal{H}_r(X)) \to \mathcal{H}_r(X)$ such that $r \circ i = \mathrm{id}$. In particular i is injective and it induces a 1–1 identification between the connected components of $\mathcal{H}_r(X)$ and the connected components of $N(\mathcal{H}_r(X))$.*

For every immersed ramified hyperplane H of X we will denote by $N(H)$ the connected component of $N(\mathcal{H}_r(X))$ containing H, and we call $N(H)$ the *polygonal neighborhood of H*. We also consider the square subdivision $N_\square(H) \subset N_\square(\mathcal{H}_r(X))$.

Lemma 2.17 *Let X be an even-gonal complex with $\mu(X) \geq 4$. For any immersed ramified hyperplane H the combinatorial maps $H \to N_\square(H)$, $N_\square(H) \to X_\square$ and $H \to X_\square$ are local isometries.*

Proof The retraction $r \colon N_\square(\mathcal{H}_r(X)) \to \mathcal{H}_r(X)$ is 1–Lipschitz, thus in fact $H \to N_\square(H)$ is a global isometric embedding.

For every vertex $\overline{v}$ of $N(\mathcal{H}_r(X))$ let v be its image inside X, and let P be a polygon of X with a ramified hyperplane h such that $\overline{v}$ is the image of a vertex of $P \times \{h\}$. Then v is contained in precisely one edge a of P that intersects h. The natural map $N(\mathcal{H}_r(X)) \to X$ then induces an identification of $\operatorname{link}(\overline{v}, N(\mathcal{H}_r(X)))$ with $\operatorname{Star}(a, \operatorname{link}(v, X))$. This is a full subgraph of $\operatorname{link}(v, X)$ since $\mu(X) \geq 4$. Thus $N(\mathcal{H}_r(X)) \to X$ is a local isometry.

It follows by Lemma 2.7 that $N_\square(H) \to X_\square$ is a local isometry and by composition so is $H \to X_\square$. $\qquad \square$

Corollary 2.18 *Let X be a CAT(0) even-gonal complex with $\mu(X) \geq 4$.*

- (i) *Each ramified hyperplane H embeds as a convex subcomplex of $X_\square$. Moreover $N_\square(H)$ is a convex subcomplex of $X_\square$, and $N(H)$ is also a convex subcomplex of X.*

- (ii) *Let a, b be two distinct edges at a vertex v. Then the (ramified) hyperplanes H, K cutting a, b are distinct. More precisely if a, b are linked in $\operatorname{link}(v, X)$ then $H \cap K$ is the center of the polygon spanned by a and b, and if a, b are not linked in $\operatorname{link}(v, X)$ then $H \cap K = \varnothing$.*

Proof (i) By Lemma 2.17 the map $H \to X_\square$ is a local isometry, and by assumption its target is CAT(0). It follows that H embeds as a convex subcomplex of $X_\square$.

Similarly we get that $N_\square(H)$ is a convex subcomplex of $X_\square$, and we deduce by Lemma 2.7 that $N(H)$ is a convex subcomplex of X.

(ii) Assume first a, b span a polygon P. Let r, s be the radial segments of P at a, b. So H is the (ramified) hyperplane that contains r and K is the (ramified) hyperplane that contains r. By convexity of $H \subset X_\square$ we see that H cannot contain s. It follows that $H \cap P$ is the (ramified) hyperplane of P containing r. Similarly $K \cap P$ is the (ramified) hyperplane of P containing s. So $H \cap K \cap P$ is the center p of P. Since $H \cap K$ is convex and contains p as an isolated point it follows that $H \cap K = \{p\}$.

Assume now that a, b are unlinked in $\operatorname{link}(v, X)$. Thus the union π of the two half-edges of a, b containing v is a geodesic segment of X. The endpoints of π are in H, K but π itself is not contained in a (ramified) hyperplane since it contains a vertex. By convexity of (ramified) hyperplanes we deduce that $H \neq K$. $\qquad \square$

We note that the hexagonal tesselation of the euclidean plane has only three ramified hyperplanes, which are not at all simply-connected. So the assumption $\mu(X) \geq 4$ is essential in the Lemma above.

2.4 Intersection and osculation of hyperplanes

Proposition 2.19 (Intersections) *Let X be a* CAT(0) *even-gonal complex (with $\mu(X) \geq 4$), and let H, H' be two (ramified) hyperplanes of X. Then either:*

(i) $N(H) \cap N(H') = \varnothing$.

(ii) $N(H) \cap N(H')$ *is a non empty subgraph of* X^1.

(iii) $N(H) \cap N(H')$ *is the union of a single polygon P with a subgraph of X^1.*

(iv) $H = H'$.

In case (ii) *we have $H \cap H' = \varnothing$. In case* (iii) *we have $H \cap H' = \{p\}$, where p is the center of P. Moreover under the assumption $\mu(X) \geq 4$ we have $N(H) \cap N(H') = P$.*

Proof   Assume $N(H) \cap N(H') \neq \varnothing$. Both (ramified) hyperplanes H, H' are convex subspaces of X (of $X_\square$). Thus $H \cap H'$ is a convex subtree of H (the edges are the radial segments). We note that $N(H) \cap N(H')$ contains a polygon if and only if $H \cap H'$ contains the center of this polygon: in particular if $H \cap H' = \varnothing$ then $N(H) \cap N(H')$ is a subgraph of X^1.

If $H \cap H'$ contains a point that is not the center of a polygon, then $H \cap H'$ contains a radial segment and thus $H = H'$. So if $H \cap H'$ contains two distinct points then $H = H'$. It follows that if $H \cap H' \neq \varnothing$ then either $H = H'$ or there is a polygon P with center p such that $H \cap H' = \{p\}$, in which case P is obviously the single polygon contained inside $N(H) \cap N(H')$.

Assume $\mu(X) \geq 4$ and $H \cap H'$ is the center p of a polygon P. We know P is contained in the convex subcomplex $N(H) \cap N(H')$. We claim that P is a connected component of $N(H) \cap N(H')$, which implies $N(H) \cap N(H') = P$. So assume by contradiction that a is an edge of $N(H) \cap N(H')$ such that $a \cap P$ is a vertex v. Since $a \subset N(H)$ there is a polygon Q of $N(H)$ such that $a \subset Q$. Since P, Q are non-disjoint polygons of $N(H)$ it follows that $P \cap Q$ is an edge b (with $b \cap H \neq \varnothing$). Similarly there is a polygon Q' of $N(H')$ which contains a and such that $P \cap Q'$ is an edge b' with $b' \cap H' \neq \varnothing$. If $b = b'$ then $H = H'$, contradicting $H \cap H' = \{p\}$. So we have found a cycle of length 3 in link(v, X): this contradicts $\mu(X) \geq 4$. $\square$

Definition 2.20 (Intersecting, crossing, osculating) Let H, H' be two (ramified) hyperplanes of a CAT(0) even-gonal complex (with $\mu(X) \geq 4$). We say H, H' *intersect* if $H \cap H' \neq \varnothing$. We say H, H' *cross* if they intersect and are distinct. We say H, H' *osculate* if $N(H) \cap N(H') \neq \varnothing$, but H, H' do not intersect.

By Proposition 2.19 H, H' cross if and only if $H \cap H'$ is the center of a polygon and H, H' osculate if and only if $N(H) \cap N(H')$ is a non-empty subgraph of X^1.

Lemma 2.21 (Nearby hyperplanes) *Let X be a CAT(0) even-gonal complex with $\mu(X) \geq 4$, and let a, b be two distinct edges of X adjacent to some vertex v. Let H, K denote the (ramified) hyperplanes dual to a, b. Then $H \neq K$ and moreover:*

(i) *H and K intersect if and only if a, b are adjacent vertices of $\text{link}(v, X)$.*

(ii) *H and K osculate if and only if a, b are non-adjacent vertices of $\text{link}(v, X)$.*

Proof Clearly $N(H) \cap N(K)$ contains v. So H, K either intersect or they osculate, and the Lemma follows from Corollary 2.18(ii). □

Remark 2.22 (The dual cubulation) Let X be a CAT(0) even-gonal complex. Using the family of hyperplanes of X there is a naturally defined embedding of graphs $X^1 \to C^1$, where C is a CAT(0) cube complex (see Nica [23] and Chatterji–Niblo [7]). Moreover the hyperplanes of X are in 1–1 correspondence with the hyperplanes of C, through the identification of parallelism classes of edges in both X and C. The cubes of C correspond to certain configurations of pairwise intersecting hyperplanes.

It is easy to describe C when X is locally large. For each $2m$–gon P of X the m hyperplanes of X cutting P are pairwise intersecting, and in C we must add an m–cube containing ∂P as a combinatorial equator. These are the only (maximal) cubes to add to X^1 in order to get C in that case.

Indeed since X is locally large the hyperplanes are convex subcomplexes of the CAT(0) complex $X_\square$ and when they intersect they meet with right angle. We deduce that if $H_1, H_2, \ldots, H_k$ is any family of pairwise intersecting hyperplanes of X, then there is a polygon P such that all hyperplanes H_i of the family cut P.

2.5 Two-dimensional and two-spherical Coxeter groups

Definition 2.23 (The Cayley graph, the Cayley 2–complex and the Davis complex) Let (W, S) be a Coxeter system.

The *Cayley graph of* (W, S) is the graph $\mathcal{G}(W, S)$ defined as follows: The set of vertices is W. There is an edge between w and ws_i for any $w \in W$ and any generator

$s_i \in S$. Note that edges are labeled in S (or in I). Observe this version of Cayley graphs is adapted to a group generated by involutions since it has a single edge between two adjacent vertices (unlike in the classical definition).

We now describe the *Cayley 2–complex of* (W, S). This is an even-gonal complex $\Sigma^2(W, S)$ obtained as follows. The 1–skeleton of $\Sigma^2(W, S)$ is the Cayley graph $\mathcal{G}(W, S)$. For any pair i, j with $m_{ij} < \infty$ and any $2m_{ij}$–cycle c whose labels alternate between i and j there is a polygon bounding c.

The *Davis complex* of (W, S) is a polyhedral complex $\Sigma(W, S)$ whose 2–skeleton is the Cayley complex $\Sigma^2(W, S)$, with additional higher-dimensional cells corresponding to (cosets of) finite subgroups generated by subsets of S (of cardinality > 2). For more details see Davis [8].

Observe that the Cayley 2–complex is connected and simply-connected. In his thesis [20], Moussong defined a piecewise euclidean metric on the Davis complex, and showed that it is indeed CAT(0). In particular the Davis complex is contractible.

Observe that the left multiplication induces an action of W by polygonal automorphisms onto $\Sigma^2(W, S)$. This action is simply-transitive on vertices.

Definition 2.24 (2–dimensional) We say a Coxeter system (W, S) is 2–*dimensional* provided for any subset $T \subset S$ with $|T| \geq 3$ the Coxeter system (W_T, T) is infinite. This is equivalent to requiring that the Cayley 2–complex equals the Davis complex. In this case $\Sigma^2(W, S)$ is already a CAT(0) even-gonal complex.

When (W, S) is 2–dimensional we will thus omit the superscript 2, write $\Sigma(W, S)$ instead of $\Sigma^2(W, S)$, and refer to this polygonal complex as to the Davis complex.

Definition 2.25 (2–spherical) A Coxeter system (W, S) where all the m_{ij} are finite is called 2–*spherical*. This is equivalent to demanding that the vertex links in $\Sigma^2(W, S)$ be complete graphs.

Without using Moussong's criterion a very simple systole computation in the links of vertices gives:

Lemma 2.26 Let (W, S) be a Coxeter system such that $m_{ij} \geq 3$ for all i, j. Then (W, S) is 2–dimensional (in other words $\Sigma^2(W, S)$ is CAT(0) and equal to $\Sigma(W, S)$).

Moreover (W, S) is word-hyperbolic iff $\Sigma(W, S)$ is negatively curved, which happens exactly when there is no subset $\{i, j, k\} \subset S$ with $m_{ij} = m_{jk} = m_{ki} = 3$.

We now recall the key-point about the combinatorial geometry in Coxeter systems (for these classical results see Bourbaki [3]). Recall first that for each edge e of the Cayley graph $\mathcal{G}(W, S)$ joining elements w and ws, the element $t := wsw^{-1}$ is called a *reflection of* (W, S). The set of fixed points of the reflection t is the set of midpoints of certain edges (including e), it is called the *hyperplane dual to* e. We will denote it by H_e or H_t. Then $\mathcal{G}(W, S) \setminus H_e$ has two connected components.

Theorem 2.27 (Hyperplane characterization of combinatorial geodesics) *Suppose that $(e_1, e_2, \ldots, e_n)$ is an edge-path in the Cayley graph of a Coxeter system (W, S), and let $H_1, \ldots, H_n$ be the hyperplanes of $\mathcal{G}(W, S)$ dual to $e_1, \ldots, e_n$.*

Then $(e_1, e_2, \ldots, e_n)$ is a combinatorial geodesic if and only if there is no repetition in the sequence $(H_1, \ldots, H_n)$. The combinatorial distance between two vertices is the number of hyperplanes which separate them.

We conclude this section by describing geometrically a natural finite extension of a Coxeter group. So let (W, S) be a 2–dimensional Coxeter system, so that the Davis complex $\Sigma(W, S)$ is a nice CAT(0) even-gonal complex.

We denote by $\mathrm{Aut}_{\mathrm{diag}}(W, S)$ the finite group of permutations $f \colon S \to S$ such that for each i, j we have $m_{f(s_i)f(s_j)} = m_{ij}$. There is a natural embedding $\mathrm{Aut}_{\mathrm{diag}}(W, S) \to \mathrm{Aut}(W)$, and the corresponding automorphisms of W are called *diagram automorphisms*. We denote by $\widehat{W}$ the semi-direct product of W with $\mathrm{Aut}_{\mathrm{diag}}(W, S)$ (this is an abuse of notation since $\widehat{W}$ depends on S). Note the diagram automorphisms preserve the set of generating reflections, so the action of $\mathrm{Aut}_{\mathrm{diag}}(W, S)$ onto W extends to an action on the Davis complex $\Sigma(W, S)$.

