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Convergence properties of end invariants

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We prove a continuity property for ending invariants of convergent sequences of Kleinian surface groups. We also analyze the bounded curve sets of such groups and show that their projections to non-annular subsurfaces lie a bounded Hausdorff distance from geodesics joining the projections of the ending invariants.

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1 Introduction

The solution (Minsky [26], and Brock, Canary and Minsky [11]) of Thurston's ending lamination conjecture (together with that of Marden's tameness conjecture; see Agol [1], and Calegari and Gabai[13]) gives a complete classification of finitely generated Kleinian groups in terms of their topological type and their *end invariants*. This classification leaves an incomplete picture, however, because it does not describe the *topology* of the deformation space of hyperbolic structures associated to a given group (with the natural topology induced from representation spaces). In particular, the end invariant data does not vary continuously with deformations in any of the usual topologies that have arisen historically (Brock [8], and Anderson and Canary [2]). Moreover, such deformation spaces can fail to be locally connected (Bromberg [12], and Magid [20]). In this article, we describe how end invariants do converge in limiting families of hyperbolic structures. In the process, we produce a number of important structural refinements to the geometric picture developed in [26; 11].

We restrict ourselves to *Kleinian surface groups*, which are discrete, faithful representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ where S is an oriented compact surface (a parabolicity condition is imposed on ∂S if it is nonempty). Let $\mathrm{AH}(S)$ denote the space of conjugacy classes of such representations, viewed as a subset of the $\mathrm{PSL}_2(\mathbb{C})$ character variety of $\pi_1(S)$. The end invariants of $[\rho] \in \mathrm{AH}(S)$ are a pair of data $\nu^{\pm}(\rho)$, each a union of marked Riemann surface structures and geodesic laminations supported on essential subsurfaces of S (see Section 2 for details). The orientations of S and of the

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quotient manifold $N_\rho = \mathbb{H}^3/\operatorname{Im}(\rho)$ give N_ρ a "top" and "bottom" side or *end*, with asymptotic geometry encoded by ν^+ and ν^-, respectively.

Limits of projections of end invariants The primary objective of [26; 11], as well as their precursors by Minsky [24; 25], is to obtain coarse information about N_ρ using the projections of ν^+ and ν^- to the *curve complexes* $\mathcal{C}(W)$, where $W \subseteq S$ denotes an essential subsurface of S. Let $\pi_W(\nu^\pm)$ denote these projections. (We emphasize that we allow the possibility that $W = S$.) We recall that $\mathcal{C}(W)$ is a δ–hyperbolic metric space (Masur and Minsky [21]), and that (for W non-annular) its Gromov boundary can be identified with $\mathcal{EL}(W)$, the space of *unmeasured filling laminations* in W (Hamenstädt [17] and Klarreich [18]). Moreover $\mathcal{EL}(W)$ is also the set of laminations that can occur as components of the end invariants $\nu^\pm$ supported on non-annular W. Our first theorem describes a sense in which the end invariants in a convergent sequence of representations can be said to converge, establishing a continuity property for the projections of end invariants to subsurfaces.

Theorem 1.1 *Let $\rho_n \to \rho$ in $\operatorname{AH}(S)$. If $W \subseteq S$ is an essential subsurface of S other than an annulus or a pair of pants, and $\lambda \in \mathcal{EL}(W)$ is a lamination supported on W, the following statements are equivalent:*

(1) *λ is a component of $\nu^+(\rho)$.*

(2) *$\{\pi_W(\nu^+(\rho_n))\}$ converges to λ.*

Furthermore, we have:

(a) *If $\{\pi_W(\nu^+(\rho_n))\}$ accumulates on $\lambda \in \mathcal{EL}(W)$, then it converges to λ.*

(b) *The sequences $\{\nu^+(\rho_n)\}$ and $\{\nu^-(\rho_n)\}$ do not converge to a common $\lambda \in \mathcal{EL}(S)$.*

(c) *If $W \subsetneq S$ is a proper subsurface, then convergence of $\{\pi_W(\nu^+(\rho_n))\}$ to $\lambda \in \mathcal{EL}(W)$ implies $\{\pi_W(\nu^-(\rho_n))\}$ does not accumulate on $\mathcal{EL}(W)$.*

The same statements hold with "$+$" replaced by "$-$".

Leininger and Schleimer [19, Theorem 6.5] previously established Theorem 1.1 in the setting of doubly degenerate Kleinian surface groups, ie, Kleinian surface groups whose ending invariants are a pair of filling laminations on S.

Ohshika has independently obtained a similar result in [27, Theorem 2], phrased in the equivalent language of Hausdorff limits. One can make the hybrid objects $\nu^\pm(\rho_n)$ into laminations by replacing each Riemann surface component of $\nu^\pm(\rho_n)$ with bounded length pants decompositions on the associated hyperbolic metric. We then let $\lambda^\pm$ denote the Hausdorff limit of these sequences. The statement that $\pi_W(\nu^+(\rho_n))$ converges to $\lambda \in \mathcal{EL}(W)$ is then equivalent to the condition that λ^+ contains λ as a component.

This convergence behavior was presaged in the examples of [8] for representations in a *Bers slice*, and those examples also indicate how the Hausdorff topology on end invariants *necessarily* fails to predict the full end-invariant of the limit. In particular, one might hope to find that the parabolic components of $\nu^\pm(\rho)$ always arise either as components of the Hausdorff limits λ^+ or λ^-, or as boundary components of subsurfaces W filled by components λ of these Hausdorff limits. However, in [8] examples are given in which parabolic curves in the limiting invariants are not related to the Hausdorff limit in either of these ways.

In the other direction, the phenomenon of *wrapping* explored in detail in [2] gives examples in which *both* Hausdorff limits λ^+ and λ^- contain the same curve as a component, but the curve can only appear in the end invariant of one side or the other in the limit.

We do not address these subtleties here, but note that all parabolics and ending laminations are predicted in full by recording the full collection of subsurfaces for which such projections diverge, or equivalently, by studying the sequence of *hierarchies* associated to the end invariants. The precise behavior and its connection to end invariants will be described in Brock, Bromberg, Canary and Lecuire [9].

Theorem 1.1, together with Theorem 1.2 below, is used in our related paper [10] to analyze (and rule out) "bumping" phenomena on the boundary of $\mathrm{AH}(S)$, and in particular to identify boundary points where $\mathrm{AH}(S)$ is locally connected. Theorem 1.1 will also be applied, together with Theorems 1.2 and 1.3, in [9], which gives a complete characterization, in terms of end invariants, of convergence and divergence of sequences of Kleinian surface groups.

Controlling the bounded curve sets The second theme of this paper involves improving our understanding of the *bounded curve sets* associated to a Kleinian surface group. In [26; 11] we applied the notion of a *hierarchy of geodesics* as developed in Masur and Minsky [22]. This combinatorial device connects the two end invariants with a family of *markings*, curve systems on S, in a combinatorially efficient way. A crucial step in [26; 11] is to establish a priori bounds on the geodesic lengths of all simple closed curves that appear in such a hierarchy. On the other hand one can simply ask to understand the full set

$$\mathcal{C}(\rho, L) = \{\alpha \in \mathcal{C}(S) : \ell_\rho(\alpha) \le L\}$$

of simple closed curves in S whose ρ–length is bounded by L (for a given L). Our second theorem gives a description of this set in terms of its *subsurface projections*.

We denote by $\mathrm{hull}_W(\nu^+,\nu^-)$ the union of geodesics in $\mathcal{C}(W)$ connecting $\pi_W(\nu^+)$ to $\pi_W(\nu^-)$. (Hyperbolicity of $\mathcal{C}(W)$ implies that this union lies in a uniform neighborhood of any one of its members). The set of curves appearing in the hierarchy has the property that its projections into each $\mathcal{C}(W)$ lie in a uniformly bounded neighborhood of $\mathrm{hull}_W(\nu^\pm)$. The next theorem shows that the same holds for the bounded curve set. Let d_{Haus} denote Hausdorff distance for subsets of a metric space, applied below to $\mathcal{C}(W)$. We also use $d_W(x,y)$ as an abbreviation for $d_{\mathcal{C}(W)}(\pi_W(x),\pi_W(y))$.

Theorem 1.2 *Given S, there exists L_0 such that for all $L\geq L_0$ there exists $D=D(S,L)$, such that given $\rho\in\mathrm{AH}(S)$ with end invariants $\nu^\pm$ and an essential subsurface $W\subset S$ that is not an annulus or a pair of pants,*

$$d_{\mathrm{Haus}}\big(\pi_W(\mathcal{C}(\rho,L)),\mathrm{hull}_W(\nu^\pm(\rho))\big)\leq D.$$

Moreover, if $d_W(\nu^+(\rho),\nu^-(\rho))>D$, then $\mathcal{C}(\rho,L)\cap\mathcal{C}(W)$ is nonempty and

$$d_{\mathrm{Haus}}\big(\mathcal{C}(\rho,L)\cap\mathcal{C}(W),\mathrm{hull}_W(\nu^\pm(\rho))\big)\leq D.$$

Our third theorem relates the projections of bounded-length curves to their topological ordering in the manifold (in the sense described in Section 2.5). It states that when the geodesic representative α^* of a curve $\alpha\in\mathcal{C}(\rho,L)$ lies above the geodesic representative β^* of some component β of the boundary of a subsurface W that it overlaps, then its projection to $\mathcal{C}(W)$ is uniformly close to $\pi_W(\nu^+)$. (We recall that α^* lies above β^* if α^* can be pushed arbitrarily far upward, in the complement of β^*, in the product structure on $N_\rho\cong S\times\mathbb{R}$.) This property follows directly from the machinery of [11] in the case of curves that arise in the hierarchy of N_ρ (see Lemma 2.6).

Theorem 1.3 *Given S and $L>0$ there exists c such that, given $\rho\in\mathrm{AH}(S)$, an essential subsurface $W\subset S$ that is not a pair of pants, and a curve $\alpha\in\mathcal{C}(\rho,L)$ such that α^* lies above the geodesic representative of some component of ∂W that it overlaps, then*

$$d_W(\alpha,\nu^+(\rho))\leq c.$$

Furthermore, if W is not an annulus or a pair of pants, $\alpha\in\mathcal{C}(\rho,L)$ overlaps ∂W, and

$$d_W(\alpha,\nu^-)>c$$

then α^* lies above the geodesic representative of every component of ∂W that it overlaps.

The same holds when replacing "above" with "below" and ν^+ with ν^-.

We note that the conclusion of Theorem 1.2 need not hold in the case that W is an annulus. It is possible that N_ρ contains a bounded geometry pleated surface that is "wrapped" several, say n, times about the Margulis tube $\mathbb{T}(\beta)$ associated to the core curve of W. If α is a curve on the pleated surface of bounded length, say L, that overlaps β, then α may be concatenated with n copies of the meridian of the Margulis tube of β to obtain a curve α' of length roughly $L+nC$ whose geodesic representative lies above or below β^*. Moreover, $d_W(\alpha,\alpha')$ is roughly $n d_W(\nu^+,\nu^-)$. For any given value of n, one may construct families of examples where $d_W(\nu^+,\nu^-)$ is arbitrarily large, but one may make uniform choices of L and C. (This wrapping construction was introduced in [2]; see also McMullen [23, Lemma A.4] or Canary [15].)

Outline of the paper In Section 2 we review background on curve complexes, hierarchies, Kleinian surface groups and their end invariants. We also review some material from our previous work in [26; 11], particularly the structure of model manifolds associated to hierarchies, and some consequences. In Section 2.5, and particularly Lemma 2.6, we discuss the relationship between combinatorial order relations in a hierarchy, and its connection to a topological ordering in the corresponding 3–manifold. In Section 2.7 we discuss W–product regions, which are submanifolds of either the model manifold or the hyperbolic manifold that are homeomorphic to $W\times[0,1]$ (for some subsurface W) and so that $\partial W\times[0,1]$ is identified with a submanifold of the boundaries of the tubes associated to ∂W. Lemma 2.13 provides criteria on a hierarchy that imply the existence of "large" W–product regions in the associated 3–manifolds.

In Section 3 we study the question of which curves from a hierarchy are "visible" in a pleated surface (or any Lipschitz surface) in a Kleinian surface group. Lemma 3.1 provides a bounded-length system of hierarchy curves in every such surface, satisfying some additional bounded-projection properties. This lemma plays a central role in each of the main theorems.

In Section 4 we prove Theorem 1.2. The main new ingredient here is provided by Lemma 3.1.

In Section 5 we prove Theorem 1.1. We remark that the principal difficulty in the proof involves showing that a component of the limiting lamination corresponding to the top invariants of a sequence is in fact a top invariant for the limit, and not a bottom invariant in the limit. The issue of such possible "flipped ends" in the limit has a long history in this subject, arising first in the work of Thurston [28] on strong limits of quasi-Fuchsian groups. It arises in our proof in [11] of the bi-Lipschitz model theorem as well, and the relevant arguments there contain echos of Thurston's original interpolation argument. In the present paper, we rely primarily on properties of the bi-Lipschitz model, with

Lemma 2.6 on topological ordering and Lemma 2.13 on the existence of thick product regions playing a central role.

In Section 6 we give the proof of Theorem 1.3. We remark that the conclusion of the theorem is already known from the properties of hierarchies and models, when the curves in question are hierarchy curves. Thus we must answer the question of how close the given curves of bounded length are to being hierarchy curves, and Lemma 3.1 provides the needed connection via pleated surfaces. The product region from Lemma 2.13 then gives the necessary control of these pleated surfaces. This argument, for the case of a non-annular surface, is detailed in Section 6.1, whereas for the case of annuli, a fairly different argument is needed, which appears in Section 6.2.

2 Background

2.1 Curve complexes and laminations

We briefly recall definitions and terminology from [11; 22] and related papers. We will denote by $\mathcal{C}(S)$ the curve complex of a surface S of finite type, recalling that it is a locally infinite complex that is δ–hyperbolic with respect to a natural path metric [21]. Vertices of $\mathcal{C}(S)$ are isotopy classes of essential closed curves in S, and simplices correspond to systems of disjoint curves (with a few standard exceptions). The *curve and arc complex* $\mathcal{A}(S)$ is formed similarly, with vertices corresponding to essential properly embedded arcs (up to isotopy rel boundary) as well as curves.

Klarreich's theorem [17; 18] states that the Gromov boundary $\partial\mathcal{C}(S)$ is naturally identified with $\mathcal{EL}(S)$, the set of *filling geodesic laminations* in S, with topology inherited from the space of measured laminations.

Markings A *marking* on S, in the sense of [22], is a system of curves (ie, a simplex of $\mathcal{C}(S)$) together with a selection of *transversal curves*, at most one for each curve in the system. Each transversal intersects the curve it is associated with at most two times, and is disjoint from the others. The simplex of a marking μ is denoted $\mathrm{base}(\mu)$. If the base is a pants decomposition of S and every curve has a transversal we call the marking *complete*.

A *generalized marking* on S is a similar object, except that $\mathrm{base}(\mu)$ is allowed to have components that are minimal geodesic laminations, not just simple closed curves.

Subsurface projections Given an essential non-annular subsurface $W \subset S$, there is a natural map $\pi_{\mathcal{A}(W)}\colon \mathcal{C}(S) \to \mathcal{A}(W) \cup \{\varnothing\}$, which assigns to a curve system in S

the barycenter of the span of the components of its essential intersection with W (or $\varnothing$ if there are none).

A natural construction takes vertices of $\mathcal{A}(W)$ to points in $\mathcal{C}(W)$: Given a proper arc (or curve) $a \subset W$, take the essential components of a regular neighborhood of $a \cup \partial W$. Composing this with $\pi_{\mathcal{A}(W)}$ we obtain a map π_W, which takes vertices of $\mathcal{C}(S)$ to (finite sets of) vertices in $\mathcal{C}(W)$.

For a marking μ in S we can define $\pi_W(\mu) \subset \mathcal{C}(W)$ as the union of $\pi_W(\beta)$ over the curves β in μ. The union has uniformly bounded diameter. For a generalized marking we need to allow π_W to take values in $\mathcal{C}(W) \cup \mathcal{EL}(W)$. If μ contains a minimal component $\lambda \in \mathcal{EL}(W)$ then $\pi_W(\mu) = \lambda$. If not then as above $\pi_W(\mu)$ is the union of $\pi_W(\beta)$ over the closed curves β in μ.

