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Contact manifolds with symplectomorphic symplectizations

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We provide examples of contact manifolds of any odd dimension greater than or equal to 5 which are not diffeomorphic but have exact symplectomorphic symplectizations.

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1 Introduction

Symplectization provides a bridge between contact and symplectic geometry. It associates to any contact manifold (M,ξ) (namely any manifold M equipped with a co-oriented contact structure ξ) an exact symplectic manifold $(\mathrm{S}_\xi M,\lambda_\xi)$ (that is $\omega_\xi = d\lambda_\xi$ is a symplectic form on $\mathrm{S}_\xi M$) diffeomorphic to $\mathbb{R}\times M$. Most known contact invariants (such as those arising from symplectic field theory; see Eliashberg, Givental and Hofer [5]) are defined using symplectizations. Therefore, one might think that if two contact manifolds have symplectomorphic symplectizations then they are contactomorphic (see Cieliebak and Eliashberg [2, page 239] where the problem is addressed). In this paper, we prove the following theorem which shows that this is not true (see Section 3 for the definition of exact symplectomorphism).

Theorem 1.1 *Let M and M' be closed manifolds of dimension greater than or equal to 5 such that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic. Then for every contact structure ξ on M, there exists a contact structure ξ' on M' such that the symplectizations $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are exact symplectomorphic.*

As a concrete example, consider $M = L(7,1)\times S^{2n}$ and $M' = L(7,2)\times S^{2n}$ for $n\geq 1$, where $L(p,q)$ denotes the three-dimensional lens space of type (p,q). In [8], J Milnor showed using Reidemeister torsion that M and M' are not diffeomorphic, but proved that they are h–cobordant. The s–cobordism theorem then implies that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic (see Section 2). On the other hand M admits a contact structure ξ. Indeed, for $n=1$, M is diffeomorphic to the unit tangent bundle of $L(7,1)$ and in general, M is the boundary of $L(7,1)\times D^3\times D^{2n-2}$ (after smoothing corners) which is a product of Liouville domains. Theorem 1.1 above then provides a

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contact structure ξ' on M' such that $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are exact symplectomorphic, though M and M' are not even diffeomorphic.

The main ingredients in the proof are the flexibility properties of certain Weinstein cobordisms, first discovered by Y Eliashberg [4] and developed with K Cieliebak [2] on the base of E Murphy's work [11].

This paper is organized as follows. Section 2 contains some recollections about Morse–Smale theory and the s–cobordism theorem. In Section 3, we discuss symplectization of contact manifolds, Weinstein cobordisms, and quote two theorems from [2] about so-called *flexible* Weinstein cobordisms. Section 4 contains our results and Section 5 discusses a few open questions.

2 h–cobordisms

Since we look for contact manifolds with symplectomorphic symplectizations, we must first tackle the following well-known problem from differential topology:

> *If M and M' are closed oriented manifolds such that $\mathbb{R}\times M$ and $\mathbb{R}\times M'$ are diffeomorphic, what can we say about M and M'?*

Certainly M and M' must have the same homotopy type. However, as we shall see, M and M' need not be diffeomorphic (see the example in the introduction). Let us introduce some terminology. A *cobordism from M to M'* is a triple $(W; M, M')$, where W is a compact oriented manifold together with a decomposition of its boundary as $\partial W = \partial_+ W \sqcup \partial_- W$ and orientation-preserving diffeomorphisms $\partial_- W \to -M$ and $\partial_+ W \to M'$. Here, as customary, ∂W is oriented with outer normal first convention and $-M$ means M with opposite orientation. We insist that the identification of the boundary is part of the data (as in Milnor [9]). Given two cobordisms $(W; M, M')$ and $(W'; M', M'')$, we can *compose* them by gluing along M' and get another cobordism denoted by $(W \odot W'; M, M'')$. Producing an actual smooth structure on $W \odot W'$ requires some choices but the result is independent of these choices up to a diffeomorphism relative to the boundary. A cobordism $(W; M, M')$ is called an *h–cobordism* if both inclusion maps $M \to W$ and $M' \to W$ are homotopy equivalences. A *product cobordism* $([0,1]\times M; M, M)$ is an obvious example. A *Morse function* on a cobordism $(W; M, M')$ is a smooth function $\phi\colon W \to \mathbb{R}$ which is constant on the boundary, satisfies $\mathrm{d}\phi > 0$ on inward pointing vectors at M and outward pointing vectors at M', and whose critical points are nondegenerate. A *pseudogradient vector field* for a Morse function ϕ is a vector field X such that $X.\phi > 0$ outside of the critical points of ϕ and such that at each critical point p, the linearized vector field X_p^{lin}