Lemma 2.28 (Identifying $\widehat{W}$) *The group $\mathrm{Aut}_{\mathrm{diag}}(W, S)$ normalizes W inside the group $\mathrm{Aut}(\Sigma(W, S))$, and the action by conjugation of $f \in \mathrm{Aut}_{\mathrm{diag}}(W, S)$ onto $W < \mathrm{Aut}(\Sigma(W, S))$ coincides with the natural action by diagram-automorphisms.*

The normalizer of W in $\mathrm{Aut}(\Sigma(W, S))$ is generated by W and $\mathrm{Aut}_{\mathrm{diag}}(W, S)$, and it is isomorphic with $\widehat{W}$.

Proof For $f \in \mathrm{Aut}_{\mathrm{diag}}(W, S)$ let $\widehat{f}$ denote the corresponding automorphism of $\Sigma(W, S)$. Let $w \in W$ and let $x \in \Sigma(W, S)$. Then

$$\widehat{f}(wx) = f(w)\,\widehat{f}(x).$$

Indeed this relation is true for $x \in \Sigma^0(W, S) = W$ and moreover the automorphisms of $\Sigma(W, S)$ are entirely determined by their restriction to W.

We deduce the following conjugation formula:

For $v \in W$, $\quad (\widehat{f} \circ w \circ \widehat{f}^{-1})(vx) = \widehat{f}(wf^{-1}(v)\widehat{f}^{-1}(x)) = f(w)vx.$

In other words we have $\widehat{f} \circ w \circ \widehat{f}^{-1} = f(w)$ (in $\mathrm{Aut}_{\mathrm{diag}}(W, S)$).

The subgroup generated by W and $\mathrm{Aut}_{\mathrm{diag}}(W, S)$ inside $\mathrm{Aut}(\Sigma(W, S))$ is then isomorphic with $\widehat{W}$ since using the simple-transitivity of W on $\Sigma^0(W, S)$ we see that $W \cap \mathrm{Aut}_{\mathrm{diag}}(W, S) = \{1\}$.

It remains to prove that any automorphism $\varphi \colon \Sigma(W, S) \to \Sigma(W, S)$ that normalizes W is inside $W. \mathrm{Aut}_{\mathrm{diag}}(W, S)$. Let $w_0 \in W = \Sigma^0(W, S)$ be the image under φ of the origin $1 \in W = \Sigma^0(W, S)$. We set $\varphi_0 = (w_0)^{-1} \circ \varphi$. So φ_0 normalizes W and fixes the origin 1. In particular φ_0 induces a permutation f of the edges at 1. These edges are labelled by generators $s \in S$, so f is in fact a permutation of S. Since φ_0 is a polygonal automorphism it preserves the number of sides of the polygons adjacent to the origin. In other words $f \in \mathrm{Aut}_{\mathrm{diag}}(W, S)$.

Clearly $\varphi_0 = \widehat{f}$ on the star of 1 in $\Sigma(W, S)$. So the automorphism of W induced by f and by φ_0 coincides on the generating set S, and therefore for any vertex $w \in W$ we have

$$\widehat{f}(w) = \varphi_0(w).$$

We conclude that $\varphi_0 = \widehat{f}$ on the whole of $\Sigma(W, S)$. $\qquad\square$

3 Quasi-isometric embedding CAT(0) large-gonal complexes

Definition 3.1 (Corners) A *corner of order k* in a polygonal complex X is a connected subgraph of the boundary of a polygonal face of X that contains $k + 1$ edges. For $k = 1$ we simply say a *corner*, and for $k = 2$ we say a *double corner*.

Let $f \colon X \to Y$ be a polygonal map of polygonal complexes.

We say $f \colon X \to Y$ has *no missing corner* provided for any combinatorial path (a, b) of X with $a \neq b$, if the edges $f(a), f(b)$ form a corner of Y, then a, b form a corner of X.

We say $f \colon X \to Y$ has *no missing double corner* provided for any combinatorial path (a, b, c) of X with $a \neq b, b \neq c$, if the edges $f(a), f(b), f(c)$ form a double corner of Y, then a, b, c form a double corner of X.

We say $f \colon X \to Y$ has *no missing half-cell* provided for any edge-path π of X of length $k + 1$, if $f(\pi)$ is a corner of order k inside a $2(k+1)$-gon of Y, then π is a corner of order k inside a $2(k+1)$-gon of X.

Observe a polygonal map $f: X \to Y$ is an isometric embedding for the CAT(0) metrics if and only if it has no missing corner.

Proposition 3.2 Let X, Y denote two CAT(0) *even-gonal complexes. Let* $f: X \to Y$ *be a polygonal map.*

Assume f *is locally injective and furthermore* f *has no missing half-cell.*

Then $f: X^{(1)} \to Y^{(1)}$ *is an isometric embedding (where the 1–skeleta are equipped with the combinatorial distance).*

Proof Assume by contradiction for some integer $n \geq 1$ there is a geodesic edge path $(\vec{e}_1, \ldots, \vec{e}_n)$ of X that f maps to a non-geodesic path, and let n be the smallest such integer. Since f is locally injective we have $n \geq 2$.

Two of the edges $f(e_1), \ldots, f(e_n)$ are dual to the same hyperplane K of Y. By minimality of n both subpaths $(f(\vec{e}_1), \ldots, f(\vec{e}_{n-1}))$ and $(f(\vec{e}_2), \ldots, f(\vec{e}_n))$ are geodesic. Thus $f(e_1)$ and $f(e_n)$ are dual to the same hyperplane K. By the combinatorial convexity of the polygonal neighborhood $N(K)$ the combinatorial geodesic $(f(\vec{e}_1), \ldots, f(\vec{e}_{n-1}))$ is entirely contained inside $N(K)$.

Since f is locally injective, the edge $f(e_2)$ is not dual to the hyperplane. Since $f(e_2)$ is contained inside $N(K)$ there is a polygon Q of Y that contains both $f(e_1)$ and $f(e_2)$. Let e' be the edge of Q which is opposite to $f(e_1)$. Then the locally injective path $(f(\vec{e}_1), \ldots, f(\vec{e}_{n-1}))$ must touch e': there is some integer $m \leq n-1$ such that $(f(\vec{e}_1), \ldots, f(\vec{e}_m))$ describes half of the polygon Q and ends in an extremity of e'.

Since f has no missing half-cell there is a polygon P of X which contains the path $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m)$. Then $f(P) = Q$ and moreover $f(e) = e'$, where e denotes the edge of P that is opposite to e_1. Note $e_{m+1} \neq e$ since $(\vec{e}_1, \ldots, \vec{e}_n)$ is a geodesic edge path. Let $\vec{e}$ be the orientation of e such that the endpoint is the vertex of $e_m \cap e_{m+1}$.

The path $(\vec{e}, \vec{e}_{m+1}, \ldots, \vec{e}_n)$ is still geodesic in X^1, its image is not a geodesic, contradicting the minimality of our n. □

Corollary 3.3 Let X, Y denote two CAT(0) *even-gonal complexes with* $\mu(Y) \geq 6$ *and let* $f: X \to Y$ *be a locally injective polygonal map with no missing double corner. Then* f *induces an isometric embedding* $X^1 \to Y^1$ *(thus it is a quasi-isometric embedding).*

Proof Indeed since $\mu(Y) \geq 6$ we note that no missing corner implies no missing half-cell. We then apply Proposition 3.2. □

We will use Corollary 3.3 above in case when all vertex links of Y are complete graphs, and $\mu(X) \geq 4$: so any combinatorial map $X \to Y$ has missing corners, but it will be possible to find combinatorial maps with no missing double corner. Thus even when there cannot exist any local isometry $X \to Y$, it may possible to find an isometric embedding $X^1 \to Y^1$.

Remark 3.4 There is a similar statement for CAT(0) cube complexes.

Namely let $f\colon X \to Y$ be a combinatorial locally injective map. Assume there exists an integer $R \geq 1$ such that f has no missing rectangle $[0, R] \times [0, 1]$ of size $R \times 1$ and is an isometric embedding on combinatorial geodesics of length $\leq R + 1$. Then $f\colon X^1 \to Y^1$ is an isometric embedding. The case $R = 1$ corresponds to the assumption that $f\colon X \to Y$ be a local isometry of CAT(0) cube complex.

4 Real hyperbolic convex cocompact Coxeter groups

Coxeter groups correspond to certain kinds of presentations (see Section 1). In this section we study methods to produce real hyperbolic convex cocompact Coxeter groups.

4.1 Reflection subgroups in $\mathbb{H}^p$

A major source of examples of Coxeter groups is given by discrete *reflection groups* in real hyperbolic space (meaning discrete subgroups of $\mathrm{Isom}(\mathbb{H}^p)$ generated by reflections).

We will identify $\mathbb{H}^p$ with

$$\{(x_1, \ldots, x_p, x_{p+1}) \in \mathbb{R}^{p+1} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 = -1,\ x_{p+1} > 0\}.$$

For vectors $\vec{u} = (x_1, \ldots, x_p, x_{p+1})$, $\vec{v} = (y_1, \ldots, y_p, y_{p+1})$ we denote by $\langle\!\langle \vec{u}, \vec{v} \rangle\!\rangle$ the inner product in $\mathbb{R}^{p,1}$: so $\langle\!\langle \vec{u}, \vec{v} \rangle\!\rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1}$. And $\langle \vec{u}, \vec{v} \rangle$ denotes the standard inner product in $\mathbb{R}^{p+1}$.

Definition 4.1 (Hyperbolic polyhedra, Poincaré polyhedra) A *hyperbolic polyhedron* P is a (non empty) intersection of finitely many closed half-spaces of some real hyperbolic space $\mathbb{H}^p$ (we allow $P = \mathbb{H}^p$). A *face* of P is a hyperbolic polyhedron F such that $F \subset P$ and $P \setminus F$ is connected. The *span* of a face F is the smallest totally geodesic subspace of $\mathbb{H}^p$ containing F. The *dimension* of F is the dimension of its span. We will always assume that the span of P is the whole ambient space $\mathbb{H}^p$.

Note P has finitely many faces. The *relative boundary* of a face F is the union $\partial_{\mathrm{rel}} F$ of all faces $F' < F$ with $F' \neq F$. The *relative interior* of F is then the subspace $\mathrm{Int}_{\mathrm{rel}} F := F \setminus \partial_{\mathrm{rel}} F$.

The *boundary hyperplanes* of P are the finitely many hyperplanes $H_1, \ldots, H_r$ of $\mathbb{H}^p$ such that $H_i \cap P$ is a codimension one face of P, which we will always denote by $P_1, \ldots, P_r$. The *defining half-spaces* of P are the half-spaces $X_1, \ldots, X_r$ of $\mathbb{H}^p$ such that $\partial X_i = H_i$ and $P \subset X_i$. We will always denote by $\vec{n}_i$ the vector of $\mathbb{R}^{p,1}$ that is orthogonal to H_i, of unit length, and that points outside the half-space X_i. Note that for any $p \in H_i$ the vector $\vec{n}_i$ belongs to the tangent space at p.

The *Gram matrix* of P consists in the $r \times r$–matrix $\mathrm{Gram}(P)$ of all inner products $\langle\langle \vec{n}_i, \vec{n}_j \rangle\rangle$.

We say P is a *Poincaré polyhedron* provided any two codimension one faces P_i, P_j are either disjoint, or intersect along a codimension two face of P, and moreover the angle between the unit normal vectors $\vec{n}_i, -\vec{n}_j$ is of the form $\frac{\pi}{m_{ij}}$ (for some integer $m_{ij} \geq 2$ depending on the intersecting faces P_i, P_j).

Note in our definition a Poincaré polyhedron P is a simple polyhedron: the links of faces are simplices. Note also that when two codimension-one faces P_i, P_j of a hyperbolic polyhedron P intersect at some point p then the unit normal outgoing vectors $\vec{n}_i, \vec{n}_j$ are both in the tangent space of p, and then $\langle\langle \vec{n}_i, \vec{n}_j \rangle\rangle = \langle \vec{n}_i, \vec{n}_j \rangle$. The angle between the unit normal vectors $\vec{n}_i, -\vec{n}_j$ is nothing else than the dihedral angle of P between H_i and H_j at p. So the condition on P for being a Poincaré polyhedron can be rephrased as follows: for any pair of intersecting codimension 1 faces P_i, P_j, the coefficient $\langle\langle \vec{n}_i, \vec{n}_j \rangle\rangle$ of the Gram matrix is $-\cos(\frac{\pi}{m_{ij}})$ (which is > -1).

Theorem 4.2 (Poincaré's Theorem) Let $P \subset \mathbb{H}^p$ denote a Poincaré polyhedron, and let $H_1, \ldots, H_r$ denote the boundary hyperplanes. Let $s_1, \ldots, s_r$ denote the reflections along $H_1, \ldots, H_r$, and let $W = W(P) < \mathrm{Isom}(\mathbb{H}^p)$ be the group generated by $s_1, \ldots, s_r$.

Then P is a strict fundamental domain for the action of W onto $\mathbb{H}^p$ (any W–orbit intersect P in a single point) and in particular every relation in W on the generators $s_1, \ldots, s_r$ can be deduced from the relations

$$(s_i s_j)^{m_{ij}} = 1.$$

In other words $(W, \{s_1, \ldots, s_r\})$ is a Coxeter system.

Moreover the stabilizer of any face F of P is the finite subgroup W_F generated by all reflections s_i along a boundary hyperplane H_i that contains F.

The above theorem contains the spherical and euclidean versions as special cases. For an argument see Ratcliffe [24, Theorems 7.1.3 and 7.1.4]. Note that the subgroup W_F is the reflection group associated with the Poincaré polyhedron P_F obtained by intersecting the closed half-spaces containing P and bounded by some boundary hyperplane H_i which contains F. Any finite group of isometries of $\mathbb{H}^p$ has a fixed point, thus each finite subgroup of W is conjugate in one of the finite subgroups W_F.