Complexes and projections can be defined for annuli also, with some care. If A is an annulus and γ its core curve, we consider the annular lift of S associated to A, which has a natural compactification coming from the circle at infinity of $\widetilde{S}$. Vertices of $\mathcal{A}(S)$ are essential arcs in this annulus, up to homotopy fixing endpoints. Given a curve α in S that crosses an annulus A essentially, lift α to the annular cover and keep only those components that cross the annulus (or select one arbitrarily) to obtain $\pi_A(\alpha)$.

Given generalized markings (or curves) α and β that intersect W essentially, we regularly use the shorthand $d_W(\alpha, \beta)$ to denote $d_{\mathcal{C}(W)}(\pi_W(\alpha), \pi_W(\beta))$. If γ is the core of an annulus A, we write d_A and d_γ interchangeably.

2.2 Kleinian surface groups and end invariants

Let $\mathrm{AH}(S)$ denote the space of Kleinian surface groups, ie, discrete faithful representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ taking peripheral elements to parabolics, and considered up to conjugacy in the image. The *end invariants* of $\rho \in \mathrm{AH}(S)$ are two hybrid objects $\nu^{\pm}(\rho)$, each a combination of laminations and conformal structures on subsurfaces of S. We sketch a description here, referring to [26; 11] and the references therein for more details.

Let $N = N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ be the quotient 3–manifold, and let N^0 denote N minus the (open) cusp neighborhoods associated to the parabolic subgroups of $\rho(\pi_1(S))$ (which we note include one cusp for each component of ∂S). This manifold with boundary has a *relative compact core* $\mathcal{K} \subset N^0$, which meets each cusp boundary in one core annulus. Thus $\mathcal{K}$ can be identified with $S \times [-1, 1]$, and $\mathcal{K} \cap \partial N^0$ is a union P of annuli in $\partial\mathcal{K}$ that includes $P_0 = \partial S \times [-1, 1]$. We further decompose $P - P_0$ into the union P^+ of components of P in $S \times \{1\}$ and the remaining components P^-.

This decomposition has the property that no two annuli in P are pairwise isotopic in $S \times I$.

The closure of each component W of $\partial\mathcal{K} \setminus P$ bounds a component U_W of $N^0 \setminus \mathcal{K}$, which is a *neighborhood of an end of* N^0. We say that W *faces* this end and vice versa, and there are two possibilities for its geometry:

- **Geometrically finite** It corresponds to a component of the boundary at infinity of N_ρ, and W inherits a finite-type conformal structure, ie, a point in $\mathrm{Teich}(W)$. The convex core of N intersects U_W in a bounded set.
- **Simply degenerate** It is described by an *ending lamination*, which is a filling geodesic lamination in W, ie, an element of $\mathcal{EL}(W)$. This lamination is the support of the limit (in Thurston's projective lamination space) of any sequence of curves in W whose geodesic representatives exit every bounded subset of U_W.

The end invariant $\nu^+(\rho)$ is a list of the following data: The core curves of the annuli of P^+ that lie in $S \times \{1\}$, the conformal structures associated to geometrically finite ends facing subsurfaces in $S \times \{1\} \setminus P^+$, and the laminations associated to simply degenerate ends facing subsurfaces in $S \times \{1\} \setminus P^+$. The invariant $\nu^-(\rho)$ is defined similarly; the ends associated to ν^+ and to ν^- are called upward-pointing and downward-pointing, respectively.

We recall here Thurston's notion of a *pleated surface* (or map), which is a map $f\colon X \to N$ where X is a hyperbolic surface and N a hyperbolic 3–manifold, such that f is length-preserving and totally geodesic on the strata of a geodesic lamination on X. In the setting of a Kleinian surface group $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, we typically consider pleated maps with underlying surface S, in the homotopy class determined by ρ. We say that such a map *realizes* a lamination λ if it maps the leaves of λ geodesically.

The laminations and parabolic components of the end invariants are exactly those laminations that are *unrealizable* in ρ. So for example if $\nu^+(\rho)$ is a single lamination that fills S, there is no pleated map that carries ν^+ geodesically, and moreover if γ_n is a sequence of closed curves converging to ν^+ then a sequence of pleated surfaces realizing γ_n will necessarily escape every compact subset of N_ρ and converge to the end associated to ν^+.

End markings In order to have a more topological object to work with, in [26, Section 7.1] we convert the end invariants $\nu^\pm$ to a pair of generalized markings $\mu^\pm$ as follows: for each conformal structure on a subsurface W we select a minimal-length complete marking on W. The union of these with core curves of the annuli P^+ and the lamination components of ν^+ will be the generalized marking μ^+; define μ^-

similarly. Note that the total length of $\operatorname{base}(\mu^{\pm})$ is bounded by the *Bers constant*, L_B, which bounds the length of the minimal curve system in any hyperbolic structure on S (Bers [4]).

With this in mind we can define the projections $\pi_Y(\nu^{\pm}) = \pi_Y(\mu^{\pm})$ for any essential non-annular subsurface Y that is not a core curve of a component of the parabolic locus $P^{\pm}$.

There is a bit of flexibility in this definition, as the choice of markings in the geometrically finite subsurfaces may not be unique. For our purposes this will not matter, as the different choices have π_Y images differing by a uniform amount, and moreover convergence conditions of the type $\pi_Y(\nu_n^+) \to \lambda \in \mathcal{EL}(Y)$ are unaffected by the choices in the definition.

We also record a consequence of these definitions and the basic properties of pleated surfaces: for an essential subsurface W, we have

$$\pi_W(\mu^{\pm}) \cap \overline{\pi_W(\mathcal{C}(\rho, L))} \neq \varnothing \tag{2-1}$$

provided L is at least the Bers constant L_B. (Recall from the introduction that $\mathcal{C}(\rho, L)$ is the set of essential simple closed curves in S whose ρ–length is bounded by L.) If W intersects a closed curve component β of $\operatorname{base}(\mu)$, then $l_\rho(\beta) \le L_B$ and so $\pi_W(\beta) \in \pi_W(\mathcal{C}(\rho, L))$. If λ is a lamination component of $\operatorname{base}(\mu^{\pm})$ and $Z = \operatorname{supp}(\mu)$, then there exists a family of pleated surfaces $\{f_n\colon X_n \to N_\rho\}$ with base surface Z that exit the end associated to λ (see Bonahon [5]). If we choose shortest curves β_n on Z_n, then $l_\rho(\beta_n) \le L_B$, so $\{\beta_n\} \subset \mathcal{C}(\rho, L)$ and $\beta_n \to \lambda$. If W overlaps Z, then $\pi_W(\beta_n) \to \pi_W(\lambda)$, so:

$$\pi_W(\lambda) \in \pi_W(\mu^{\pm}) \cap \overline{\pi_W(\mathcal{C}(\rho, L))}.$$

Margulis tubes We fix throughout a Margulis constant ϵ_1 for $\mathbb{H}^3$, which it will be convenient to take to be the same choice of Margulis constant as in [26] (see page 19) and [11]. In particular, this number is sufficiently small that the ϵ_1–thin part of a hyperbolic 3–manifold is a disjoint union of cusps and solid-torus neighborhoods of geodesics.

If α is a curve in S and ρ is a given Kleinian surface group we let $\mathbb{T}(\alpha)$ denote the component of the ϵ_1–thin part $(N_\rho)_{\mathrm{thin}(\epsilon_1)}$ whose core is in the homotopy class of α. If $\epsilon < \epsilon_1$, then we define:

$$\mathbb{T}_\epsilon(\alpha) = \mathbb{T}(\alpha) \cap (N_\rho)_{\mathrm{thin}(\epsilon)}.$$

2.3 Hierarchies

Given two generalized markings μ^+ and μ^-, we let $H(\mu^+,\mu^-)$ or $H(\mu^\pm)$ denote the hierarchy connecting them, in the sense of [22; 26; 11]. We also denote this by $H(\nu^\pm)$, if $\mu^\pm$ are obtained from a pair of end invariants $\nu^\pm$. We give an impressionist discussion here, referring the reader to those three articles for the details. A hierarchy is a collection of *tight geodesics* supported on subsurfaces of S, and interlocked in a structure that encodes certain nesting and ordering properties. Each tight geodesic is essentially a directed geodesic in the curve complex of the subsurface it is supported on. We typically denote such a geodesic k_W if W is the supporting surface, and we let ι_W and τ_W denote the initial and terminal vertices.

We will use $\mathcal{CH}(\mu^\pm)$, or sometimes $\mathcal{CH}$, to denote the set of all vertices of $\mathcal{C}(S)$ that occur in the (non-annular) geodesics in a hierarchy $H(\mu^\pm)$.

A *resolution* of $H(\mu^\pm)$ is a (possibly infinite) sequence of markings (μ_n), separated by elementary moves, and connecting μ^- to μ^+ (in the sense that μ_n is either equal to μ^+ for the last n, or converges to it as $n\to\infty$ if μ^+ has a lamination component, and similarly for μ^-). Each marking is composed of curves that occur as vertices in a nested collection of geodesics of H, which is known as a "slice" of H, and successive markings are separated by elementary moves, which correspond in a specific way to forward motion along the geodesics of H.

Hierarchies and projections We will make crucial use of [22, Lemma 6.2], sometimes called the "large link lemma". Given markings $\mu^\pm$ and a subsurface $W\subset S$ we let $\mathrm{hull}_W(\mu^\pm)$ denote a geodesic in $\mathcal{C}(W)$ joining $\pi_W(\mu^+)$ to $\pi_W(\mu^-)$ (there may be more than one such geodesic but hyperbolicity implies that all such choices are within uniform Hausdorff distance of each other).

Lemma 2.1 *There exists $A=A(S)$ such that if $H(\mu^\pm)$ is a hierarchy, $W\subset S$ is an essential subsurface, and $d_W(\mu^+,\mu^-)>A$, then $H(\mu^\pm)$ contains a geodesic k_W with domain W and*

$$d_{\mathrm{Haus}}(k_W,\mathrm{hull}_W(\mu^\pm))\le A.$$

Moreover

$$d_W(\tau_W,\mu^+)\le A \quad \text{and} \quad d_W(\iota_W,\mu^-)\le A,$$

where τ_W and ι_W are the terminal and initial vertices of k_W.

In fact the first inequality of Lemma 2.1 can be strengthened to something that holds in the setting where a geodesic k_W may not necessarily exist:

Lemma 2.2 *Given S there exists M, such that for any pair of generalized markings and any essential $W \subset S$,*

$$d_{\text{Haus}}\big(\pi_W(\mathcal{C}H(\mu^\pm)), \text{hull}_W(\mu^\pm)\big) \leq M.$$

This result, which is established in the proof of [26, Lemma 5.14], follows from [22, Lemmas 6.1 and 6.9], which are part of the same machinery used in the proof of the large link lemma.

2.4 Model manifolds

To each hierarchy $H = H(\nu^\pm)$ we associate (in [26]) a *model manifold* $M = M(\nu^\pm)$, which is equipped with an orientation-preserving embedding into $S \times \mathbb{R}$ (which we treat as inclusion), a path metric and a disjoint collection of *tubes*, one for each vertex of H. The tube associated to $v \in \mathcal{C}H$ is an open solid torus of the form $U(v) \equiv \text{collar}(v) \times I$, where $\text{collar}(v) \subset S$ is an annulus whose core is v, and I is an interval (sometimes infinite). Each tube $U(v)$ is isometric to a standard Margulis tube (possibly parabolic, for finitely many of the v). Let $\mathcal{U} \subset M$ denote the union of all the tubes. The complement, $M \setminus \mathcal{U}$, decomposes into a union of *blocks*, which (with the exception of a bounded number of *boundary blocks*) are submanifolds that fall into a fixed finite number of isometry classes. The boundary of each block is a union of annuli on tube boundaries and *level* 3*–holed spheres*, where the latter have the form $Y \times \{t\}$, for a three-holed sphere $Y \subset S$ obtained as a complementary component of $S \setminus \text{collar}(\Gamma)$ for a curve system Γ. (The boundary blocks, whose structure is slightly more complicated, are all adjacent to the boundary of M, if any, and will not affect the rest of our arguments.)

The model contains a collection of *split-level surfaces*, each associated to markings or partial markings that occur in resolutions. Suppose μ is such a marking, restricted to a subsurface W (so that $\partial W \subset \text{base}(\mu)$ and $\text{base}(\mu)$ determines a pants decomposition of W). The split-level surface $F_\mu \subset M \setminus \mathcal{U}$ is a disjoint union of level three-holed spheres $Y \times \{t_Y\}$, where Y runs over the components of $W \setminus \text{collar}(\text{base}(\mu))$. Each three-holed sphere is properly embedded in $(M \setminus \mathcal{U}, \partial\mathcal{U})$, and in the induced metric they are all isometric to a single standard 3–holed sphere. Moreover, if F_Y intersects a tube $U(v)$, then $F_Y \cap U(v)$ is a geodesic in the metric on $\partial U(V)$.

An *extended* split-level surface $\widehat{F}_\mu$ is obtained from F_μ by adding, for every v in $\text{base}(\mu) \cap \text{int}(W)$, an annulus in the corresponding tube $U(v)$. These annuli are identified with the corresponding collars in a way that extends the identification of $W \setminus \text{collar}(\text{base}(\mu))$ with F_μ to an identification of W with $\widehat{F}_\mu$. In particular $\widehat{F}_\mu$ is an isotope of $W \times \{0\}$.

The annulus in each $U(v)$ is chosen so that it has a CAT(-1) metric: If $U(v)$ is the Margulis tube with geodesic core then this can be done by extending the boundaries of the annuli radially to the core, and if $U(v)$ is parabolic we can simply rule the annulus by geodesics connecting the boundaries.

If the domain surface W of an (extended) split-level surface is all of S, we call it maximal.

The maximal extended split-level surfaces $\widehat{F}_{\mu_n}$ associated to a resolution are isotopes of $S\times\{0\}$ and are *monotonically arranged* in the sense that the transition from $\widehat{F}_{\mu_n}$ to $\widehat{F}_{\mu_{n+1}}$ always involves isotoping a subsurface upward in the $\mathbb{R}$ direction of $S\times\mathbb{R}$. This provides a connection between topological ordering in M and the directionality of the hierarchy, aspects of which we will state more precisely below.

Bi-Lipschitz model map The main theorem of [11] provides a bi-Lipschitz homeomorphism between the model manifold associated to the end invariants of a hyperbolic 3–manifold N, and the *augmented convex core* of N, denoted $\widehat{C}_N$. This is the union of a 1–neighborhood of the convex hull of N with the thin part of N. (An extension of this theorem gives a model that covers all of N, but we will not need it.) We give here a statement that combines this bi-Lipschitz map with other structural facts derived in that and related papers:

Theorem 2.3 *Given S, there exists $K_h>1$, $\epsilon_h>0$ and $L_h>0$ such that, if $\rho\in \mathrm{AH}(S)$ has end invariants $\nu^\pm$, then:*

(1) *There exists a K_h–bi-Lipschitz homeomorphism $h\colon M(\nu^+,\nu^-)\to\widehat{C}_{N_\rho}$ that is orientation-preserving.*

(2) $\mathcal{CH}(\nu^\pm)\subset\mathcal{C}(\rho,L_h)$.

(3) $\mathcal{C}(\rho,\epsilon_h)\subset\mathcal{CH}(\nu^\pm)$.

(4) *If $l(\alpha)<\epsilon_h$, then $h(U(\alpha))=\mathbb{T}(\alpha)$.*

Remark Part (2) is a formal consequence of (1), but is established [26, Lemma 7.9] as part of the proof of (1).

An important additional feature of the model manifold is that, for any resolution (μ_n), every point of $M\setminus\mathcal{U}$ lies within uniformly bounded distance of at least one split level surface $\widehat{F}_{\mu_n}$, since every block intersects some $\widehat{F}_{\mu_n}$.

Lemma 2.4 *There exists $c_0>0$ such that if S is a compact surface, $\rho\in\mathrm{AH}(S)$ has end invariants $\nu^\pm$ with associated model manifold $M=M(\nu^\pm)$ and (μ_n) is a* *resolution sequence of the associated hierarchy* $H = H(\nu^{\pm})$*, then if* $x \in M \setminus \mathcal{U}$*, there exists* n *such that*

$$d(x, F_{\mu_n}) < c_0.$$

Let us also record the following useful fact, relating the appearance of short curves in N_ρ with high subsurface projections.