has no eigenvalue with vanishing real part. We call (X,ϕ) a *Morse pair*. A *Morse homotopy* is a smooth path (X_s,ϕ_s) which is generic in the sense that ϕ_s encounters only birth-death type singularities. There are finitely many parameters s where ϕ_s has a degenerate critical points, for any other parameter s, (X_s,ϕ_s) is a Morse pair. S Smale showed in [13] that simply connected h–cobordisms of dimension greater than or equal to 6 are diffeomorphic to product cobordisms. The nonsimply connected case is the subject of the s–cobordism theorem, proved by D Barden, B Mazur and J Stallings, which provides a complete classification of h–cobordisms $(W;M,-)$ up to diffeomorphism relative to M in terms of so-called *Whitehead torsion*. These theorems are proved using what is now called *Morse–Smale theory*. This consist in simplifying Morse pairs by canceling critical points. For example, if we are able to cancel all the critical points of a Morse function on a cobordism, the latter must be diffeomorphic to a product cobordism.

Here are two lemmas from Morse–Smale theory which are building blocks for the proof of the s–cobordism theorem (see Kervaire [7]). We will use them in Section 4.

Lemma 2.1 (Normal form) *Let $(W;M,M')$ be an h–cobordism of dimension greater than or equal to 6. Then there is a Morse pair with only critical points of index 2 and 3.*

We briefly indicate why it is not always possible to cancel the remaining critical points (see [7] for more details). Take a Morse pair (X,ϕ) given by Lemma 2.1 and lift it to a Morse pair $(\widetilde{X},\widetilde{\phi})$ on a universal cover $\widetilde{M}\to M$. The Morse complex (C_i,∂_i) associated to $(\widetilde{X},\widetilde{\phi})$ is a chain complex over $\mathbb{Z}[\pi_1 M]$ which is only nonzero in degree 2 and 3. Moreover, since W is an h–cobordism, this complex is acyclic. Therefore we get a matrix $A\in \mathrm{GL}(\mathbb{Z}[\pi_1 M])$ which represents the boundary operator $\partial_3\colon C_3\to C_2$. It turns out that the class of A in a quotient group $\mathrm{Wh}(\pi_1 M)$ of $\mathrm{GL}(\mathbb{Z}[\pi_1 M])$, called the *Whitehead group* of $\pi_1 M$ is an actual invariant of the h–cobordism, called *Whitehead torsion*. The remaining critical points can be cancelled if and only if the Whitehead torsion vanishes.

Lemma 2.2 *Let $(W;M,M')$ be an h–cobordism of dimension greater than or equal to 6 with vanishing Whitehead torsion. Let (X,ϕ) be a Morse pair with only critical points of index 2 and 3. Then there is a Morse homotopy (X_s,ϕ_s) fixed near the boundary, with only critical points of index 2 and 3, such that $(X_0,\phi_0)=(X,\phi)$ and (X_1,ϕ_1) has no critical points.*

We now state the s–cobordism theorem.

Theorem 2.3 (Barden, Mazur, Stallings; 1965) *Let M be a closed oriented manifold of dimension greater than or equal to 5. Whitehead torsion $\tau(W, M) \in \mathrm{Wh}(\pi_1 M)$ of a cobordism W from M induces a bijective correspondence:*

$$\tau\colon \{h\text{–cobordisms } (W; M, -) \text{ up to diffeomorphism relative to } M\} \to \mathrm{Wh}(\pi_1 M).$$

The reader may consult Kervaire [7], Milnor [10] and Ranicki [12] for more information about Whitehead torsion and the s–cobordism theorem. We do not go further in this topic since we will only need the following corollary.

