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Geometry & Topology 17 (2013) 2689–2729

Uniqueness of Lagrangian self-expanders

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We show that zero-Maslov class Lagrangian self-expanders in $\mathbb{C}^n$ that are asymptotic to a pair of planes intersecting transversely are locally unique if $n > 2$ and unique if $n = 2$.

53D12; 53C44

1 Introduction

Self-similar solutions to mean curvature flow model the flow behaviour near a singularity. If the initial condition for the flow is a zero-Maslov class Lagrangian in $\mathbb{C}^n$, it is well known (Neves [17, Corollary 3.5]) that self-shrinkers are trivial (ie, stationary solutions) and so the ones left to study are self-expanders. These are Lagrangians $L \subset \mathbb{C}^n$ so that $L_t = \sqrt{2t}\, L$ is a solution to mean curvature flow.

Moreover, it is shown in Neves and Tian [19] that blow-downs of eternal solutions to Lagrangian mean curvature flow (like translating solutions for instance) are self-expanders for positive time. Thus if one wants to understand whether or not non-trivial translating solutions can occur as blow-ups of finite time singularities of Lagrangian mean curvature flow, it is important that we understand self-expanders first.

Another related perspective on self-expanders is that they are the simplest solutions to mean curvature flow that start on cones and hence could be seen as models for starting the flow with singular initial condition.

The first examples of Lagrangian self-expanders were constructed in Anciaux [1] and Lee and Wang [13; 14]. In [10], Joyce, Lee and Tsui generalized these constructions and in particular they found, for any two Lagrangian planes $P_1, P_2 \subset \mathbb{C}^n$ satisfying an angle criterion, explicit examples of zero-Maslov class Lagrangians asymptotic to these planes. They are diffeomorphic to $S^{n-1} \times \mathbb{R}$ and can be seen as the equivalent of Lawlor necks for the self-expander equation. The construction in [10] is quite general and provides examples that are asymptotic to non-stationary cones and examples that have Maslov class. Further examples were constructed in Castro and Lerma [3].

Published: 30 September 2013 DOI: 10.2140/gt.2013.17.2689

Given a Lagrangian cone in $\mathbb{C}^n$ that is graphical over a real plane and such that the Hessian of the potential function has eigenvalues uniformly in $(-1, 1)$, Chau, Chen and He [4] showed there is a unique graphical Lagrangian self-expander asymptotic to that cone.

Let $P_1, P_2 \subset \mathbb{C}^n$ be two Lagrangian planes intersecting transversely, denote the space of bounded smooth functions with compact support by $C_0^\infty(\mathbb{C}^n)$ and let $\mathcal{H}^n$ be n–dimensional Hausdorff measure. We let $P_1 + P_2$ denote the varifold whose support is given by the union of the planes $P_1 \cup P_2$.

Definition 1.1 We say the self-expander L is asymptotic to $L_0 = P_1 + P_2$ if

$$\lim_{t\to 0} \int_{\sqrt{2t}\,L} \phi \, \mathrm{d}\mathcal{H}^n = \int_{L_0} \phi \, \mathrm{d}\mathcal{H}^n$$

for all $\phi \in C_0^\infty(\mathbb{C}^n)$.

In this paper we first show local uniqueness.

Theorem A *Assume that neither $P_1 + P_2$ nor $P_1 - P_2$ are area-minimizing.*

Let L be a smooth zero-Maslov class Lagrangian self-expander in $\mathbb{C}^n$ asymptotic to $P_1 + P_2$.

There exist $R_0 > 0$ and $\varepsilon > 0$ so that any smooth zero-Maslov class Lagrangian self-expander that is

- *asymptotic to $P_1 + P_2$,*
- *ε–close in C^2 to L in B_{R_0},*

coincides with L.

The idea for the proof of Theorem A is classical. We show that the linearization of the self-expander equation defines a Banach space isomorphism and then we apply the Inverse Function Theorem to obtain local uniqueness.

When $n = 2$ we improve this result and show global uniqueness.

Theorem B *Assume that neither $P_1 + P_2$ nor $P_1 - P_2$ are area-minimizing.*

Smooth zero-Maslov class Lagrangian self-expanders asymptotic to $P_1 + P_2$ are unique and thus coincide with one of the examples found by Joyce, Lee and Tsui.

Remark 1.2 It is known that special Lagrangians in $\mathbb{C}^2$ that are asymptotic to a pair of planes are unique modulo scaling and rigid motions. This uses the fact that, after a hyperkähler rotation of the complex structure, special Lagrangian surfaces become holomorphic curves. No similar characterization holds for Lagrangian self-expanders, and hence the need for a different idea in Theorem B.

Moreover, without the smoothness assumption the uniqueness statement does not hold, as can be seen in Nakahara [15].

We now briefly describe the idea behind the proof of Theorem B.

The key result is to show Theorem 6.1, which says that the set of self-expanders in $\mathbb{C}^2$ that are asymptotic to a pair of transverse planes is compact. Assuming this result, the idea, given a pair of planes P_1, P_2 as in Theorem B, is to deform P_2 into a plane Q_2 so that the Lagrangian angle remains constant and P_1, Q_2 become *equivariant*, ie, share the same S^1–symmetry. From Theorem A we can accompany the deformation of the planes with a (local) deformation of any self-expander L asymptotic to $P_1 + P_2$. Theorem 6.1 ensures that this local deformation can be carried all the way until we obtain a self-expander Q asymptotic to $P_1 + Q_2$. Since P_1, Q_2 are equivariant, it is simple to show that Q is unique (see Lemma 7.1) and hence L must have been unique as well.

Roughly speaking, the proof of Theorem 6.1 rests on the fact that every non-trivial special Lagrangian cone in $\mathbb{C}^2$ has area-ratios not smaller than two, but the area-ratios of any self-expander as in Theorem B are strictly smaller than two, ie, the area-ratios are too small for a singularity to develop.

Organization In Section 2 we introduce the basic concepts.

In Section 3 we show that zero-Maslov class self-expanders asymptotic to a transverse intersection of planes have exponential decay outside a compact set.

In Section 4 we develop the Fredholm theory for the linearization of the self-expander equation.

In Section 5 we show that zero-Maslov class self-expanders in $\mathbb{C}^n$ that are asymptotic to a transverse intersection of a non-area-minimizing pair of planes are locally unique. This implies Theorem A.

In Section 6 we show that, given a compact set of transversely intersecting non-area-minimizing pairs of planes in $\mathbb{C}^2$, the family of zero-Maslov class self-expanders in $\mathbb{C}^2$ asymptotic to those pairs of planes is also compact.

In Section 7 we use the work of the previous section and Section 4 to show global uniqueness for zero-Maslov class self-expanders in $\mathbb{C}^2$ that are asymptotic to a transverse intersection of a non-area-minimizing pair of planes. This proves Theorem B.

Acknowledgements

Both authors would like to thank Dominic Joyce for comments on an earlier version of this paper and the referee for the extensive comments that helped improve the exposition of the paper.

The first author was supported by an EPSRC Career Acceleration Fellowship. The second author was partly supported by Marie Curie IRG Grant and ERC Start Grant.

2 Basic theory and notation

Consider $\mathbb{C}^n$ with its usual complex coordinates $z_j = x_j + iy_j$, for $j = 1, \dots, n$, complex structure J, Kähler form $\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$ and holomorphic volume form $\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$. Observe that the Liouville form $\lambda = \sum_{j=1}^n (x_j \mathrm{d}y_j - y_j \mathrm{d}x_j)$ satisfies $\mathrm{d}\lambda = 2\omega$.

Let L be a connected *Lagrangian* in $\mathbb{C}^n$; that is, L is a (real) n–dimensional submanifold of $\mathbb{C}^n$ such that $\omega|_L \equiv 0$. Let $\boldsymbol{x}$ denote the position vector on L, let ∇ be the (induced) Levi-Civita connection on L and let H be the mean curvature vector on L. We adopt the convention that H is the trace of the second fundamental form on L. Standard Euclidean differentiation is denoted by $\overline{\nabla}$.

Notice that λ is trivially a closed 1–form on L. We say that L is *exact* if there exists $\beta \in C^\infty(L)$ such that $\mathrm{d}\beta = \lambda|_L$.

Since $\Omega|_L$ is a unit complex multiple of the volume form at each point on L, we may define the *Lagrangian angle* θ on L by the formula $\Omega|_L = e^{i\theta}\mathrm{vol}_L$. We also have the relation $H = J\nabla\theta$ (cf Thomas and Yau [22, Lemma 2.1]). The Maslov class on L is defined by the cohomology class of $\mathrm{d}\theta$, so L has *zero-Maslov class* if θ is a single-valued function.

Observe that, since $T\mathbb{C}^n|_L = TL \oplus NL$, we may decompose any vector $\boldsymbol{v}$ on L into tangential and normal components, denoted $\boldsymbol{v}^\top$ and $\boldsymbol{v}^\perp$ respectively.

Definition 2.1 We say that L is a *self-expander* if $H = \kappa \boldsymbol{x}^\perp$ for some $\kappa > 0$. By rescaling L we may assume that $\kappa = 1$.

The importance of self-expanders L with $H = \boldsymbol{x}^{\perp}$ is that $L_t = \sqrt{2t}\, L$ for $t > 0$ solves mean curvature flow.

We have the following basic properties of self-expanders.

Lemma 2.2

(i) *Lagrangian self-expanders with zero-Maslov class are exact.*

(ii) *Let L be a zero-Maslov class self-expander. Then L is a self-expander with $H = \boldsymbol{x}^{\perp}$ if and only if $\beta + \theta$ is constant.*

Proof Let L satisfy $H = \boldsymbol{x}^{\perp}$. Since $H = J\nabla\theta$, one sees that

$$\nabla\theta = -J(\boldsymbol{x}^{\perp}) = -(J\boldsymbol{x})^{\top},$$

so $\lambda|_L = -\mathrm{d}\theta$. This proves the first property.

To prove the second property note that

$$H - \boldsymbol{x}^{\perp} = 0 \Longleftrightarrow \nabla\theta + J\boldsymbol{x}^{\perp} = 0 \Longleftrightarrow \nabla\theta + (J\boldsymbol{x})^{\top} = 0 \Longleftrightarrow \nabla(\theta + \beta) = 0. \quad \square$$

Let P_1, P_2 be two Lagrangian planes intersecting transversely. From Ilmanen [7, Section 7.2], there exists a constant C_0 such that whenever a self-expander L is asymptotic to $P_1 + P_2$ then

$$\mathcal{H}^n\big(L \cap B_R\big) \leq C_0 R^n \quad \text{for all } R > 0, \tag{1}$$

where B_R will always denote $B_R(0)$, the ball of radius R about 0 in $\mathbb{C}^n$.

In this paper, all self-expanders L we consider have the following properties:

- L is Lagrangian with zero-Maslov class.
- L has $H = \mathbf{x}^{\perp}$.
- L is asymptotic to $L_0 = P_1 + P_2$, where P_1, P_2 are transversely intersecting Lagrangian planes.

We abuse notation and often identity the varifold $L_0 = P_1 + P_2$ with its support $L_0 = P_1 \cup P_2$.

A key tool in studying self-expanders is the backwards heat kernel.

Definition 2.3 Given any (x_0, l) in $\mathbb{C}^n \times \mathbb{R}$, we consider the backwards heat kernel

$$\Phi(x_0, l)(x, t) = \frac{\exp\left(-\dfrac{|x - x_0|^2}{4(l-t)}\right)}{(4\pi(l-t))^{n/2}}. \tag{2}$$

Given a solution $(L_t)_{t>0}$ to mean curvature flow and $x_0 \in \mathbb{C}^n$, $l > 0$, we consider

$$\Theta_t(x_0, l) = \int_{L_t} \Phi(x_0, l)(x, 0)\, \mathrm{d}\mathcal{H}^n. \tag{3}$$

Note that when $L_t = \sqrt{2t}\, L$, where L is a self-expander, we have that $\Theta_t(x_0, l)$ is finite for $0 < t < l$ due to (1) (see Ecker [5, Lemma C.3]).

Definition 1.1 implies that for all $x_0 \in \mathbb{C}^n$ and $l > 0$

$$\lim_{t\to 0} \Theta_t(x_0, l) = \int_{L_0} \Phi(x_0, l)\, \mathrm{d}\mathcal{H}^n = \Theta_0(x_0, l). \tag{4}$$

Moreover, we have from Huisken's monotonicity formula [6] that

$$\Theta_t(x_0, l) \leq \Theta_0(x_0, l + t) \quad \text{for all } x_0 \in \mathbb{C}^n, t > 0, l > 0. \tag{5}$$

We conclude this section with the following observation. Given P_1, P_2 transverse planes we have

$$\Theta_0(x_0, l) = \int_{P_1+P_2} \Phi(x_0, l)\, \mathrm{d}\mathcal{H}^n < 2 \quad \text{for all } l > 0 \tag{6}$$

unless $x_0 = 0$. One consequence of this observation is the following:

Lemma 2.4

- *The self-expander L is embedded.*
- *There is c_1 depending only on L_0 so that*

$$\mathcal{H}^n\big(L \cap B_r(x)\big) \leq c_1 r^n \quad \text{for all } r > 0 \text{ and } x \in \mathbb{C}^n.$$

Proof Suppose that L is immersed. Then there exists $x_0 \in L$ where

$$\lim_{\delta\to 0} \Theta_{\frac{1}{2}}(x_0, \delta) \geq 2.$$

By (5), making $t = \frac{1}{2}$ and $l \to 0$, we obtain

$$2 \leq \Theta_0(x_0, \tfrac{1}{2})$$

and so, by (6), $x_0 = 0$ and equality holds in the equation above. In this case,

$$2 \leq \lim_{\delta\to 0} \Theta_{\frac{1}{2}}(0, \delta) \leq \Theta_0(x_0, \tfrac{1}{2}) = 2$$

and so equality holds in Huisken's monotonicity formula. Hence L is also a self-shrinker (ie, $H = -\boldsymbol{x}^\perp$) and thus $H = \boldsymbol{x}^\perp = 0$ as L is a self-expander. Therefore, L must be a cone but, because L is asymptotic to L_0, this is only possible if $L = L_0$, which then contradicts the assumption that L is smooth. This proves the first property.

In what follows c denotes a constant depending only on n. By (5) and (6)

$$\frac{\mathcal{H}^n\big(L \cap B_r(x)\big)}{r^n} \leq c\Theta_{1/2}(x, r^2) \leq c\Theta_0(x, r^2 + \tfrac{1}{2}) \leq 2c. \qquad \square$$

3 Exponential decay

In this section we show that the self-expander L converges exponentially fast to L_0 outside a compact set. This naturally coincides with the behaviour of the relevant self-expanders in [10], but is in marked contrast to special Lagrangian Lawlor necks, which only converge at rate $O(r^{1-n})$ to their asymptotic planes.