Theorem 2.5 [25, Theorem B] *Given a surface* S*,* $\epsilon > 0$ *and* $L > 0$*, there exists* $K = K(S, \epsilon, L)$ *such that if* $\rho \in \mathrm{AH}(S)$ *and* W *is an essential subsurface of* S*, then* $l_\rho(\partial W) < \epsilon$ *if* $\mathrm{diam}(\pi_W(C(\rho, L))) \geq K$*.*

2.5 Ordering

In a product $S \times \mathbb{R}$ there is a natural notion of topological ordering induced by the projection $q\colon S \times \mathbb{R} \to \mathbb{R}$ to the second factor. The details are however slightly messy so we take some care with the definitions.

Given two maps $f\colon A \to S \times \mathbb{R}$ and $g\colon B \to S \times \mathbb{R}$, we say that f lies *above* g if f extends to a map $F\colon A \times [0, \infty) \to S \times \mathbb{R}$ such that $F(\cdot, 0) = f$, the image of F is disjoint from $g(B)$, and $q \circ F(\cdot, t)$ goes uniformly to $+\infty$ as $t \to +\infty$. We define *below* similarly with $+\infty$ replaced by $-\infty$. If g lies above f, f lies below g, and the opposite statements are false, we write $f \prec_{\mathrm{top}} g$ (in spite of the notation, however, this relation is not a partial order). We will also apply this terminology to subsets of $S \times \mathbb{R}$ where the map is presumed to be the inclusion map.

If A and B are subsets of S and f and g are homotopic to the inclusions $A \to A \times \{0\}$ and $B \to B \times \{0\}$, then we say that f and g *overlap* if A and B intersect essentially (ie, cannot be made disjoint by isotopy). Note that in this case if f lies above g then g cannot lie above f, and so on. If f and g are overlapping *level embeddings*, ie, of the form $a \mapsto (a, t)$ and $b \mapsto (b, s)$, then $f \prec_{\mathrm{top}} g$ if and only if $t < s$.

This notion of ordering can usefully be applied to the tubes in a model manifold, where it is closely related to the ordering of the geodesics in the hierarchy. (For an extensive discussion of topological ordering and its relationship to the hierarchy, see [11, Sections 3 and 4].)

Given a geodesic g in $\mathcal{C}(W)$, let π_g denote the composition of the projection π_W with a nearest-point projection $\mathcal{C}(W) \to g$.

For a directed geodesic g, we can fix an orientation-preserving identification with an interval of $\mathbb{Z}$, so that addition makes sense, and $a < b$ means a occurs earlier than b. This lemma describes the relation between topological order of tubes in a model manifold, and the order of projections along hierarchy geodesics.

Lemma 2.6 *Let $H = H(\mu^{\pm})$ be a hierarchy and $M = M(\mu^{\pm})$ the associated model manifold. Suppose that k is a geodesic in H, supported in a non-annular $W \subset S$. There is a constant $r = r(S)$ such that:*

(1) *For any two vertices $u, v \in \mathcal{C}H(\mu^{\pm})$ that overlap W and each other,*

$$U(u) \prec_{\text{top}} U(v) \Longrightarrow \pi_k(u) \le \pi_k(v) + r.$$

(2) *If $\gamma \in \mathcal{C}H(\mu^{\pm})$ overlaps a component β of ∂W, then*

$$U(\beta) \prec_{\text{top}} U(\gamma) \Longrightarrow d_W(\gamma, \mu^+) \le r$$

and similarly

$$U(\gamma) \prec_{\text{top}} U(\beta) \Longrightarrow d_W(\gamma, \mu^-) \le r.$$

Proof Fix a resolution (μ_n) of the hierarchy. Following the notation in [11, Section 4], for any vertex or simplex a in $\mathcal{C}H(\mu^{\pm})$ define $J(a) \subset \mathbb{Z}$ to be the set of n such that base(μ_n) contains a. There is also a subset $J(k_W)$ that consists of those indices for which the geodesic k_W is "active" in the resolution in a certain sense. Rather than give the full definition we will note that

$$J(k_W) \subset J([\partial W]),$$

ie, the geodesic is only active when ∂W is visible in the marking, and that for each $n \in J(k_W)$ there must be some vertex x of k_W such that $x \in \text{base}(\mu_n)$ – in other words, $n \in J(x)$. [11, Lemma 4.9] states that $J(a)$ and $J(k_W)$ are *intervals in* $\mathbb{Z}$.

Note that if a and b overlap then $J(a)$ and $J(b)$ are disjoint. Because the split-level surfaces F_{μ_n} move monotonically upward in $S \times \mathbb{R}$, we have immediately that

$$U(a) \prec_{\text{top}} U(b) \Longrightarrow \max J(a) < \min J(b). \tag{2-2}$$

Another aspect of the monotonicity property of resolutions is that the vertices of k_W are traversed monotonically. That is, if u and v are vertices in k_W, then

$$\max J(u) < \min J(v) \Longrightarrow u < v. \tag{2-3}$$

We will also need the following: If $n \in J(a)$, then

$$n < \min J(k_W) \Longrightarrow d_W(\mu_n, \mu^-) \le r_0 \tag{2-4}$$

and similarly

$$\max J(k_W) < n \Longrightarrow d_W(\mu_n, \mu^+) \le r_0 \tag{2-5}$$

for some uniform choice of r_0. In other words, the projection to W of everything in the hierarchy that happens "before" k_W is frozen, and similarly for everything afterwards.

This is a consequence of [22, Lemmas 6.1 and 6.9], in a way similar to Lemma 2.2. (The discussion in [22] identifies a certain sequence of geodesic segments in H that connect k_W to μ^- and μ^+ in such a way that every vertex in the sequence projects nontrivially to $\mathcal{C}(W)$, and [22, Lemma 6.1] shows that the parts of this sequence before and after k_W, respectively, have bounded projections to $\mathcal{C}(W)$. [22, Lemma 6.9] shows that every slice in the resolution meets some part of this sequence.)

Proof of part (1) Since $U(u) \prec_{\text{top}} U(v)$, we can choose $s_u \in J(u)$ and $s_v \in J(v)$ such that $s_u < s_v$ (by (2-2)).

If $s_u \in J(k_W)$, then $\text{base}(\mu_{s_u})$ contains a vertex u' in k_W, and in particular $s_u \in J(u')$. Note that u' is within 1 of u in $\mathcal{C}(W)$, and hence within 2 of $\pi_{k_W}(u)$.

If $s_v \in J(k_W)$ as well, then we similarly have v' in k_W, so that $s_v \in J(v')$ and v' is within 2 of $\pi_{k_W}(v)$. It therefore suffices to show that $u' \le v' + 1$.

If $d_W(u', v') \le 1$ then we are done, and otherwise u' and v' overlap, so that $J(u')$ and $J(v')$ are disjoint. Since $s_u < s_v$, it must be that $\max J(u') < \min J(v')$, so that u' must appear before v' in k_W (by (2-3)). Again we are finished in this case.

If one of s_u and s_v is not in $J(k_W)$, suppose without loss of generality it is s_u.

If $s_u < \min J(k_W)$ then, by (2-4), $\pi_W(u)$ is within r_0 of the initial point of k_W. In this case the conclusion holds trivially no matter where $\pi_{k_W}(v)$ is.

If $\max J(k_W) < s_u$, then by (2-5), $\pi_W(u)$ is within r_0 of the final point of k_W. Since $s_u < s_v$ the same holds for $\pi_W(v)$, and again we are done.

For the proof of part (2), we first note that since $\gamma, \beta \in \mathcal{C}H(\mu^{\pm})$ and γ and β overlap, $U(\gamma)$ and $U(\beta)$ are topologically ordered (see [11, Lemma 4.9]). Moreover, since $\beta \in [\partial W]$, $J(\gamma)$ and $J(k_W)$ are disjoint. If $U(\beta) \prec_{\text{top}} U(\gamma)$ then $\max J(k_W) < \min J(\gamma)$, and as above, (2-5) implies that $d_W(\gamma, \mu^+)$ is bounded. The proof of the opposite case is similar. □

2.6 Topological lemmas

In this section, we collect topological lemmas concerning ordering of curves and surfaces in $S \times \mathbb{R}$, which will be applicable to split-level and pleated surfaces in our hyperbolic manifolds.

We begin by showing that a proper homotopy equivalence whose image is disjoint from a level curve $\gamma_0 = \gamma \times \{0\}$ lies above γ_0 if and only if there is an essential curve on the surface that intersects γ whose image lies above γ_0.

Lemma 2.7 *Let α and γ be overlapping curves on S and let*

$$f\colon (S, \partial S) \to (S \times \mathbb{R}, \partial S \times \mathbb{R})$$

be a homotopy equivalence with image disjoint from $\gamma_0 = \gamma \times \{0\}$. Then $f|_\alpha$ lies above γ_0 if and only if f lies above γ_0.

Proof Clearly if f lies above γ_0 then $f|_\alpha$ lies above γ_0.

Suppose that $f|_\alpha$ lies above γ_0. Let $A = \gamma \times (-\infty, 0]$. We can homotope $f|_\alpha$, in the complement of γ_0, to a map whose image is disjoint from A. This homotopy can be extended to all of S where the homotopy is supported on a neighborhood of α and the image of the homotopy is disjoint from γ_0. Let $g\colon (S, \partial S) \to (S \times \mathbb{R}, \partial S \times \mathbb{R})$ be the new map. We can assume $g(S)$ intersects A transversely. Then $\Gamma = g^{-1}(A)$ will be a collection of disjoint curves on S. Since g is a homotopy equivalence every curve in Γ will either be homotopic to γ or will bound a disk. However, any curve that is homotopic to γ must intersect α and since $g(\alpha)$ is disjoint from A we must have that all curves in Γ bound disks. Using the standard innermost disk argument and the fact that $(S \times \mathbb{R}) - \gamma_0$ is irreducible we can then homotope g, in the complement of γ_0, to a map whose image is disjoint from A. Such a map will lie above γ_0 so g and therefore f lies above γ_0. □

We next observe that a proper homotopy equivalence whose image is disjoint from an essential non-annular level subsurface lies either above or below that subsurface

Lemma 2.8 *Let W be a non-annular subsurface of a compact surface S. If*

$$f\colon (S, \partial S) \to (S \times \mathbb{R}, \partial S \times \mathbb{R})$$

is a homotopy equivalence with image disjoint from $W_0 = W \times \{0\}$, then either f lies above or below W_0.

Proof Much as in the proof above, we may homotope f (in the complement of W_0) so that $f^{-1}(W \times \mathbb{R}) = f^{-1}(W \times (\mathbb{R} \setminus \{0\}))$ is a union of essential subsurfaces of S. Since f is a homotopy equivalence, these subsurfaces must consist of one isotope of W and a (possibly empty) collection of disjoint annuli.

Each annulus maps either to $W \times (0, \infty)$ or $W \times (-\infty, 0)$, where it must be homotopic rel boundary to $\partial W \times (0, \infty)$ or $\partial W \times (-\infty, 0)$, respectively. Thus, after homotopy we may assume that $f^{-1}(W \times \mathbb{R})$ is just W, and $f(W)$ lies either in $W \times (0, \infty)$ or in $W \times (-\infty, 0)$. It follows that we can homotope f to $+\infty$ or $-\infty$, respectively, in the complement of W_0. □

We say a map of a curve system in S to $S\times\mathbb{R}$ is *unknotted* if it is isotopic to a level embedding. It will also be useful to recall that knotting of the boundary is the only obstruction to extending an embedding of an essential subsurface to a proper embedding of the entire surface that is isotopic to a level surface. This result is a special case of [11, Lemma 3.10], although the proof of just this case is not hard.

Lemma 2.9 *Let W be a compact essential subsurface of S. If $h\colon W\to S\times\mathbb{R}$ is an embedding homotopic to a level embedding, such that $h(\partial W)$ is unknotted, then h extends to an embedding of S in $S\times\mathbb{R}$ whose image is isotopic to $S\times\{0\}$.*

Our final topological lemma is a degree computation, which will be used to complete the proof of Theorem 1.3 in the case that the essential subsurface is an annulus.

Lemma 2.10 *Let γ be an essential curve in S and $\mathcal{N}(\gamma)$ be an open regular neighborhood of $\gamma\times\{1/2\}$ in $S\times[0,1]$ with boundary the torus T. Suppose that*

$$f\colon ((S\times[0,1])\backslash\mathcal{N}(\gamma),\partial S\times[0,1],T)\to((S\times[0,1])\backslash\mathcal{N}(\gamma),\partial S\times[0,1],T)$$

is a continuous map of triples such that $f|_{S\times\{0\}}$ and $f|_{S\times\{1\}}$ are homotopic in the complement of $\mathcal{N}(\gamma)$.

Then the restriction of f to T has degree zero.

Proof Since $f|_{S\times\{0\}}$ and $f|_{S\times\{1\}}$ are homotopic to each other in the complement of $\mathcal{N}(\gamma)$, we may assume, possibly adjusting f by homotopy, that $f(x,0)=f(x,1)$ for all $x\in S$. Thus, f descends to a map

$$F\colon (S\times S^1)\setminus\mathcal{N}(\gamma)\to(S\times[0,1])\setminus\mathcal{N}(\gamma)$$

with $F(\partial S\times S^1)\subseteq\partial S\times[0,1]$. Since F defines a relative 3–chain in

$$((S\times[0,1])\backslash\mathcal{N}(\gamma),\partial S\times[0,1])$$

whose boundary is $F|_T$, we see that

$$[F|_T]=0\in H_2((S\times[0,1])\backslash\mathcal{N}(\gamma),\partial S\times[0,1]).$$

However, $[T]$ is a non-trivial homology class in $H_2((S\times[0,1])\backslash\mathcal{N}(\gamma),\partial S\times[0,1])$ and

$$F_*([T])=[F|_T]=d[T],$$

where d is the degree of the restriction of F to T or, equivalently, the degree of the restriction of f to T. Therefore, this degree is zero. □

2.7 Thick distance, bounded diameter lemmas and subsurface product regions

A simple but useful feature of hyperbolic geometry is the fact that the thick part of a surface of bounded area has components of uniformly bounded diameter. This, together with the observation that a π_1–injective Lipschitz map of a hyperbolic surface into a hyperbolic 3–manifold takes the thick part to the thick part (with slightly different constants), is a useful tool that recurs, for example, in the work of Thurston and Bonahon, and others. We develop some notation in order to discuss and apply these ideas in our context.

If X is a path-metric space and $A \subset X$ a subset, we denote by $\boldsymbol{d}|_{A,X}(x,y)$ the infimum over paths α in X from x to y of the length of $\alpha \cap A$. This is a pseudometric that assigns distance 0 to pairs of points in the same component of $X \setminus A$. We let $\mathbf{diam}|_{A,X}$ denote diameter with respect to this pseudometric, and also use the abbreviation $\mathbf{diam}|_A(X) \equiv \mathbf{diam}|_{A,X}(X)$. It will also be useful, for a subset $Y \subset X$, to let $\boldsymbol{d}|_{A,Y}$ denote the same as $\boldsymbol{d}|_{A\cap Y,Y}$.

We will use this notation when X is a hyperbolic manifold N and $A = N_{\mathrm{thick}(\epsilon)}$, and when X is a model M and $A = M \setminus \mathcal{U}$. In particular, it is easy to express the bounded diameter lemma for surfaces in this language.

Lemma 2.11 *Given a compact surface S and $\epsilon > 0$, there exists $b = b(S,\epsilon)$ such that:*

(1) *If X is a finite volume surface homeomorphic to S, then*

$$\mathbf{diam}|_{X_{\mathrm{thick}(\epsilon)}}(X) < b.$$

(2) *If M is a model manifold associated to a hierarchy and $\widehat{F}$ is an extended split-level surface in M, then*

$$\mathbf{diam}|_{\widehat{F}_{\mathrm{thick}(\epsilon)}}(\widehat{F}) < b.$$

For hyperbolic surfaces, this is a consequence of the thick-thin decomposition (see Bonahon [5, Proposition 1.5]). For split-level surfaces, it follows from the fact that each such surface is a union of three-holed spheres whose metric is standard and a bounded number of CAT(-1) annuli each of whose intersection with the ϵ–thick part consists of one or two annuli whose diameter is uniformly bounded in terms of ϵ.