Observe also that two disjoint faces P_i, P_j of a Poincaré polyhedron P span disjoint hyperplanes. This follows since in a Coxeter system (W, S) the order of $s_i s_j$ is infinite if and only if m_{ij} is infinite. It is then easy to check that the unit normal vector $\vec{n}_i, \vec{n}_j$ satisfy $\langle\langle \vec{n}_i, \vec{n}_j \rangle\rangle \leq -1$. So we may characterize Poincaré polyhedra by a criterion using the Gram matrix:

> *A simple hyperbolic polyhedron P is a Poincaré polyhedron if and only if the off-diagonal entries of $\mathrm{Gram}(P)$ are either ≤ -1 or of the form $-\cos(\frac{\pi}{m_{ij}})$.*

Definition 4.3 (Non-obtuse polyhedra) Let $P \subset \mathbb{H}^p$ be a polyhedron with boundary hyperplanes $H_1, \ldots, H_r$. We say P is *non-obtuse* provided for any pair of intersecting facets P_i, P_j the dihedral angle of H_i, H_j is $\leq \frac{\pi}{2}$.

Poincaré polyhedra are examples of non-obtuse polyhedra.

Lemma 4.4 (Geometry of non-obtuse polyhedra) *Let $P \subset \mathbb{H}^p$ be a non-obtuse polyhedron and let H be a boundary hyperplane of P. We denote by Q the corresponding codimension one face: $Q := P \cap H$.*

  (i)   *For any point $x \in P$ the orthogonal projection of x onto H belongs to Q.*

  (ii) *We set $Q^\perp = \{ x \in \mathbb{H}^p \mid \text{the orthogonal projection of } x \text{ onto } H \text{ belongs to } Q \}$. Then $P \subset Q^\perp$ and $Q^\perp$ is a hyperbolic polyhedron.*

  (iii) *For any point $x \in H$ the orthogonal projection of x onto P belongs to Q.*

Proof (i) Let x_i be the point of the closed subset P_i which is nearest to x. We must show that $d(x, x_i) = d(x, H_i)$.

We may and will assume that $x \neq x_i$. So let σ denote the (non-trivial) geodesic segment from x_i to x. By convexity $\sigma \subset P$. We must prove that σ is perpendicular to H_i at x_i. Let $\vec{u}$ be the unit tangent vector of σ at x_i, let S_i be the hypersphere of the unit sphere of the tangent space of $\mathbb{H}^p$ at x_i corresponding to the hyperplane H_i, and finally let $\vec{v}_i$ be the unit vector at x_i that is orthogonal to H_i and points toward P.

Assume by contradiction that the distance between $\vec{u}$ and S_i is $< \frac{\pi}{2}$. Let then Π be the totally geodesic plane of $\mathbb{H}^p$ through x_i, whose tangent space at x_i is generated

by $(\vec{u}, \vec{v}_i)$. Let also $\vec{v}$ be the unit vector of the tangent space of Π at x_i that is perpendicular to $\vec{v}_i$ and such that $\vec{u}$ lies in the cone generated by $\vec{v}_i$ and $\vec{v}$.

The open half-line R of Π starting at x_i and directed by $\vec{v}$ is disjoint from P_i, otherwise σ wouldn't be minimizing. Since $R \subset H_i$ we must also have $R \cap P = \varnothing$. So let H_k be a boundary hyperplane of P through x_i that separates R from σ. Let $\vec{v}_j$ be the unit vector at x_i that is normal to H_k and points inside P.

The vector $\vec{v}_j$ is not orthogonal to Π and moreover the orthogonal projection $\vec{v}_j'$ of $\vec{v}_j$ onto the plane generated by $(\vec{u}, \vec{v}_i)$ has positive coordinate along $\vec{v}_i$ and negative coordinate along $\vec{v}$.

This implies that the dihedral angle between H_i and H_k is obtuse, a contradiction.

(ii) The inclusion $P \subset Q^{\perp}$ follows by the first part of the Lemma. Either $Q = H$, in which case $Q^{\perp} = \mathbb{H}^p$. Or Q is an intersection of H–half-spaces $G_1, \ldots, G_s$. We then have $Q^{\perp} = \bigcap_{k=1}^{s}(G_k)^{\perp}$, and each $(G_k)^{\perp}$ is a closed half-space of $\mathbb{H}^p$.

(iii) We now prove the third statement. For $x \in \mathbb{H}$ let y denote the orthogonal projection of x onto P, and let z denote the orthogonal projection of x onto $Q^{\perp}$.

Clearly $Q^{\perp}$ is invariant under the orthogonal reflection along H. It follows that $z \in H$. So $z \in H \cap Q^{\perp}$. We note $H \cap Q^{\perp} = Q$ so that $z \in Q \subset P$. Since $P \subset Q^{\perp}$ we conclude that $y = z$. $\square$

Corollary 4.5 *Let $P \subset \mathbb{H}^p$ be a non-obtuse polyhedron with boundary hyperplanes $H_1, \ldots, H_r$.*

(i) *Any face $F < P$ is a non-obtuse polyhedron (of $\mathrm{Span}(F)$).*

(ii) *For any point $x \in P$ and any face $F < P$, the orthogonal projection of x onto $\mathrm{Span}(F)$ belongs to F.*

(iii) *For any two faces F, G of P, either $F \cap G \neq \varnothing$, or $\mathrm{Span}(F) \cap \mathrm{Span}(G) = \varnothing$. In the latter case either $d(F, G) = 0$ or $d(F, G) > 0$ and there exists a geodesic segment $\sigma \subset P$ connecting orthogonally the faces F, G so that $d(\mathrm{Span}(F), \mathrm{Span}(G)) = d(F, G) > 0$.*

Proof (i) To show that a face F is non-obtuse it suffices by induction to handle the case when F has codimension one. So let H be a boundary hyperplane of P such that $F = H \cap P$. The boundary hyperplanes of F in its span H are the intersections with H of the boundary hyperplanes H_i of P which intersect transversally H. Let $\vec{n}, \vec{n}_i, \vec{n}_j$ be the unit vectors at some point $x \in F \cap H_i \cap H_j$, such that $\vec{n}, \vec{n}_i, \vec{n}_j$ are orthogonal to H, H_i, H_j and are going out of P. Consider the decomposition $\vec{n}_i = \vec{v}_i + \lambda_i \vec{n}$

where $\vec{v}_i \perp \vec{n}$ and $\lambda_i \in \mathbb{R}$. The non-obtuse condition yields $\lambda_i \leq 0$. Similarly we have $\vec{n}_j = \vec{v}_j + \lambda_j \vec{n}$ with $\vec{v}_j \perp \vec{n}$ and $\lambda_j \leq 0$. Now $\langle \vec{v}_i, \vec{v}_j \rangle = \langle \vec{n}_i, \vec{n}_j \rangle - \lambda_i \lambda_j$ and it follows that the dihedral angle between $H \cap H_i$ and $H \cap H_j$ inside F is non-obtuse.

(ii)   The second statement follows by induction on $\mathrm{codim}(F)$, using Lemma 4.4 (since faces are non-obtuse polyhedra by (i)), as well as the following standard fact:

> Let F_1 and F_2 be two totally geodesic subspaces of $\mathbb{H}^p$, and let p_i denote the orthogonal projection onto F_i. If $F_1 \subset F_2$, then $p_1 = p_1 \circ p_2$.

(iii)   Let us prove the last stated trichotomy. So assume F, G are faces of P with $F \cap G = \varnothing$. Assume by contradiction that $\mathrm{Span}(F) \cap \mathrm{Span}(G)$ contains a point x. Let y denote the orthogonal projection of x onto P. For every boundary hyperplane H of P containing F we have $y \in H \cap P$ by the third part of Lemma 4.4. It follows that $y \in F$. By symmetry $y \in G$, which contradicts $F \cap G = \varnothing$.

Assume moreover $d(F, G) > 0$. Since F, G are closed convex subsets of $\mathbb{H}^p$ it follows that there are points $p, q \in F, G$ such that $d(p, q) = d(F, G)$. Let $\sigma \subset P$ be the geodesic segment joining p and q. By (ii) the point q is the orthogonal projection of p onto $\mathrm{Span}(G)$, and similarly p is the orthogonal projection of q onto $\mathrm{Span}(F)$, so we are done. $\square$

In the situation of Poincaré's Theorem we obtain a geometrically finite Coxeter group $W(P)$: P has only finitely many faces. In general $W(P)$ might contain parabolic elements. We will now give conditions under which $W(P)$ is convex-cocompact.

Lemma 4.6   Let $P \subset \mathbb{H}^p$ denote a Poincaré polyhedron, let $H_1, \ldots, H_r$ denote the boundary hyperplanes, and let W be the group generated by the reflections s_i along H_i. Let F, G be disjoint faces of P.

(i)   If $d(F, G) > 0$ then $\partial_\infty \mathrm{Span}(F) \cap \partial_\infty \mathrm{Span}(G) = \varnothing$.

(ii)   If $d(F, G) = 0$ then $\partial_\infty \mathrm{Span}(F) \cap \partial_\infty \mathrm{Span}(G) \neq \varnothing$ and W is not convex cocompact.

In either cases $\mathrm{Span}(F) \cap \mathrm{Span}(G) = \varnothing$.

Here as usual for a subspace $X \subset \mathbb{H}^p$ we denote by $\partial_\infty X$ the set of points $\xi \in \mathbb{S}^{p-1} = \partial_\infty \mathbb{H}^p$ which are in the closure of X.

Proof The third part of Corollary 4.5 tells us that $\mathrm{Span}(F) \cap \mathrm{Span}(G) = \varnothing$.

(i) When $d(F, G) > 0$ the third assertion in Corollary 4.5 insures that

$$d\big(\mathrm{Span}(F), \mathrm{Span}(G)\big) = d(F, G) > 0$$

and thus $\partial_\infty \mathrm{Span}(F) \cap \partial_\infty \mathrm{Span}(G) = \varnothing$.

(ii) Assume now $d(F, G) = 0$. Since $d(F, G) = 0$ and $F \cap G = \varnothing$ it follows that $\partial_\infty \mathrm{Span}(F) \cap \partial_\infty \mathrm{Span}(G) \neq \varnothing$. So let $\xi \in \partial_\infty \mathrm{Span}(F) \cap \partial_\infty \mathrm{Span}(G)$. The whole subgroup Π generated by W_F and W_G fixes ξ and in fact preserves all horospheres based at ξ.

Observe that the fixed point set of Π equals $\mathrm{Span}(F) \cap \mathrm{Span}(G)$, so it is empty. It follows that Π is infinite. Thus Π contains an element g of infinite order, which preserves all horospheres based at ξ. We deduce that no orbital map $\langle g \rangle \to \mathbb{H}^p, g^n \mapsto g^n x$ is a quasi-isometric embedding, and so W is not convex-cocompact. $\qquad\square$

Two faces F, G of a hyperbolic polyhedron P are said to be *asymptotic* if $F \cap G = \varnothing$ and $d(F, G) = 0$.

Theorem 4.7 (Convex cocompact reflection groups of $\mathbb{H}^p$) *Let $P \subset \mathbb{H}^p$ denote a Poincaré polyhedron and let W be the group generated by the reflections along the boundary hyperplanes of P.*

Then $W < \mathrm{Isom}(\mathbb{H}^p)$ is convex cocompact if and only if P has no pair of asymptotic faces.

We are certain that this result is familiar to certain people working in the field but we couldn't find a reference. So we provide a proof of the theorem in the next section.

4.2 Proof of Theorem 4.7: convex cocompactness when there is no pair of asymptotic faces

The existence of an asymptotic pair of faces in the Poincaré polyhedron P prevents $W = W(P)$ from being convex cocompact: this has been established in Lemma 4.6.

We now assume that P has no pair of asymptotic faces. In fact we only assume that no face is asymptotic with a codimension-one face: the convex-cocompactness statement holds true under this weaker assumption. We denote by $H_1, \ldots, H_r$ the boundary hyperplanes of P.

We choose a point x in the interior of P. Then by Poincaré's Theorem the orbital map $W \ni w \mapsto w.x \in \mathbb{H}^p$ is injective, and we identify W with the orbit $\{wx\}_{w \in W}$. We

show that W is convex cocompact, by proving that the orbital map is a quasi-isometric embedding (W is equipped with its word metric $|\cdot|_S$).

We work with the *tiling* of $\mathbb{H}^p$ by W–translates of P: $\mathbb{H}^p = \bigcup_{w \in W} wP$. A *hyperplane of the tiling* is a hyperplane of the form $wH_i \subset \mathbb{H}^p$. Since H_i is the fixed point set of the hyperbolic reflection s_i, the conjugate ws_iw^{-1} is again a hyperbolic reflection and its fixed point set is precisely wH_i.

At the end of this section we establish the following three results:

Lemma 4.8 (The word distance is the hyperplane distance) *The number of hyperplanes of the tiling separating wx from $w'x$ equals $|w'^{-1}w|_S$.*

Lemma 4.9 (Pencil of hyperplanes) *There exists a positive integer $M > 0$ such that for any $w, w' \in W$, the set R of hyperplanes of the tiling separating wx from $w'x$ contains a subset T with $|T| \geq \frac{1}{M}|R|$, and moreover two distinct hyperplanes in T are disjoint.*

Lemma 4.10 (No pair of asymptotic hyperplanes) *There exists a positive constant $D = D(P)$ such that for any two hyperplanes H, H' of the tiling we have either $H \cap H' \neq \varnothing$ or $d(H, H') \geq D$.*

We now use these three results to finish the argument.

We will show that there is a positive constant $c > 0$ such that for $w, w' \in W$ we have

$$(*) \qquad\qquad d_{\mathbb{H}^p}(wx, w'x) \geq c|w'^{-1}w|_S$$

(the reverse inequality is not a problem). It is enough to work with $w \neq w'$.

