

Toric LeBrun metrics and Joyce metrics

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We show that, on the connected sum of complex projective planes, any toric LeBrun metric can be identified with a Joyce metric admitting a semi-free circle action through an explicit conformal equivalence. A crucial ingredient of the proof is an explicit connection form for toric LeBrun metrics.

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1 Introduction

The subject of self-dual metrics on four-manifolds has rapidly developed since the discovery by Poon of a 1-parameter family of self-dual conformal classes on $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ [8]. We do not attempt to give a complete review of subsequent developments here; in this short note we are concerned only with two classes of self-dual metrics on

$$n\mathbb{C}\mathbb{P}^2 := \mathbb{C}\mathbb{P}^2 \# \dots \# \mathbb{C}\mathbb{P}^2 \quad (n \text{ times}).$$

First, in 1991, Claude LeBrun [6] produced explicit examples with $U(1)$ -symmetry on $n\mathbb{C}\mathbb{P}^2$, using a hyperbolic ansatz inspired by the Gibbons–Hawking ansatz [2]. LeBrun’s construction depends on the choice of n points in hyperbolic 3-space \mathcal{H}^3 . For $n = 2$, the only invariant of the configuration is the distance between the monopole points, and LeBrun conformal classes are the same as the 1-parameter family found by Poon. The Poon metrics are toric, that is, they admit a smooth effective action by a real torus $U(1) \times U(1)$. For $n > 2$, a LeBrun metric admits a torus action if and only if the monopole points belong to a common hyperbolic geodesic. These form a sub-class of LeBrun metrics, which we call *toric LeBrun metrics*.

The second class of metrics we are concerned with are the metrics on $n\mathbb{C}\mathbb{P}^2$ discovered by Dominic Joyce in [4]. Joyce’s construction depends on the choice of $n + 2$ points on the *boundary* of hyperbolic 2-space. These metrics are always toric. It was subsequently shown by Fujiki that any compact toric self-dual four-manifold with non-zero Euler characteristic is necessarily diffeomorphic to $n\mathbb{C}\mathbb{P}^2$, and furthermore the self-dual structure is of Joyce-type [1].

We recall that a *semi-free* action is a non-trivial action of a group G on a connected space M such that for every $x \in M$, the corresponding isotropy subgroup is either all of G or is trivial. Many of the families of metrics constructed by Joyce are not of LeBrun-type. However, if the torus action contains a circle subgroup that acts semi-freely on $n\mathbb{C}\mathbb{P}^2$, they are the same (for each n , such a torus action is unique). This coincidence was stated in [4], but without proof. This fact follows from Fujiki’s Theorem mentioned above, however we feel it is useful to have a direct proof. Recently, the authors determined the conformal automorphism groups of LeBrun’s monopole metrics [3]. In the course of that work an explicit connection for any toric LeBrun metric was found, which we use in this paper to prove the following:

Theorem 1.1 *On $n\mathbb{C}\mathbb{P}^2$, the class of toric LeBrun metrics and the class of Joyce metrics admitting a semi-free circle action are the same, and any metric of the first class can be identified with a metric of the second class through an explicit conformal equivalence.*

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2 An explicit global connection

First we quickly recall the construction of LeBrun’s self-dual hyperbolic monopole metrics. Let $\mathcal{H}^3 = \{(x, y, z) \mid z > 0\}$ be equipped with the usual hyperbolic metric $g_{\mathcal{H}^3} := (dx^2 + dy^2 + dz^2)/z^2$. Let n be any non-negative integer and $P = \{p_1, \dots, p_n\}$ be distinct points in \mathcal{H}^3 . Let Γ_{p_α} be the fundamental solution for the hyperbolic Laplacian based at p_α with normalization $\Delta\Gamma_{p_\alpha} = -2\pi\delta_{p_\alpha}$, and define

$$(2-1) \quad V = 1 + \sum_{\alpha=1}^n \Gamma_{p_\alpha}.$$

Then $*dV$ is a closed 2-form on $\mathcal{H}^3 \setminus P$, and $[*dV]/2\pi$ belongs to an integral class $H^2(\mathcal{H}^3 \setminus P, \mathbb{Z})$. Let $\pi: X_0 \rightarrow \mathcal{H}^3 \setminus P$ be the unique principal $U(1)$ -bundle determined by this integral class. By Chern–Weil theory, there is a connection form $\omega \in H^1(X_0, i\mathbb{R})$ such that $d\omega = i(*dV)$. Then LeBrun’s metric is defined by

$$(2-2) \quad g_{\text{LB}} = z^2 \left(V \cdot g_{\mathcal{H}^3} - \frac{1}{V} \omega \odot \omega \right).$$