The following remark will be useful for us. Let $f\colon X \to N$ be a π_1–injective K–Lipschitz map. Then, since $f(X_{\mathrm{thin}(\epsilon)}) \subset N_{\mathrm{thin}(K\epsilon)}$, we obtain

$$\mathbf{diam}|_{N_{\mathrm{thick}(K\epsilon)},N}(f(X)) \le K\,\mathbf{diam}|_{X_{\mathrm{thick}(\epsilon)}}(X). \tag{2-6}$$

Finally, let us also observe:

Lemma 2.12 *If N is a hyperbolic 3–manifold and N^0 the complement of cusp neighborhoods in N, then $\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon)},N^0}$ is a proper pseudometric on N^0, when ϵ is less than the Margulis constant.*

This follows immediately from the fact that, with ϵ less than the Margulis constant, the distance between any two components of $N_{\mathrm{thin}(\epsilon)}$ is uniformly bounded below.

Subsurface product regions Another important feature of Kleinian surface groups, and their bi-Lipschitz models, is the presence of "thick product regions", namely regions in $S\times\mathbb{R}$ that are topologically products $W\times J$, and geometrically anchored on Margulis tubes (or model tubes), and bounded by split-level surfaces. We introduce some notation for discussing these regions, and indicate their interaction with the topological ordering relation and the structure of the hierarchy.

If W is an essential non-annular subsurface of S and M is a model manifold associated to $\rho\in\mathrm{AH}(S)$, we say that $Q\subset M$ is a *W–product region* if there exists an orientation-preserving homeomorphism $g\colon W\times[0,1]\to Q$ so that $g(\partial W\times[0,1])\subset U(\partial W)$ and, if $g_0\colon W\to Q$ is the inclusion map given by $g_0(x)=g(x,0)$, then $(g_0)_*$ is conjugate to $\rho|_{\pi_1(W)}$. In this case, $\partial_0 Q=g(W\times\{0\})$ and $\partial_1 Q=G(W\times\{1\})$ are called the *horizontal* boundary components of Q. Similarly, if $N=N_\rho$ and $h\colon M\to N$ is the model map, we say that $R\subset N$ is a W–product region for N if and only if $h^{-1}(Q)$ is a W–product region for M. In this case, $\partial_0 R=h(\partial_0 Q)$ and $\partial_1 R=h(\partial_1 Q)$ are called the horizontal boundary components of R.

The following lemma shows that if one has a long geodesic $k_W\subset H$ associated to a level subsurface, then one can find thick W–product regions in the model manifold.

Lemma 2.13 *Let $\rho\in\mathrm{AH}(S)$ have associated hierarchy H and model map $h\colon M\to N_\rho$. Let $W\subset S$ be the support of a geodesic k_W in H.*

For every simplex $v\in k_W$ there is an extended split-level surface $\widehat{F}_v\subset M$ in the isotopy class of W, passing through $U(v)$, such that, if $u,v\in k_W$ and $d_W(u,v)\geq 5$, then:

(1) *$\widehat{F}_u$ and $\widehat{F}_v$ are disjoint and comprise the horizontal boundaries of a W–product region for M. Moreover, if $u<v$, then $\widehat{F}_u\prec_{\mathrm{top}}\widehat{F}_v$.*
(2) *There exists $c_1=c_1(S)>0$ such that*
$$\boldsymbol{d}|_{M\setminus\mathcal{U},Q}(F_u,F_v)>c_1 d_W(u,v).$$
(3) *There exists $c_3=c_3(S)$ such that for $R=h(Q)$ and $G_x=h(F_x)$,*
$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon_h)},R}(G_u,G_v)>c_3 d_W(u,v).$$

Proof Each simplex v in k_W can be extended to a marking $\mu(v)$ in W, and we can let $\hat{F}_v$ denote $\hat{F}_{\mu(v)}$ with a slight abuse of notation. For u, v separated by at least 5, the pair $\hat{F}_u$ and $\hat{F}_v$ form a special case of the "cut systems" described in [11, Section 4]. [11, Proposition 4.15] implies that the surfaces are disjoint, form the horizontal boundary of a W–product region, and that the topological order agrees with the ordering of vertices. This gives us (1).

To prove (2), we first observe that there is a definite lower bound b_0 on separation between surfaces in the product region, namely

$$\boldsymbol{d}|_{M\setminus\mathcal{U},Q}(\hat{F}_u, \hat{F}_v) > b_0 > 0 \tag{2-7}$$

when $d_W(u,v) \geq 5$. To see this, note that in M every tube is separated by a definite distance $b_1 > 0$ from every other tube – this is a consequence of the uniform geometry of the blocks that compose $M \setminus \mathcal{U}$. Similarly, each level 3–holed sphere in a split-level surface has a b_2–neighborhood that meets no other 3–holed spheres. We conclude from this that the union of $\hat{F}_u$ with all the tubes associated to the base of its marking has a regular neighborhood of definite width within $M \setminus \mathcal{U}$. Since $d_W(u,v) \geq 5$, the two markings cannot share any base curves, which implies that these regular neighborhoods are disjoint. This gives (2-7).

Now suppose that $d_W(u,v) \geq 5n$. Then by suitably subdividing the interval $[u,v]$ in k_W we can subdivide their product region into a sequence of n product regions, each with the definite separation given by (2-7). This suffices to give (2).

Because h is K–bi-Lipschitz, we have the inequality

$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon_h)},R} \geq \frac{1}{K}\boldsymbol{d}|_{h^{-1}(N_{\mathrm{thick}(\epsilon_h)}),Q}.$$

Now since (by Theorem 2.3) $h^{-1}(N_{\mathrm{thick}(\epsilon_h)})$ contains $M \setminus \mathcal{U}$, we have

$$\boldsymbol{d}|_{h^{-1}(N_{\mathrm{thick}(\epsilon_h)}),Q} \geq \boldsymbol{d}|_{M\setminus\mathcal{U},Q}.$$

We conclude that

$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon_h)},R}(G_u, G_v) > \frac{1}{K}\boldsymbol{d}|_{M\setminus\mathcal{U},Q}(F_u, F_v)$$

which gives (3). □

We next observe that any large enough W–product region gives rise to a pants decomposition of W consisting of hierarchy curves.

Lemma 2.14 *Given a compact surface S and $\epsilon > 0$, there exist $d_0 = d_0(S,\epsilon) > 0$ and $d_1 = d_1(S,\epsilon) > 0$ with the following properties. Let $\rho \in \mathrm{AH}(S)$ have end invariants $\nu^{\pm}$ and associated model manifold $M = M(\nu^{\pm})$. Then, if W is an essential non-annular subsurface of S, Q is a W–product region for M, $z \in Q \setminus \mathcal{U}$ and*

$$\boldsymbol{d}|_{M_{\mathrm{thick}(\epsilon)},Q}(z, W \times \{0,1\}) > d_0$$

then there exists a pants decomposition Γ of W so that $\Gamma \subset \mathcal{CH}$ and $U(\Gamma) \subset Q$.

Similarly, if $h\colon M \to N = N_\rho$ is the model map, R is a W–product region for N, $z \in R \cap N_{\mathrm{thick}(\epsilon)}$ and

$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon)},R}(z, h(W \times \{0,1\})) > d_1$$

then there exists a pants decomposition Γ of W so that $\Gamma \subset \mathcal{CH}$ and $h(U(\Gamma)) \subset R$.

Proof We first prove our claim in the setting of the model manifold. Lemma 2.4 gives a maximal split-level surface F_μ in M whose image comes within c_0 of z. Let Δ be the collection of components of $\mathrm{base}(\mu)$ that are components of ∂W. Let Z be the component of $\widehat{F}_\mu \setminus U(\Delta)$ that comes within c_0 of z.

Since Z is an extended split level surface it meets model tubes only if they are associated to its base, and by construction no base curve in Z can be a component of ∂W. Hence, Z cannot meet $U(\partial W)$. Lemma 2.11 implies that

$$\mathbf{diam}|_{Z_{\mathrm{thick}(\epsilon)}}(Z) < b = b(S,\epsilon).$$

Therefore, if we choose $d_0 > c_0 + b$, then $Z \subset Q$.

We conclude from this that (the underlying subsurface associated to) Z is homotopic into W, which implies that Z is isotopic to W since ∂Z is isotopic into ∂W. This implies that $\mathrm{base}(\mu)$ contains a pants decomposition of W and the lemma follows when M is a model manifold.

We now assume we are in the hyperbolic manifold setting. Let $Q = h^{-1}(R)$. If $z \in N_{\mathrm{thick}(\epsilon)} \cap R$, then $h^{-1}(z) \subset M_{\mathrm{thick}(\epsilon/K_h)}$ where K_h is the constant from Theorem 2.3. Therefore, there exists $z_0 \in M \setminus \mathcal{U}$ that may be joined to $h^{-1}(z)$ by a path in $M - U(\partial W)$ of length at most c_1 (where c_1 depends only on S and on ϵ/K_h). If

$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon)},R}(z, h(W \times \{0,1\})) > d_1 = K_h d_0(S,\epsilon/K_h) + K_h c_1$$

then $z_0 \in Q$ and

$$\boldsymbol{d}|_{M_{\mathrm{thick}(\epsilon/K_h)},Q}(z_0, W \times \{0,1\}) > d_0(S,\epsilon/K_h).$$

Then the model manifold case guarantees that there exists a pants decomposition Γ of W so that $\Gamma \subset \mathcal{C}H$ and $U(\Gamma) \subset Q$. It follows that $h(U(\Gamma)) \subset R$ and our proof is complete. □

3 Controlled hierarchy curve systems

In this section we provide a tool for directly relating bounded-length curves on pleated surfaces in a Kleinian surface group to the curves that occur in the associated hierarchy. Lemma 3.1 says that in any such pleated surface (or more generally a Lipschitz surface with fixed bounds) there is a maximal collection of disjoint hierarchy curves, all of uniformly bounded length, such that in the curve complexes of their complementary subsurfaces the projection of the entire hierarchy is within bounded distance from the set of bounded-length curves.

Lemma 3.1 *Given a compact surface S and $K > 0$ there exists $B = B(S, K) > 0$ such that if $X \in \mathcal{T}(S)$ is a finite area hyperbolic surface,*

$$f\colon X \to N$$

is a K–Lipschitz homotopy equivalence, $\rho = f_* \in \mathcal{D}(S)$ has end invariants $\nu^{\pm}$ and $H = H(\nu^{\pm})$ is the associated hierarchy, then there exists a curve system Γ on X such that if $\gamma \in \Gamma$, then

$$\gamma \in \mathcal{C}H \quad \text{and} \quad l_X(\gamma) \le B.$$

Moreover, if W is a component of $X \setminus \Gamma$ that is not a thrice-punctured sphere, then:

(1) *$\mathcal{C}(W)$ contains no curves in $\mathcal{C}H$.*

(2) *There exists $\beta \in \mathcal{C}(W)$ such that $l_X(\beta) \le B$ and*

$$\operatorname{diam}(\pi_W(\beta \cup \mathcal{C}H)) \le B.$$

The proof will proceed by contradiction. We assume that we have a sequence $\{\rho_n\}$ where it is not possible to choose appropriate collections of hierarchy curves for any uniform choice of constants. We then re-mark and pass to a subsequence so that there is a maximal collection Γ_1 of curves so that $l_{\rho_n}(\Gamma_1) \to 0$ and if Y_1 is a component of $S - \Gamma_1$, then $\{\rho_n|_{\pi_1(Y_1)}\}$ is convergent. For large enough values of n, Γ_1 are hierarchy curves in H_n. If $\lim \rho_n|_{\pi_1(Y_1)}$ is geometrically infinite, we pull-back a wide Y_1–product region from the corresponding end of the limit manifold and consider a split-level surface passing through the middle of the product region in the approximates to find a pants decomposition of Y_1 by hierarchy curves for all large enough values of n. If $\lim \rho_n|_{\pi_1(Y_1)}$ is geometrically finite, then the set of projections of bounded length curves to any subsurface of Y_1 is finite and our result follows as well.

Proof We suppose that, for some S and K, it is not possible to choose such a value of B and proceed to find a contradiction. So assume there exists a sequence $\{f_n\colon X_n \to N_n\}$ of K–Lipschitz homotopy equivalences with associated representations $\{\rho_n = (f_n)_*\}$ and hierarchies $\{H_n\}$, and $B_n \to \infty$ such that, for each n, one cannot find a disjoint collection of curves in H_n that have length at most B_n on X_n whose complementary regions that are not thrice-punctured spheres have properties (1) and (2) with constant B_n.

By passing to a subsequence and re-marking the X_n we can assume that there is a curve system Γ_0 on S such that $\ell_{X_n}(\Gamma_0) \to 0$ and there is a uniform lower bound on the length of any homotopically non-trivial, non-peripheral curve disjoint from Γ_0. We may further re-mark the X_n by homeomorphisms of S fixing Γ_0 to guarantee that if γ is any fixed curve in $S \setminus \Gamma_0$, then $\{\ell_{X_n}(\gamma)\}$ is bounded (where the bound depends on γ). Notice that, by Theorem 2.3, each curve in Γ_0 eventually lies in the set $\mathcal{C}H_n$ of hierarchy curves. If each component of $S \setminus \Gamma_0$ is a thrice-punctured sphere, then we have already achieved a contradiction.

We now focus on a component Y of $S \setminus \Gamma_0$ that is not a thrice-punctured sphere. Since each curve on Y has bounded length on X_n for all n, we may pass to a subsequence so that $\{\rho_n|_{\pi_1(Y)}\}$ converges (up to conjugation). Let $\rho_Y \in \mathrm{AH}(Y)$ be the limit of (the subsequence) $\{\rho_n|_{\pi_1(Y)}\}$.

Let Γ_1 be a maximal collection of disjoint simple closed curves on Y such that if $\gamma \in \Gamma_1$, then $\rho_Y(\gamma)$ is parabolic. Notice that if $\gamma \in \Gamma_1$, then eventually $\gamma \in \mathcal{C}H_n$ and $\ell_{X_n}(\gamma)$ is bounded independent of n. We are again finished if every component of $Y \setminus \Gamma_1$ is a thrice-punctured sphere.

Given a component Y_1 of $Y \setminus \Gamma_1$ that is not a thrice-punctured sphere, we will construct a further subsequence and a curve system Γ_2 in Y_1 consisting of hierarchy curves in $\mathcal{C}H_n$ for all n such that if W is a component of $Y_1 \setminus \Gamma_2$ that is not a thrice punctured sphere, then $\mathcal{C}(W)$ contains no curves in $\mathcal{C}H_n$ and if $\beta \in \mathcal{C}(W)$ then there is an upper bound on $\mathrm{diam}(\pi_W(\beta \cup \mathcal{C}H_n))$ for all n. As we can apply this procedure iteratively to each component of $Y \setminus \Gamma_1$ that is not a thrice-punctured sphere, this will achieve the desired contradiction and complete the proof.

Fix, then, a component Y_1 of $Y \setminus \Gamma_1$ that is not a thrice-punctured sphere. We first claim that, for each $\alpha \in \mathcal{C}H_n$ that overlaps Y_1, there exists a curve $\psi_n(\alpha)$ in $\mathcal{C}(Y_1)$ such that $\ell_{\rho_n}(\psi_n(\alpha))$ and $d_{Y_1}(\psi_n(\alpha), \alpha)$ are uniformly bounded (independently of α and n).

Recall that $\ell_{\rho_n}(\alpha) \le L_0$ for all $\alpha \in \mathcal{C}H_n$ (see Theorem 2.3). Hence we can let $\psi_n(\alpha) = \alpha$ if α is already in $\mathcal{C}(Y_1)$. In general, [25, Lemma 4.1] provides for each

ρ_n and each $\alpha \in \mathcal{CH}_n$ a pleated surface $g\colon Z \to N_n$ (in the homotopy class of f_n, where Z denotes a hyperbolic structure on S) mapping ∂Y_1 geodesically, so that every minimal Z–length proper arc τ in Y_1 satisfies a bound $d_{\mathcal{A}(Y_1)}(\tau,\alpha) \le c$, where c depends only on S and L_0.

Now given our upper bound on $\ell_{\rho_n}(\partial Y_1)$ (and hence on $\ell_Z(\partial Y_1)$), we can combine arcs in τ and $\partial\,\mathrm{collar}(\partial Y_1)$ to obtain an essential simple closed curve α' in Y_1 with $\ell_{\rho_n}(\alpha') \le \ell_Z(\alpha') \le d'$ and $d_{Y_1}(\alpha,\alpha') \le c'$, for uniform c' and d'. This is the desired $\psi_n(\alpha)$.

