

Commuting tuples in reductive groups and their maximal compact subgroups

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Let G be a reductive algebraic group and $K \subset G$ a maximal compact subgroup. We consider the representation spaces $\text{Hom}(\mathbb{Z}^k, K)$ and $\text{Hom}(\mathbb{Z}^k, G)$ with the topology induced from an embedding into K^k and G^k , respectively. The goal of this paper is to prove that $\text{Hom}(\mathbb{Z}^k, K)$ is a strong deformation retract of $\text{Hom}(\mathbb{Z}^k, G)$.

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1 Introduction[†]

By a commuting k -tuple in a group G we mean a k -tuple $(a_1, \dots, a_k) \in G^k$ with $a_i a_j = a_j a_i$ for all i and j . Observe that any such commuting k -tuple determines uniquely a homomorphism

$$\mathbb{Z}^k \rightarrow G, \quad (n_1, \dots, n_k) \mapsto a_1^{n_1} \cdots a_k^{n_k}.$$

In fact, we obtain in this way a bijection between $\text{Hom}(\mathbb{Z}^k, G)$ and the set of commuting k -tuples in G .

Suppose now that G is a Lie group and that $K \subset G$ is a closed subgroup. Endowing $\text{Hom}(\mathbb{Z}^k, G) \subset G^k$ and $\text{Hom}(\mathbb{Z}^k, K) \subset K^k$ with the subspace topology, we consider the embedding

$$(1.1) \quad \text{Hom}(\mathbb{Z}^k, K) \hookrightarrow \text{Hom}(\mathbb{Z}^k, G)$$

induced by the inclusion of K into G . The goal of this paper is to prove that the map (1.1) is a homotopy equivalence if G is a reductive algebraic group and $K \subset G$ is a maximal compact subgroup.

Before stating our main result precisely, recall that a *weak retraction* of a topological space X to a subspace A is a continuous map

$$F: [0, 1] \times X \rightarrow X, \quad (t, x) \mapsto F_t(x)$$

[†]Per me si va ne la città dolente, per me si va ne l'eterno dolore, per me si va tra la perduta gente.

with $F_0 = \text{Id}_X$ and $F_1(X) \subset A$, and such that $F_t(A) \subset A$ for all t .

Theorem 1.1 *Let G be the group of complex or real points of a reductive algebraic group, defined over \mathbb{R} in the latter case, and $K \subset G$ a maximal compact subgroup. For all k there is a weak retraction of $\text{Hom}(\mathbb{Z}^k, G)$ to $\text{Hom}(\mathbb{Z}^k, K)$.*

Before going further, recall that the presence of a weak retraction implies that the inclusion of $\text{Hom}(\mathbb{Z}^k, K)$ into $\text{Hom}(\mathbb{Z}^k, G)$ is a homotopy equivalence (Hatcher [18, page 18]). Notice also that $\text{Hom}(\mathbb{Z}^k, K)$ and $\text{Hom}(\mathbb{Z}^k, G)$ are algebraic varieties and hence homeomorphic to CW-complexes (Whitney [38]). It follows thus from Whitehead's Theorem [18, page 346] that in fact $\text{Hom}(\mathbb{Z}^k, K)$ is a strong deformation retract of $\text{Hom}(\mathbb{Z}^k, G)$.

Corollary 1.2 *If G is the group of complex or real points of a reductive algebraic group, defined over \mathbb{R} in the latter case, and $K \subset G$ a maximal compact subgroup, then $\text{Hom}(\mathbb{Z}^k, K)$ is a strong deformation retract of $\text{Hom}(\mathbb{Z}^k, G)$ for all k . \square*

If we replace the free abelian group \mathbb{Z}^k by the free group \mathbb{F}_k , then the analogues of Theorem 1.1 and Corollary 1.2 follow directly from the well-known fact that G retracts to K . It is worth noticing that such an argument cannot be applied to prove Theorem 1.1 because in general there is no retraction of G to K that preserves commutativity (Souto [34]).

It goes without saying that for a general finitely generated group Γ , the inclusion of $\text{Hom}(\Gamma, K)$ into $\text{Hom}(\Gamma, G)$ is not a homotopy equivalence. For example, if $\Gamma \subset \text{SL}_2 \mathbb{C}$ is a cocompact lattice then it follows from Mostow's Rigidity Theorem – or already from Weil's Local Rigidity Theorem – that the component of $\text{Hom}(\Gamma, \text{SL}_2 \mathbb{C})$ containing the tautological representation $\Gamma \hookrightarrow \text{SL}_2 \mathbb{C}$ consists of conjugates of this representation and hence is disjoint from $\text{Hom}(\Gamma, \text{SU}_2)$. Also, invariance of the Euler number implies that there are components of $\text{Hom}(\pi_1(\Sigma), \text{PSL}_2 \mathbb{R})$ that do not intersect $\text{Hom}(\pi_1(\Sigma), \text{SO}_2)$, where Σ is a closed orientable surface of genus at least 2 (Goldman [14]).

Remark The proof of Theorem 1.1 applies, with only minor modifications, if we replace \mathbb{Z}^k by any other finitely generated abelian group. In a subsequent paper, Lior Silberman and the second author use the results in the paper to show that Theorem 1.1 still holds true if \mathbb{Z}^k is replaced by for instance a free nilpotent group or a Heisenberg group.

Consequences of Theorem 1.1 Recall that the space $\text{Hom}(\mathbb{Z}^k, G)$ may be identified with the space of (framed) *flat* G -connections on the k -torus modulo based gauge transformations. As such, these and other related spaces play a role in, for example, gauge theory, mathematical physics and symplectic geometry; see Jeffrey's nice survey describing these different points of view [21]. In fact, it was the work of Witten [40; 41] which motivated Borel, Friedman and Morgan [9], and Kac and Smilga [22] to initiate the systematic study of the topological properties of the spaces $\text{Hom}(\mathbb{Z}^k, G)$. While the papers [9] and [22] concentrate on understanding the number of connected components of $\text{Hom}(\mathbb{Z}^k, G)$ for certain values of k , higher topological invariants of these spaces, such as cohomology groups, were first investigated by Adem and Cohen in [1]. Starting with the latter paper, there have been a wealth of results in this direction: Adem, Cohen and Gómez [2; 3], Adem, Cohen and Torres-Giese [4], Baird [5], Baird, Jeffrey and Selik [6], Crabb [12], Higuera [19], Gómez, Pettet and Souto [15] and Torres-Giese and Sjerve [37].

At this point it should be mentioned that all results proved in the papers just mentioned concern, almost without exception, the space of commuting tuples in compact groups. Combining these results with Theorem 1.1 we obtain hence the corresponding results for reductive groups. We give just a few examples of this strategy:

Corollary 1.3 *Suppose that G is the set of complex points of a semisimple algebraic group. The space $\text{Hom}(\mathbb{Z}^k, G)$ is connected if and only if one of the following conditions is satisfied:*

- (1) $k = 1$ and G is connected,
- (2) $k = 2$ and G is connected and simply connected, or
- (3) $k \geq 3$ and G is a product of copies of $\text{SL}_n \mathbb{C}$ and $\text{Sp}_n \mathbb{C}$.

Proof Let K be a maximal compact subgroup of G , and recall that K is homotopy equivalent to G . It is due to Kac and Smilga [22] that $\text{Hom}(\mathbb{Z}^k, K)$ is connected if and only if $k = 1$ and K is connected, $k = 2$ and K is connected and simply connected, or $k \geq 3$ and K is a product of copies of SU_n and Sp_n for some $n \geq 1$. As K determines G up to isomorphism (see Serre [31, VIII]), and as SU_n and Sp_n are maximal compact subgroups of $\text{SL}_n \mathbb{C}$ and $\text{Sp}_n \mathbb{C}$, the claim follows from Theorem 1.1. \square

Remark In the proof of Corollary 1.3 we have used that complex reductive groups are determined by their maximal compact subgroups; this is not true in the real case: $\text{O}_n(\mathbb{R})$ is a maximal compact subgroup of both $\text{GL}_n \mathbb{R}$ and $\text{SO}_{(n,1)} \mathbb{R}$. Theorem 1.1 implies that the varieties of commuting k -tuples of these two groups are homotopy equivalent; at first glance, the authors find this somehow surprising.

If K is a compact group, then the connected component $\text{Hom}(\mathbb{Z}^k, K)_\perp$ of the trivial representation in $\text{Hom}(\mathbb{Z}^k, K)$ is much better understood than is the whole space. For instance, its fundamental group [15] and rational cohomology groups [5] are known.

Corollary 1.4 *If G is a reductive group, complex or real, then $\pi_1(\text{Hom}(\mathbb{Z}^k, G)_\perp) = \pi_1(G)^k$ for all k . Here, $\text{Hom}(\mathbb{Z}^k, G)_\perp$ is the connected component in $\text{Hom}(\mathbb{Z}^k, G)$ of the trivial representation.*

Proof Let K be a maximal compact subgroup of G . It follows from Theorem 1.1 that $\text{Hom}(\mathbb{Z}^k, G)_\perp$ and $\text{Hom}(\mathbb{Z}^k, K)_\perp$ are homotopy equivalent and hence have isomorphic fundamental groups. By [15] we have that $\pi_1(\text{Hom}(\mathbb{Z}^k, K)_\perp)$ and $\pi_1(K)^k$ are isomorphic. Since K and G are also homotopy equivalent, the result follows. \square

Corollary 1.5 *Let G be a connected reductive group, complex or real, $T \subset G$ a maximal torus, and $W = \mathcal{N}_G(T)/\mathcal{Z}_G(T)$ the Weyl-group of G . We have*

$$H^*(\text{Hom}(\mathbb{Z}^k, G)_\perp, F) \simeq H^*(G/T \times T^k, F)^W,$$

where F is any field with characteristic either 0 or relatively prime to the order W .

Proof Let K be a maximal compact subgroup of G such that $T_K = T \cap K$ is a maximal torus in K ; notice that this implies that the inclusion $T_K \rightarrow T$ is a homotopy equivalence. In particular, the inclusion

$$K/T_K \times T_K^k \rightarrow G/T \times T^k$$

is a W -equivariant homotopy equivalence and hence we have:

$$(1.2) \quad H^*(K/T_K \times T_K^k, F)^W \simeq H^*(G/T \times T^k, F)^W.$$

It is due to Baird [5, Theorem 4.3] that the left side of (1.2) is isomorphic to $H^*(\text{Hom}(\mathbb{Z}^k, K)_\perp, F)$ and hence to $H^*(\text{Hom}(\mathbb{Z}^k, G)_\perp, F)$ by Theorem 1.1. \square

Remark From the proof of Theorem 1.1 we obtain the following description of $\text{Hom}(\mathbb{Z}^k, G)_\perp$: a commuting tuple (a_1, \dots, a_k) belongs to $\text{Hom}(\mathbb{Z}^k, G)_\perp$ if and only if there is a torus $T \subset G$ containing the semisimple part of the Jordan decomposition of a_i for all i .

Topological considerations aside, observe that, with notation as before, we can also consider $\text{Hom}(\mathbb{Z}^k, G)$ as an algebraic variety. In [29], Richardson proved that $\text{Hom}(\mathbb{Z}^2, G)$ is an irreducible variety whenever G is the group of complex points of a connected, simply connected and semisimple algebraic group. Combining Richardson’s result with the fact that an irreducible complex variety is connected in the Hausdorff topology, we deduce from Corollary 1.3:

Corollary 1.6 *Let G be the group of complex points of a semisimple algebraic group. The variety of commuting pairs in G is irreducible if and only if G is connected and simply connected.* □

Similarly, we obtain from Corollary 1.3 that for G simple and $k \geq 3$, the variety $\text{Hom}(\mathbb{Z}^k, G)$ is reducible, unless possibly for $G = \text{SL}_n \mathbb{C}$ and $G = \text{Sp}_n \mathbb{C}$. This result is far from optimal: it follows easily from the work of Gerstenhaber [13] that $\text{Hom}(\mathbb{Z}^k, \text{SL}_n \mathbb{C})$ is reducible for $k, n \geq 4$; see also Guralnick [17] and the references therein for more recent results.

Remark For comparison, it is easy to see that $\text{Hom}(\mathbb{Z}^k, \text{SU}_n)$ is an irreducible analytic space for all k and n , meaning that it is not the union of two proper analytic subsets. To see that this is the case, let \mathbb{T} be the group of diagonal matrices in SU_n and observe that the analytic map

$$\text{SU}_n \times \mathbb{T}^k \rightarrow \text{Hom}(\mathbb{Z}^k, \text{SU}_n), \quad (g, a_1, \dots, a_k) \mapsto (ga_1g^{-1}, \dots, ga_kg^{-1})$$

is surjective. The domain is a connected analytic manifold and hence irreducible as an analytic space; the image of an irreducible analytic space under an analytic map is irreducible.

Outline of the paper We will divide the proof of Theorem 1.1, and with it the whole of this paper, into two distinct parts, as the arguments lend themselves naturally to this partition. In the second part of the paper (Sections 7–8), we prove that there is a commutativity-preserving retraction of G to G^s , the subset consisting of semisimple elements. It follows directly that there is a retraction of $\text{Hom}(\mathbb{Z}^k, G)$ to $\text{Hom}(\mathbb{Z}^k, G^s)$, the space of commuting semisimple k -tuples. In the first part of the paper (Sections 2–6), we prove that $\text{Hom}(\mathbb{Z}^k, K)$ is a retract of $\text{Hom}(\mathbb{Z}^k, G^s)$, completing the proof of Theorem 1.1.

Sections 2–6 The goal of these sections is to prove that the space $\text{Hom}(\mathbb{Z}^k, G^s)$ of commuting semisimple tuples retracts to $\text{Hom}(\mathbb{Z}^k, K)$.

We stratify $\text{Hom}(\mathbb{Z}^k, G^s)$ according to the conjugacy class of the stabilizer under the action by conjugation $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G)$ and note that this stratification has finitely many strata; this is the content of Section 2.

To any stratum Z of $\text{Hom}(\mathbb{Z}^k, G^s)$ we associate an abelian subgroup $A_Z \subset G$ and a surjective map

$$(1.3) \quad (G \times A_Z^k) / \mathcal{N}_G(A_Z) \rightarrow Z,$$

where $\mathcal{N}_G(A_Z)$ is the normalizer of A_Z in G . The groups A_Z are described in Section 3, where we discuss the *well-centered groups*: these are the diagonalizable subgroups of G that are equal to the center of their centralizer. The main result of this section is that the sets \mathcal{G} and \mathcal{K} of well-centered groups in G and K are manifolds with finitely many connected components, and that there is an injective homotopy equivalence $\mathcal{K} \hookrightarrow \mathcal{G}$. The map (1.3) and some of its properties are discussed in Section 4. The central observation there is that the domain of (1.3) is a flat bundle over a connected component of the manifold \mathcal{G} of well-centered groups. The flatness of this bundle allows horizontal lifting of any retraction of \mathcal{G} to \mathcal{K} . We prove that one can choose the retractions of \mathcal{G} to \mathcal{K} in such a way that the associated horizontal lifts descend under (1.3) to a retraction of $\text{Hom}(\mathbb{Z}^k, G^s)$ to $\text{Hom}(\mathbb{Z}^k, K)$. These steps make up Sections 5 and 6; some of the painful details of Section 5 are suppressed until the Appendix.

Sections 7–8 These two sections are devoted to proving that there is a commutativity-preserving retraction of G into the set G^s of semisimple elements. The basic idea is to associate to every element of G its multiplicative Jordan decomposition and to homotope away the unipotent part using the logarithm. In doing so, we face the problem that the Jordan decomposition is not continuous. We bypass this difficulty by constructing a continuous approximation of the Jordan decomposition. More concretely, letting G act on itself and on $G \times G$ by conjugation, we prove:

Theorem 1.7 *Let $G \subset \text{SL}_n \mathbb{C}$ be an algebraic group and $H_1, \dots, H_r \subset G$ a finite collection of algebraic subgroups of G . For every $\epsilon > 0$ there is a continuous G -equivariant map*

$$G \rightarrow G \times G, \quad g \mapsto (g_s^\epsilon, g_u^\epsilon)$$

satisfying the following properties for all $g \in G$:

- (1) $g = g_s^\epsilon g_u^\epsilon = g_u^\epsilon g_s^\epsilon$.
- (2) g_s^ϵ and g_u^ϵ are polynomials in g , ie, they belong to $\mathbb{C}[g] \subset M_n \mathbb{C}$.
- (3) If $h \in G$ commutes with $g \in G$, then h also commutes with g_s^ϵ and g_u^ϵ .
- (4) g_s^ϵ is semisimple and $g_u^\epsilon - \text{Id}$ has spectral radius at most ϵ .
- (5) If $K \subset G$ is a maximal compact group then $g_s^\epsilon, g_u^\epsilon \in K$ for all $g \in K$.
- (6) $g_s^\epsilon, g_s^u \in H_i$ for all $g \in H_i$.
- (7) If moreover G is defined over \mathbb{R} , then $\overline{g_s^\epsilon} = \overline{g_s^\epsilon}$ and $\overline{g_u^\epsilon} = \overline{g_u^\epsilon}$, where $\overline{\cdot}$ is complex conjugation.

Remark Notice that equivariance in Theorem 1.7 means that

$$(hgh^{-1})_s^\epsilon = hg_s^\epsilon h^{-1} \quad \text{and} \quad (hgh^{-1})_u^\epsilon = hg_u^\epsilon h^{-1}$$

for all $g, h \in G$. Observe also that the closure of $K^G = \bigcup_{g \in G} gKg^{-1}$ in G contains non-trivial unipotent elements. In particular, Theorem 1.7(5) cannot be strengthened to get $g_s^\epsilon = g_s$ for all $g \in K$. A similar argument shows that (6) cannot be strengthened to allow infinite collections of Zariski closed subgroups.

We refer to the decomposition $g = g_s^\epsilon g_u^\epsilon$ provided by Theorem 1.7 as the *approximated Jordan decomposition* of $g \in G$. It shares the commutativity properties of the standard Jordan decomposition $g = g_s g_u$, and likewise it behaves well with respect to conjugation, and has the property that both components are polynomials in g . The distinct advantage of the approximated version is that it is continuous. The main compromise is that we can only say that the linear transformation $g_u^\epsilon - \text{Id}$ has spectral radius bounded above by ϵ ; in the standard decomposition, the spectral radius of g_u is vanishing, ie, g_u is unipotent. Recall that the *spectral radius* of a matrix is the maximum of the norms of its eigenvalues.

Concluding remark The strategy of the proof of Theorem 1.1 can likely be adapted to shed some light on whether or not $\text{Hom}(\Gamma, G)$ retracts to $\text{Hom}(\Gamma, K)$, when Γ is a right angled Artin group (Charney [11]). The latter problem is particularly interesting as right angled Artin groups in some sense interpolate between free groups and free abelian groups; and although for these latter two classes of groups, $\text{Hom}(\Gamma, G)$ retracts to $\text{Hom}(\Gamma, K)$, each does so for its own distinct reason. Finally, much of the content of Sections 2–6 seems well posed for the study of other problems, such as computing cohomological invariants of $\text{Hom}(\mathbb{Z}^k, G)$.

Real vs complex Most of the work in this paper is devoted to prove Theorem 1.1 for complex groups. The proof for real groups is almost identical; it suffices to add only a few remarks here and there. When needed, we will discuss the real case of the results in a section only after the complex case has been completely treated.

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2 Action by conjugation

By an *algebraic group* G we mean the set of complex points of a linear algebraic group; we say that G is connected if it is connected in the Zariski topology. If G is defined over \mathbb{R} , then we denote by $G(\mathbb{R})$ the set of real points. Note that G and $G(\mathbb{R})$ are both Lie groups satisfying

$$\dim_{\mathbb{C}}(G) = \dim_{\mathbb{R}}(G(\mathbb{R})).$$

As Lie groups, both G and $G(\mathbb{R})$ have finitely many connected components. In fact, G is connected as a Lie group if and only if it is connected in the sense above. Notice however that the set $G(\mathbb{R})$ of real points of a connected algebraic group defined over \mathbb{R} does not need to be a connected Lie group.

We will denote by K a maximal compact subgroup of G and by $K(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$. It is well-known that maximal compact subgroups exist and that any two are conjugate to each other; in fact, any two maximal compact subgroups are conjugate by elements in the identity component G_0 (resp. $G(\mathbb{R})_0$) of the ambient group G (resp. $G(\mathbb{R})_0$). It is also well known that the inclusion $K \hookrightarrow G$ (resp. $K(\mathbb{R}) \hookrightarrow G(\mathbb{R})$) of a maximal compact subgroup into the ambient group is a homotopy equivalence.

Recall that the unipotent radical $\mathcal{R}_u(G)$ of an algebraic group G is the largest normal connected unipotent subgroup thereof. If $\mathcal{R}_u(G)$ is trivial, then G is said to be *reductive*. The identity component G_0 of G is itself an algebraic group. A real reductive group is the group of real points of a reductive algebraic group defined over \mathbb{R} .

Algebraic varieties will be, unless explicitly stated, complex affine varieties; in fact, varieties will be more often than not actually embedded inside affine space. In particular, variety is often just synonymous to Zariski closed subset of \mathbb{C}^n for some n .

Remark Throughout most of this paper we will be interested in topological properties of groups and spaces with respect to the analytic topology. Whenever we work with the Zariski topology we will make this explicit by writing Zariski closed, Zariski open, etc.

We refer to Borel [7] and Shafarevich [33] for basic facts about algebraic groups and varieties. See Borel [8] for the relation between complex and real algebraic groups.

2.1 The variety of commuting tuples

Given a linear algebraic group G , we identify the set $\text{Hom}(\mathbb{Z}^k, G)$ of all homomorphisms $\mathbb{Z}^k \rightarrow G$ with the variety of commuting k -tuples in G :

$$\text{Hom}(\mathbb{Z}^k, G) = \{(a_1, \dots, a_k) \in G^k \mid a_i a_j = a_j a_i \ \forall i, j\}.$$

In this way, $\text{Hom}(\mathbb{Z}^k, G)$ becomes an affine complex algebraic variety. The identification between $\text{Hom}(\mathbb{Z}^k, G)$ and the variety of commuting k -tuples in G depends on the choice of basis of \mathbb{Z}^k . It is well-known, and otherwise easy to see, that the structure of $\text{Hom}(\mathbb{Z}^k, G)$ as an algebraic variety does not depend on the chosen embedding.

As an analytic space, the variety $\text{Hom}(\mathbb{Z}^k, G)$ also has the associated Hausdorff topology. Notice that this topology agrees with the topology induced by embedding $\text{Hom}(\mathbb{Z}^k, G)$ into G^k , when considering the algebraic group G as a complex Lie group.

Remark We will denote elements in $\text{Hom}(\mathbb{Z}^k, G)$ by ρ if we consider them as homomorphisms and by (a_1, \dots, a_k) if we consider them as tuples. With the hope that no confusion arises, we feel free to pass from one form of notation to the other.

The group G acts on $\text{Hom}(\mathbb{Z}^k, G)$ by conjugation:

$$\begin{aligned} G \times \text{Hom}(\mathbb{Z}^k, G) &\rightarrow \text{Hom}(\mathbb{Z}^k, G), \\ (g, (a_1, \dots, a_k)) &\mapsto (ga_1g^{-1}, \dots, ga_kg^{-1}). \end{aligned}$$

In this section we study a few properties of this action. However, we first remind the reader of a few facts on algebraic varieties and algebraic actions of algebraic groups.

2.2 Quotients

Recall that it is a theorem of Cartan that any closed subgroup H of a Lie group G is also a Lie group. It is well known that the coset space G/H is a manifold that admits a unique smooth structure with respect to which the quotient map

$$\pi: G \rightarrow G/H$$

is the projection associated to a principal bundle with structure group H .

Similarly, any Zariski closed subgroup H of a (complex) algebraic group G is also an algebraic group. In this setting, the coset space G/H has a unique structure as a quasiprojective variety such that:

- the map $\pi: G \rightarrow G/H$ is a morphism,

- the Zariski topology of G/H is the quotient topology, and
- for any Zariski open set $U \subset G/H$, the pull-back map is an isomorphism between the algebras of functions on U and of H -invariant functions on $\pi^{-1}(U)$.

While it is important to keep in mind that G/H is in general not affine, but rather only quasiprojective, the following result due to Richardson [28] will have the consequence that in all cases of interest in this paper, the coset space is in fact affine:

Theorem 2.1 (Richardson [28]) *Suppose G is a reductive algebraic group and $H \subset G$ is a Zariski closed subgroup. Then the following are equivalent:*

- *The quasi-projective variety G/H is isomorphic to an affine variety.*
- *H is reductive.*

Still assuming that H is a Zariski closed subgroup of G , it is worth mentioning that the subadjacent analytic space of the variety G/H is a smooth manifold which, as one would expect, agrees with the manifold structure mentioned above.

We refer the reader to [16] and [7; 26] for basic properties of quotient spaces in the smooth and algebraic categories.

2.3 Actions of reductive groups

Suppose that G is an algebraic group and X an affine variety, both complex. An *algebraic action* $G \curvearrowright X$ of G on X is a morphism of varieties

$$G \times X \rightarrow X$$

satisfying the usual properties of an action of a group on a set. We refer to [10] for a gentle introduction to the basic properties of algebraic actions.

Given an (algebraic) action $G \curvearrowright X$ and a point $x \in X$ we denote by $\text{Orbit}_G(x)$ and $\text{Stab}_G(x)$ the orbit and the stabilizer of x , respectively. We summarize a few properties of the orbits and stabilizers of algebraic groups:

- (1) $\text{Stab}_G(x)$ is Zariski closed in G ; hence $\text{Stab}_G(x)$ is an algebraic subgroup of G .
- (2) $\text{Orbit}_G(x)$ is a smooth quasi-projective variety.
- (3) The map $G/\text{Stab}_G(x) \rightarrow \text{Orbit}_G(x)$ given by $g\text{Stab}_G(x) \mapsto gx$ is an isomorphism of quasi-projective varieties.

Notice that if $\text{Orbit}_G(x)$ happens to be a Zariski closed subset of the affine variety X , then $\text{Orbit}_G(x)$ is itself an affine variety. We deduce hence from (3) and Theorem 2.1:

Lemma 2.2 *Suppose that $G \curvearrowright X$ is an algebraic action of a reductive group G on an affine variety X . If the orbit $\text{Orbit}_G(x)$ of some $x \in X$ is Zariski closed, then $\text{Stab}_G(x)$ is reductive.*

It should be remarked that there are actions of reductive groups with reductive stabilizers for which the orbits are not closed; this is so because the image of an affine variety under a morphism is in general not Zariski closed.

2.4 Orbit types and principal orbits

Suppose that $G \curvearrowright X$ is an action of a group on a set. We say that $x, y \in X$ have *the same orbit type* if the stabilizers $\text{Stab}_G(x)$ and $\text{Stab}_G(y)$ are conjugate in G .

Remark Suppose for the sake of concreteness that G is an algebraic group, X is an algebraic variety, and $G \curvearrowright X$ is an algebraic action. Notice that $x, y \in X$ have the same orbit type if and only if their orbits $\text{Orbit}_G(x)$ and $\text{Orbit}_G(y)$ are G -equivariantly isomorphic.

A topological (resp. algebraic) action $G \curvearrowright X$ of a Lie group (resp. algebraic group) G on a topological space (resp. algebraic variety) X is said to have a *principal orbit* if there is a dense open (resp. Zariski dense Zariski open) set $U \subset X$ such that any two $x, y \in U$ have the same orbit type.

It is well-known that actions of compact Lie groups on compact manifolds have principal orbits. As a general rule, reductive groups are the algebraic analogue of compact Lie groups. In this spirit, the following result due to Richardson [27] asserts that under suitable conditions, algebraic actions of reductive groups on affine varieties have principal orbits as well:

Theorem 2.3 (Richardson [27]) *Let $G \curvearrowright X$ be an algebraic action of a reductive algebraic group G on an irreducible affine algebraic variety X . If there is a smooth point $x \in X$ with $\text{Stab}_G(x)$ reductive, then the action $G \curvearrowright X$ has a principal orbit.*

In the setting of actions of compact Lie groups on compact manifold M , Mostow [24] proved that the stabilizers of points near $x \in M$ are conjugate to subgroups of the stabilizer of x . In [27], Richardson also obtained an analogue of this assertion for algebraic actions of reductive groups:

Theorem 2.4 (Richardson [27]) *Let $G \curvearrowright X$ be an algebraic action of a reductive algebraic group G on an affine algebraic variety X . Suppose that $x \in X$ is such that its orbit $\text{Orbit}_G(x)$ is closed. Then there is a Zariski open set $U \subset X$ such that the stabilizer $\text{Stab}_G(y)$ of every $y \in U$ is conjugate to a subgroup of $\text{Stab}_G(x)$.*

Theorem 2.4 is also a consequence of Luna's Slice Theorem [23], which asserts that there is a $\text{Stab}_G(x)$ -invariant affine subvariety $Y \subset X$ with $x \in Y$ such that the morphism

$$(Y \times G) / \text{Stab}_G(x) \rightarrow X$$

is *étale*. In complex analytic terms, “*étale*” is synonymous with “local analytic isomorphism” (Mumford [25, III.5 Corollary 2]). In particular, Theorem 2.4 can be strengthened as follows: *if $x_n \rightarrow x$ converges in the analytic topology, then there are $g_n \rightarrow \text{Id}$ with $g_n \text{Stab}_G(x_n) g_n^{-1} \subset \text{Stab}_G(x)$.*

2.5 Centralizers of semisimple tuples

Assume that G is a reductive algebraic group and recall that we identify elements ρ in $\text{Hom}(\mathbb{Z}^k, G)$ with commuting k -tuples (a_1, \dots, a_k) in G . As we mentioned earlier, the group G acts on $\text{Hom}(\mathbb{Z}^k, G)$ by conjugation. The stabilizer of a representation ρ under this action of G is the centralizer of its image:

$$\text{Stab}_G(\rho) = \mathcal{Z}_G(\rho(\mathbb{Z}^k)),$$

where, for $X \subset G$:

$$\mathcal{Z}_G(X) = \{g \in G \mid gx = xg \ \forall x \in X\}.$$

A tuple $(a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, G)$ is *semisimple* if each a_i is. Since every abelian subgroup of G generated by semisimple elements consists solely of semisimple elements [7] we have:

Lemma 2.5 $\rho \in \text{Hom}(\mathbb{Z}^k, G)$ is semisimple if and only if $\rho(\mathbb{Z}^k)$ is an abelian subgroup consisting of semisimple elements. \square

We prove that the action $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G)$ has only finitely many orbit types of semisimple tuples.