Let $G_L(n, \mathbb{C}^n)$ denote the set of all Lagrangian planes in $\mathbb{C}^n$, which has a natural topology when viewed as the homogeneous space $U(n)/O(n)$. Consider the open subset of $G_L(n, \mathbb{C}^n) \times G_L(n, \mathbb{C}^n)$ given by

$$G_n = \{(P_1, P_2) \in G_L(n, \mathbb{C}^n) \times G_L(n, \mathbb{C}^n) \mid P_1 \cap P_2 = \{0\}\}.$$

Given a compact set $K \subset G_n$, we denote by $\mathcal{S}(K)$ the set of all self-expanders which are asymptotic to $L_0 = P_1 + P_2$, with $(P_1, P_2) \in K$.

Theorem 3.1 *For every compact set $K \subset G_n$ and $k \in \mathbb{N}$, there is $R_1 > 0$, C and b so that for all $L \in \mathcal{S}(K)$ we find $\psi \in C^\infty(L_0 \setminus B_{R_1})$ satisfying*

$$L \setminus B_{2R_1} \subset \{x + J\,\overline{\nabla}\psi(x) \mid x \in L_0 \setminus B_{R_1}\} \subset L \setminus B_{R_1/2}$$

and

$$\|\psi\|_{C^k(L_0 \setminus B_R)} \leq Ce^{-bR^2} \quad \text{for all } R \geq R_1.$$

Proof The next proposition says that if L is locally graphical over $L_0 \setminus B_{R_0}$ for some R_0 and the local graph is asymptotic to zero in the C^{k+1}–norm, then we can find R_1 large so that L is a global graph over $L_0 \setminus B_{R_1}$ and the graph has its C^k–norm decaying exponentially fast.

Proposition 3.2 *Fix $R_0 > 2r_0 > 0$, $k \in \mathbb{N}$, a compact set $K \subset G_n$, and a decreasing function $D_k = D_k(r)$ tending to zero at infinity.*

Suppose that for every $L \in \mathcal{S}(K)$ and $y_0 \in L \setminus B_{R_0}$ we can find $x_0 \in L_0$ and

$$\phi\colon L_0 \cap B_{2r_0}(x_0) \to \mathbb{C}^n$$

so that

- *the $C^{k+1}\big(B_{2r_0}(x_0)\big)$–norm of ϕ is bounded by $D_k(|y_0|)$,*
- *$L \cap \widehat{B}_{r_0}(y_0) \subset \{x + J\,\overline{\nabla}\phi(x) \mid x \in L_0 \cap B_{2r_0}(x_0)\}$, where $\widehat{B}_{r_0}(y_0)$ denotes the intrinsic ball of L with radius r_0 and centered at y_0.*

Then there exist R_1, C, b, and an open set $B \subset \mathbb{C}^n$ with compact closure, depending on r_0, R_0, D_k and K, such that for every $L \in \mathcal{S}(K)$ we can find $\psi \in C^\infty(L_0 \setminus B_{R_1})$ with

$$L \setminus B \subset \{x + J\,\overline{\nabla}\psi(x) \mid x \in L_0 \setminus B_{R_1}\} \tag{7}$$

and

$$\|\psi\|_{C^k(L_0 \setminus B_R)} \leq Ce^{-bR^2} \quad \text{for all } R \geq R_1. \tag{8}$$

Proof From the hypotheses of the proposition, for every $L \in \mathcal{S}(K)$ and $\widetilde{R}_1$ sufficiently large we can find an open set $\widetilde{B} \subset \mathbb{C}^n$ with compact closure and a projection map

$$\pi_L \colon L \setminus \widetilde{B} \longrightarrow L_0 \setminus B_{\widetilde{R}_1}.$$

We claim we can choose $R_1 = R_1(r_0, R_0, C_k)$ so that π_L is a diffeomorphism when restricted to $\pi_L^{-1}(L_0 \setminus B_{R_1})$.

Suppose not. Then we can find $L^i \in \mathcal{S}(K)$ with $L_0^i = P_1^i + P_2^i$ tending to $P_1 + P_2$, x_i in $L_0^i \setminus B_{R_0}$ such that $|x_i| \to \infty$ as $i \to \infty$, and $\pi_{L^i}^{-1}(x_i) \supset \{y_i, z_i\}$ where $y_i \neq z_i$ for all i.

By hypothesis, there exists $\delta_0 > 0$ such that $L^i \cap B_{2\delta_0}(x_i)$ contains the graphs of functions f_i, g_i on $B_{\delta_0}(x_i) \cap L_0^i$ with $f_i(x_i) = y_i$ and $g_i(x_i) = z_i$ for all i. Therefore, recalling Φ given in (2), Huisken's monotonicity formula implies

$$\begin{aligned} \int_{L_0^i} \Phi(x_i, \delta^2 + \tfrac{1}{2})\,\mathrm{d}\mathcal{H}^n &\geq \int_{L^i} \Phi(x_i, \delta^2)\,\mathrm{d}\mathcal{H}^n \\ &\geq \int_{L^i \cap B_{2\delta_0}(x_i)} \Phi(x_i, \delta^2)\,\mathrm{d}\mathcal{H}^n \\ &\geq \int_{\mathrm{graph}(f_i) \cup \mathrm{graph}(g_i)} \Phi(x_i, \delta^2)\,\mathrm{d}\mathcal{H}^n \\ &\geq 2 \int_{B_{\delta_0}(x_i) \cap L_0^i} \Phi(x_i, \delta^2)\,\mathrm{d}\mathcal{H}^n, \end{aligned} \tag{9}$$

where the last inequality comes from the fact that L_0^i is a union of planes.

Since L_0^i tends to $P_1 + P_2$ with $P_1 \cap P_2 = \{0\}$ and $|x_i| \to \infty$, we have from the hypothesis of the proposition that $L_0^i - x_i$ tends to either P_1 or P_2. We assume that the first case occurs and so, after translating by $-x_i$, $B_{\delta_0}(x_i) \cap L_0^i$ tends to $B_{\delta_0}(0) \cap P_1$. Thus, letting $i \to \infty$ and then $\delta \to 0$, we obtain from (9) that

$$1 = \int_{P_1} \Phi(0, \tfrac{1}{2})\,\mathrm{d}\mathcal{H}^n \geq 2.$$

This proves the claim.

The fact that π_L is a diffeomorphism implies the existence of a smooth vector field X on $L_0 \setminus B_{R_1}$ that is tangent to $L_0 \setminus B_{R_1}$ so that, for some open set B,

$$L \setminus B = \{x + JX(x) : x \in L_0 \setminus B_{R_1}\}.$$

Since L is Lagrangian, it is standard to see that $X = \overline{\nabla}\psi$ locally. We need to make sure that ψ is defined globally on $L_0 \setminus B_{R_1}$. Recall the primitive β for the Liouville form λ on L. Set $\bar{\beta} = \beta \circ \pi_L^{-1}$ and define ψ on $L_0 \setminus B_{R_1}$ by

$$\psi(x) = \tfrac{1}{2}\big(\langle X(x), x\rangle - \bar{\beta}(x)\big),$$

using the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Then, for every vector v tangent to L_0 we have

$$\begin{aligned}\langle \overline{\nabla}\psi(x), v\rangle &= \tfrac{1}{2}\big(\langle \overline{\nabla}_v X, x\rangle + \langle X(x), v\rangle - \langle \overline{\nabla}\bar{\beta}(x), v\rangle\big)\\ &= \tfrac{1}{2}\big(\langle \overline{\nabla}_v X, x\rangle + \langle X(x), v\rangle - \lambda(v + J\,\overline{\nabla}_v X(x))\big)\end{aligned}$$

since $\overline{\nabla}\bar{\beta}(x) = \overline{\nabla}\beta(x + JX(x))$ and $\mathrm{d}\beta = \lambda|_L$, and hence

$$\begin{aligned}\langle \overline{\nabla}\psi(x), v\rangle &= \tfrac{1}{2}\big(\langle \overline{\nabla}_v X, x\rangle + \langle X(x), v\rangle - \langle Jx - X(x), v + J\,\overline{\nabla}_v X(x)\rangle\big)\\ &= \langle X(x), v\rangle - \tfrac{1}{2}\langle Jx, v\rangle + \tfrac{1}{2}\langle X(x), J\,\overline{\nabla}_v X(x)\rangle.\end{aligned}$$

Observe that $\langle Jx, v\rangle = 0$ since L_0 is a pair of Lagrangian planes and thus x is tangent to L_0 and orthogonal to Jv. Furthermore, $\langle X(x), J\,\overline{\nabla}_v X(x)\rangle = 0$ because $\overline{\nabla}_v X$ has no component orthogonal to L_0 as L_0 is a union of planes. Thus $\overline{\nabla}\psi = X$, the decomposition (7) holds, and for $R \geq R_1$,

$$\|\nabla\psi\|_{C^{k+1}(L_0 \setminus B_R)} \leq D_k(R-1), \tag{10}$$

where D_k is our given decreasing function by hypothesis.

We now wish to prove (8). Let S_0 be a connected component of $L_0 \setminus \{0\}$, ie, S_0 is a Lagrangian plane minus the origin, and let S be the connected component of $L \setminus B_{R_1}$ asymptotic to S_0 given by the graphical decomposition in (7). After changing coordinates so that S_0 is identified with a real plane, we consider the following vector-valued function defined on S:

$$\boldsymbol{y} = (J\,\overline{\nabla}\psi) \circ \pi_L = i \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} \in \mathbb{C}^n.$$

Let $\Delta = -\mathrm{d}^*\mathrm{d}$ be the analyst's Laplacian (or Laplace–Beltrami operator) acting on functions on L. For $j = 1, \dots, n$ we have on S that

$$\Delta y_j^2 = 2y_j \left\langle i\frac{\partial}{\partial y_j}, H \right\rangle + 2|\partial_{y_j}^{\top}|^2 \Longrightarrow \Delta|\boldsymbol{y}|^2 \geq 2\langle \boldsymbol{y}, H\rangle.$$

Calculating with respect to the induced metric on S and recalling that $\boldsymbol{x}$ is the position vector on L, we see that, since L is a self-expander,

$$|\boldsymbol{y}|^2 = \langle \boldsymbol{y}, \boldsymbol{x}\rangle = \langle \boldsymbol{y}, \boldsymbol{x}^{\perp}\rangle + \langle \boldsymbol{y}, \boldsymbol{x}^{\top}\rangle = \langle \boldsymbol{y}, H\rangle + \langle \boldsymbol{y}, \boldsymbol{x}^{\top}\rangle = \langle \boldsymbol{y}, H\rangle + \tfrac{1}{2}\langle \nabla|\boldsymbol{y}|^2, \boldsymbol{x}\rangle.$$

Thus, if we define

$$\mathcal{L}(\phi) = \Delta\phi + \langle \boldsymbol{x}, \nabla\phi\rangle - 2\phi \tag{11}$$

for suitably differentiable functions ϕ on S, where all quantities are computed with respect to the induced metric on S, we see that

$$\mathcal{L}(|\boldsymbol{y}|^2) \geq 0.$$

We are now in the position to construct a barrier for $|\boldsymbol{y}|^2$ and deduce (8).

Set $\rho(x) = \exp(-|x|^2/2)$, an ambient function on $\mathbb{C}^n$. Since L is a self-expander, we have that

$$\Delta(|x|^2) = 2n + 2\langle H, \boldsymbol{x}\rangle = 2n + 2|\boldsymbol{x}^{\perp}|^2.$$

Therefore one sees that, on L,

$$\Delta\rho = -\frac{\rho}{2}\Delta(|x|^2) + \frac{\rho}{4}|\nabla|x|^2|^2 = \rho(|\boldsymbol{x}^{\top}|^2 - n - |\boldsymbol{x}^{\perp}|^2).$$

Moreover,

$$\langle \nabla\rho, \boldsymbol{x}\rangle = -\rho|\boldsymbol{x}^{\top}|^2$$

and so

$$\mathcal{L}(\rho) = \rho(|\boldsymbol{x}^{\top}|^2 - n - 2 - |\boldsymbol{x}|^2) \leq -(n+2)\rho. \tag{12}$$

Let $\varepsilon > 0$ and using (10) choose $C = C(r_0, R_0, D_k)$ so that

$$|\boldsymbol{y}|^2 = |(\overline{\nabla}\psi)\circ\pi_L|^2 < C\exp(-|x|^2/2) \quad \text{on } \partial S.$$

Set $\widetilde{\rho} = \varepsilon + C\rho$. For all R sufficiently large we have $|\boldsymbol{y}|^2 < \widetilde{\rho}$ on $S \cap \partial\{R_1 < |x| < R\}$ because $|\boldsymbol{y}|^2$ tends to zero at infinity by hypothesis. Furthermore, using $\mathcal{L}(|\boldsymbol{y}|^2) \geq 0$ and (12), we have

$$\mathcal{L}(\widetilde{\rho} - |\boldsymbol{y}|^2) = \mathcal{L}(\rho - |\boldsymbol{y}|^2) - 2\varepsilon < 0$$

and thus the maximum principle implies that

$$|\boldsymbol{y}|^2 < \varepsilon + C\exp(-|x|^2/2) \quad \text{on } S \cap \{R_1 < |x| < R\}.$$

Letting $R \to \infty$ and then $\varepsilon \to 0$, we conclude that

$$|\boldsymbol{y}|^2 \le C\exp(-|x|^2/2) \quad \text{on } S.$$

Recall that on S we have $\boldsymbol{y} = (J\overline{\nabla}\psi)\circ\pi_L$. Therefore we can add a constant to ψ and find some other constant $C = C(r_0, R_0, D_k)$ so that, after integration,

$$|\psi(x)| \le C\exp\big(-\tfrac{1}{4}|x|^2\big) \quad \text{for all } x \in S_0 \setminus B_{R_1}.$$

As the C^{k+1} norm of ψ in $S_0 \setminus B_{R_1}$ is bounded, it follows from standard interpolation inequalities for Hölder spaces (see eg Krylov [11, Theorem 3.2.1]) that, for some further constant $C = C(r_0, R_0, D_k)$,

$$\|\psi\|_{C^k(L_0\setminus B_R)} \le C\exp(-aR^2)$$

for some constant $a > 0$ and any $R \ge R_1$.

We can argue in the same manner for each connected component of $L_0 \setminus B_{R_1}$ and conclude the desired result. □

We now make an observation concerning Θ given in (3). Given $L^i \in \mathcal{S}(K)$, we denote by $\Theta^i_t(x_0, l)$ the Gaussian density ratios (3) evaluated at L^i_t.

Lemma 3.3 *Let L^i be a sequence in $\mathcal{S}(K)$ and $(x_i)_{i\in\mathbb{N}}$ a sequence of points in $\mathbb{C}^n$ with $|x_i|$ tending to infinity. Then, for all $l > 0$,*

$$\lim_{i\to\infty} \Theta^i_0(x_i, l) \le 1$$

with equality only if $\lim_{i\to\infty} \operatorname{dist}(x_i, L^i_0) = 0$.