This is anti-self-dual with respect to a Kähler orientation of X_0 . By attaching points \tilde{p}_α over each p_α , we obtain a complete, Kähler scalar-flat (and therefore anti-self-dual) ALE manifold, which can be conformally compactified by adding a point at infinity, yielding a self-dual conformal class on $n\mathbb{CP}^2$. The $U(1)$ -action of the principal $U(1)$ -bundle naturally extends to $n\mathbb{CP}^2$, and the resulting $U(1)$ -action on $n\mathbb{CP}^2$ is semi-free. See [6; 7] for details.

From the construction LeBrun metrics always admit a $U(1)$ -action, and they admit an effective $U(1) \times U(1)$ -action if and only if all the n points belong to a common geodesic. We call these latter metrics *toric LeBrun metrics*. By applying a hyperbolic isometry, without loss of generality we may assume that the geodesic is the z -axis, and we let $p_\alpha = (0, 0, c_\alpha)$ with $0 < c_1 < c_2 < \dots < c_n$. We also define $c_0 = 0$ and $c_{n+1} = \infty$.

For toric LeBrun metrics, we shall explicitly write down the connection form ω on the $U(1)$ -bundle $\pi: X_0 \rightarrow \mathcal{H}^3$. For this, we first let $U = \mathcal{H}^3 \setminus \{z\text{-axis}\}$ and take cylindrical coordinates on U as

$$(2-3) \quad U = \{(x, y, z) = (r \cos \tau, r \sin \tau, z) \mid z > 0, 0 \leq \tau < 2\pi\}$$

(we use τ for the angular coordinate here, since θ will be used below as the angular coordinate on the circle bundle). Also on the z -axis we define an interval

$$(2-4) \quad I_\alpha := \{(0, 0, z) \mid c_{\alpha-1} < z < c_\alpha\}, \quad 1 \leq \alpha \leq n + 1,$$

and we let $U_\alpha := U \cup I_\alpha$ for each $1 \leq \alpha \leq n + 1$. Then we obtain an open covering

$$(2-5) \quad \mathcal{H}^3 \setminus \{p_1, p_2, \dots, p_n\} = U_1 \cup U_2 \cup \dots \cup U_{n+1}.$$

Finally, for any positive real number c , we define a function f_c by

$$(2-6) \quad f_c(r, z) = \frac{r^2 + z^2 - c^2}{2\sqrt{(c^2 + r^2 + z^2)^2 - 4c^2z^2}} - \frac{1}{2}.$$

We note that $(c^2 + r^2 + z^2)^2 - 4c^2z^2 \geq 0$ and is zero only at $(0, 0, c)$. Therefore, f_c is a function defined on all of $\mathcal{H}^3 \setminus \{(0, 0, c)\}$.

Theorem 2.1 *Using the above notation, define a function on $\mathcal{H}^3 \setminus \{p_1, p_2, \dots, p_n\}$ by $f := f_{c_1} + f_{c_2} + \dots + f_{c_n}$. Then f satisfies*

$$(2-7) \quad d(f \, d\tau) = *dV,$$

in U . That is, the 1-form $if \, d\tau$ is a local connection form in U . Next, for each α with $1 \leq \alpha \leq n + 1$, the 1-form

$$(2-8) \quad \omega_\alpha = i(f + n + 1 - \alpha) \, d\tau,$$

is well-defined on U_α . Together, these 1-forms define a global connection form (with values in $\mathfrak{u}(1) = i\mathbb{R}$) on the total space $X_0 \rightarrow M$. That is, there is a global connection ω on X_0 , such that $\omega = \omega_\alpha + i \cdot d\theta$ over U_α , where θ is an angular coordinate on the fiber.