Let R be the set of hyperplanes of the tiling which separate wx from $w'x$. By Lemma 4.8 we have

$$(1) \qquad\qquad |R| = |w'^{-1}w|_S.$$

By Lemma 4.9 there exists a subset $T \subset R$ such that

$$(2) \qquad\qquad |T| \geq \frac{1}{M}|R|,$$

and moreover two distinct hyperplanes in T are disjoint. The geodesic segment σ of $\mathbb{H}^p$ from wx to $w'x$ crosses successively the hyperplanes of T, so we may write $T = \{H_1, \ldots, H_n\}$ and σ meets first H_1, then H_2 and so on. It follows that

$$d_{\mathbb{H}^p}(wx, w'x) \geq d_{\mathbb{H}^p}(wx, H_1) + d_{\mathbb{H}^p}(H_1, H_2)$$
$$+ \cdots + d_{\mathbb{H}^p}(H_{n-1}, H_n) + d_{\mathbb{H}^p}(H_n, w'x)$$
$$\geq 2d(x, \partial P) + (n-1)D,$$

where D is the constant of Lemma 4.10. Up to replacing D by $\min(D, d(x, \partial P))$ we have thus

$$(3) \qquad\qquad d_{\mathbb{H}^p}(wx, w'x) \geq nD.$$

Combining (1), (2) and (3) we obtain the relation $(*)$ with $c = \frac{D}{M}$.

We finally turn to the proof of the remaining statements. We will consider an embedded copy of the Cayley graph $\mathcal{G}(W, S)$ inside $\mathbb{H}^p$. We first embed the set of vertices by the orbital map $w \mapsto wx$. Note that the injectivity of this map follows by Poincaré's Theorem 4.2, since x is in the interior of P. We then map each edge $[w, ws]$ by a constant speed geodesic segment joining wx to wsx, thus getting a W–equivariant map $\mathcal{G}(W, S) \to \mathbb{H}^p$. We now check that this map is injective. The midpoint q of $[w, ws]$ maps to the midpoint m of $[wx, wsx]$. Observe that m belongs to both tiles wP and wsP since by Lemma 4.4 the orthogonal projection of wx onto the hyperplane of fixed points of wsw^{-1} belongs to wP. It follows that the intersection of $[wx, wsx]$ with the interior of wP is the segment $[wx, m[$, and similarly the intersection of $[wx, wsx]$ with the interior of wsP is the segment $]m, wsx]$.

Assume now two points y, y' inside two edges $[w, ws]$ and $[w', w's']$ of $\mathcal{G}(W, S)$ get identified inside $\mathbb{H}^p$, and let z denote their common image. We may assume $y \in]w, q]$ and $y' \in]w', q']$ (with q, q' the midpoints of $[w, ws], [w', w's']$). If $y \neq q$ (or $y' \neq q'$) then wx and $w'x$ are contained in the interior of the same tile and thus $wx = w'x$, so in fact $w = w'$. Moreover $[wx, z] = [w'x, z]$ and it follows by extending the geodesic segment until we reach the boundary of the tile that $m = m'$. By extending further this geodesic we get that $wsx = w's'x$, so that $ws = w's'$, and finally $y = y'$.

To finish the argument assume $y = q$ and $y' = q'$. Since the orthogonal projection of the whole polyhedron wP onto the hyperplane $H = \text{Fix}(wsw^{-1})$ belongs to the codimension-one face $F = H \cap wP$ (Lemma 4.4), and because x is interior to P we deduce that the point m (midpoint of $[wx, wsx]$) belongs to the relative interior of F. Similarly m is in the relative interior of $H \cap wsP$. Moreover wP and wsP are the only two tiles containing m. It follows that $\{w, ws\} = \{w', w's'\}$, and so $y = y'$. This concludes the proof of the injectivity of the map $\mathcal{G}(W, S) \to \mathbb{H}^p$.

Proof of Lemma 4.8 Let $(w_0 = w, w_1, \ldots, w_n = w')$ be a combinatorial geodesic in (W, S) from w to w'. For $i = 1, \ldots, n$ set $t_i = w_i w_{i-1}^{-1}$. The t_i are pairwise distinct reflections of (W, S). We now consider the sequence of tiles $P_0 = w_0 P$, $P_1 = w_1 P, \ldots, P_n = w_n P$. Two consecutive tiles intersect. So if a hyperplane H of the tiling separates wx from $w'x$ then it must separate the interior of two consecutive tiles P_{i-1}, P_i. It then follows that H contains the relative interior of the codimension-one

face $P_{i-1} \cap P_i$. Thus the reflection along H is t_i. So the number of hyperplanes of the tiling separating wx from $w'x$ is the combinatorial distance $d_S(w, w')$. □

Proof of Lemma 4.9 As we have just seen the hyperplanes of R correspond to the half-spaces of (W, S) which separate w from w'. Moreover two hyperplanes of the tiling are disjoint iff the product of the corresponding reflections is of infinite order. We now let X be the CAT(0) cube complex associated to the wall-space (W, S), as in Niblo–Reeves [22]. In particular the 1–skeleton of X contains the Cayley graph of (W, S) as a totally geodesic subgraph.

Niblo and Reeves showed that there exists a positive constant $M = M(W, S)$ such that any family of $M + 1$ hyperplanes of the Cayley graph contains a pair of nested hyperplanes (see [22, Lemma 3]). This shows that the Niblo–Reeves cube complex has dimension $\leq M$. Recall that the hyperplanes (resp, half-spaces) of X correspond bijectively to the walls (resp, half-spaces) of (W, S). Moreover in this construction, crossing walls correspond to intersecting hyperplanes.

So to conclude the argument it is enough to prove the following:

Lemma 4.11 (Pencils in finite-dimensional cube complexes) *Let X be a CAT(0) cube complex of dimension M. Then for any two vertices $v, v' \in X^0$ at distance n there is a family T of hyperplanes of X separating v from v' such that any two hyperplanes of T are disjoint and moreover $|T| \geq \frac{1}{M}n$.*

We will in fact construct T as a sequence of hyperplanes with the additional property that two consecutive hyperplanes have non-disjoint cubical neighborhoods.

Proof of the cubical statement Let $e_1, \ldots, e_d$ be the edges of X at v which are the initial edges of combinatorial geodesics from v to v'. Then it is well known that $e_1, \ldots, e_d$ span a cube Q_1 at v. In particular $d \leq M$. Let v_1 be the vertex opposite to v inside Q.

If $v_1 = v'$ we are done. Otherwise, let R (resp, R_1) be the set of hyperplanes of X separating v from v' (resp, v_1 from v'). Then $R = R_1 \sqcup \{H_1, \ldots, H_d\}$, where H_i is the hyperplane dual to e_i. By induction there is a set $T_1 \subset R_1$ such that any two hyperplanes of T_1 are disjoint, T_1 contains a hyperplane K_1 adjacent to v_1, and

$$|T_1| \geq \frac{1}{M}|R_1| = \frac{1}{M}(|R| - d) \geq \frac{1}{M}|R| - 1.$$

One of the hyperplanes H_i cutting Q must be disjoint from K_1, otherwise there would exist a $(d+1)$–cube Q' at v_1 cut by all hyperplanes $H_1, \ldots, H_d, K_1$. Then there would be one more edge at v sitting on a geodesic from v to v' (through v_1). The set $T = T_1 \cup H_1 \cup \cdots \cup H_d$ has the desired properties. □

This completes the proof of Lemma 4.9. □

Proof of Lemma 4.10 Let ε_1 denote the smallest of the distances $d(F, G)$ where F, G are disjoint faces of P, and F has codimension one. Under the assumption on P we note that ε_1 is positive.

Let H, H' be two hyperplanes of the tiling such that $H \cap H' = \varnothing$. We show that for $(x, x') \in H \times H'$ we have $d(x, x') \geq D = \min(\alpha', \frac{\varepsilon_1}{2})$ where $\alpha' = \alpha'(P)$ is some positive constant that we will choose later.

So let $(x, x') \in H \times H'$. Up to letting W act on the situation we may assume $x \in P$ and H is a boundary hyperplane of P. The corresponding face is $F = P \cap H$. Up to moving slightly both x and x' we may assume each point of the segment $[x, x']$ is contained in at most one of the hyperplanes of the tiling. Note that under this genericity assumption F has codimension one. Up to applying the reflection along H we may also assume that H separates the interior of P from x'.

We can order the hyperplanes of the tiling which cut $[x, x']$, namely $H_0, H_1, \ldots, H_\ell$, in such a way that the sequence $x_i = H_i \cap [x, x']$ is increasing from x to x'. Note $H_0 = H$ and $H_\ell = H'$. None of the open geodesic segments $]x_i, x_{i+1}[$ meets a hyperplane of the tiling, so there exists a (unique) tile P_{i+1} such that $[x_i, x_{i+1}] \subset P_{i+1}$. The face spanned by x_i in both P_i and P_{i+1} has codimension one, and we denote it by F_i. For example $F_0 = F$.

We also consider the non-increasing sequence of faces defined inductively by $G_0 = F_0$, $G_{i+1} = G_i \cap F_{i+1}$. Let $k + 1$ be the integer such that $G_k \neq \varnothing$ and $G_{k+1} = \varnothing$ (recall $H_\ell \cap H_0 = \varnothing$).

We may assume $k > 0$. For $k = 0$ means that $[x_0 x_1]$ is a geodesic segment joining two disjoint codimension-one faces of P_1. It follows that $d(x_0, x_1) \geq \varepsilon_1$ and thus $d(x, x') \geq \varepsilon_1$.

Let N be the total number of hyperplanes of the tiling whose intersection with P is non-empty (note N is bounded by a number depending only on the isomorphism class of the Coxeter system (W, S)). Let also $\alpha > 0$ denote some positive constant, that we will specify later.

Either one the distances $d(x_i, x_{i+1}), i = 0, 1, \ldots, k - 1$ is at least $\frac{\alpha}{N}$. In that case we have $d(x, x') \geq \alpha'$ where we define $\alpha' = \frac{\alpha}{N}$.

So from now on we assume that all distances $d(x_i, x_{i+1}), i = 0, 1, \ldots, k - 1$ are $< \frac{\alpha}{N}$. Thus $d(x_0, x_i) < \alpha$ for $i = 1, \ldots, k$. We will prove that in this case x is near to G_k. We first retract the situation inside P as follows. For each $i = 0, 1, \ldots, k$ the reflection

s_{H_i} maps the tile P_{i+1} onto P_i, and fixes pointwise both faces G_i and G_{i+1}. The retraction map $\rho_i \colon P_i \to P$ is then defined inductively by $\rho_1 = s_{H_0|P_1}, \ldots, \rho_{i+1} = \rho_i \circ s_{H_i|P_{i+1}}, \ldots$. Note each ρ_i is the restriction of an isometry. The retractions ρ_i and ρ_{i+1} agree on F_i. Each face F_i is sent to a codimension-one face F_i' of P. We note that for each $i = 0, 1, \ldots, k$ we have $G_i = F_0' \cap F_1' \cap \cdots \cap F_i'$. In particular $G_k = F_0' \cap F_1' \cap \cdots \cap F_k'$.

By retracting the points x_i we find that $d(x, F_i') \leq \alpha$ and thus $d(x, H_i') \leq \alpha$ where H_i' denote the boundary hyperplane of P spanned by F_i'. The Poincaré polyhedron P is simple so the dimension of G is $p - (k+1)$, and the span of G coincides with $H_0' \cap \cdots \cap H_k'$. We argue that there is a function $\alpha_G \colon (0, +\infty) \to (0, +\infty)$ such that if a point y of $\mathbb{H}^p$ is at distance $\leq \alpha(r)$ of each hyperplane $H_0', H_1', \ldots, H_k'$ then y is r–near to the intersection of the H_i.

We can now specify the value of α: we choose $\alpha = \min_{G \text{ a proper face of } P} \alpha_G(\frac{\varepsilon_1}{2})$.

With this choice of α our point x is $\frac{\varepsilon_1}{2}$–near to the span of G. Since P is non-obtuse it follows that the orthogonal projection p of x onto the span of G belongs to G. The tile P_k contains both faces F_k and F_{k+1}, and $G \subset F_k$. So the distance from $p \in G$ to $x_{k+1} \in F_{k+1}$ is $\geq \varepsilon_1$ (recall F_{k+1} has codimension one). We conclude that $d(x, x_{k+1}) \geq \frac{\varepsilon_1}{2}$ and we are done. $\qquad\square$

We note that we can slightly simplify the characterization of convex cocompact reflection groups:

Theorem 4.12 (Convex cocompact reflection groups II) *Let $P \subset \mathbb{H}^p$ denote a Poincaré polyhedron and let W be the group generated by the reflections along the boundary hyperplanes of P.*

Then W is convex cocompact if and only if

  (i) *W is word-hyperbolic, and*

  (ii) *P has no pair of asymptotic codimension-one faces.*

Proof If W is convex cocompact then it is a hyperbolic group, and by Theorem 4.7 there is no pair F, G of asymptotic faces, where F has codimension one and G has arbitrary codimension.

Conversely assume W is word-hyperbolic but not convex cocompact. Then by Theorem 4.7 there is a codimension-one face F and a face G such that $F \cap G = \varnothing$ but $d(F, G) = 0$. Then G is an intersection of k codimension-one faces: $G = P \cap H_1 \cap \cdots \cap H_k$, and we may choose G so that k is minimal. Let H_0 be the

boundary hyperplane of P such that $F = H_0 \cap P$. We will show $k = 1$ which is the conclusion we need.

Let then $\xi \in \partial_\infty \mathbb{H}^{r-1}$ be a point belonging to both $\partial_\infty F$ and $\partial_\infty G$. Let us now consider a horosphere of $\mathbb{H}^{r-1}$ centered at ξ, which we denote by $\mathcal{H}_\xi$. It follows that each reflection s_{H_i} for $i = 0, 1, \ldots, k$ preserves $\mathcal{H}_\xi$, and so does the whole group W' generated by these reflections.

As a discrete isometry group of euclidean space W' is virtually abelian. But (W, S) is assumed to be hyperbolic, so either W' is finite or W' is virtually $\mathbb{Z}$.

Since $F \cap G = \varnothing$ the group W' must be infinite.