Proposition 2.6 *Let G be a reductive algebraic group with identity component G_0 . The actions of G and G_0 on $\text{Hom}(\mathbb{Z}^k, G)$ by conjugation have only finitely many orbit types of semisimple representations.*

The key ingredient in the proof of Proposition 2.6 is the following result due to Richardson:

Theorem 2.7 (Richardson [30]) *Let G be a reductive algebraic group and consider the action $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G)$ by conjugation. If $\rho \in \text{Hom}(\mathbb{Z}^k, G)$ is semisimple, then $\text{Orbit}_G(\rho)$ is Zariski closed in $\text{Hom}(\mathbb{Z}^k, G)$; hence, $\text{Stab}_G(\rho)$ is reductive.*

Proof of Proposition 2.6 Since G_0 has finite index in G , it suffices to prove that the action of G_0 has finitely many orbit types. Let \mathcal{V} be the set of all irreducible, G_0 -invariant affine subvarieties $V \subset \text{Hom}(\mathbb{Z}^k, G)$ containing as a smooth point a semisimple tuple. Arguing by induction on the dimension, we will show that each $V \in \mathcal{V}$ consists of finitely many orbit types. The base case of the induction, namely $\dim V = 0$, is clear, for these are exactly the tuples of central elements in G_0 .

So, suppose that we have proved the claim for all elements in \mathcal{V} of at most dimension $d - 1$ and let $V \in \mathcal{V}$ be of dimension d . By assumption, V contains as a smooth point a semisimple tuple ρ . By Theorem 2.7, $\text{Stab}_{G_0}(\rho)$ is reductive. Hence, we can apply Theorem 2.3 and deduce that $G_0 \curvearrowright V$ has a principal orbit. In other words, there is $U \subset V$ Zariski open such that for any two $\rho_1, \rho_2 \in U$ the stabilizers $\text{Stab}_{G_0}(\rho_1)$ and $\text{Stab}_{G_0}(\rho_2)$ are conjugate. Hence it suffices to show that the affine G_0 -invariant variety $V \setminus U$ contains finitely many orbit types of semisimple tuples; notice that $V \setminus U$ has at most dimension $d - 1$. Consider the Zariski closure V' of the set of semisimple tuples in $V \setminus U$ and let $V' = V'_1 \cup \dots \cup V'_r$ be the decomposition into irreducible components. Clearly V' is G_0 -invariant; by the connectedness of G_0 , each V'_i is also G_0 -invariant. Moreover, each V'_i has at most dimension $d - 1$ and contains a semisimple tuple as a smooth point, as the set of singular points is a subvariety. From the induction assumption, we deduce that each V'_i contains only finitely many orbit types of semisimple tuples. Since every semisimple tuple in $V \setminus U$ is contained in V' , this proves that V also contains only finitely many orbit types of semisimple tuples.

At this point we know that every affine irreducible G_0 -invariant subvariety $V \subset \text{Hom}(\mathbb{Z}^k, G)$ containing as a smooth point a semisimple tuple contains only finitely many orbit types of semisimple tuples. As in the induction step, the Zariski closure of the set of semisimple tuples in $\text{Hom}(\mathbb{Z}^k, G)$ is a finite union of such subvarieties. In particular, $\text{Hom}(\mathbb{Z}^k, G)$ contains only finitely many G_0 -orbit types of semisimple tuples, as claimed. \square

2.6 Triangulations

In later sections we will need the existence of suitable triangulations of the subset of $\text{Hom}(\mathbb{Z}^k, G)$ consisting of semisimple representations. Before stating a precise result,

we introduce some notation for the sets of semisimple elements and of elements that can be conjugated into some maximal compact subgroup $K \subset G$:

$$(2.1) \quad G^s = \{g \in G \mid g \text{ semisimple}\}, \quad K^G = \bigcup_{g \in G} gKg^{-1}.$$

We will also use the following notation:

$$(2.2) \quad \text{Hom}(\mathbb{Z}^k, G^s) = \{(a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, G) \mid a_i \in G^s \ \forall i\},$$

$$(2.3) \quad \text{Hom}(\mathbb{Z}^k, K^G) = \{(a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, G) \mid a_i \in K^G \ \forall i\}.$$

Remark Since every element in K is semisimple, we have $K^G \subset G^s$.

We can now state the needed existence of triangulations:

Proposition 2.8 *Suppose that G is a reductive algebraic group, and let $K \subset G$ be a maximal compact subgroup. There is a triangulation of $\text{Hom}(\mathbb{Z}^k, G^s)$ such that*

$$\mathcal{X}_H = \{\rho \in \text{Hom}(\mathbb{Z}^k, G^s) \mid \exists g \in G \text{ with } gHg^{-1} \subset \text{Stab}_G(\rho)\}$$

is a subcomplex for every subgroup $H \subset G$. Moreover, the subsets $\text{Hom}(\mathbb{Z}^k, K)$ and $\text{Hom}(\mathbb{Z}^k, K^G)$ are also subcomplexes.

Recall that a *triangulation* of a topological space is a homeomorphism to the geometric realization of a simplicial complex. Abusing notation, we will not distinguish between simplicial complexes, their geometric realizations, and triangulations.

Before launching into the proof of Proposition 2.8, we need a few preliminary observations.

Lemma 2.9 *The set $\{\mathcal{X}_H \mid H \subset G\}$ of subsets of $\text{Hom}(\mathbb{Z}^k, G^s)$ is finite.*

Proof By Proposition 2.6, the action $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G^s)$ has only finitely many orbit types. In other words, there is a finite set H_1, \dots, H_r of subgroups of G such that for all $\rho \in \text{Hom}(\mathbb{Z}^k, G^s)$ there are $i \in \{1, \dots, r\}$ and $g \in G$ with $gH_i g^{-1} = \text{Stab}_G(\rho)$. It follows that \mathcal{X}_H is precisely the union of the sets \mathcal{X}_{H_i} for which there is $g \in G$ with $gHg^{-1} \subset H_i$. Clearly, only finitely many subsets of $\text{Hom}(\mathbb{Z}^k, G^s)$ can be constructed as unions of subcollections of the finitely many sets $\mathcal{X}_{H_1}, \dots, \mathcal{X}_{H_r}$. □

Lemma 2.10 *\mathcal{X}_H is a closed subset of $\text{Hom}(\mathbb{Z}^k, G^s)$ for all $H \subset G$.*

It is worth noticing that in general the set of those $\rho \in \text{Hom}(\mathbb{Z}^k, G^s)$ such that there is $g \in G$ with $gHg^{-1} = \text{Stab}_G(\rho)$ is neither open nor closed.

Proof We prove that the complement of \mathcal{X}_H in $\text{Hom}(\mathbb{Z}^k, G^s)$ is open. Given $\rho \notin \mathcal{X}_H$, recall that $\text{Stab}_G(\rho)$ is reductive and that $\text{Orbit}_G(\rho)$ is Zariski closed by Theorem 2.7. Theorem 2.4 implies hence that ρ belongs to a Zariski open set $U \subset \text{Hom}(\mathbb{Z}^k, G^s)$ with the property that the stabilizer $\text{Stab}_G(\rho')$ of any $\rho' \in U$ is conjugate to a subgroup of $\text{Stab}_G(\rho)$. The assumption that H cannot be conjugated into $\text{Stab}_G(\rho)$ then implies that neither can it be conjugated into $\text{Stab}_G(\rho')$ for $\rho' \in U \cap \text{Hom}(\mathbb{Z}^k, G^s)$. This proves that $U \cap \mathcal{X}_H = \emptyset$. Since U is Zariski open, it is a fortiori open; the claim follows. \square

Lemma 2.11 *The sets $\text{Hom}(\mathbb{Z}^k, K)$ and $\text{Hom}(\mathbb{Z}^k, K^G)$ are closed in $\text{Hom}(\mathbb{Z}^k, G^s)$.*

Notice that K^G is not a closed subset of G . This implies that $\text{Hom}(\mathbb{Z}^k, K^G)$ is not closed in $\text{Hom}(\mathbb{Z}^k, G)$.

Proof Clearly $\text{Hom}(\mathbb{Z}^k, K)$ is closed because $K \subset G^s$ is compact and hence closed. Similarly, in order to prove that $\text{Hom}(\mathbb{Z}^k, K^G)$ is closed in $\text{Hom}(\mathbb{Z}^k, G^s)$, it suffices to show that K^G is closed in G^s . Choosing an embedding of G into $\text{SL}_n \mathbb{C}$ for some suitable n , it suffices to prove that the closure of $K^G \subset \text{SU}_n^{\text{SL}_n \mathbb{C}}$ within the set of diagonalizable matrices consists of diagonalizable matrices that can be conjugated into SU_n . However, a diagonalizable matrix $A \in \text{SL}_n \mathbb{C}$ can be conjugated into SU_n if and only if all of its eigenvalues have unit norm; this is a closed condition (compare with Lemma 7.3). \square

Proposition 2.8 will follow easily from the fact that semialgebraic sets are triangulizable. Recall that a subset $X \subset \mathbb{C}^N$ is *semialgebraic* if it is the finite union of sets determined by a finite collection of real polynomial equalities and inequalities.

Theorem 2.12 *Let X be a semialgebraic set and $\{X_i\}_{i \in I}$ a locally finite collection of closed semialgebraic subsets of X . Then there is a simplicial complex C and a homeomorphism $t: C \rightarrow X$ such that each X_i is the image of a subcomplex of C .*

Theorem 2.12 is a version of Whitney’s Stratification Theorem [38; 39]; see also Thom [36]. We refer the interested reader to the PhD thesis of Kyle Hofmann [20] for a very pleasant discussion of these topics.

Lemma 2.13 *The sets $\text{Hom}(\mathbb{Z}^k, G^s)$, $\text{Hom}(\mathbb{Z}^k, K^G)$, $\text{Hom}(\mathbb{Z}^k, K)$ and \mathcal{X}_H are semialgebraic.*

Proof Embedding G into $SL_n \mathbb{C}$ for some n , notice that $K \subset G$ is equal to the intersection of G with a conjugate of SU_n . It follows that K is semialgebraic. Since $\text{Hom}(\mathbb{Z}^k, K)$ is the intersection of the algebraic set $\text{Hom}(\mathbb{Z}^k, G) \subset G^k$ with the semialgebraic set K^k , it is itself semialgebraic.

Let G_0 be the identity component of G , $\mathbb{T} \subset G_0$ a maximal torus and consider the following algebraic subvariety of $(G_0 \times \mathbb{T})^k$:

$$\mathcal{X} = \{((g_1, a_1), \dots, (g_k, a_k)) \in (G_0 \times \mathbb{T})^k \mid [g_i a_i g_i^{-1}, g_j a_j g_j^{-1}] = \text{Id}_G\}.$$

Notice that \mathcal{X} is an affine variety and hence a semialgebraic set of \mathbb{C}^N for some suitable choice of N . We consider now the map:

$$\alpha: \mathcal{X} \rightarrow \text{Hom}(\mathbb{Z}^k, G_0), \quad (g_i, a_i)_{i=1, \dots, k} \mapsto (g_i a_i g_i^{-1})_{i=1, \dots, k}.$$

By definition, $\alpha(\mathcal{X}) \subset \text{Hom}(\mathbb{Z}^k, G_0^s)$. On the other hand, since G_0 is connected it follows that every semisimple element in G_0 can be conjugated into \mathbb{T} . This implies that $\alpha(\mathcal{X}) = \text{Hom}(\mathbb{Z}^k, G_0^s)$. We have proved that $\text{Hom}(\mathbb{Z}^k, G_0^s)$ is the image of a semialgebraic set under a polynomial map. A renowned theorem of Tarski [35] asserts that images under polynomial maps of semialgebraic sets are semialgebraic. This proves that $\text{Hom}(\mathbb{Z}^k, G_0^s)$ is semialgebraic. Let now d be the index of G_0 in G and consider the map:

$$\text{Hom}(\mathbb{Z}^k, G) \rightarrow \text{Hom}(\mathbb{Z}^k, G_0), \quad (a_1, \dots, a_k) \mapsto (a_1^d, \dots, a_k^d).$$

The set $\text{Hom}(\mathbb{Z}^k, G^s)$ is the preimage under this algebraic map of the semialgebraic set $\text{Hom}(\mathbb{Z}^k, G_0^s)$ and hence is semialgebraic.

Notice now that the same argument shows that $\text{Hom}(\mathbb{Z}^k, K^G)$ is also semialgebraic.

It remains to prove that \mathcal{X}_H is semialgebraic for every subgroup $H \subset G$. To begin with, observe that

$$\text{Fix}^s(H) = \{\rho \in \text{Hom}(\mathbb{Z}^k, G^s) \mid H \subset \text{Stab}_G(\rho)\}$$

is semialgebraic because it is the intersection of the fixed-point set of H on $\text{Hom}(\mathbb{Z}^k, G)$, an affine variety, with the semialgebraic set $\text{Hom}(\mathbb{Z}^k, G^s)$. Consider the polynomial map:

$$\beta: G \times \text{Fix}^s(H) \rightarrow \text{Hom}(\mathbb{Z}^k, G^s), \quad \beta: (g, \rho) \mapsto g\rho g^{-1}.$$

It follows directly from its definition that \mathcal{X}_H is equal to the image of β . Hence, Tarski's Theorem shows again that \mathcal{X}_H is semialgebraic, as claimed. □

Proposition 2.8 follows directly from Lemmas 2.9, 2.10, 2.11, 2.13 and Theorem 2.12. □

2.7 The real case

We consider now the real case. More concretely, assume that $G \subset \mathrm{SL}_n \mathbb{C}$ is a reductive algebraic group defined over \mathbb{R} and denote by

$$G(\mathbb{R}) = \{g \in G \mid \bar{g} = g\}$$

the group of real points, where \bar{g} is the complex conjugate of g . We denote by $G(\mathbb{R})_0$ the identity component of the Lie group $G(\mathbb{R})$. Notice that $G(\mathbb{R})_0$ has finite index in $G_0(\mathbb{R})$.

The following lemma is one of the key facts needed to reduce the real case to the complex case.

Lemma 2.14 *Let $G \subset \mathrm{SL}_n \mathbb{C}$ be the group of complex points of an algebraic group defined over \mathbb{R} and $H \subset G$ an algebraic subgroup also defined over \mathbb{R} . There are finitely many $G(\mathbb{R})$ -conjugacy classes of subgroups $H' \subset G$ that are defined over \mathbb{R} and conjugate to H by an element in G .*

The reader can find a proof of Lemma 2.14 in Serre [32, III 4.4]; for the sake of completeness we give a simple argument.

Proof Let $N = \mathcal{N}_G(H)$ be the normalizer of H in G and identify the smooth variety G/N with the set of subgroups of G conjugate to H . The subgroup N is a closed subgroup defined over \mathbb{R} and hence G/N and the map $G \rightarrow G/N$ are also defined over \mathbb{R} ; compare with [7, Chapter II].

Suppose that $H' \in G/N$ and let $g \in G$ with $H' = gHg^{-1}$. Taking into account that $\overline{H'} = \bar{g}\bar{H}\bar{g}^{-1} = \bar{g}H\bar{g}^{-1}$ we deduce that the set of real points

$$(G/N)(\mathbb{R}) = \{gN \in G/N \mid \bar{g}N = gN\}$$

of G/N is exactly the set of subgroups of G defined over \mathbb{R} and conjugate to H by an element in G .

Notice that the left action $G \curvearrowright G/N$ induces an action $G(\mathbb{R}) \curvearrowright (G/N)(\mathbb{R})$; the claim of Lemma 2.14 follows once we have proved that the latter action has only finitely many orbits. Taking into account that $(G/N)(\mathbb{R})$ is both a submanifold of G/N and a real algebraic variety, and that as such, $(G/N)(\mathbb{R})$ has only finitely many connected components with respect to the Hausdorff topology [38], we deduce that it suffices to prove that $G(\mathbb{R}) \curvearrowright (G/N)(\mathbb{R})$ has open orbits.

To avoid introducing new notation, we will just prove that the $G(\mathbb{R})$ -orbit of H is open; the argument is identical for any other point of $(G/N)(\mathbb{R})$. Notice that the

stabilizer of H under the $G(\mathbb{R})$ action is the group $N(\mathbb{R}) = N \cap G(\mathbb{R})$ of real points of N and that the induced map

$$(2.4) \quad G(\mathbb{R})/N(\mathbb{R}) \rightarrow (G/N)(\mathbb{R})$$

is locally injective. Observe also that the dimension of $(G/N)(\mathbb{R})$ as a manifold is at most as large as the complex dimension of G/N :

$$\dim_{\mathbb{R}}(G/N)(\mathbb{R}) \leq \dim_{\mathbb{C}} G/N.$$

On the other hand, since G and N are groups defined over \mathbb{R} , we have

$$\dim_{\mathbb{R}} G(\mathbb{R}) = \dim_{\mathbb{C}} G \quad \text{and} \quad \dim_{\mathbb{R}} N(\mathbb{R}) = \dim_{\mathbb{C}} N,$$

and hence

$$\dim_{\mathbb{R}} G(\mathbb{R})/N(\mathbb{R}) = \dim_{\mathbb{C}} G/N \geq \dim_{\mathbb{R}}(G/N)(\mathbb{R}).$$

This implies that the locally injective map (2.4) is open. □

The space $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ of representations of \mathbb{Z}^k in $G(\mathbb{R})$ is the set of fixed points under complex conjugation in the variety $\text{Hom}(\mathbb{Z}^k, G) \subset G \times \cdots \times G$; in particular, $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ is a real algebraic variety.

Notice that the action by conjugation $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G)$ induces the action by conjugation of $G(\mathbb{R})$ on $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$. We prove that this action has finitely many orbit types of semisimple representations:

Proposition 2.6 (Real case) *Let $G(\mathbb{R})$ be the set of real points of a reductive algebraic group defined over \mathbb{R} . The actions of $G(\mathbb{R})$ and $G(\mathbb{R})_0$ on $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ have only finitely many orbit types of semisimple representations.*

Proof Consider the action $G \curvearrowright \text{Hom}(\mathbb{Z}^k, G)$ by conjugation and observe that the stabilizer $\text{Stab}_G(\rho) = \mathcal{Z}_G(\rho(\mathbb{Z}^k))$ of any point $\rho \in \text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ is a closed subgroup defined over \mathbb{R} . By Proposition 2.6, the set of all stabilizers of semisimple points in $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ falls into finitely many G_0 -conjugacy classes. Lemma 2.14 implies now that they also fall into finitely many $G(\mathbb{R})$ -conjugacy classes, as we needed to prove. Since $G(\mathbb{R})_0$ has finite index in $G(\mathbb{R})$, the same is true for the action $G(\mathbb{R})_0 \curvearrowright \text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$. □

Continuing with the same notation, let $G(\mathbb{R})^s = G^s \cap G(\mathbb{R})$ be the set of real semisimple elements and consider

$$\text{Hom}(\mathbb{Z}^k, G(\mathbb{R})^s) = \{(a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, G) \mid a_i \in G(\mathbb{R})^s \ \forall i\},$$

the set of real semisimple representations. As the fixed point set under complex conjugation, $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R})^s)$ is closed semialgebraic subset of $\text{Hom}(\mathbb{Z}^k, G^s)$. This said, one can apply word-for-word the arguments used to prove Proposition 2.8 to show:

Proposition 2.8 (Real case) *Suppose that $G(\mathbb{R})$ is the group of real points of a reductive algebraic group defined over \mathbb{R} , and let $K(\mathbb{R}) \subset G(\mathbb{R})$ be a maximal compact subgroup. There is a triangulation of $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R})^s)$ such that*

$$\mathcal{X}_H = \{\rho \in \text{Hom}(\mathbb{Z}^k, G(\mathbb{R})^s) \mid \exists g \in G(\mathbb{R}) \text{ with } gHg^{-1} \subset \text{Stab}_{G(\mathbb{R})}(\rho)\}$$

is a subcomplex for every subgroup $H \subset G(\mathbb{R})$. Moreover, $\text{Hom}(\mathbb{Z}^k, K(\mathbb{R}))$ and $\text{Hom}(\mathbb{Z}^k, K(\mathbb{R})^{G(\mathbb{R})})$ are also subcomplexes. \square

3 Well-centered groups

Throughout this section we assume that G is the group of complex points of a reductive algebraic group and that $K \subset G$ is a maximal compact subgroup; the real case will only be discussed at the end. Recall that if $H \subset G$ is a subgroup, then $\mathcal{Z}_G(H)$ is the centralizer of H in G . We denote by $C(H)$ the center of H .

3.1 Diagonalizable groups

A Zariski closed subgroup $A \subset G$ is *diagonalizable* if it is abelian and consists of semisimple elements. Notice that the Zariski closure of every abelian subgroup consisting of semisimple elements is diagonalizable. Also, the center of a reductive group $C(G)$ is diagonalizable. Conversely, the centralizer $\mathcal{Z}_G(A)$ of a diagonalizable subgroup $A \subset G$ is reductive.

We refer the reader to [7, Chapter 8] for basic facts on diagonalizable groups; we will just prove a few surely well-known facts for which we have not found a reference.

Lemma 3.1 *Suppose that $G \subset \text{SL}_n \mathbb{C}$ is a reductive algebraic group and that $A \subset G$ is a diagonalizable subgroup. Then:*

- *A contains a unique maximal compact subgroup \bar{A} . Moreover, \bar{A} is Zariski dense in A .*
- *A contains a Zariski dense subgroup generated by n elements.*

Proof Denote by A_0 the connected component of the identity so that we have an exact sequence

$$0 \rightarrow A_0 \rightarrow A \rightarrow A/A_0 \rightarrow 0,$$

where A/A_0 is a finite abelian group. This exact sequence splits [7, 8.7], and hence A is the direct product of the torus A_0 and a finite subgroup F of G isomorphic to A/A_0 . Identifying A_0 with $\mathbb{C}^* \times \dots \times \mathbb{C}^*$ we obtain that

$$\bar{A} = \mathbb{S}^1 \times \dots \times \mathbb{S}^1 \times F$$

is a maximal compact group that is clearly Zariski dense. Given any $(a_0, f) \in A_0 \times F = A$, consider the powers $(a_0, f)^{n|F|} = (a_0^{n|F|}, 1)$, where $|F|$ is the order of the finite group F . It follows that if (a_0, f) belongs to some compact subgroup, then $a_0 \in \mathbb{S}^1 \times \dots \times \mathbb{S}^1$. This proves that the maximal compact subgroup \bar{A} is unique, and the first claim is proved.

Starting now with the proof of the second claim, observe that the finite subgroup $F \subset G \subset \text{SL}_n \mathbb{C}$ is contained in an $\text{SL}_n \mathbb{C}$ -conjugate of the group of diagonal matrices with determinant one. It follows that F is isomorphic to a subgroup of $\mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_{n-1}\mathbb{Z}$ with $q_i \in \mathbb{Z}$. In particular, F can be generated by $n - 1$ elements. Since the torus A_0 contains a Zariski dense cyclic subgroup, it follows that A has a Zariski dense subgroup generated by n elements. □

Suppose that $(a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, K^G)$ and recall that this just means that each a_i can be conjugated into K . Since K consists of semisimple elements, Lemma 2.5 implies that the Zariski closure $A \subset G$ of the group generated by a_1, \dots, a_k is a diagonalizable subgroup of G . Each a_i belongs to some compact subgroup of A and by Lemma 3.1, A contains a unique maximal compact subgroup \bar{A} ; hence $a_i \in \bar{A}$ for all i . Since \bar{A} can be conjugated into K , we have proven that the elements a_i can be simultaneously conjugated into K .

Lemma 3.2 *An element $\rho \in \text{Hom}(\mathbb{Z}^k, G)$ belongs to $\text{Hom}(\mathbb{Z}^k, K^G)$ if and only if there is $g \in G$ with $g\rho(\mathbb{Z}^k)g^{-1} \subset K$.* □

3.2 Trans

Given two subsets X_1, X_2 of G , we denote by

$$\text{Trans}_G(X_1, X_2) = \{g \in G \mid gX_1g^{-1} \subset X_2\}$$

the subset of G consisting of all elements that conjugate X_1 into X_2 . Notice for instance that if H is a subgroup of G , then

$$\text{Trans}_G(H, H) = \mathcal{N}_G(H)$$

is the normalizer of H in G . Abusing notation, we will denote the intersection of $\text{Trans}_G(\cdot, \cdot)$ with the identity component G_0 of G by

$$\text{Trans}_{G_0}(X_1, X_2) = \text{Trans}_G(X_1, X_2) \cap G_0,$$

and accordingly:

$$\begin{aligned} \mathcal{Z}_{G_0}(H) &= \mathcal{Z}_G(H) \cap G_0, \\ \mathcal{N}_{G_0}(H) &= \mathcal{N}_G(H) \cap G_0. \end{aligned}$$

The sets $\text{Trans}_K(\cdot, \cdot)$ and $\text{Trans}_{K_0}(\cdot, \cdot)$ are defined accordingly.

Suppose that $A, A' \subset G$ are diagonalizable. It is easy to see that $\text{Trans}_G(A, A')$ is an algebraic variety [7, Chapter I] and hence has finitely many connected components. We show now that each one of them is homeomorphic to the connected component of the identity in the centralizer $\mathcal{Z}_G(A)$ of A in G .

Lemma 3.3 *Suppose that A, A' are diagonalizable subgroups of G , and let V be a connected component of $\text{Trans}_G(A, A')$. For all $g_0 \in V$, the map*

$$\mathcal{Z}_G(A)_0 \rightarrow V, \quad h \mapsto g_0h$$

is an isomorphism. Here, $\mathcal{Z}_G(A)_0$ is the identity component of the centralizer of A in G .

In particular, the Weyl groups $W_A = \mathcal{N}_G(A)/\mathcal{Z}_G(A)$ and $W_A^0 = \mathcal{N}_{G_0}(A)/\mathcal{Z}_{G_0}(A)$ are finite for every diagonalizable subgroup A of G .

Proof With the same notation as in the statement of the lemma, observe that by Lemma 3.1 we have

$$\text{Trans}_G(A, A') = \text{Trans}_G(\bar{A}, \bar{A}'),$$

where \bar{A}, \bar{A}' are unique maximal compact subgroups of A and A' . Endowed with the compact open topology, $\text{Hom}(\bar{A}, \bar{A}')$ is a totally disconnected space. Hence, the map

$$\text{Trans}(A, A') \rightarrow \text{Hom}(A, A'), \quad g \mapsto \{a \mapsto gag^{-1}\}$$

is constant on each connected component (compare with [7, 8.10]). In other words, it follows that

$$gag^{-1} = g_0a(g_0)^{-1}$$

for all $a \in A$ and all $g \in V$. This implies in particular that $g_0^{-1}g \in \mathcal{Z}_G(A)_0$ for all $g \in V$. Since on the other hand the morphism

$$\mathcal{Z}_G(A)_0 \rightarrow V, \quad h \mapsto g_0h$$

is injective, the claim follows. □

The following result, of capital importance in this paper, relates Trans_G and Trans_K , where $K \subset G$ is a maximal compact subgroup.

Lemma 3.4 *If $X_1, X_2 \subset K$ are subsets of K , then the inclusion*

$$\text{Trans}_K(X_1, X_2) \hookrightarrow \text{Trans}_G(X_1, X_2)$$

is a homotopy equivalence. In particular we have:

- (1) $X_1, X_2 \subset K$ are conjugate in G if and only if they are in K .
- (2) If $H \subset K$ is a subgroup then the inclusion $\mathcal{N}_K(H) \hookrightarrow \mathcal{N}_G(H)$ is a homotopy equivalence.

Before proving Lemma 3.4 we need to recall a few facts about the Cartan decomposition. It is well-known that the Lie algebra \mathfrak{g} of the reductive group G contains an $\text{Ad}(K)$ -invariant linear subspace $\mathfrak{p} \subset \mathfrak{g}$ such that the map

$$\mathfrak{p} \times K \rightarrow G, \quad (X, k) \mapsto e^X k$$

is a diffeomorphism. In particular, we can write every element $g \in G$ uniquely as $g = e^{X_g} k_g$, where $X_g \in \mathfrak{p}$ and $k_g \in K$; this is the *Cartan decomposition* of g associated to K .