We can break into two cases now:

Case 1 There is a sequence $\alpha_n \in \mathcal{CH}_n$ such that $\alpha_n' = \psi_n(\alpha_n)$ takes on infinitely many values. After taking a further subsequence we can assume that $\{\alpha_n'\}$ converges in $\mathcal{PML}(Y_1)$ to a lamination μ. Let V be the support of μ. Using an argument of Kobayashi and Luo (see [21, Section 4.3]), $d_V(\beta,\alpha_n') \to \infty$, for any fixed β, and hence, by Theorem 2.5, $\ell_{\rho_n}(\partial V)$ goes to 0. Since there are no further parabolics within Y_1, this means that $V = Y_1$, and μ is filling in Y_1. Since $\ell_{\rho_n}(\alpha_n')$ is bounded while the length of α_n' in the fixed surface Y_1 goes to ∞, continuity of length (Brock [7]) implies that $\ell_{\rho_Y}(\mu) = 0$, and therefore that N_Y^0 has a degenerate end with base surface Y_1.

The idea now is that this degenerate end will correspond to a part of the model manifold from which we can extract a pants decomposition of Y_1 consisting of hierarchy curves. The details of this are a bit delicate because we have to consider how the hierarchies H_n interact with the structure of this limiting degenerate end.

We may assume, after passing to a further subsequence, that $\{N_n\}$ converges geometrically to N_G and that there is a covering map $p\colon N_Y \to N_G$. The covering theorem (Thurston [28] and Canary [14]) can then be used to show that there is a neighborhood of this degenerate end that embeds in N_G (see, for example, the proof of [11, Proposition 6.10]). Let $\mathcal{E}$ be the image of this neighborhood in N_G. If Y_1 is identified with the interior of a compact surface $\overline{Y}_1$, one may further assume that the closure of $\mathcal{E}$ is homeomorphic to $\overline{Y}_1 \times [0,\infty)$. Let $\partial_0\mathcal{E}$ be the image of $\overline{Y}_1 \times \{0\}$ and $\partial_1\mathcal{E}$ be the image of $\partial\overline{Y}_1 \times [0,\infty)$ under this homeomorphism.

For any fixed $R \subset \mathcal{E}$, which is identified with $\overline{Y}_1 \times [0,a]$ for some $a > 0$, for all large n there exist 2–bi-Lipschitz comparison maps

$$\phi_n\colon R \to N_n$$

in the homotopy class of $\rho_n \circ (\rho_Y|_{\pi_1(Y_1)})^{-1}$ such that

$$\phi_n(R \cap \partial\mathbb{T}(\partial Y_1)) \subset \partial\mathbb{T}(\partial Y_1, n),$$

where $\mathbb{T}(\partial Y_1, n)$ is the collection of Margulis tubes in N_n associated to the components of ∂Y_1. (See [11, Lemma 2.8].) Let H_n and M_n be the hierarchy and model manifold associated to N_n and let $h_n\colon M_n \to N_n$ be the model map. For all sufficiently large n, $\ell_{\rho_n}(\partial Y_1) < \epsilon_h$, so $\partial Y_1 \subset \mathcal{C}H_n$ and $\mathbb{T}(\partial Y_1, n) = h_n(U(\partial Y_1)$. Therefore, $R_n = \phi_n(R)$ is a Y_1–product region for N_n for all sufficiently large n.

Since the pseudometric $\boldsymbol{d}|_{(N_G)_{\mathrm{thick}(\epsilon_1/4)},\mathcal{E}}$ is proper (see Lemma 2.12), we may choose R so that there exists $z \in R \cap (N_G)_{\mathrm{thick}(\epsilon_1)}$ with

$$\boldsymbol{d}|_{(N_G)_{\mathrm{thick}(\epsilon_1/4)},R}(z, Y_1 \times \{0, a\})) > 2d_1(S, \epsilon_1/2),$$

where $d_1(S, \epsilon_1/2)$ is the constant from Lemma 2.14. For large enough n, $z_n = \phi_n(z) \in N_{\mathrm{thick}(\epsilon_1/2)}$ and

$$\boldsymbol{d}|_{(N_G)_{\mathrm{thick}(\epsilon_1/2)},R_n}(z, \phi_n(Y_1 \times \{0, a\})) > d_1(S, \epsilon_1/2).$$

Therefore, by Lemma 2.14, there exists a pants decomposition Γ_n of Y_1 so that $\Gamma_n \subset \mathcal{C}H_n$ and $h(U(\Gamma_n)) \subset R_n$. Since every curve in Γ_n has a representative of uniformly bounded length in R_n, it also has a representative of uniformly bounded length in R. Since there are only finitely many such curves in R, we can pass to a subsequence such that Γ_n is a fixed pants decomposition Γ. The fixed set of curves Γ will have uniformly bounded length on X_n. This completes the proof in this case.

Case 2 For some k, the union $\bigcup_{n \geq k} \psi_n(\mathcal{C}H_n)$ is finite. Taking a subsequence, we can assume that $\psi_n(\mathcal{C}H_n)$ is a constant set Ψ. Since $\mathcal{C}H_n \cap \mathcal{C}(Y_1)$ is contained in Ψ, we may assume that it is constant for all n as well. Let Γ_2 be a maximal curve system in Y_1 whose elements are in $\mathcal{C}H_n \cap \mathcal{C}(Y_1)$.

Suppose first that Γ_2 is empty. Then there are no hierarchy curves in $\mathcal{C}(Y_1)$. The projection $\pi_{Y_1}(\mathcal{C}H_n)$ lies, for all n, within a uniform distance of the finite set ψ, and hence within uniform distance of any fixed curve $\beta \in \mathcal{C}(Y_1)$. This concludes the proof in this case.

If Γ_2 is nonempty, consider any component W of $Y_1 \setminus \Gamma_2$. By maximality of Γ_2, $\mathcal{C}(W)$ contains no hierarchy curves. We can now repeat the argument replacing Y_1 by W. We construct a new collection $\psi_n(\mathcal{C}H_n)$ in $\mathcal{C}(W)$, and find ourselves either in Case 2 but with Γ_2 empty, or in Case 1.

If we are in Case 2 we can complete the proof as above, with a uniform bound on the set of projections of $\mathcal{C}H_n$ into W. If we are in Case 1, we note that the argument shows that in fact the boundary of W must consist of parabolics for ρ_Y, so that in fact $W = Y_1$, contradicting the fact that Γ_2 is nonempty. $\square$

4 Projections of the bounded curve set

In this section, we prove Theorem 1.2, which we restate here for convenience:

Theorem 1.2 *Given S, there exists L_0 such that for all $L \geq L_0$ there exists $D = D(S, L)$, such that given $\rho \in \mathrm{AH}(S)$ with end invariants $\nu^{\pm}$ and an essential subsurface $W \subset S$ that is not an annulus or a pair of pants,*

$$d_{\mathrm{Haus}}\big(\pi_W(\mathcal{C}(\rho, L)), \mathrm{hull}_W(\nu^{\pm}(\rho))\big) \leq D.$$

Moreover, if $d_W(\nu^+(\rho), \nu^-(\rho)) > D$ then $\mathcal{C}(\rho, L) \cap \mathcal{C}(W)$ is nonempty and

$$d_{\mathrm{Haus}}\big(\mathcal{C}(\rho, L) \cap \mathcal{C}(W), \mathrm{hull}_W(\nu^{\pm}(\rho))\big) \leq D.$$

The main new content of this theorem is the statement that $\pi_W(\mathcal{C}(\rho, L))$ is contained in a uniform neighborhood of $\mathrm{hull}_W(\nu^{\pm})$, and for this we use Lemma 3.1, which gives us a comparison between the short curves in pleated surfaces and hierarchy curves. The other inclusions were already known, and are essentially consequences of the bi-Lipschitz model Theorem 2.3, which relates hierarchy curves to the hyperbolic structure, and Lemma 2.2, which controls hierarchies in terms of subsurface projections.

The main new ingredient in the proof is Lemma 3.1. Given a curve $\alpha \in \mathcal{C}(\rho, L)$, we consider a pleated surface realizing α and the system Γ of hierarchy curves produced by Lemma 3.1. If some element of Γ overlaps W, then the result follows from Lemma 2.2, which is essentially a version of Theorem 1.2 for hierarchy curves. If not, then $\pi_W(\mathcal{C}H)$ has bounded diameter, and Lemma 3.1 provides a bounded length curve whose projection to W is uniformly near $\pi_W(\mathcal{C}H)$, which again allows us to complete the proof.

Proof We first recall (from (2-1) in Section 2) that if L_0 is chosen to be greater than the Bers constant L_B, then

$$\pi_W(\nu^{\pm}) \cap \overline{\pi_W(\mathcal{C}(\rho, L))} \neq \varnothing, \tag{4-1}$$

where the closure is in the Gromov closure $\overline{\mathcal{C}(W)} = \mathcal{C}(W) \cup \mathcal{EL}(W)$. From this we immediately have

$$\mathrm{diam}\, \pi_W(\mathcal{C}(\rho, L)) \geq d_W(\nu^+, \nu^-). \tag{4-2}$$

We next wish to get an inclusion in one direction,

$$\pi_W(\mathcal{C}(\rho, L)) \subset \mathcal{N}_{d_1}(\mathrm{hull}_W(\nu^+, \nu^-)) \tag{4-3}$$

for a uniform d_1. Recall that Theorem 2.3 provides $\epsilon_h > 0$ such that $\mathcal{C}(\rho, \epsilon_h) \subset \mathcal{CH}(\nu^{\pm})$. Theorem 2.5 gives a constant $K = K(S, \epsilon_h, L)$ such that if $\operatorname{diam} \pi_W(\mathcal{C}(\rho, L)) > K$, then $\ell_\rho(\gamma) < \epsilon_h$ for each component of ∂W. Thus we suppose for now that $\operatorname{diam} \pi_W(\mathcal{C}(\rho, L)) > K$, and therefore that $[\partial W] \subset \mathcal{CH}(\nu^{\pm})$.

Given $\alpha \in \mathcal{C}(\rho, L)$, let $f\colon X \to N_\rho$ be a pleated surface, in the homotopy class of ρ, realizing α. Let Γ be the curve system provided by Lemma 3.1, which consists of curves in $\mathcal{CH}(\nu^{\pm})$ whose length in X is at most $B = B(S, 1)$.

If a component $\gamma \in \Gamma$ intersects W essentially, then, since the length bound on γ and α in X implies a uniform upper bound on the intersection number of γ and α and hence an uniform upper bound on $d_W(\alpha, \gamma)$ (see [21, Lemma 2.1]), we obtain a uniform upper bound on $d_W(\alpha, \mathcal{CH}(\nu^{\pm}))$. Lemma 2.2 gives a uniform bound on the Hausdorff distance between $\pi_W(\mathcal{CH}(\nu^{\pm}))$ and $\operatorname{hull}_W(\nu^{\pm})$, so we obtain a uniform upper bound on $d_W(\alpha, \operatorname{hull}_W(\nu^{\pm}))$.

If, on the other hand, W is disjoint from Γ, consider the component Z of $S - \operatorname{collar}(\Gamma)$ containing W. Since, by Lemma 3.1, $\mathcal{C}(Z) \cap \mathcal{CH}(\nu^{\pm})$ is empty and $[\partial W] \subset \mathcal{CH}(\nu^{\pm})$, we must have $W = Z$. Moreover, again by Lemma 3.1, there exists $\beta \in \mathcal{C}(W)$ such that $l_X(\beta) \le B$ and

$$\operatorname{diam}_W(\beta \cup \pi_W(\mathcal{CH}(\nu^{\pm}))) \le B.$$

The length bounds on α and β again give a uniform upper bound on $d_W(\alpha, \beta)$, so we again obtain a uniform upper bound on $d_W(\alpha, \operatorname{hull}_W(\nu^{\pm}))$.

Since we have obtained a uniform upper bound on $d_W(\alpha, \operatorname{hull}_W(\nu^{\pm}))$ for all $\alpha \in \mathcal{C}(\rho, L)$, we have established the containment (4-3), for some uniform d_1.

Lemma 2.1 implies that there exists $A = A(S, L)$ such that if $d_W(\nu^+, \nu^-) > A$, then the hierarchy H contains a geodesic k_W supported on the subsurface W, whose initial and terminal vertices lie within A of $\pi_W(\nu^-)$ and $\pi_W(\nu^+)$. If

$$\operatorname{diam}(\pi_W(\mathcal{C}(\rho, L)) > 2d_1 + A,$$

then (4-3) implies that $d_W(\nu^+, \nu^-) > A$, and hence that we have k_W in H.

In this case, assuming $L_0 \ge L_h$, Theorem 2.3 implies that the vertices of k_W are all contained in $\mathcal{C}(\rho, L)$, and since the Hausdorff distance between k_W and $\operatorname{hull}_W(\nu^{\pm})$ is uniformly bounded (since $\mathcal{C}(W)$ is Gromov hyperbolic), we may conclude that there exists a uniform d_2 so that

$$\operatorname{hull}_W(\nu^{\pm}) \subset \mathcal{N}_{d_2}(\mathcal{C}(\rho, L) \cap \mathcal{C}(W)). \tag{4-4}$$

In particular

$$\operatorname{hull}_W(\nu^{\pm}) \subset \mathcal{N}_{d_2}(\pi_W(\mathcal{C}(\rho, L))). \tag{4-5}$$

Therefore, if $\operatorname{diam} \pi_W(\mathcal{C}(\rho, L)) > 2d_1 + K + A$, then both the Hausdorff distance between $\mathrm{hull}_W(\nu^{\pm})$ and $\pi_W(\mathcal{C}(\rho, L))$ and the Hausdorff distance between $\mathrm{hull}_W(\nu^{\pm})$ and $\mathcal{C}(\rho, L) \cap \mathcal{C}(W)$ are bounded from above by $d_1 + d_2$. Therefore, we have established our theorem in this case if we choose $D = 2d_1 + d_2 + A + K$.

It remains to consider the case that $\operatorname{diam} \pi_W(\mathcal{C}(\rho, L)) \leq 2d_1 + A + K$. However in this case the conclusion of the theorem is immediate from (4-1). $\square$

5 Ending laminations in the algebraic limit

We now prove Theorem 1.1, which asserts that ending laminations of geometrically infinite ends arise as limits of projections of end invariants.

Theorem 1.1 *Let $\rho_n \to \rho$ in $\mathrm{AH}(S)$. If $W \subseteq S$ is an essential subsurface of S, other than an annulus or a pair of pants, and $\lambda \in \mathcal{EL}(W)$ is a lamination supported on W, the following statements are equivalent:*

(1) *λ is a component of $\nu^+(\rho)$.*

(2) *$\{\pi_W(\nu^+(\rho_n))\}$ converges to λ.*

Furthermore we have:

(a) *If $\{\pi_W(\nu^+(\rho_n))\}$ accumulates on $\lambda \in \mathcal{EL}(W)$, then it converges to λ.*

(b) *The sequences $\{\nu^+(\rho_n)\}$ and $\{\nu^-(\rho_n)\}$ do not converge to a common $\lambda \in \mathcal{EL}(S)$.*

(c) *If $W \subsetneq S$ is a proper subsurface then convergence of $\{\pi_W(\nu^+(\rho_n))\}$ to $\lambda \in \mathcal{EL}(W)$ implies $\{\pi_W(\nu^-(\rho_n))\}$ does not accumulate on $\mathcal{EL}(W)$.*

The same statements hold with "+" replaced by "−".

We note that we allow the case $W = S$ unless explicitly noted otherwise.

For simplicity of notation, we let $\nu_n^+ = \nu^+(\rho_n)$ and $\nu_n^- = \nu^-(\rho_n)$. If λ is a component of $\nu^+(\rho)$, it is not difficult to show that λ is an accumulation point of either $\{\pi_W(\nu_n^+)\}$ or of $\{\pi_W(\nu_n^-)\}$. We first show, in Lemma 5.1, that it cannot be both. In order to show that it is an accumulation point of $\{\pi_W(\nu_n^+)\}$, we use the covering theorem and geometric limit arguments to pull-back larger and larger W–product regions from N_ρ to the approximates. We then consider intersections of split-level surfaces with these product regions to find pairs of hierarchy curves in $\mathcal{CH}_n$, one of which lies in a bounded set and the other of which approximates λ, such that the geodesic representative of the approximation to λ lies above the geodesic representative of the curve in the bounded set. This allows us to prove that (1) implies (2). On the other hand, if $\{\pi_W(\nu_n^+)\}$ converges to λ, we check that λ is the ending lamination of a geometrically infinite end of N_ρ. Then, using the fact that (1) implies (2), we see that the end must be upward-pointing, which establishes that (2) implies (1).