Corollary 2.4 *For any h–cobordism $(W; M, M')$ of dimension greater than or equal to 6, there is an h–cobordism $(W'; M', M)$ such that $W \odot W'$ is diffeomorphic to $[0,1] \times M$ and $W' \odot W$ is diffeomorphic to $[0,1] \times M'$.*

The reason is that, according to the s–cobordism theorem, h–cobordisms are classified by Whitehead torsion which takes value in an abelian group. The inverse h–cobordism W' in Corollary 2.4 is essentially the h–cobordism with opposite Whitehead torsion (see [10]).

Let $\Psi\colon \mathbb{R} \times M \to \mathbb{R} \times M'$ be a diffeomorphism. Consider in $\mathbb{R} \times M'$ the regions between $\{c'\} \times M'$ and $\Psi(\{c\} \times M)$, and between $\{c'\} \times M'$ and $\Psi(\{-c\} \times M)$ for c sufficiently large. These are cobordisms inverse to each other, so in particular h–cobordisms. Conversely, we have the following well-known corollary of the s–cobordism theorem.

Corollary 2.5 *Let M and M' be closed oriented manifolds of dimension greater than or equal to 5. If M and M' are h–cobordant, then $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic.*

Proof The proof is an instance of the so-called *Mazur trick* which consists of introducing parentheses in an infinite sum in two different ways.

By Corollary 2.4, there are h–cobordisms $(W; M, M')$ and $(W'; M', M)$ such that

$$W \odot W' \simeq [0,1] \times M, \quad W' \odot W \simeq [0,1] \times M'.$$

We now consider the open manifold V obtained by gluing infinitely many copies of W and W' in an alternate pattern:

$$V = \cdots \odot W' \odot W \odot W' \odot W \odot \cdots$$

Now we just write, on one hand,

$$V \simeq \bigodot_{j \in \mathbb{Z}} (W \odot W') \simeq \bigodot_{j \in \mathbb{Z}} [j, j+1] \times M \simeq \mathbb{R} \times M,$$

and on the other hand,

$$V \simeq \bigodot_{j\in\mathbb{Z}} (W' \odot W) \simeq \bigodot_{j\in\mathbb{Z}} [j, j+1] \times M' \simeq \mathbb{R} \times M'. \qquad \square$$

We finish this section by studying the extension problem of nondegenerate 2–forms on h–cobordisms.

Remark 2.6 In the case of a product cobordism $W = [0,1] \times M$, we can retract W by an isotopy to $[0,\epsilon] \times M$ with $\epsilon > 0$ as small as we want. Therefore we can extend any nongenerate 2–form defined near $\{0\} \times M$ to a nondegenerate 2–form on W in a unique way up to homotopy relative to a neighborhood of $\{0\} \times M$.

This is also true for h–cobordisms of dimension greater than or equal to 6 according to the following lemma.

Lemma 2.7 *Let $(W; M, M')$ be an h–cobordism of dimension greater than or equal to 6 with a nondegenerate 2–form η defined near M. There is a nondegenerate two-form ω on W that coincides with η near M. Moreover, the extension is unique up to homotopy relative to a neighborhood of M.*

Proof Let $(W'; M', M)$ be an inverse h–cobordism given by Corollary 2.4, so that $W \odot W' \simeq [0,1] \times M$. By Remark 2.6, there is a nondegenerate 2–form ω on $[0,1]\times M$ which coincides with η near $\{0\}\times M$. Restricting ω to W gives the required extension. Now suppose that we have two nondegenerate 2–forms ω and ω' on W which coincide with η near M. According to what we have just proved, they both extend further to W' because W' is an h–cobordism. Again by Remark 2.6, ω and ω' are homotopic on $[0,1] \times M$ relative to a neighborhood of $\{0\} \times M$, in particular they are homotopic on W relative to a neighborhood of M. $\square$