Proof This was proved in [3, Theorem 3.1] for the case of two monopole points. It is straightforward to generalize the argument to the case of n monopole points, so we only provide a brief sketch here. The Green function is given by

$$(2-9) \quad \Gamma_{(0,0,c)}(x, y, z) = -\frac{1}{2} + \frac{1}{2} \left[1 - \frac{4c^2 z^2}{(r^2 + z^2 + c^2)^2} \right]^{-1/2},$$

where $r^2 = x^2 + y^2$; see [5, Section 2]. In the proof of [3, Theorem 3.1], it is shown that

$$(2-10) \quad d(f_{c_\alpha} d\tau) = *d(\Gamma_{p_\alpha}).$$

It is easy to see that

$$(2-11) \quad f_{c_\alpha}(0, z) = \begin{cases} -1 & z < c_\alpha, \\ 0 & z > c_\alpha. \end{cases}$$

From these it follows that the sum $f := f_{c_1} + f_{c_2} + \dots + f_{c_n}$ then satisfies

$$(2-12) \quad d(f d\tau) = *dV,$$

$$(2-13) \quad f(0, z) = \alpha - n - 1, \quad z \in I_\alpha.$$

Consequently, the form

$$(2-14) \quad \omega_\alpha = i(f + n + 1 - \alpha) d\tau,$$

extends smoothly to U_α . It then follows from basic connection theory that the ω_α are the local representatives of a globally defined connection. □

Although we do not require this in the proof of our main theorem, we remark that one can use the above local connection forms to write down the transition functions of the $U(1)$ -bundle explicitly:

Proposition 2.2 *With respect to the open covering (2-5), the transition functions of the $U(1)$ -bundle $\pi: X_0 \rightarrow \mathcal{H}^3 \setminus \{p_1, p_2, \dots, p_n\}$ are given by $g_{\alpha\beta} = e^{i(\beta-\alpha)\tau}$.*

Proof From above, we have that

$$(2-15) \quad \omega_\beta - \omega_\alpha = i(f + n + 1 - \beta) d\tau - i(f + 1 + n - \alpha) d\tau = i(\alpha - \beta) d\tau.$$

The formula for the change of connection is given by

$$(2-16) \quad \omega_\beta - \omega_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha},$$

which implies that $g_{\beta\alpha} = e^{i(\alpha-\beta)\tau}$, or equivalently, $g_{\alpha\beta} = e^{i(\beta-\alpha)\tau}$. □

3 Explicit identification with Joyce metrics

In this section, we use the explicit connection forms from Section 2 to prove Theorem 1.1. As in Section 2, (r, τ, z) denotes cylindrical coordinates on $U = \mathcal{H}^3 \setminus \{z\text{-axis}\}$. We introduce another coordinate system (x_1, x_2) by setting

$$(3-1) \quad x_1 = r^2 - z^2, \quad x_2 = 2rz.$$

The map $(r, z) \mapsto (x_1, x_2)$ is a diffeomorphism from the quarter plane $\{(r, z) \mid r > 0, z > 0\}$ to the upper half-plane $\{(x_1, x_2) \mid x_2 > 0\}$. (Thus we adapt the upper half-plane model, rather than the right half-plane model used in [4].) The point $(r, z) = (0, c_\alpha)$ (on the boundary of $\{r > 0, z > 0\}$) determined from the monopole point p_α , is mapped to the point $(x_1, x_2) = (-c_\alpha^2, 0)$ (on the boundary of $\{x_2 > 0\}$). In order to save space, for each integer α with $3 \leq \alpha \leq n + 2$, we put

$$(3-2) \quad q_\alpha := -c_{\alpha-2}^2, \quad r_\alpha := \sqrt{(x_1 - q_\alpha)^2 + x_2^2}, \quad R := \sqrt{x_1^2 + x_2^2}$$

(we adopt this un-natural numbering for a later purpose). Then we have $0 > q_3 > q_4 > \dots > q_{n+2}$, and also

$$(3-3) \quad r^2 = \frac{1}{2}(R + x_1), \quad z^2 = \frac{1}{2}(R - x_1).$$

$$(3-4) \quad dx_1^2 + dx_2^2 = 4(r^2 + z^2)(dr^2 + dz^2) = 4R(dr^2 + dz^2).$$

Under the coordinates (r, z, τ, θ) , the metric g_{LB} multiplied by a conformal factor $(z^2V)^{-1}$ can be written as

$$(3-5) \quad \begin{aligned} \frac{g_{LB}}{z^2V} &= \frac{dr^2 + r^2 d\tau^2 + dz^2}{z^2} + \frac{1}{V^2} (d\theta + f d\tau)^2 \\ &= \frac{dr^2 + dz^2}{z^2} + \frac{r^2}{z^2} d\tau^2 + \frac{1}{V^2} (d\theta + f d\tau)^2, \end{aligned}$$

and noting $q_\alpha < 0$ the functions V and f can be computed, in terms of the coordinates (x_1, x_2) , as

$$(3-6) \quad V = 1 - \frac{n}{2} + \sum_{\alpha=3}^{n+2} \frac{R - q_\alpha}{2r_\alpha}, \quad f(x_1, x_2) = -\frac{n}{2} + \sum_{\alpha=3}^{n+2} \frac{R + q_\alpha}{2r_\alpha}.$$