Lemma 3.5 *Suppose that $k \in K$ and $g = e^{X_g} k_g \in G$ are such that $g k g^{-1} \in K$. Then $k_g k k_g^{-1} = g k g^{-1}$ and $\text{Ad}(g k g^{-1})(X_g) = X_g$.*

Proof We have $e^{X_g} k_g k k_g^{-1} e^{-X_g} = g k g^{-1} = k' \in K$ and hence that:

$$e^{X_g} k_g k k_g^{-1} = k' e^{X_g} = e^{\text{Ad}(k')(X_g)} k'.$$

The claim follows now from the uniqueness of the Cartan decomposition. □

Proof of Lemma 3.4 By Lemma 3.5 we have for $g = e^X k \in G$ that $g X_1 g^{-1} = X_2$ if and only if $k X_1 k^{-1} = X_2$ and $X \in \text{Fix}(\text{Ad}(X_2)) = \{X \in \mathfrak{p} \mid \text{Ad}(g)(X) = X \ \forall g \in X_2\}$. It follows that there is a diffeomorphism:

$$\text{Trans}_G(X_1, X_2) \simeq \text{Trans}_K(X_1, X_2) \times \text{Fix}(\text{Ad}(X_2)).$$

Notice that $\text{Fix}(\text{Ad}(X_2))$ is contractible because it is a linear subspace of \mathfrak{p} . It follows that the inclusion

$$\text{Trans}_K(X_1, X_2) \rightarrow \text{Trans}_G(X_1, X_2)$$

is a homotopy equivalence. □

Remark With the notation of Lemma 3.5, let G_0 and K_0 be, respectively, the identity connected components of G and K . Notice that $g \in G_0$ if and only if $k_g \in K_0$. In particular, the argument given to prove Lemma 3.4 also shows that the inclusion

$$\text{Trans}_{K_0}(X_1, X_2) \rightarrow \text{Trans}_{G_0}(X_1, X_2)$$

is a homotopy equivalence for any two subsets $X_1, X_2 \subset K$.

3.3 Well-centered subgroups

We will be interested in a particular class of diagonalizable subgroups of the reductive group G and its maximal compact subgroup K .

A diagonalizable subgroup $A \subset G$ is *well-centered in G* if it is the center of its centralizer: $A = C(\mathcal{Z}_G(A))$. Similarly, an abelian subgroup $H \subset K$ is *well-centered in K* if $H = C(\mathcal{Z}_K(H))$. Recall that the Zariski closure in G of every abelian semisimple subgroup is diagonalizable.

Well-centered groups will play a central role in this paper; we discuss here briefly what they are in the particular case that $G = \text{SL}_n \mathbb{C}$.

Example: well-centered groups in $\text{SL}_n \mathbb{C}$ It is well-known that every diagonalizable subgroup A of $\text{SL}_n \mathbb{C}$ is “diagonalizable”, meaning that there is a direct sum decomposition

$$(3.1) \quad \mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$$

such that every element $a \in A$ preserves each one of the factors V_i and acts on it by a homothety. In fact, taking the direct sum decomposition (3.1) with the smallest possible number of factors, we have the following description of the centralizer of A :

$$(3.2) \quad \mathcal{Z}_{\text{SL}_n \mathbb{C}}(A) = \text{SL}_n \mathbb{C} \cap (\text{GL}(V_1) \times \cdots \times \text{GL}(V_r)).$$

We might refer to the unique direct sum decomposition satisfying (3.2) as the *canonical direct sum decomposition diagonalizing A* ; notice that its factors are not ordered.

The center of the centralizer of A is then the subgroup of $\text{SL}_n \mathbb{C}$ consisting of all elements diagonalized by the canonical direct sum decomposition (3.1) associated to A :

$$(3.3) \quad C(\mathcal{Z}_{\text{SL}_n \mathbb{C}}(A)) = \{g \in \text{SL}_n \mathbb{C} \mid \forall i = 1, \dots, r \exists \lambda_i \text{ with } g v_i = \lambda_i v_i \ \forall v_i \in V_i\}$$

From this discussion we deduce that:

There is a natural correspondence between the sets of well-centered groups in $\text{SL}_n \mathbb{C}$ and of unordered direct sum decompositions of \mathbb{C}^n .

Returning now to the general setting, we state for further reference a few general facts about well-centered groups:

Lemma 3.6

- (1) We have $C(\mathcal{Z}_G(A)) = \mathcal{Z}_G(\mathcal{Z}_G(A))$ for any $A \subset G$ abelian.
- (2) Suppose that A_1 and A_2 are well-centered groups in G . Then $A_1 \supset A_2$ if and only if $\mathcal{Z}_G(A_1) \subset \mathcal{Z}_G(A_2)$.
- (3) Suppose that $A \subset G$ is an abelian subgroup consisting of semisimple elements. Then $\hat{A} = C(\mathcal{Z}_G(A))$ is the smallest well-centered group containing A .

The proof of Lemma 3.6 is a straightforward computation in elementary group theory; notice only that in (3), the proviso that A consist of semisimple elements is just to ensure that \hat{A} is diagonalizable. We leave the proof of Lemma 3.6 to the reader. \square

The following is the central observation of this section:

Lemma 3.7 *Suppose that $H \subset K$ is well-centered in K . Then H is the unique maximal compact subgroup in the well-centered group $C(\mathcal{Z}_G(H))$. In particular*

$$H = K \cap C(\mathcal{Z}_G(H))$$

is Zariski dense in $C(\mathcal{Z}_G(H))$.

Proof By Lemma 3.6, $\hat{H} = C(\mathcal{Z}_G(H))$ is well-centered and in particular diagonalizable. Lemma 3.1 implies hence that \hat{H} has a unique maximal compact subgroup $\bar{H} \subset \hat{H}$. Since \bar{H} is also Zariski dense, by Lemma 3.1 it will suffice to prove that $H = \bar{H}$. To begin with, notice that

$$H \subset \bar{H}.$$

We have to prove the opposite inclusion. Since \bar{H} is compact, there is $g = e^X k \in G$ with $g\bar{H}g^{-1} \subset K$. By Lemma 3.5 we have $ghg^{-1} = khk^{-1}$ for all $h \in H$. So, replacing g by $k^{-1}g$, we may assume that $g \in \mathcal{Z}_G(H)$. Now, since \hat{H} is the center of $\mathcal{Z}_G(H)$ we have that $g\hat{H}g^{-1} = \hat{H}$. By uniqueness of the maximal compact subgroup \bar{H} of \hat{H} , we have $g\bar{H}g^{-1} = \bar{H}$. But this proves that $\bar{H} \subset K$. Hence

$$\bar{H} \subset K \cap \mathcal{Z}_G(H) = \mathcal{Z}_K(H).$$

Now, \bar{H} is central in $\mathcal{Z}_G(H)$ and hence a fortiori in $\mathcal{Z}_K(H)$. This shows that

$$\bar{H} \subset C(\mathcal{Z}_K(H)) = H.$$

This proves the claim. \square

Example: well-centered groups in $SL_n \mathbb{C}$ and polynomials As a matter of example, we discuss the third claim of Lemma 3.6 for $G = SL_n \mathbb{C}$ and A generated by a single semisimple element $a \in SL_n \mathbb{C}$. Denote by (3.1) the canonical direct sum decomposition associated to the Zariski closure of $A = \langle a \rangle$ and consider

$$\mathcal{A} = \{g \in M_n \mathbb{C} \mid \forall i = 1, \dots, r \exists \lambda_i \text{ with } g v_i = \lambda_i v_i \ \forall v_i \in V_i\},$$

ie, the linear subspace of the vector space of all $n \times n$ matrices which are diagonalized by (3.1). The space \mathcal{A} has dimension r and the Vandermonde determinant implies that the elements $\text{Id}, a, a^2, \dots, a^{r-1} \in \mathcal{A}$ are linearly independent. It follows that \mathcal{A} can be identified with the subring $\mathbb{C}[a]$ of $M_n \mathbb{C}$ generated by a . From (3.3) we deduce hence that

$$C(\mathcal{Z}_{SL_n \mathbb{C}}(a)) = \mathbb{C}[a] \cap SL_n \mathbb{C}.$$

Taking into account that $\mathcal{Z}_G(a) = \mathcal{Z}_{SL_n \mathbb{C}}(a) \cap G$ for every subgroup G of $SL_n \mathbb{C}$ we deduce:

Lemma 3.8 *Suppose that $G \subset SL_n \mathbb{C}$ is an algebraic group and that $a \in G$ is semisimple. Then $C(\mathcal{Z}_G(a)) = \mathbb{C}[a] \cap G$ is the smallest well-centered group containing a .*

□

3.4 Manifolds of well-centered groups

Continuing with the same notation, let

$$\begin{aligned} \mathcal{G} &= \{A \subset G \mid A \text{ well-centered in } G\}, \\ \mathcal{K} &= \{H \subset K \mid H \text{ well-centered in } K\}. \end{aligned}$$

The groups G and K act, from the left, on \mathcal{G} and \mathcal{K} by conjugacy.

Lemma 3.9 *Let G_0 and K_0 be the identity components of G and K , respectively. The actions $G_0 \curvearrowright \mathcal{G}$ and $K_0 \curvearrowright \mathcal{K}$ have finitely many orbits.*

Proof By Lemma 3.1, there is some fixed n such that every well-centered subgroup A of G contains a Zariski dense subgroup B generated by n elements. Observe that A and B have the same centralizer. Since B is generated by n elements, there is a semisimple representation $\mathbb{Z}^n \rightarrow G$ with image B . By Proposition 2.6, there are only finitely many G_0 -conjugacy classes of centralizers of images of semisimple representations $\mathbb{Z}^n \rightarrow G$. This proves that there are finitely many G_0 -conjugacy classes of centralizers of well-centered groups.

Suppose now that A and A' have G_0 -conjugate centralizers. Then we have

$$A' = C(\mathcal{Z}_G(A')) = C(g\mathcal{Z}_G(A)g^{-1}) = g(C(\mathcal{Z}_G(A)))g^{-1} = gAg^{-1}.$$

It follows that A and A' belong to the same orbit of G_0 under conjugation. We have proved that $G_0 \curvearrowright \mathcal{G}$ has only finitely many orbits.

The claim for $K_0 \curvearrowright \mathcal{K}$ follows from the validity of the claim for $G_0 \curvearrowright \mathcal{G}$ and Lemma 3.4, the remark following the proof of Lemma 3.4, and Lemma 3.7. \square

Give \mathcal{G} the topology in which each G_0 -orbit has the obvious structure as a connected G_0 -homogenous manifold. More precisely, we consider the connected component of \mathcal{G} to which some well-centered group A belongs as a smooth manifold via the canonical identification as a G_0 -set with the homogenous space $G_0/\mathcal{N}_{G_0}(A)$; here $\mathcal{N}_{G_0}(A)$ is the intersection of G_0 with the normalizer $\mathcal{N}_G(A)$ of A in G . Having a topology on each G_0 -orbit we endow \mathcal{G} with the disjoint union topology.

Similarly, we consider \mathcal{K} with the topology in which each K_0 -orbit is a connected K_0 -homogenous manifold.

Example: manifolds of well-centered groups in $SL_n \mathbb{C}$ As mentioned above, there is a natural correspondence between the sets of well-centered groups in $SL_n \mathbb{C}$ and of unordered direct sum decompositions of \mathbb{C}^n . In particular, the manifolds of well-centered groups in $SL_n \mathbb{C}$ are disjoint unions of quotients of products of open sets in Grassmannians. Similarly, the manifolds of well-centered groups in SU_n are disjoint unions of quotients of compact flag manifolds.

From the proof of Lemma 3.9 we deduce that two well-centered groups $A, A' \in \mathcal{G}$ are in the same connected component of \mathcal{G} if and only if they, or equivalently their centralizers, are G_0 -conjugate. We state this fact for later reference:

Lemma 3.10 *For any two well-centered groups $A, A' \in \mathcal{G}$ the following are equivalent:*

- A and A' are in the same connected component of \mathcal{G} .
- A and A' are conjugate by an element in G_0 .
- $\mathcal{Z}_G(A)$ and $\mathcal{Z}_G(A')$ are conjugate by an element in G_0 . \square

We come now to the main result of this section:

Proposition 3.11 *The map $\iota: \mathcal{K} \rightarrow \mathcal{G}$, $\iota(H) = C(\mathcal{Z}_G(H))$, is an injective homotopy equivalence.*

Proof Suppose that $H, H' \in \mathcal{K}$ are such that $\iota(H) = \iota(H')$. By Lemma 3.7, H and H' are each the unique maximal compact subgroup of the well-centered group $\iota(H) = \iota(H')$; in other words, $H = H'$. This proves that ι is injective.

We prove next that ι induces a bijection between the connected components of \mathcal{K} and \mathcal{G} . Suppose that $H, H' \in \mathcal{K}$ are such that $\iota(H)$ and $\iota(H')$ belong to the same connected component of \mathcal{G} . In other words, there is $g \in G_0$ with $\iota(H') = g\iota(H)g^{-1}$. We deduce again from Lemma 3.7 that $H' = gHg^{-1}$. The remark following the proof of Lemma 3.4 implies then that H and H' are K_0 -conjugate as well. This shows that H and H' belong to the same connected component of \mathcal{K} . We have proven that the map between connected components is injective. Given now $A \in \mathcal{G}$, let \bar{A} be the unique maximal compact subgroup of A , and let $g \in G_0$ be such that $g\bar{A}g^{-1} \subset K$. Since $g\bar{A}g^{-1}$ is the unique maximal compact subgroup of the well-centered group gAg^{-1} , it follows from Lemma 3.7 that $g\bar{A}g^{-1} \in \mathcal{K}$ and that

$$\iota(g\bar{A}g^{-1}) = gAg^{-1}.$$

Since A and gAg^{-1} belong to the same connected component of \mathcal{G} we deduce that the map between the sets of connected components is also surjective.

It remains to prove that the restriction of ι to each connected component $\mathcal{K}_0 \subset \mathcal{K}$ is a homotopy equivalence into the corresponding component $\mathcal{G}_0 \subset \mathcal{G}$. Fix $H \in \mathcal{K}_0$ and let $A = \iota(H) \in \mathcal{G}_0$. By definition we have that $\mathcal{G}_0 = \text{Orbit}_{G_0}(A)$, that $\mathcal{K}_0 = \text{Orbit}_K(H)$, and that

$$\text{Stab}_{K_0}(H) = \mathcal{N}_{K_0}(H_0), \quad \text{Stab}_{G_0}(A) = \mathcal{N}_{G_0}(A) = \mathcal{N}_{G_0}(H),$$

where the last equality holds by Lemma 3.1 and Lemma 3.7.

The remark following Lemma 3.4 implies that the inclusion $\text{Stab}_{K_0}(H) \rightarrow \text{Stab}_{G_0}(A)$ is a homotopy equivalence. In particular we have a diagram as follows

$$\begin{array}{ccccc} \text{Stab}_{K_0}(H) & \longrightarrow & K_0 & \longrightarrow & \mathcal{K}_0 \\ \downarrow & & \downarrow & & \downarrow \iota \\ \text{Stab}_{G_0}(A) & \longrightarrow & G_0 & \longrightarrow & \mathcal{G}_0, \end{array}$$

where the rows are fibrations and where the first two vertical arrows are homotopy equivalences. It follows that the third vertical arrow is also a homotopy equivalence, as claimed. □

3.5 Orderings on $\pi_0(\mathcal{G})$ and $\pi_0(\mathcal{K})$

We introduce a partial order on the set $\pi_0(\mathcal{G})$ of connected components of \mathcal{G} as follows: if \mathcal{G}_1 and \mathcal{G}_2 are connected components of \mathcal{G} then we say that

$$\mathcal{G}_1 \geq \mathcal{G}_2$$

if for some, and hence for every, $A_1 \in \mathcal{G}_1$ there is $A_2 \in \mathcal{G}_2$ with $A_1 \supset A_2$. Recall that Lemma 3.6 implies that this is equivalent to asking that for every $A_1 \in \mathcal{G}_1$ there is $A_2 \in \mathcal{G}_2$ with $\mathcal{Z}_G(A_1) \subset \mathcal{Z}_G(A_2)$.

On $\pi_0(\mathcal{K})$ we consider the analogously defined partial ordering: if \mathcal{K}_1 and \mathcal{K}_2 are connected components of \mathcal{K} , then we say that

$$\mathcal{K}_1 \geq \mathcal{K}_2$$

if for some, and hence for every, $H_1 \in \mathcal{K}_1$, there is $H_2 \in \mathcal{K}_2$ with $H_1 \supset H_2$. Lemmas 3.1, 3.6 and 3.7 imply that, as above, this is equivalent to asking that for every $H_1 \in \mathcal{K}_1$ there is a $H_2 \in \mathcal{K}_2$ with $\mathcal{Z}_G(H_1) \subset \mathcal{Z}_G(H_2)$.

The following claim follows directly from the definitions:

Lemma 3.12 *The homotopy equivalence $\iota: \mathcal{K} \rightarrow \mathcal{G}$ induces an order-preserving bijection $\pi_0(\iota): \pi_0(\mathcal{K}) \rightarrow \pi_0(\mathcal{G})$. □*

Henceforth we will identify \mathcal{K} and $\iota(\mathcal{K})$ and so feel free to write $\mathcal{K} \subset \mathcal{G}$.

Example: the orderings if $G = \text{SL}_n \mathbb{C}$ As mentioned above, there is a natural correspondence between the manifolds of well-centered groups in $\text{SL}_n \mathbb{C}$ and of unordered direct sum decompositions of \mathbb{C}^n . If \mathcal{G}_1 and \mathcal{G}_2 are connected components of the manifolds of unordered direct sum decompositions then $\mathcal{G}_1 \geq \mathcal{G}_2$ if and only if for every direct sum decomposition $\mathbb{C}^n = C_1 \oplus \dots \oplus V_r$ in \mathcal{G}_1 there is a direct sum decomposition $\mathbb{C}^n = W_1 \oplus \dots \oplus W_s$ in \mathcal{G}_2 such that for each $i \in \{1, \dots, r\}$ there is $j \in \{1, \dots, s\}$ with $V_i \subset W_j$.

3.6 The set $\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$

Suppose that we have two components $\mathcal{G}_1, \mathcal{G}_2$ of \mathcal{G} with $\mathcal{G}_1 \geq \mathcal{G}_2$ and consider the set

$$(3.4) \quad \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} = \{(A_1, A_2) \in \mathcal{G}_1 \times \mathcal{G}_2 \mid A_1 \supset A_2\}.$$

Our next goal is to prove that the projections of $\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ onto \mathcal{G}_1 and \mathcal{G}_2 are respectively a finite cover and a locally trivial fibration. Before doing so, notice that

for any $(A_1, A_2) \in \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ the groups $\mathcal{N}_{G_0}(A_1), \mathcal{N}_{G_0}(A_2)$ act from the right on $\text{Trans}_{G_0}(A_2, A_1)$:

$$\begin{aligned} \text{Trans}_{G_0}(A_2, A_1) \times \mathcal{N}_{G_0}(A_2) &\rightarrow \text{Trans}_{G_0}(A_2, A_1), & (g, h) &\mapsto gh, \\ \text{Trans}_{G_0}(A_2, A_1) \times \mathcal{N}_{G_0}(A_1) &\rightarrow \text{Trans}_{G_0}(A_2, A_1), & (g, h) &\mapsto h^{-1}g. \end{aligned}$$

We prove:

Lemma 3.13 Fixing $(A_1, A_2) \in \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ the following holds:

- The projection $p_2: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_2$ on the second factor is a locally trivial fiber bundle with fiber $\text{Trans}_{G_0}(A_2, A_1)/\mathcal{N}_{G_0}(A_1)$.
- The projection $p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$ on the first factor is a finite covering with fiber $\text{Trans}_{G_0}(A_2, A_1)/\mathcal{N}_{G_0}(A_2)$.

Proof Identifying \mathcal{G}_1 and \mathcal{G}_2 with the G_0 -orbits of A_1 and A_2 , we obtain identifications

$$\mathcal{G}_1 = G_0/\mathcal{N}_{G_0}(A_1), \quad \mathcal{G}_2 = G_0/\mathcal{N}_{G_0}(A_2).$$

Consider the standard right action of $\mathcal{N}_{G_0}(A_1) \times \mathcal{N}_{G_0}(A_2)$ on $G_0 \times G_0$

$$\begin{aligned} (G_0 \times G_0) \times (\mathcal{N}_{G_0}(A_1) \times \mathcal{N}_{G_0}(A_2)) &\rightarrow G_0 \times G_0, \\ ((g_1, g_2), (h_1, h_2)) &\mapsto (g_1h_1, g_2h_2), \end{aligned}$$

and notice that the set

$$\tilde{\mathcal{R}} = \{(g_1, g_2) \in G_0 \times G_0 \mid g_1A_1g_1^{-1} \supset g_2A_2g_2^{-1}\}$$

is $\mathcal{N}_{G_0}(A_1) \times \mathcal{N}_{G_0}(A_2)$ -invariant. In fact we have that

$$\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} = \tilde{\mathcal{R}}/(\mathcal{N}_{G_0}(A_1) \times \mathcal{N}_{G_0}(A_2)).$$

The projection $\tilde{p}_2: \tilde{\mathcal{R}} \rightarrow G_0$, $\tilde{p}_2(g_1, g_2) = g_2$ has fibers

$$\tilde{p}_2^{-1}(g_2) = \{(g_1, g_2) \in G_0 \times G_0 \mid g_1^{-1}g_2 \in \text{Trans}_{G_0}(A_2, A_1)\}.$$

It follows that $\tilde{p}_2: \tilde{\mathcal{R}} \rightarrow G_0$ is a locally trivial fiber bundle over G_0 , with fiber $\text{Trans}_{G_0}(A_2, A_1)$. From here we obtain that the projection

$$p_2: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_2$$

is also a locally trivial fiber bundle with fibers $\text{Trans}_{G_0}(A_2, A_1)/\mathcal{N}_{G_0}(A_1)$.

The same reasoning yields that the projection

$$p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$$

is also a locally trivial fiber bundle with fibers $\text{Trans}_{G_0}(A_2, A_1)/\mathcal{N}_{G_0}(A_2)$. It follows from Lemma 3.3 that the quotient $\text{Trans}_{G_0}(A_2, A_1)/\mathcal{N}_{G_0}(A_2)$ is finite. As it is a locally trivial fiber bundle with discrete fibers, $p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$ is a cover, as claimed. \square

Recall that we are identifying \mathcal{K} with a subset of \mathcal{G} . Given $\mathcal{G}_1, \mathcal{G}_2 \in \pi_0(\mathcal{G})$ with $\mathcal{G}_1 \geq \mathcal{G}_2$, let $\mathcal{K}_1 \subset \mathcal{G}_1$ and $\mathcal{K}_2 \subset \mathcal{G}_2$ be the corresponding connected components of \mathcal{K} and set

$$(3.5) \quad \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} = \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \cap (\mathcal{K}_1 \times \mathcal{K}_2) = \{(H_1, H_2) \in \mathcal{K}_1 \times \mathcal{K}_2 \mid H_1 \supset H_2\}.$$

The same argument used to prove Lemma 3.13 yields:

Lemma 3.14 *Fixing $(H_1, H_2) \in \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2}$, the following holds:*

- *The projection $p_2: \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{K}_2$ on the second factor is a locally trivial fiber bundle with fiber $\text{Trans}_{K_0}(H_2, H_1)/\mathcal{N}_{K_0}(H_1)$.*
- *The projection $p_1: \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{K}_1$ on the first factor is a finite covering with fiber $\text{Trans}_{K_0}(H_2, H_1)/\mathcal{N}_{K_0}(H_2)$.* \square

3.7 The real case

Suppose now that the reductive group G is defined over \mathbb{R} , let $G(\mathbb{R})$ be the group of real points and $K(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$. Denote by $G(\mathbb{R})_0$ and $K(\mathbb{R})_0$, as always, the identity components of $G(\mathbb{R})$ and $K(\mathbb{R})$.

We will say that $A \subset G(\mathbb{R})$ is *diagonalizable* if its Zariski closure $\text{Zar}(A) \subset G$ is diagonalizable and satisfies $A = \text{Zar}(A)(\mathbb{R})$. A diagonalizable subgroup A of $G(\mathbb{R})$ (resp. $K(\mathbb{R})$) is *well-centered in $G(\mathbb{R})$* (resp. in $K(\mathbb{R})$) if it is equal to the center of its centralizer in $G(\mathbb{R})$ (resp. in $K(\mathbb{R})$). As above, we denote by \mathcal{G} and \mathcal{K} the set of all well-centered subgroups of $G(\mathbb{R})$ and $K(\mathbb{R})$, respectively.

We state now a few facts whose proofs we leave to the reader:

- (1) If $A \subset G(\mathbb{R})$ is an abelian subgroup consisting of semisimple elements, then $\text{Zar}(A)(\mathbb{R})$ (resp. $C(\mathcal{Z}_{G(\mathbb{R})}(A))$) is the smallest diagonalizable subgroup of $G(\mathbb{R})$ (resp. smallest well-centered group in $G(\mathbb{R})$) containing A .
- (2) If $A \subset G(\mathbb{R})$ is diagonalizable, then the intersection of A and the unique maximal compact subgroup of $\text{Zar}(A)$ is the unique maximal compact subgroup of A . Moreover, the Zariski closures of A and its unique maximal compact subgroup agree.

- (3) If $A, A' \subset G(\mathbb{R})$ are diagonalizable, then $\text{Trans}_{G(\mathbb{R})}(A, A')$ is the subset of $\text{Trans}_G(\text{Zar}(A), \text{Zar}(A'))$ fixed by complex conjugation and hence a real algebraic variety; as such, it has only finitely many connected components.
- (4) If $A, A' \subset G(\mathbb{R})$ are diagonalizable, $V \subset \text{Trans}_{G(\mathbb{R})}(A, A')$ is a connected component, and $g_0 \in V$, then the map $\mathcal{Z}_{G(\mathbb{R})}(A)_0 \rightarrow V, h \mapsto g_0 h$ is an isomorphism; here, $\mathcal{Z}_{G(\mathbb{R})}(A)_0$ is the connected component of the identity in the Lie group $\mathcal{Z}_{G(\mathbb{R})}(A)$.
- (5) If $H_1, H_2 \subset K(\mathbb{R})$ are subgroups, then the maps

$$\begin{aligned} \text{Trans}_{K(\mathbb{R})}(H_1, H_2) &\hookrightarrow \text{Trans}_{G(\mathbb{R})}(H_1, H_2), \\ \text{Trans}_{K(\mathbb{R})_0}(H_1, H_2) &\hookrightarrow \text{Trans}_{G(\mathbb{R})_0}(H_1, H_2), \end{aligned}$$

are homotopy equivalences.

- (6) If $H \in \mathcal{K}$ is well-centered in $K(\mathbb{R})$, then H is the unique maximal compact subgroup of the well-centered group $C(\mathcal{Z}_{G(\mathbb{R})})(H) \in \mathcal{G}$.
- (7) The actions $G(\mathbb{R}) \curvearrowright \mathcal{G}$ and $K(\mathbb{R}) \curvearrowright \mathcal{K}$ have finitely many orbits.

Remarks about the proofs of (1)–(7) Claims (1), (2) and (3) are just elementary considerations; compare with the proof of Lemma 2.14 for (3). Lemma 3.3 and (3) imply (4). The group $G(\mathbb{R})$ has a Cartan decomposition, now called *polar decomposition*; this said, Lemma 3.5 holds and hence the proofs of Lemma 3.4 and Lemma 3.7 transfer word-for-word to prove (5) and (6). Claim (7) follows from Lemma 3.9 and Lemma 2.14.

Before going any further, observe that (4) implies:

Lemma 3.3 (Real case) *The Weyl groups $W_A = \mathcal{N}_{G(\mathbb{R})}(A)/\mathcal{Z}_{G(\mathbb{R})}(A)$ and $W_A^0 = \mathcal{N}_{G(\mathbb{R})_0}(A)/\mathcal{Z}_{G(\mathbb{R})_0}(A)$ are finite for every diagonalizable subgroup A of $G(\mathbb{R})$. \square*

Endow \mathcal{G} (resp. \mathcal{K}) with the manifold structure induced by the action of $G(\mathbb{R})_0$ (resp. $K(\mathbb{R})_0$). With facts (1)–(6) on hand, we can repeat word-for-word the proof of Proposition 3.11 and show:

Proposition 3.11 (Real case) *The map $\iota: \mathcal{K} \rightarrow \mathcal{G}, \iota(H) = C(\mathcal{Z}_{G(\mathbb{R})}(H))$ is an injective homotopy equivalence. \square*

At this point we can define orderings on $\pi_0(\mathcal{G})$ and $\pi_0(\mathcal{K})$ exactly as we did in Section 3.5 in the complex case. We have then:

Lemma 3.12 (Real case) *The homotopy equivalence $\iota: \mathcal{K} \rightarrow \mathcal{G}$ induces an order-preserving bijection $\pi_0(\iota): \pi_0(\mathcal{K}) \rightarrow \pi_0(\mathcal{G})$. \square*

As for the complex case, we identify \mathcal{K} and $\iota(\mathcal{K}) \subset \mathcal{G}$. For any two components $\mathcal{G}_1, \mathcal{G}_2$ of \mathcal{G} with $\mathcal{G}_1 \geq \mathcal{G}_2$ containing components $\mathcal{K}_1 \subset \mathcal{G}_1$ and $\mathcal{K}_2 \subset \mathcal{G}_2$ of \mathcal{K} we define the sets $\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ and $\mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2}$ exactly as we did in (3.4) and (3.5). Armed with Facts (1)–(7) we can repeat word-for-word the same arguments as in Section 3.6 to prove:

Lemma 3.13 (Real case) *Fixing $(A_1, A_2) \in \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$, the following hold:*

- The projection $p_2: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_2$ on the second factor is a locally trivial fiber bundle with fiber $\text{Trans}_{G(\mathbb{R})_0}(A_2, A_1)/\mathcal{N}_{G(\mathbb{R})_0}(A_1)$.
- The projection $p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$ on the first factor is a finite covering with fiber $\text{Trans}_{G(\mathbb{R})_0}(A_2, A_1)/\mathcal{N}_{G(\mathbb{R})_0}(A_2)$. □

Lemma 3.14 (Real case) *Fixing $(H_1, H_2) \in \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2}$, the following holds:*

- The projection $p_2: \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{K}_2$ on the second factor is a locally trivial fiber bundle with fiber $\text{Trans}_{K(\mathbb{R})_0}(H_2, H_1)/\mathcal{N}_{K(\mathbb{R})_0}(H_1)$.
- The projection $p_1: \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{K}_1$ on the first factor is a finite covering with fiber $\text{Trans}_{K(\mathbb{R})_0}(H_2, H_1)/\mathcal{N}_{K(\mathbb{R})_0}(H_2)$. □

4 The canonical bundle over \mathcal{G}

As before, let G be the group of either complex or real points of a reductive algebraic group (defined over \mathbb{R} in the latter case), let $K \subset G$ be a maximal compact subgroup, and recall that \mathcal{G} is the manifold of all well-centered subgroups of G . As always G_0 and K_0 are the identity components of G and K . In this section we study a certain flat bundle

$$\pi: \mathcal{D} \rightarrow \mathcal{G}.$$

We take a flat bundle to mean a bundle endowed with a complete flat connection; in more concrete terms, the total space \mathcal{D} is foliated in such a way that the restriction of π to each leaf of the foliation is a covering of \mathcal{G} .