3 Contact manifolds and Weinstein cobordisms

Let (M, ξ) be a *contact manifold*, we mean ξ is a co-oriented hyperplane field which is maximally nonintegrable. We always endow M with the orientation induced by ξ. An *exact symplectic manifold* is a manifold V together with a 1–form λ such that $d\lambda$ is a symplectic form. There are at least two notions of isomorphism between exact symplectic manifolds. If (V, λ) and (V', λ') are exact symplectic manifold, a diffeomorphism $\Psi\colon V \to V'$ is said to be

- an *exact symplectomorphism* if $\Psi^*\lambda' - \lambda$ is an exact 1–form on W,
- a *symplectomorphism* if $\Psi^*\lambda' - \lambda$ is a closed 1–form on W.

The *symplectization* of a contact manifold (M, ξ) is an exact symplectic manifold that can be described as follows. The space of cotangent vectors of M vanishing on ξ is a one-dimensional subbundle of the cotangent bundle T^*M. Restricting our attention to nonzero cotangent vectors which induce the right co-orientation of ξ yields a principal $\mathbb{R}_+^*$–bundle that we denote by $\mathrm{S}_\xi M$. Since ξ is co-oriented, this bundle admits global sections which correspond to *contact forms* for ξ. In particular, $\mathrm{S}_\xi M$ is diffeomorphic to $\mathbb{R} \times M$. The canonical 1–form λ of T^*M induces a 1–form denoted by λ_ξ on $\mathrm{S}_\xi M$ called the *Liouville form*, whose exterior derivative $\omega_\xi = \mathrm{d}\lambda_\xi$ is a symplectic form (this is equivalent to ξ being a contact structure). The principal bundle structure can be recovered from the 1–form λ_ξ. Indeed, the *Liouville vector field* X_ξ, defined by $X_\xi \lrcorner \omega_\xi = \lambda_\xi$, is the infinitesimal generator of the $\mathbb{R}_+^*$–action. The flow $\varphi_{X_\xi}^t$ of X_ξ satisfies $(\varphi_{X_\xi}^t)^* \lambda_\xi = e^t \lambda_\xi$, so it preserves $\ker \lambda_\xi$. Hence the projection map

$$(\mathrm{S}_\xi M / \mathbb{R}_+^*, \ker \lambda_\xi) \to (M, \xi)$$

is a contactomorphism. In particular, we have that the symplectization $(\mathrm{S}_\xi M, \lambda_\xi)$ entirely recovers the contact manifold (M, ξ). In other words, any diffeomorphism $\Psi\colon (\mathrm{S}_\xi M, \lambda_\xi) \to (\mathrm{S}_{\xi'} M', \lambda_{\xi'})$ such that $\Psi^* \lambda_{\xi'} = \lambda_\xi$ induces a contactomorphism $(M, \xi) \to (M', \xi')$. However, Theorem 1.1 shows that if $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are only exact symplectomorphic, then M and M' need not even be diffeomorphic.

Remark 3.1 If we choose a contact form α for ξ, the symplectization naturally splits as

$$(\mathrm{S}_\xi M, \lambda_\xi) = (\mathbb{R} \times M, e^t \alpha).$$

A *Weinstein structure* on a cobordism $(W; M, M')$ is a triple (ω, X, ϕ), where (X, ϕ) is a Morse pair and ω is a symplectic form (positive with respect to the orientation of W) such that $X.\omega = \omega$. We call X the *Liouville vector field.* It gives rise to a *Liouville form* $\lambda = X \lrcorner \omega$. In fact, (ω, X) and λ are equivalent pieces of data, often called a *Liouville structure*. The Liouville form λ induces contact structures ξ on M and ξ' on M' with contact forms $\alpha = \iota^* \lambda$ and $\alpha' = \iota'^* \lambda$, where $\iota\colon M \to W$ and $\iota'\colon M' \to W$ are the inclusion maps. We sometimes say that (W, ω, X, ϕ) is a Weinstein cobordism from (M, ξ) to (M', ξ').