Hence, writing $g_{\mathcal{H}^2} := (dx_1^2 + dx_2^2)/x_2^2$, we have:

$$(3-7) \quad \frac{g_{\text{LB}}}{z^2 V} = \frac{\frac{dx_1^2 + dx_2^2}{4R}}{\frac{1}{2}(R - x_1)} + \frac{R + x_1}{R - x_1} d\tau^2 + \frac{(d\theta + f d\tau)^2}{V^2}$$

$$(3-8) \quad = \frac{x_2^2}{2R(R - x_1)} \left[g_{\mathcal{H}^2} + \frac{2R^2}{x_2^2} \left\{ \left(1 + \frac{x_1}{R}\right) d\tau^2 + \left(1 - \frac{x_1}{R}\right) \frac{(d\theta + f d\tau)^2}{V^2} \right\} \right].$$

From (3-6), this expresses a toric LeBrun metric in terms of the coordinates (x_1, x_2, τ, θ) . In the following, for simplicity of notation, we denote by \tilde{g}_{LB} the quantity in the brackets [] in (3-8); namely we define

$$(3-9) \quad \tilde{g}_{\text{LB}} := \frac{2R(R - x_1)}{x_2^2 z^2 V} g_{\text{LB}}.$$

Next we explain the explicit form of Joyce metrics on $n\mathbb{C}\mathbb{P}^2$ of arbitrary type, following [4]. Let $k := n + 2$, and $q_1 > q_2 > \dots > q_k$ be the set of elements in $\mathbb{R} \cup \{\infty\}$ involved in the construction of Joyce metrics ([4, Theorem 3.3.1], where the letter p_i was used instead of q_α). For each α with $1 \leq \alpha \leq k$ and $q_\alpha \neq 0, \infty$, let $\rho_\alpha := \{(x_1 - q_\alpha)^2 + x_2^2\}^{1/2}$, and let $u^{(q_\alpha)}$ be an \mathbb{R}^2 -valued function defined by

$$(3-10) \quad u^{(q_\alpha)}(x_1, x_2) = \begin{pmatrix} u_1^{(q_\alpha)}(x_1, x_2) \\ u_2^{(q_\alpha)}(x_1, x_2) \end{pmatrix} \quad \text{where } u_1^{(q_\alpha)} = \frac{x_2}{\rho_\alpha}, \quad u_2^{(q_\alpha)} = \frac{x_1 - q_\alpha}{\rho_\alpha}$$

(in [4] the notation $f^{(p_i)}$ is used instead of $u^{(q_\alpha)}$). When $q_\alpha = \infty$ or $q_\alpha = 0$, we let

$$(3-11) \quad u^{(\infty)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad u^{(0)} = \begin{pmatrix} x_2/R \\ x_1/R \end{pmatrix}.$$

Let $\{(m_\alpha, n_\alpha) \mid 1 \leq \alpha \leq k\}$ be the set of pairs of coprime integers determined from the $U(1) \times U(1)$ -action on $n\mathbb{C}\mathbb{P}^2$ we are considering (namely, the stabilizer data). Without loss of generality, we can always suppose that $m_\alpha n_{\alpha+1} - m_{\alpha+1} n_\alpha = -1$ for α with $1 \leq \alpha < k$, $(m_1, n_1) = (0, 1)$ and $(m_k, n_k) = (1, 0)$, and also $m_\alpha > 0, n_\alpha > 0$ for any α with $1 < \alpha < k$. After this normalization, we let

$$(3-12) \quad \phi = \sum_{\alpha=1}^{k-1} \frac{u^{(q_\alpha)} - u^{(q_{\alpha+1})}}{2} \otimes (m_\alpha, n_\alpha) + \frac{u^{(q_k)} + u^{(q_1)}}{2} \otimes (m_k, n_k).$$