We also construct for all k a map

$$\Psi: \mathcal{D}(k) \rightarrow \text{Hom}(\mathbb{Z}^k, G^s),$$

where $\mathcal{D}(k) \rightarrow \mathcal{G}$ is the k -fold direct product of the bundle $\mathcal{D} \rightarrow \mathcal{G}$ with itself. Finally, we describe how Ψ is related to the ordering of $\pi_0(\mathcal{G})$ and the isotopies induced by the flat connection.

4.1 Canonical bundle

Endow $\mathcal{D} = \{(A, a) \in \mathcal{G} \times G \mid a \in A\}$ with the subspace topology and consider the projection

$$\pi: \mathcal{D} \rightarrow \mathcal{G}, \quad (A, a) \mapsto A$$

onto the first factor. We claim that $\pi: \mathcal{D} \rightarrow \mathcal{G}$ is a locally trivial bundle.

Fix a connected component \mathcal{G}_0 of \mathcal{G} set

$$\mathcal{D}_{\mathcal{G}_0} = \pi^{-1}(\mathcal{G}_0) = \{(A, a) \in \mathcal{G}_0 \times G \mid a \in A\}$$

and consider the restriction of π to $\mathcal{D}_{\mathcal{G}_0}$. In order to prove that $\pi: \mathcal{D} \rightarrow \mathcal{G}$ is a locally trivial bundle, it suffices to prove it for $\pi: \mathcal{D}_{\mathcal{G}_0} \rightarrow \mathcal{G}_0$.

Fixing a base point $A \in \mathcal{G}_0$ recall that, by construction, \mathcal{G}_0 is equal to the orbit of A under the action of G_0 by conjugation. It follows that the map

$$\sigma_0: G_0 \times A \rightarrow \mathcal{D}_{\mathcal{G}_0}, \quad (g, a) \mapsto (gAg^{-1}, gag^{-1})$$

is surjective. The normalizer $\mathcal{N}_{G_0}(A)$ of A in G_0 acts from the right on $G_0 \times A$ as follows:

$$(G_0 \times A) \times \mathcal{N}_{G_0}(A) \rightarrow G_0 \times A, \quad ((g, a), h) \mapsto (g, a) \cdot h = (gh, h^{-1}ah).$$

A simple computation shows that the fibers of the map σ_0 agree with the $\mathcal{N}_{G_0}(A)$ -orbits on $G_0 \times A$. It follows that σ_0 descends to a homeomorphism

$$\tau_0: (G_0 \times A)/\mathcal{N}_{G_0}(A) \rightarrow \mathcal{D}_{\mathcal{G}_0}.$$

Notice that, as G_0 acts transitively on \mathcal{G}_0 , the base point A yields a canonical identification $\mathcal{G}_0 = G_0/\mathcal{N}_{G_0}(A)$. The projection $G_0 \times A \rightarrow G_0$ onto the first factor induces a projection

$$\pi': (G_0 \times A)/\mathcal{N}_{G_0}(A) \rightarrow G_0/\mathcal{N}_{G_0}(A) = \mathcal{G}_0.$$

Moreover, the projection π' and the homeomorphism τ_0 are such that the following diagram commutes:

$$\begin{array}{ccc} (G_0 \times A)/\mathcal{N}_{G_0}(A) & \xrightarrow{\tau_0} & \mathcal{D}_{\mathcal{G}_0} \\ & \searrow \pi' & \swarrow \pi \\ & G_0/\mathcal{N}_{G_0}(A) & \end{array}$$

In particular, in order to prove that $\pi: \mathcal{D}_{\mathcal{G}_0} \rightarrow \mathcal{G}_0$ is a locally trivial bundle it will be sufficient to prove it for

$$\pi': (G_0 \times A)/\mathcal{N}_{G_0}(A) \rightarrow G_0/\mathcal{N}_{G_0}(A).$$

Recall that by Lemma 3.3, the group

$$W_A^0 = \mathcal{N}_{G_0}(A) / \mathcal{Z}_{G_0}(A)$$

is finite. Taking into account that

$$\begin{aligned} (G_0 \times A) / \mathcal{N}_{G_0}(A) &= ((G_0 / \mathcal{Z}_{G_0}(A)) \times A) / W_A^0, \\ G_0 / \mathcal{N}_{G_0}(A) &= (G_0 / \mathcal{Z}_{G_0}(A)) / W_A^0, \end{aligned}$$

we obtain a commuting diagram as follows:

$$\begin{array}{ccc} G_0 / \mathcal{Z}_{G_0}(A) \times A & \xrightarrow{\mathfrak{r}_{W_A^0}} & (G_0 \times A) / \mathcal{N}_{G_0}(A) \\ \downarrow & & \downarrow \pi' \\ G_0 / \mathcal{Z}_{G_0}(A) & \xrightarrow{\mathfrak{r}_{W_A^0}} & G_0 / \mathcal{N}_{G_0}(A) \end{array}$$

Here the left arrow is the projection onto the first factor, and the horizontal arrows are the orbit maps of the action of W_A^0 .

By construction, W_A^0 acts on the trivial bundle $G_0 / \mathcal{Z}_{G_0}(A) \times A \rightarrow G_0 / \mathcal{Z}_{G_0}(A)$ by bundle homomorphisms. Moreover, the action is free and a fortiori discrete because W_A^0 is finite. It follows that the quotient $\pi': (G_0 \times A) / \mathcal{N}_{G_0}(A) \rightarrow G_0 / \mathcal{N}_{G_0}(A)$ is a locally trivial bundle. We have proved that

$$\pi: \mathcal{D} \rightarrow \mathcal{G}$$

is a locally trivial bundle. We will refer to this bundle as *the canonical bundle over \mathcal{G}* .

Continuing with the same notation, endow now the trivial bundle $(G_0 / \mathcal{Z}_{G_0}(A)) \times A \rightarrow G_0 / \mathcal{Z}_{G_0}(A)$ with the flat connection induced by the product structure. In other words, the leaves of the foliation associated to this flat connection are the sets of the form $(G_0 / \mathcal{Z}_{G_0}(A)) \times \{a\}$. Since the group W_A^0 preserves the product structure, it follows that this flat connection descends to a flat connection on the canonical bundle. We have hence:

Proposition 4.1 *The canonical bundle $\pi: \mathcal{D} \rightarrow \mathcal{G}$ is a flat locally trivial bundle. \square*

4.2 Horizontal lifts

The use of having a flat connection on the bundle $\pi: \mathcal{D} \rightarrow \mathcal{G}$ is that to every curve $\gamma: [0, 1] \rightarrow \mathcal{G}$ and to every point $a \in \pi^{-1}(\gamma(0)) \subset \mathcal{D}$ we can associate the *horizontal lift of γ starting at a* . This is namely the unique curve

$$\tilde{\gamma}: [0, 1] \rightarrow \mathcal{D}$$

with $\tilde{\gamma}(0) = a$ and $\pi(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [0, 1]$, and whose image is contained in a single leaf of the foliation associated to the flat connection.

Our next goal is to give a concrete description of the horizontal lifts of curves $\gamma: [0, 1] \rightarrow \mathcal{G}$. To do so, let \mathcal{G}_0 be the connected component of \mathcal{G} containing the image of γ , fix $A \in \mathcal{G}_0$, and recall that the map

$$G_0 \rightarrow \mathcal{G}_0, \quad g \mapsto gAg^{-1}$$

is a fiber bundle with fiber $\mathcal{N}_{G_0}(A)$. Hence, every curve $\gamma: [0, 1] \rightarrow \mathcal{G}$ admits a (far from unique) lift $[0, 1] \rightarrow G_0, t \mapsto g_t$. The condition that $t \mapsto g_t$ is a lift of γ amounts to

$$\gamma(t) = g_t A g_t^{-1} = (g_t g_0^{-1}) \gamma(0) (g_t g_0^{-1})^{-1} \quad \text{for all } t.$$

Notice that any element in $\pi^{-1}(\gamma(0)) \in \mathcal{D}$ is of the form $(\gamma(0), a)$, where $a \in \gamma(0) = g_0 A g_0^{-1} \subset G$. The curve

$$\tilde{\gamma}: [0, 1] \rightarrow \mathcal{D}, \quad t \mapsto ((g_t g_0^{-1}) \gamma(0) (g_t g_0^{-1})^{-1}, (g_t g_0^{-1}) a (g_t g_0^{-1})^{-1})$$

is both parallel and a lift of γ ; in other words, $\tilde{\gamma}$ is the horizontal lift of γ with $\tilde{\gamma}(0) = (\gamma(0), a)$. We have proved:

Lemma 4.2 (Horizontal lift) *Let $\gamma: [0, 1] \rightarrow \mathcal{G}$ be a continuous curve and fix a point A in the connected component of \mathcal{G} containing the image of γ .*

For any $a \in \gamma(0) \in \mathcal{G}$, the horizontal lift $\tilde{\gamma}: [0, 1] \rightarrow \mathcal{D}$ of γ starting at $\tilde{\gamma}(0) = (\gamma(0), a)$ is given by

$$\tilde{\gamma}(t) = ((g_t g_0^{-1}) \gamma(0) (g_t g_0^{-1})^{-1}, (g_t g_0^{-1}) a (g_t g_0^{-1})^{-1}),$$

where $[0, 1] \rightarrow G_0, t \mapsto g_t$ is an arbitrary curve with $\gamma(t) = g_t A g_t^{-1}$ for all t . □

4.3 The evaluation map

Fix now k , recall that $\text{Hom}(\mathbb{Z}^k, G^s)$ is the set of commuting semisimple k -tuples in G , and denote by

$$\pi: \mathcal{D}(k) \rightarrow \mathcal{G}$$

the k^{th} fibered product of the bundle $\mathcal{D} \rightarrow \mathcal{G}$. In more concrete terms, the total space $\mathcal{D}(k)$ is given by

$$\mathcal{D}(k) = \{(A, a_1, \dots, a_k) \in \mathcal{G} \times G \times \dots \times G \mid a_1, \dots, a_k \in A\}.$$

As a fibered product of flat bundles, $\mathcal{D}(k)$ is naturally endowed with a flat connection.

Forgetting for a moment the bundle structure of $\mathcal{D}(k) \rightarrow \mathcal{G}$, we consider the following continuous map defined on the total space:

$$(4.1) \quad \Psi: \mathcal{D}(k) \rightarrow \text{Hom}(\mathbb{Z}^k, G^s), \quad (A, a_1, \dots, a_k) \mapsto (a_1, \dots, a_k).$$

Here we are identifying representations of \mathbb{Z}^k with k -commuting tuples in G . Notice that the *evaluation map* Ψ is well-defined because every well-centered group $A \in \mathcal{G}$ is, by definition, abelian and consists of semisimple elements.

Lemma 4.3 *The evaluation map $\Psi: \mathcal{D}(k) \rightarrow \text{Hom}(\mathbb{Z}^k, G^s)$ is surjective.*

Proof Suppose that we are given $\rho \in \text{Hom}(\mathbb{Z}^k, G^s)$. By Lemma 2.5, the abelian group $\rho(\mathbb{Z}^k)$ consists of semisimple elements. In particular, its Zariski closure is diagonalizable. It follows hence from Lemma 3.6 that $A = C(\mathcal{Z}_G(\rho(\mathbb{Z}^k)))$ is a well-centered group containing $\rho(\mathbb{Z}^n)$. Let $a_i = \rho(e_i)$ where e_1, \dots, e_k is the standard basis of \mathbb{Z}^k . Then $(A, a_1, \dots, a_k) \in \mathcal{D}(k)$ and $\rho = \Psi(A, a_1, \dots, a_k)$ by construction. □

Before moving on we observe the following useful fact:

Lemma 4.4 *We have $\Psi(gAg^{-1}, g\alpha g^{-1}) = g\Psi(A, \alpha)g^{-1}$ for all $(A, \alpha) \in \mathcal{D}(k)$ and $g \in G$.* □

Example: horizontal lifting and evaluation map for $\text{SL}_n \mathbb{C}$ In the case of $\text{SL}_n \mathbb{C}$, the combination of horizontal lifting and the evaluation map just means the following: Suppose that a matrix $A \in \text{SL}_\mathbb{C}$ is diagonalized by a direct sum decomposition $\mathbb{C}^n = V_1 \oplus \dots \oplus V_r$ and let λ_i be the homothety factor of A when restricted to V_i . If we perturb each one of the spaces V_i slightly to get some other space V'_i , then there is a unique matrix $A' \in \text{SL}_n \mathbb{C}$, which, restricted to V'_i , is a homothety with factor λ_i .

4.4 Compatibility of horizontal lifts

Suppose that we are given $\mathcal{G}_1, \mathcal{G}_2 \in \pi_0(\mathcal{G})$ with $\mathcal{G}_1 \geq \mathcal{G}_2$, and recall that this means that some, and hence every, $A_2 \in \mathcal{G}_2$ can be conjugated into some, and hence into every, $A_1 \in \mathcal{G}_1$. Recall that by Lemma 3.13, the projections of the set

$$\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} = \{(A_1, A_2) \in \mathcal{G}_1 \times \mathcal{G}_2 \mid A_1 \supset A_2\}$$

onto \mathcal{G}_1 and \mathcal{G}_2 are respectively a finite cover and a locally trivial fibration.

Lemma 4.5 (Compatibility of horizontal lifts) *Suppose that we are given a curve*

$$[0, 1] \rightarrow \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}, \quad t \mapsto (\gamma_1(t), \gamma_2(t))$$

and two points $(\gamma_1(0), \alpha_1) \in \mathcal{D}_{\mathcal{G}_1}(k)$ and $(\gamma_2(0), \alpha_2) \in \mathcal{D}_{\mathcal{G}_2}(k)$, and consider the horizontal lifts

$$\tilde{\gamma}_1: [0, 1] \rightarrow \mathcal{D}_{\mathcal{G}_1}(k), \quad \tilde{\gamma}_2: [0, 1] \rightarrow \mathcal{D}_{\mathcal{G}_2}(k)$$

of γ_1 and γ_2 , with $\tilde{\gamma}_i(0) = (\gamma_i(0), \alpha_i)$ for $i = 1, 2$.

If $\Psi(\gamma_1(0), \alpha_1) = \Psi(\gamma_2(0), \alpha_2)$, then $\Psi(\tilde{\gamma}_1(t)) = \Psi(\tilde{\gamma}_2(t))$ for all t .

Proof Setting $(A_1, A_2) = (\gamma_1(0), \gamma_2(0))$, we obtain, as so often above, the identifications

$$\mathcal{G}_1 = G_0/\mathcal{N}_{G_0}(A_1), \quad \mathcal{G}_2 = G_0/\mathcal{N}_{G_0}(A_2).$$

Consider $G_0 \rightarrow \mathcal{G}_1$ as a fiber bundle, and let $t \rightarrow g_t$ be any continuous lift of the curve $t \mapsto \gamma_1(t)$ with $g_1(0) = \text{Id}$. Recall that by Lemma 3.13, the projection

$$p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$$

is a finite covering. The two curves

$$\begin{aligned} [0, 1] &\rightarrow \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}, \quad t \mapsto (\gamma_1(t), \gamma_2(t)), \\ [0, 1] &\rightarrow \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}, \quad t \mapsto (g_t A_1 g_t^{-1}, g_t A_2 g_t^{-1}) = (\gamma_1(t), g_t A_2 g_t^{-1}), \end{aligned}$$

are lifts of $t \mapsto \gamma_1(t)$ under the covering $p_1: \mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \rightarrow \mathcal{G}_1$; since both curves agree for $t = 0$, it follows that $\gamma_2(t) = g_t A_2 g_t^{-1}$ for all t . This proves that the curve $t \mapsto g_t$ is also a lift of the curve $t \mapsto \gamma_2(t)$ for the fiber bundle $G_0 \rightarrow \mathcal{G}_2$.

Lemma 4.2 shows that the horizontal lift $\tilde{\gamma}_i: [0, 1] \rightarrow \mathcal{D}_i$ of γ_i with $\tilde{\gamma}_i(0) = (\gamma_i(0), \alpha_i)$ is given by

$$\tilde{\gamma}_i(t) = (g_t \gamma_i(0) g_t^{-1}, g_t \alpha_i g_t^{-1})$$

for $i = 1, 2$. Lemma 4.4 implies that

$$\Psi(\tilde{\gamma}_i(t)) = g_t \Psi(\tilde{\gamma}_i(0)) g_t^{-1}.$$

The claim follows. □

4.5 An interesting subbundle

Continuing with the same notation, recall that by Lemma 3.1, every well-centered group $A \in \mathcal{G}$ contains a uniquely determined maximal compact subgroup \bar{A} . Consider the subset

$$\bar{\mathcal{D}} = \{(A, a) \in \mathcal{G} \times G \mid a \in \bar{A}\} \subset \mathcal{D}$$

and notice that the uniqueness of the maximal compact subgroups implies that $\overline{gAg^{-1}} = g\bar{A}g^{-1}$ for all $A \in \mathcal{G}$ and all $g \in G$. It follows hence from Lemma 4.2 that for any $(A, a) \in \bar{\mathcal{D}}$ and any curve $\gamma: [0, 1] \rightarrow \mathcal{G}$ with $\gamma(0) = A$, we have $\tilde{\gamma}(t) \in \bar{\mathcal{D}}$ for all t , where $\tilde{\gamma}$ is the horizontal lift of γ with $\tilde{\gamma}(0) = (A, a)$. In other words, the subset $\bar{\mathcal{D}}$ of the total space of the bundle $\mathcal{D} \rightarrow \mathcal{G}$ is parallel.

This implies directly that the subset

$$\bar{\mathcal{D}}(k) = \{(A, a_1, \dots, a_k) \in \mathcal{G} \times G \times \dots \times G \mid a_1, \dots, a_k \in \bar{A}\}$$

is a parallel subset of the bundle $\pi: \mathcal{D}(k) \rightarrow \mathcal{G}$. Observe now that any parallel subset of a flat bundle is a flat bundle in its own right. We have proved:

Lemma 4.6 $\pi: \bar{\mathcal{D}}(k) \rightarrow \mathcal{G}$ is a flat subbundle of $\pi: \mathcal{D}(k) \rightarrow \mathcal{G}$. □

Recall at this point that \mathcal{K} is the manifold of well-centered subgroups of K , and that by Proposition 3.11 the map

$$\iota: \mathcal{K} \rightarrow \mathcal{G}, \quad \iota(H) = C(\mathcal{Z}_G(H))$$

is an injective homotopy equivalence. Recall also the definition (2.3) of $\text{Hom}(\mathbb{Z}^k, K^G)$.

Lemma 4.7 *With the same notation as above, we have:*

- (1) $\Psi(\bar{\mathcal{D}}(k)) = \text{Hom}(\mathbb{Z}^k, K^G)$.
- (2) $\Psi(\bar{\mathcal{D}}(k) \cap \pi^{-1}\mathcal{K}) = \text{Hom}(\mathbb{Z}^k, K)$.

Proof Recall that $(A, a_1, \dots, a_k) \in \bar{\mathcal{D}}(k)$ if and only if $a_1, \dots, a_k \in \bar{A}$ where \bar{A} is the unique maximal compact subgroup of A . Since \bar{A} is compact, there is $g \in G$ with $\bar{A} \subset gKg^{-1}$. In particular, $a_1, \dots, a_k \in K^G = \bigcup_{g \in G} gKg^{-1}$; this proves that

$$\Psi(A, a_1, \dots, a_k) = (a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, K^G)$$

and hence that $\Psi(\mathcal{D}(k)) \subset \text{Hom}(\mathbb{Z}^k, K^G)$. The opposite inclusion follows from a similar, simpler, argument (compare with Lemma 3.2); this proves (1).

Continuing with the same notation assume now that $A \in \mathcal{K} \subset \mathcal{G}$ and recall that this means that there is a well-centered $H \subset K$ with $A = C(\mathcal{Z}_G(H))$. Lemma 3.7 shows that H is the maximal compact subgroup of A , ie, $H = \bar{A}$. Since $H \subset K$, it follows that $\Psi(A, a_1, \dots, a_k) = (a_1, \dots, a_k) \in \text{Hom}(\mathbb{Z}^k, K)$ for any $a_1, \dots, a_k \in \bar{A} = H$. This proves that $\Psi(\bar{D}(k) \cap \pi^{-1}\mathcal{K}) \subset \text{Hom}(\mathbb{Z}^k, K)$. The opposite inclusion follows again by a similar, again simpler, argument. This proves (2). \square

5 The homotopic framework

In this section we introduce some terminology and state a topological result, Proposition 5.3, which will play a key role in the proof of Theorem 1.1. We defer its proof to the Appendix.

All topological spaces in this section will be CW-complexes, or more precisely, homeomorphic to CW-complexes.

5.1 Regular coverings of simplicial complexes

Suppose that X is a (finite-dimensional, locally finite) simplicial complex and denote by $C(X)$ the set of all closed cells of X . Abusing terminology, we will not distinguish between the abstract complex X and its geometric realization. We denote by $X^{(d)}$ the d -skeleton of X , ie, the simplicial complex consisting of all the cells of X of dimension at most d ; the set of closed d -dimensional cells we denote by $C_d(X)$. Recall also that an open covering of a space is *good* if all the possible intersections are either empty or contractible.

Definition An open covering $\mathcal{U} = \{U_c\}_{c \in C(X)}$ of a simplicial complex X with components indexed by the set of closed cells of X is *regular* if the following are satisfied:

- \mathcal{U} is a good covering.
- For all d , $\bigcup_{c \in C_d(X)} \bar{U}_c$ is a closed regular neighborhood in $X \setminus X^{(d-1)}$ of $X^{(d)} \setminus \bigcup_{c \in C(X^{(d-1)})} U_c$. Here \bar{U}_c is the closure of U_c in X .
- If $c, c' \in C(X)$ are cells with $U_c \cap U_{c'} \neq \emptyset$ then either $c \subset c'$ or $c' \subset c$. In particular, if c and c' are distinct cells of the same dimension then $U_c \cap U_{c'} = \emptyset$.
- For any cell $c \in C(X)$ we have $c \subset \bigcup_{c' \subset c} U_{c'}$.

The reader can find in Figure 1 an example of a regular covering of the 2-dimensional simplex. Clearly, every simplicial complex has a regular covering. Later we will need the following lemma, whose proof we leave to the reader:

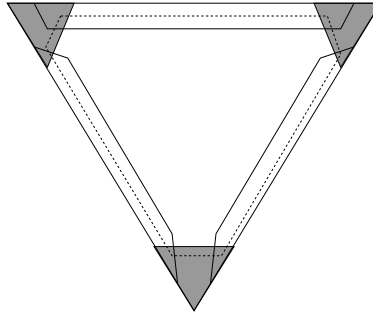


Figure 1: The shaded regions are the sets U_c for the vertices. The solid lines bound the neighborhoods U_c for the edges and the dashed line is the boundary of U_c for the 2-dimensional cell.

Lemma 5.1 (Smashing Lemma) *Let X be simplicial complex and $\mathcal{U} = \{U_c\}_{c \in C(X)}$ a regular covering. There is a continuous (non-simplicial) map*

$$\Phi: X \rightarrow X$$

with $\Phi(U_c) = \Phi(c) = c$ for any closed cell $c \in C(X)$. □

In the particular case represented in Figure 1, the map Φ provided by Lemma 5.1 sends the shaded regions to the vertices, the regions bounded by solid lines to the edges, and the region bounded by the dashed line onto the whole simplex.

5.2 Fibered relations

Recall that a fibration is a continuous surjective map $\pi: S \rightarrow S'$ with the homotopy lifting property. In this paper we are not lucky enough to find ourselves working with fibrations. What we encounter are *fibered relations*.

Definition A subset $\mathcal{R} \subset S \times S'$ of the cartesian product $S \times S'$ of two CW-complexes is a *fibered relation* if $\mathcal{R} \rightarrow S'$ is a fibration.

If $K \subset S$ and $K' \subset S'$ are such that $\mathcal{O} = \mathcal{R} \cap (K \times K')$ is a fibered relation in $K \times K'$, then we say that $(\mathcal{R}, \mathcal{O})$ is a *fibered relation pair*.

Remark In all cases of interest in this paper, the spaces S , S' , K , K' , \mathcal{R} and \mathcal{O} are smooth manifolds, and the projections are locally trivial fiber bundles.

It is easy to construct examples of fibered relations. For instance, $\mathcal{R} = S \times S'$ is a fibered relation. The graph $\{(s, \pi(s)) \mid s \in S\} \subset S \times S'$ of any fibration $\pi: S \rightarrow S'$ is

also a fibered relation. Finally, if $\tilde{\mathcal{R}} \subset \tilde{S} \times \tilde{S}'$ is a fibered relation and $\alpha: \tilde{S} \rightarrow S$ and $\beta: \tilde{S}' \rightarrow S'$ are coverings, then $\mathcal{R} = (\alpha \times \beta)(\tilde{\mathcal{R}}) \subset S \times S'$ is a fibered relation.

We will work simultaneously with various fibered relations. Suppose that we are given the following data:

- (*) Let $\mathcal{P} = (\mathcal{P}, \geq)$ be a partially ordered set, $\mathcal{S} = \{S_\Delta \mid \Delta \in \mathcal{P}\}$ and $\mathcal{K} = \{K_\Delta \mid \Delta \in \mathcal{P}\}$ set of CW-complexes indexed by \mathcal{P} with $K_\Delta \subset S_\Delta$ for all Δ . For all $\Delta, \Delta' \in \mathcal{P}$ with $\Delta \geq \Delta'$ let

$$\mathcal{R}_{\Delta, \Delta'} \subset S_\Delta \times S_{\Delta'} \quad \text{and} \quad \mathcal{O}_{\Delta, \Delta'} = \mathcal{R}_{\Delta, \Delta'} \cap (K_\Delta \times K_{\Delta'}) \subset K_\Delta \times K_{\Delta'}$$

such that $(\mathcal{R}_{\Delta, \Delta'}, \mathcal{O}_{\Delta, \Delta'})$ is fibered relation pair. Set also

$$\mathcal{R} = \{\mathcal{R}_{\Delta, \Delta'} \mid \Delta \geq \Delta', \Delta, \Delta' \in \mathcal{P}\},$$

$$\mathcal{O} = \{\mathcal{O}_{\Delta, \Delta'} \mid \Delta \geq \Delta', \Delta, \Delta' \in \mathcal{P}\}.$$

Assuming (*), let X be a simplicial complex and recall that the set of all closed cells is denoted by $C(X)$. A \mathcal{P} -labeling of a simplicial complex X is a map

$$\delta: C(X) \rightarrow (\mathcal{P}, \geq)$$

satisfying $\delta(c) \geq \delta(c')$ for all faces $c, c' \in C(X)$ with $c \supset c'$. A \mathcal{P} -labeled simplicial complex is a pair (X, δ) where δ is a \mathcal{P} -labeling of X .

Continuing with notation as in (*), suppose that (X, δ) is a \mathcal{P} -labeled simplicial complex. To every regular cover $\mathcal{U} = \{U_c\}_{c \in C(X)}$ of X we associate the bundle

$$\mathcal{S} \times_\delta \mathcal{U} := \bigsqcup_{c \in C(X)} (S_{\delta(c)} \times U_c) \rightarrow \bigsqcup_{c \in C(X)} U_c,$$

where \bigsqcup stands for disjoint union. As a bundle $\mathcal{S} \times_\delta \mathcal{U}$ has no interest: it is just the disjoint union of trivial bundles. However, a rather non-trivial structure is added if we take into account the intersections of the components of \mathcal{U} and that we have the set \mathcal{R} of fibered relations.

Definition A section of $(\mathcal{S} \times_\delta \mathcal{U}, \mathcal{R})$ is a collection of continuous maps

$$\{\sigma_c\}_{c \in C(X)} = \{\sigma_c: U_c \rightarrow S_{\delta(c)} \mid c \in C(X)\}$$

with $(\sigma_c(x), \sigma_{c'}(x)) \in \mathcal{R}_{\delta(c), \delta(c')}$ for all $c, c' \in C(X)$, $c \supset c'$, and all $x \in U_c \cap U_{c'}$.

Two sections $\{\sigma_c\}_{c \in C(X)}$ and $\{\sigma'_c\}_{c \in C(X)}$ of $(\mathcal{S} \times_\delta \mathcal{U}, \mathcal{R})$ are homotopic if for every $c \in C(X)$ there is a continuous map

$$[0, 1] \times U_c \rightarrow S_{\delta(c)}, \quad (t, x) \mapsto \sigma_c^t(x)$$

with $\sigma_c^0 = \sigma_c$ and $\sigma_c^1 = \sigma'_c$, such that $\{\sigma_c^t\}_{c \in C(X)}$ is a section for all t . If $W \subset X$ is such that $\sigma_c^t(x) = \sigma_c(x)$ for all $c \in C(X)$, all $x \in W \cap U_c$, and all t , then we say that the homotopy $\{\sigma_c^t\}_{c \in C(X)}$ is *constant on W* .