Remark 3.2 Let (W, ω, X, ϕ) be a Weinstein cobordism from (M, ξ) to (M', ξ') and (W', ω', X', ϕ') be a Weinstein cobordism from (M', ξ') to (M'', ξ''). We now explain how to compose them in a Weinstein cobordism from (M, ξ) to (M'', ξ''). Suppose that the Liouville forms λ and λ' induce the same contact form α' on M'. The flow of the Liouville vector fields X and X' define collar neighborhoods $[-\epsilon, 0] \times M'$ in W and $[0, \epsilon] \times M'$ in W' where λ and λ' both read $e^{t'} \alpha'$ (t' is the coordinate in $\mathbb{R}$). Using these collar neighborhoods, we can glue W and W' along M' and get a smooth cobordism $(W \odot W'; M, M'')$ with a Liouville structure (ω'', X'') that restricts to (ω, X) and to (ω', X') respectively on W and W'. Even if $\phi = \phi'$ on M', they do not necessarily glue to a smooth function on $W \odot W'$. This can be arranged by composing ϕ with a diffeomorphism of W which is the identity on M' and supported in an arbitrary small neighborhood of M'. For example, it is enough to arrange that $X.\phi = 1$ and $X'.\phi' = 1$ in a neighborhood of M'. Finally, we get a Weinstein cobordism $(W \odot W', \omega'', X'', \phi'')$ from (M, ξ) to (M'', ξ'').

The easiest example is the following: let M be a closed manifold together with a contact form α. For any two smooth functions f_-, f_+ on M with $\max f_- < \min f_+$, we consider the part of symplectization

$$W = \{(t, x) \in \mathbb{R} \times M \mid f_-(x) \leq t \leq f_+(x)\}.$$

It admits a Liouville structure $(\omega = \mathrm{d}(e^t \alpha), X = \partial/\partial t)$. By choosing a Morse function ϕ (constant on the boundary, as always) without critical points such that $X.\phi > 0$, we get a Weinstein cobordism (W, ω, X, ϕ).

Remark 3.3 If $(W; M, M')$ is a cobordism with Weinstein structure (ω, X, ϕ). This induces contact forms α and α' respectively on M and M'. By multiplying ω by a positive number, and composing with parts of symplectizations as above, we can change the contact forms α and α' for any contact forms $e^k e^{-f} \alpha$ and $e^k e^{f'} \alpha'$ with $k \in \mathbb{R}$, and smooth functions $f \colon M \to [0, +\infty[$ and $f' \colon M' \to [0, +\infty[$.

A *Weinstein homotopy* on W is a smooth path (ω_s, X_s, ϕ_s), such that (X_s, ϕ_s) is a Morse homotopy and for all but finitely many parameters s (where (X_s, ϕ_s) encounters a birth-death singularity) (ω_s, X_s, ϕ_s) is a Weinstein structure.

For a cobordism to admit a Weinstein structure, it is necessary that it carries a nondegenerate 2–form. But there are more severe topological constraints due to the following (see [2, page 242] for a proof).

Proposition 3.4 *If (W, ω, X, ϕ) is a Weinstein cobordism of dimension $2n$, then the critical points of ϕ have index less than or equal to n.*

A Weinstein cobordism (W, ω, X, ϕ) of dimension $2n$ is called *subcritical* if the critical points of ϕ have index less than n. It is known for some time that subcritical Weinstein cobordisms exhibit remarkable flexibility properties (see [4]). Yet a larger class of Weinstein cobordisms with flexibility properties was recently discovered. A Weinstein cobordism (W, ω, X, ϕ) is called *flexible* if it is the composition of finitely many Weinstein cobordisms $(W^i, \omega^i, X^i, \phi^i)$ which are elementary (that is X^i has no trajectory joining critical points) and whose attaching spheres of Lagrangian handles form a *loose* Legendrian link in the lower boundary of W^i (see [2, pages 250–251]). Notice that it is clear from the definition that the composition of two flexible Weinstein cobordisms is still a flexible Weinstein cobordism.