Proof We have $L^i_0 = P^i_1 + P^i_2$ and write $x_i = a^1_i + b^1_i = a^2_i + b^2_i$, where $a^j_i \in P^i_j$ and $\langle a^j_i, b^j_i\rangle = 0$ for $j = 1, 2$. We set $Q_i = L^i_0 - x_i$, where we mean that we translate L^i_0 by the vector given by x_i. We have

$$\min\{|b^1_i|, |b^2_i|\} = \operatorname{dist}(x_i, L^i_0) = \operatorname{dist}(0, Q_i). \tag{13}$$

Suppose first that $\limsup_{i\to\infty} |b^1_i| = \limsup_{i\to\infty} |b^2_i| = +\infty$. We then have $\operatorname{dist}(0, Q_i)$ tending to infinity and so

$$\lim_{i\to\infty} \Theta_0(x_i, l) = \lim_{i\to\infty} \int_{Q_i} \Phi(0, l)\, \mathrm{d}\mathcal{H}^n = 0.$$

Otherwise, without loss of generality, $\limsup_{i\to\infty}|b_i^1|<\infty$, and necessarily $\liminf_{i\to\infty}|a_i^1|=+\infty$. Note that we must also have $\liminf_{i\to\infty}|b_i^2|=+\infty$ because otherwise we could extract a subsequence of (P_1^i, P_2^i) converging to a pair of planes intersecting along a line.

Therefore Q_i sequentially converges to P_1+b, ie, an affine plane parallel to some plane P_1, where $b=\lim_{i\to\infty} b_i^1$ is orthogonal to P_1. Thus

$$\begin{aligned}\lim_{i\to\infty}\Theta_0^i(x_i,l)&=\int_{P_1+b}\Phi(0,l)\,\mathrm{d}\mathcal{H}^n=\int_{P_1}\Phi(-b,l)\,\mathrm{d}\mathcal{H}^n\\&=\exp\left(-\frac{|b|^2}{4l}\right)\int_{P_1}\Phi(0,l)\,\mathrm{d}\mathcal{H}^n=\exp\left(-\frac{|b|^2}{4l}\right)\le 1,\end{aligned}$$

with equality only if $b=0$. This proves the desired result. □

We can now finish the proof of Theorem 3.1. The idea is to show that the hypotheses of Proposition 3.2 are satisfied for all $L\in\mathcal{S}(K)$.

First we claim that

$$\lim_{R\to\infty}\sup\{\operatorname{dist}(x,L_0)\mid L\in\mathcal{S}(K),\, x\in L\setminus B_R\}=0. \tag{14}$$

Indeed, if we choose any sequence $L^i\in\mathcal{S}(K)$ and pick $x_i\in L^i$ with $|x_i|$ tending to infinity, we have from (5) that

$$1=\lim_{r\to 0}\Theta^i_{1/2}(x_i,r)\le\lim_{r\to 0}\Theta^i_0(x_i,r+\tfrac12)=\Theta^i_0(x_i,\tfrac12)$$

and thus

$$\lim_{i\to\infty}\Theta^i_0(x_i,\tfrac12)\ge 1.$$

The claim now follows from Lemma 3.3.

Second we claim the existence of $R_1>0$ so that the $C^{2,\alpha}$ norm of $L\setminus B_{R_1}$ is uniformly bounded for all $L\in\mathcal{S}(K)$. Indeed, with $\varepsilon_0>0$ fixed, we obtain from Lemma 3.3 the existence of R_1 so that for all $L\in\mathcal{S}(K)$ we have

$$\Theta_0(x,2)\le 1+\varepsilon_0\quad\text{for all } |x|\ge R_1/2$$

and so, for all $t,l\in[0,1]$ we have

$$\Theta_t(x,l)\le\Theta_0(x,l+t)\le\Theta_0(x,2)\le 1+\varepsilon_0\quad\text{for all } |x|\ge R_1/2.$$

White's Regularity Theorem [23, Theorem 3.1] implies the desired claim.

From the first and second claim we see that given $r>0$ and $\varepsilon>0$ we can find R_2 so that for all $L\in\mathcal{S}(K)$ and $x\in L\setminus B_{R_2}$ we have that $\widehat{B}_r(x)\cap L$ is ε–close in $C^{2,\alpha}$ to

a ball of radius r in L_0. Elliptic regularity implies that for every $k \in \mathbb{N}$ we can choose R_2 larger so that $\widehat{B}_r(x) \cap L$ is ε–close in $C^{k+1,\alpha}$ to a ball of radius r in L_0. Thus, for every $k \in \mathbb{N}$, we can find r_0, R_0, and D_k so that the hypotheses of Proposition 3.2 are satisfied for all $L \in \mathcal{S}(K)$. This implies the desired result. □

4 Fredholm theory

In this section we develop the Fredholm theory for the operator

$$\mathcal{L}(\phi) = \Delta\phi + \langle \boldsymbol{x}, \nabla\phi \rangle - 2\phi$$

defined on the self-expander L, which already arose in (11). The relevant spaces to consider are given below.

Definition 4.1 For $k \in \mathbb{Z}^+$, let $H^k(L)$ denote the Sobolev space $W^{k,2}(L)$ with norm

$$\|\phi\|_{H^k} = \Big(\sum_{j=0}^{k} \int_L |\nabla^j \phi|^2 \, \mathrm{d}\mathcal{H}^n \Big)^{\frac{1}{2}}.$$

Let $H^k_*(L)$ denote the subspace of $H^k(L)$ such that the norm

$$\|\phi\|_{H^k_*} = \Big(\sum_{j=0}^{k} \int_L |\nabla^j \phi|^2 \, \mathrm{d}\mathcal{H}^n + \sum_{j=1}^{k-1} \int_L \langle \boldsymbol{x}^\top, \nabla^j \phi \rangle^2 \, \mathrm{d}\mathcal{H}^n \Big)^{\frac{1}{2}}$$

is finite. Here the symbol $\langle \boldsymbol{x}^\top, \nabla^j \phi \rangle$ means that we form a tensor by contracting $\boldsymbol{x}^\top$ and $\nabla^j \phi$ using the Euclidean metric; for example, $\langle \boldsymbol{x}^\top, \nabla\phi \rangle$ is given by viewing $\nabla\phi$ as a vector using the Euclidean metric and then taking its inner product with $\boldsymbol{x}^\top$. Thus $\langle \boldsymbol{x}^\top, \nabla^j \phi \rangle^2$ is the square norm of the tensor $\langle \boldsymbol{x}^\top, \nabla^j \phi \rangle$. Both H^k and H^k_* are Banach spaces (in fact, Hilbert spaces).

Our first result, which shows the utility of Definition 4.1, is the following.

Proposition 4.2 *The map $\mathcal{L}\colon H^{k+2}_*(L) \to H^k(L)$ is well-defined and continuous.*

Proof From Theorem 3.1 we have that every derivative of the second fundamental form of L is uniformly bounded. Thus if T is a tensor on L we have from the Bochner formula and Gauss equation that

$$\nabla \Delta T = \Delta \nabla T + C_1 \star \nabla T + C_0 \star T, \tag{15}$$

where C_0, C_1 are two uniformly bounded tensors (depending only on L) and $A \star B$ denotes any contraction of tensor A with tensor B. Moreover, using the fact that $H = \boldsymbol{x}^\perp$ on L we also have

$$\nabla\langle \boldsymbol{x}^\top, T\rangle = \langle \boldsymbol{x}^\top, \nabla T\rangle + C_0 \star T, \tag{16}$$

where C_0 is another uniformly bounded tensor which depends only on L.

We can use (15) and (16) inductively to conclude that for all $i \in \mathbb{N}$ and $\phi \in C_0^\infty(L)$ we have

$$\nabla^i \mathcal{L}(\phi) = \Delta \nabla^i \phi + \langle \boldsymbol{x}^\top, \nabla^{i+1}\phi\rangle + \sum_{j=0}^{i} C_j \star \nabla^j \phi, \tag{17}$$

where the C_j are uniformly bounded tensors. Thus for all $i \in \mathbb{N}$ we can find a positive constant $c_i = c_i(L)$ so that

$$\int_L |\nabla^i \mathcal{L}(\phi)|^2 \,\mathrm{d}\mathcal{H}^n \le c_i\Big(\sum_{j=0}^{i+2}\int_L |\nabla^j\phi|^2 \,\mathrm{d}\mathcal{H}^n + \int_L \langle \boldsymbol{x}^\top, \nabla^{i+1}\phi\rangle^2 \,\mathrm{d}\mathcal{H}^n\Big).$$

It follows that $\mathcal{L}$ defines a continuous map from $H_*^{k+2}(L)$ to $H^k(L)$. □

Theorem 4.3 *The map $\mathcal{L}\colon H_*^2(L) \longrightarrow L^2(L)$ is an isomorphism.*

Proof We start by proving the existence of a positive constant $C_0 = C_0(L, n)$ so that for all ϕ in $H_*^2(L)$ we have

$$\|\phi\|^2_{H^2_*} \le C_0 \int_L |\mathcal{L}(\phi)|^2 \,\mathrm{d}\mathcal{H}^n = C_0 \|\mathcal{L}(\phi)\|^2_{L^2}. \tag{18}$$

Using the fact that $H = \boldsymbol{x}^\perp$ on L, we have

$$\operatorname{div} \boldsymbol{x}^\top = n + |\boldsymbol{x}^\perp|^2 \tag{19}$$

and so direct computation shows that

$$2\phi\langle \boldsymbol{x}, \nabla\phi\rangle = \operatorname{div}(\boldsymbol{x}^\top \phi^2) - (n + |\boldsymbol{x}^\perp|^2)\phi^2. \tag{20}$$

Thus

$$-\int_L \mathcal{L}(\phi)\phi \,\mathrm{d}\mathcal{H}^n = \int_L |\nabla\phi|^2 + (2 + n/2 + |\boldsymbol{x}^\perp|^2/2)\phi^2 \,\mathrm{d}\mathcal{H}^n$$

and so, since $-\mathcal{L}(\phi)\phi \le |\mathcal{L}(\phi)|^2 + \frac{1}{4}\phi^2$, we obtain

$$\int_L |\nabla\phi|^2 \,\mathrm{d}\mathcal{H}^n \le \int_L |\mathcal{L}(\phi)|^2 \,\mathrm{d}\mathcal{H}^n. \tag{21}$$

Moreover, if A is the second fundamental form of L,

$$\Delta\phi\langle \boldsymbol{x}, \nabla\phi\rangle = \operatorname{div}\left(\nabla\phi\langle \boldsymbol{x}^\top, \nabla\phi\rangle - \boldsymbol{x}^\top\frac{|\nabla\phi|^2}{2}\right) + \frac{|\nabla\phi|^2}{2}(n-2+|\boldsymbol{x}^\perp|^2) - \langle H, A(\nabla\phi,\nabla\phi)\rangle \tag{22}$$

and combining (22) with (20) we obtain

$$\int_L |\mathcal{L}(\phi)|^2\,\mathrm{d}\mathcal{H}^n = \int_L (\Delta\phi)^2 + \langle \boldsymbol{x}, \nabla\phi\rangle^2 + (n+2+|\boldsymbol{x}^\perp|^2)|\nabla\phi|^2\,\mathrm{d}\mathcal{H}^n + 2\int_L (n+2+|\boldsymbol{x}^\perp|^2)\phi^2\,\mathrm{d}\mathcal{H}^n - 2\int_L \langle H, A(\nabla\phi,\nabla\phi)\rangle\,\mathrm{d}\mathcal{H}^n. \tag{23}$$

From the Bochner formula and Gauss equation there is a constant $c = c(n)$ so that on L we have

$$|\nabla\Delta\phi - \Delta\nabla\phi| \le c|A|^2|\nabla\phi|.$$

Observing that

$$\operatorname{div}\big(\nabla^2\phi(\nabla\phi,\cdot)\big) = \langle\Delta\nabla\phi,\nabla\phi\rangle + |\nabla^2\phi|^2,$$
$$\operatorname{div}(\Delta\phi\nabla\phi) = \langle\nabla\Delta\phi,\nabla\phi\rangle + (\Delta\phi)^2,$$

we thus see that

$$|\nabla^2\phi|^2 \le (\Delta\phi)^2 + \operatorname{div}\big(\nabla^2\phi(\nabla\phi,\cdot) - \Delta\phi\nabla\phi\big) + c|A|^2|\nabla\phi|^2. \tag{24}$$

Inserting (24) in (23), we have

$$\int_L |\nabla^2\phi|^2 + \langle \boldsymbol{x}, \nabla\phi\rangle^2 + |\nabla\phi|^2 + \phi^2\,\mathrm{d}\mathcal{H}^n \le \int_L |\mathcal{L}(\phi)|^2\,\mathrm{d}\mathcal{H}^n + (c+2)\int_L |A|^2|\nabla\phi|^2\,\mathrm{d}\mathcal{H}^n. \tag{25}$$

From Theorem 3.1 we know that $|A|^2$ is uniformly bounded on L and so we have from (21) the existence of a constant $C = C(L,n)$ so that

$$\int_L |A|^2|\nabla\phi|^2\,\mathrm{d}\mathcal{H}^n \le C\int_L |\mathcal{L}(\phi)|^2\,\mathrm{d}\mathcal{H}^n.$$

This last inequality and (25) imply (18) at once.

The immediate consequence of (18) is that $\mathcal{L}$ is injective. It also follows that its range is closed for the following reason. If $v_i = \mathcal{L}(\phi_i)$ is a sequence converging in $L^2(L)$ to v, then (18) implies $(\phi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $H^2_*(L)$ and thus

sequentially converging to $\phi \in H^2_*(L)$. Naturally, $\mathcal{L}(\phi) = v$ since $\mathcal{L}$ is continuous by Proposition 4.2.

We now argue that $\mathcal{L}$ is surjective. In order to do so we compute the formal adjoint $\mathcal{L}^*$ of $\mathcal{L}$. Using (19) we have, for every $u, v \in C^\infty(L)$,

$$\langle \boldsymbol{x}, \nabla u\rangle v = \operatorname{div}(\boldsymbol{x}^\top uv) - (n + |\boldsymbol{x}^\perp|^2)uv - \langle \boldsymbol{x}, \nabla v\rangle u. \tag{26}$$

Hence, if $\phi \in C^\infty_0(L)$ and $\eta \in C^\infty(L)$,

$$\int_L \mathcal{L}(\phi)\eta \,\mathrm{d}\mathcal{H}^n = \int_L \phi\big(\Delta\eta - \langle \boldsymbol{x}, \nabla\eta\rangle - (n + |\boldsymbol{x}^\perp|^2 + 2)\eta\big)\,\mathrm{d}\mathcal{H}^n,$$

which means

$$\mathcal{L}^*(\eta) = \Delta\eta - \langle \boldsymbol{x}, \nabla\eta\rangle - (n + |\boldsymbol{x}^\perp|^2 + 2)\eta. \tag{27}$$

Suppose that $\mathcal{L}$ is not surjective. Since its range is closed we can find $\eta \in L^2(L)$ non-zero such that

$$\int_L \mathcal{L}(\phi)\eta \,\mathrm{d}\mathcal{H}^n = 0 \quad \text{for all } \phi \in C^\infty_0(L).$$

Elliptic regularity implies that η is smooth and hence a solution to $\mathcal{L}^*(\eta) = 0$. The next lemma implies that $\eta = 0$ which is a contradiction.

Lemma 4.4 *If $\eta \in C^\infty(L) \cap L^2(L)$ and $\mathcal{L}^*(\eta) = 0$, then $\eta = 0$.*

Proof We start by arguing that $\eta \in H^1(L)$. For each $R > 0$ consider a cut-off function $\phi_R \in C^\infty_0(\mathbb{C}^n)$ so that

$$0 \le \phi_R \le 1, \quad \phi_R|_{B_R} = 1, \quad \operatorname{supp}(\phi_R) \subseteq B_{2R}, \quad |\nabla\phi_R| \le c_0 R^{-1}, \tag{28}$$

for some universal constant c_0.