Proof We first observe that if $\{\pi_W(\nu_n^+)\}$ converges to λ, then $\{\pi_W(\nu_n^-)\}$ cannot accumulate at λ.

Lemma 5.1 *Suppose that $\{\rho_n\}$ is a convergent sequence in* $\mathrm{AH}(S)$. *If $W \subseteq S$ is a (not necessarily proper) subsurface of S and $\{\pi_W(\nu_n^+)\}$ converges to $\lambda \in \mathcal{EL}(W)$, then $\pi_W(\nu_n^-)$ does not accumulate at λ. Similarly, if $\{\pi_W(\nu_n^-)\}$ converges to $\lambda \in \mathcal{EL}(W)$, then $\{\pi_W(\nu_n^+)\}$ does not accumulate at λ.*

Proof Let W be any non-annular subsurface of S that is not a pair of pants. Suppose that $\{\pi_W(\nu_n^+)\}$ converges to $\lambda \in \mathcal{EL}(W)$. If $\{\pi_W(\nu_n^-)\}$ also accumulates at λ, then we can pass to a subsequence, still called $\{\rho_n\}$, such that both $\{\pi_W(\nu_n^+)\}$ and $\{\pi_W(\nu_n^-)\}$ converge to λ. Let $\rho = \lim \rho_n$.

Let α be a curve on W. Then the distance of α to any geodesic joining $\pi_W(\nu_n^+)$ and $\pi_W(\nu_n^+)$ diverges to ∞. Since there exists some $L \geq L_0$ such that $l_{\rho_n}(\alpha) \leq L$ for all n, this contradicts Theorem 1.2.

The proof of the other case is exactly the same. □

Suppose that $\lambda \in \mathcal{EL}(W)$ is a component of $\nu^+(\rho)$ – that is, N_ρ^0 has an upward-pointing end with base surface W and ending lamination λ. In order to show that (1) implies (2), it suffices to show that any subsequence of $\{\rho_n\}$ has a further subsequence $\{\rho_{n_k}\}$ such that $\pi_W(\nu_{n_k}^+) \to \lambda$.

Given any subsequence of $\{\rho_n\}$, we may pass to a subsequence (still denoted $\{\rho_n\}$) such that $\{N_n\} = \{N_{\rho_n}\}$ converges geometrically to a manifold N_G, which is covered by N_ρ. It is a consequence of the covering theorem [28; 14] (see [3, Proposition 5.2]) that there is a neighborhood of the geometrically infinite end of N_ρ^0 with ending lamination λ that embeds in N_G. Let $\mathcal{E}$ be the image of this neighborhood in N_G. We may identify $\mathcal{E}$ in an orientation-preserving way with $W \times [0, \infty)$.

After passing to a subsequence, we may assume that there exist 2–bi-Lipschitz maps

$$\phi_n \colon W \times [0, n+1] \to N_n$$

so that $\phi_n(\partial W \times [0, n]) \subset \mathbb{T}(\partial W, n)$ where $\mathbb{T}(\partial W, n)$ is the collection of Margulis tubes in N_n associated to the curves in ∂W. After passing to a further subsequence we can adjust the product structure on $\mathcal{E}$ and choose points

$$z_n \in (N_G)_{\mathrm{thick}(\epsilon_1)} \cap (W \times [n, n+1])$$

so that $\phi_n(W\times[k,k+1])$ is a W–product region, $\phi_n(z_k)\in(N_n)_{\mathrm{thick}(\epsilon_1/2)}$, and

$$\boldsymbol{d}|_{(N_n)_{\mathrm{thick}(\epsilon_1/2)},\phi_n(W\times[k,k+1])}\big(\phi_n(z_k),\phi_n(W\times\{k,k+1\})\big)>d_1(S,\epsilon_1/2)$$

for all $k=0,\dots,n$. In other words, $(\phi_n(W\times[k,k+1]),\phi_n(z_k))$ satisfies the assumptions of Lemma 2.14. (See the proof of Lemma 3.1.)

Let $h_n\colon M_n\to N_n$ be the model map provided by Theorem 2.3. Lemma 2.14 guarantees that for all n there exists $\alpha_n\in\mathcal{C}H_n$ with $h_n(U(\alpha_n))\subset\phi_n(W\times[0,1])$. Each α_n has a representative in $W\times[0,1]\subset\mathcal{E}$ of uniformly bounded length. As in the proof of Lemma 3.1 there will be a finitely many such curves and we can pass to a further subsequence such that α_n is a fixed curve α.

Similarly, for all n, Lemma 2.14 implies that there exists a curve $\beta_n\in\mathcal{C}H_n$ with $h_n(U(\beta_n))\subset\phi_n(W\times[n,n+1])$. Then $\{\phi_n^{-1}(\beta_n)\}$ is a sequence of bounded length curves exiting $\mathcal{E}$, so we have that $\beta_n\to\lambda$. In particular, for large n, α and β_n overlap. Furthermore, by construction $\phi_n^{-1}(h_n(U(\beta_n)))$ lies above $\phi_n^{-1}(h_n(U(\alpha)))$ in $\mathcal{E}$. Lemma 2.9 implies that $\phi_n(W\times\{1\})$ extends to an embedded surface X isotopic to a level surface in N_n. Since $h_n(U(\alpha_n))$ and $h_n(U(\beta_n))$ lie in a collar neighborhood of $\phi_n(W\times\{1\})$ we may assume they are disjoint from X. Since each ϕ_n is orientation-preserving, this implies that

$$h_n(U(\alpha))\prec_{\mathrm{top}}h_n(U(\beta_n))$$

in N_n. Since h_n is orientation-preserving, we may conclude that $U(\alpha)\prec_{\mathrm{top}}U(\beta_n)$ in M_n.

Since $d_W(\alpha,\beta_n)\to\infty$ and $\alpha,\beta_n\in\pi_W(C(\rho_n,L_0))$, Theorem 1.2 implies that $d_W(\nu_n^+,\nu_n^-)\to\infty$. We may pass to a subsequence so that $d_W(\nu_n^+,\nu_n^-)\geq A(S)$ for all n, so Lemma 2.1 implies that H_n contains a geodesic k_n with domain W. Lemma 2.2 implies that $d_W(\pi_{k_n}(\alpha),\alpha)$ and $d_W(\pi_{k_n}(\beta_n),\beta_n)$ are uniformly bounded, where π_{k_n} is the projection from $\mathcal{C}(S)$ to k_n through $\mathcal{C}(W)$. After identifying k_n with an interval, Lemma 2.6 implies that there exists r so that

$$\pi_{k_n}(\alpha)<\pi_{k_n}(\beta_n)+r.$$

Now, since $\beta_n\to\lambda\in\mathcal{EL}(W)$, it follows that $\pi_{k_n}(\beta_n)\to\lambda$. Since

$$\pi_{k_n}(\alpha)<\pi_{k_n}(\beta_n)+k,$$

β_n lies between α and the terminal vertex τ_n of k_n for all large enough n. Recalling that all the $\pi_{k_n}(\alpha)$ lie in a finite diameter set in $\mathcal{C}(W)$, we may conclude that the terminal vertex τ_n also converges to λ. (Here, we use that $\mathcal{C}(W)$ is hyperbolic and Klarreich's theorem [18] identifying $\partial\mathcal{C}(W)$ with $\mathcal{EL}(W)$.) Since, by Lemma 2.1,

$d_W(\tau_n, \nu_n^+)$ is uniformly bounded, we further conclude that $\{\pi_W(\nu_n^+)\}$ converges to λ, as desired. This completes the proof that (1) implies (2).

Now suppose that $\{\pi_W(\nu_n^+)\}$ converges to $\lambda \in \mathcal{EL}(W)$. Lemma 5.1 then implies that $\{\pi_W(\nu_n^-)\}$ does not accumulate at λ. Therefore, $d_W(\nu_n^+, \nu_n^-) \to \infty$. Lemma 2.1 implies that, for all large enough n, H_n contains a geodesic k_n with base surface W. For each n choose a vertex β_n of k_n so that $\beta_n \to \lambda$. By Klarreich's theorem, there is a subsequence so that $\{\beta_n\}$ converges projectively to a measured lamination ν on W whose support is λ, that is, $\{\beta_n / l_X(\beta_n)\}$ converges to ν, where X is a fixed finite area hyperbolic metric on W. By continuity of length [7], $l_\rho(\nu) = \lim l_{\rho_n}(\beta_n)/l_X(\beta_n) = 0$, so λ is unrealizable in ρ. This implies that λ is an ending lamination for an end based on W (see Section 2).

If this end were downward-pointing, then, since (1) implies (2) (applied in the downward-pointing case), $\{\pi_W(\nu_n^-)\}$ would converge to λ, which would contradict Lemma 5.1. Therefore, the end must be upward-pointing. This completes the proof that (2) implies (1).

In order to establish Claim (a), we assume that $\{\pi_W(\nu_n^+)\}$ accumulates at $\lambda \in \mathcal{EL}(W)$. By applying (2) $\Longrightarrow$ (1) to a subsequence $\{\rho_{n_j}\}$ where $\{\pi_W(\nu_{n_j}^+)\}$ converges to λ, we see that λ is a component of $\nu^+(\rho)$, and so, applying implication (1) $\Longrightarrow$ (2), we see that the entire sequence $\{\pi_W(\nu_n^+)\}$ converges to λ. Claim (b) in the statement follows immediately from Lemma 5.1. Finally, for Claim (c), note that for W a proper subsurface of S, if $\{\pi_W(\nu_n^+)\}$ converges to $\lambda^+ \in \mathcal{EL}(W)$ and $\{\pi_W(\nu_n^-)\}$ converges to $\lambda^- \in \mathcal{EL}(W)$, then λ^+ is a component of $\nu^+(\rho)$ and λ^- is a component of $\nu^-(\rho)$ by an application of (2) $\Longrightarrow$ (1). Therefore, for some boundary component γ of W that is non-peripheral in S, $\gamma \times \{0\}$ is isotopic in $S \times I$ into the annuli P^+ and P^- determined by the relative compact core for N_ρ, contradicting that no two components of P^+ and P^- are isotopic. $\square$

6 Overlapping curves

In this section we prove Theorem 1.3, which states that if a curve $\alpha \in \mathcal{C}(\rho, L)$ overlaps and lies above ∂W in N_ρ, then $\pi_W(\alpha)$ is uniformly close to $\pi_W(\nu^+(\rho))$.

Theorem 1.3 *Given S and $L > 0$ there exists c such that, given $\rho \in \mathrm{AH}(S)$, an essential subsurface $W \subset S$ that is not a pair of pants, and a curve $\alpha \in \mathcal{C}(\rho, L)$ such that α^* lies above the geodesic representative of some component of $(\partial W)^*$ that it overlaps, then*

$$d_W(\alpha, \nu^+(\rho)) \le c.$$

Furthermore, if W is not an annulus or a pair of pants, $\alpha \in \mathcal{C}(\rho, L)$ overlaps ∂W, and

$$d_W(\alpha, \nu^-) > c$$

then α^* lies above the geodesic representative of every component of ∂W that it overlaps.

The same holds when replacing "above" with "below" and ν^+ with ν^-.

When $\rho(\alpha)$ or $\rho(\beta)$ is parabolic we interpret the statement "α^* lies above β^*" to mean that all sufficiently short representatives of α in N_ρ lie above all sufficiently short representatives of β. Equivalently, α^* lies above β^* if α is in the top end invariant or β is in the bottom end invariant.

In the course of the proof we will notice that if $l_\rho(\alpha) \leq L$, W is not an annulus, and $d_W(\nu^+, \nu^-)$ is sufficiently large, then α^* either lies above or below all geodesic representatives of components of ∂W that it overlaps (see Proposition 6.1).

We will first prove the theorem when W is not an annulus. The result will be straightforward if $d_W(\nu^+, \nu^-)$ is small. We sketch the argument when $d_W(\nu^+, \nu^-)$ is large and therefore all the components of ∂W lie in $\mathcal{CH}$, are very short, and Lemma 2.13 gives us a wide W–product region. If α is a curve in the hierarchy the theorem follows from statement (2) of Lemma 2.6. If not we may realize α by a pleated surface $f\colon X \to N$ and then replace α with a hierarchy curve γ that overlaps W given to us by Lemma 3.1. If α^* overlaps and lies above a component β^* of $(\partial W)^*$ then the pleated surface also lies above β^*. Therefore, if γ and β^* overlap, $f(\gamma)$ also lies above β^* and the theorem again follows from (2) of Lemma 2.6. Most of the work in the proof involves the case when there is no such β^* that both α and γ overlap. For example, it is possible that γ lies in W. Here we employ the observation that the pleated surface has bounded penetration into the wide W–product region, and an application of part (1) of Lemma 2.6 completes the proof.

In the case that W is an annulus, note that the argument above works perfectly well as long as the curve $\gamma \in \Gamma$ overlaps the core of W. However, the possibility that γ is equal to the core of W, together with the phenomenon of wrapping, force a completely different approach. Given a curve α of bounded length such that α^* lies above γ^* (where γ is now the core of W), we first note that there does exist a curve β of bounded length such that β^* also lies above γ^*, and that moreover $d_\gamma(\beta, \nu^+)$ is bounded (β is essentially obtained from the top ending data of N_ρ, in Lemma 6.2). We therefore have to bound $d_\gamma(\alpha, \beta)$. To do this we consider a "model manifold" constructed with ending data α and β. If $d_\gamma(\alpha, \beta)$ is large enough then this model will have a deep tube $U(\gamma)$, and a Lipschitz model map constructed in Lemma 6.4 using a variation on the work in [26] will take $\partial U(\gamma)$ to the boundary of the corresponding Margulis tube $\mathbb{T}(\gamma)$. The fact that α^* and β^* are both above this tube will be used to show that the map $\partial U(\gamma) \to \partial \mathbb{T}(\gamma)$ has degree 0, but a Lipschitz map of degree 0 between tori of bounded geometry cannot have kernel generated by a very long curve (Lemma 6.5). This bounds the length of the meridian of $U(\gamma)$, and hence bounds $d_\gamma(\alpha, \beta)$.

6.1 Proof in the non-annular case

When W is not an annulus the theorem will follow from this proposition:

Proposition 6.1 *If S is a compact surface and $L > 0$, then there exists $c_2 = c_2(S, L)$ such that if $\rho \in \mathrm{AH}(S)$, W is a non-annular subsurface, $d_W(\nu^+, \nu^-) > c_2$ and $\alpha \in \mathcal{C}(S, L)$ overlaps ∂W, then either α^* lies above the geodesic representative in N_ρ of every component of ∂W it overlaps or α^* lies below the geodesic representative of every component of ∂W it overlaps.*

Moreover, if α^* lies above the geodesic representative in N_ρ of every component of ∂W it overlaps, then

$$d_W(\alpha, \nu^+) \le c_2.$$

The same holds when replacing "above" with "below" and "$+$" with "$-$".

Proof of Theorem 1.3 in non-annular case, given Proposition 6.1 If $d_W(\nu^+, \nu^-) > c_2$, then the second claim of Proposition 6.1 is exactly the first claim of the theorem. For the second claim of the theorem we note that the first claim implies that α^* cannot lie below $(\partial W)^*$. Proposition 6.1 then implies that α^* lies above $(\partial W)^*$.

If $d_W(\nu^+, \nu^-) \le c_2$ it is convenient to assume, without loss of generality, that $L \ge L_0$ where L_0 is the constant from Theorem 1.2. Theorem 1.2 then implies, since $l_\rho(\alpha) \le L$, that $\pi_W(\alpha)$ lies within $D = D(S, L)$ of $\mathrm{hull}_W(\nu^\pm)$. Therefore,

$$d_W(\alpha, \nu^+) \le D + d_W(\nu^+, \nu^-) \le D + c_2$$

regardless of whether or not α^* lies above any component of $(\partial W)^*$. In particular, we have both claims of the theorem. □

Proof of Proposition 6.1 We first assume that $c_2 > A$, where $A = A(S)$ is the constant given in Lemma 2.1. Therefore, the assumption that $d_W(\nu^+, \nu^-) > c_2$ implies that W is the support of a geodesic k_W in the hierarchy $H = H(\nu^+, \nu^-)$.