If we write this as

$$\phi = \begin{pmatrix} a_1(x_1, x_2) & b_1(x_1, x_2) \\ a_2(x_1, x_2) & b_2(x_1, x_2) \end{pmatrix},$$

then on the dense open subset $\mathcal{H}^2 \times U(1) \times U(1)$ of $n\mathbb{C}\mathbb{P}^2$ the Joyce metric with the given $U(1) \times U(1)$ -action is expressed as

$$(3-13) \quad g_J = g_{\mathcal{H}^2} + \frac{(a_1^2 + a_2^2)dy_1^2 + (b_1^2 + b_2^2)dy_2^2 - 2(a_1b_1 + a_2b_2)dy_1dy_2}{(a_1b_2 - a_2b_1)^2},$$

where y_1, y_2 are coordinates with period 2π on $U(1) \times U(1)$.

Proposition 3.1 *If a Joyce metric has a $U(1)$ -subgroup of $U(1) \times U(1)$ acting semi-freely on $n\mathbb{C}\mathbb{P}^2$, then the stabilizer data can be supposed to be*

$$(3-14) \quad (m_1, n_1) = (0, 1), \quad \text{and} \quad (m_\alpha, n_\alpha) = (1, k - \alpha) \quad \text{for} \quad 2 \leq \alpha \leq k.$$

Proof For this, as in Proposition 3.1.1 of [4], by choosing appropriate \mathbb{Z} -basis of \mathbb{Z}^2 , we can always normalize the stabilizer data $\{(m_\alpha, n_\alpha)\}$ in a way that they satisfy

$$(3-15) \quad (m_1, n_1) = (0, 1), \quad (m_k, n_k) = (1, 0), \quad m_\alpha n_{\alpha+1} - m_{\alpha+1} n_\alpha = -1 \quad \text{for} \quad 1 \leq \alpha < k.$$

The last condition in particular means that (m_α, n_α) moves in the clockwise direction as α increases. Therefore $m_\alpha > 0$ and $n_\alpha > 0$ hold for $1 < \alpha < k$. As in [4] for mutually coprime integers m and n define a $U(1)$ -subgroup $G(m, n)$ of $U(1) \times U(1)$ by

$$G(m, n) = \{(e^{2\pi i\phi}, e^{2\pi i\psi}) \mid e^{2\pi i(m\phi + n\psi)} = 1\}.$$

Then in $n\mathbb{C}\mathbb{P}^2$ for each $1 \leq \alpha \leq n+2$ there exists a distinguished $U(1) \times U(1)$ -invariant 2-sphere whose stabilizer is exactly $G(m_\alpha, n_\alpha)$. Let S_α^2 be this 2-sphere. (The union of all these spheres are exactly the complement of $\mathcal{H}^2 \times U(1) \times U(1)$ in $n\mathbb{C}\mathbb{P}^2$.) It is elementary to see from (3-15) that if $G(m, n) \subset U(1) \times U(1)$ is a $U(1)$ -subgroup that acts semi-freely on the first sphere S_1^2 , then $(m, n) = (0, 1)$ or otherwise $m = 1$, up to simultaneous inversion of the sign. Similarly, if $G(m, n) \subset U(1) \times U(1)$ is a $U(1)$ -subgroup that acts semi-freely on the last sphere S_k^2 , then $(m, n) = (1, 0)$ or otherwise $n = 1$, up to simultaneous inversion of the sign. Taking the intersection of these, any $U(1)$ -subgroup acting semi-freely on $n\mathbb{C}\mathbb{P}^2$ has to be of the form $G(1, 1), G(1, 0)$ or $G(0, 1)$. But again it is elementary to see that the subgroup $G(1, 1)$ cannot act semi-freely on $S_\alpha^2, 1 < \alpha < k$. Therefore the two subgroups $G(1, 0)$ and $G(0, 1)$ are the only subgroups that can act semi-freely on $n\mathbb{C}\mathbb{P}^2$. If $G(1, 0)$ (resp. $G(0, 1)$) acts semi-freely on $S_\alpha^2 (1 < \alpha < k)$, it follows that $n_\alpha = 1$ (resp. $m_\alpha = 1$). Thus the stabilizer data must be

$$(3-16) \quad (m_1, n_1) = (0, 1), \quad (m_k, n_k) = (1, 0), \quad (m_\alpha, n_\alpha) = (m_\alpha, 1) \quad \text{for} \quad 1 < \alpha < k,$$

for some $m_\alpha > 0$, or

$$(3-17) \quad (m_1, n_1) = (0, 1), \quad (m_k, n_k) = (1, 0), \quad (m_\alpha, n_\alpha) = (1, n_\alpha) \quad \text{for } 1 < \alpha < k,$$

for some $n_\alpha > 0$. But of course these represent the same $U(1) \times U(1)$ -action on $n\mathbb{C}\mathbb{P}^2$, so we dispose of the former. Then the final condition in (3-15) means $m_\alpha = k - \alpha$, and we are done. □