We now state two theorems about flexible Weinstein structures that are relevant to our purpose [2, page 279].

Theorem 3.5 (Cieliebak, Eliashberg) *Let $(W; M, M')$ be a cobordism of dimension $2n \geq 6$ together with a nondegenerate 2–form η and a Morse pair (Y, ϕ) with critical points of index less than or equal to n such that (η, Y, ϕ) is a Weinstein structure near M. Then there is a flexible Weinstein structure (ω, X, ϕ) on W such that $\omega = \eta$ near M.*

Theorem 3.6 (Cieliebak, Eliashberg) *Let $(W; M, M')$ be a cobordism of dimension $2n \geq 6$ together with a flexible Weinstein structure (ω, X, ϕ). Then for any Morse homotopy (Y_s, ϕ_s) fixed near the boundary, with critical points of index less than or equal to n, such that $(Y_0, \phi_0) = (X, \phi)$, there is a Weinstein homotopy (ω_s, X_s, ϕ_s) satisfying*

- $(\omega_0, X_0, \phi_0) = (\omega, X, \phi)$,
- *(X_s, ϕ_s) is fixed near ∂W, ω_s is fixed near $\partial_- W$ and $\omega_s = e^{c_s}\omega_0$ near $\partial_+ W$ for a smooth real-valued function $s \mapsto c_s$.*

4 Main results

4.1 Symplectomorphic symplectizations

We start by a lemma which shows that Theorem 3.5 can be applied to any h–cobordism of dimension greater than or equal to 6 from a closed contact manifold.

Lemma 4.1 *Let (M, ξ) be a closed contact manifold of dimension greater than or equal to 5 and let $(W; M, M')$ be an h–cobordism. Then there is a flexible Weinstein structure (ω, X, ϕ) on W that induces a contact structure isotopic to ξ on M and which has only critical points of index 2 and 3.*

Proof Take a collar neighborhood $[0, \epsilon] \times M$ of M in W. Consider the standard Weinstein structure $(\mathrm{d}(e^t\alpha), \partial/\partial t, t)$ in this collar. By Lemma 2.7, the 2–form $\mathrm{d}(e^t\alpha)$ extends to W as a nondegenerate 2–form. By Lemma 2.1, the Morse pair $(\partial/\partial t, t)$

extends to a Morse pair (Y,ϕ) on W with only critical points of index 2 and 3. We now apply Theorem 3.5 to get a flexible Weinstein structure (ω, X, ϕ) such that $\omega = \eta$ near M. Then the induced contact structure on M is isotopic to ξ. $\square$

Remark 4.2 By Gray's stability theorem, any two isotopic contact structures are contactomorphic. So after applying Lemma 4.1, we may compose the identification of $\partial_- W$ with M by such a contactomorphism to actually get a Weinstein cobordism from (M,ξ). We will do this implicitly in the proof of Theorem 4.3 below.

We now turn to our main result which can be thought of as a symplectic analogue of Corollary 2.5.

Theorem 4.3 *Let (M,ξ) be a closed contact manifold of dimension greater than or equal to 5. Then for any h–cobordism $(W; M, M')$ there is a contact structure ξ' on M' such that $(\mathrm{S}_\xi M, \lambda_\xi)$ and $(\mathrm{S}_{\xi'} M', \lambda_{\xi'})$ are exact symplectomorphic.*