Using (26) with $u = \eta^2/2$ and $v = \phi_R^2$ we have

$$0 = -\int_L \mathcal{L}^*(\eta)\eta\phi_R^2 \,\mathrm{d}\mathcal{H}^n = \int_L |\nabla\eta|^2\phi_R^2 + \left(\frac{n + |\boldsymbol{x}^\perp|^2}{2} + 2\right)\eta^2\phi_R^2 \,\mathrm{d}\mathcal{H}^n$$
$$+ \int_L \langle \nabla\eta, \nabla\phi_R^2\rangle\eta \,\mathrm{d}\mathcal{H}^n - \int_L \langle \boldsymbol{x}, \nabla\phi_R^2\rangle\frac{\eta^2}{2}\,\mathrm{d}\mathcal{H}^n.$$

We have for some universal constant c and all $\varepsilon > 0$ that

$$\int_L \langle \nabla\eta, \nabla\phi_R^2\rangle\eta \,\mathrm{d}\mathcal{H}^n \le c\varepsilon \int_L |\nabla\eta|^2\phi_R^2 \,\mathrm{d}\mathcal{H}^n + \frac{c}{\varepsilon}\int_L \eta^2 \,\mathrm{d}\mathcal{H}^n$$

and thus we find another uniform constant c so that

$$\int_L |\nabla\eta|^2\phi_R^2\,\mathrm{d}\mathcal{H}^n \le c\int_L \langle \boldsymbol{x}, \nabla\phi_R^2\rangle\frac{\eta^2}{2}\,\mathrm{d}\mathcal{H}^n + c\int_L \eta^2\,\mathrm{d}\mathcal{H}^n.$$

The term $\langle \boldsymbol{x}, \nabla\phi_R^2\rangle$ is uniformly bounded (independent of R) by (28) and so

$$\int_L |\nabla\eta|^2\phi_R^2\,\mathrm{d}\mathcal{H}^n \le c\int_L \eta^2\,\mathrm{d}\mathcal{H}^n$$

for some other universal constant c. Letting R tend to infinity we obtain that $\eta \in H^1(L)$.

We now show that $\eta = 0$. Set

$$f(r) = \int_{L\cap B_r} \eta^2\,\mathrm{d}\mathcal{H}^n + \Big(2+\frac{n}{2}\Big)^{-1}\int_{L\cap B_r} |\nabla\eta|^2\,\mathrm{d}\mathcal{H}^n.$$

The idea is to show that

$$\Big(2+\frac{n}{2}\Big)f(r) \le rf'(r) \quad \text{for almost all } r \ge 2+\frac{n}{2}\,. \tag{29}$$

If true, we obtain from integrating (29) that, for all $r \ge r_1 \ge 2+\frac{n}{2}$,

$$f(r) \ge f(r_1)\Big(\frac{r}{r_1}\Big)^{2+(n/2)}.$$

As $\eta \in H^1(L)$ the function f is uniformly bounded, which contradicts the inequality above unless $f \equiv 0$, which means $\eta = 0$.

Hence to complete the proof we need to show (29). Applying integration by parts to the identity

$$\int_{L\cap B_r} \mathcal{L}^*(\eta)\eta\,\mathrm{d}\mathcal{H}^n = 0$$

and using (20) with $\phi = \eta$ we obtain

$$\int_{L\cap B_r} |\nabla\eta|^2 + \Big(2+\frac{n}{2}+\frac{|\boldsymbol{x}^\perp|^2}{2}\Big)\eta^2\,\mathrm{d}\mathcal{H}^n = -\frac{1}{2}\oint_{\partial(L\cap B_r)} \big(\langle \boldsymbol{x}^\top, \nu\rangle\eta^2 - \partial_\nu\eta^2\big)\,\mathrm{d}\mathcal{H}^{n-1},$$

where ν is the outward unit normal to the boundary of $L\cap B_r$. Therefore

$$\begin{aligned}\Big(2+\frac{n}{2}\Big)f(r) &\le \frac{r}{2}\oint_{\partial(L\cap B_r)} \eta^2\,\mathrm{d}\mathcal{H}^{n-1} + \oint_{\partial(L\cap B_r)} |\eta||\nabla\eta|\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \frac{r+1}{2}\oint_{\partial(L\cap B_r)} \eta^2\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\oint_{\partial(L\cap B_r)} |\nabla\eta|^2\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{30}$$

On the other hand using the co-area formula we have, for almost all r,

$$f'(r) = \oint_{\partial(L\cap B_r)} \eta^2 \frac{|\boldsymbol{x}|}{|\boldsymbol{x}^\top|}\,\mathrm{d}\mathcal{H}^{n-1} + \Big(2+\frac{n}{2}\Big)^{-1} \oint_{\partial(L\cap B_r)} |\nabla\eta|^2 \frac{|\boldsymbol{x}|}{|\boldsymbol{x}^\top|}\,\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq \oint_{\partial(L\cap B_r)} \eta^2\,\mathrm{d}\mathcal{H}^{n-1} + \Big(2+\frac{n}{2}\Big)^{-1} \oint_{\partial(L\cap B_r)} |\nabla\eta|^2\,\mathrm{d}\mathcal{H}^{n-1}.$$

Combining this inequality with (30) we obtain that (29) holds for almost all $r \geq 2+\frac{n}{2}$. □

Applying Lemma 4.4 shows that $\mathcal{L}$ is surjective, completing the proof of Theorem 4.3. □

Corollary 4.5 *The map $\mathcal{L}\colon H^{k+2}_*(L) \longrightarrow H^k(L)$ is an isomorphism for all $k \in \mathbb{N}$.*

Proof We proceed by induction where the case $k=0$ follows from Theorem 4.3. Assume Corollary 4.5 holds for some $k \in \mathbb{N}$. Thus $\mathcal{L}$ is injective and given $v \in H^{k+1}(L) \subset H^k(L)$ there is $\phi \in H^{k+2}_*(L)$ so that $\mathcal{L}(\phi) = v$. We need to show that $\phi \in H^{k+3}_*(L)$ in order to prove the corollary.

Set $T = \nabla^{k+1}\phi$ and recall that Δ acting on T is given by the trace of $\nabla^2 T$. From (17) there exists a constant $C_1 = C_1(L,n,k)$ so that

$$\|\Delta T + \langle \boldsymbol{x}^\top, \nabla T\rangle\|^2_{L^2} \leq C_1\big(\|\phi\|^2_{H^{k+1}_*} + \|\mathcal{L}(\phi)\|^2_{H^{k+1}}\big). \tag{31}$$

Next we argue that, for some constant $C_2 = C_2(L,n,k)$,

$$\left|\int_L \langle \Delta T, \nabla_{\boldsymbol{x}^\top} T\rangle\,\mathrm{d}\mathcal{H}^n\right| \leq C_2 \int_L |\nabla T|^2 + |T|^2\,\mathrm{d}\mathcal{H}^n. \tag{32}$$

Reasoning as in (22), there is a universal constant $c = c(n,k)$ so that

$$\left|\langle \Delta T, \nabla_{\boldsymbol{x}^\top} T\rangle - \operatorname{div}\left(\langle \nabla T, \nabla_{\boldsymbol{x}^\top} T\rangle - \boldsymbol{x}^\top \frac{|\nabla T|^2}{2}\right)\right|$$
$$\leq c|\nabla T|^2(1+|A|^2+|\boldsymbol{x}^\perp|^2) + c|T||\nabla T||A|^2|\boldsymbol{x}^\top|.$$

From Theorem 3.1 we have that $|A|^2$ and $|H|^2 = |\boldsymbol{x}^\perp|^2$ have exponential decay, so $|A|^2|\boldsymbol{x}|$ and $|A|^2+|\boldsymbol{x}^\perp|^2$ are uniformly bounded on L. Thus we can find a constant $C_2 = C_2(L,n,k)$ so that

$$\left|\langle \Delta T, \nabla_{\boldsymbol{x}^\top} T\rangle - \operatorname{div}\left(\langle \nabla T, \nabla_{\boldsymbol{x}^\top} T\rangle - \boldsymbol{x}^\top \frac{|\nabla T|^2}{2}\right)\right| \leq C_2(|\nabla T|^2 + |T|^2),$$

which implies, after integration, inequality (32).

Combining (31) with (32), we have

$$\|\Delta T\|^2_{L^2} + \|\langle \boldsymbol{x}^\top, \nabla T\rangle\|^2_{L^2} \leq C_3\big(\|\phi\|^2_{H^{k+2}_*} + \|\mathcal{L}(\phi)\|^2_{H^{k+1}}\big) \tag{33}$$

for some $C_3 = C_3(C_1, C_2)$. Using the Bochner formula and the fact that $|A|^2$ is uniformly bounded on L we can find a constant $C_4 = C_4(L, n, k)$ so that

$$\|\nabla^2 T\|^2_{L^2} \leq \|\Delta T\|^2_{L^2} + C_4\|\nabla T\|^2_{L^2}.$$

Combining this inequality with (33) we obtain

$$\|\phi\|^2_{H^{k+3}_*} \leq C_5\big(\|\phi\|^2_{H^{k+2}_*} + \|\mathcal{L}(\phi)\|^2_{H^{k+1}}\big)$$

for some constant C_5, which implies that $\phi \in H^{k+3}_*(L)$ as required. □

5 Local uniqueness

In this section we prove the local uniqueness of zero-Maslov class Lagrangian self-expanders L asymptotic to transverse pairs of multiplicity one planes $L_0 = P_1 + P_2$.

Recall the definition of G_n in Section 3 and consider a smooth path $(P_1(s), P_2(s)) \in G_n$ with $P_1(0) = P_1$ and $P_2(0) = P_2$. We assume that the difference of the Lagrangian angles $\theta\big(P_1(s)\big) - \theta\big(P_2(s)\big)$ is constant.

Theorem 5.1 *Given a zero-Maslov class self-expander L asymptotic to $L_0 = P_1 + P_2$, there is $R_0 > 0$, $s_0 > 0$ and $\varepsilon > 0$ so that zero-Maslov class self-expanders L^s that satisfy*

- *L^s is asymptotic to $L^s_0 = P_1(s) + P_2(s)$ for some $|s| \leq s_0$,*
- *L^s is ε–close in C^2 to L in B_{R_0},*

exist and are unique. The family $(L^s)_{|s|\leq s_0}$ is continuous in $C^{2,\alpha}$.

We achieve this by studying the deformation theory of L and applying the Implicit Function Theorem.

We start by constructing the tubular neighbourhoods of L_0 and L that we require and derive some basic properties.

Symplectic preliminaries

The cotangent bundle of a Lagrangian N has a natural symplectic structure, which is exact, meaning that there is a tautological one form $\tau \in \Lambda^1(T^*N)$ so that if ω_N is the tautological symplectic form on T^*N, then $\mathrm{d}\tau = -\omega_N$. The form τ is determined by the following property: if $\Xi \in \Lambda^1(N)$ and we consider the natural map $\Xi\colon N \to T^*N$, then $\Xi^*\tau = \Xi$. We remark that on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, $\tau = \sum_{i=1}^n y_i\,\mathrm{d}x_i$ and hence τ is different from the Liouville form λ.

For $\Xi \in \Lambda^1(N)$, we let Γ_Ξ denote the section of T^*N given by $x \mapsto \big(x, \Xi(x)\big)$.

A symplectomorphism $\Phi\colon (M_1, \mathrm{d}\lambda_1) \to (M_2, \mathrm{d}\lambda_2)$ between exact symplectic manifolds is called exact if $\Phi^*(\lambda_2) - \lambda_1$ is exact.

In particular, in the case of $\mathbb{R}^n \subseteq \mathbb{C}^n$, the map from $T^*\mathbb{R}^n$ to $\mathbb{C}^n$ given by

$$\left(x = (x_1, \dots, x_n), \sum_{j=1}^n y_j(x)\,\mathrm{d}x_j\right) \mapsto (x_1 + iy_1, \dots, x_n + iy_n) \tag{34}$$

is an exact symplectomorphism identifying the zero section with the real $\mathbb{R}^n$ in $\mathbb{C}^n$.

The construction of the tubular neighbourhood of $L_0 = P_1 \cup P_2$ is elementary. Without loss of generality we may assume that P_1 is the real $\mathbb{R}^n \subseteq \mathbb{C}^n$ and that $P_2 = A \cdot \mathbb{R}^n$ where $A = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Naturally, we may define symplectomorphisms $\Psi_j\colon T^*P_j \to \mathbb{C}^n$ for $j = 1, 2$, where Ψ_1 is given by (34) and $\Psi_2 = A \circ \Psi_1$. Clearly, there exists $\zeta > 0$ so that if

$$V_j = \big\{\big(x, \xi(x)\big) \in T^*(P_j \setminus \{0\}) : |\xi(x)| < 2\zeta|x|\big\} \tag{35}$$

then $\Psi_1(V_1) \cap \Psi_2(V_2) = \varnothing$. This choice ensures that we can allow for rotations of P_1 and P_2 in a tubular neighbourhood that is symplectomorphic to an open neighbourhood in $T^*(L_0 \setminus \{0\})$, so we have the following.

Lemma 5.2 *Set:*

- $V_0 = V_1 \cup V_2 \subseteq T^*(L_0 \setminus \{0\})$.
- $T_0 = \Psi_1(V_1) \cup \Psi_2(V_2) \subseteq \mathbb{C}^n$.
- $\Psi_0\colon V_0 \to T_0$ *defined by* $\Psi_0|_{V_j} = \Psi_j$.

Then V_0 and T_0 are open tubular neighbourhoods of $L_0 \setminus \{0\}$ and Ψ_0 is a symplectomorphism preserving the Liouville form. Moreover, any small rotation of either of the planes P_1 or P_2 remains in T_0.

We now use V_0, T_0 and Ψ_0 to construct our tubular neighbourhoods of L. The point will be to ensure that the symplectomorphism we construct is compatible with the standard symplectomorphism (34) over the planes.

By Theorem 3.1, there is $R_1 > 0$ and an open set $B \subset \mathbb{C}^n$ with compact closure so that

$$L \setminus B = \{x + J\,\overline{\nabla}\psi(x) : x \in L_0 \setminus B_{R_1}\}$$

for some $\psi \in C^\infty(L_0 \setminus B_{R_1})$ whose $C^{2,\alpha}$ norm decays exponentially. In particular, we may assume by making R_1 larger if necessary that $|\mathrm{d}\psi(x)| < \zeta|x|$ for all $x \in L_0 \setminus B_{R_1}$. Let

$$\pi\colon L_0 \setminus B_{R_1} \to L \setminus B, \quad \pi(x) = x + J\,\overline{\nabla}\psi(x) \tag{36}$$

so that $\pi^*\colon T^*(L \setminus B) \to T^*(L_0 \setminus B_{R_1})$ is an isomorphism.

