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Erratum for "An elementary construction of Anick's fibration"

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A misstatement in the key proof in our paper "An elementary construction of Anick's fibration" led to an erroneous proof. This is repaired by a slightly longer argument.

55P35, 55P40, 55P45; 55Q51, 55Q52

In [3] we gave an elementary construction of Anick's space T_{2n-1}. This is a space that lies in a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}, \tag{1}$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is the $(p^r)^{\text{th}}$ power map. This construction was carried out for any $p \geqslant 3$ and $r \geqslant 1$.

We also proved in [3, Theorem 4.3] that there is an H–space structure on T_{2n-1} such that (1) is an H–fibration. The proof of 4.3 involved induction over the skeleton of T_{2n-1} and cycled through 14 steps $((\mathrm{a}), \dots, (\mathrm{n}))$. The argument given for the proof of [3, Theorem 4.3(l)] contained an incorrect statement and is not valid. The purpose of this note is to supply a correct proof for 4.3(l).

We will abbreviate T_{2n-1} as T and write T^m for the m skeleton of T. Recall that $\mathcal{W}_a^b$ is the collection of all spaces that are of the homotopy type of a simply connected locally finite wedge of mod p^s Moore spaces for $a \leq s \leq b$.

At the point in the induction that we need to prove 4.3(l), we have established the following facts:

(A) $\Sigma T^{2np^k} \simeq G_k \vee W_k$ with $W_k \in \mathcal{W}_r^{r+k-1}$ (4.3(j))

(B) $G_k = G_{k-1} \cup_{\alpha_k} \mathrm{CP}^{2np^k}(p^{r+k})$ (4.3(c))

(C) $\alpha_k \colon P^{2np^k}(p^{r+k}) \to G_{k-1}$ is divisible by p^{r+k-1} (4.3(c), (e))

(D) $\Sigma^2 T^{2np^k} \in \mathcal{W}_r^{r+k}$ (4.3(b))

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We will also use various steps in the induction at level $k-1$. Our task here is to prove:

Theorem 4.3(l) $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$

In the discussion of 4.3(l), steps 1 and 2 correctly conclude that $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$. To complete the proof of 4.3(l), we need to analyze the cofibration sequence from (B) above:

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \longrightarrow G_k \wedge T^{2np^k}.$$

$P^{2np^k}(p^{r+k})$ is a double suspension and consequently $P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$ by (D) above. Since $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$ it suffices to show that $\alpha_k \wedge 1$ is null homotopic. A key ingredient in establishing this is the fact that $\alpha_k \wedge 1$ is divisible by p^{r+k-1}.

To clarify the situation, we recall from [3, 4.2] the following:

Lemma 1 [3] *Suppose $W \in \mathcal{W}_a^b$ and $f\colon P^k(p^s) \to W$ is divisible by p^b. Write $W \simeq W_1 \vee W_2$ with $W_1 \in \mathcal{W}_a^{b-1}$ and $W_2 \in \mathcal{W}_b^b$. Then:*

(a) *f factors through W_2 up to homotopy.*

(b) *Suppose that W_2 is $(d-1)$ connected and $k < pd$. Then $f \sim *$.*

Proof This follows from the results of Cohen, Moore and Neisendorfer [1] and uses the Hilton–Milnor Theorem. Details are in [3]. □

In order to apply this, we need to know the exponents of the torsion in the integral cohomology of T.

Lemma 2 *Let $\nu_p(m)$ be the number of powers of p in m. Then*

$$H^k(T) = \begin{cases} Z/p^{r+\nu_p(m)} & \text{if } k = 2mn, \\ 0 & \text{otherwise.} \end{cases}$$

Proof This is implicit in [3] and follows from the integral cohomology Serre spectral sequence for the fibration

$$S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$$

using the divided power relations in $H^*(\Omega S^{2n+1})$. □

We now apply the Lemma 1 with $a = r$, $b = r + k - 1$ and $W = G_{k-1} \wedge T^{2np^k}$. We write $W = W_1 \vee W_2$ and, by Lemma 2, we have

$$W_2 = P^{2np^{k-1}+1}(p^{r+k-1}) \wedge \Big(\bigvee_{i=1}^{p-1} P^{2np^{k-1}i}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k}) \Big)$$

and by Lemma 1(a), $\alpha_k \wedge 1$ factors through W_2.