Notation In order to lighten the burden of notation we will from now on write

$$S \times_\delta U = (S \times_\delta U, \mathcal{R})$$

dropping any reference to the set \mathcal{R} of fibered relations.

5.3 Fibered relations associated to a group G

Let G be the group of either complex or real points of a reductive algebraic group (defined over \mathbb{R} in the latter case) and $K \subset G$ a maximal compact subgroup. As in Section 3.4 let \mathcal{G} be the manifold of all well-centered subgroups of G and consider $\pi_0(\mathcal{G})$ with the partial order introduced in Section 3.5. Recall that every component \mathcal{G}_i contains a unique component \mathcal{K}_i of the manifold \mathcal{K} of well-centered groups in K . The following is a direct consequence of Lemmas 3.13 and 3.14:

Lemma 5.2 *Let $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{K}_1, \mathcal{K}_2$ be connected components of the manifolds \mathcal{G} and \mathcal{K} of well-centered groups in G and K , respectively; assume that $\mathcal{G}_1 \geq \mathcal{G}_2$ and that $\mathcal{K}_i \subset \mathcal{G}_i$ for $i = 1, 2$. The pair*

$$(\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}, \mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2}) \subset (\mathcal{G}_1 \times \mathcal{G}_2, \mathcal{K}_1 \times \mathcal{K}_2),$$

where $\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ and $\mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2}$ are as in (3.4) and (3.5), is a fibered relation pair. □

To conclude the analogy with (*), set

$$(5.1) \quad \mathcal{R}_G = \{\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2} \mid \mathcal{G}_1 \geq \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2 \in \pi_0(\mathcal{G})\},$$

$$(5.2) \quad \mathcal{O}_G = \{\mathcal{O}_{\mathcal{G}_1, \mathcal{G}_2} \mid \mathcal{G}_1 \geq \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2 \in \pi_0(\mathcal{G})\}.$$

Given now a $\pi_0(\mathcal{G})$ -labeled simplicial complex (X, δ) and a regular cover \mathcal{U} of X we consider the bundle $\mathcal{G} \times_\delta \mathcal{U}$ and sections therein as defined above.

Remark To clarify the nature of the different objects, when working with $\mathcal{G} \times_\delta \mathcal{U}$ we will denote by $\mathcal{G}_{\delta(c)}$ the connected component $\delta(c) \in \pi_0(\mathcal{G})$ for $c \in C(X)$.

The following result asserts that, up to slightly reducing the domain, every section of $\mathcal{G} \times_\delta \mathcal{U}$ can be homotoped to a section of $\mathcal{K} \times_\delta \mathcal{U}$.

Proposition 5.3 *Let G be the group of either complex or real points of a reductive algebraic group, $K \subset G$ a maximal compact subgroup, \mathcal{G} and \mathcal{K} the manifolds of well-centered groups in G and K , and consider the sets of fibered relations \mathcal{R}_G and \mathcal{O}_G defined in (5.1) and (5.2).*

Let (X, δ) be a $\pi_0(\mathcal{G})$ -labeled complex, and $\mathcal{U} = \{U_c\}_{c \in C(X)}$ and $\mathcal{V} = \{V_c\}_{c \in C(X)}$ regular covers of X with V_c relatively compact in U_c for all $c \in C(X)$. Finally, let $\{\sigma_c\}_{c \in C(X)}$ be a section of $\mathcal{G} \times_\delta \mathcal{U}$, and $W \subset X$ with $\sigma_c(w) \in \mathcal{K}_{\delta(c)}$ for all $c \in C(X)$ and all $w \in U_c \cap W$.

The restriction of $\{\sigma_c\}_{c \in C(X)}$ to $\mathcal{G} \times_\delta \mathcal{V}$ is homotopic to a section $\{\sigma'_c\}_{c \in C(X)}$ with $\sigma'_c(V_c) \subset \mathcal{K}_{\delta(c)}$ for all $c \in C(X)$; moreover the homotopy is constant on W .

Recall that by Proposition 3.11 the manifold \mathcal{G} retracts to the submanifold \mathcal{K} . In light of this fact, the reader may be willing to believe that the proof of Proposition 5.3 is a very tedious and painful exercise in topology; we thereby defer it to the Appendix. Essentially the idea is to observe that the set of fibered relation pairs $(\mathcal{R}_G, \mathcal{O}_G)$ satisfies some conditions (transitivity, reflexivity, simultaneous lifting and relative retraction properties) that allow us to work with sets of fibered relations very much as one would with single fibrations.

6 Proof of Theorem 1.1

In this section we prove Theorem 1.1 assuming the following result:

Proposition 6.1 *Let G be the group of complex points of a connected reductive algebraic group, K a maximal compact subgroup of G , and recall that G acts on $\text{Hom}(\mathbb{Z}^k, G)$ by conjugation. There is a continuous G -equivariant map*

$$\Phi: [0, 1] \times (\text{Hom}(\mathbb{Z}^k, G), \text{Hom}(\mathbb{Z}^k, K)) \rightarrow (\text{Hom}(\mathbb{Z}^k, G), \text{Hom}(\mathbb{Z}^k, K))$$

with the following properties:

- $\Phi(0, \rho) = \rho$ and $\Phi(1, \rho) \in \text{Hom}(\mathbb{Z}^k, K^G)$ for all $\rho \in \text{Hom}(\mathbb{Z}^k, G)$.
- There is an open covering \mathcal{W} of $\text{Hom}(\mathbb{Z}^k, K^G)$ such that for every $W \in \mathcal{W}$ one of the following non-exclusive alternatives holds:
 - (1) All the elements in $\Phi(\{1\} \times W)$ have the same orbit type.
 - (2) All the elements in W have the same orbit type.

If moreover G is defined over \mathbb{R} , then Φ is equivariant under complex conjugation.

Recall that by Lemma 3.2, $\text{Hom}(\mathbb{Z}^k, K^G)$ is the set of representations whose image is contained in a conjugate of K . In particular, $\text{Hom}(\mathbb{Z}^k, K^G)$ is contained in the set $\text{Hom}(\mathbb{Z}^k, G^s)$ of semi-simple representations and contains $\text{Hom}(\mathbb{Z}^k, K)$. Recall also that two elements ρ, ρ' in $\text{Hom}(\mathbb{Z}^k, G)$ have *the same orbit type* if their stabilizers $\text{Stab}_G(\rho)$ and $\text{Stab}_G(\rho')$ are conjugate in G .

We will prove Proposition 6.1 in Section 8.

Theorem 1.1 *Let G be the group of complex or real points of a reductive algebraic group, defined over \mathbb{R} in the latter case, and $K \subset G$ a maximal compact subgroup. For all k there is a weak retraction of $\text{Hom}(\mathbb{Z}^k, G)$ to $\text{Hom}(\mathbb{Z}^k, K)$.*

For the sake of concreteness, suppose that G is the group of complex points. With the same notation as in Proposition 6.1, let

$$(6.1) \quad \phi: (\text{Hom}(\mathbb{Z}^k, K^G), \text{Hom}(\mathbb{Z}^k, K)) \rightarrow (\text{Hom}(\mathbb{Z}^k, K^G), \text{Hom}(\mathbb{Z}^k, K))$$

be the restriction of $\Phi(1, \cdot)$ to $\text{Hom}(\mathbb{Z}^k, K^G)$ and notice that G -equivariance implies that

$$\text{Stab}_G(\phi(\rho)) \supset \text{Stab}_G(\rho)$$

for all $\rho \in \text{Hom}(\mathbb{Z}^k, G)$.

Theorem 1.1 will follow once we show that ϕ is homotopic, as a map of pairs, to a map whose image is contained in $\text{Hom}(\mathbb{Z}^k, K)$. To see why this is the case suppose that

$$[0, 1] \times \text{Hom}(\mathbb{Z}^k, K^G) \rightarrow \text{Hom}(\mathbb{Z}^k, K^G), \quad (t, \rho) \mapsto \phi^t(\rho)$$

is a homotopy with $\phi^0 = \phi$, with $\phi^1(\text{Hom}(\mathbb{Z}^k, K^G)) \subset \text{Hom}(\mathbb{Z}^k, K)$, and with $\phi^t(\text{Hom}(\mathbb{Z}^k, K)) \subset \text{Hom}(\mathbb{Z}^k, K)$ for all t . Then the map

$$f: [0, 1] \times \text{Hom}(\mathbb{Z}^k, G) \rightarrow \text{Hom}(\mathbb{Z}^k, G)$$

given by

$$\begin{aligned} f(t, \rho) &= \Phi(3t, \rho) && \text{for } t \in [0, \frac{1}{3}], \\ f(t, \rho) &= \Phi(3t - 1, \phi(\rho)) && \text{for } t \in [\frac{1}{3}, \frac{2}{3}], \\ f(t, \rho) &= \phi^{3t-2}(\phi(\rho)) && \text{for } t \in [\frac{2}{3}, 1], \end{aligned}$$

is a continuous homotopy between $f(0, \rho) = \rho$ and $f(1, \rho) = \phi^1(\phi(\rho))$. Clearly, the image of $f(1, \cdot)$ is contained in $\text{Hom}(\mathbb{Z}^k, K)$ because the image of $\phi^1(\cdot)$ is. Also, since $\Phi(t, \cdot)$ and $\phi^t(\cdot)$ preserve $\text{Hom}(\mathbb{Z}^k, K)$ for all t it follows that the same is true for $f(t, \cdot)$. In other words, $f(\cdot, \cdot)$ is a weak retraction of $\text{Hom}(\mathbb{Z}^k, G)$ to $\text{Hom}(\mathbb{Z}^k, K)$.

It remains to prove that the homotopy ϕ^t exists. We sketch briefly how we will prove its existence, but before doing so refer to Section 3.4, Section 4.3 and Section 4.5 for the definitions of \mathcal{G} , \mathcal{K} , $\pi: \mathcal{D}(k) \rightarrow \mathcal{G}$ and $\pi: \bar{\mathcal{D}}(k) \rightarrow \mathcal{G}$.

The strategy Considering $X = \text{Hom}(\mathbb{Z}^k, K^G)$ as a simplicial complex with set of cells $C(X)$, we will associate to the map ϕ

- a regular covering $\mathcal{U} = \{U_c\}_{c \in C(X)}$ of X , and
- a labeling $\delta: C(X) \rightarrow \pi_0(\mathcal{G})$.

In order to remind the reader of the nature of the different objects we will denote by $\mathcal{G}_{\delta(c)}$ the connected component $\delta(c)$ of \mathcal{G} ; accordingly, we also set $\bar{\mathcal{D}}_{\delta(c)}(k) = \pi^{-1}(\mathcal{G}_{\delta(c)})$.

Besides the covering \mathcal{U} and the labeling δ , we also associate to the map ϕ

- a section $\bigsqcup_{c \in C(X)} \sigma_c: \bigsqcup_{c \in C(X)} U_c \rightarrow \bigsqcup_{c \in C(X)} \mathcal{G}_{\delta(c)}$ of $\mathcal{G} \times_{\delta} \mathcal{U}$, and
- a map $\bigsqcup_{c \in C(X)} \tilde{\sigma}_c: \bigsqcup_{c \in C(X)} U_c \rightarrow \bigsqcup_{c \in C(X)} \bar{\mathcal{D}}_{\delta(c)}(k)$

such that the following diagram commutes:

$$(6.2) \quad \begin{array}{ccccc} & & \bigsqcup_{c \in C(X)} \bar{\mathcal{D}}_{\delta(c)}(k) & & \\ & \nearrow \bigsqcup \tilde{\sigma}_c & \downarrow \pi & \searrow \Psi & \\ & \bigsqcup_{c \in C(X)} U_c & \xrightarrow{\bigsqcup \sigma_c} & \bigsqcup_{c \in C(X)} \mathcal{G}_{\delta(c)} & \\ \downarrow & & & & \searrow \phi \\ X & & & & X \end{array}$$

Here $\Psi: \bar{\mathcal{D}}(k) \rightarrow \text{Hom}(\mathbb{Z}^k, G)$ is the evaluation map defined in Section 4.3 and the unlabeled vertical arrow is the obvious map $\bigsqcup U_c \rightarrow \bigcup U_c = X$.

These objects in place, we apply Proposition 5.3 to obtain a homotopy of the section $\bigsqcup \sigma_c$ to a section which takes values in $\mathcal{K} \times_{\delta} \mathcal{U}$. The flatness of the bundle $\pi: \bar{\mathcal{D}} \rightarrow \mathcal{G}$ allows us to lift this homotopy to a homotopy of the map $\bigsqcup \tilde{\sigma}_c$. Composing the elements of this last homotopy with the evaluation map Ψ , we obtain a one parameter family of maps $\bigsqcup U_c \rightarrow X$ with the property that the image of the final map is contained in $\text{Hom}(\mathbb{Z}^k, K)$. The compatibility of horizontal lifts, Lemma 4.5, implies then that this one parameter family of maps descends to a one parameter family of maps $X \rightarrow X$

starting with ϕ and ending with a map whose image is $\text{Hom}(\mathbb{Z}^k, K)$; this is the desired homotopy.

Still assuming that G is the group of complex points, we now put the proof of Theorem 1.1 into full swing:

The simplicial complex Consider the triangulation \mathcal{T}_0 of $\text{Hom}(\mathbb{Z}^k, K^G)$ provided by Proposition 2.8 and notice that by Proposition 6.1 there is a refinement \mathcal{T} of \mathcal{T}_0 with the property that every closed cell $c \in \mathcal{T}$ has an open neighborhood W_c such that:

- (1) either any two $\rho, \rho' \in \phi(W_c)$ have the same orbit type, or
- (2) any two $\rho, \rho' \in W_c$ have the same orbit type.

Consider from now on

$$X = (\text{Hom}(\mathbb{Z}^k, K^G), \mathcal{T})$$

as a simplicial complex and notice that by Proposition 2.8, $\text{Hom}(\mathbb{Z}^k, K) \subset X$ is a subcomplex.

Remark As a triangulation, \mathcal{T} is locally finite. In particular, we can assume without loss of generality that $W_{c'} \subset W_c$ for any two closed cells c, c' of X with $c' \subset c$.

The labeling With the same notation as in Section 3, let \mathcal{G} and \mathcal{K} be the manifolds of all well-centered groups in G and K and recall that by Proposition 3.11 the map

$$\iota: \mathcal{K} \rightarrow \mathcal{G}, \quad H \mapsto C(\mathcal{Z}_G(H))$$

is an injective homotopy equivalence. Recall also that the inclusion relation between well-centered groups yields an order on the sets $\pi_0(\mathcal{G})$ and $\pi_0(\mathcal{K})$ of connected components of \mathcal{G} and \mathcal{K} and that the map ι induces an order-preserving bijection $\pi_0(\mathcal{G}) \rightarrow \pi_0(\mathcal{K})$ by Lemma 3.12.

Our next goal is to define a $\pi_0(\mathcal{G})$ -labeling on the complex X , but first we remind the reader of a few facts:

- $\text{Stab}_G(\rho) = \mathcal{Z}_G(\rho(\mathbb{Z}^k))$ for all $\rho \in \text{Hom}(\mathbb{Z}^k, G)$.
- By Lemma 3.6, the group $C(\mathcal{Z}_G(\rho(\mathbb{Z}^k))) = C(\text{Stab}_G(\rho))$ is well-centered, ie, an element of \mathcal{G} , for any $\rho \in \text{Hom}(\mathbb{Z}^k, K^G)$.
- By Lemma 3.10, $\rho, \rho' \in \text{Hom}(\mathbb{Z}^k, K^G)$ have the same G_0 -orbit type if and only if $C(\text{Stab}_G(\rho))$ and $C(\text{Stab}_G(\rho'))$ belong to the same connected component of \mathcal{G} .

Notice that it follows from these observations, together with (1) and (2) above, that for every cell $c \in C(X)$ one of the following non-mutually exclusive alternatives holds:

(1') Either there is a component \mathcal{G}_0 of \mathcal{G} with

$$C(\text{Stab}_G(\phi(\rho))), C(\text{Stab}_G(\phi(\rho'))) \in \mathcal{G}_0$$

for all $\rho, \rho' \in W_c$, or

(2') there is a component \mathcal{G}_1 of \mathcal{G} with

$$C(\text{Stab}_G(\rho)), C(\text{Stab}_G(\rho')) \in \mathcal{G}_1$$

for all $\rho, \rho' \in W_c$.

Keeping the notation above, we define a function $\delta: C(X) \rightarrow \pi_0(\mathcal{G})$ by

$$\delta(c) = \mathcal{G}_0 \quad \text{if (1') is satisfied, and}$$

$$\delta(c) = \mathcal{G}_1 \quad \text{otherwise.}$$

We claim that (X, δ) is a labeled complex.

Lemma 6.2 *The function $\delta: C(X) \rightarrow \pi_0(\mathcal{G})$ is a labeling, ie, $\delta(c) \geq \delta(c')$ for all $c, c' \in C(X)$ with $c \supset c'$.*

Proof Suppose first that (1') holds for c . The assumption that $W_{c'} \subset W_c$ for all $c' \subset c$ implies that (1') also holds for c' . This proves that $\delta(c') = \delta(c)$.

Assume now that (1') is not satisfied for c . In particular

$$C(\text{Stab}_G(\rho)), C(\text{Stab}_G(\rho')) \in \delta(c)$$

for all $\rho, \rho' \in W_c$. Notice that if (1') is also not satisfied for c' , then we again have $\delta(c') = \delta(c)$. So, suppose that (1') is satisfied for c' . Recall that for $\rho \in W_{c'} \subset W_c$ we have

$$\text{Stab}_G(\rho) \subset \text{Stab}_G(\phi(\rho))$$

by Proposition 6.1. It follows hence from Lemma 3.6 that

$$C(\text{Stab}_G(\rho)) \supset C(\text{Stab}_G(\phi(\rho))).$$

Since $C(\text{Stab}_G(\rho)) \in \delta(c)$ and $C(\text{Stab}_G(\phi(\rho))) \in \delta(c')$ this means that $\delta(c) \geq \delta(c')$, as we needed to prove. □

From now on we consider $X = (X, \delta)$ as a $\pi_0(\mathcal{G})$ -labeled complex. We will denote the connected component $\delta(c)$ of \mathcal{G} by $\mathcal{G}_{\delta(c)}$ for $c \in C(X)$.

The cover and the section Fix once and for all a regular cover $\mathcal{U} = \{U_c\}_{c \in C(X)}$ of X with $U_c \subset W_c$ for all $c \in C(X)$. Our next goal is to associate to the map ϕ a section $\{\sigma_c\}_{c \in C(X)}$ of

$$\mathcal{G} \times_{\delta} \mathcal{U} = \bigsqcup_{c \in C(X)} \mathcal{G}_{\delta(c)} \times U_c \rightarrow \bigsqcup_{c \in C(X)} U_c.$$

For $\rho \in U_c$ we set, with the same notation as before,

$$\begin{aligned} \sigma_c(\rho) &= C(\text{Stab}_G(\phi(\rho))) \quad \text{if (1') is satisfied, and} \\ \sigma_c(c) &= C(\text{Stab}_G(\rho)) \quad \text{otherwise,} \end{aligned}$$

and notice that by definition $\sigma_c(\rho) \in \mathcal{G}_{\delta(c)}$. We have hence for all $c \in C(X)$ a map

$$\sigma_c: U_c \rightarrow \mathcal{G}_{\delta(c)}.$$

We summarize in the following lemma a few properties of the maps σ_c :

Lemma 6.3 *Each one of the maps $\sigma_c: U_c \rightarrow \mathcal{G}_{\delta(c)}$ is continuous. Moreover, the collection of maps $\{\sigma_c\}_{c \in C(X)}$ is a section of $\mathcal{G} \times_{\delta} \mathcal{U}$.*

Proof Suppose that we have a sequence (ρ_n) in U_c converging to $\rho \in U_c$ and suppose for the sake of concreteness that (1') holds for c . Observe that the continuity of ϕ implies that $\phi(\rho_n) \rightarrow \phi(\rho)$. Noticing that

$$\phi(\rho) \in \text{Hom}(\mathbb{Z}^k, K^G) \subset \text{Hom}(\mathbb{Z}^k, G^s)$$

is a semisimple representation, it follows from Theorem 2.7 that the orbit of $\phi(\rho)$ under G is closed in the affine variety $\text{Hom}(\mathbb{Z}^k, G)$. Hence, Theorem 2.4 and its following remark imply there is a sequence (g_n) in g converging to Id such that

$$g_n \text{Stab}_G(\phi(\rho_n))g_n^{-1} \subset \text{Stab}_G(\phi(\rho)).$$

On the other hand, since we are assuming that (1') is satisfied, we know $\text{Stab}_G(\phi(\rho))$ and $\text{Stab}(\phi(\rho_n))$ are conjugate for all n , hence

$$g_n \text{Stab}_G(\phi(\rho_n))g_n^{-1} = \text{Stab}_G(\phi(\rho)).$$

It follows now directly from $g_n \rightarrow \text{Id}$ and the definition of σ_c that

$$\sigma_c(\rho_n) \rightarrow \sigma_c(\rho)$$

as we needed to prove. This proves that the map σ_c is continuous if (1') is satisfied for c . If (1') is not satisfied then the argument is word-for-word the same, once we remove the appearance of ϕ everywhere.

It remains to prove that the collection of maps $\{\sigma_c\}_{c \in C(X)}$ is a section of $\mathcal{G} \times_\delta \mathcal{U}$. The argument is the same as that we used in Lemma 6.2 to prove that δ is a labeling, and we leave it to the reader to make the necessary modifications. \square

Before moving on, notice that it follows directly from the definitions that

$$(6.3) \quad \sigma_c(U_c \cap \text{Hom}(\mathbb{Z}^k, K)) \subset \mathcal{K}_{\delta(c)}$$

for all $c \in C(X)$.

The lift Consider as in Section 4 the k^{th} power

$$\pi: \mathcal{D}(k) \rightarrow \mathcal{G}$$

of the canonical bundle and set $\mathcal{D}_{\delta(c)}(k) = \pi^{-1}(\mathcal{G}_{\delta(c)})$ for $c \in C(X)$. In more concrete terms:

$$\mathcal{D}_{\delta(c)}(k) = \{(A, a_1, \dots, a_k) \in \mathcal{G}_{\delta(c)} \times G \times \dots \times G \mid a_i \in A \forall i\}$$

Denote also by $\bar{\mathcal{D}}_{\delta(c)}(k)$ the intersection of $\mathcal{D}_{\delta(c)}(k)$ with the total space of the flat subbundle $\pi: \bar{\mathcal{D}}(k) \rightarrow \mathcal{G}$ introduced in Section 4.5. Explicitly, $\bar{\mathcal{D}}_{\delta(c)}$ is the set of those $(A, a_1, \dots, a_k) \in \mathcal{D}_{\delta(c)}$ such that the elements a_1, \dots, a_k belong to the unique maximal compact subgroup of A .

Notice now that for $c \in C(X)$ and $\rho \in U_c$ we have that

$$(\phi(\rho))(\mathbb{Z}^n) \subset C(\text{Stab}_G(\phi(\rho))) \subset C(\text{Stab}_G(\rho))$$

and hence that $(\phi(\rho))(\mathbb{Z}^n) \subset \sigma_c(\rho)$. Also, as $\phi(\rho) \in \text{Hom}(\mathbb{Z}^k, K^G)$ it follows that $(\phi(\rho))(e_i)$ is contained in the unique maximal compact subgroup of $\sigma_c(\rho)$, where e_1, \dots, e_k is the standard basis of \mathbb{Z}^k . It follows from these two considerations that for all $c \in C(X)$ the map

$$\tilde{\sigma}_c: U_c \rightarrow \bar{\mathcal{D}}_{\delta(c)}(k), \quad \tilde{\sigma}_c(\rho) = (\sigma_c(\rho), (\phi(\rho))(e_1), \dots, (\phi(\rho))(e_k))$$

is well-defined.

It follows directly from the construction and the definitions that

$$(\pi \circ \tilde{\sigma}_c)(\rho) = \sigma_c(\rho) \quad \text{and} \quad \phi(\rho) = (\Psi \circ \tilde{\sigma}_c)(\rho)$$

for all $\rho \in U_c$. Here, $\Psi: \mathcal{D}(k) \rightarrow \text{Hom}(\mathbb{Z}^k, G)$ is the evaluation map defined in Section 4.3; recall that $\Psi(\bar{\mathcal{D}}(k)) \subset \text{Hom}(\mathbb{Z}^k, K^G)$ by Lemma 4.7.

At this point we have constructed all the objects in the commuting diagram (6.2) above. It remains to construct the homotopy of ϕ .

The homotopy Proposition 5.3 asserts that, up to reducing each one of the open sets U_c slightly, there is a continuous map

$$\bigsqcup_{c \in C(X)} ([0, 1] \times U_c) \rightarrow \bigsqcup_{c \in C(X)} \mathcal{G}_{\delta(c)}$$

sending $(t, \rho) \in [0, 1] \times U_c$ to $\sigma_c^t(\rho)$ such that the following properties are satisfied:

- (i) For all t the collection of maps $\{\sigma_c^t\}_{c \in C(X)}$ is a section of $\mathcal{G} \times_{\delta} \mathcal{U}$.
- (ii) $\sigma_c^0 = \sigma_c$ for all $c \in C(X)$.
- (iii) $\sigma_c^1(U_c) \subset \mathcal{K}_{\delta(c)}$ for all $c \in C(X)$.
- (iv) $\sigma_c^t(\rho) = \sigma_c(\rho)$ for all $c \in C(X)$ and all $\rho \in U_c$ with $\sigma_c(\rho) \in \mathcal{K}_{\delta(c)}$.

At this point, recall that for all $c \in C(X)$, the bundle $\pi: \bar{\mathcal{D}}_{\delta(c)} \rightarrow \mathcal{G}_{\delta(c)}$ is flat, and that we have the map $\tilde{\sigma}_c: U_c \rightarrow \bar{\mathcal{D}}_{\delta(c)}$ with

$$\pi \circ \tilde{\sigma}_c = \sigma_c = \sigma_c^0.$$

Consider for all $c \in C(X)$ the horizontal lift

$$[0, 1] \times U_c \rightarrow \bar{\mathcal{D}}_{\delta(c)}, \quad (t, \rho) \mapsto \tilde{\sigma}_c^t(\rho)$$

of the homotopy $[0, 1] \times U_c \rightarrow \mathcal{G}_{\delta(c)}$, $(t, \rho) \mapsto \sigma_c^t(\rho)$ satisfying $\tilde{\sigma}_c^0(\cdot) = \tilde{\sigma}_c(\cdot)$. As the horizontal lift of a continuous homotopy, the map $(t, \rho) \mapsto \tilde{\sigma}_c^t(\rho)$ is continuous as well.

Lemma 6.4 *The map*

$$\bigsqcup_{c \in C(X)} ([0, 1] \times U_c) \rightarrow X, \quad [0, 1] \times U_c \ni (t, \rho) \mapsto \Psi(\tilde{\sigma}_c^t(\rho))$$

descends to a continuous map

$$[0, 1] \times X \rightarrow X, \quad (t, \rho) \mapsto \phi^t(\rho)$$

with $\phi^0 = \phi$. Moreover, $\phi^t(\rho) = \phi(\rho)$ for all t and all $\rho \in \text{Hom}(\mathbb{Z}^k, K)$.

Proof Suppose that we have two cells c, c' with $U_c \cap U_{c'} \neq \emptyset$, and notice that, by the definition of a regular covering, one of them is contained in the other, say, $c' \subset c$. This implies, hence, that $\delta(c) \geq \delta(c')$. The condition that the maps $\{\sigma_c^t\}_{c \in C(X)}$ form a section of $\mathcal{G} \times_{\delta} \mathcal{U}$ implies that for all $\rho \in U_c \cap U_{c'}$ we have

$$(\sigma_c^t(\rho), \sigma_{c'}^t(\rho)) \in \mathcal{R}_{\delta(c), \delta(c')}$$

for all t . Recall now that we have

$$\Phi(\tilde{\sigma}_c(\rho)) = \phi(\rho) = \Phi(\tilde{\sigma}_{c'}(\rho))$$

and that $\tilde{\sigma}_c^t(\rho)$ and $\tilde{\sigma}_{c'}^t(\rho)$ are both obtained through horizontal lifting. Lemma 4.5 asserts that we have

$$\Phi(\tilde{\sigma}_c^t(\rho)) = \Phi(\tilde{\sigma}_{c'}^t(\rho))$$

for all t . We have proved that the map

$$\bigsqcup_{c \in C(X)} ([0, 1] \times U_c) \rightarrow X,$$

$$[0, 1] \times U_c \ni (t, \rho) \mapsto \tilde{\Phi}(\sigma_c^t(\rho)),$$

descends to a map

$$[0, 1] \times X \rightarrow X, \quad (t, \rho) \mapsto \phi^t(\rho)$$

given by $\phi^t(\rho) = \Phi(\tilde{\sigma}_c^t(\rho))$ for any $c \in C(X)$ with $\rho \in U_c$; notice that by construction $\phi^0 = \phi$. The map $(t, \rho) \mapsto \phi^t(\rho)$ is continuous because its restriction to each one of the sets $[0, 1] \times U_c$ is the continuous map $(t, \rho) \mapsto \Phi(\tilde{\sigma}_c^t(\rho))$.