So the only remaining possibility is that W' is virtually $\mathbb{Z}$. Assume by contradiction that $k > 1$. Since $d(H_0, H_i) = 0$ it follows by minimality of k that $H_0 \cap H_i \neq \varnothing$. Note $(W', S' = \{s_{H_0}, s_{H_1}, \ldots, s_{H_k}\})$ is a Coxeter system where any two reflections s_{H_i}, s_{H_j} generate a finite group. By the work of Davis and Meier [9] the infinite Coxeter system (W', S') is one-ended, a contradiction.   □

4.3 Constructions using the Witt–Tits quadratic form

In this section we use the so-called Witt–Tits quadratic form for two opposite purposes:

   (i) Construct families of convex cocompact reflection groups in hyperbolic space.

  (ii) Construct new families of word-hyperbolic Coxeter groups which cannot be represented as discrete reflection group in hyperbolic spaces.

Definition 4.13 (The Witt–Tits quadratic form) For any (abstract) Coxeter system $(W, S = \{s_1, \ldots, s_r\})$ with Coxeter matrix $(m_{ij})_{1 \le i, j \le r}$ let B denote the bilinear form on $\mathbb{R}^r$ defined by $B(e_i, e_j) = -\cos(\frac{\pi}{m_{ij}})$. The corresponding quadratic form $q_{(W,S)}$ is called the *Witt–Tits form*. When the signature of the Witt–Tits form is $(r-1, 1)$ we say the Witt–Tits quadratic form is *hyperbolic*. In this case there is a Poincaré polyhedron P inside $\mathbb{R}^{r-1,1}$ whose Gram matrix is precisely given by the $B(e_i, e_j)$. Unit vectors normal to the codimension-one faces of P are just the vectors of the standard basis of $\mathbb{R}^{r-1,1}$. We call P the *Witt–Tits polyhedron* of (W, S) (see also Vinberg [25]). Using the reflections along the sides of P we get a representation of (W, S) into $\mathrm{Isom}(\mathbb{H}^{r-1})$, which we call the *Witt–Tits representation*. So having a hyperbolic Witt–Tits quadratic form is one way to get a reflection group.

Lemma 4.14 (Pairs of asymptotic hyperplanes in the Witt–Tits polyhedron) *Assume the Witt–Tits form is hyperbolic and $r \ge 3$. Then the hyperplanes H_i, H_j are asymptotic exactly when $B(e_i, e_j) = -1$.*

Proof The non-zero vector $e_i + e_j$ is isotropic (so it defines a point ξ in $\partial_\infty \mathbb{H}^{r-1}$) and orthogonal to both e_i and e_j (so $\xi \in \partial_\infty H_i \cap \partial_\infty H_j$). □

So the Witt–Tits representation is convex cocompact only if all coefficients m_{ij} are finite, in other words (W, S) is *2–spherical*.

Proposition 4.15 (Convex cocompact Witt–Tits representation) *Suppose $(W, S = \{s_1, \ldots, s_r\})$ is a Coxeter system with a Witt–Tits quadratic form of signature $(r-1, 1)$.*

Then the Witt–Tits representation is convex cocompact if and only if

(i) *W is 2–spherical, and*

(ii) *W is word-hyperbolic.*

Proof The two conditions are clearly necessary. The converse holds by applying Theorem 4.12. □

We note that the Witt–Tits polyhedron may have to appear in any representation by reflection of a Coxeter system.

Lemma 4.16 *Let $P \subset \mathbb{H}^p$ be a Poincaré polyhedron with boundary hyperplanes $H_1, \ldots, H_r$ and let W be the group generated by the set $S = \{s_{H_1}, \ldots, s_{H_r}\}$ of reflections. Let $\vec{n}_1, \ldots, \vec{n}_r$ denote the unit vectors of $\mathbb{R}^{p,1}$ which are orthogonal to $H_1, \ldots, H_r$ and point outwards of P. Let $\vec{V} \subset \mathbb{R}^{p,1}$ be the linear subspace spanned by $\vec{n}_1, \ldots, \vec{n}_r$. Set $V = \mathbb{H}^p \cap \vec{V}$. We denote by G the Gram matrix of P and by B the matrix of the Witt–Tits quadratic form.*

(i) *$V \subset \mathbb{H}^p$ is empty or a totally geodesic W–invariant subspace. The action of W on the orthogonal complement of $\vec{V}$ is trivial. The matrices G and B have the same coefficient in the place i, j provided $H_i \cap H_j \neq \varnothing$ (in other words: m_{ij} is finite). In particular if (W, S) is 2–spherical then $G = B$.*

(ii) *$V \neq \varnothing$ iff W is infinite and not virtually euclidean. In that case we obtain a hyperbolic Poincaré polyhedron $Q \subset V$ by setting $Q = V \cap P$, which we call the core of P. The r boundary hyperplanes of the polyhedron $Q \subset V$ are $H_1 \cap V, \ldots, H_r \cap V$ and the Gram matrix of Q is G.*

(iii) *Assume W is not virtually abelian. Then the Gram matrix G is non-degenerate, and the dimension of $\vec{V}$ is r if and only if G is non singular.*

Proof (i) This is obvious.

(ii) By homogeneity $V = \varnothing$ precisely when the quadratic form of $\mathbb{R}^{p,1}$ is ≥ 0 on $\vec{V}$. Either it is positive on $\vec{V} \setminus \{0\}$, in which case W is finite. Or $\vec{V}$ intersects the set of isotropic vectors along a line, corresponding to a unique point $\xi \in \partial_\infty \mathbb{H}^p$. Thus all reflections s_{H_i} fix ξ, and W is virtually abelian.

So assume now that $V \neq \varnothing$. Since V is W–invariant each hyperplane H_i has a non-empty intersection with V. For $x \in V \cap H_i$ the plane spanned by x and $\vec{u}_i$ is contained inside $\vec{V}$ and it intersects $\mathbb{H}^p$ along an infinite geodesic line that leaves H_i orthogonally. So in fact $H_i \cap V$ is a hyperplane of V. For $x \in P$ we consider the decomposition $x = \vec{v} + \vec{n}$ with $\vec{v} \in \vec{V}$ and $\vec{n}$ orthogonal to $\vec{V}$. Since $V \cap \mathbb{H}^p \neq \varnothing$ the quadratic form of $\mathbb{R}^{p,1}$ has hyperbolic signature on $\vec{V}$, and thus it is positive definite on the subspace orthogonal to $\vec{V}$. It follows that $\vec{n}$ has non-negative norm and so $\vec{v}$ has negative norm. In other words $\vec{v}$ defines a point v in V. Moreover for each i we have $\langle\langle \vec{v}, \vec{n}_i \rangle\rangle = \langle\langle x, \vec{n}_i \rangle\rangle$, so $v \in P$. The intersection $Q = P \cap V$ is thus non-empty, and it is bounded by the V–hyperplanes $V \cap H_i$. We note that the vectors $\vec{n}_i$ are in $\vec{V}$, are normal to the V–hyperplanes $V \cap H_i$, and are pointing outwards Q. It follows that $\mathrm{Gram}\, Q = G$.

(iii) Assume W is not virtually abelian. Then either $\vec{V}$ is a euclidean subspace, or it is a hyperbolic subspace of $(\mathbb{R}^{p,1}, \langle\langle - \rangle\rangle)$. In both cases the restriction of the bilinear form $\langle\langle - \rangle\rangle$ is non degenerate.

Assume first one of the $\vec{n}_i$ is a linear combination of the others. Then the ith line of G is the corresponding linear combination of the others lines of G, so G is singular. Conversely assume the ith line – say, the first line – is a linear combination of the others lines of G. It means that there are real numbers $\lambda_2, \ldots, \lambda_r$ such that $\vec{u} := \vec{n}_1 - \sum_{k>1} \lambda_k \vec{n}_k$ satisfies $\langle\langle \vec{u}, \vec{n}_i \rangle\rangle = 0$ for each i. Since $\vec{V}$ is spanned by the $\vec{n}_i$ it follows that $\vec{u}$ is in the kernel of the hyperbolic bilinear form of $\mathbb{R}^{p,1}$ restricted to $\vec{V}$. Under our assumption we get $\vec{u} = \vec{0}$, so the dimension of $\vec{V}$ is $< r$. $\qquad\square$

Corollary 4.17 (2–spherical implies rigidity of the core) *Let $P \subset \mathbb{H}^p$ be a Poincaré polyhedron with boundary hyperplanes $H_1, \ldots, H_r$ and let W be the group generated by the set of reflections $S = \{s_{H_1}, \ldots, s_{H_r}\}$. Assume W is infinite, not virtually abelian and (W, S) is 2–spherical. Assume moreover that B is non-singular. Then the Witt–Tits quadratic form is hyperbolic, the core Q of P is non empty and it is isometric to the Witt–Tits polyhedron. The action of W onto the span V of Q in $\mathbb{H}^p$ is conjugate to the Witt–Tits representation.*

So under the assumptions of the Corollary there is essentially one possible representation of (W, S) as a discrete reflection group.

Corollary 4.18 Let (W, S) be an abstract 2–spherical Coxeter system with non degenerate Witt–Tits form. Assume W is word-hyperbolic and non elementary.

Then (W, S) can be realized as a discrete reflection group in some real hyperbolic space $\mathbb{H}^p$ if and only if the signature of the Witt–Tits form is $(r - 1, 1)$.

In that case W preserves a totally geodesic subspace V of $\mathbb{H}^p$ of dimension $r - 1$, the action of W onto V is conjugate to the Witt–Tits representation and the action on the orthogonal complement of V is trivial. Moreover W is convex cocompact on both V and $\mathbb{H}^p$.

The equivalence stated in this corollary appears in Felikson–Tumarkin [11] who also refer to Vinberg [25, Lemma 12]. We thank Anna Felikson for telling us about the rigidity phenomenon for 2–spherical Poincaré polyhedra in hyperbolic space.

In the rest of this section we will compute the signature of the Witt–Tits quadratic form under certain type of assumptions which will usually imply that all coefficients m_{ij} are large (for $i \neq j$).

So for $r \geq 3$ we consider a $r \times r$ Coxeter matrix $M = (m_{ij})$. (Recall this means M is symmetric, $m_{ii} = 1$ and for $i \neq j$ we have $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$.) Let $(W, S = \{s_1, \dots, s_r\})$ be the associated Coxeter system. Let B be the matrix of the Witt–Tits quadratic form.

Definition 4.19 (m–large, m–small) Let $m \geq 2$ be some natural number. We say the Coxeter matrix M is m–large if $m_{ij} \geq m$ for any pair (i, j) with $i \neq j$. We say M is m–small if $m_{ij} \leq m$ for any pair (i, j) with $i \neq j$.

The following is clear:

Lemma 4.20 Assume the matrix M is 4–large, $r \geq 3$ and all the m_{ij} are finite. Then (W, S) is a hyperbolic non elementary 2–spherical Coxeter system.

In fact it suffices to assume that M is 3–large and there is no triple $\{i, j, k\}$ with $m_{ij} = m_{jk} = m_{ik} = 3$.

Proposition 4.21 (Examples of hyperbolic Witt–Tits form)

(i) Assume all m_{ij} are infinite for $i \neq j$ (and $r \geq 3$ as usual). Then B is hyperbolic.

(ii) For any $r \geq 3$ there is a natural number $m(r) \geq 4$ such that if the $r \times r$ Coxeter matrix M is $m(r)$–large then B is hyperbolic.

(iii) Assume all m_{ij} are equal to some fixed number $m \geq 4$ for $i \neq j$. Then B is hyperbolic. This is still true if for $i \neq j$ all the m_{ij} are equal to 3 and $r \geq 4$.

Combining Corollary 4.18, Lemma 4.20 and Proposition 4.21(iii) we obtain:

Corollary 4.22 (The Coxeter group $W(p,m)$) *For $p \in \mathbb{N}$, $p \geq 3$ and $m \in \mathbb{N}$, $m \geq 3$, let $M(p,m)$ be the $p \times p$ Coxeter matrix all of whose off-diagonal entries are m. Let $W(p,m)$ be the associated Coxeter group.*

The signature of the corresponding Witt–Tits form is $(p-1, 1)$ unless $m = p = 3$.

For $m \geq 4$ the Witt–Tits representation of $W(p,m)$ is convex cocompact in $\mathbb{H}^{p-1}$. For $m = 3$ and $p \geq 4$ it is not.

In Kapovich [19] another reflection representation for $W(p,m)$ is considered, but it occurs in $\mathbb{H}^p$. Kapovich needs this specific representation because it allows the use of right-angled polygons, which in turn lead to quasi-isometric embedding results. Nevertheless the core of the polyhedron considered in [19] is isometric to the Witt–Tits polyhedron.

For the next result let us denote by $\widehat{W(p,m)}$ the semi-direct product of $W(p,m)$ with $\mathfrak{S}_p$. (Note the action of $\mathfrak{S}_p$ on the generating set $\{s_1, \ldots, s_p\}$ extends to an action by automorphisms of $W(p,m)$ because all the m_{ij} are equal to the same number m.)

Proposition 4.23 (Extending the representation) *Let $p \in \mathbb{N}$, $p \geq 3$ and $m \in \mathbb{N}$, $m \geq 4$. The Witt–Tits representation of $W(p,m)$ on $\mathbb{H}^{p-1}$ extends to a convex cocompact representation of $\widehat{W(p,m)}$.*

This completes the proof of Lemma 1.2.

Proof Since $W(p,m) \subset \mathrm{Isom}(\mathbb{H}^{p-1})$ is already convex cocompact by Corollary 4.22 and

$$\left[\widehat{W(p,m)} : W(p,m) \right] < \infty$$

it is sufficient to prove that the action of $W(p,m)$ extends to $\widehat{W(p,m)}$.