Define $A = P^{2np^{k-1}+1}(p^{r+k-1}) \wedge P^{2np^{k-1}}(p^{r+k-1})$ and write $W_2 \simeq A \vee B$, where B is $6np^{k-1}-2$ connected. We now apply the splitting:

$$\Omega(A \vee B) \simeq \Omega A \times \Omega(B \rtimes \Omega A).$$

Since $B \rtimes \Omega A$ is $6np^{k-1}-2$ connected, the component of $\alpha_k \wedge 1$ in $B \rtimes \Omega A$ is null homotopic by part (b) of Lemma 1; this implies that $\alpha_k \wedge 1$ factors through A, ie

$$P^{2np^{k-1}+1}(p^{r+k-1}) \wedge P^{2np^{k-1}}(p^{r+k-1}) \\ \simeq P^{4np^{k-1}}(p^{r+k-1}) \vee P^{4np^{k-1}+1}(p^{r+k-1}).$$

Lemma 3 *Suppose $f\colon P^m(p^{s+1}) \to P^{2n}(p^s)$ has order p^{s+1}. Then $m \geqslant (4n-2)p$.*

Proof By [1], there is a decomposition

$$\Omega P^{2n}(p^s) \simeq S^{2n-1}\{p^s\} \times \Omega\Big(\bigvee_{k \geqslant 2} P^{(2n-2)k+3}(p^s) \Big).$$

Since the identity of $S^{2n-1}\{p^s\}$ has order p^s (see [4]), it follows that some component of the second factor has order p^{s+1}. By Lemma 1(b) we have $m \geqslant ((2n-2)k + 2)p$ for some $k \geqslant 2$. □

It now follows that $\alpha_k \wedge 1$ has no essential component in $P^{4np^{k-1}}(p^{r+k-1})$ and we conclude that $\alpha_k \wedge 1$ factors as

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha} P^{4np^{k-1}+1}(p^{r+k-1}) \xrightarrow{\beta} G_{k-1} \wedge T^{2np^k},$$

where β induces a monomorphism in mod p homology.

We will describe a map $\gamma\colon G_{k-1} \wedge T^{2np^k} \to P^{4np^{k-1}+1}(p^{r+k-1})$ with the property that $\gamma\beta$ is a homotopy equivalence and $\gamma(\alpha_k \wedge 1) \sim *$. This will complete the proof that $\alpha_k \wedge 1$ is null homotopic and $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$.

The map γ is the composition

$$G_{k-1} \wedge T^{2np^k} \longrightarrow G_{k-1} \wedge \Omega S^{2n-1} \xrightarrow{1 \wedge H_{p^{k-1}}} G_{k-1} \wedge \Omega S^{2np^{k-1}+1} \\ \xrightarrow{\xi} G_{k-1} \wedge S^{2np^{k-1}} \longrightarrow P^{4np^{k-1}+1}(p^{r+k-1}).$$

The first map in the composition comes from the projection in the fibration (1), and the second is the James Hopf invariant. We will describe ξ in the next proposition. The fourth map comes from the splitting of ΣG_{k-1} (from (B) and (D)).

Proposition 4 *Suppose G is a co–H space. Then the inclusion*

$$G \wedge X \longrightarrow G \wedge \Omega\Sigma X$$

has a left homotopy inverse ξ: $G \wedge \Omega\Sigma X \to G \wedge X$ and ξ commutes with co–H maps.

Proof Let ξ be the composition

$$G \wedge \Omega\Sigma X \xrightarrow{\nu\wedge 1} \Sigma\Omega G \wedge \Omega\Sigma X \simeq \Omega G \wedge \Sigma\Omega\Sigma X \xrightarrow{1\wedge\epsilon} \Omega G \wedge \Sigma X \xrightarrow{\epsilon\wedge 1} G \wedge X,$$

where ν is the co–H structure map and ϵ is an evaluation. Clearly ξ is a left homotopy inverse to the inclusion. If ϕ: $G \to H$ is a co–H map, there is a homotopy commutative square:

$$\begin{tikzcd} G \arrow[r, "\nu"] \arrow[d, "\phi"'] & \Sigma\Omega G \arrow[d, "\Sigma\Omega\phi"] \\ H \arrow[r, "\nu'"] & \Sigma\Omega H \end{tikzcd}$$

(see [2]), so the map ξ is natural for co–H maps. □

We now show that γ induces an isomorphism in $\bmod p$ homology in dimension $4np^{k-1}+1$. We first note that

$$H_{4np^{k-1}+1}(G_{k-1} \wedge T^{2np^k}) \cong H_{2np^{k-1}+1}(G_{k-1}) \otimes H_{2np^k}(T^{2np^k}) \cong Z/p.$$

From the description of γ and the fact that ξ_* is an epimorphism, it follows that γ induces an isomorphism in this dimension. Since β induces a monomorphism, $\gamma\beta$ induces an isomorphism in dimension $4np^{k-1}+1$. This implies that $\gamma\beta$ is a homotopy equivalence.