Proposition 5.3 *Recall the notation of Lemma 5.2. There exist*

- *open neighbourhoods $\widehat{V} \subset V$ of L in T^*L,*
- *open tubular neighbourhoods $\widehat{T} \subset T$ of L in $\mathbb{C}^n$,*
- *an exact symplectomorphism $\Psi\colon V \to T$ with $\Psi|_L = \mathrm{id}_L$ and $\widehat{T} = \Psi(\widehat{V})$,*

such that

$$\begin{aligned} \pi^*(V) &= \big\{\big(x, \xi(x)\big) \in T^*(L_0 \setminus B_{R_1}) : |\xi(x)| < \zeta|x|\big\} \subseteq V_0, \\ \pi^*(\widehat{V}) &= \big\{\big(x, \xi(x)\big) \in T^*(L_0 \setminus B_{R_1}) : |\xi(x)| < \tfrac{1}{2}\zeta|x|\big\}, \end{aligned} \tag{37}$$

and

$$\Psi \circ (\pi^*)^{-1}\big(x, \xi(x)\big) = \Psi_0\big(x, \mathrm{d}\psi(x) + \xi(x)\big) \quad \text{for all } (x, \xi) \in \pi^*(V). \tag{38}$$

Proof Recall that L is embedded by Lemma 2.4.

Since L is Lagrangian, the normal bundle NL of L is isomorphic to T^*L so we may consider the exponential map exp acting on T^*L. Given any compact $K \subset L$, we may apply the usual tubular neighbourhood theorem to give open neighbourhoods of K in T^*K and $\mathbb{C}^n$ that are diffeomorphic via exp. Moreover, exp and its derivative act as the identity on K. In particular, $\exp^* \lambda = \lambda_L$ on K, where λ_L is the Liouville form on T^*L.

We may define V and Ψ over $L \setminus B$ via (37) and (38) so that $\Psi(V)$ is a tubular neighbourhood of $L \setminus B$. Using the exponential map over the remainder of L, we can extend to open neighbourhoods V, T of L in T^*L and $\mathbb{C}^n$ and a diffeomorphism $\Phi\colon V \to T$ such that Φ and Ψ agree over $L \setminus B$.

As π in (36) is a diffeomorphism, the map $\pi^*\colon T^*(L\setminus B)\to T^*(L_0\setminus B_{R_1})$ is a symplectomorphism preserving the tautological 1–form (see, for example, Cannas da Silva [20, Theorem 2.1]). Since Ψ_0 preserves the Liouville form, we see that on $L\setminus B$ we have $\Phi^*\lambda-\lambda_L=\mathrm{d}\alpha_L$ for some smooth function α_L. We can smoothly cut-off the function α_L so that it is defined on T^*L and vanishes over a compact subset of L. Hence Φ is an exact symplectomorphism outside some compact set. The idea now is to essentially use Moser's trick to perturb Φ over a compact set to a global exact symplectomorphism.

Define

$$\lambda_t=(1-t)(\lambda_L+\mathrm{d}\alpha_L)+t\Phi^*\lambda,$$

so that

$$\mathrm{d}\lambda_t=2(1-t)\omega_L+2t\Phi^*\omega$$

is a closed nondegenerate 2–form on V for all $t\in[0,1]$. Using nondegeneracy, we can uniquely solve

$$X_t\lrcorner\,\mathrm{d}\lambda_t=\lambda_L+\mathrm{d}\alpha_L-\Phi^*\lambda$$

pointwise for X_t. Since $\Phi^*\lambda=\lambda_L+\mathrm{d}\alpha_L$ over $L\setminus B$, we see that X_t is zero outside a compact set, so we may solve for a smooth vector field X_t on V for all $t\in[0,1]$ by shrinking V if necessary. Moreover, L (viewed as the zero section) is Lagrangian with respect to ω_L and $\Phi^*\omega$, so we may deduce that $\mathrm{d}\lambda_t|_L=0$ and hence $X_t|_L=0$ as well.

Define diffeomorphisms f_t on V such that $f_0=\mathrm{id}$ and $\frac{\mathrm{d}}{\mathrm{d}t}f_t=X_t\circ f_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t^*\lambda_t=f_t^*\big(\Phi^*\lambda-\lambda_L-\mathrm{d}\alpha_L+\mathrm{d}(X_t\lrcorner\lambda_t)+X_t\lrcorner\,\mathrm{d}\lambda_t\big)=\mathrm{d}\,f_t^*(X_t\lrcorner\lambda_t).$$

We deduce that $f_1^*\Phi^*\lambda-\lambda_L=f_1^*\lambda_1-f_0^*\lambda_0+\mathrm{d}\alpha_L$ is exact. Moreover the diffeomorphism f_1 acts as the identity on L and on V over $L\setminus B$. Hence we have an exact symplectomorphism $\Psi=\Phi\circ f_1\colon V\to T$ which satisfies (37) and (38). The remainder of the proposition follows by taking appropriate open subsets of V and T. □

Write $C^2_{\mathrm{loc}}(V)$ and $C^2_{\mathrm{loc}}(\widehat{V})$ for the spaces of locally C^2 1–forms Ξ with graph $\Gamma_\Xi\subseteq V$ and $\Gamma_\Xi\subseteq\widehat{V}$ respectively. We use similar notation for $C^\infty_{\mathrm{loc}}(V)$. For $\Xi\in C^2_{\mathrm{loc}}(V)$ we define a C^2–embedding $f^\Xi\colon L\to T$ by

$$f^\Xi(x)=\Psi\big(x,\Xi(x)\big)$$

so that $f^\Xi(L)$ is the deformation of L given by Ξ.

We note that $f^\Xi(L)$ is Lagrangian if and only if $\mathrm{d}\Xi=0$.

However, we want to restrict ourselves to exact zero-Maslov class deformations $f^{\Xi}(L)$ since, by Lemma 2.2, we know that if $f^{\Xi}(L)$ is a self-expander it must be exact. This motivates the next lemma.

Lemma 5.4 *Let $L' = f^{\Xi}(L)$ with $\Xi \in C^2_{\mathrm{loc}}(V)$. Then L' is exact and zero-Maslov class if and only if $\Xi = \mathrm{d}\phi$ for some $\phi \in C^3_{\mathrm{loc}}(L)$ with $\Gamma_{\mathrm{d}\phi} \subseteq V$.*

Moreover, if we set

- *$\bar{\phi}$ defined on T so that $\bar{\phi}(\Psi(x,\xi)) = \phi(x)$,*
- *$H(L^s)$ the mean curvature of $L^s = f^{s\mathrm{d}\phi}(L)$ and Δ^s the pullback to L of the Laplacian on L^s,*
- *θ^{ϕ} the pullback to L of the Lagrangian angle of $f^{\mathrm{d}\phi}(L)$, and*
- *β^{ϕ} the pullback to L of the primitive of the Liouville form of $f^{\mathrm{d}\phi}(L)$,*

then θ^{ϕ} and β^{ϕ} can be given, respectively, by

$$\theta^{\phi}(x) = \theta(x) + \int_0^1 \Delta^s\phi(x) - \langle H(L^s), \overline{\nabla}\bar{\phi}\rangle|_{f^{s\,\mathrm{d}\phi}(x)}\,\mathrm{d}s$$

and

$$\beta^{\phi}(x) = \beta(x) - 2\phi(x) + \int_0^1 \langle \boldsymbol{x}, \overline{\nabla}\bar{\phi}\rangle|_{f^{s\,\mathrm{d}\phi}(x)}\,\mathrm{d}s.$$

We also have

$$\left.\frac{\mathrm{d}\theta^{s\phi}}{\mathrm{d}s}\right|_{s=0} = \Delta\phi - \langle H, \overline{\nabla}\bar{\phi}\rangle,$$
$$\left.\frac{\mathrm{d}\beta^{s\phi}}{\mathrm{d}s}\right|_{s=0} = -2\phi + \langle \boldsymbol{x}, \nabla\phi\rangle + \langle H, \overline{\nabla}\bar{\phi}\rangle.$$

Proof We first show that exactness of $f^{\Xi}(L)$ corresponds to exactness of Ξ. It suffices to see that $(f^{\Xi})^*(\sum_{i=1}^n y_i\,\mathrm{d}x_i)$ is exact because that differs from $-(f^{\Xi})^*(\lambda/2)$ by an exact form. Since Ψ is an exact symplectomorphism there exists a function α_L on T^*L such that $\Psi^*\big(\sum_{i=1}^n y_i\,\mathrm{d}x_i\big) = \tau + \mathrm{d}\alpha_L$. By definition, $f^{\Xi} = \Psi \circ \Xi$ so

$$(f^{\Xi})^*\left(\sum_{i=1}^n y_i\,\mathrm{d}x_i\right) = \Xi^* \circ \Psi^*\left(\sum_{i=1}^n y_i\,\mathrm{d}x_i\right) = \Xi^*(\tau + \mathrm{d}\alpha_L) = \Xi + \mathrm{d}\,\Xi^*(\alpha_L).$$

Hence $f^{\Xi}(L)$ is exact if and only if Ξ is exact.

We now compute the stated identities for β^{ϕ}. Consider the vector field on T^*L given by

$$\widetilde{X}|_{(x,\xi)} = \big(0, \mathrm{d}\phi(x)\big) \in T_{(x,\xi)}(T^*L),$$

the function $\widetilde{\phi}$ on T^*L given by $\widetilde{\phi}(x,\xi)=\phi(x)$, and $X=\Psi_*(\widetilde{X})$ a vector field on T. We have $\widetilde{X}\lrcorner\Psi^*\omega=-\mathrm{d}\widetilde{\phi}$ and so, because $\overline{\phi}\circ\Psi=\widetilde{\phi}$, we obtain $X=J\,\overline{\nabla}\overline{\phi}$ on T. As a result we have

$$\frac{\mathrm{d}}{\mathrm{d}s}f^{s\mathrm{d}\phi}(x)=\Psi_*|_{(x,s\mathrm{d}\phi)}(\widetilde{X})=X|_{f^{s\mathrm{d}\phi}(x)}=J\,\overline{\nabla}\overline{\phi}|_{f^{s\mathrm{d}\phi}(x)}.$$

Therefore we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(f^{s\mathrm{d}\phi})^*\lambda=(f^{s\mathrm{d}\phi})^*\mathcal{L}_{J\,\overline{\nabla}\overline{\phi}}\lambda&=(f^{s\mathrm{d}\phi})^*(\mathrm{d}(J\,\overline{\nabla}\overline{\phi}\lrcorner\lambda)+J\,\overline{\nabla}\overline{\phi}\lrcorner 2\omega)\\&=(f^{s\mathrm{d}\phi})^*\mathrm{d}(\langle\boldsymbol{x},\overline{\nabla}\overline{\phi}\rangle)-2\,\mathrm{d}\phi.\end{aligned}$$

In light of this formula we see that if we define

$$\beta^{\phi}(x)=\beta(x)-2\phi(x)+\int_0^1\langle\boldsymbol{x},\overline{\nabla}\overline{\phi}\rangle|_{f^{s\,\mathrm{d}\phi}(x)}\,\mathrm{d}s,$$

we have $(f^{\mathrm{d}\phi})^*\lambda=\mathrm{d}\beta^{\phi}$ and, since L is a self-expander,

$$\begin{aligned}\left.\frac{\mathrm{d}\beta^{s\phi}}{\mathrm{d}s}\right|_{s=0}=-2\phi+\langle\boldsymbol{x},\overline{\nabla}\overline{\phi}\rangle&=-2\phi+\langle\boldsymbol{x},\nabla\phi\rangle+\langle\boldsymbol{x}^{\perp},\overline{\nabla}\overline{\phi}\rangle\\&=-2\phi+\langle\boldsymbol{x},\nabla\phi\rangle+\langle H,\overline{\nabla}\overline{\phi}\rangle.\end{aligned}$$

We now show that the zero-Maslov class condition imposes no condition on $\Xi=\mathrm{d}\phi$. We have $(f^{s\mathrm{d}\phi})^*(\Omega)=e^{i\theta^s}(f^{s\mathrm{d}\phi})^*\mathrm{vol}_{L^s}$ where $e^{i\theta^s}$ is an S^1–valued function on L.

If the deformation vector X were orthogonal to L^s, we would have by [22, Lemma 2.3] that

$$\frac{\mathrm{d}\theta^s}{\mathrm{d}s}=\Delta^s\phi.$$

The fact that X might have a tangential component along L^s implies

$$\frac{\mathrm{d}\theta^s}{\mathrm{d}s}=\Delta^s\phi+\langle X,(f^{s\mathrm{d}\phi})_*\nabla\theta^s\rangle=\Delta^s\phi+\langle JX,H(L^s)\rangle=\Delta^s\phi-\langle H(L^s),\overline{\nabla}\overline{\phi}\rangle.$$

Therefore, integrating this equation for θ^s together with the initial condition $\theta^0=\theta$ allows us to define a (single-valued) function on each L^s that is the Lagrangian angle. In particular, $f^{\mathrm{d}\phi}(L)$ has zero-Maslov class and we can set

$$\theta^{\phi}(x)=\theta(x)+\int_0^1\Delta^s\phi(x)-\langle H(L^s),\overline{\nabla}\overline{\phi}\rangle|_{f^{s\,\mathrm{d}\phi}(x)}\,\mathrm{d}s.$$

The equation for $\frac{\mathrm{d}}{\mathrm{d}s}\theta^{s\phi}$ was computed above. □

Proof of Theorem 5.1

Consider smooth rotations $P_1(s)$ and $P_2(s)$ of the planes P_1 and P_2 respectively, so that the planes remain Lagrangian and the difference of their Lagrangian angles stays the same. In that case we find a one parameter family of matrices $B_1(s), B_2(s) \in U(n)$ with

- $P_1(s) = B_1(s) \cdot P_1$ and $P_2(s) = B_2(s) \cdot P_2$,
- $B_1(s)B_2^{-1}(s) \in \mathrm{SU}(n)$ and $B_1(0) = B_2(0) = \mathrm{id}$,
- $\det B_1(s) = \det B_2(s) = e^{i\theta(s)}$, where $\theta(s)$ is a smooth function with $\theta(0) = 0$.

Consider $G_s\colon \mathbb{C}^n \to \mathbb{C}^n$, a one-parameter group of diffeomorphisms with $G_0 = \mathrm{id}$,

$$G_s(x) = B_1(s)(x) \text{ on } \Psi_1(V_1) \setminus B_{R_1} \quad \text{and} \quad G_s(x) = B_2(s)(x) \text{ on } \Psi_2(V_2) \setminus B_{R_1}.$$

From Proposition 5.3 we can find s_0 so that for all $|s| \le s_0$ and $\mathrm{d}\phi \in C^3_{\mathrm{loc}}(\widehat{V})$ we have $G_s \circ f^{\mathrm{d}\phi}(L) \subset T$. In this case we define the C^2 embedding

$$f^{\mathrm{d}\phi,s}\colon L \to T, \quad f^{\mathrm{d}\phi,s}(x) = G_s \circ \Psi\big(x, \mathrm{d}\phi(x)\big).$$

Given $\phi \in C^3_{\mathrm{loc}}(L)$ so that $\mathrm{d}\phi \in C^2_{\mathrm{loc}}(\widehat{V})$ and $|s| \le s_0$, we set $L^{\phi,s} = f^{\mathrm{d}\phi,s}(L)$. Thus, by Lemma 5.4 we know that $L^{\phi,s}$ is an exact zero-Maslov class Lagrangian and we consider its Lagrangian angle $\theta^{\phi,s}$ and primitive for the Liouville form $\beta^{\phi,s}$ pulled-back to L, which are given by Lemma 5.4.