We note that the permutation group $\mathfrak{S}_p$ acts on the standard basis by preserving the Witt–Tits quadratic form of $W(p,m)$. This induces an isometric action of $\mathfrak{S}_p$ on $\mathbb{H}^{p-1}$ that preserves the Witt–Tits polyhedron by permuting the codimension-one faces. Clearly $\mathfrak{S}_p \cap W(p,m) = \{\mathrm{Id}_{\mathbb{H}^{p-1}}\}$ and $\mathfrak{S}_p$ normalizes $W(p,m)$. Moreover the action of $\mathfrak{S}_p$ by conjugation on $W(p,m)$ is the action by diagram automorphisms. It follows that the subgroup of $\mathrm{Isom}(\mathbb{H}^{p-1})$ generated by $\mathfrak{S}_p$ and $W(p,m)$ is isomorphic to $\widehat{W(p,m)}$. $\qquad\square$

4.4 Computation of signatures, proof of [Proposition 4.21](#) and new examples

In this section we fix a natural number $r \geq 3$. We consider a regular graph $\mathcal{G}$ with vertex set $\{1, \ldots, r\}$, and we denote by k its valency (note that $k \leq r - 1$). Let A be the adjacency matrix of $\mathcal{G}$, and let $\bar{A}$ be the adjacency matrix of the complementary graph. Thus $I_r + A + \bar{A}$ is the $r \times r$ matrix J whose entries are all equal to 1.

Now let $a, \bar{a} \in [0; 1], a \leq \bar{a}$ be fixed real numbers, and let us consider the $r \times r$ real symmetric matrix $B = I_r - aA - \bar{a}\bar{A}$. We study the signature of the quadratic form

$$Q_{\mathcal{G},a,\bar{a}} \colon X \mapsto{}^t XBX$$

or in other words we study the sign of the eigenvalues of B. We use the decomposition

$$B = (1 + \bar{a})I_r + (\bar{a} - a)A - \bar{a}J.$$

When $a = \cos(\frac{\pi}{m})$ and $a = \bar{a}$ (or equivalently $\mathcal{G}$ is the complete graph) the matrix B is the matrix of the Witt–Tits form of $W(r, m)$. So the computation below will prove [Proposition 4.21](#). But we want to study also Coxeter systems where the off-diagonal coefficients m_{ij} can take two distinct values. Our examples are in the same spirit as the examples of Benoist and de la Harpe [2], but our Coxeter systems will not be right-angled.

Let U denote the r–column all of whose entries are equal to 1. Since $\mathcal{G}$ is k–regular we have

$$BU = (1 + \bar{a})U + (\bar{a} - a)kU - r\bar{a}U = [(1 - ak + (1 + k - r)\bar{a})]U.$$

So U is an eigenvector and the corresponding eigenvalue is

$$\lambda_U = -[(ak - 1) + \bar{a}((r - 1) - k)].$$

Note $\lambda_U < 0$ in the cases we will be interested in. For example since $\bar{a} \geq a$ we have $(ak - 1) + \bar{a}((r - 1) - k) \geq a(r - 1)$, so when $a \geq \frac{1}{2}$ and $r \geq 4$ we have $\lambda_U < 0$. When $r = 3$ and either $a > \frac{1}{2}$ or $a = \frac{1}{2}$ and $\bar{a} > a$ we still have $\lambda_U < 0$. We could also assume $\bar{a} > \frac{1}{(r-1)-k}$.

Now if V is any column whose sum of components is zero (in other words: if V is in the kernel of J), we have

$$BV = (1 + \bar{a})V + (\bar{a} - a)AV.$$

Note that since $\mathcal{G}$ is regular we have $JA = kJ$ and so the term AV is still in the kernel of J. We deduce that the kernel of J is invariant under B. Since on $\ker J$ we have

$$BV = (1 + \bar{a})V + (\bar{a} - a)AV$$

our initial eigenvalue problem for B reduces to the corresponding problem for the matrix $T = (1 + \bar{a})I_r + (\bar{a} - a)A$ (restricted to the invariant subspace $\ker J$). In other words up to an affine transformation we are really studying the spectrum of the adjacency matrix of $\mathcal{G}$.

For instance when $\mathcal{G}$ is the complete graph on r vertices, then $A = J - I_r$, and we deduce that for $V \in \ker J$ we have $BV = (1 + a)V$, so that the signature of Q is $(r - 1, 1)$.

This proves part (i) and (iii) of Proposition 4.21. Part (ii) follows from (i) by a standard continuity argument.

We now assume $\mathcal{G}$ is bipartite, and let V be the vector with entries 1 on black vertices and -1 on white vertices. Then $AV = -kV$ so that

$$BV = (1 + \bar{a})V + (\bar{a} - a)AV = (1 + \bar{a} - k(\bar{a} - a))V,$$

so A has a second negative eigenvalue, provided k is large enough with respect to $\bar{a} - a$: specifically $k > (1 + \bar{a})/(\bar{a} - a)$. The discussion above leads to the following:

Lemma 4.24 *For fixed numbers $\frac{1}{2} \leq a < \bar{a} \leq 1$ let k be any natural number such that $k > (1 + \bar{a})/(\bar{a} - a)$. Let $r = 2k$ and let $\mathcal{G}$ be the complete bipartite graph on $k + k$ vertices. Precisely: in $\mathcal{G}$ the integers $i < j$ are connected by an edge iff $1 \leq i \leq k$ and $k + 1 \leq j \leq 2k$.*

Then the signature of the quadratic form $Q_{\mathcal{G},a,\bar{a}}$ is $(r - 2, 2)$. In particular $Q_{\mathcal{G},a,\bar{a}}$ is not degenerate and not hyperbolic.

Proof Let Π^- be the plane spanned by the vector U all of whose entries are 1, and the vector V whose entries are 1 on the first k coordinates, and -1 on the last k coordinates. By the previous discussion and the assumption $k > (1 + \bar{a})/(\bar{a} - a)$ we have $Q_{\mathcal{G},a,\bar{a}} < 0$ on Π^-.

Now let E^+ be the subspace consisting in vectors X such that the sum of the k first coordinates is 0, and the sum of the k last coordinates is 0 too. Observe $\dim(E^+) = 2(k - 1) = r - 2$. Moreover since $X \in \ker(J)$ we have

$$BX = (1 + \bar{a})X + (\bar{a} - a)AX.$$

Clearly $AX = 0$ so that $BX = (1 + \bar{a})X$. Thus $Q_{\mathcal{G},a,\bar{a}} > 0$ on E^+. $\qquad\square$

Corollary 4.25 Let m, m' be natural numbers with $4 \le m < m'$. Let k be any natural number such that

$$k > \frac{1 + \cos\left(\frac{\pi}{m'}\right)}{\cos\left(\frac{\pi}{m'}\right) - \cos\left(\frac{\pi}{m}\right)}.$$

Let $r = 2k$ and let $\mathcal{G}$ be the complete bipartite graph on $k + k$ vertices. Consider the Coxeter system (W, S) of rank r with $m_{ij} = m$ if i, j are linked inside $\mathcal{G}$, and $m_{ij} = m'$ when i, j are linked inside the complementary graph of $\mathcal{G}$.

Then (W, S) is word hyperbolic, but does not act by reflection on $\mathbb{H}^p$ in any dimension.

Note the contrast with the case when all off-diagonal entries are the same number $m \ge 4$, that is the case of $W(r, m)$.

5 Faithful representation of large even-gonal groups into two-dimensional Coxeter groups

5.1 Various complications for the action of a group on the set of hyperplanes

Let X be a simply-connected even-gonal complex such that all polygons have at least four sides, and all vertex links have girth ≥ 4. Let Γ be an automorphism group of X.

In this section we adapt the various definitions leading to the notion of a special action on a CAT(0) cube complex (see Haglund–Wise [16]) to the present context, where ramified hyperplanes are allowable. We then study variations around it.

Since X is even-gonal and CAT(0) we may consider the family $\mathcal{H}$ of its straight hyperplanes. And since X is furthermore large at vertices we may also consider the family $\mathcal{R}$ of its ramified hyperplanes.

For H a hyperplane in either $\mathcal{H}$ or $\mathcal{R}$:

  (i) The group Γ has a *self-intersection at H* provided there is some $\gamma \in \Gamma$ such that γH and H intersect.

  (ii) The group Γ has a *self-osculation at H* provided there is some $\gamma \in \Gamma$ such that γH and H osculate.

For H, K two hyperplanes both in either $\mathcal{H}$ or $\mathcal{R}$:

  (iii) The group Γ has an *inter-osculation at H, K* provided H and K osculate and there is some $\gamma \in \Gamma$ such that γK and H intersect.

(iv) (Here we assume H, K are ramified hyperplanes) The group Γ has an *ambiguous intersection at H, K* provided H and K intersect at the center of a $2m$–gon and there is some $\gamma \in \Gamma$ such that γK and H intersect at the center of a $2m'$–gon with $m' \neq m$.

Definition 5.1 (Special action) The action of Γ on $\mathcal{R}$ (resp, $\mathcal{H}$) is *special* provided none of the above complications arise: Γ has no self-intersection, no self-osculation, no inter-osculation and no ambiguous intersection (this is required only for pairs of ramified hyperplanes).

When X is $2m$–gonal (for example a square complex) then any automorphism group $\Gamma \subset \mathrm{Aut}(X)$ acts without ambiguous intersections.

Note also that when X is a CAT(0) square complex we have $\mathcal{R} = \mathcal{H}$ so the two notions of special actions coincide, and we recover the notion of a C–special group as in Haglund–Wise [16]. In fact considering the action of Γ onto the Sageev CAT(0) cube complex Y associated with the (straight) hyperplanes we note that the Γ–action on Y is cubically special if and only if the Γ–action on $\mathcal{H}$ is special. Indeed the hyperplanes of X and Y are in 1–1 correspondence, intersections of pairs of hyperplane occur in X iff they occurs in Y, and the same equivalence holds true for osculations (since $\mu(X) \geq 4$).

We will consider another kind of situation that is relevant to our context:

For H a hyperplane in $\mathcal{R}$:

(i′) The group Γ has a *self-osculation at distance 1 at H* provided there is some $\gamma \in \Gamma$ such that the polygonal neighborhoods of γH and H are disjoint but connected by an edge.

(ii′) The group Γ has a *self-osculation at distance $\leq n$ at H* provided there is some $\gamma \in \Gamma$ such that $\gamma H \neq H$ but the polygonal neighborhoods of γH and H are connected by a combinatorial path of length $\leq n$.

Thus for example self-intersection or self-osculation at H amounts to self-osculation at distance 0 at H.

Definition 5.2 (Strongly clean) The action of the group Γ on $\mathcal{R}$ has no self-osculation at distance $\leq n$ provided there is no ramified hyperplane H at which it has a self-osculation at distance $\leq n$.

The action of the group Γ on $\mathcal{R}$ is *strongly clean* provided it has no self-osculation at distance ≤ 1.

Lemma 5.3 (Separability properties imply virtually special) *Let X be a locally compact simply-connected even-gonal complex with $\mu(X) \geq 4$. Let Γ act cocompactly on X.*

(i) *If the stabilizers of the ramified hyperplanes are separable subgroups of Γ then for any integer $n \geq 0$ the group Γ has a finite index subgroup Γ' whose action on $\mathcal{R}$ has no self-intersection and no self-osculation at distance $\leq n$.*

(ii) *If double cosets of crossing ramified hyperplane stabilizers are closed in the profinite topology then Γ has a finite index subgroup Γ'' whose action has no inter-osculation and no ambiguous intersection.*

Note the corresponding statement for straight hyperplanes appears in [16] (except of course for the non ambiguous intersections). We provide a complete argument for the convenience of the reader.

Proof (i) Let $H \in \mathcal{R}$ be a ramified hyperplane, and let $N(H)$ be its polygonal neighborhood. We define $B_n(H, \Gamma) = \{g \in \Gamma \mid \text{there is a combinatorial path of length} \leq n \text{ connecting } N(H) \text{ with } g(N(H))\}$.

Then $B_n(H, \Gamma)$ is a union of double cosets modulo the stabilizer Γ_H of H in Γ:

$$B_n(H, \Gamma) = \Gamma_H b_1 \Gamma_H \sqcup \cdots \sqcup \Gamma_H b_k \Gamma_H.$$

Note the union is finite because under the assumptions Γ_H is cocompact on H. By separability, let Γ_1' be a finite index subgroup containing Γ_H and disjoint from $\{b_1, \ldots, b_k\}$. Clearly $B_n(H, \Gamma_1') = \varnothing$. This means that Γ_1' does not self-intersect H, and has no self-osculation at distance $\leq n$ at H. The same properties hold for any finite index subgroup of Γ_1'; in particular we may replace Γ_1' by a finite index subgroup $\Gamma'(H)$ which is normal in Γ.

Choose such a finite index normal subgroup $\Gamma'(H)$ for each ramified hyperplane of a finite family that intersects each orbit of Γ in $\mathcal{R}$ (recall Γ is cocompact on X).

Then the intersection of the finitely many subgroups $\Gamma'(H)$ yields a finite index subgroup Γ' whose action on $\mathcal{R}$ has no self-intersection and no self-osculation at distance $\leq n$.

(ii) We now assume that for each crossing pair (H, K) of ramified hyperplanes the double coset $\Gamma_H \Gamma_K$ is closed in the profinite topology.

So let H, K be crossing ramified hyperplanes and define $\text{Cross}(H, K, \Gamma) = \{g \in \Gamma \mid gK \text{ and } H \text{ are crossing}\}$. Note $1 \in \text{Cross}(H, K, \Gamma)$, and by cocompactness there are finitely many elements $c_0 = 1, c_1, \ldots, c_\ell$ such that

$$\text{Cross}(H, K, \Gamma) = \Gamma_H c_0 \Gamma_K \sqcup \Gamma_H c_1 \Gamma_K \sqcup \cdots \sqcup \Gamma_H c_\ell \Gamma_K.$$

By separability let N be a finite index normal subgroup such that

$$\Gamma_H \Gamma_K \cap \left(\bigcup_{i=1}^{\ell} c_i N \right) = \varnothing.$$

We define $\Gamma_2'' = \Gamma_H N$ and observe that Γ_2'' is a finite index subgroup of Γ that contains Γ_H. Moreover we claim that

$$\text{Cross}\left(H, K, \Gamma_2''\right) = \Gamma_H \left(\Gamma_K \cap \Gamma_2''\right).$$

Indeed for $g \in \text{Cross}(H, K, \Gamma_2'')$ we may write $g = h c_i k$ for some $i = 0, 1, \ldots, \ell$ and with $h \in \Gamma_H, k \in \Gamma_K$, and since $g \in \Gamma_2''$ we also have $g = h'n$ with $h' \in \Gamma_H$ and $n \in N$. It then follows that $c_i k n^{-1} k^{-1} = h^{-1} h' k^{-1}$. Thus $c_i N \cap \Gamma_H \Gamma_K$ is not disjoint, $i = 0$ and $k = h^{-1} h' n \in \Gamma_2'' \cap \Gamma_K$.