Finally, suppose that $\rho \in \text{Hom}(\mathbb{Z}^k, K)$ and that $c \in C(X)$ is a cell with $\rho \in U_c$. By (6.3) we have $\sigma_c(\rho) \in \mathcal{K}_{\delta(c)}$ and hence, by (iv) above, we have that $\sigma_c^t(\rho) = \sigma_c(\rho)$ for all t . This in turn implies that $\tilde{\sigma}_c^t(\rho) = \tilde{\sigma}_c(\rho)$ for all t . Hence

$$\phi^t(\rho) = \Phi(\tilde{\sigma}_c^t(\rho)) = \Phi(\tilde{\sigma}_c(\rho)) = \phi(\rho)$$

again for all t . This concludes the proof of Lemma 6.4. □

The end At this point, we are nearly done with the proof of Theorem 1.1. It only remains to prove that the final map ϕ^1 of the homotopy provided by Lemma 6.4 takes values in $\text{Hom}(\mathbb{Z}^k, K)$. In order to see that this is the case, recall that by construction we have $\sigma_c^1(U_c) \subset \mathcal{K}_{\delta(c)}$ for all $c \in C(X)$, hence

$$\tilde{\sigma}_c^1(U_c) \subset \bar{\mathcal{D}}_{\delta(c)} \cap \pi^{-1}(\mathcal{K}_{\delta(c)}).$$

Lemma 4.7 implies now that

$$\phi^1(U_c) = \Phi(\tilde{\sigma}_c^1(U_c)) \subset \text{Hom}(\mathbb{Z}^k, K).$$

The claim follows because the sets U_c cover $\text{Hom}(\mathbb{Z}^k, K^G)$.

We have proved that the map (6.1) is homotopic as a map of pairs to a map that takes values in $\text{Hom}(\mathbb{Z}^k, K)$. This concludes the proof of Theorem 1.1 if G is the group of complex points.

The real case Suppose now that $G = G(\mathbb{R})$ is the group of real points, and let $K(\mathbb{R})$ and K be maximal compact subgroups of $G(\mathbb{R})$ and G with $K(\mathbb{R}) \subset K$. The map

$\Phi(1, \cdot)$ provided by Proposition 6.1 is equivariant under complex conjugation. In particular, $\Phi(1, \cdot)$ maps $\text{Hom}(\mathbb{Z}^k, G(\mathbb{R}))$ to $\text{Hom}(\mathbb{Z}^k, K(\mathbb{R})^{G(\mathbb{R})})$, and $\text{Hom}(\mathbb{Z}^k, K(\mathbb{R}))$ to itself. Consider the map

$$\phi: \text{Hom}(\mathbb{Z}^k, K(\mathbb{R})^{G(\mathbb{R})}) \rightarrow \text{Hom}(\mathbb{Z}^k, K(\mathbb{R})^{G(\mathbb{R})})$$

obtained by restricting $\Phi(1, \cdot)$ to $\text{Hom}(\mathbb{Z}^k, K(\mathbb{R})^{G(\mathbb{R})})$. Now we can repeat word-for-word, referring to the appropriate results in Section 2.7 and Section 3.7, the argument given in the complex case. \square

The next two sections are devoted to the proof of Proposition 6.1.

7 Approximated Jordan decomposition

The main tool needed to prove Proposition 6.1 is what we call the *approximated Jordan decomposition*. This construction is the goal of the present section.

7.1 Jordan decomposition

Recall that $g \in \text{SL}_n \mathbb{C}$ is *unipotent* if one of the following three equivalent conditions are satisfied:

- g has characteristic polynomial $\chi_g(T) = (T - 1)^n$.
- $g - \text{Id}$ is nilpotent, meaning that $(g - \text{Id})^n = 0$.
- $g - \text{Id}$ has spectral radius $\rho(g - \text{Id}) = 0$ where the *spectral radius* of a linear transformation is the maximum of the norms of its eigenvalues.

Given arbitrary $g \in \text{SL}_n \mathbb{C}$, there is $h \in \text{SL}_n \mathbb{C}$ such that hgh^{-1} is in Jordan normal form. Denote by a the diagonal part of hgh^{-1} , and set

$$g_s = h^{-1}ah \quad \text{and} \quad g_u = gg_s^{-1}.$$

By construction, g_s is semisimple and g_u is unipotent. Moreover, it is well-known that g_s and g_u are polynomials in g , meaning that $g_s, g_u \in \mathbb{C}[g] \subset M_n \mathbb{C}$. In particular, if $h \in \text{SL}_n \mathbb{C}$ commutes with g , then it also commutes with g_s and g_u . Suppose now that $G \subset \text{SL}_n \mathbb{C}$ is an algebraic group. A priori, the semisimple part g_s and the unipotent part g_u of $g \in G$ only belong to the ambient group $\text{SL}_n \mathbb{C}$, but in fact $g_s, g_u \in G$ as well [7]. To summarize, we have:

Jordan decomposition *Let G be an algebraic group. For every $g \in G$ there are uniquely determined elements, $g_s \in G$ semisimple and $g_u \in G$ unipotent, with $g = g_s g_u$, and such that if $h \in G$ commutes with g , then it also commutes with each of g_s and g_u .*

It is a rather unlucky fact that the Jordan decomposition $g \mapsto (g_s, g_u)$ is not continuous; the goal of this section is to construct a continuous approximation of this decomposition. Letting G act on itself and on $G \times G$ by conjugation, we prove:

Theorem 1.7 *Let $G \subset \mathrm{SL}_n \mathbb{C}$ be an algebraic group and $H_1, \dots, H_r \subset G$ a finite collection of algebraic subgroups of G . For every $\epsilon > 0$ there is a continuous G -equivariant map*

$$G \rightarrow G \times G, \quad g \mapsto (g_s^\epsilon, g_u^\epsilon)$$

satisfying the following properties for all $g \in G$:

- (1) $g = g_s^\epsilon g_u^\epsilon = g_u^\epsilon g_s^\epsilon$.
- (2) g_s^ϵ and g_u^ϵ are polynomials in g , ie, they belong to $\mathbb{C}[g] \subset M_n \mathbb{C}$.
- (3) If $h \in G$ commutes with $g \in G$, then h also commutes with g_s^ϵ and g_u^ϵ .
- (4) g_s^ϵ is semisimple and $g_u^\epsilon - \mathrm{Id}$ has spectral radius at most ϵ .
- (5) If $K \subset G$ is a maximal compact group then $g_s^\epsilon, g_u^\epsilon \in K$ for all $g \in K$.
- (6) $g_s^\epsilon, g_u^\epsilon \in H_i$ for all $g \in H_i$.
- (7) If moreover G is defined over \mathbb{R} , then $\overline{g_s^\epsilon} = g_s^\epsilon$ and $\overline{g_u^\epsilon} = g_u^\epsilon$, where $\bar{\cdot}$ is complex conjugation.

We refer to the decomposition

$$g = g_s^\epsilon g_u^\epsilon$$

provided by Theorem 1.7 as the *approximated Jordan decomposition* for the following reason. The elements g_s^ϵ and g_u^ϵ have the same commutativity properties as those of the Jordan decomposition, and while g_u^ϵ is not unipotent, it is almost so: recall that $u \in \mathrm{SL}_n \mathbb{C}$ is unipotent if and only if $\rho(u - \mathrm{Id}) = 0$.

To prove Theorem 1.7 we will work in the ambient group $\mathrm{SL}_n \mathbb{C}$: we construct the desired map there and then restrict it to G . In this setting, it is easy to understand the reason for the lack of continuity of the Jordan decomposition and essentially the idea behind the construction of the proof of Theorem 1.7 is to preempt its discontinuities. The precise statement we will prove is the following:

Proposition 7.1 *For every $\epsilon > 0$ there is a continuous $\mathrm{SL}_n \mathbb{C}$ -equivariant map*

$$(7.1) \quad \mathrm{SL}_n \mathbb{C} \rightarrow \mathrm{SL}_n \mathbb{C} \times \mathrm{SL}_n \mathbb{C}, \quad g \mapsto (g_s^\epsilon, g_u^\epsilon)$$

satisfying the following properties for all $g \in \mathrm{SL}_n \mathbb{C}$:

- (1) $g = g_s^\epsilon g_u^\epsilon = g_u^\epsilon g_s^\epsilon$.

- (2) g_s^ϵ and g_u^ϵ are polynomials in g , ie, they belong to $\mathbb{C}[g] \subset M_n\mathbb{C}$.
- (3) g_s^ϵ is semisimple and $g_u^\epsilon - \text{Id}$ has at most spectral radius ϵ .
- (4) $g_s^\epsilon, g_u^\epsilon \in \text{SU}_n$ for all $g \in \text{SU}_n$.
- (5) $\bar{g}_s^\epsilon = \overline{g_s^\epsilon}$ and $\bar{g}_u^\epsilon = \overline{g_u^\epsilon}$, where $\bar{\cdot}$ is complex conjugation.

Moreover, if $G_1, \dots, G_r \subset \text{SL}_n\mathbb{C}$ is a finite collection of algebraic groups then the map (7.1) can be chosen in such a way that $g_s^\epsilon, g_u^\epsilon \in G_i$ for all $g \in G_i$.

With the same notation as in Theorem 1.7, notice that we may conjugate $G \subset \text{SL}_n\mathbb{C}$ so that the maximal compact subgroup K is contained in SU_n . In particular, Theorem 1.7 follows immediately once we have proved Proposition 7.1. Before proving the latter, we require a few preparatory remarks, but first we introduce some notation used throughout the remainder of this section.

We denote by $\mathbb{S}_n \subset \text{SL}_n\mathbb{C}$ the standard maximal torus of $\text{SL}_n\mathbb{C}$, ie, the group of diagonal matrices with determinant 1. Recall that the Weyl-group

$$W(\mathbb{S}_n, \text{SL}_n\mathbb{C}) = \mathcal{N}_{\text{SL}_n\mathbb{C}}(\mathbb{S}_n) / \mathcal{Z}_{\text{SL}_n\mathbb{C}}(\mathbb{S}_n)$$

is the symmetric group Σ_n on n symbols. Here, $\mathcal{N}_{\text{SL}_n\mathbb{C}}(\mathbb{S}_n)$ and $\mathcal{Z}_{\text{SL}_n\mathbb{C}}(\mathbb{S}_n)$ are, as always, the normalizer and the centralizer of \mathbb{S}_n in $\text{SL}_n\mathbb{C}$.

We identify \mathbb{S}_n with a subset of \mathbb{C}^n :

$$(7.2) \quad \mathbb{S}_n \simeq \{(\lambda_i)_{i=1, \dots, n} \in \mathbb{C}^n \mid \lambda_1 \cdots \lambda_n = 1\}, \quad \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \simeq (\lambda_1, \dots, \lambda_n)$$

Under this identification, the action of the Weyl group Σ_n on \mathbb{S}_n is just the restriction of the standard action on \mathbb{C}^n .

7.2 Diagonal matrices conjugated into an algebraic group

It is well-known that if G is a connected algebraic group then every semisimple element in G can be conjugated into every maximal torus $\mathbb{T} \subset G$. We discuss now an extension of this observation for disconnected groups.

Lemma 7.2 *Let $G \subset \text{SL}_n\mathbb{C}$ be an algebraic group. There is a finite collection \mathcal{T} of algebraic subgroups of the diagonal group $\mathbb{S}_n \subset \text{SL}_n\mathbb{C}$ such that for every $g \in G$ and $h \in \text{SL}_n\mathbb{C}$ with $hgh^{-1} \in \mathbb{S}_n$ there is $T \in \mathcal{T}$ such that $g \in h^{-1}Th \subset G$.*

Proof Let $R_u(G)$ be the unipotent radical of G and observe that by the Levi decomposition theorem there is $G' \subset G$ reductive with $G = R_u(G) \rtimes G'$. Recall also that every semisimple element in G can be conjugated into G' . Since every element that can be conjugated into \mathbb{S}_n is semisimple, it suffices to prove that the statement of the lemma holds for G' . In other words, we may assume without loss of generality that G is reductive to begin with.

By Theorem 2.1, the smooth variety $\mathrm{SL}_n \mathbb{C} / G$ is affine. Consider the left action

$$(7.3) \quad \mathbb{S}_n \times \mathrm{SL}_n \mathbb{C} / G \rightarrow \mathrm{SL}_n \mathbb{C} / G, \quad (t, hG) \mapsto thG$$

and observe that it has a principal orbit by Theorem 2.4. In fact, since every subgroup of \mathbb{S}_n is reductive, the argument in the proof of Proposition 2.6 shows that the action (7.3) has finitely many orbit types. Observe however that since \mathbb{S}_n is abelian, finiteness of orbit types implies in fact that the collection \mathcal{T} of stabilizers of points under the action (7.3) is finite.

Given now $g \in G$ and $h \in \mathrm{SL}_n \mathbb{C}$ suppose that $hgh^{-1} = t \in \mathbb{S}_n$. This means that $h^{-1}th \in G$ and hence that $thG = hG$, so $t \in \mathrm{Stab}_{\mathbb{S}_n \curvearrowright \mathrm{SL}_n \mathbb{C} / G}(hG)$. This proves that there is $T = \mathrm{Stab}_{\mathbb{S}_n \curvearrowright \mathrm{SL}_n \mathbb{C} / G}(hG) \in \mathcal{T}$ with $g \in h^{-1}Th$. Continuing with the same notation, observe that for every $t' \in T$ we have $t'hG = hG$ and hence $h^{-1}t'h \in G$. This proves that $h^{-1}Th \subset G$, as claimed. \square

7.3 Musings on sets of eigenvalues

Suppose that we are given $g \in \mathrm{SL}_n \mathbb{C}$, and let g_s be its semisimple part with respect to the Jordan decomposition. As g_s is diagonalizable, there is $h \in \mathrm{SL}_n \mathbb{C}$ with $hg_s h^{-1} \in \mathbb{S}_n$. Notice that if $h' \in \mathrm{SL}_n \mathbb{C}$ is another element with $h'g_s h'^{-1} \in \mathbb{S}_n$, then $hg_s h^{-1}$ and $h'g_s h'^{-1}$ differ by an element of the Weyl group Σ_n . Hence we can associate to g the uniquely determined element $\mathcal{E}_g \in \mathbb{S}_n / \Sigma_n$.

Identifying \mathbb{S}_n as above with a subset of \mathbb{C}^n , let $[\lambda_1, \dots, \lambda_n] \in \mathbb{C}^n / \Sigma_n$ be an unordered n -tuple of complex numbers representing \mathcal{E}_g . It follows from the definition that $\mathcal{E}_g = [\lambda_1, \dots, \lambda_n]$ is the unordered tuple of the eigenvalues of g , counted with multiplicity.

We can also consider the unordered tuple $\mathcal{E}_g = [\lambda_1, \dots, \lambda_n]$ as an atomic measure on \mathbb{C} with total measure n :

$$\mathcal{E}_g = \delta_{\lambda_1} + \dots + \delta_{\lambda_n},$$

where δ_{λ_i} is the Dirac measure in \mathbb{C} with atom in λ_i .

Below we will pass freely between these three incarnations of the *spectral measure* \mathcal{E}_g : as an element in \mathbb{S}_n / Σ_n , as an unordered tuple of n complex numbers, and as a measure.

7.4 Continuity properties of eigenspaces

Keeping notation as before, consider \mathcal{E}_g as a measure and denote its support by $|\mathcal{E}_g|$. Given $\lambda \in |\mathcal{E}_g|$, ie, an eigenvalue of g , denote by

$$E_\lambda^g = \ker(g - \lambda \text{Id}) \quad \text{and} \quad \bar{E}_\lambda^g = \ker(g - \lambda \text{Id})^n$$

the λ -eigenspace and generalized λ -eigenspace of g . Observe that the \mathcal{E}_g -measure of the singleton $\{\lambda\}$ is given by $\mathcal{E}_g(\lambda) = \dim \bar{E}_\lambda^g$.

As \mathbb{C} is algebraically closed, we have that \mathbb{C}^n is the direct sum of the generalized eigenspaces \bar{E}_λ^g :

$$\mathbb{C}^n = \bigoplus_{\lambda \in |\mathcal{E}_g|} \bar{E}_\lambda^g.$$

We list a few well known linear algebraic facts:

- (1) Suppose that $g, h \in \text{SL}_n \mathbb{C}$ commute, and let $\lambda \in |\mathcal{E}_g|$ be an eigenvalue of g . Then $hE_\lambda^g = E_\lambda^g$ and $h\bar{E}_\lambda^g = \bar{E}_\lambda^g$.
- (2) Suppose that $h \in \text{SL}_n \mathbb{C}$ is such that for every $\lambda \in |\mathcal{E}_g|$ we have
 - (a) $h\bar{E}_\lambda^g = \bar{E}_\lambda^g$, and
 - (b) $h|_{\bar{E}_\lambda^g}$ is a homothety.
 Then every $g' \in \text{SL}_n \mathbb{C}$ that commutes with g also commutes with h , meaning that $h \in C(\mathcal{Z}_{\text{SL}_n \mathbb{C}}(g))$.
- (3) The semisimple part g_s of $g \in \text{SL}_n \mathbb{C}$ is the unique linear transformation with $g_s v = \lambda v$ for all $v \in \bar{E}_\lambda^g$ and $\lambda \in |\mathcal{E}_g|$.

It is easy to find examples showing that the eigenspaces of g , generalized or not, do not depend continuously on g . The following well known lemmas show the kind of continuity we do have:

Lemma 7.3 *The map $g \mapsto \mathcal{E}_g$ is continuous.*

Proof The characteristic polynomial depends continuously on g ; and the set of zeroes, counted with multiplicity, of a complex polynomial depends also continuously on the coefficients. □

Lemma 7.4 *Suppose that the sequence (g_i) in $\text{SL}_n \mathbb{C}$ converges to g . Suppose also that $\delta > 0$ is such that any two distinct points in $|\mathcal{E}_g|$ are at distance at least 2δ . Then for all $\lambda \in |\mathcal{E}_g|$ we have*

$$\lim_{i \rightarrow \infty} \bigoplus_{\lambda' \in B_\delta(\lambda)} \bar{E}_{\lambda'}^{g_i} = \bar{E}_\lambda^g.$$

Here $B_\delta(\lambda)$ is the ball in \mathbb{C} of radius δ centered at λ .

Sketch of proof If $\lambda_i \in |\mathcal{E}_{g_i}|$ is any sequence of eigenvalues of g_i converging to $\lambda \in |\mathcal{E}_g|$, then, up to passing to a subsequence, the generalized eigenspaces $\bar{E}_{\lambda_i}^{g_i}$ converge to a subspace of \bar{E}_λ^g . It follows that

$$\lim_{i \rightarrow \infty} \bigoplus_{\lambda' \in B_\delta(\lambda)} \bar{E}_{\lambda'}^{g_i} \subset \bar{E}_\lambda^g.$$

The equality follows as both spaces have the same dimension by Lemma 7.3. □

7.5 Smashing eigenvalues

In the construction of the approximated Jordan decomposition we will need a technical lemma, proved next. Consider the following function $d: \mathbb{S}_n \times \mathbb{S}_n \rightarrow \mathbb{R}$:

$$d \left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \begin{pmatrix} \lambda'_1 & & \\ & \ddots & \\ & & \lambda'_n \end{pmatrix} \right) = \max_{i=1, \dots, n} \left| \frac{\lambda_i}{\lambda'_i} - 1 \right|$$

The function $d(\cdot, \cdot)$ is not a distance, but we will think of it anyway as such. In this section we prove the following lemma:

Lemma 7.5 *For all $\epsilon > 0$ there is a $\delta \in (0, \epsilon)$ and a Σ_n -equivariant continuous map*

$$\begin{aligned} \Theta: (\mathbb{S}_n, \mathbb{S}_n \cap \text{SU}_n) &\rightarrow (\mathbb{S}_n, \mathbb{S}_n \cap \text{SU}_n), \\ \Theta: \lambda = (\lambda_i) &\mapsto \Theta(\lambda) = (\Theta_i(\lambda)), \end{aligned}$$

with $\overline{\Theta(\cdot)} = \Theta(\overline{\cdot})$ and such that for all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{S}_n$ and all $i, j \in \{1, \dots, n\}$ the following holds:

- (1) $d(\lambda, \Theta(\lambda)) \leq \epsilon$, and
- (2) if $\left| \frac{\lambda_i}{\lambda_j} - 1 \right| \leq \delta$ then $\Theta_i(\lambda) = \Theta_j(\lambda)$.

Moreover, if $T_1, \dots, T_r \subset \mathbb{S}_n$ are Zariski closed subgroups, then Θ can be chosen so that $\Theta(T_i) \subset T_i$ for all i .

Proof Let $\bar{\Sigma}_n$ be the finite group of transformations of \mathbb{S}_n generated by the symmetric group Σ_n and by complex conjugation $(\lambda_i) \mapsto (\bar{\lambda}_i)$. Choose a $\bar{\Sigma}_n$ -invariant triangulation \mathcal{T} of \mathbb{S}_n that contains both $\mathbb{S}_n \cap \text{SU}_n$ and the given subgroups T_1, \dots, T_r as subcomplexes, and consider $X = \mathbb{S}_n / \bar{\Sigma}_n$ with the induced structure of a simplicial

complex. We may assume without loss of generality that for any simplex $c \in \mathcal{T}$ we have

$$(7.4) \quad \max_{\lambda, \lambda' \in c} d(\lambda, \lambda') \leq \epsilon.$$

Let now $\mathcal{U} = \{U_c\}_{c \in C(X)}$ be a regular covering of X and choose $\delta \in (0, \frac{\epsilon}{2})$ small enough so that

$$(7.5) \quad \{x \in X \mid \min_{y \in c} d(x, y) \leq 2\delta\} \subset \bigcup_{c' \subset c} U_{c'}$$

for any cell $c \in C(X)$. For further use we also assume that $\delta < 10^{-10}$.

Let now $\Theta': X \rightarrow X$ be the map provided by Lemma 5.1 and recall that Θ' sends every closed simplex to itself. In particular, Θ' induces uniquely a $\bar{\Sigma}_n$ -equivariant map

$$\Theta: \mathbb{S}_n \rightarrow \mathbb{S}_n$$

which sends every closed simplex of \mathcal{T} to itself. It follows from (7.4) that Θ satisfies (1). Notice also that as Θ maps every simplex to itself, it preserves every subcomplex in \mathbb{S}_n . It follows that $\Theta(\mathbb{S}_n \cap \text{SU}_n) \subset \mathbb{S}_n \cap \text{SU}_n$ and $\Theta(T_i) \subset T_i$ for all i .

It remains to prove that Φ satisfies (2). Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{S}_n$, suppose that for some i and j we have

$$\left| \frac{\lambda_i}{\lambda_j} - 1 \right| \leq \delta < 10^{-10}$$

and let $r \in \mathbb{C}$ be the unique complex number in the ball of radius $\frac{1}{2}$ around 1 with

$$r^2 = \frac{\lambda_i}{\lambda_j}.$$

We consider now the element $\lambda' = (\lambda'_1, \dots, \lambda'_n) \in \mathbb{S}_n$ with

$$\begin{aligned} \lambda'_k &= \lambda_k \quad \text{if } k \neq i, j, \\ \lambda'_i &= \lambda'_j = r\lambda_j, \end{aligned}$$

and notice that

$$d(\lambda, \lambda') = \max\{|r - 1|, |r^{-1} - 1|\} < 2\delta.$$

Denote by c the minimal closed simplex in \mathcal{T} containing λ' and observe that there is a cell $c' \subset c$ with $\lambda \in U_{c'}$ by (7.5). Since the map Θ maps $U_{c'}$ to c' , it follows that $\Theta(\lambda) \in c' \subset c$. Observe at this point that c is pointwise fixed by the stabilizer of λ' inside $\Sigma_n \subset \bar{\Sigma}_n$. In other words, for every $\lambda'' = (\lambda''_1, \dots, \lambda''_n) \in c$, we have

$\lambda''_i = \lambda''_j$. This applies in particular to the point $\Theta(\lambda)$; this proves that Θ satisfies (2) and concludes the proof of Lemma 7.5. \square

7.6 Proof of Proposition 7.1

Suppose that $G_1, \dots, G_r \subset \mathrm{SL}_n \mathbb{C}$ are algebraic groups, let \mathcal{T}_i be the collection of subgroups of \mathbb{S}_n associated by Lemma 7.2 to the subgroup G_i , and set $\mathcal{T} = \bigcup_i \mathcal{T}_i$. Suppose from now on that

$$\Theta: \mathbb{S}_n \rightarrow \mathbb{S}_n$$

is the map provided by Lemma 7.5 satisfying $\Theta(T) \subset T$ for all $T \in \mathcal{T}$.

Given $g \in \mathrm{SL}_n \mathbb{C}$ with semisimple part g_s , let $h \in \mathrm{SL}_n \mathbb{C}$ be such that $hg_s h^{-1} \in \mathbb{S}_n$. We define

$$g_s^\epsilon = h^{-1}(\Theta(hg_s h^{-1}))h.$$

Notice that the equivariance of Θ under the action of the Weyl group of \mathbb{S}^n implies that g_s^ϵ does not depend on the choice of h . In other words, the map

$$\mathrm{SL}_n \mathbb{C} \rightarrow \mathrm{SL}_n \mathbb{C}, \quad g \mapsto g_s^\epsilon$$

is well defined. Once we have defined g_s^ϵ , we have no choice but to define g_u^ϵ as follows:

$$\mathrm{SL}_n \mathbb{C} \rightarrow \mathrm{SL}_n \mathbb{C}, \quad g \mapsto g_u^\epsilon = g(g_s^\epsilon)^{-1}.$$

Notice that by construction, g_s^ϵ and g_u^ϵ behave well with respect to conjugation, meaning that $(hgh^{-1})_s^\epsilon = hg_s^\epsilon h^{-1}$ and $(hgh^{-1})_u^\epsilon = hg_u^\epsilon h^{-1}$ for all $h \in \mathrm{SL}_n \mathbb{C}$. In other words, $g \mapsto (g_s^\epsilon, g_u^\epsilon)$ is $\mathrm{SL}_n \mathbb{C}$ -equivariant.

Suppose now that $g, h \in \mathrm{SL}_n \mathbb{C}$ are such that $hg_s h^{-1} \in \mathbb{S}_n$ and notice that

$$\bar{g}_s = \overline{g_s} \quad \text{and} \quad \bar{h} \bar{g}_s \bar{h}^{-1} \in \mathbb{S}_n.$$

Taking into account that Θ is equivariant under complex conjugation, we have that:

$$\begin{aligned} \bar{g}_s^\epsilon &= \bar{h}^{-1} \Theta(\overline{h g_s h^{-1}}) \bar{h} = \bar{h}^{-1} \Theta(\bar{h} \bar{g}_s \bar{h}^{-1}) \bar{h} \\ &= \bar{h}^{-1} \Theta(h g_s h^{-1}) \bar{h} = \bar{h}^{-1} \Theta(h g h^{-1}) \bar{h} \\ &= \overline{h^{-1} \Theta(h g h^{-1}) h} = \overline{g_s^\epsilon}. \end{aligned}$$

This implies that $\bar{g}_u^\epsilon = \overline{g_u^\epsilon}$ for any $g \in \mathrm{SL}_n \mathbb{C}$.

Suppose now that we are given an element $g \in G_i$. The semisimple part g_s also belongs to G_i . Since g_s is diagonalizable, there is $h \in \mathrm{SL}_n \mathbb{C}$ with $hg_s h^{-1} \in \mathbb{T}$. By

Lemma 7.2 there is $T \in \mathcal{T}_i \subset \mathcal{T}$ such that $hg_s h^{-1} \in T$. Since Θ preserves T we have that $\Theta(hg_s h^{-1})$ also belongs to T . Hence

$$g_s^\epsilon = h^{-1} \Theta(hg_s h^{-1}) h \in h^{-1} T h \subset G_i,$$

where the last inclusion holds by Lemma 7.2. So, $g_u^\epsilon = g(g_s^\epsilon)^{-1}$ belongs to G_i as well. A similar argument shows that $g_s^\epsilon, g_u^\epsilon \in \text{SU}_n$ for $g \in \text{SU}_n$.

It remains to prove that the maps $g \mapsto g_s^\epsilon$ and $g \mapsto g_u^\epsilon$ are continuous and satisfy (1), (2) and (3) in Proposition 7.1. Before doing so, we give a different description of g_s^ϵ . Continuing with the same notation, identify \mathbb{S}_n with a subset of \mathbb{C}^n as in (7.2), let $\mathcal{E}_g = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be the tuple associated to $hg_s h^{-1} \in \mathbb{S}_n$, and let $(\Theta_1(\mathcal{E}_g), \dots, \Theta_n(\mathcal{E}_g)) \in \mathbb{C}^n$ be the tuple associated to $\Theta(hg_s h^{-1})$. Then g_s^ϵ is the unique linear map with the following property

$$(7.6) \quad g_s^\epsilon(v) = \Theta_i(\mathcal{E}_g) \cdot v \quad \text{for all } v \in \bar{E}_{\lambda_i}^g.$$

Notice now that (7.6) implies that g_s^ϵ commutes with every element $h \in \text{SL}_n \mathbb{C}$ that commutes with the semisimple part g_s of g . In other words,

$$g_s^\epsilon \in C(\mathcal{Z}_{\text{SL}_n \mathbb{C}}(g_s)).$$

It follows hence from Lemma 3.8 that g_s^ϵ and its inverse are polynomials in g_s . Since g_s is in turn a polynomial in g we deduce that

$$g_s^\epsilon, g_u^\epsilon = g(g_s^\epsilon)^{-1} \in \mathbb{C}[g].$$

This proves (2), and (1) follows immediately.