Next, by the usual $\text{PSL}(2, \mathbb{R})$ -action, we may suppose that $q_1 = \infty$ and $q_2 = 0$. From these normalizations, we compute

$$(3-18) \quad \begin{aligned} \phi &= \frac{u^{(q_1)} - u^{(q_2)}}{2} \otimes (0, 1) + \sum_{\alpha=2}^{k-1} \frac{u^{(q_\alpha)} - u^{(q_{\alpha+1})}}{2} \otimes (1, n + 2 - \alpha) \\ &\quad + \frac{u^{(q_k)} + u^{(q_1)}}{2} \otimes (1, 0) \\ &= \frac{1}{2} \left(u^{(q_1)} + u^{(q_2)}, u^{(q_1)} + (k - 3)u^{(q_2)} - \sum_{\alpha=3}^k u^{(q_\alpha)} \right) \\ &= \frac{1}{2} \begin{pmatrix} \frac{x_2}{R} & (k - 3)\frac{x_2}{R} - \sum_{\alpha=3}^k \frac{x_2}{\rho_\alpha} \\ \frac{x_1}{R} - 1 & (k - 3)\frac{x_1}{R} - \sum_{\alpha=3}^k \frac{x_1 - q_\alpha}{\rho_\alpha} - 1 \end{pmatrix} \left(= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \right). \end{aligned}$$

Substituting these into (3-13), we obtain the explicit form of Joyce metrics that admit a semi-free $U(1)$ -action.

We next have the following:

Theorem 3.2 *With respect to the above coordinates and the identification $q_\alpha = q_\alpha$ so that $r_\alpha = \rho_\alpha$ ($3 \leq \alpha \leq n + 2$), the toric LeBrun metric \tilde{g}_{LB} (defined in (3-9)) and the Joyce metric g_J (defined in (3-13) with (3-18)) are isometric under the map*

$$(3-19) \quad (x_1, x_2, \theta, \tau) \longmapsto (x_1, x_2, y_1, y_2) = (x_1, x_2, \theta, \tau).$$

For the proof, we begin with the following:

Lemma 3.3 *As functions on $\mathcal{H}^2 = \{(x_1, x_2) \mid x_2 > 0\}$, we have the following relationship*

$$(3-20) \quad a_1 b_2 - a_2 b_1 = -\frac{x_2}{2R} V.$$

Remark 3.4 The negativity of $a_1b_2 - a_2b_1$ seemingly contradicts [4, Lemma 3.3.3], but this is not a problem, since the sign comes from the difference of the orientation on the right half-plane used in Joyce’s paper and that on the upper half-plane used in this paper.

Proof By (3-18), we have

$$4(a_1b_2 - a_2b_1) = \frac{x_2}{R} \left\{ (k-3) \frac{x_1}{R} - \sum_{\alpha=3}^k \frac{x_1 - q_\alpha}{\rho_\alpha} - 1 \right\} - \left(\frac{x_1}{R} - 1 \right) \left\{ (k-3) \frac{x_2}{R} - \sum_{\alpha=3}^k \frac{x_2}{\rho_\alpha} \right\},$$

and after several cancellations, this equals

$$\frac{x_2}{R} \left\{ k - 4 - \sum_{\alpha=3}^k \frac{R - q_\alpha}{\rho_\alpha} \right\}$$

which is exactly $-(2x_2V)/R$, under our assumption $q_\alpha = \rho_\alpha$. Dividing by 4 gives the claim of the lemma. □

Proof of Theorem 3.2 We write the two metrics as

$$\begin{aligned} \tilde{g}_{LB} &= g_{\mathcal{H}^2} + \tilde{g}_{11} d\theta^2 + 2\tilde{g}_{13} d\theta d\tau + \tilde{g}_{33} d\tau^2, \\ g_J &= g_{\mathcal{H}^2} + g_{11} dy_1^2 + 2g_{12} dy_1 dy_2 + g_{22} dy_2^2. \end{aligned}$$