As a consequence Γ_2'' has the following property: for $g \in \Gamma_2''$ such that gK crosses H at the center of a polygon P', there exists $\gamma \in \Gamma_H$ such that $\gamma P = P'$ (in particular P and P' have the same number of sides);

Note the above property remains true for any finite index subgroup of Γ_2''. In particular we may replace Γ_2'' by a finite index subgroup $\Gamma''(H, K)$ that is normal in Γ.

Choose such a finite index normal subgroup $\Gamma''(H, K)$ for each pair of crossing ramified hyperplanes of a finite family that intersects each orbit of Γ in $\{(H, K) \in \mathcal{R} \times \mathcal{R}$ such that H, K cross $\}$.

Then the intersection of the finitely many subgroups $\Gamma''(H, K)$ yields a finite index subgroup Γ'' whose action on $\mathcal{R}$ has no ambiguous intersections.

In the construction of Γ_2'' above we may also choose the finite index normal subgroup N so that $\Gamma_H \Gamma_K \cap (\bigcup_{i=1}^{s} d_i N) = \varnothing$, where $d_1, \ldots d_s$ are finitely many elements such that if gK osculates H then $g \in \bigcup_{i=1}^{i=s} \Gamma_H d_i \Gamma_K$. The same argument as above shows that Γ_2'' has no inter-osculation at (H, K). It follows that Γ'' has no inter-osculation on $\mathcal{R}$. $\square$

Corollary 5.4 (Cubically special implies polygonally special) *Let X be a locally compact simply-connected even-gonal complex with $\mu(X) \geq 4$. Assume X is Gromov-hyperbolic. Let $\Gamma \subset \text{Aut}(X)$ be a discrete cocompact subgroup.*

If Γ is virtually cubically special then Γ has a finite index normal subgroup whose action on the set of ramified hyperplanes of X is special and strongly clean.

Proof Since Γ is virtually cubically special and Gromov-hyperbolic it follows by [16] that each quasi-convex subgroup and each double coset of quasiconvex subgroups is separable. We conclude by applying Lemma 5.3. $\square$

5.2 The Coxeter group associated to an action without ambiguous inter-section

In this section we assume X is a CAT(0) even-gonal complex with $\mu(X) \geq 4$. Let $\Gamma \subset \mathrm{Aut}(X)$ be a group acting without ambiguous intersection. We explain how to associate to the Γ–action onto X a Coxeter system (W, S) $(= (W(\Gamma, X), S(\Gamma, X)))$.

5.2.1 Generators For each ramified hyperplane $H \in \mathcal{R}$ we denote by $[H]$ the orbit of H in $\mathcal{R}$ under Γ. For each such orbit let $s_{[H]}$ denote a generating involution. We let $S(X, \Gamma) = \{s_{[H]} \mid H \in \mathcal{R}\}$. So when Γ is cocompact the generating set $S(X, \Gamma)$ is finite.

5.2.2 Relations Consider the map $m: \{H, K\} \mapsto \mathbb{N} \cup \{\infty\}$ that sends a pair of distinct ramified hyperplanes H, K to the number $m_{H,K} = k$ if $N(H) \cap N(K)$ is a $2k$–gon, and to $m_{H,K} = \infty$ otherwise (that is: when $H \cap K = \varnothing$).

Let H, K be two crossing ramified hyperplanes, so that $N(H) \cap N(K)$ consists in some polygon P with $2m$ sides. Since Γ acts without ambiguous intersection for any two crossing ramified hyperplanes H', K' with $H' \in [H], K' \in [K]$ the intersection $N(H') \cap N(K')$ consists in a single polygon P' having again $2m$ sides. This means that the map $\{H, K\} \mapsto m_{H,K}$ induces a map on the set of pairs of distinct orbits of ramified hyperplanes. We denote it by $m_{[H],[K]}$, and extend as usual this map on the full set of pairs by setting $m_{[H],[H]} = 1$.

We then impose the relations

$$(s_{[H]} s_{[K]})^{m_{[H],[K]}} = 1$$

and denote by $(W(\Gamma, X), S(\Gamma, X))$ the corresponding Coxeter system.

5.2.3 The W–distance For each combinatorial path $\pi = (v_0, v_1, \ldots, v_n)$ we define the W–*length* of π (denoted $\mathrm{length}_W(\pi)$) to be the product $s_{(v_0, v_1)} s_{(v_1, v_2)} \cdots s_{(v_{n-1}, v_n)}$ where $s_{(v_i, v_{i+1})} = 1$ if $v_i = v_{i+1}$ and otherwise $s_{(v_i, v_{i+1})}$ is the generator $s_{[H]}$ where $[H]$ is the Γ–orbit of the ramified hyperplane dual to the edge joining v_i to v_{i+1}.

Lemma 5.5 (Properties of the W–length)

   (i) *For $g \in \Gamma$ we have $\mathrm{length}_W(g\pi) = \mathrm{length}_W(\pi)$.*

   (ii) *For concatenable paths π_1, π_2 we have*

$$\mathrm{length}_W(\pi_1 \pi_2) = \mathrm{length}_W(\pi_1) \, \mathrm{length}_W(\pi_2).$$

   (iii) *If π and π' are homotopic with fixed extremities then*

$$\mathrm{length}_W(\pi) = \mathrm{length}_W(\pi').$$

Proof We give an argument only for the third property. Since X is simply-connected there is a sequence $(\pi_0 = \pi, \pi_1, \ldots, \pi_k = \pi')$ such that π_i and π_{i+1} differ by one of the three types of elementary moves:

(i) $(\ldots, v, v, \ldots) \leftrightarrow (\ldots, v, \ldots)$,

(ii) $(\ldots, v, w, v, \ldots) \leftrightarrow (\ldots, v, \ldots)$,

(iii) polygonal move (or "exchange condition"); see below.

The homotopy invariance of the W–length is obvious in the case of the two first moves. Thus it suffices to consider the case when π and π' are two complementary paths of the boundary of a polygon P. Let H, K be the two ramified hyperplanes dual two edges of P. Then $(s_{[H]}s_{[K]})^{m_{H,K}} = 1$ and this can be reformulated as $\mathrm{length}_W(\pi) = \mathrm{length}_W(\pi')$. □

Corollary 5.6 (W–distance)   *There is a map $\delta \,(= \delta_W)\colon X^0 \times X^0 \to W$ such that*

(i)   *for any combinatorial path π from p to q we have $\delta(p,q) = \mathrm{length}_W(\pi)$,*

(ii) *for any three vertices p, q, r we have $\delta(p,r) = \delta(p,q)\delta(q,r)$, and*

(iii) *for any two vertices p, q and any element $g \in \Gamma$ we have $\delta(gp, gq) = \delta(p,q)$.*

The above map δ_W we call the W–*distance* on X^0.

5.2.4   Morphisms $\Gamma \to W$ and polygonal maps $X \to \Sigma(W, S)$ For any vertex $p \in X^0$ we consider two maps:

$$f_p\colon X^0 \longrightarrow W \qquad\qquad \varphi_p\colon \Gamma \longrightarrow W$$
$$q \longmapsto \delta_W(p,q) \qquad\qquad \gamma \longmapsto \delta_W(p, \gamma p)$$

Lemma 5.7   (i)   *φ_p is a morphism and $\varphi_q = \delta(p,q)^{-1}\varphi_p\delta(p,q)$.*

(ii) *f_p defines a simplicial map $X^1 \to \Sigma^1(W, S)$.*

(iii) *f_p is φ_p–equivariant.*

Proof   We use Corollary 5.6.

(i)   We have

$$\varphi_p(\gamma_1\gamma_2) = \delta_W(p, \gamma_1\gamma_2 p) = \delta_W(p, \gamma_1 p)\delta_W(\gamma_1 p, \gamma_1\gamma_2 p) = \varphi_p(\gamma_1)\varphi_p(\gamma_2).$$

For two vertices p, q in X and for $\gamma \in \Gamma$ we have

$$\varphi_q(\gamma) = \delta_W(q, \gamma q) = \delta_W(q, p)\delta_W(p, \gamma p)\delta_W(\gamma p, \gamma q) = \delta_W(p,q)^{-1}\varphi_p(\gamma)\delta_W(p,q).$$

(ii) Assume q, q' are the endpoints of an edge e. If H is the ramified hyperplane dual to e then $\delta_W(q, q') = s_{[H]}$. Thus

$$f_p(q') = \delta_W(p, q') = \delta_W(p, q)\delta_W(q, q') = f_p(q)s_{[H]}.$$

It follows that $f_p(q), f_p(q')$ are the endpoints of an edge of $\Sigma(W, S)$.

(iii) We have

$$f_p(\gamma q) = \delta_W(p, \gamma q) = \delta_W(p, \gamma p)\delta_W(\gamma p, \gamma q) = \varphi_p(\gamma) f_p(q). \qquad \square$$

Thus for each choice of a basepoint p in X we get a morphism $\varphi_p\colon \Gamma \to W$, which we call a *special representation*.

5.2.5 Naturality of the construction

Theorem 5.8 (Normalizer extension) *Let X be a* CAT(0) *even-gonal complex with $\mu(X) \geq 4$. Assume $\overline{\Gamma} \subset \operatorname{Aut}(X)$ has a normal subgroup Γ whose action on X has no ambiguous intersections.*

Let (W, S) be the Coxeter group associated with the Γ–action on X. Then for each choice p of a base vertex in X there is a natural action of $\overline{\Gamma}$ onto $\Sigma(W, S)$ with the following properties:

- (i)   *The morphism $\overline{\varphi}_p\colon \overline{\Gamma} \to \operatorname{Aut}(\Sigma(W, S))$ extends $\varphi_p\colon \Gamma \subset \operatorname{Aut}(X) \to W \subset \operatorname{Aut}(\Sigma(W, S))$.*
- (ii) *The image of $\overline{\Gamma}$ in $\operatorname{Aut}(\Sigma(W, S))$ normalizes W.*
- (iii) *$f_p\colon X^1 \to \Sigma^1(W, S)$ is $\overline{\varphi}_p$–equivariant.*

Proof Since Γ is normal in $\overline{\Gamma}$ it follows that $\overline{\Gamma}$ acts on the set S of Γ–orbits in the set $\mathcal{R}$ of ramified hyperplanes. For each $\overline{\gamma} \in \overline{\Gamma}$ let $\rho(\overline{\gamma})$ be the corresponding permutation of S. If $[H], [K]$ have representative H, K such that $N(H) \cap N(K)$ is a $2m$–gon P then for $\overline{\gamma} \in \overline{\Gamma}$ we have $N(\overline{\gamma}H) \cap N(\overline{\gamma}K) = \overline{\gamma}P$, a $2m$–gon again. Thus the permutation morphism $\rho\colon \overline{\Gamma} \to \mathfrak{S}(S)$ extends to a morphism $\rho\colon \overline{\Gamma} \to \operatorname{Aut}(W, S)$. Note $\rho = \operatorname{id}_W$ on Γ. Note also that for any combinatorial path π and any element $\overline{\gamma} \in \overline{\Gamma}$ we have

$$\operatorname{length}_W(\overline{\gamma}\pi) = \rho(\overline{\gamma})(\operatorname{length}_W(\pi)).$$

It follows that $\delta_W(\overline{\gamma}p, \overline{\gamma}q) = \rho(\overline{\gamma})(\delta_W(p, q))$.

Let p denote a fixed vertex inside X. For each $\overline{\gamma} \in \overline{\Gamma}$ we define an automorphism $\overline{\varphi}_p(\overline{\gamma}) \in \operatorname{Aut}(\Sigma(W, S))$. We first define $\overline{\varphi}_p(\overline{\gamma})$ as a permutation of $\Sigma^0(W, S) = W$, by setting

$$\overline{\varphi}_p(\overline{\gamma})(w) = \delta_W(p, \overline{\gamma}p)\rho(\overline{\gamma})(w).$$

Observe that

$$
\begin{aligned}
\bar{\varphi}_p(\bar{\gamma}_1\bar{\gamma}_2)(w) &= \delta_W(p, \bar{\gamma}_1\bar{\gamma}_2 p)\rho(\bar{\gamma}_1\bar{\gamma}_2)(w) \\
&= \delta_W(p, \bar{\gamma}_1 p)\delta_W(\bar{\gamma}_1 p, \bar{\gamma}_1\bar{\gamma}_2 p)\rho(\bar{\gamma}_1)\rho(\bar{\gamma}_2) \\
&= \delta_W(p, \bar{\gamma}_1 p))\rho(\bar{\gamma}_1)[\delta_W(p, \bar{\gamma}_2 p)]\rho(\bar{\gamma}_1)[\rho(\bar{\gamma}_2)(w)] \\
&= \bar{\varphi}_p(\bar{\gamma}_1)[\bar{\varphi}_p(\bar{\gamma}_2)(w)].
\end{aligned}
$$

Thus $\bar{\varphi}_p\colon \Gamma \to \mathfrak{S}(W)$ is indeed a morphism. We then note that for each $\bar{\gamma} \in \overline{\Gamma}$ the permutation $\bar{\varphi}_p(\bar{\gamma})$ defines a simplicial automorphism of $\Sigma^1(W, S)$. Indeed for $s = s_{[H]} \in S$ we have

$$
\begin{aligned}
\bar{\varphi}_p(\bar{\gamma})(ws) &= \delta_W(p, \bar{\gamma} p)\rho(\bar{\gamma})(ws) \\
&= \delta_W(p, \bar{\gamma} p)\rho(\bar{\gamma})(w)\rho(\bar{\gamma})(s) \\
&= \bar{\varphi}_p(\bar{\gamma})(w)s_{[\bar{\gamma} H]}.
\end{aligned}
$$

This computation indeed shows that two edges having the same S–label (say: s) are mapped to two edges with the same label ($\rho(\bar{\gamma})(s)$). Thus $\bar{\varphi}_p(\bar{\gamma})$ extends to a unique automorphism of the Davis complex $\Sigma(W, S)$, and moreover $\bar{\varphi}_p\colon \overline{\Gamma} \to \operatorname{Aut}(\Sigma(W, S))$ is a morphism.