If ϕ has strong enough decay then $L^{\phi,s}$ is asymptotic to $L^s_0 = P_1(s) + P_2(s)$. For simplicity we write $L^\phi = L^{\phi,0}$, $\beta^\phi = \beta^{\phi,0}$ and $\theta^\phi = \theta^{\phi,0}$.

Lemma 2.2 shows that $L^{\phi,s}$ is a self-expander with $H = \boldsymbol{x}^\perp$ if and only if $\beta^{\phi,s} + \theta^{\phi,s}$ is constant. This motivates our definition of a deformation map.

Definition 5.5 We define a function $\widehat{F}$ on functions $\phi \in C^3_{\mathrm{loc}}(L)$ such that $\mathrm{d}\phi \in C^2_{\mathrm{loc}}(\widehat{V})$ and $|s| < s_0$ by

$$\widehat{F}(\phi, s) = \beta^{\phi,s} + \theta^{\phi,s} - \theta(s).$$

We also let $F(\phi) = \widehat{F}(\phi, 0)$.

We now compute the linearisations of F and $\widehat{F}$ at zero, whose kernels will govern infinitesimal deformations of L. We have in fact already encountered the key operator in (11).

Lemma 5.6 *For $\phi \in C^2_{\mathrm{loc}}(L)$ and $s \in \mathbb{R}$,*

$$\mathrm{d}F|_0(\phi) = \mathcal{L}(\phi) = \Delta\phi + \langle \boldsymbol{x}, \nabla\phi\rangle - 2\phi,$$
$$\mathrm{d}\widehat{F}|_{(0,0)}(\phi, s) = \mathcal{L}(\phi) + s\gamma,$$

where γ is a smooth function with compact support.

Proof Since $\widehat{F}$ acts on a pair (ϕ, s), we may decompose

$$\mathrm{d}\widehat{F}|_{(0,0)}(\phi, s) = \mathrm{d}_1\widehat{F}|_{(0,0)}(\phi) + \mathrm{d}_2\widehat{F}|_{(0,0)}(s).$$

From Lemma 5.4 we have

$$\mathrm{d}F|_0(\phi) = \frac{\mathrm{d}}{\mathrm{d}t}(\beta^{t\phi} + \theta^{t\phi})|_{t=0} = \Delta\phi + \langle \boldsymbol{x}, \nabla\phi\rangle - 2\phi.$$

Thus $\mathrm{d}_1\widehat{F}|_{(0,0)}(\phi) = \mathcal{L}(\phi)$.

We note that we can find $\alpha_s \in C^\infty_{\mathrm{loc}}(L)$ so that $L^{0,s} = f^{\mathrm{d}\alpha_s}(L)$ and thus, using Lemma 5.4 again, we obtain

$$\mathrm{d}_2\widehat{F}|_{(0,0)}(s) = \frac{\mathrm{d}}{\mathrm{d}t}(\beta^{0,st} + \theta^{0,st})|_{t=0} - s\theta'(0) = s(\mathcal{L}(\alpha) - \theta'(0))$$

for some $\alpha \in C^\infty_{\mathrm{loc}}(L)$. We now argue that $\gamma = \mathcal{L}(\alpha) - \theta'(0)$ has compact support and this finishes the proof of the lemma.

On each connected component of $T \setminus B_{R_1}$, G_s belongs to $U(n)$ and we have $\theta^{0,s} = \theta + \theta(s)$ and $\beta^{0,s} - \beta$ a constant $c(s)$ for all $|s| < s_0$. Next we argue that $c(s) = 0$ for all $|s| < s_0$, which implies that

$$\widehat{F}(0, s) = \theta^{0,s} + \beta^{0,s} - \theta(s) = \theta + \theta(s) + \beta - \theta(s) = 0$$

outside a compact set, so γ indeed has compact support.

Recall the diffeomorphism π given in (36). From Proposition 5.3 we see that, by choosing a larger R_1 if necessary, we can find $\chi_s, \psi_s \in C^\infty(L_0 \setminus B_{R_1})$ with $\chi_0 = 0$, $\psi_0 = \psi$, respectively, so that, for all $x \in L_0 \setminus B_{R_1}$ and $|s| \le s_0$, we have

$$f^{0,s}\big(\pi(x)\big) = G_s \circ \Psi_0(x, \mathrm{d}\psi(x)) = \Psi_0\big(x, \mathrm{d}\chi_s + \mathrm{d}\psi_s(x)\big).$$

Note that χ_s is a homogeneous quadratic polynomial and the C^2 norm of ψ_s decays exponentially. Therefore we have from Lemma 5.4 that

$$c(s) = \beta^{0,s}\big(\pi(x)\big) - \beta\big(\pi(x)\big) = -2\chi_s + \langle x, \nabla\chi_s\rangle + o(|x|^{-1}).$$

Since χ_s is a homogeneous quadratic, we have $-2\chi_s + \langle x, \nabla\chi_s\rangle = 0$ and thus $c(s) = 0$ for all $|s| < s_0$. □

We show that the nonlinear map F is well-defined for ϕ in some open ball about zero in $H^{k+2}_*(L)$ for all k large.

Proposition 5.7 *If $k > 1 + \frac{n}{2}$, there exists $\varepsilon_0 > 0$ so that*

$$F\colon B_{\varepsilon_0}(0) \subset H^{k+2}_*(L) \to H^k(L) \quad \text{and} \quad \widehat{F}\colon B_{\varepsilon_0}(0) \times (-s_0, s_0) \to H^k(L)$$

are well-defined.

Proof If $k > 1+\frac{n}{2}$ then the Sobolev Embedding Theorem implies that $H^{k+2} \hookrightarrow C^{3,\alpha}$ for some $\alpha \in (0,1)$ with $\alpha < k-1-\frac{n}{2}$, so there is an open set containing zero in H^{k+2} on which F is defined. The existence of ε_0 so that $F(\phi)$ and $\widehat{F}(\phi, s)$ are defined for $\phi \in B_{\varepsilon_0}(0)$ and $|s| < s_0$ is then immediate as $H^{k+2}_* \subset H^{k+2}$.

We need to show that F and $\widehat{F}$ take $B_{\varepsilon_0}(0)$ into $H^k(L)$. Note that the conditions satisfied by $G_s \in U(n)$ in $T \setminus B_{R_1}$ imply that, outside B_{R_1}, $\theta^{\phi,s} = \theta^\phi + \theta(s)$ and $\beta^{\phi,s}$ differs from β^ϕ by a constant $c(\phi, s)$. Since ϕ tends to zero at infinity in C^3, we have that $c(s,\phi) = c(s)$, and we saw in the proof of Lemma 5.6 that $c(s) = 0$. Thus $\beta^{\phi,s}$ is identical to β^ϕ and so $\widehat{F}(\phi, s)$ and $F(\phi)$ are identical functions outside B_{R_1}. Hence, we only need to argue that F takes $B_{\varepsilon_0}(0)$ into $H^k(L)$.

Notice that θ^ϕ depends only on the tangent space of L^ϕ, and thus on $\nabla\phi$ and $\nabla^2\phi$. Thus we can consider a smooth function of its arguments $Q_\theta(x, y, z)$ so that

$$\theta^\phi(x) = \theta(x) + \Delta\phi(x) - \langle H, \overline{\nabla}\overline{\phi}\rangle + Q_\theta\big(x, \nabla\phi(x), \nabla^2\phi(x)\big).$$

Using the expression for the linearisation of θ^ϕ given in Lemma 5.4 and arguing as in Joyce [9, Proposition 2.10], we conclude that Q_θ, $\partial_y Q_\theta$ and $\partial_z Q_\theta$ vanish at $(x, 0, 0)$.

From the expression for β^ϕ given in Lemma 5.4 we see that we can find a smooth function of its arguments $Q_\beta(x, y)$ so that

$$\beta^\phi(x) = \beta(x) + \langle \boldsymbol{x}, \nabla\phi(x)\rangle - 2\phi(x) + \langle H, \overline{\nabla}\overline{\phi}\rangle + Q_\beta\big(x, \nabla\phi(x)\big).$$

Using the expression for the linearisation of β^ϕ given in Lemma 5.4 and again arguing as in [9, Proposition 2.10], we conclude that Q_β and $\partial_y Q_\beta$ vanish at $(x, 0)$.

Therefore, since $F(0) = \theta + \beta = 0$, we see that

$$F(\phi)(x) = \mathcal{L}(\phi)(x) + Q\big(x, \nabla\phi(x), \nabla^2\phi(x)\big),$$

where $Q = Q(x, y, z)$ is a smooth function of its arguments such that Q, $\partial_y Q$ and $\partial_z Q$ all vanish at $(x, 0, 0)$. Observe that $Q(x, y, z)$ does not directly depend on $\phi(x)$. By Proposition 4.2 it is now enough to show that Q takes $B_{\varepsilon_0}(0)$ into $H^k(L)$.

Let $\phi \in B_{\varepsilon_0}(0)$ be a smooth function with compact support. We derive estimates for η given by $\eta(x) = Q\big(x, \nabla\phi(x), \nabla^2\phi(x)\big)$. Since Q and its first derivatives in y and z vanish when $\phi = 0$,

$$|\eta(x)| \leq C(x)(|\nabla\phi(x)|^2 + |\nabla^2\phi(x)|^2) \tag{39}$$

for some non-negative function C on L. Our first objective is to show that C is bounded.

Recall we have a diffeomorphism $\pi\colon L_0 \setminus B_{R_1} \to L \setminus B$ and set $\phi_0 = \pi^*(\phi)$, $F_0(\phi) = \pi^*\big(F(\phi)\big)$. From Proposition 5.3 we have

$$f^{\phi,0}\big(\pi(x)\big) = \Psi_0\big(x, \mathrm{d}\psi(x) + \mathrm{d}\phi_0(x)\big)$$

and so on $L_0 \setminus B_{R_1}$ we have $F_0(\phi) = \theta_0^\phi + \beta_0^\phi$, where θ_0^ϕ and β_0^ϕ are the pull backs of the Lagrangian angle and primitive for λ on the graph of $\mathrm{d}\phi_0 + \mathrm{d}\psi$ over $L_0 \setminus B_{R_1}$. Thus,

$$F_0(\phi)(x) = \mathcal{L}_0(\phi_0 + \psi)(x) + Q_0\big(x, \nabla(\phi_0 + \psi)(x), \nabla^2(\phi_0 + \psi)(x)\big),$$

where $\mathcal{L}_0$ is the operator given in (11) calculated on L_0 and Q_0 is a function with the same properties as Q. Since we are working over the planes L_0, we compute

$$\beta_0^\phi = \beta_0 + \langle \boldsymbol{x}, \nabla(\phi_0 + \psi)\rangle - 2(\phi_0 + \psi) \quad \text{and} \quad \theta_0^\phi = \theta_0 + \sum_{j=1}^n \tan^{-1}(\mu_j)$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of Hess $(\phi_0 + \psi)$, and θ_0, β_0 are the Lagrangian angle and primitive for the Liouville form on L_0. Thus, because we have chosen $\theta_0 + \beta_0 = 0$, we have

$$Q_0\big(x, \nabla(\phi_0 + \psi)(x), \nabla^2(\phi_0 + \psi)(x)\big) = \sum_{j=1}^n \tan^{-1}(\mu_j)(x) - \Delta(\phi_0 + \psi)(x).$$

From this explicit formula, we deduce that Q_0 and all its derivatives are bounded on $L_0 \setminus B_{R_1}$. Moreover, the decay of $|(\partial_x)^a Q_0(x, y, z)|$ is controlled by $|y|^2 + |z|^2$. The exponential decay of ψ in Theorem 3.1 implies Q and Q_0 differ by terms with exponential decay and so we have that Q and all its derivatives are bounded on L, and that the x derivatives of Q satisfy

$$\big|(\partial_x)^a Q\big(x, \nabla\phi(x), \nabla^2\phi(x)\big)\big| \leq C^a(|\nabla\phi(x)|^2 + |\nabla^2\phi(x)|^2) \tag{40}$$

for some constants C^a. In particular, we can choose $C(x) = C$ independent of x in (39) and we deduce that

$$\|\eta\|_{L^2} \leq C\|\phi\|_{C^2}\|\phi\|_{H^2}.$$

Since any element of H^{k+2}_* has bounded C^2 norm and lies in H^2, we deduce that Q maps $B_{\varepsilon_0}(0)$ into L^2.

Now let $j \in \{1,\dots,k\}$. Our aim is to show that $\nabla^j \eta$ lies in L^2. By the chain rule,

$$|\nabla^j \eta| \le j! \sum_{\substack{a,b,c\ge 0\\ a+b+c\le j}} |(\partial_x)^a(\partial_y)^b(\partial_z)^c Q| \times \sum_{\substack{m_1,\dots,m_b,n_1,\dots,n_c\ge 1\\ a+m_1+\cdots+m_b+n_1+\cdots+n_c=j}} \prod_{i=1}^{b} |\nabla^{m_i}(\nabla\phi)| \prod_{l=1}^{c} |\nabla^{n_l}(\nabla^2\phi)|. \tag{41}$$

If $j = a$ in the sum in (41), the products are trivial and we can use (40) to show that the corresponding terms lie in L^2. Therefore we now assume that $j > a$. Let $q_1,\dots,q_b,r_1,\dots,r_c$ be positive constants so that

$$\sum_{i=1}^{b} \frac{1}{q_i} + \sum_{l=1}^{c} \frac{1}{r_l} = 1. \tag{42}$$

Applying Hölder's inequality to (41), we see that

$$\begin{aligned}\int_L |\nabla^j \eta|^2 \,\mathrm{d}\mathcal{H}^n &\le \sum_{\substack{a,b,c\ge 0\\ a+b+c\le j}} C(a,b,c) \sum_{\substack{m_1,\dots,m_b,n_1,\dots,n_c\ge 1\\ a+m_1+\cdots+m_b+n_1+\cdots+n_c=j}} \prod_{i=1}^{b} \left(\int_L |\nabla^{m_i}(\nabla\phi)|^{2q_i} \,\mathrm{d}\mathcal{H}^n\right)^{1/q_i} \\ &\quad \times \prod_{l=1}^{c} \left(\int_L |\nabla^{n_l}(\nabla^2\phi)|^{2r_l} \,\mathrm{d}\mathcal{H}^n\right)^{1/r_l}\end{aligned} \tag{43}$$

for some constants $C(a,b,c)$ determined by j and the derivatives of Q, which are bounded.