Then the claim of Theorem 3.2 is equivalent to the three identities

$$(3-21) \quad g_{11} = \tilde{g}_{11}, \quad g_{12} = \tilde{g}_{13}, \quad g_{22} = \tilde{g}_{33}.$$

In the following, for simplicity of notation, we write

$$\sum_{\alpha=3}^{n+2} =: \sum' \quad \text{and} \quad \sum_{3 \leq \alpha < \beta \leq n+2} =: \sum''$$

First, we readily have

$$(3-22) \quad \tilde{g}_{11} = \frac{2R^2}{x_2^2} \frac{1 - \frac{x_1}{R}}{V^2},$$

and also, by using Lemma 3.3,

$$(3-23) \quad g_{11} = \frac{a_1^2 + a_2^2}{(a_1b_2 - a_2b_1)^2} = \frac{\frac{1}{2} \left(1 - \frac{x_1}{R} \right)}{\left(\frac{x_2}{2R} V \right)^2} = \frac{2R^2}{x_2^2} \frac{1 - \frac{x_1}{R}}{V^2}.$$

Therefore we obtain $g_{11} = \tilde{g}_{11}$.

Second, from (3-8), we have

$$(3-24) \quad \tilde{g}_{13} = \frac{2R^2}{x_2^2} \left(1 - \frac{x_1}{R}\right) \frac{f}{V^2} = \frac{2R^2}{x_2^2} \left(1 - \frac{x_1}{R}\right) \frac{-\frac{n}{2} + \frac{1}{2} \sum' \frac{R + q_\alpha}{r_\alpha}}{V^2}.$$

On the other hand, from (3-18) we can compute, by using the relation $x_1^2 + x_2^2 = R^2$ twice,

$$(3-25) \quad \begin{aligned} &4(a_1b_1 + a_2b_2) \\ &= \frac{x_2}{R} \left\{ (n-1) \frac{x_2}{R} - \sum' \frac{x_2}{\rho_\alpha} \right\} + \left(\frac{x_1}{R} - 1 \right) \left\{ (n-1) \frac{x_1}{R} - \sum' \frac{x_1 - q_\alpha}{\rho_\alpha} - 1 \right\} \\ &= \left(n - \sum' \frac{R + q_\alpha}{\rho_\alpha} \right) \left(1 - \frac{x_1}{R} \right). \end{aligned}$$

Hence again by using Lemma 3.3 we obtain

$$(3-26) \quad g_{12} = -\frac{a_1b_1 + a_2b_2}{(a_1b_2 - a_2b_1)^2} = -\frac{\frac{1}{4} \left(n - \sum' \frac{R + q_\alpha}{\rho_\alpha} \right) \left(1 - \frac{x_1}{R} \right)}{\left(\frac{x_2}{2R} V \right)^2}.$$

By comparing (3-24) and (3-26), we obtain $g_{12} = \tilde{g}_{13}$.

Finally, for the remaining coefficients \tilde{g}_{33} and g_{22} , we have, by (3-8),

$$(3-27) \quad \begin{aligned} \tilde{g}_{33} &= \frac{2R^2}{x_2^2} \left\{ \left(1 + \frac{x_1}{R} \right) + \left(1 - \frac{x_1}{R} \right) \frac{f^2}{V^2} \right\} \\ &= \frac{2R^2}{x_2^2 V^2} \left\{ (V^2 + f^2) + \frac{x_1}{R} (V^2 - f^2) \right\}. \end{aligned}$$

Further by (3-6) we compute

$$(3-28) \quad \begin{aligned} V^2 + f^2 &= 1 - n + \frac{n^2}{2} + \sum' \frac{R - q_\alpha}{r_\alpha} - n \sum' \frac{R}{r_\alpha} \\ &\quad + \frac{1}{2} \left(\sum' \frac{R^2 + q_\alpha^2}{r_\alpha^2} + 2 \sum'' \frac{R^2 + q_\alpha q_\beta}{r_\alpha r_\beta} \right), \end{aligned}$$

$$(3-29) \quad V^2 - f^2 = 1 - n + \sum' \frac{R - q_\alpha}{r_\alpha} + n \sum' \frac{q_\alpha}{r_\alpha} - R \left(\sum' \frac{q_\alpha}{r_\alpha^2} + \sum'' \frac{q_\alpha + q_\beta}{r_\alpha r_\beta} \right).$$