Starting with the proof of (3), notice that by construction, g_s^ϵ is conjugate to an element in \mathbb{S}_n and hence is semisimple. To prove that $g_u^\epsilon - \text{Id}$ has very small spectral radius let, with the same notation as above,

$$\mathcal{E}_g = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$$

be the tuple corresponding to $hg_s h^{-1} \in \mathbb{S}_n$. Considering g in Jordan normal form we see that the eigenvalues of g_u^ϵ are given by $\lambda_i / \Theta_i(\mathcal{E}_g)$. It follows from Lemma 7.5 that

$$\rho(g_u^\epsilon - \text{Id}) = \max \left\{ \left| \frac{\lambda_i}{\Theta_i(\mathcal{E}_g)} - 1 \right|, i = 1, \dots, n \right\} = d(\mathcal{E}_g, \Theta(\mathcal{E}_g)) < \epsilon.$$

This proves (3).

At this point we have proven all the claims of Proposition 7.1, except for the continuity of the map $g \mapsto (g_s^\epsilon, g_u^\epsilon) = (g_s^\epsilon, g(g_s^\epsilon)^{-1})$. It obviously suffices to prove that $g \mapsto g_s^\epsilon$ is continuous.

Suppose that (g_i) is a sequence in $SL_n \mathbb{C}$ converging to some g . Let δ be as in Lemma 7.5, and choose $\delta_0 < \delta$ positive, such that any two atoms of the measure \mathcal{E}_g are at distance at least $2\delta_0$. Recall that, associated to g and g_i , we have direct sum decompositions of \mathbb{C}^n into generalized eigenspaces. Given λ in the support of \mathcal{E}_g , we define for all i :

$$V_{\lambda,i} = \bigoplus_{\substack{\lambda' \in B_{\delta_0}(\lambda) \\ \mathcal{E}_{g_i}(\lambda') > 0}} \bar{E}_{\lambda'}^{g_i}$$

It follows from Lemma 7.4 that

$$\bar{E}_{\lambda}^g = \lim_{i \rightarrow \infty} V_{\lambda,i}$$

for each eigenvalue λ of g .

By construction the map $(g_i)_s^\epsilon$ is a homothety on each one of the eigenspaces of g_i . Moreover, as δ_0 is assumed to be smaller than δ , it follows from Lemma 7.5(2) that in fact the factor of the homothety of $(g_i)_s^\epsilon$ is equal for any two generalized eigenspaces of g_i that are contained in the same $V_{\lambda,i}$; in other words, the restriction of $(g_i)_s^\epsilon$ to $V_{\lambda,i}$ is a homothety. Finally, the continuity of Θ ensures that the homothety factors of $(g_i)_s^\epsilon$ on $V_{\lambda,i}$ converge to the homothety factor of g_s^ϵ on E_{λ}^g . It follows that the maps $(g_i)_s^\epsilon$ converge to g_s^ϵ , as claimed.

This concludes the proof of Proposition 7.1. □

Remark Notice that if G is an algebraic group then we have by construction that $g_s^\epsilon = (g_s)_s^\epsilon$ for every $g \in G$ and every ϵ positive.

Before continuing, we wish to make a technical observation that will be important later on. Its proof, which we leave it to the reader, follows directly from the construction of the approximated Jordan decomposition:

Proposition 7.6 *Suppose that $V \subset \mathbb{C}^n$ is a linear subspace, and suppose $g \in SL_n \mathbb{C}$ is such that $gV = V$ and that the restriction $g|_V$ of g to V is a homothety. If (g_j) is a sequence of semisimple elements in $SL_n \mathbb{C}$ converging to g , with*

$$g_j V = V$$

for all j , then there is j_0 such that for all $j \geq j_0$ the following holds:

- $(g_j)_s^\epsilon V = V$, and
- the restriction $(g_j)_s^\epsilon|_V$ of $(g_j)_s^\epsilon$ to V is a homothety. □

8 Retraction into $\text{Hom}(\mathbb{Z}^k, K^G)$

In this section we prove Proposition 6.1. However, before doing so we need to make a few observations on the exponential and the logarithm of a matrix.

8.1 The exponential and the logarithm

Recall the definition of the exponential and the logarithm as formal power series:

$$e^X = \sum_{m=0}^{\infty} \frac{1}{m!} X^m, \quad \log(X) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (X - \text{Id})^m.$$

If we substitute X for an arbitrary square matrix, then the exponential series converges absolutely. The logarithmic series converges absolutely if the matrix $X - \text{Id}$ has spectral radius $\rho(X - \text{Id}) < 1$.

Lemma 8.1 *Given two commuting square matrices $g, h \in M_n(\mathbb{C})$ such that all the involved series converge absolutely, we have:*

- $e^{\log(g)} = g$.
- $e^{g+h} = e^g e^h$.
- $\log(gh) = \log(g) + \log(h)$.
- g commutes with both e^h and $\log(h)$.
- $\overline{e^g} = e^{\bar{g}}$ and $\overline{\log(g)} = \log(\bar{g})$. □

Aside from the simple facts listed in Lemma 8.1, we will also need the following fact, which is surely well-known to experts:

Lemma 8.2 *Let $G \subset \text{SL}_n \mathbb{C}$ be either an algebraic group or a compact group, and denote its Lie algebra by \mathfrak{g} . There is $\epsilon \in (0, 1)$ so that $\log(g) \in \mathfrak{g}$ for all $g \in G$ with $\rho(g - \text{Id}) < \epsilon$.*

It is easy to construct examples showing that the constant ϵ cannot be chosen, even after fixing the ambient group $\text{SL}_n \mathbb{C}$ and the abstract group G , to be independent of the embedding of G in $\text{SL}_n \mathbb{C}$.

Proof It is well-known that whenever $g \in G$ is unipotent, then $\log(g) \in \mathfrak{g}$. We consider now the semisimple case. To begin with, let \mathcal{T} be the finite set of subgroups of the diagonal group $\mathbb{S}_n \subset \text{SL}_n \mathbb{C}$ provided by Lemma 7.2 and for each $T_i \in \mathcal{T}$ let

$\widehat{T}_i \subset G$ be a conjugate of T_i and $\mathfrak{t}_i \subset \mathfrak{g}$ the Lie algebra of \widehat{T}_i . Since the topology of \widehat{T}_i is the same as the induced topology, and since \log is a local inverse of

$$\mathfrak{g} \rightarrow G, \quad X \mapsto e^X$$

near Id , it follows that there is small neighborhood $U \subset G$ of Id with

$$\log(\widehat{T}_i \cap U) \subset \mathfrak{t}_i$$

for all i . On the other hand, \widehat{T}_i consists of semisimple commuting elements. Hence there is a basis v_1, \dots, v_n of \mathbb{C}^n consisting of eigenvectors of $t \in \widehat{T}_i$ for all t . It follows that any element $t \in \widehat{T}_i$ with eigenvalues very close to 1 is very close to the identity. In other words, there is $\epsilon \in (0, 1)$ so that for all i , any $t \in \widehat{T}_i$ with $\rho(t - \text{Id}) < \epsilon$ is in U . We have proved the claim for elements in the finite collection of subgroups $\{\widehat{T}_i\}$. Suppose now that $g \in G$ is an arbitrary semisimple element. By Lemma 7.2 there are i and some $h \in G$ with $hgh^{-1} \in \widehat{T}_i$. Noticing that $\rho(hgh^{-1} - \text{Id}) < \epsilon$ whenever $\rho(g - \text{Id}) < \epsilon$, we deduce that

$$\log(g) = h^{-1} \log(hgh^{-1})h = \text{Ad}_{h^{-1}}(\log(hgh^{-1})) \in \text{Ad}_{h^{-1}} \mathfrak{t}_i \subset \mathfrak{g}$$

for any $g \in G$ semisimple with $\rho(g - \text{Id}) < \epsilon$.

To conclude the proof we consider now the case of a general element $g \in G$ with $\rho(g - \text{Id}) < \epsilon$. Consider the Jordan decomposition $g = g_s g_u$ of g , and notice that $\rho(g_s - \text{Id}) = \rho(g - \text{Id}) < \epsilon < 1$, and that $\rho(g_u - \text{Id}) = 0$. Hence, $\log(g_s)$ and $\log(g_u)$ are also absolutely converging power series. Since g_s and g_u commute, and since $\log(g_u), \log(g_s) \in \mathfrak{g}$, we deduce from Lemma 8.1 that

$$\log(g) = \log(g_s) + \log(g_u) \in \mathfrak{g}.$$

This concludes the proof of Lemma 8.2. □

From Lemma 8.2 we obtain immediately the following corollary:

Corollary 8.3 *Let $G \subset \text{SL}_n \mathbb{C}$ be either an algebraic group or a compact group. There is $\epsilon \in (0, 1)$ with $e^{t \log(g)} \in G$ for all $g \in G$ with $\rho(g - \text{Id}) < \epsilon$ and all t .* □

8.2 Retraction on the level of G

Recall that for n sufficiently large, there is no retraction of $\text{SL}_n \mathbb{C}$ to SU_n preserving commutativity [34]. What we prove now is that if $G \subset \text{SL}_n \mathbb{C}$ is an algebraic group, then there is a weak retraction, preserving commutativity, of G into the set G^s of semisimple elements in G .

Proposition 8.4 *Suppose that $G \subset \text{SL}_n \mathbb{C}$ is an algebraic group, and let $K \subset G$ be a maximal compact subgroup. For all $\epsilon > 0$ small enough, there is a G -equivariant continuous map*

$$F: [0, 1] \times (G, K) \rightarrow (G, K), \quad (t, g) \mapsto F_t(g)$$

with

$$F_0(g) = g \quad \text{and} \quad F_1(g) = g_s^\epsilon$$

for all $g \in G$, where g_s^ϵ is as provided by Theorem 1.7. Moreover, if G is defined over \mathbb{R} , then F is equivariant under complex conjugation.

In particular,

- $F_1(G) \subset G^s$, and
- if $g, h \in G$ commute then so do $F_t(g)$ and $F_t(h)$ for all t .

Here, as always, G acts on itself by conjugation.

Proof Suppose that ϵ is positive and smaller than the constant provided by Corollary 8.3 for the groups G and K . For $g \in G$ consider the approximated Jordan decomposition $g = g_s^\epsilon g_u^\epsilon$ provided by Theorem 1.7 and recall that $\rho(g_u^\epsilon - \text{Id}) < \epsilon$. We define the map F as follows:

$$F: [0, 1] \times G \rightarrow \text{SL}_n \mathbb{C}, \quad F_t(g) = g_s^\epsilon e^{(1-t) \cdot \log g_u^\epsilon}.$$

The G -equivariance and continuity of F follow directly from Theorem 1.7 and the absolute convergence of the involved series. As $g_s^\epsilon, g_u^\epsilon \in G$, we deduce from the choice of ϵ and Corollary 8.3 that $F_t(g)$ belongs to G for all t . The same argument implies that whenever $g \in K$ we have $F_t(g) \in K$ for all t . Also, if G is defined over \mathbb{R} , then equivariance of the approximated Jordan decomposition under complex conjugation implies that F is equivariant as well.

Notice that the G -equivariance of F implies that if $g, h \in G$ commute then so do $F_t(g)$ and h for all t . This implies that $F_t(g)$ and $F_t(h)$ also commute for all t . \square

8.3 Scaling

As above, let S_n be the diagonal group of $\text{SL}_n \mathbb{C}$, identified with a subset of \mathbb{C}^n as in (7.2). Consider the map

$$\bar{\sigma}: [0, 1] \times S_n \rightarrow S_n, \quad (t, (\lambda_i)) \mapsto \bar{\sigma}_t((\lambda_i)) = (e^{-t \log |\lambda_i|} \lambda_i),$$

where $\log(\cdot)$ is the standard real logarithm. Notice that $\bar{\sigma}$ is continuous and equivariant under the action of the Weyl group Σ_n , and that we have

$$\bar{\sigma}_0(g) = g \quad \text{and} \quad \bar{\sigma}_1(g) \in \mathbb{S}_n \cap \text{SU}_n$$

for all $g \in \mathbb{S}_n$.

Given now $g \in \text{SL}_n \mathbb{C}$ semisimple, there is h with $hgh^{-1} \in \mathbb{S}_n$. Invariance of $\bar{\sigma}$ under the Weyl group implies that for all t the element

$$\sigma_t(g) = h^{-1} \bar{\sigma}_t(hgh^{-1})h$$

is independent of the choice of h . In particular, we obtain a well defined map

$$(8.1) \quad \sigma: [0, 1] \times \text{SL}_n^s \mathbb{C} \rightarrow \text{SL}_n^s \mathbb{C}, \quad (t, g) \mapsto \sigma_t(g),$$

where $\text{SL}_n^s \mathbb{C}$ is the set of semisimple elements in $\text{SL}_n \mathbb{C}$. It is easy to see that σ is continuous and that it satisfies the following properties:

- (1) $\sigma_0(g) = g$ and $\sigma_1(g) \in \bigcup_{h \in \text{SL}_n \mathbb{C}} h \text{SU}_n h^{-1}$ for all $g \in \text{SL}_n^s \mathbb{C}$.
- (2) $\sigma_t(g) = g$ for all $g \in \text{SU}_n$.
- (3) σ is $\text{SL}_n \mathbb{C}$ -equivariant; in particular, if g and h commute, then so do $\sigma_t(g)$ and $\sigma_t(h)$ for all t .
- (4) If $g \in \text{SL}_n \mathbb{R}$, then $\sigma_t(g) \in \text{SL}_n \mathbb{R}$ for all t .

Suppose now that $G \subset \text{SL}_n \mathbb{C}$ is an algebraic group, let \mathcal{T} be finite collection of subgroups of the diagonal groups $\mathbb{S}_n \subset \text{SL}_n \mathbb{C}$ provided by Lemma 7.2, and let K be a maximal compact subgroup in G . Up to conjugating in $\text{SL}_n \mathbb{C}$, we can assume that $K \subset \text{SU}_n$. By Lemma 7.2, every semisimple element $g \in G$ there is $h \in \text{SL}_n \mathbb{C}$ with $hgh^{-1} \in T_i$ for some i . Since the map $\bar{\sigma}_t(\cdot)$ preserves T_i for all t , and since $h^{-1}T_i h \subset G$, we deduce from that $\sigma_t(g) \in G$ for all t and all $g \in G$ semisimple. With notation as in (2.1), we have hence proved:

Lemma 8.5 *Suppose that G is a linear algebraic group, and let $K \subset G$ be a maximal compact subgroup. There is a G -equivariant continuous map*

$$\sigma: [0, 1] \times G^s \rightarrow G^s, \quad (t, g) \mapsto \sigma_t(g)$$

with:

- (1) $\sigma_0(g) = g$ and $\sigma_1(g) \in K^G$ for all $g \in G^s$.
- (2) $\sigma_t(g) = g$ for all t and all $g \in K^G$.

In particular, if $g, h \in G$ commute, then so do $\sigma_t(g)$ and $\sigma_t(h)$ for all t .

Moreover, if G is defined over \mathbb{R} , then σ is equivariant under complex conjugation. \square

8.4 Proof of Proposition 6.1

At this point we are finally ready to prove:

Proposition 6.1 *Let G be the group of complex points of a reductive algebraic group, K a maximal compact subgroup of G , and recall that G acts on $\text{Hom}(\mathbb{Z}^k, G)$ by conjugation. There is a continuous G -equivariant map*

$$\Phi: [0, 1] \times (\text{Hom}(\mathbb{Z}^k, G), \text{Hom}(\mathbb{Z}^k, K)) \rightarrow (\text{Hom}(\mathbb{Z}^k, G), \text{Hom}(\mathbb{Z}^k, K))$$

with the following properties:

- $\Phi(0, \rho) = \rho$ and $\Phi(1, \rho) \in \text{Hom}(\mathbb{Z}^k, K^G)$ for all $\rho \in \text{Hom}(\mathbb{Z}^k, G)$.
- There is an open covering \mathcal{W} of $\text{Hom}(\mathbb{Z}^k, K^G)$ such that for every $W \in \mathcal{W}$ one of the following non-exclusive alternatives holds:
 - (1) All the elements in $\Phi(\{1\} \times W)$ have the same orbit type.
 - (2) All the elements in W have the same orbit type.

If moreover G is defined over \mathbb{R} , then Φ is equivariant under complex conjugation.

Recall that two elements ρ, ρ' in $\text{Hom}(\mathbb{Z}^k, G)$ have the same orbit type if their stabilizers $\text{Stab}_G(\rho)$ and $\text{Stab}_G(\rho')$ are conjugate in G .

Proof Choose $\epsilon > 0$ small enough for Proposition 8.4 to apply, and let

$$F: [0, 1] \times G \rightarrow G, \quad (t, g) \mapsto F_t(g)$$

be the map provided by that proposition. Recall that $F_1(g) = g_s^\epsilon$, where $g_s^\epsilon \in G^s$ is the approximated semisimple part of g provided by Theorem 1.7. Let also

$$\sigma: [0, 1] \times G^s \rightarrow G^s, \quad (t, g) \mapsto \sigma_t(g)$$

be the map provided by Lemma 8.5.

Considering elements in $\text{Hom}(\mathbb{Z}^k, G)$ as commuting k -tuples in G , we define Φ as follows:

$$\begin{aligned} \Phi(t, (a_1, \dots, a_k)) &= (F_{2t}(a_1), \dots, F_{2t}(a_k)) && \text{for } t \leq \frac{1}{2}, \\ \Phi(t, (a_1, \dots, a_k)) &= (\sigma_{2t-1}((a_1)_s^\epsilon), \dots, \sigma_{2t-1}((a_k)_s^\epsilon)) && \text{for } t \geq \frac{1}{2}. \end{aligned}$$

By construction, the homotopy Φ is continuous, preserves $\text{Hom}(\mathbb{Z}^k, K)$, and we have $\Phi(0, \rho) = \rho$ and $\Phi(1, \rho) \in \text{Hom}(\mathbb{Z}^k, K^G)$ for all $\rho \in \text{Hom}(\mathbb{Z}^k, G)$. Moreover, it follows from the G -equivariance of F and σ that Φ is G -equivariant as well. Also,

if G is defined over \mathbb{R} , then Φ is equivariant under complex conjugation because F and σ are.

We have proved all claims of Proposition 6.1 but the existence of the covering \mathcal{W} . Suppose now that $H \subset G$ is a subgroup, and consider the set

$$\mathcal{X}_H = \{\rho \in \text{Hom}(\mathbb{Z}^k, K^G) \mid \exists g \in G \text{ with } gHg^{-1} \subset \text{Stab}_G(\rho)\}.$$

It follows from Proposition 2.8 that there is a triangulation of $\text{Hom}(\mathbb{Z}^k, K^G)$ such that \mathcal{X}_H is a (closed) subcomplex for all $H \subset G$.

Noting that G -equivariance implies $\Phi(1, \mathcal{X}_H) \subset \mathcal{X}_H$, observe that the existence of the open cover \mathcal{W} follows once we have established the following claim:

Claim For every $H \subset G$ there is an open neighborhood U of \mathcal{X}_H with $\Phi(1, U) \subset \mathcal{X}_H$.

Proof of the claim Given a sequence (a_1^j, \dots, a_k^j) in $\text{Hom}(\mathbb{Z}^k, K^G)$ converging to $(a_1, \dots, a_k) \in \mathcal{X}_H$, notice that it follows from Theorem 2.7, Theorem 2.4 and the remark after the latter theorem, that there is a sequence (g_j) in G converging to Id_G and with

$$\begin{aligned} \text{Stab}_G(g_j a_1^j g_j^{-1}, \dots, g_j a_k^j g_j^{-1}) &= g_j \text{Stab}_G(a_1^j, \dots, a_k^j) g_j^{-1} \\ &\subset \text{Stab}_G(a_1, \dots, a_k) \end{aligned}$$

for all sufficiently large j . Observe also that by the G -equivariance of Φ and the G -invariance of \mathcal{X}_H , it suffices to prove that

$$\Phi(1, g_j a_1^j g_j^{-1}, \dots, g_j a_k^j g_j^{-1}) \in \mathcal{X}_H$$

for all j large enough. We may hence assume, without loss of generality, that

$$(a_1^j, \dots, a_k^j) \rightarrow (a_1, \dots, a_k), \quad \text{Stab}_G(a_1^j, \dots, a_k^j) \subset \text{Stab}_G(a_1, \dots, a_k) \supset H$$

for all j .

By assumption, the elements $a_1, \dots, a_k \in G \subset \text{SL}_n \mathbb{C}$ commute and are diagonalizable. In particular, they are simultaneously diagonalizable. Let

$$\mathbb{C}^n = V_1 \oplus \dots \oplus V_r$$

be the coarsest direct sum decomposition of \mathbb{C}^n such that for all $i = 1, \dots, k$ and all $s = 1, \dots, r$ we have:

- $a_i(V_s) = V_s$, and
- the restriction $a_i|_{V_s}$ of a_i to V_s is a homothety.

Notice that the assumption that $\mathbb{C}^n = V_1 \oplus \dots \oplus V_r$ is the coarsest direct sum decomposition with these properties implies that

$$\text{Stab}_G(a_1, \dots, a_k) = G \cap (\text{GL}(V_1) \times \dots \times \text{GL}(V_r)).$$

For sufficiently large j we have hence that the elements

$$a_1^j, \dots, a_k^j \in \text{Stab}_G(a_1^j, \dots, a_k^j) \subset \text{Stab}_G(a_1, \dots, a_k)$$

preserve V_1, \dots, V_r . Proposition 7.6 implies that for all i and s , and for all sufficiently large j , we have

- $(a_i^j)_s^\epsilon V_s = V_s$, and
- the restriction $(a_i^j)_s^\epsilon|_{V_s}$ of $(a_i^j)_s^\epsilon$ to V_s is a homothety.

This implies that $\mathcal{Z}_G((a_i^j)_s^\epsilon) \supset G \cap (\text{GL}(V_1) \times \dots \times \text{GL}(V_r))$ for all j . At this point notice that the G -equivariance of the map σ provided by Lemma 8.5 implies that we also have

$$\mathcal{Z}_G(\sigma_1((a_i^j)_s^\epsilon)) \supset G \cap (\text{GL}(V_1) \times \dots \times \text{GL}(V_r))$$

for all j . This shows that

$$\begin{aligned} \text{Stab}_G(\Phi(1, (a_1^j, \dots, a_k^j))) &= \text{Stab}_G(\sigma_1((a_1^j)_s^\epsilon), \dots, \sigma_1((a_k^j)_s^\epsilon)) \\ &= \bigcap_{i=1}^j \mathcal{Z}_G((a_i^j)_s^\epsilon) \\ &\supset G \cap (\text{GL}(V_1) \times \dots \times \text{GL}(V_r)) \\ &= \text{Stab}_G(a_1, \dots, a_k) \supset H \end{aligned}$$

implying that $\Phi(1, (a_1^j, \dots, a_k^j)) \in \mathcal{X}_H$, as we needed to prove. □

Having established the claim, we have proved Proposition 6.1. □

Appendix

While reading this appendix the reader should always keep in mind the terminology and notation introduced in Section 5. More concretely, notation will always be as in (*) in Section 5.

We will identify four properties of the set $(\mathcal{R}, \mathcal{O})$ of fibered relation pairs that ensure that if (X, δ) is a \mathcal{P} -labeled complex and \mathcal{U} is a regular cover of X , then every section of $\mathcal{S} \times_\delta \mathcal{U}$ is homotopic to a section of $\mathcal{K} \times_\delta \mathcal{U} \subset \mathcal{S} \times_\delta \mathcal{U}$. More precisely, we will

define *transitivity*, *reflexivity*, and the *simultaneous lifting* and the *relative retraction* properties so that the following holds:

Proposition A.1 *With notation as in (*), suppose that \mathcal{R} is transitive and reflexive, that it satisfies the simultaneous lifting property, and that $(\mathcal{R}, \mathcal{O})$ satisfies the relative retraction property. Let also (X, δ) be a \mathcal{P} -labeled locally finite and finite-dimensional simplicial complex, and $\mathcal{U} = \{U_c\}_{c \in C(X)}$ and $\mathcal{V} = \{V_c\}_{c \in C(X)}$ regular covers of X with V_c relatively compact in U_c for all $c \in C(X)$. Finally, let $\{\sigma_c\}_{c \in C(X)}$ be a section of $\mathcal{S} \times_\delta \mathcal{U}$, and $W \subset X$ with $\sigma_c(w) \in K_{\delta(c)}$ for all $c \in C(X)$ and all $x \in U_c \cap W$.*

The restriction of $\{\sigma_c\}_{c \in C(X)}$ to $\mathcal{S} \times_\delta \mathcal{V}$ is homotopic to a section $\{\sigma'_c\}_{c \in C(X)}$ with $\sigma'_c(V_c) \subset K_{\delta(c)}$ for all $c \in C(X)$; moreover the homotopy is constant on W .

Once Proposition A.1 is proved, we will show that the sets of fibered relation pairs $(\mathcal{R}_G, \mathcal{O}_G)$ associated in (5.1) and (5.2) to a group G satisfies these four properties. Proposition 5.3 follows immediately.

A.1 Transitivity and reflexivity

The set \mathcal{R} is *transitive* if

$$\mathcal{R}_{\Delta, \Delta''} \supset \{(x, z) \in S_\Delta \times S_{\Delta''} \mid \exists y \in S_{\Delta'} \text{ with } (x, y) \in \mathcal{R}_{\Delta, \Delta'}, (y, z) \in \mathcal{R}_{\Delta', \Delta''}\}$$

for all $\Delta, \Delta', \Delta'' \in \mathcal{P}$ with $\Delta \geq \Delta' \geq \Delta''$; \mathcal{R} is *reflexive* if

$$\mathcal{R}_{\Delta, \Delta} = \{(x, x) \in S_\Delta \times S_\Delta \mid x \in S_\Delta\}$$

for all $\Delta \in \mathcal{P}$.

Notice that transitivity and reflexivity of \mathcal{R} imply that \mathcal{O} is also transitive and reflexive.

A.2 Simultaneous lifting

Suppose now that (X, δ) is a \mathcal{P} -labeled simplicial complex, and let $\mathcal{U} = (U_c)_{c \in C(X)}$ be a regular cover. Consider the bundle

$$\mathcal{S} \times_\delta \mathcal{U} := \bigsqcup_{c \in C(X)} (S_{\delta(c)} \times U_c) \rightarrow \bigsqcup_{c \in C(X)} U_c.$$

Our goal is to prove that, under certain conditions, one can homotope sections $\{\sigma_c\}_{c \in C(X)}$ of $\mathcal{S} \times_\delta \mathcal{U}$ to sections $\{\sigma'_c\}_{c \in C(X)}$ such that σ'_c takes values in the subset $K_{\delta(c)} \subset S_{\delta(c)}$ for all $c \in C(X)$. In order to do so we want to be able to construct homotopies of sections skeleton by skeleton, ie, starting with the maps $\sigma_c: U_c \rightarrow S_{\delta(c)}$

corresponding to the vertices of X , continuing with those corresponding to the edges, etc... We will see shortly that the following is the property allowing us to do so.

The set \mathcal{R} has the *simultaneous lifting property* if:

(SL) Suppose that $\Delta_0, \Delta_1, \dots, \Delta_r \in \mathcal{P}$ are such that $\Delta_i \geq \Delta_0$ for all i and that W_0, \dots, W_r are contractible open sets in a CW-complex with $W_i \subset W_0$ for all i . Finally suppose that we have for each $i = 0, \dots, r$ a continuous map $\sigma_i: W_i \rightarrow S_{\Delta_i}$ with $(\sigma_i(x), \sigma_j(x)) \in \mathcal{R}_{\Delta_i, \Delta_j}$ for all $i, j \in \{0, \dots, r\}$ with $\Delta_i \geq \Delta_j$ and all $x \in W_i \cap W_j$.

For any continuous map $\phi_0: [0, 1] \times W_0 \rightarrow S_{\Delta_0}$ with $\phi_0(0, x) = \sigma_0(x)$ there are continuous maps $\phi_i: [0, 1] \times W_i \rightarrow S_{\Delta_i}$ for $i = 1, \dots, r$ with $\phi_i(0, x) = \sigma_i(x)$ and with

$$(\phi_i(t, x), \phi_j(t, x)) \in \mathcal{R}_{\Delta_i, \Delta_j}$$

for all t , all $i, j \in \{0, \dots, r\}$ with $\Delta_i \geq \Delta_j$ and all $x \in W_i \cap W_j$.

Moreover, if the homotopy ϕ_0 is constant outside of some compact set $C \subset W_0$, then ϕ_i is also constant outside of $C \cap W_i$.

Lemma A.2 (Lifting Lemma) *Suppose that \mathcal{R} is transitive and reflexive, and satisfies the simultaneous lifting property and let $\mathcal{U} = \{U_c\}_{c \in C(X)}$ be a regular cover of a \mathcal{P} -labeled complex (X, δ) .*

Suppose moreover that $\{\sigma_c\}_{c \in C(X)}$ is a section of $S \times_\delta \mathcal{U}$ and that for some $c_0 \in C(X)$ we have a continuous map

$$[0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}, \quad (t, x) \rightarrow \bar{\sigma}_{c_0}^t(x)$$

with $\bar{\sigma}_{c_0}^0 = \sigma_{c_0}$, such that:

- (1) $(\bar{\sigma}_{c_0}^t(x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $c_0 \supsetneq c$ and all $x \in U_{c_0} \cap U_c$.
- (2) $\bar{\sigma}_{c_0}^t(x) = \sigma_{c_0}(x)$ for all $x \notin C$, where $C \subset U_{c_0}$ is compact.

Then there is a homotopy of sections $\{\sigma_c^t\}_{c \in C(X)}$ of $S \times_\delta \mathcal{U}$ satisfying

- $\sigma_c^0 = \sigma_c$ for all $c \in C(X)$,
- $\sigma_{c_0}^t = \bar{\sigma}_{c_0}^t$ for all t ,
- $\sigma_c^t = \sigma_c$ for all $c \in C(X)$ with $c_0 \not\supseteq c$ and all t , and
- the homotopy $\{\sigma_c^t\}_{c \in C(X)}$ is constant on $X \setminus C$.