Given a section σ of a vector bundle with a connection D over L that lies in $H^s \cap L^\infty$ we have by Cantor [2, Theorem 3] that

$$\int_L |D^p \sigma|^{2s/p} \,\mathrm{d}\mathcal{H}^n \le C \|\sigma\|_\infty^{2(s/p)-2} \|\sigma\|_{H^s}^2 \tag{44}$$

for some constant C independent of σ, whenever $s \ge p$. (Notice that the results in [2] apply since L is complete, has injectivity radius bounded away from zero and bounded sectional curvature.) Choosing $q_i = (j-a)/m_i$ and $r_l = (j-a)/n_l$, we see that (42) holds and we can apply (44) to deduce that there exists some constant C, independent

of ϕ, so that

$$\int_L |\nabla^{m_i}(\nabla\phi)|^{2q_i}\,\mathrm{d}\mathcal{H}^n \le C\|\nabla\phi\|_\infty^{2q_i-2}\|\phi\|^2_{H^{j-a+1}}, \tag{45}$$

$$\int_L |\nabla^{n_l}(\nabla^2\phi)|^{2r_l}\,\mathrm{d}\mathcal{H}^n \le C\|\nabla^2\phi\|_\infty^{2r_l-2}\|\phi\|^2_{H^{j-a+2}}. \tag{46}$$

Therefore, substituting (45) and (46) into (43) we see that there exists a constant $C(j, Q, \|\phi\|_{C^2})$ so that

$$\int_L |\nabla^j\eta|^2\,\mathrm{d}\mathcal{H}^n \le C(j, Q, \|\phi\|_{C^2})\|\phi\|^2_{H^{j+2}}.$$

Since this holds for all smooth compactly supported $\phi \in B_\varepsilon(0)$ we see that $\eta \in H^j$ whenever $\phi \in H^{j+2}\cap C^2$ for $j = 1,\dots,k$. The result for F follows. □

We can now prove the following local uniqueness result.

Theorem 5.8 *Let $k > 1+\frac{n}{2}$. There exist $0<\varepsilon_1\le\varepsilon_0$ and $0<s_1\le s_0$ so that for each $|s|<s_1$ there exists a unique $\phi(s)\in B_{\varepsilon_1}(0)\subseteq H^{k+2}_*(L)$ so that $\widehat{F}\big(\phi(s),s\big)=0$, where $s\mapsto\phi(s)$ is continuous.*

Moreover, if $(\phi,s)\in B_{\varepsilon_1}(0)\times(-s_1,s_1)$ then $L^{\phi,s}$ is a self-expander with $H=\boldsymbol{x}^\perp$ if and only if $\phi=\phi(s)$.

Proof By Proposition 5.7, $\widehat{F}\colon B_{\varepsilon_0}(0)\times(-s_0,s_0)\to H^k(L)$ is well-defined and $L^{\phi,s}$ is a self-expander if and only if $\widehat{F}(\phi,s)$ is constant. However, if $\phi\in B_{\varepsilon_0}(0)$ then $\widehat{F}(\phi,s)\in C^1(L)\cap L^2(L)$ by the Sobolev Embedding Theorem and so $|\widehat{F}(\phi,s)(x)|\to 0$ as $|x|\to\infty$. Hence $L^{\phi,s}$ is a self-expander if and only if $\widehat{F}(\phi,s)=0$.

By Lemma 5.6, $\mathrm{d}\widehat{F}|_{(0,0)}(\phi,s)=\mathcal{L}(\phi)+s\gamma$ and γ has compact support. Corollary 4.5 implies that $\mathrm{d}_1\widehat{F}|_{(0,0)}=\mathcal{L}\colon H^{k+2}_*(L)\to H^k(L)$ is an isomorphism. Thus

$$\mathrm{d}\widehat{F}|_{(0,0)}\colon H^{k+2}_*(L)\times\mathbb{R}\to H^k(L)$$

is surjective. Moreover, there exists unique $\Gamma\in H^{k+2}_*(L)$ such that $\mathcal{L}(\Gamma)=-\gamma$, so $\mathrm{d}\widehat{F}|_{(0,0)}$ has a 1–dimensional kernel.

Applying the Implicit Function Theorem for Banach spaces (Lang [12, Chapter XIV, Theorem 2.1]), we see that there exist $\varepsilon_1\le\varepsilon_0$, $s_1\le s_0$ and a unique continuous map $s\mapsto\phi(s)$ so that

$$\widehat{F}^{-1}(0)\cap\big(B_{\varepsilon_1}(0)\times(-s_1,s_1)\big)=\big\{\big(\phi(s),s\big):|s|<s_1\big\}.$$

The result follows. □

We now finish the proof of Theorem 5.1.

From Theorem 5.8 we obtain, for all $|s| < s_1$, the existence of a zero-Maslov class self-expander L^s asymptotic to $L^s_0 = P_1(s) + P_2(s)$. The family $(L_s)_{|s|<s_1}$ is continuous in $C^{2,\alpha}$.

To show uniqueness, apply Theorem 3.1 with $k = 3$ and

$$K = \big\{\big(P_1(s), P_2(s)\big)\big\}_{|s|\le s_1} \subset G_n$$

to obtain the existence of ε and R_0 so that if N^s is a self-expander asymptotic to L^s_0 that is ε–close in C^2 to L in B_{R_0}, then $N^s = L^{\phi,s}$ for some $\phi \in B_{\varepsilon_1}$. Theorem 5.8 implies N^s is unique and equal to L^s. □

6 Compactness theorem in $\mathbb{C}^2$

We now restrict to the situation where the self-expander is asymptotic to transverse planes in $\mathbb{C}^2$. The reason is that it is only in $\mathbb{C}^2$ where a Lagrangian cone with density strictly less than 2 must be a plane. For $n > 2$, the Harvey–Lawson $U(1)^{n-1}$–invariant special Lagrangian cone in $\mathbb{C}^n$ has density strictly between 1 and 2.

Consider $M = G_L(2, \mathbb{C}^2) \times G_L(2, \mathbb{C}^2)$, where $G_L(2, \mathbb{C}^2)$ denotes the set of all multiplicity one Lagrangian planes in $\mathbb{C}^2$. Define

$$SL = \{(P_1, P_2) \in M \mid P_1 + P_2 \text{ or } P_1 - P_2 \text{ is area-minimizing}\}$$

and

$$\Lambda = \{(P_1, P_2) \in M \mid P_1 \cap P_2 = \{0\}\} \setminus SL. \tag{47}$$

Since $(P_1, P_2) \in M$ lies in SL if and only if the sum of the angles between P_1 and P_2 is an integer multiple of π, we see that Λ is an open subset of M.

The aim of this section is to prove the following compactness result.

Theorem 6.1 *Pick a compact set $K \subset \Lambda$. The set*

$$\mathcal{S}(K) = \{L \subseteq \mathbb{C}^2 \mid L \text{ is a zero-Maslov class Lagrangian self-expander that is asymptotic to } P_1 + P_2, \text{ where } (P_1, P_2) \in K\}$$

is compact in the $C^{2,\alpha}$ topology.

Proof Let $(L^i)_{i\in\mathbb{N}}$ be a sequence of self-expanders in $\mathcal{S}(K)$ asymptotic to $L^i_0 = P^i_1 + P^i_2$. Setting $L^i_t = \sqrt{2t}\, L^i$, we thus have a sequence $(L^i_t)_{t\ge 0}$ of solutions to

Lagrangian mean curvature flow that are smooth for all $t > 0$. From Lemma 2.4 we have uniform area bounds for $(L^i_t)_{t\geq 0}$ and so Ilmanen [8, Theorem 7.1] implies that we can consider a subsequence that converges weakly to an integral Brakke motion $(L_t)_{t\geq 0}$.

It also follows from [8, Theorem 7.1] that, for almost all $t > 0$, L^i_t admits a subsequence that converges to L_t as an integral varifold and so $2tH = \boldsymbol{x}^\perp$ on L_t. Furthermore, Radon measure convergence implies that $L_t = \sqrt{2t}\, L_{1/2}$ for all $t > 0$. In particular, $L_{1/2}$ is an integral varifold with $H = \boldsymbol{x}^\perp$ and we denote it simply by L.

Compactness of K implies that, after passing to another subsequence, (P^i_1, P^i_2) converges to $(P_1, P_2) \in K$. Our objective is to show that $L \in \mathcal{S}(K)$.

We first show that L is asymptotic to $L_0 = P_1 + P_2$.

Lemma 6.2 *There is $R_0 > 0$ and $\psi \in C^\infty(L_0 \setminus B_{R_0})$ so that*

$$L \setminus B_{2R_0} \subset \{x + J\overline{\nabla}\psi(x) \mid x \in L_0 \setminus B_{R_0}\} \subset L \setminus B_{R_0/2}$$

and, for some $b > 0$,

$$\|\psi\|_{C^{3,\alpha}(L_0\setminus B_R)} = O(e^{-bR^2}) \quad \text{as } R \to \infty.$$

Moreover, $L^i \setminus B_{R_0}$ converges to $L \setminus B_{R_0}$ in $C^{2,\alpha}$ as $i \to \infty$.

Proof The lemma follows from Theorem 3.1 applied to $(L^i)_{i\in\mathbb{N}}$. □

We can now deduce that L is asymptotic to L_0.

Lemma 6.3 *As Radon measures, $L_t \to L_0$ as $t \to 0$.*

Proof Given $\varepsilon > 0$ small we obtain from Lemma 6.2 that, for all t sufficiently small,

$$L_t \setminus B_{2\varepsilon} \subset \{x + JY_t(x) \mid x \in L_0 \setminus B_\varepsilon\} \subset L_t \setminus B_{\varepsilon/2},$$

where the $C^{2,\alpha}$ norm of the vector field Y_t tends to zero as $t \to 0$. Thus,

$$\lim_{t\to 0^+} \int_{L_t} \phi \, \mathrm{d}\mathcal{H}^2 = \int_{L_0} \phi \, \mathrm{d}\mathcal{H}^2$$

for all $\phi \in C^\infty_0(\mathbb{C}^2)$. □

The next proposition is one of the key steps to ensure that L is smooth.

Proposition 6.4 *L is not a stationary varifold.*

Proof Assume L is stationary. Then L needs to be a cone because $\boldsymbol{x}^\perp = H = 0$. Thus $L_t = \sqrt{2t}\, L$ has $H = 0$ for all $t > 0$ and we obtain from Lemma 6.3 that $L = P_1 + P_2$. The goal for the rest of this proof is to show that L must be area-minimizing and this gives us a contradiction because $(P_1, P_2) \notin SL$.

Since L^i is a self-expander we have (from varifold convergence) that for every $r > 0$

$$\lim_{i\to\infty} \int_{L^i \cap B_r} |\boldsymbol{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^2 = \int_{L\cap B_r} |\boldsymbol{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^2 = 0$$

and thus, for all $r > 0$,

$$\lim_{i\to\infty} \int_{L^i \cap B_r} \big(|H|^2 + |\boldsymbol{x}^\perp|^2\big) \, \mathrm{d}\mathcal{H}^2 = \lim_{i\to\infty} \int_{L^i \cap B_r} \big(2|\boldsymbol{x}^\perp|^2\big) \, \mathrm{d}\mathcal{H}^2 = 0. \tag{48}$$

Lemma 6.5 *The following properties hold:*

(i) *There is $d_0 > 0$ so that for every $R > 0$, every i sufficiently large, and every open subset A of $L^i \cap B_{4R}$ with rectifiable boundary we have*

$$\big(\mathcal{H}^2(A)\big)^{\frac{1}{2}} \le d_0 \mathcal{H}^1(\partial A).$$

(ii) *There is $R_1 > 0$ so that for all $R > R_1$ and all i sufficiently large*

$$L^i \cap B_{2R} \text{ is connected and } \partial(L^i \cap B_{3R}) \subset \partial B_{3R}.$$

(iii) *There is $c > 0$ so that for all i sufficiently large we have*

$$\sup_{L^i} |\theta^i| = \sup_{L^i} |\beta^i| \le c.$$

Proof We first prove (i). From the Michael–Simon Sobolev inequality (see Simon [21, Theorem 18.6])

$$\big(\mathcal{H}^2(A)\big)^{\frac{1}{2}} \le c_0 \int_A |H| + c_0 \mathcal{H}^1(\partial A)$$

for some universal constant $c_0 > 0$. In this case we have

$$\big(\mathcal{H}^2(A)\big)^{\frac{1}{2}} \le c_0 \big(\mathcal{H}^2(A)\big)^{1/2} \bigg(\int_A |H|^2 \bigg)^{\frac{1}{2}} + c_0 \mathcal{H}^1(\partial A)$$

and so we get the desired claim because for all i sufficiently large we have (due to (48))

$$c_0^2 \int_{L^i \cap B_{4R}} |H|^2 \le \frac{1}{4}.$$

Property (ii) follows from Lemma 6.2.

Finally, we prove property (iii). Given $y_i \in L^i$, denote by $\widehat{B}_r(y_i)$ the intrinsic ball in L^i of radius r and set $\psi_i(r) = \mathcal{H}^2(\widehat{B}_r(y_i))$. From (i) we see that for almost all r

$$\big(\psi_i(r)\big)^{\frac{1}{2}} \leq d_0 \mathcal{H}^1\big(\partial \widehat{B}_r(y_i)\big) = d_0 \psi_i'(r).$$

Integrating the above inequality implies the existence of $d_1 > 0$, depending only on d_0, so that for all $R > 0$

$$\mathcal{H}^2\big(\widehat{B}_r(y_i)\big) \geq d_1 r^2 \quad \text{for all } y_i \in B_{3R} \cap L^i \text{ and } r < R. \tag{49}$$

Choose β^i, the primitive for the Liouville form $\lambda|_{L_i}$, so that $\beta^i + \theta^i = 0$ (L^i is a self-expander). Combining the uniform area bounds given in Lemma 2.4 with (49), we have that the intrinsic diameter of $L^i \cap B_R$ is uniformly bounded for all i sufficiently large. Hence, if $x, y \in L^i \cap B_R$ and γ is a path in $L^i \cap B_R$ connecting x to y, we have

$$\beta^i(x) - \beta^i(y) = \int_\gamma \lambda \leq R \operatorname{length}(\gamma).$$

Thus the oscillation of β^i in $L^i \cap B_R$ is uniformly bounded. The angle $\theta^i = -\beta^i$ can always be chosen so that its range in $L^i \cap B_R$ intersects the interval $[0, 2\pi]$ and so we obtain that $\theta^i = -\beta^i$ is uniformly bounded in B_R.

From Lemma 6.2 we know that $\theta^i = -\beta^i$ are uniformly bounded outside a large ball and thus are uniformly bounded on L^i. □

We can now finish the proof of Proposition 6.4. Recall that $L = P_1 + P_2$ in the varifold sense and, if necessary, we can change the orientation of one of the planes so that the identity also holds in the current sense. We want to show that $L = P_1 + P_2$ is area-minimizing.