From these we obtain

$$(3-30) \quad \tilde{g}_{33} = \frac{2R^2}{x_2^2 V^2} \left\{ \frac{1}{2} \sum' \frac{R^2 + q_\alpha^2 - 2x_1 q_\alpha}{r_\alpha^2} + \sum'' \frac{R^2 + q_\alpha q_\beta - x_1(q_\alpha + q_\beta)}{r_\alpha r_\beta} \right. \\ \left. + \sum' \frac{(1-n)R - q_\alpha}{r_\alpha} + \frac{x_1}{R} \sum' \frac{R + (n-1)q_\alpha}{r_\alpha} \right. \\ \left. + (1-n)\frac{x_1}{R} + 1 - n + \frac{n^2}{2} \right\}.$$

Noting that $R^2 + q_\alpha^2 - 2x_1 q_\alpha = r_\alpha^2$, the first summation becomes just n , so this equals

$$(3-31) \quad \frac{2R^2}{x_2^2 V^2} \left\{ \sum'' \frac{R^2 + q_\alpha q_\beta - x_1(q_\alpha + q_\beta)}{r_\alpha r_\beta} + \sum' \frac{(1-n)R - q_\alpha}{r_\alpha} \right. \\ \left. + \frac{x_1}{R} \sum' \frac{R + (n-1)q_\alpha}{r_\alpha} + (1-n)\frac{x_1}{R} + 1 - \frac{n}{2} + \frac{n^2}{2} \right\}.$$

On the other hand, by (3-18), we can compute

$$(3-32) \quad 4(b_1^2 + b_2^2) \\ = (n-1)^2 - 2(n-1) \left(\frac{x_2^2}{R} \sum' \frac{1}{\rho_\alpha} + \frac{x_1}{R} \sum' \frac{x_1 - q_\alpha}{\rho_\alpha} + \frac{x_1}{R} \right) \\ + x_2^2 \left(\sum' \frac{1}{\rho_\alpha} \right)^2 + \left(\sum' \frac{x_1 - q_\alpha}{\rho_\alpha} \right)^2 + 2 \sum' \frac{x_1 - q_\alpha}{\rho_\alpha} + 1 \\ = (n-1)^2 - 2(n-1) \left\{ \frac{1}{R} \sum' \frac{R^2 - x_1 q_\alpha}{\rho_\alpha} + \frac{x_1}{R} \right\} + \sum' \frac{x_2^2 + (x_1 - q_\alpha)^2}{\rho_\alpha^2} \\ + 2 \sum'' \frac{x_2^2 + (x_1 - q_\alpha)(x_1 - q_\beta)}{\rho_\alpha \rho_\beta} + 2 \sum' \frac{x_1 - q_\alpha}{\rho_\alpha} + 1 \\ = 2 \sum'' \frac{R^2 + q_\alpha q_\beta - x_1(q_\alpha + q_\beta)}{\rho_\alpha \rho_\beta} + \frac{2x_1}{R} \sum' \frac{R + (n-1)q_\alpha}{\rho_\alpha} \\ + 2 \sum' \frac{(1-n)R - q_\alpha}{\rho_\alpha} + 2(1-n)\frac{x_1}{R} + n^2 - n + 2.$$

Under the identification $q_\alpha = \alpha_\alpha$, this is exactly twice the quantity in the braces { } in (3-31). Consequently,

$$(3-33) \quad \tilde{g}_{33} = \frac{2R^2}{x_2^2 V^2} \cdot 2(b_1^2 + b_2^2) = \frac{4R^2}{x_2^2 V^2} (b_1^2 + b_2^2).$$

On the other hand by (3-13) and Lemma 3.3 we have

$$g_{22} = \frac{b_1^2 + b_2^2}{(a_1 b_2 - a_2 b_1)^2} = \frac{b_1^2 + b_2^2}{\left(\frac{x_2}{2R} V\right)^2}.$$

Hence from (3-33) we obtain

$$(3-34) \quad \tilde{g}_{33} = \frac{4R^2}{x_2^2 V^2} \left(\frac{x_2}{2R} V\right)^2 g_{22} = g_{22},$$

as required, which completes the proof of Theorem 3.2. □

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