Proof Let $c_1, \dots, c_s \in C(X) \setminus \{c_0\}$ be cells in X with $U_{c_i} \cap U_{c_0} \neq \emptyset$ and observe that these are exactly the cells c_i in $C(X)$ with either $c_0 \subsetneq c_i$ or with $c_i \subsetneq c_0$. We may assume that c_1, \dots, c_r are those containing c_0 and c_{r+1}, \dots, c_s those contained in c_0 . In particular $\delta(c_i) \geq \delta(c_0)$ for all $i = 1, \dots, r$. Applying the simultaneous lifting property to

- $\delta(c_0), \dots, \delta(c_r)$,
- the contractible open sets $U_{c_i} \cap U_{c_0}$ for $i = 0, \dots, r$,
- the restrictions of σ_{c_i} to $U_{c_i} \cap U_{c_0}$ again for $i = 0, \dots, r$, and
- the map $[0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$, $(t, x) \mapsto \bar{\sigma}_{c_0}^t(x)$,

we obtain for $i = 1, \dots, r$ a map

$$[0, 1] \times (U_{c_0} \cap U_{c_i}) \rightarrow S_{\delta(c_i)}, \quad (t, x) \mapsto \bar{\sigma}_{c_i}^t(x)$$

satisfying $(\bar{\sigma}_{c_i}^t(x), \bar{\sigma}_{c_j}^t(x)) \in \mathcal{R}_{\delta(c_i), \delta(c_j)}$ for all $i, j \in \{0, \dots, r\}$ with $c_i \supset c_j$ and all $x \in U_{c_i} \cap U_{c_j}$. A priori, the maps $\bar{\sigma}_{c_i}^t$ are only defined on $U_{c_i} \cap U_{c_0}$. However, since $\bar{\sigma}_{c_0}^t(x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0} \setminus C$, we can extend $\bar{\sigma}_{c_i}^t$ continuously to U_{c_i} by $\bar{\sigma}_{c_i}^t(x) = \sigma_{c_i}(x)$ for all $x \in U_{c_i} \setminus C$.

Suppose now that $c_i \subsetneq c_0$, ie, $i \in \{s+1, \dots, r\}$. By the assumptions in the lemma we have that $(\bar{\sigma}_{c_0}^t(x), \sigma_{c_i}(x)) \in \mathcal{R}_{\delta(c_0), \delta(c_i)}$ for all $x \in U_{c_0} \cap U_{c_i}$. The transitivity of \mathcal{R} implies thus that for all $j \in \{1, \dots, r\}$ we have also $(\bar{\sigma}_{c_j}^t(x), \sigma_{c_i}(x)) \in \mathcal{R}_{\delta(c_j), \delta(c_i)}$ for all $x \in U_{c_j} \cap U_{c_i}$. It follows that $\{\sigma_c^t\}_{c \in C(X)}$ with

$$\sigma_c^t = \bar{\sigma}_c^t \quad \text{for } c \in \{c_0, \dots, c_r\}, \quad \sigma_c^t = \sigma_c \quad \text{for } c \notin \{c_0, \dots, c_r\}$$

is a section of $\mathcal{S} \times_{\delta} \mathcal{U}$ for all t . It has the desired properties by construction. □

A.3 Relative retractions

We will be interested later in pairs (S_{Δ}, K_{Δ}) with the property that the inclusion $K_{\Delta} \rightarrow S_{\Delta}$ is a homotopy equivalence. However, since we are not working with single spaces but with sets of spaces, we need that the homotopy equivalence between K_{Δ} and S_{Δ} behaves well with respect to other pairs $(S_{\Delta'}, K_{\Delta'})$.

The set $(\mathcal{R}, \mathcal{O})$ satisfies the *relative retraction property* if:

- (RR1) For all $\Delta \in \mathcal{P}$ the inclusion $K_{\Delta} \rightarrow S_{\Delta}$ is a homotopy equivalence.
- (RR2) For all $\Delta, \Delta' \in \mathcal{P}$ with $\Delta \geq \Delta'$ there is a retraction of $\mathcal{R}_{\Delta, \Delta'} \cap (S_{\Delta} \times K_{\Delta'})$ to $\mathcal{O}_{\Delta, \Delta'} = \mathcal{R}_{\Delta, \Delta'} \cap (K_{\Delta} \times K_{\Delta'})$ that preserves the fibers of the projection $\mathcal{R}_{\Delta, \Delta'} \cap (S_{\Delta} \times K_{\Delta'}) \rightarrow K_{\Delta'}$.

We formulate the second condition in more concrete, but also more obscure terms:

Lemma A.3 (Reformulation of (RR2)) *Suppose that $(\mathcal{R}, \mathcal{O})$ satisfies the relative retraction property. Given $\Delta_0, \Delta_1 \in \mathcal{P}$ with $\Delta_0 \geq \Delta_1$, an open set W in a CW-complex, continuous maps $\sigma_i: W \rightarrow S_{\Delta_i}$ with $(\sigma_0(x), \sigma_1(x)) \in \mathcal{R}_{\Delta_0, \Delta_1}$ for all $x \in W$ and such that $\sigma_1(W) \subset K_{\Delta_1}$, and finally $U \subset W$ open and relatively compact, then the following holds:*

There is a continuous map $\phi_0: [0, 1] \times W \rightarrow S_{\Delta_0}$ satisfying:

- (1) $\phi_0(0, x) = \sigma_0(x)$ for all $x \in W$,
- (2) $(\phi_0(t, x), \sigma_1(x)) \in \mathcal{R}_{\Delta_0, \Delta_1}$ for all $x \in W$ and all t ,
- (3) $\phi_0(t, x) = \sigma_0(x)$ for all x outside some compact set $C \subset W$ and all $x \in W$ with $\sigma_0(x) \in K_{\Delta_0}$, and
- (4) $\phi_0(1, x) \in K_{\Delta_0}$ for all $x \in U$. □

Before the relative retraction property can be of any use to us, we need to extract from it something that takes into account the interplay between multiple members of the set $(\mathcal{R}, \mathcal{O})$ of fibered relations, and not only two of them. This is done in the following lemma:

Lemma A.4 (Retraction Lemma) *Suppose that \mathcal{R} is transitive and reflexive, and that $(\mathcal{R}, \mathcal{O})$ satisfies the relative retraction property. Let $\mathcal{U} = \{U_c\}_{c \in C(X)}$ and $\mathcal{V} = \{V_c\}_{c \in C(X)}$ be regular covers of a \mathcal{P} -labeled complex (X, δ) with V_c relatively compact in U_c for all $c \in C(X)$.*

Suppose moreover that $\{\sigma_c\}_{c \in C(X)}$ is a section of $S \times_{\delta} \mathcal{U}$ and that for some $c_0 \in C(X)$ the following holds:

$$\sigma_c(U_c) \subset K_{\delta(c)} \quad \forall c \in C(X) \text{ with } c \subsetneq c_0$$

There is a continuous map $\phi_{c_0}: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$ satisfying:

- (1) $\phi_{c_0}(0, x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0}$,
- (2) $(\phi_{c_0}(t, x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $c_0 \supsetneq c$, $x \in V_{c_0} \cap V_c$ and t ,
- (3) $\phi_{c_0}(t, x) = \sigma_{c_0}(x)$ for all x outside some compact set $C \subset U_{c_0}$ and all $x \in U_{c_0}$ with $\sigma_{c_0}(x) \in K_{\delta(c_0)}$, and
- (4) $\phi_{c_0}(1, x) \in K_{\delta(c_0)}$ for all $x \in V_{c_0}$.

Before proving Lemma A.4 we establish some terminology. Suppose that $\phi: X \rightarrow Y$ is a map between two CW-complexes and that for some (countable) set of indices i, j, \dots we have homotopies

$$\phi_i: [0, 1] \times X \rightarrow Y$$

with $\phi_i(0, x) = \phi(x)$ for all $x \in X$. We say that the homotopies ϕ_i do *not interfere with each other* if for each i there is a compact set $C_i \subset X$ of satisfying the following two properties:

- $C_i \cap C_j = \emptyset$ for all $i \neq j$ and no sequence (x_i) with $x_i \in C_i$ has a convergent subsequence.
- $\phi_i(t, x) = \phi(x)$ for all $x \notin C_i$ and all t .

If the homotopies ϕ_i do not interfere with each other, then the map

$$\Phi: [0, 1] \times X \rightarrow Y$$

given by $\Phi(t, x) = \phi_i(t, x)$ for $x \in C_i$ and $\Phi(t, x) = \phi(x)$ for $x \notin \bigcup_i C_i$ is continuous. We will refer to Φ as the *combination* of the homotopies ϕ_i .

Continuing with the same notation suppose that we have maps

$$\Phi, \Phi': [0, 1] \times X \rightarrow Y$$

with $\Phi(1, x) = \Phi'(0, x)$ for all $x \in X$. The continuous map $\Phi'': [0, 1] \times X \rightarrow Y$ given by $\Phi''(t, x) = \Phi(2t, x)$ for $t \leq \frac{1}{2}$ and $\Phi''(t, x) = \Phi'(2t - 1, x)$ for $t \geq \frac{1}{2}$ is the *juxtaposition* of Φ and Φ' .

Proof of Lemma A.4 Let $D = \dim(c_0)$ be the dimension of c_0 and choose for $i = 0, \dots, D$ a regular cover $\mathcal{U}^i = \{U_c^i\}_{c \in C(X)}$ with $\mathcal{U}^0 = \mathcal{U}$, $\mathcal{U}^D = \mathcal{V}$ and such that U_c^{i+1} is relatively compact in U_c^i for all $c \in C(X)$.

Suppose that $c \subsetneq c_0$ is a codimension 1 face of c_0 . Since the relative retraction property is satisfied, we can apply Lemma A.3 to $\delta(c_0) \geq \delta(c)$, the open set $U_{c_0}^0 \cap U_c^0$ and the restrictions of σ_{c_0} and σ_c to $U_{c_0}^0 \cap U_c^0$. Noticing that $U_{c_0}^1 \cap U_c^1$ is relatively compact in $U_{c_0}^0 \cap U_c^0$ we obtain hence a continuous map

$$\phi_c: [0, 1] \times (U_{c_0}^0 \cap U_c^0) \rightarrow S_{\delta(c_0)}$$

satisfying:

- (1) $\phi_c(0, x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0}^0 \cap U_c^0$,
- (2) $(\phi_c(t, x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $x \in U_{c_0}^0 \cap U_c^0$ and t ,

- (3) $\phi_c(t, x) = \sigma_{c_0}(x)$ for all x outside some compact set $C \subset U_{c_0}^0 \cap U_c^0$ and all $x \in U_{c_0}^0 \cap U_c^0$ with $\sigma_0(x) \in K_{\delta(c_0)}$, and
- (4) $\phi_c(1, x) \in K_{\delta(c_0)}$ for all $x \in U_{c_0}^1 \cap U_c^1$.

Notice that defining $\phi_c(t, x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0}^0 \setminus U_c^0$, we obtain a continuous map ϕ_c defined on the whole of $[0, 1] \times U_{c_0}^0$. Suppose now that we are given a further proper face \bar{c} of c_0 with $U_{c_0}^0 \cap U_c^0 \cap U_{\bar{c}}^0 \neq \emptyset$ and notice that this implies that $c_0 \supset c \supset \bar{c}$. We obtain from (2) the assumption that $\{\sigma_c\}_{c \in C(X)}$ is a section, and from the transitivity of \mathcal{R} that

$$(A.1) \quad (\phi_c(t, x), \sigma_{\bar{c}}(x)) \in \mathcal{R}_{\delta(c_0), \delta(\bar{c})}$$

for all $x \in U_{c_0}^0 \cap U_c^0 \cap U_{\bar{c}}^0$. On the other hand, if $x \in (U_{c_0}^0 \setminus U_c^0) \cap U_{\bar{c}}^0$ we have by definition that $\phi_c(t, x) = \sigma_{c_0}(x)$; it follows that (A.1) is actually satisfied for all $x \in U_{c_0}^0 \cap U_{\bar{c}}^0$ and hence:

$$(2') \quad (\phi_c(t, x), \sigma_{\bar{c}}(x)) \in \mathcal{R}_{\delta(c_0), \delta(\bar{c})} \text{ for all } c_0 \supsetneq \bar{c}, x \in U_{c_0}^0 \cap U_{\bar{c}}^0 \text{ and } t.$$

Suppose that we are given a second codimension one face c' of c_0 . Since $U_c^0 \cap U_{c'}^0 = \emptyset$ the homotopies ϕ_c and $\phi_{c'}$ do not interfere with each other. Combining all the homotopies corresponding to all the codimension one faces of c_0 , we have then that for $d = 1$ there is a map

$$\psi^d: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$$

satisfying:

- (a) $\psi^d(0, x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0} = U_{c_0}^0$,
- (b) $(\psi^d(t, x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $c_0 \supsetneq c, x \in U_{c_0}^{d-1} \cap U_c^{d-1}$ and t ,
- (c) $\psi^d(t, x) = \sigma_{c_0}(x)$ for all x outside some compact set $C \subset U_{c_0}$ and for all $x \in U_{c_0}$ with $\sigma_{c_0}(x) \in K_{\delta(c_0)}$, and
- (d) $\psi^d(1, x) \in K_{\delta(c)}$ for all $x \in U_{c_0}^d \cap U_c^d$ where $c \subsetneq c_0$ has at most codimension d .

Arguing by induction, we may assume that for some $d \geq 1$ we have a map $\psi^d: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$ satisfying (a), (b), (c) and (d). Suppose that $c \subset c_0$ is a codimension $d + 1$ face. Applying Lemma A.3 to $\delta(c_0) \geq \delta(c)$, the open set $U_{c_0}^d \cap U_c^d$ and the restrictions of $\psi^d(1, \cdot)$ and σ_c to $U_{c_0}^d \cap U_c^d$, we obtain again a map

$$\phi_c: [0, 1] \times (U_{c_0}^d \cap U_c^d) \rightarrow S_{\delta(c_0)}$$

satisfying:

$$(\hat{1}) \quad \phi_c(0, x) = \psi^d(1, x) \text{ for all } x \in U_{c_0}^d \cap U_c^d,$$

- ($\widehat{2}$) $(\phi_c(t, x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $x \in U_{c_0}^d \cap U_c^d$ and t ,
- ($\widehat{3}$) $\phi_c(t, x) = \psi^d(1, x)$ for all x outside some compact set $C \subset U_{c_0}^d \cap U_c^d$ and all $x \in U_{c_0}^d \cap U_c^d$ with $\psi^d(1, x) \in K_{\delta(c_0)}$, and
- ($\widehat{4}$) $\phi_c(1, x) \in K_{\delta(c_0)}$ for all $x \in U_{c_0}^{d+1} \cap U_c^{d+1}$.

As above, we set $\phi_c(t, x) = \psi_d(1, x)$ for all $x \in U_{c_0} \setminus U_c^d$ obtaining thus a continuous extension:

$$\phi_c: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$$

We claim that ϕ_c satisfies the following strengthening of ($\widehat{2}$):

$$(\widehat{2}') \quad (\phi_c(t, x), \sigma_{\bar{c}}(x)) \in \mathcal{R}_{\delta(c_0), \delta(\bar{c})} \text{ for all } c_0 \supsetneq \bar{c}, x \in U_{c_0}^d \cap U_{\bar{c}}^d \text{ and } t.$$

If $\bar{c} \subset c$ then, as above, the transitivity of the set \mathcal{R} implies that ($\widehat{2}'$) holds for all $x \in U_{c_0}^d \cap U_{\bar{c}}^d$ and t . Suppose now that $c \subsetneq \bar{c}$. By (d) we have that $\psi^d(1, x) \in K_{\delta(c_0)}$ for all $x \in U_{c_0}^d \cap U_{\bar{c}}^d$. It follows hence from ($\widehat{3}$) and (b) that $(\phi_c(t, x), \sigma_{\bar{c}}(x)) \in \mathcal{R}_{\delta(c_0), \delta(\bar{c})}$ for all $x \in U_{c_0}^d \cap U_{\bar{c}}^d$ and t . Finally observe that $U_c^d \cap U_{\bar{c}}^d = \emptyset$ if \bar{c} is neither contained in nor contains c . It follows that for any such c we have $\phi_c(t, x) = \psi^d(1, x)$ for all $x \in U_{c_0} \cap U_c$ and all t . The validity of ($\widehat{2}'$) for $x \in U_{c_0}^d \cap U_{\bar{c}}^d$ follows hence from (b). We have proved that ϕ^c satisfies ($\widehat{2}'$).

As above, the maps ϕ_c and $\phi_{c'}$ do not interfere with each other for any two distinct co-dimension $d + 1$ faces c and c' of c_0 . The juxtaposition

$$\psi^{d+1}: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c)}$$

of ψ^d and the combination of all the homotopies ϕ_c with c a co-dimension $d + 1$ face of c_0 satisfies (a), (b), (c) and (d) for $d + 1$.

Recall that $\mathcal{U}^D = \mathcal{V}$; repeating this process D times we obtain hence a map

$$\psi = \psi^D: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$$

satisfying:

- (a') $\psi(0, x) = \sigma_{c_0}(x)$ for all $x \in U_{c_0}$,
- (b') $(\psi(t, x), \sigma_c(x)) \in \mathcal{R}_{\delta(c_0), \delta(c)}$ for all $c_0 \supsetneq c, x \in V_{c_0} \cap V_c$ and t ,
- (c') $\psi(t, x) = \sigma_{c_0}(x)$ for all x outside some compact set $C \subset U_{c_0}$ and for all $x \in U_{c_0}$ with $\sigma_{c_0}(x) \in K_{\delta(c_0)}$, and
- (d') $\psi(1, x) \in K_{\delta(c_0)}$ for all $x \in V_{c_0} \cap (\bigcup_{c \subsetneq c_0} V_c)$.

So far, we have only used the second part of the relative retraction property (RR2). Taking now into account (d') and that the inclusion of $K_{\delta(c_0)} \rightarrow S_{\delta(c_0)}$ is a homotopy equivalence, we can find a homotopy

$$\psi': [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$$

relative to $V_{c_0} \cap (\bigcup_{c \neq c_0} V_c)$, to the complement of a compact subset of U_{c_0} , starting with $\psi'(0, x) = \psi(1, x)$, such that $\psi'(1, x) \in K_{\delta(c_0)}$ for all $x \in V_{c_0}$. Juxtaposing ψ and ψ' we obtain finally a continuous map $\phi_{c_0}: [0, 1] \times U_{c_0} \rightarrow S_{\delta(c_0)}$ satisfying conditions (1), (2), (3) and (4) in the statement of Lemma A.4. \square

A.4 Proof of Proposition A.1

Starting with the proof of Proposition A.1, let $D = \dim X$ and choose a regular cover $U^i = \{U_c^i\}_{c \in C(X)}$, for $i = 0, \dots, D$ with $U^0 = \mathcal{U}$, $U^D = \mathcal{V}$, such that U_c^{i+1} is relatively compact in U_c^i for all $c \in C(X)$. We will denote by $X^{(d)}$ the d -skeleton of X , ie, the simplicial complex consisting of all the cells of dimension at most d ; the set of cells of dimension precisely d is denoted by $C_d(X)$.

Again we will argue by induction. Observing that $X^{(-1)} = \emptyset$, we may suppose that for some $d \geq 0$ we have $\sigma_c(U_c^{d-1}) \subset K_{\delta(c)}$ for all $c \in C(X)$ of dimension at most $d - 1$.

We will show that there is a homotopy of sections, constant on W , which starts with the restriction of $\{\sigma_c\}_{c \in C(X)}$ to U^d and ends with a section $\{\sigma'_c\}_{c \in C(X)}$ of $S \times_{\delta} U^d$ with $\sigma'_c(U_c^d) \subset K_{\delta(c)}$ for all $c \in C(X^{(d)})$.

Let $c_0 \in C_d(X)$ be a d -dimensional cell. Consider the cells $c_1, \dots, c_r \in C(X)$ that are properly contained in c_0 and notice that $\delta(c_0) \geq \delta(c_i)$ for all i . The retraction lemma, Lemma A.4, yields a map

$$\bar{\sigma}_{c_0}: [0, 1] \times U_{c_0}^{d-1} \rightarrow S_{\delta(c_0)}, \quad (t, x) \mapsto \bar{\sigma}_{c_0}^t(x)$$

with $\bar{\sigma}_{c_0}^0 = \sigma_{c_0}$, $\bar{\sigma}_{c_0}^1(U_{c_0}^d) \subset K_{\delta(c_0)}$, such that:

- $(\bar{\sigma}_{c_0}^t(x), \sigma_{c_i}(x)) \in \mathcal{R}_{\delta(c_0), \delta(c_i)}$ for all $i > 0$, $x \in U_{c_0}^d \cap U_{c_i}^d$, and t .
- $\bar{\sigma}_{c_0}^t(t, x) = \sigma_{c_0}(x)$ for all x outside some compact set $C \subset U_{c_0}^{d-1}$ and all $x \in W \cap U_{c_0}^{d-1}$.

The Lifting Lemma A.2 implies that there is a section $\{\sigma'_c\}_{c \in C(X)}$ of $S \times_{\delta} U^d$ with $\sigma'_{c_0} = \bar{\sigma}_{c_0}^1$ and a homotopy $\{\sigma_c^t\}_{c \in C(X)}$ supported by K between $\{\sigma_c\}_{c \in C(X)}$ and $\{\sigma'_c\}_{c \in C(X)}$. Moreover, the homotopy $t \mapsto \sigma_c^t$ is constant unless $c_0 \subset c$. In particular, $\sigma'_c = \sigma_c$ for all $c \in C(X^{(d-1)})$.

Since $U_c^d \cap U_{c'}^d = \emptyset$ for every other $c'_0 \in C_d(X)$, we deduce that the homotopy just constructed for c_0 and the corresponding one for c'_0 do not interfere with each other. Combining the homotopies corresponding to all the d -dimensional cells, we obtain that the restriction to \mathcal{U}^d of the section $\{\sigma_c\}$ is homotopic, through a homotopy constant on W , to a section $\{\sigma'_c\}$ of $\mathcal{S} \times_{\delta} \mathcal{U}^d$, with $\sigma_c(U_c^d) \subset K_{\delta(c)}$ for all $c \in C(X)$ with $\dim(c) \leq d$. This concludes the proof of the induction step and hence the proof of Proposition A.1. \square

A.5 Proof of Proposition 5.3

Let G be the group of complex or real points of a reductive algebraic group (defined over \mathbb{R} in the latter case), $K \subset G$ a maximal compact subgroup, and let $G_0 \subset G$ and $K_0 \subset G$ the respective identity components. Let also \mathcal{G} and \mathcal{K} be the manifolds of all well-centered subgroups of G and K . Consider $\pi_0(\mathcal{G})$ with the partial ordering introduced in Section 3.5, and \mathcal{R}_G and \mathcal{O}_G as defined in (5.1) and (5.2). By Lemma 5.2, $(\mathcal{R}_G, \mathcal{O}_G)$ is the set of fibered relation pairs. It follows directly from the definitions that \mathcal{R}_G is transitive and reflexive:

Lemma A.5 *The set \mathcal{R}_G of fibered relations is transitive and reflexive.* \square

We prove that \mathcal{R}_G and $(\mathcal{R}_G, \mathcal{O}_G)$ satisfy, respectively, the simultaneous lifting and relative retraction properties.

Lemma A.6 *The set \mathcal{R}_G satisfies the simultaneous lifting property.*

Proof Suppose that $\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_r \in \pi_0(\mathcal{G})$ are such that $\mathcal{G}_i \geq \mathcal{G}_0$ for all i and that W_0, \dots, W_r are contractible open sets in a CW-complex with $W_i \subset W_0$ for all i . Finally suppose that we have for each $i = 0, \dots, r$ a continuous map $\sigma_i: W_i \rightarrow \mathcal{G}_i$ with $(\sigma_i(x), \sigma_j(x)) \in \mathcal{R}_{\mathcal{G}_i, \mathcal{G}_j}$ for all $i, j \in \{0, \dots, r\}$ with $\mathcal{G}_i \geq \mathcal{G}_j$ and all $x \in W_i \cap W_j$.

Let also $\phi_0: [0, 1] \times W_0 \rightarrow \mathcal{G}_0$ be continuous with $\phi_0(0, x) = \sigma_0(x)$. Choosing a base point $A \in \mathcal{G}_0$ we consider the map

$$p: G_0 \rightarrow \mathcal{G}_0, \quad g \mapsto gAg^{-1}.$$

Notice that $p: G_0 \rightarrow \mathcal{G}_0$ is a fiber bundle with fiber $\mathcal{N}_{G_0}(A) = \mathcal{H}_G(A) \cap G_0$; hence p is a fibration. It follows from the contractibility of W_0 that there is map

$$\alpha: [0, 1] \times W_0 \rightarrow G_0$$

such that the following diagram commutes:

$$\begin{array}{ccc}
 & & G_0 \\
 & \nearrow \alpha & \downarrow p \\
 [0, 1] \times W_0 & \xrightarrow{\phi_0} & \mathcal{G}_0
 \end{array}$$

Moreover, if there is a compact set $K \subset W_0$ with $\phi_0(t, x) = \phi_0(0, x)$ for all $x \notin K$ and all t then we may assume also that $\alpha(t, x) = \alpha(0, x)$ for all $x \notin K$ and all t .

We consider now the map

$$\beta: [0, 1] \times W_0 \rightarrow G_0, \quad \beta(t, x) = \alpha(t, x)\alpha(0, x)^{-1}.$$

Observe that $\beta(t, x) = \text{Id}$ for all $x \notin K$ and all t , that $\beta(0, x) = \text{Id}$ for all $x \in W_0$ and that

$$\begin{aligned}
 \phi(t, x) &= \alpha(t, x)A\alpha(t, x)^{-1} = \alpha(t, x)\alpha(0, x)^{-1}\sigma_0(x)\alpha(0, x)\alpha(t, x)^{-1} \\
 &= \beta(t, x)\sigma_0(x)\beta(t, x)^{-1}
 \end{aligned}$$

for all $x \in W_0$ and all t .

For $i = 1, \dots, r$ we consider the map

$$\phi_i: [0, 1] \times W_i \rightarrow \mathcal{G}_i, \quad \phi_i(t, x) = \beta(t, x)\sigma_i(x)\beta(t, x)^{-1}$$

and notice that $t \mapsto \phi_i(t, x)$ is constant for x outside of $K \cap W_i$. Also, it follows directly from (5.1), the definition of $\mathcal{R}_{\mathcal{G}_i, \mathcal{G}_j}$, that

$$(\phi_i(t, x), \phi_j(t, x)) \in \mathcal{R}_{\mathcal{G}_i, \mathcal{G}_j}$$

for all t , all $i, j \in \{0, \dots, r\}$ with $\mathcal{G}_i \geq \mathcal{G}_j$ and all $x \in W_i \cap W_j$. We have proved that \mathcal{R}_G has the simultaneous lifting property. □

Lemma A.7 *The set $(\mathcal{R}_G, \mathcal{O}_G)$ satisfies the relative retraction property.*

Proof To begin with recall that for every $\mathcal{G}_0 \in \pi_0(\mathcal{G})$ the inclusion $\mathcal{K}_0 \rightarrow \mathcal{G}_0$ is a homotopy equivalence by Proposition 3.11; in other words, the first half (RR1) of the relative retraction property is satisfied. Fixing a base point $(A_0, A_1) \in \mathcal{R}_{\mathcal{G}_0, \mathcal{G}_1} \cap (\mathcal{K}_0 \times \mathcal{K}_1)$ we obtain from Lemma 3.13 and Lemma 3.14 that the fibrations

$$\mathcal{R} \cap (\mathcal{G}_0 \times \mathcal{K}_1) \rightarrow \mathcal{K}_1 \quad \text{and} \quad \mathcal{R} \cap (\mathcal{K}_0 \times \mathcal{K}_1) \rightarrow \mathcal{K}_1$$

are locally trivial fiber bundles with fibers

$$\text{Trans}_{\mathcal{G}_0}(A_1, A_0)/\mathcal{N}_{\mathcal{G}_0}(A_1) \quad \text{and} \quad \text{Trans}_{\mathcal{K}_0}(A_1, A_0)/\mathcal{N}_{\mathcal{K}_0}(A_1).$$

In order to prove that (RR2) is satisfied, it suffices to show that the inclusion

$$\mathrm{Trans}_{K_0}(A_1, A_0)/\mathcal{N}_{K_0}(A_1) \rightarrow \mathrm{Trans}_{G_0}(A_1, A_0)/\mathcal{N}_{G_0}(A_1)$$

is a homotopy equivalence. Consider the commuting diagram:

$$\begin{array}{ccccc} \mathcal{N}_{K_0}(A_1) & \longrightarrow & \mathrm{Trans}_{K_0}(A_1, A_0) & \longrightarrow & \mathrm{Trans}_{K_0}(A_1, A_2)/\mathcal{N}_{K_0}(A_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_{G_0}(A_1) & \longrightarrow & \mathrm{Trans}_{G_0}(A_1, A_0) & \longrightarrow & \mathrm{Trans}_{G_0}(A_1, A_2)/\mathcal{N}_{G_0}(A_1) \end{array}$$

The rows are fibrations and the first two columns are homotopy equivalences by Lemma 3.4. It follows that the third column is also a homotopy equivalence, as we needed to show. This concludes the proof of Lemma A.7. \square

Proposition 5.3 follows directly from Lemmas A.5, A.6 and A.7 and Proposition A.1. \square

The second author wishes to quote Günter Netzer, who, to the best of his understanding, was paraphrasing Berti Vogts: *Deutsche Tugenden haben wir immer noch am besten*. Gemeint waren Kampf, Kondition, nimmermüde Einsatzbereitschaft, Durchsetzungsvermögen und Siegeswille.

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