Proof Let $(W'; M', M)$ be an inverse h–cobordism of $(W; M, M')$ as given by Corollary 2.4. By Lemma 4.1, there is a flexible Weinstein structure (ω, X, ϕ) on W which induces the contact structure ξ on M. It also induces a contact structure ξ' on M'. Again by Lemma 4.1, there is a flexible Weinstein structure (ω', X', ϕ') on W' that induces the contact structure ξ' on M'. Denote by α and α' the contact forms respectively on M and M' induced by (W, ω, X, ϕ). According to Remark 3.3, we can arrange W' so that the contact form induced on M' equals α'. Up to composing ϕ and ϕ' by affine transformations of $\mathbb{R}$, we can assume that $\phi = 0$ on M, $\phi = 1$ on M', $\phi' = 1$ on M' and $\phi' = 2$ on M. After arranging the functions ϕ and ϕ' as in Remark 3.2, we can compose W and W' to get a smooth cobordism $W'' = W \odot W'$ together with a Weinstein structure (ω'', X'', ϕ'') which restricts to (ω, X, ϕ) on W and to (ω', X', ϕ') on W'. The function ϕ'' has only critical points of index 2 and 3. Since $W \odot W'$ is diffeomorphic to a product cobordism, Lemma 2.2 implies that there is a Morse homotopy (Y_s, ϕ''_s) fixed near the boundary, with only critical points of index 2 and 3, such that $(Y_0, \phi''_0) = (X'', \phi'')$ and ϕ''_1 has no critical points. Now by Theorem 3.6, there is a Weinstein homotopy $(\omega''_s, X''_s, \phi''_s)$ such that

- $(\omega''_0, X''_0, \phi''_0) = (\omega'', X'', \phi'')$,
- (X''_s, ϕ''_s) is fixed near $\partial W''$, ω''_s is fixed near $\partial_- W''$ and $\omega_s = e^{c_s}\omega''_0$ near $\partial_+ W''$ for a smooth real-valued function $s \mapsto c_s$.

Near $\partial_+ W''$, X''_s is fixed and ω''_s is fixed up to a constant, so in particular, the contact structure ξ''_s induced on $\partial_+ W'' = M$ is fixed during the homotopy. The holonomy of the Liouville vector field X''_1 defines a contactomorphism (M,ξ) to (M,ξ''_1). In the cobordism W', we now change the identification of $\partial_+ W'$ with M by composing it with this contactomorphism (as in Remark 4.2), so that the contact structure on M induced by (W',ω',X',ϕ') is equal to ξ. According to Remark 3.3, we may compose W' with a part of the symplectization of M so that it induces the contact forms $e^k\alpha$ for some $k>0$. The Weinstein homotopy $(\omega''_s, X''_s, \phi''_s)$ obviously extends to this slightly enlarged cobordism since (X''_s,ϕ''_s) is fixed near $\partial_+ W''$ and $\omega'' s = e^{c_s}\omega''_0$ near $\partial_+ W''$. Up to composing ϕ''_s with a diffeomorphism of $\mathbb{R}$, assume that $\phi''_s = 2$ on $\partial_+ W''$ still holds.

In the spirit of the proof of Corollary 2.5, we will construct an exact symplectic manifold V by gluing infinitely many copies of W and W' and show that V is exact symplectomorphic to both $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$.

We now define translates of W and W' as follows, for $j\in\mathbb{Z}$,

$$(W^j,\omega^j,X^j,\phi^j) = (W, e^{jk}\omega, X, \phi+2j),$$
$$(W'^j,\omega'^j,X'^j,\phi'^j) = (W', e^{jk}\omega', X', \phi'+2j),$$

and consider

$$V = \cdots \odot W^{-1} \odot W'^{-1} \odot W^0 \odot W'^0 \odot W^1 \odot W'^1 \odot \cdots .$$

According to Remark 3.2, this is well-defined and carries a Weinstein structure (ω, X, ϕ) that restricts to the given one on each W^i and W'^i.

We now prove that V is exact symplectomorphic to $\mathrm{S}_\xi M$.

We want to repeat the homotopy $(\omega''_s, X''_s, \phi''_s)$ on the whole V by translation. We just need to take care of the scaling factor e^{c_s} near the top boundary. So define, for $j\in\mathbb{Z}$, on $W^j \odot W'^j$,

$$(\omega_s, X_s, \phi_s) = (e^{jc_s} e^{jk}\omega''_s, X''_s, \phi''_s + 2j).$$

This gives a Weinstein homotopy of V during which the vector field X_s is complete (it is invariant by translation in j) and is transverse to the hypersurfaces $M^j = \phi_s^{-1}(2j) = \phi^{-1}(2j) \simeq M$ for all $j\in\mathbb{Z}$. Note that this homotopy is fixed near $\phi^{-1}(0)\simeq M$ (we will make use of this in Section 4.2).