We may further assume that $c_2 > K + 2D$, where $K = K(S, \epsilon_h, L)$ is the constant from Theorem 2.5 and $D = D(S, L)$ is the constant from Theorem 1.2. Then Theorem 1.2 implies $\mathrm{diam}_W(\mathcal{C}(\rho, L)) > K$, so that $l_\rho(\partial W) < \epsilon_h$.

Let $N = N_\rho$ and let $h\colon M \to N$ be the model map provided by Theorem 2.3. Since $l_\rho(\partial W) < \epsilon_h$, $\partial W \subset \mathcal{C}H$ (where $H = H(\nu^\pm)$) and $h(U(\partial W)) = \mathbb{T}(\partial W)$.

If $d_W(\nu^+, \nu^-) = \infty$ then W supports a geometrically infinite end and every component of $(\partial W)^*$ will be parabolic and will either lie in ν^+ or ν^-. The ordering claim of the proposition then follows. For the remainder of the proof we assume that if W supports a geometrically infinite end it is downward-pointing and therefore k_W, the hierarchy geodesic supported on W, has a terminal vertex τ_W and α^* lies above every component of ∂W that it overlaps.

Let $f\colon X \to N$ be a pleated surface, in the homotopy class of ρ, realizing α. Let Γ be the system of hierarchy curves and $B = B(S, 1)$ the constant provided by Lemma 3.1, which bounds the length on X of every curve in Γ. Since we know that W is the support of a geodesic in H, and there are no hierarchy curves in the complement of Γ, there exists a hierarchy curve $\gamma \in \Gamma$ that overlaps W. Notice that, since $l_X(\alpha) \le L$ and $l_X(\gamma) \le B$, $d_W(\alpha, \gamma) \le a$ for some uniform constant a. Therefore

$$d_W(\alpha, \nu^+) \le d_W(\gamma, \tau_W) + a + A. \tag{6-1}$$

Thus our goal now is to bound $d_W(\gamma, \tau_W)$.

We will define a constant a_1 and require that $c_2 > 2a_1 + 2A$, so that, by Lemma 2.1, k_W has length at least $2a_1$. The constant a_1 will be chosen so that the W–product region provided by Lemma 2.13 will be "thick" enough for our purposes.

We begin by giving the argument in a simpler case, where it is easier to understand the structure of the argument.

Simplified ordering argument Consider the case in which the following hold:

(S1) $f(X)$ is disjoint from $\mathbb{T}(\partial W)$.

(S2) $f(\gamma) \subset h(U(\gamma))$.

Recall from Lemma 2.11 that there exists a constant $b = b(S, \epsilon_h)$, so that

$$\mathbf{diam}|_{X_{\mathrm{thick}(\epsilon_h)}}(X) \le b.$$

Since f is 1–Lipschitz it follows (see (2-6)) that

$$\mathbf{diam}|_{N_{\mathrm{thick}(\epsilon_h)}}(f(X)) \le b.$$

Let $c_3 = c_3(S)$ be the constant given by Lemma 2.13, and assume that a_1 has been chosen so that

$$a_1 \ge \lceil b/c_3 \rceil.$$

Let $v_0 < v_1 < v_2 = \tau_W$ be three vertices in k_W such that $d_W(v_i, v_{i+1}) = a_1$. Lemma 2.13 gives us images of extended split-level surfaces $G_i = h(\widehat{F}_{v_i})$ and two W–product regions R_1 and R_2 where R_1 has horizontal boundary $G_0 \cup G_1$ and R_2 has horizontal boundary $G_1 \cup G_2$. If we let $R = R_1 \cup R_2$, then $G_1 \subset R$, and

$$d|_{N_{\mathrm{thick}(\epsilon_h)}, R}(G_1, G_j) > b \tag{6-2}$$

for $j = 0$ and $j = 2$.

We claim now that $f(X)$ is disjoint from G_1. If not, then, since $f(X)$ is disjoint from $\mathbb{T}(\partial W)$ and

$$\mathbf{diam}|_{N_{\mathrm{thick}(\epsilon_h)}}(f(X)) \leq b,$$

$f(X)$ would have to be contained in R. However, this is impossible, since f is a homotopy equivalence.

Since $f(X)$ is disjoint from G_1, it lies above it or below it by Lemma 2.8. By Lemma 2.7, α^* lies either above or below every component of $(\partial W)^*$ that it overlaps. This proves the ordering statement of the proposition.

Assume that the former holds: $f(\alpha) = \alpha^*$ lies above the core curve of $\mathbb{T}(\beta)$ for every component $\beta \subset \partial W$ that α overlaps. Since $f(\alpha)$ is disjoint from the tube $\mathbb{T}(\beta)$, it also lies above the corresponding boundary component of G_1. Hence $G_1 \prec_{\mathrm{top}} f(X)$. We therefore can conclude that $G_1 \prec_{\mathrm{top}} f(\gamma)$, and finally, since $f(\gamma) \subset h(U(\gamma))$, that

$$G_1 \prec_{\mathrm{top}} h(U(\gamma)) \tag{6-3}$$

provided G_1 is disjoint from $h(U(\gamma))$.

Note that G_1 intersects $h(U(\gamma))$ only if $\widehat{F}_{v_1}$ intersects $U(\gamma)$. Since $\widehat{F}_{v_1}$ is an extended split-level surface, this can only happen if γ is one of the base curves of $\widehat{F}_{v_1}$, and in particular if γ is disjoint from v_1 (as curves on S). In this case $d_W(\gamma, v_1) \leq 1$, so

$$d_W(\gamma, \tau_W) \leq a_1 + 1n.$$

If γ does intersect v_1, then G_1 and $h(U(\gamma))$ are disjoint and so

$$U(v_1) \prec_{\mathrm{top}} U(\gamma).$$

By Lemma 2.6, $\pi_{k_W}(v_1) \leq \pi_{k_W}(\gamma) + r$, which implies that

$$d_W(\gamma, \tau_W) \leq a_1 + r.$$

We have uniformly bounded $d_W(\gamma, \tau_W)$ in all cases. In combination with (6-1), this completes the proof of the second claim in our simplified case.

General ordering argument We now adapt the above argument to hold in the general setting where (S1) and (S2) may not hold. It will be convenient to divide ∂W into the collection $\partial_1 W$ of components that do not overlap α, and the collection $\partial_0 W$ of components that do overlap α. Moreover, we will assume that the pleated surface f realizes $\alpha \cup \partial_1 W$. As in the simplified case, we will construct extended split level surfaces G_0, G_1 and G_2 such that G_0 and G_2 are the horizontal boundary components of a W–product region R that contains G_1. We will choose the spacing constant to guarantee that:

(1) $f(X)$ is disjoint from G_1.

(2) There exists a collection of annuli A_0 joining (suitable components of) ∂G_1 to $(\partial_0 W)^*$ such that $f(X)$ misses A_0.

(3) There exists a homotopy from $f(\gamma)$ into $h(U(\gamma))$ that is disjoint from G_1.

By (1) and (2), $f(X)$ is disjoint from $\overline{G}_1 = G_1 \cup A_0$ and therefore, by Lemma 2.8, $f(X)$ lies above or below $\overline{G}_1$ and, as in the simplified case, Lemma 2.7 implies that α^* lies either above or below every component of $(\partial W)^*$ that it overlaps. This proves the ordering statement. Since $f(X)$ lies above $\overline{G}_1$ and hence above G_1, then (3) implies that $h(U(\gamma))$ lies above G_1 if it is disjoint from G_1. Therefore, once we have established (1)–(3), we can use Lemma 2.6 to complete the proof just as we did in the simplified case.

We remark that with a little more work, we could homotope f to a uniformly Lipschitz map g such that $g(\gamma) \subset h(U(\gamma))$ and $g(X)$ is disjoint from $\mathbb{T}(\partial_0 W)$. Our proof would then resemble the simplified case even more closely.

We now establish (1)–(3). Notice that our assumptions imply that

$$f(\mathrm{collar}(\partial_1 W)) \subset \mathbb{T}(\partial_1 W).$$

Since $l_X(\alpha) \leq L$ and α overlaps every component of $\partial_0 W$, there exists $L_2 = L_2(S, L)$ so that passing through every point in $X - \mathrm{collar}(\partial_1(W))$ there is a curve of length at most L_2 that is not homotopic into $\mathrm{collar}(\partial W)$. Moreover, there exists $\epsilon_3 = \epsilon_3(\max\{L_2, B, K_h\})$ so that any curve of length at most $\max\{L_2, B, K_h\}$ that intersects $\mathbb{T}_{\epsilon_3}(\partial W)$ is contained in $\mathbb{T}(\partial W)$. It follows that $f(X - \mathrm{collar}(\partial_1 W))$ is disjoint from $\mathbb{T}_{\epsilon_3}(\partial W)$.

Applying Theorem 2.5, we may assume that $d_W(\nu^+, \nu^-)$ is large enough that $l_\rho(\partial W) \leq \epsilon_3/4$. Therefore, there exists a constant K_3 so that the radial projection

$$r_W\colon \mathbb{T}(\partial W) - \mathbb{T}_{\epsilon_3}(\partial W) \to \partial \mathbb{T}(\partial W)$$

is K_3–Lipschitz. We may extend r_W to a K_3–Lipschitz map defined on $N - \mathbb{T}_{\epsilon_3}(\partial W)$ by setting it to be the identity off of $\mathbb{T}(\partial W)$.

Let Z be a component of $X - \mathrm{collar}(\partial_1 W)$. Lemma 2.11 implies that there exists $b = b(S, \epsilon_3)$ so that

$$\mathbf{diam}|_{X_{\mathrm{thick}(\epsilon_3)}}(Z) \le b.$$

Since $f(\gamma)$ has length at most B and the core curve of $h(U(\gamma))$ has length at most K_h, one may homotope $f(\gamma)$ into $h(U(\gamma))$ through a family of curves of length at most $\max\{B, K_h\}$. It follows that this homotopy may be completed in the complement of $\mathbb{T}_{\epsilon_3}(\partial W)$. Therefore, there exists a constant $b_1 = b_1(B, \epsilon_3)$ so that there is a homotopy from $f(\gamma)$ into $h(U(\gamma))$ all of whose tracks have length at most b_1 and are disjoint from $\mathbb{T}_{\epsilon_3}(\partial W)$. (If $\ell_\rho(\gamma) < \epsilon_h$, then $h(U(\gamma)) = \mathbb{T}(\gamma)$ and this follows from [11, Lemma 2.6]. If not, then the ruled homotopies from $f(\gamma)$ to γ^* and from the core curve of $h(U(\gamma))$ to γ^* each have uniformly bounded length tracks and avoid $\mathbb{T}_{\epsilon_3}(\partial W)$, so they may be concatenated to produce the desired homotopy.)

Let $c_3 = c_3(S)$ be the constant given by Lemma 2.13, and assume that

$$a_1 \ge \lceil K_3(b + b_1)/c_3 \rceil.$$

As in the simplified case, this implies that k_W has length at least $2a_1$.

Let $v_0 < v_1 < v_2 = \tau_W$ be three vertices in k_W such that $d_W(v_i, v_{i+1}) = a_1$. Lemma 2.13 provides extended split level surfaces $G_i = h(\widehat{F}_{v_i})$ and a W–product region R with horizontal boundary $G_0 \cup G_2$ such that $G_1 \subset R$ and

$$\boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon_3)}, R}(G_1, G_j) \ge \boldsymbol{d}|_{N_{\mathrm{thick}(\epsilon_h)}, R}(G_1, G_j) > K_3(b + b_1) \tag{6-4}$$

for $j = 0$ and $j = 2$.

We first claim that if Z is a component of $X - \mathrm{collar}(\partial_1 W)$, then $f(Z)$ is disjoint from G_1. Since $f(\mathrm{collar}(\partial_1 W)) \subset \mathbb{T}(\partial_1(W))$, this implies that $f(X)$ is disjoint from G_1, which establishes (1). Notice that Z is not homotopic into W, so $f(Z)$ cannot be contained in $R \cup \mathbb{T}(\partial W)$. So, if $f(Z)$ intersects G_1, there is a path ν in Z so that $\nu \cap Z_{\mathrm{thick}(\epsilon_3)}$ has length at most b and $f(\nu)$ joins G_1 to a point outside of $R \cup \mathbb{T}(\partial W)$. Since $f(Z_{\mathrm{thin}(\epsilon_3)})$ is disjoint from $\mathbb{T}_{\epsilon_3}(\partial W)$ and $f(Z_{\mathrm{thin}(\epsilon_3)}) \subset N_{\mathrm{thin}(\epsilon_3)}$, $\overline{\nu} = r_W(f(\nu))$ is a path contained in $N - \mathbb{T}(\partial W)$ joining a point on G_1 to a point outside of R such that $\overline{\nu} \cap N_{\mathrm{thin}(\epsilon_3)}$ has length at most $K_3 b$. However, this would contradict (6-4).

A very similar argument establishes (2). Let A_0 be the collection of radial annuli in $\mathbb{T}(\partial_0 W)$ joining components of $(\partial_0 W)^*$ to the appropriate components of ∂G_1. If Z is a component of $X - \mathrm{collar}(\partial_1 W)$ and $f(Z)$ intersects A_0, then there is a path ν

in Z so that $\nu \cap Z_{\mathrm{thick}(\epsilon_3)}$ has length at most b and $f(\nu)$ joins A_0 to a point outside of $R \cup \mathbb{T}(\partial_0 W)$. Then $\overline{\nu} = r_W(f(\nu))$ would be a path contained in $N - \mathbb{T}(\partial W)$ joining a point on G_1 to a point outside of R such that $\overline{\nu} \cap N_{\mathrm{thin}(\epsilon_3)}$ has length at most $K_3 b$. Again, this contradicts (6-4). Therefore, $f(X)$ is disjoint from A_0 and we have established (2).

To establish (3), consider the homotopy with track lengths at most b_1 joining $f(\gamma)$ to a curve in $h(U(\gamma))$ (in the complement of $\mathbb{T}_{\epsilon_3}(\partial W)$). If the homotopy intersects G_1 then there is a path η of length b_1 joining G_1 to $f(X)$. Then $r_W(\eta)$ is a path of length Kb_1 in $N - \mathbb{T}(\partial W)$ joining G_1 to a point in $z \in f(X)$. One may then apply the construction in (1) to find a path $\overline{\nu}$ in $N - \mathbb{T}(\partial W)$ joining $r_W(z)$ to a point in $N - R$ such that $\overline{\nu} \cap N_{\mathrm{thin}(\epsilon_3)}$ has length at most $K_3 b$. Concatenating η and $\overline{\nu}$ would again contradict (6-4). This completes the proof of the first claim in the non-annular case. □

6.2 Proof in the annular case

We now proceed to give a proof of Theorem 1.3 in the case when W is an annulus. Let γ be the core of W. Assume that $\alpha \in C(\rho, L)$ and that α^* lies above γ^*.

We first observe that we may assume that there is a bounded length curve β such that $d_W(\beta, \nu^+) \le 1$ whose geodesic representative lies above γ^*.

Lemma 6.2 *Let S be a compact surface and $L \ge L_0$. There exists $K_3 = K_3(S, L)$ and $L_1 = L_1(S, L)$ such that if $\rho \in \mathrm{AH}(S)$ has end invariants $\nu^\pm$, $\alpha \in \mathcal{C}(\rho, L)$, $\gamma \in \mathcal{C}(S)$, α^* lies above γ^* and*

$$d_\gamma(\alpha, \nu^+) > K_3,$$

then there exists $\beta \in \mathcal{C}(\rho, L_1)$ such that $d_\gamma(\beta, \nu^+) \le 1$ and β^* lies above γ^*.

Proof First notice that if γ overlaps a simple closed curve component of ν^+ then we can choose β to be that curve. (Recall that, by convention, we say that β^* lies above γ^* if β is an upward-pointing parabolic).

If γ overlaps a lamination component λ of ν^+ with support Y, then there exists a sequence $\{\beta_n\} \subset \mathcal{C}(\rho, L_h) \cap \mathcal{C}(Y)$ so that $\beta_n \to \lambda$ in $\mathcal{C}(Y)$. For all sufficiently large n, $d_W(\beta_n, \nu^+) \le 1$ and β_n^* lies above γ^*, so we may choose $\beta = \beta_n$ for some specific large enough n.