It suffices, then, to show that $\gamma(\alpha_k \wedge 1)$ is null homotopic. We appeal to the construction of γ. We will show that the following diagram is homotopy commutative:

$$\begin{tikzcd} P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \arrow[r] \arrow[d, "\alpha_k\wedge 1"'] & P^{2np^k} \wedge \Omega S^{2n+1} \arrow[r] \arrow[d, "\alpha_k\wedge 1"'] & P^{2np^k} \wedge \Omega S^{2np^{k-1}+1} \arrow[r, "\xi"] \arrow[d, "\alpha_k\wedge 1"'] & P^{2np^k} \wedge S^{2np^{k-1}} \arrow[d, "\alpha_k\wedge 1"'] \\ G_{k-1} \wedge T^{2np^k} \arrow[r] & G_{k-1} \wedge \Omega S^{2n+1} \arrow[r] & G_{k-1} \wedge \Omega S^{2np^{k-1}+1} \arrow[r, "\xi"] & G_{k-1} \wedge S^{2np^{k-1}} \end{tikzcd}$$

The composition of the bottom row with the splitting

$$G_{k-1} \wedge S^{2np^{k-1}} \longrightarrow P^{4np^{k-1}+1}(p^{r+k})$$

is the map γ, and the right hand vertical arrow is a suspension of α_k. By (B) and (D) in case $k-1$, ΣG_{k-1} is a retract of ΣG_k so $\Sigma\alpha_k$ is null homotopic. Consequently it suffices to show that the diagram is homotopy commutative. The only issue is resolved by:

Proposition 5 *α_k is a co–H map.*

The proof of this result relies on:

Lemma 6 $\Omega G_{k-1} * \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$

Proof We use (A) in case $k-1$ to see that the space $\Omega G_{k-1} * \Omega G_{k-1}$ is a retract of $\Omega\Sigma T^{2np^{k-1}} * \Omega\Sigma T^{2np^{k-1}}$. Using the James splitting of $\Sigma\Omega\Sigma X$, we have for any X

$$\Omega\Sigma X * \Omega\Sigma X \simeq \bigvee_{\substack{i\geqslant 1\\ j\geqslant 1}} \Sigma X^{(i)} \wedge X^{(j)},$$

so it suffices to show that $\Sigma T^{2np^{k-1}} \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$. However by (A) in case $k-1$,

$$\begin{aligned}\Sigma T^{2np^{k-1}} \wedge T^{2np^{k-1}} &\simeq (G_{k-1} \vee W_{k-1}) \wedge T^{2np^{k-1}}\\ &\simeq G_{k-1} \wedge T^{2np^{k-1}} \vee W_{k-1} \wedge T^{2np^{k-1}},\end{aligned}$$

which is in $\mathcal{W}_r^{r+k-1}$ by 4.3(l) and (D) in case $k-1$. □

Proof of Proposition 5 It is required to show that there is a homotopy commutative diagram:

$$\begin{tikzcd} P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k"] \arrow[d, "\nu'"'] & G_{k-1} \arrow[d, "\nu_{k-1}"'] \\ P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k}) \arrow[r, "\alpha_k\vee\alpha_k"] & G_{k-1} \vee G_{k-1} \end{tikzcd}$$

Let Δ: $P^{2np^k}(p^{r+k}) \to G_{k-1} \vee G_{k-1}$ be the difference between the two sides. Since ν' is a suspension, Δ is divisible by p^{r+k-1}. The composition

$$p^{2np^k}(p^{r+k}) \xrightarrow{\Delta} G_{k-1} \vee G_{k-1} \longrightarrow G_{k-1} \times G_{k-1}$$

is null homotopic, since each component is $\alpha_k - \alpha_k$. However there is a splitting [2]

$$\Omega(G_{k-1} \vee G_{k-1}) \simeq \Omega(G_{k-1} \times G_{k-1}) \times \Omega(\Omega G_{k-1} * \Omega G_{k-1})$$

so Δ factors through $\Omega G_{k-1} * \Omega G_{k-1}$ and is divisible by p^{r+k-1} in this space:

$$P^{2np^k}(p^{r+k}) \xrightarrow{p^{r+k-1}} P^{2np^k}(p^{r+k-1}) \longrightarrow \Omega G_{k-1} * \Omega G_{k-1} \longrightarrow G_{k-1} \vee G_{k-1}$$

Since $\Omega G_{k-1} * \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$ by Lemma 6, we can apply Lemma 1 with $a = r$ and $b = r + k - 1$. In this case

$$W_2 = \Omega P^{2np^{k-1}+1}(p^{r+k-1}) * \Omega P^{2np^{k-1}+1}(p^{r+k-1}),$$

which is $4np^{k-1}-2$ connected. Since $2np^k < p(4np^{k-1}-1)$, Δ is null homotopic. □

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