We know that for all $R > 0$,

$$\lim_{i\to\infty} \int_{L^i \cap B_R} |H|^2 + |\boldsymbol{x}^\perp|^2 \, \mathrm{d}\mathcal{H}^2 = 0$$

and $|\boldsymbol{x}^\perp| = |\nabla \beta^i|$ is uniformly bounded in B_R. From Lemma 6.5 we have that all conditions necessary to apply Neves' [16, Proposition A.1] are met and so we conclude the existence of a constant $\overline{\beta}$ and R_2 such that, for all $\phi \in C_0^\infty(\mathbb{C}^2)$,

$$\lim_{i\to\infty} \int_{L^i \cap B_{R_2}} (\beta^i - \overline{\beta})^2 \phi \, \mathrm{d}\mathcal{H}^2 = 0.$$

Hence

$$\lim_{i\to\infty} \int_{L^i \cap B_{R_2}} (\theta^i + \overline{\beta})^2 \phi \, \mathrm{d}\mathcal{H}^2 = 0$$

for all $\phi \in C_0^\infty(\mathbb{C}^2)$. We deduce, from [16, Proposition 5.1], that L has constant Lagrangian angle $-\overline{\beta}$ and is thus area-minimizing, providing our required contradiction. □

Using the fact that L is not stationary, we now show that L satisfies the conditions of White's Regularity Theorem. It is in this lemma that we use the fact that $n = 2$ in a crucial way. Recall the definition of Gaussian density in (3).

Lemma 6.6 *Given $\varepsilon_0 > 0$ small, there is $\delta > 0$ so that*

$$\Theta_t(y, l) \leq 1 + \frac{\varepsilon_0}{2} \quad \text{for every } l \leq \delta t,\ y \in \mathbb{C}^2 \text{ and } t > 0.$$

Remark 6.7 We briefly sketch the idea. The first step is to find δ so that $\Theta_{1/2}(y, l) < 2$ for all $y \in \mathbb{C}^n$ and $l \leq \delta$. This follows because the monotonicity formula implies that $\Theta_{1/2}(y, l) \leq \Theta_0(y, l + \frac{1}{2}) \leq 2$ with equality only if L is a self-shrinker centered at the origin. In the latter case, because L is a self-expander, we obtain that L must be stationary, which contradicts Proposition 6.4. Thus, the strict inequality holds as claimed.

The second step is to show that if $\Theta_{1/2}(y_i, \delta_i) \geq 1 + \frac{\varepsilon_0}{2}$ for some sequence δ_i tending to zero, then we can blow-up L and obtain a stationary Lagrangian varifold $\widetilde{L}$ that is not a plane. Then we blow-down $\widetilde{L}$ to obtain a stationary Lagrangian cone C that must have Gaussian density at the origin bigger than $1 + \frac{\varepsilon_0}{2}$. Since we are in $\mathbb{C}^2$, this forces the Gaussian density at the origin to be at least two, which we then show contradicts the first step.

Proof It suffices to prove the lemma for $t = \frac{1}{2}$ because $L_t = \sqrt{2t}\, L$.

In what follows we will constantly use the fact that, because P_1 intersects P_2 transversely,

$$\int_{P_1+P_2} \Phi(y, l)\, \mathrm{d}\mathcal{H}^2 < 2 \quad \text{for all } l > 0 \text{ and } y \neq 0,$$

with equality holding if $y = 0$.

First step We start by arguing the existence of $c_1 > 0$ such that, for every $l \leq 2$ and $y \in \mathbb{C}^2$,

$$\int_L \Phi(y, l)\, \mathrm{d}\mathcal{H}^2 \leq 2 - c_1^{-1}. \tag{50}$$

From the monotonicity formula for Brakke flows [7, Lemma 7],

$$\int_L \Phi(y,l)\,\mathrm{d}\mathcal{H}^2 + \int_0^{\frac{1}{2}} \int_{L_t} \left| H + \frac{(x-y)^\perp}{2(l+\frac{1}{2}-t)} \right|^2 \Phi(y, l+\tfrac{1}{2}-t)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t = \int_{P_1+P_2} \Phi(y, l+\tfrac{1}{2})\,\mathrm{d}\mathcal{H}^2 \le 2. \tag{51}$$

Suppose there is a sequence y_i and l_i with $0 \le l_i \le 2$ such that

$$\int_L \Phi(y_i, l_i)\,\mathrm{d}\mathcal{H}^2 \ge 2 - \frac{1}{i}\,.$$

Then, by (51),

$$\int_{P_1+P_2} \Phi(y_i, l_i + \tfrac{1}{2})\,\mathrm{d}\mathcal{H}^2 \ge 2 - \frac{1}{i}$$

and so y_i must converge to zero.

Assuming without loss of generality that l_i converges to $\overline{l}$, we have again from (51) that

$$\begin{aligned}\int_0^{\frac{1}{2}} \int_{L_t} \left| H + \frac{x^\perp}{2(\overline{l}+\frac{1}{2}-t)} \right|^2 \Phi\big(0, \overline{l}+\tfrac{1}{2}-t\big)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \\ = \lim_{i\to\infty} \int_0^{\frac{1}{2}} \int_{L_t} \left| H + \frac{(x-y_i)^\perp}{2\big(l_i+\frac{1}{2}-t\big)} \right|^2 \Phi\big(y_i, l_i+\tfrac{1}{2}-t\big)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \\ \le 2 - \lim_{i\to\infty} \int_L \Phi(y_i, l_i)\,\mathrm{d}\mathcal{H}^2 = 0\end{aligned}$$

and thus

$$H + \frac{\boldsymbol{x}^\perp}{2\big(\overline{l}+\frac{1}{2}-t\big)} = 0 \quad \text{on } L_t \text{ for almost all } t \in [0, \tfrac{1}{2}].$$

Combining this with the fact that $H = \boldsymbol{x}^\perp/(2t)$ on L_t we obtain that L must be stationary, which contradicts Proposition 6.4. Thus (50) must hold.

Second step To finish the proof we argue again by contradiction and assume the existence of sequences $(y_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^2$ and $(\delta_j)_{j\in\mathbb{N}}$ converging to zero so that

$$\Theta_{\frac{1}{2}}(y_j, \delta_j) \ge 1 + \frac{\varepsilon_0}{2}\,. \tag{52}$$

From the monotonicity formula for Brakke flows [7, Lemma 7] we have

$$\int_{P_1+P_2} \Phi(y_j, \delta_j + \tfrac{1}{2})\,\mathrm{d}\mathcal{H}^2 \ge \Theta_{1/2}(y_j, \delta_j) \ge 1 + \frac{\varepsilon_0}{2}\,.$$

Note that the sequence $(|y_j|)_{j\in\mathbb{N}}$ is bounded by a positive constant M_0, because otherwise we could find a subsequence so that

$$\lim_{j\to\infty}\int_{P_1+P_2}\Phi(y_j,\delta_j+\tfrac{1}{2})\le 1.$$

Consider the sequence of blow-ups

$$\widetilde{L}^{j,i}_s=\delta_j^{-1/2}\big(L^i_{1/2+s\delta_j}-y_j\big)\quad\text{with } s\ge 0.$$

A standard diagonalization argument allows us to consider a subsequence $\widetilde{L}^j_s=\widetilde{L}^{j,i(j)}_s$ such that, for all $1\le l\le j$,

$$-\frac{1}{j}\le\int_{\widetilde{L}^j_0}\Phi(0,l)\,\mathrm{d}\mathcal{H}^2-\int_L\Phi(y_j,l\delta_j)\,\mathrm{d}\mathcal{H}^2\le\frac{1}{j}\,. \tag{53}$$

Thus, for every $r>0$,

$$\begin{aligned}\int_0^1\int_{\widetilde{L}^j_s\cap B_r}|H|^2\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}s&=\delta_j^{-1}\int_{1/2}^{1/2+\delta_j}\int_{L^{i(j)}_t\cap B_{\sqrt{\delta_j}r}(y_j)}|H|^2\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\&=\delta_j^{-1}\int_{1/2}^{1/2+\delta_j}\int_{L^{i(j)}_t\cap B_{\sqrt{\delta_j}r}(y_j)}\Big|\frac{\boldsymbol{x}^\perp}{2t}\Big|^2\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\le c_2\delta_j,\end{aligned}$$

where c_2 depends on r and M_0. Therefore

$$\lim_{j\to\infty}\int_0^1\int_{\widetilde{L}^j_s\cap B_r}|H|^2\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}s=0$$

and so $(\widetilde{L}^j_s)_{0\le s\le 1}$ converges to an integral Brakke flow $(\widetilde{L}_s)_{0\le s\le 1}$ with $\widetilde{L}_s=\widetilde{L}$ for all s and $\widetilde{L}$ a stationary varifold. From [16, Proposition 5.1] we conclude that $\widetilde{L}$ is a union of special Lagrangian currents (the Lagrangian angle is uniformly bounded by Lemma 6.5(iii)).

From (52) and (53) we have

$$\int_{\widetilde{L}}\Phi(0,1)\,\mathrm{d}\mathcal{H}^2\ge 1+\frac{\varepsilon_0}{2}$$

and so $\widetilde{L}$ cannot be a plane with multiplicity one. The blow-down

$$C=\lim_{i\to 0}\varepsilon_i\widetilde{L},\quad\text{where } \varepsilon_i\to 0,$$

is a union of Lagrangian planes (as these are the only special Lagrangian cones in $\mathbb{C}^2$) and so

$$\lim_{i\to\infty}\int_{\widetilde{L}}\Phi(0,\varepsilon_i^{-2})\,\mathrm{d}\mathcal{H}^2=\lim_{i\to\infty}\int_{\varepsilon_i\widetilde{L}}\Phi(0,1)\,\mathrm{d}\mathcal{H}^2=\int_C\Phi(0,1)\,\mathrm{d}\mathcal{H}^2\geq 2.$$

From (53) this implies that one can find l such that for every j sufficiently large we have

$$2-\frac{1}{2c_1}\leq\int_{\widetilde{L}_0^j}\Phi(0,l)\,\mathrm{d}\mathcal{H}^2\leq\int_L\Phi(y_j,l\delta_j)\,\mathrm{d}\mathcal{H}^2+\frac{1}{j}\,.$$

This contradicts (50). □

We may now complete the proof of Theorem 6.1. From Lemmas 6.2 and 6.6 we have that, for all i sufficiently large,

$$\Theta_t^i(y,l)\leq 1+\varepsilon_0\quad\text{for every } l\leq\delta t,\ \ y\in\mathbb{C}^2 \text{ and } t>0,$$

where $\Theta_t^i(y,l)$ is the Gaussian density (3) of L_t^i.

White's Regularity Theorem [23] implies uniform $C^{2,\alpha}$ bounds for $L^i_{1/2}$ and so $L=L_{1/2}$ is a smooth multiplicity one self-expander asymptotic to P_1+P_2 with $(P_1,P_2)\in K$ and $L^i_{1/2}$ converges to L in $C^{2,\alpha}$. Hence, $L\in\mathcal{S}(K)$ as we wanted to show. □

7 Uniqueness theorem in $\mathbb{C}^2$

We first prove the uniqueness for self-expanders that are asymptotic to planes P_1+P_2, where P_1 and P_2 share the same S^1–symmetry.

Equivariant case

We say a Lagrangian surface $N\subset\mathbb{C}^2$ is equivariant if there is a curve $\gamma\colon\mathbb{R}\to\mathbb{C}$ or $\gamma\colon[0,\infty)\to\mathbb{C}$ so that

$$N=\{(\gamma(s)\cos\alpha,\gamma(s)\sin\alpha)\mid s\in\mathbb{R},\alpha\in[0,2\pi]\}\subset\mathbb{C}^2.$$

Consider the ambient function $\mu=x_1y_2-y_2x_1$. The relevance of this function is that an embedded Lagrangian N is equivariant if and only if $N\subset\mu^{-1}(0)$ (see Neves [18, Lemma 7.1] for instance).

Studying the o.d.e. arising from $H=\boldsymbol{x}^\perp$, Anciaux [1] showed that given two equivariant planes P_1,P_2, there is a unique equivariant Lagrangian self-expander L asymptotic to $L_0=P_1+P_2$.

Lemma 7.1 *Suppose that L is a zero-Maslov class Lagrangian self-expander asymptotic to $L_0 = P_1 + P_2$, where P_1, P_2 are equivariant planes. Then L is equivariant. In particular, it is unique.*

Proof From [17, Lemma 3.3] we know that along $L_t = \sqrt{2t}\, L$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu^2 = \Delta\mu^2 - 2|\nabla\mu|^2.$$

Using the evolution equation above in Huisken's monotonicity formula we have that, for $t < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{L_t} \mu^2 \Phi(0, 1-t)\, \mathrm{d}\mathcal{H}^2 \leq 0.$$

From Theorem 3.1 we see that

$$\lim_{t\to 0}\int_{L_t} \mu^2 \Phi(0, 1-t)\, \mathrm{d}\mathcal{H}^2 = \int_{P_1+P_2} \mu^2 \Phi(0, 1)\, \mathrm{d}\mathcal{H}^2 = 0$$

because P_1, P_2 are equivariant. Thus

$$\int_{L_t} \mu^2 \Phi(0, 1-t)\, \mathrm{d}\mathcal{H}^2 = 0$$

for all $t < 1$ and this implies the desired result. □

General case

Consider the set $\Lambda \subset G_L(2, \mathbb{C}^2) \times G_L(2, \mathbb{C}^2)$ defined in (47):

$$\Lambda = \{(P_1, P_2) \mid P_1 \cap P_2 = \{0\}\} \setminus SL.$$

Theorem 7.2 *Given $(P_1, P_2) \in \Lambda$ there is a unique zero-Maslov class self-expander L that is asymptotic to $L_0 = P_1 + P_2$.*

Remark 7.3 The existence of such self-expanders was proven by Joyce, Lee and Tsui in [10]. Explicit formulae for the self-expanders are given in [10, Theorem C].

Proof Choose a basis of $\mathbb{C}^2$ so that P_1 is the real plane and

$$P_2 = \{(e^{i\theta_1}x, e^{i\theta_2}y) \mid x, y \in \mathbb{R}\}.$$

Set

$$P_2(s) = \{(e^{i\theta_1 - is(\theta_1-\theta_2)/2}x, e^{i\theta_2 + is(\theta_1-\theta_2)/2}y) \mid x, y \in \mathbb{R}\}.$$

The key properties of $(P_1, P_2(s))$ are that

- $(P_1, P_2(s)) \in \Lambda$ for all $0 \leq s \leq 1$ and the Lagrangian angle of $P_2(s)$ is constant,
- $P_1 \subset \mu^{-1}(0)$ and $P_2(1) \subset \mu^{-1}(0)$.

Consider the compact subset of Λ given by $K = \{(P_1, P_2(s))\}_{0 \leq s \leq 1}$ and recall

$$\mathcal{S}(K) = \{L \subseteq \mathbb{C}^2 \mid L \text{ is a zero-Maslov class Lagrangian self-expander that is asymptotic to } P_1 + P_2, \text{ where } (P_1, P_2) \in K\}.$$

Consider the obvious projection map $\pi\colon \mathcal{S}(K) \to [0, 1]$.

From [10, Theorem C] we know that π is surjective. By Theorem 5.1, one may choose a suitable topology on $\mathcal{S}(K)$ so that π is a local diffeomorphism. By Theorem 6.1, $\mathcal{S}(K)$ is also compact with respect to this topology. Therefore π is a covering map. However, by Lemma 7.1, we have that $\pi^{-1}(1)$ consists of a single element and so π is a diffeomorphism. In particular, $\pi^{-1}(0)$ consists of a single element. □

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Proposed: Tobias H Colding Received: 14 August 2012
Seconded: Richard Thomas, Leonid Polterovich Accepted: 29 April 2013