In the remaining case, γ is contained in Y where Y is the support of an upward pointing geometrically finite end. Let Y_h be the component of the (upper) convex hull boundary associated to Y, and let Y_∞ be the corresponding component of the boundary at infinity, with its Poincaré metric. Recall that there is a 2–Lipschitz map $r\colon Y_\infty \to Y_h$ in the correct homotopy class, called the nearest point retraction (Epstein, Marden and Markovic [16, Theorem 3.1]).

There exists $\epsilon_4 = \epsilon_4(L)$ such that any curve of length at most L that intersects $\mathbb{T}_{\epsilon_4}(\gamma)$ is homotopic to a power of γ. If $l_{Y_h}(\gamma) < \epsilon_4$, then there is an annulus A in $\mathbb{T}_{\epsilon_4}(\gamma)$ joining γ^* to its representative on Y_h. But then α^*, which lies in $C(N)$ and above γ^*, would be forced to intersect A, contradicting the assumption that $l(\alpha^*) < L$. Therefore, we may assume that $l_{Y_h}(\gamma) \ge \epsilon_4$, which implies (via the map r) that $l_{Y_\infty}(\gamma) \ge \epsilon_4/2$. It follows that the minimal Y_∞–length marking $\nu^+|_Y$ contains a curve β of length at most $L_1 = L_1(S, \epsilon_4)$ that overlaps γ.

It is clear that $r(\beta)$ lies above γ^*, unless it intersects γ^*, since $\gamma^* \subset C(N)$. There exists $\epsilon_5 = \epsilon_5(L_1)$ so that there is a homotopy from $r(\beta)$ to β^* that is disjoint from $\mathbb{T}_{\epsilon_5}(\delta)$ for any curve $\delta \in \mathcal{C}(S) - \{\beta\}$. Therefore, if $l_\rho(\gamma) < \epsilon_5$, then β^* lies above γ^*. Theorem 2.5 gives $K_3 = K(S, L, \epsilon_5)$ so that if $\mathrm{diam}_\gamma(\mathcal{C}(\rho, L)) \ge K_3$, then $l_\rho(\gamma) < \epsilon_5$. But, since

$$\mathrm{diam}_\gamma((C(\rho, L)) \ge d_\gamma(\alpha, \nu^+) \ge K_3$$

by assumption, we may conclude that β^* lies above γ^* in this case as well. □

The annular case of Theorem 1.3 then follows quickly from the following result:

Proposition 6.3 *Given a compact surface S and $L, D > 0$, there exists $F = F(D, L)$ such that if $\rho \in \mathrm{AH}(S)$ and α, β and γ are curves in $\mathcal{C}(S)$ such that*

(1) *α and β overlap γ,*
(2) *$\ell_\rho(\alpha) \le L$ and $\ell_\rho(\beta) \le L$,*
(3) *α^* and β^* lie above γ, and*
(4) *if Y is non-annular essential subsurface with $\gamma \in [\partial Y]$ then $d_Y(\alpha, \beta) \le D$,*

then $d_\gamma(\alpha, \beta) \le F$.

Proof of the annular case of Theorem 1.3 We may assume that $d_\gamma(\alpha, \nu^+) > K_3$ where $K_3 = K_3(S, L)$ is the constant from Lemma 6.2. Let β be the curve provided by Lemma 6.2. Recall that $l_\rho(\beta) < L_1 = L_1(S, L)$, $d_\gamma(\beta, \nu^+) \le 1$ and β^* lies above γ^*.

If $D = D(S, L)$ is the constant from the non-annular case of Theorem 1.3, then for any non-annular surface Y with $\gamma \in [\partial Y]$, we have

$$d_Y(\alpha, \beta) \le 2D.$$

We may then apply Proposition 6.3 to conclude that

$$d_\gamma(\alpha,\beta) \le F = F(2D, \max\{L, L_1\}).$$

It follows that $d_\gamma(\alpha,\nu^+) \le F+1$, and the proof is complete. □

We now turn to the proof of Proposition 6.3. The first lemma we need is a mild variation of the model manifold theorem from [26], in which the end invariants have been replaced by the bounded-length curves α and β. Our statement of this lemma forgets most of the structure of the model and the Lipschitz properties of the map, remembering only the properties of the map concerning the tube $U(\gamma)$ and the images of α and β. We will have to take a bit of care because the statements in [26] are given in the setting where the initial and terminal markings of the hierarchy are actually the end invariants of ρ, although the proofs go through verbatim in this setting. In Bowditch [6], simpler proofs of the a priori length bounds of [26] are given, in the more general setting, and this will simplify the discussion. We will point out the details as we go.

Lemma 6.4 *Given a compact surface S and $L, D > 0$ there exist $K_1 = K_1(S, L, D)$ and $K_2 = K_2(S, L, B)$ such that if $\rho \in \mathrm{AH}(S)$, α and β are intersecting curves in $\mathcal{C}(S)$ and $\gamma \in \mathcal{C}(S)$ intersects both α and β,*

- $\ell_\rho(\alpha), \ell_\rho(\beta) \le L$,
- $d_\gamma(\alpha,\beta) > K_1$, *and*
- $d_Y(\alpha,\beta) < D$ *if $\gamma \in [\partial Y]$ and Y is non-annular,*

then there exists a map of pairs

$$f_{\alpha,\beta}\colon (S\times[0,1], \partial S\times[0,1]) \to \big(N_\rho^0, \partial N_\rho^0\big),$$

in the homotopy class determined by ρ, such that:

(1) *The preimage $f_{\alpha,\beta}^{-1}(\mathbb{T}(\gamma))$ is a regular neighborhood $U(\gamma)$ of $\gamma\times\{1/2\}$.*

(2) *The restriction of $f_{\alpha,\beta}$ to $\partial U(\gamma)$, as a map to $\partial\mathbb{T}_\rho(\gamma)$, is K_2–Lipschitz with respect to a Euclidean metric σ on $\partial U(\gamma)$, which has area at most K_2, and meridian length bounded below by $d_\gamma(\alpha,\beta)/K_2$.*

(3) *If α^* lies above (respectively below) γ^*, then $f|_{S\times\{0\}}$ lies above (respectively below) γ^*.*

(4) *If β^* lies above (respectively below) γ^*, then $f|_{S\times\{1\}}$ lies above (respectively below) γ^*.*

Proof We first extend α to a pants decomposition P_α each of whose curves has length at most $L' = L'(S, L)$. We do so by considering a pleated surface $f\colon X \to N$ realizing α and letting P_α be a minimal length pants decomposition of X that contains α. We similarly extend β to a pants decomposition P_β each of whose curves has length at most $L' = L'(S, L)$.

Construct a hierarchy H whose initial and terminal markings are P_α and P_β, respectively. [26, Section 7], as well as the main theorem of Bowditch [6], give us an uniform upper bound on the lengths of all curves in $\mathcal{C}H$.

Choose $K_1 > K(S, \epsilon_h, L)$, where $K(S, \epsilon_h, L)$ is the constant from Theorem 2.5, so that, with our assumptions, $\ell_\rho(\gamma) < \epsilon_h$. Moreover, we may choose $K_1 > A(S)$ where $A(S)$ is the constant from Lemma 2.1, so that γ must lie in $\mathcal{C}H$.

We assume, for the moment, that P_α and P_β have no curves in common. The construction in [26, Section 8] produces a "model manifold" from the hierarchy H. This is a manifold, equipped with a path metric, homeomorphic to $S \times [0,1]$ minus the curves $P_\alpha \times \{0\}$ and $P_\beta \times \{1\}$. To each curve v in H is associated a "tube" $U(v)$, which is a solid torus regular neighborhood of a level curve isotopic to v. This applies in particular to γ and the curves of P_α and P_β. After removing $\overline{U(v)}$ for each curve v in P_α and P_β, and taking the closure of what remains (this has the effect of removing the part of $\partial U(v)$ that is in the boundary of the model), we obtain a subset M of the model that is homeomorphic to $S \times [0,1]$, and such that P_α and P_β are realized with uniformly bounded length on its boundary (we again label these curves $P_\alpha \times \{0\}$ and $P_\beta \times \{1\}$). This is the manifold on which we will define our map.

The boundary of the tube $U(\gamma)$ is a Euclidean torus, whose geometry is controlled in terms of the coefficients $\{d_Y(P_\alpha, P_\beta)\}_{\gamma \subset \partial Y}$, by [26, Theorem 9.1] (and the discussion in [26, Section 9]). In particular, given our uniform upper bound on $d_Y(P_\alpha, P_\beta)$ over all non-annular Y with $\gamma \subset \partial Y$, we obtain an uniform upper bound on the area of $\partial U(\gamma)$. The same theorem gives a lower bound of the form $d_\gamma(\alpha, \beta)/K_2$ for the length of the meridian of this torus. Together these bounds give us conclusion (2).

We now note that the construction of a map from M to N_ρ carried out in [26, Section 10] depends only on the length bounds on hierarchy curves. Thus the proof of the Lipschitz model theorem in that section, carried out in our setting, yields a continuous map $f\colon M \to N_\rho$ such that $f^{-1}(\mathbb{T}(\gamma)) = U(\gamma)$ (conclusion (3) of that theorem), that f takes $\alpha \times \{0\}$ and $\beta \times \{1\}$ to curves of uniformly bounded length, and that f is K_2–Lipschitz on $\partial \mathbb{T}(\gamma)$ (for some uniform choice of K_2 depending only on S and L).

Since $f(\alpha \times \{0\})$ has bounded length, there exists $\epsilon_6 > 0$ so that the ruled homotopy from $f(\alpha \times \{0\})$ to α^* cannot intersect $\mathbb{T}_{\epsilon_6}(\alpha)$. We again use Theorem 2.5 to observe that we may choose K_1 large enough that $l_\rho(\gamma) < \epsilon_6/2$, so that the ruled homotopy is disjoint from γ^*. Therefore, if α^* lies above (below) γ^*, then $f(\alpha \times \{0\})$ lies above (below) γ^*, which implies, by Lemma 2.7, that $f|_{S\times\{0\}}$ lies above (below) γ^*. Similarly, if β^* lies above (below) γ^*, then $f(\beta \times \{1\})$ lies above (below) γ^*, which implies, by Lemma 2.7, that $f|_{S\times\{1\}}$ lies above (below) γ^*.

Thus we obtain a map satisfying all the conditions of the lemma. This completes the proof under our assumption that P_α and P_β have no common curves.

We now explain how to handle the case when P_α and P_β do have common curves. Let Δ denote the union of their shared components.

A hierarchy between P_α and P_β still exists, and the a priori length bounds on the hierarchy curves are obtained in exactly the same way. The construction of the model, however, is now slightly different: In [26], the initial and terminal markings are allowed to have common components only if at least one of them comes with a transversal. This is not the case here, so we must use a variation of the construction.

The construction in [26] proceeds just as before, except at the stage where a tube is inserted for a component δ of Δ. This tube will now be of the form $(\textit{annulus}) \times \mathbb{R}$, and so the removal of the tubes associated to Δ will produce a model naturally homeomorphic to $R \times [0,1]$, where R is the (possibly disconnected) surface $S \setminus \mathrm{collar}(\Delta)$.

The construction of the Lipschitz map f proceeds as before on each component of $R \times [0,1]$. We note that the restriction of the model map to $R \times \{1\}$ may be extended to a map defined on $S \times \{1\}$ by appending ruled annuli connecting $f(\partial R)$ to ∂R^*. Since there is an uniform upper bound on the length of $f(\partial R)$, there exists $\epsilon_7 > 0$ so that these appended annuli cannot intersect $\mathbb{T}_{\epsilon_7}(\gamma)$. We may assume that K_1 is chosen large enough that $l_\rho(\gamma) < \epsilon_7/2$, so that a retraction can be used to adjust the map so that the pre-image of $\mathbb{T}(\gamma)$ is just $U(\gamma)$. We can extend this to a slight thickening of $S \times \{1\}$, so that our final model is homeomorphic to $S \times [0,1]$, and again the proof is complete. □

We next establish a lower bound on the Lipschitz constant of a degree zero map between two Euclidean tori, which depends on the minimal length of a homotopically non-trivial geodesic generating the kernel of the associated map between the fundamental groups.

Lemma 6.5 *If T_0 and T_1 are Euclidean tori and $f\colon T_0 \to T_1$ is a Lipschitz map such that the kernel of $f_*\colon \pi_1(T_0) \to \pi_1(T_1)$ is infinite cyclic and is generated by an* *element whose geodesic representative has length L, then*

$$\mathrm{Lip}(f) \geq \frac{2\,\mathrm{inj}(T_1)L}{\mathrm{Area}(T_0)},$$

where $\mathrm{Lip}(f)$ *is the minimal Lipschitz constant for* f.

Proof Let a be a generator of $\ker(f_*)$. Then a stabilizes a line ℓ_0 in the universal cover $\widetilde{T}_0 = \mathbb{R}^2$. Choose $b \in \pi_1(T_0)$ so that a and b generate and let $\ell_n = b^n(\ell_0)$. Notice that $\mathrm{Area}(T_0) = L d(\ell_0, \ell_1)$. It follows that

$$d(\ell_0, \ell_n) = \frac{n\,\mathrm{Area}(T_0)}{L}.$$

The lift $\widetilde{f}\colon \widetilde{T}_0 \to \widetilde{T}_1$ factors through a map to $\widetilde{T}_0/\langle a\rangle$ and in $\widetilde{T}_0/\langle a\rangle$ the image of ℓ_0 is compact. Notice that

$$\mathrm{diam}(\widetilde{f}(\ell_0)) = \mathrm{diam}(\widetilde{f}(\ell_n)).$$

Since $f_*(b)$ acts by translation on $\widetilde{T}_1 = \mathbb{R}^2$ with translation distance at least $2\,\mathrm{inj}(T_1)$, we see that:

$$d(\widetilde{f}(\ell_0), \widetilde{f}(\ell_n)) \geq 2\,\mathrm{inj}(T_1)n - 2\,\mathrm{diam}(\widetilde{f}(\ell_0))$$

Therefore

$$\mathrm{Lip}(f) \geq \mathrm{Lip}(\widetilde{f}) \geq \frac{\big(2\,\mathrm{inj}(T_1)n - 2\,\mathrm{diam}(\widetilde{f}(\ell_0))\big)L}{n\,\mathrm{Area}(T_0)}.$$

Letting $n \to \infty$ gives the desired estimate. □

Proof of Proposition 6.3 We may assume that $d_\gamma(\alpha, \beta) > K_1$, where K_1 is the constant from Lemma 6.4, since otherwise we are done.

Let $f_{\alpha,\beta}$ be the map given by Lemma 6.4 and let $f\colon \partial U(\gamma) \to \partial\mathbb{T}(\gamma)$ be the restriction of $f_{\alpha,\beta}$ to the torus $\partial U(\gamma)$. Since $f_{\alpha,\beta}|_{S\times\{0\}}$ and $f_{\alpha,\beta}|_{S\times\{1\}}$ both lie above γ^* and have image disjoint from $\mathbb{T}(\gamma)$, they are homotopic in the complement of $\mathbb{T}(\gamma)$. Lemma 2.10 then implies that $\deg(f) = 0$ and thus that $\ker(f) \neq \{\mathrm{id}\}$. Since $f_{\alpha,\beta}$ is a homotopy equivalence the kernel of f has to be contained in the kernel of the inclusion of $\partial U(\gamma)$ in $S \times [0,1]$. The kernel of this second map is generated by the meridian of $\partial U(\gamma)$ and therefore $\ker(f)$ is generated by a power of the meridian.

By Lemma 6.4 there exists a constant $K_2 > 0$ such that $\partial U(\gamma)$ is a Euclidean torus with area bounded above by K_2, the length of the meridian is bounded below by $d_\gamma(\alpha,\beta)/K_2$ and f is a K_2–Lipschitz map. The boundary of the Margulis tube $\mathbb{T}(\gamma)$ is also a Euclidean torus and its injectivity radius is uniformly bounded below by some constant C_1. We then apply Lemma 6.5 to conclude that

$$K_2 > \frac{2C_1(d_\gamma(\alpha,\beta)/K_2)}{K_2} = \frac{2C_1 d_\gamma(\alpha,\beta)}{(K_2)^2}.$$

Rearranging we have

$$d_\gamma(\alpha,\beta) < \frac{(K_2)^3}{2C_1}$$

which completes the proof. □

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