Note that for any $\gamma \in \Gamma$ and any $w \in W$ we have

$$
\bar{\varphi}_p(\gamma)(w) = \delta_W(p, \gamma p)\rho(\gamma)(w) = \delta_W(p, \gamma p)w = \varphi_p(\gamma)w.
$$

Thus $\bar{\varphi}_p$ extends φ_p.

It remains to check that $\bar{\varphi}_p(\bar{\gamma})$ normalizes $W \subset \operatorname{Aut}(\Sigma(W, S))$. So let $w \in W \subset \operatorname{Aut}(\Sigma(W, S))$ and let us compute the $\bar{\varphi}_p(\bar{\gamma})$–conjugate of w. For $g \in W = \Sigma^0(W, S)$ we have

$$
\begin{aligned}
[\bar{\varphi}_p(\bar{\gamma}) \circ w \circ \bar{\varphi}_p(\bar{\gamma})^{-1}](g) &= \bar{\varphi}_p(\bar{\gamma})\big(w\bar{\varphi}_p(\bar{\gamma}^{-1})(g)\big) \\
&= \bar{\varphi}_p(\bar{\gamma})\big(w\delta_W(p, \bar{\gamma}^{-1} p)\rho(\bar{\gamma}^{-1})(g)\big) \\
&= \delta_W(p, \bar{\gamma} p)\rho(\bar{\gamma})\big(w\delta_W(p, \bar{\gamma}^{-1} p)\rho(\bar{\gamma}^{-1})(g)\big) \\
&= \delta_W(p, \bar{\gamma} p)\rho(\bar{\gamma})(w)\rho(\bar{\gamma})\big(\delta_W(p, \bar{\gamma}^{-1} p)\big)g.
\end{aligned}
$$

Thus $\bar{\varphi}_p(\bar{\gamma}) \circ w \circ \bar{\varphi}_p(\bar{\gamma})^{-1}$ acts on $\Sigma^0(W, S)$ as the multiplication by an element of W (namely $\delta_W(p, \bar{\gamma} p)\rho(\bar{\gamma})(w)\rho(\bar{\gamma})(\delta_W(p, \bar{\gamma}^{-1} p)))$. It follows that $\bar{\varphi}_p(\bar{\gamma})$ normalizes W inside $\operatorname{Aut}(\Sigma(W, S))$. $\qquad\square$

Remark 5.9 The naturality statement above holds also for virtually special actions on CAT(0) cube complexes. In other words if $\overline{\Gamma}$ acts geometrically on a CAT(0) cube

complex X and has a finite index subgroup Γ whose action is special (C–special in the sense of [16]), then each special representation $\varphi_p\colon \Gamma \to W(\Gamma, X)$ extends naturally to a morphism $\overline{\varphi}_p\colon \overline{\Gamma} \to \mathrm{Aut}(\Sigma(W(\Gamma, X))$ whose image is contained in the normalizer of $W(\Gamma, X)$.

5.3 The special representation is faithful and convex cocompact when the action is special

Proposition 5.10 *Let X be a CAT(0) even-gonal complex with $\mu(X) \geq 4$. Let $\Gamma \subset \mathrm{Aut}(X)$ be a group acting without ambiguous intersections. Let (W, S) $(= (W(\Gamma, X), S(\Gamma, X)))$ be the associated Coxeter system.*

Then (for each vertex $p \in X^0$) the map $f_p\colon X^1 \to \Sigma^1(W, S)$ extends to a local isometry $f_p\colon X \to \Sigma(W, S)$ if and only if the Γ–action is special.

Proof The argument is the same as for actions on CAT(0) cube complexes. Here are more details.

Claim 1 $f_p\colon X^1 \to \Sigma^1(W, S)$ *is locally injective iff the Γ–action on the set of ramified hyperplanes has no self-intersection and no self-osculation.*

Indeed let a, b be two distinct edges through some vertex $q \in X^0$. Let x, y denote the endpoints of a, b distinct from q. Recall $f_p(x) = f_p(q)\delta_W(q, x)$ and $f_p(y) = f_p(q)\delta_W(q, y)$. Thus $f(a) = f(b)$ if and only if $\delta_W(q, x) = \delta_W(q, y)$. And by construction of δ_W we have $\delta_W(q, x) = \delta_W(q, y)$ if and only if the ramified hyperplanes dual to a, b are in the same Γ–orbit. According to Lemma 2.21 this corresponds either to a self-intersection (when a, b are adjacent in $\mathrm{link}(q, X)$) or to a self-osculation (when a, b are not adjacent in $\mathrm{link}(q, X)$).

Now assume that $f_p\colon X^1 \to \Sigma^1(W, S)$ is locally injective. For any $2m$–gon P of X let H, K be the two ramified hyperplanes through the center of P. Since Γ has no self-intersection we have $[H] \neq [K]$, and thus a boundary path π winding once around ∂P is mapped under f_p to a closed edge path of length $2m = 2m_{[H][K]}$, whose edge-labels alternate between $s_{[H]}$ and $s_{[K]}$. Thus $f_p(\pi)$ is the boundary path of some 2–cell of $\Sigma(W, S)$. In other words $f_p\colon X^1 \to \Sigma^1(W, S)$ extends to a polygonal map $f_p\colon X \to \Sigma(W, S)$.

Claim 2 *Assuming that $f_p\colon X^1 \to \Sigma^1(W, S)$ is locally injective, the polygonal extension $f_p\colon X \to \Sigma(W, S)$ is a local isometry if and only if the Γ–action on the set of ramified hyperplanes has no inter-osculation.*

Indeed f_p fails to be a local isometry at some vertex q exactly if there are two edges a, b containing q such that a, b are not contained in a polygon of X but $f_p(a), f_p(b)$ are contained in a polygon of $\Sigma(W, S)$. This latter condition exactly means that there exists $g \in \Gamma$ such that gH crosses K, where H, K denote the ramified hyperplanes dual to a, b. Since a, b are not contained in a polygon of X the ramified hyperplanes H, K are osculating by Lemma 2.21). Thus Γ has an inter-osculation at H, K. □

Combining Corollary 5.4, Proposition 5.10 and Theorem 5.8 we obtain:

Corollary 5.11 *Let X be a Gromov-hyperbolic CAT(0) even-gonal complex with $\mu(X) \geq 4$. Assume $\overline{\Gamma} \subset \text{Aut}(X)$ is a discrete cocompact subgroup and that $\overline{\Gamma}$ is virtually cubically special. Then:*

  (i) *$\overline{\Gamma}$ has a finite index normal subgroup Γ whose action on X has no ambiguous intersections and whose action on the set of ramified hyperplanes of X is special and strongly clean.*

  (ii) *Let (W, S) be the Coxeter group associated with the Γ–action on X. Then (for each base vertex $p \in X$) there is a monomorphism $\bar{\varphi}_p \colon \overline{\Gamma} \to \text{Aut}(\Sigma(W, S))$ and there is a $\bar{\varphi}_p$–equivariant isometric embedding $f_p \colon X \to \Sigma(W, S)$. Moreover the image of $\overline{\Gamma}$ is contained in the normalizer of W inside $\text{Aut}(\Sigma(W, S))$.*

Observe that so far we have proved the first part of Theorem 1.4. Yet the target Coxeter group W is not 2–spherical in general. The additional property of strong cleanliness will be used to remedy this. But let us first make the following:

Remark 5.12 (Equivalence of the two notions of virtually special) Let X be a CAT(0) even-gonal complex with $n(X) \geq 8$ and $\mu(X) \geq 4$ and let Γ be a discrete cocompact subgroup of $\text{Aut}(X)$. Then Γ is virtually cubically special if and only if Γ has a finite index subgroup whose action on X has no ambiguous intersections and whose action on the set of ramified hyperplanes of X is special.

The $\Rightarrow$ direction follows from Corollary 5.11 above. The $\Leftarrow$ direction follows since by Proposition 5.10 the group Γ is virtually convex cocompact in a Coxeter group with all m_{ij} at least 4, and by Haglund–Wise [17] every hyperbolic Coxeter group is virtually cubically special.

5.4 Constructing a convex cocompact 2–spherical representation

The goal of this section is to complete the proof Theorem 1.4.

Lemma 5.13 (Embedding a subcomplex in a Coxeter quotient) *Let (W, S) be a Coxeter system all of whose finite m_{ij} are ≥ 3, so that $\Sigma(W, S)$ is a CAT(0) polygonal complex.*

Let $X \subset \Sigma(W, S)$ be a convex subcomplex. Assume that for any i, j with $m_{ij} = +\infty$ the connected components of the forest $X \cap \Sigma^1_{ij}(W, S)$ have diameter $\leq D_{ij} - 1$.

Choose arbitrary numbers $M_{ij} \geq D_{ij}$ for each pair i, j with $m_{ij} = +\infty$. If $m_{ij} < \infty$ set $M_{ij} = m_{ij}$. Let (V, S) be the Coxeter system with Coxeter matrix (M_{ij}). Then the polygonal map $\Sigma(W, S) \to \Sigma(V, S)$ induces an isometric embedding $X^1 \to \Sigma^1(V, S)$.

Here $\Sigma^1_{ij}(W, S)$ denotes the subgraph of the 1–skeleton $\Sigma^1(W, S)$ whose edges have label either i or j. Since $m_{ij} = \infty$ each component of $\Sigma^1_{ij}(W, S)$ is a line.

Proof The polygonal map $X \to \Sigma(V, S)$ has no missing half-cell and thus we may apply [Proposition 3.2](). $\qquad \square$

Lemma 5.14 (Geometry of strongly clean) *Let Γ denote a uniform lattice of a CAT(0) even-gonal complex X, with $\mu(X) \geq 4$. Assume Γ acts on $\mathcal{R}$ in a special and strongly clean way.*

Let (W, S) be the Coxeter system associated with the Γ–action and consider the polygonal embedding $f_p \colon X \to \Sigma(W, S)$ (for some fixed vertex $p \in X^0$).

Then for any i, j with $m_{ij} = +\infty$ the connected components of the forest $f_p(X) \cap \Sigma^1_{ij}(W, S)$ have diameter ≤ 2.

Proof Let $(\bar{a}, \bar{b}, \bar{c})$ be an edge path of length 3 in $\Sigma^1_{ij}(W, S)$. Assume the labels of $\bar{a}, \bar{c}$ are i, and the label of $\bar{b}$ is j, and assume moreover that $\bar{a} = f_p(a), \bar{b} = f_p(b)$ are contained in $f_p(X)$. Let v be the vertex of b not contained in a. We denote by H (H') the ramified hyperplanes dual to a (b). The label i (j) corresponds to the Γ–orbit of H (H').

To conclude we claim that the ramified hyperplane K dual to any edge e containing v does not receive the label i. Indeed if $a \cup e$ is contained in some polygon of X, then $a \cup b$ is contained in the same polygon and this contradicts $m_{ij} = \infty$. So $a \cup e$ is contained in no polygon of X, and (since e is connected to a through the edge b) this implies that e is not in the polygonal neighborhood $N(H)$ of H. Then since the action of Γ is strongly clean and $N(K)$ is connected to $N(H)$ by the edge b we deduce that K is not in the same Γ–orbit as H. $\qquad \square$

Proof of Theorem 1.4 Let $\overline{\Gamma}$ denote a uniform lattice of a CAT(0) even-gonal complex X, with $\mu(X) \geq 4$ and $n(X) \geq 6$. Assume $\overline{\Gamma}$ is virtually cubically special.

Apply Corollary 5.11 to produce a finite index normal subgroup Γ whose action on X is special (with associated Coxeter system (W, S)) and strongly clean. Extend the Γ–action on $\Sigma(W, S)$ to a $\overline{\Gamma}$–action as in Corollary 5.11.

Choose some finite natural number $m \geq 3$: for example set $m = n(X)/2$. Observe that when all polygons of X have the same number of sides – say $2m \geq 6$ – the previous formula yields precisely m.

Let (V, S) be the 2–spherical Coxeter system obtained by replacing by m each infinite m_{ij} in the Coxeter matrix of (W, S). Any automorphism of the diagram of (W, S) is still an automorphism of the diagram of (V, S), and thus the quotient map $W \to V$ induces a surjection of the semi-direct product of W with $\text{Aut}_{\text{diag}}(W, S)$ onto the semi-direct product of V with $\text{Aut}_{\text{diag}}(V, S)$. In other words the morphism $W \to V$ extends to a morphism $\text{Normalizer}(W, \text{Aut}(\Sigma(W, S))) \to \text{Normalizer}(V, \text{Aut}(\Sigma(V, S)))$ (see Lemma 2.28). And thus we can extend the composition $\Gamma \to W \to V$ to a morphism $\overline{\Gamma} \to \text{Normalizer}(W, \text{Aut}(\Sigma(W, S))) \to \text{Normalizer}(V, \text{Aut}(\Sigma(V, S)))$. By applying successively Lemma 5.14 and Lemma 5.13 we deduce that the composition $X \to \Sigma(W, S) \to \Sigma(V, S)$ is injective. By equivariance it follows that the morphism $\overline{\Gamma} \to \text{Aut}(\Sigma(V, S))$ is also injective.

When $n(X) \geq 8$ then (V, S) is hyperbolic and thus convex cocompactness follows since $\overline{\Gamma}$ preserves the quasi-isometrically embedded subcomplex X. □

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