We now look for an isotopy Ψ_s of V such that $\Psi_s^*\lambda_s - \lambda_0$ is exact (here $\lambda_s = X_s \lrcorner \omega_s$). We will find it using Moser's lemma (see [2, pages 240–241] for a similar argument). Take $C > \max(0, \max c_s)$ and consider $\widetilde{M}^j = \varphi_{X_0}^{jC}(M^j)$ (φ_X^t denotes the flow at time t of a vector field X).

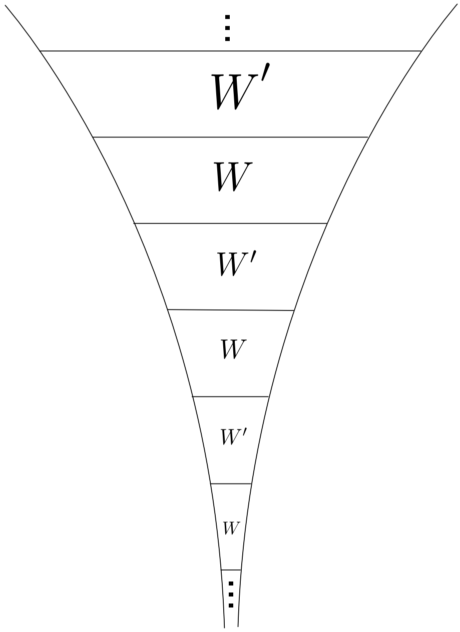

Figure 1: The symplectic manifold V obtained by Mazur's trick

Since X_s is complete for all $s \in [0,1]$, we can define

$$\Theta_s^{2j} = \varphi_{X_s}^{j(C-c_s)} \circ \varphi_{X_0}^{-jC} \colon \widetilde{M}^j \longrightarrow V.$$

And we have

$$\begin{aligned}(\Theta_s^{2j})^*\lambda_s &= (\varphi_{X_0}^{-jC})^* \circ (\varphi_{X_s}^{j(C-c_s)})^*(\lambda_s)\\ &= (\varphi_{X_0}^{-jC})^*(e^{-j(C-c_s)}\lambda_s)\\ &= (\varphi_{X_0}^{-jC})^*(e^{jC}\lambda_0) = \lambda_0.\end{aligned}$$

We can extend Θ_s^{2j} near $\widetilde{M}^j$ in a unique way so that $(\Theta_s^{2j})^*\lambda_s = \lambda_0$. The image of Θ_s^{2j} is $\varphi_{X_s}^{j(C-c_s)}(M^j)$, so they are all disjoint. Hence we can find an isotopy $\Theta_s \colon V \to V$ that coincides with Θ_s^{2j} near $\widetilde{M}^j$ for all j. The path $\Theta_s^*\lambda_s$ is now fixed near each $\widetilde{M}^j$ and Moser's lemma applied to each region between $\widetilde{M}^j$ and $\widetilde{M}^{j+1}$ gives an isotopy $\Psi^s \colon V \to V$ such that $\Psi_s^*\lambda_s - \lambda_0$ is exact.

Since X_1 is complete and nowhere vanishing, its flow defines a diffeomorphism $\Xi \colon \mathbb{R} \times M \to V$ which satisfies $\Xi^*\lambda_1 = e^t\alpha$. The map $\Xi^{-1} \circ \Theta_1$ is the required exact symplectomorphism from (V,λ) to $(\mathrm{S}_\xi M, \lambda_\xi) = (\mathbb{R} \times M, e^t\alpha)$.

Since $(W'^{-1} \odot W^0; M', M')$ is a product cobordism, we can apply exactly the same reasoning and find another Weinstein homotopy of V, which we then turn into an exact symplectomorphism from (V, λ) to $(\mathrm{S}_{\xi'} M', \lambda_{\xi'})$. □

Theorem 4.3 implies Theorem 1.1 stated in the introduction because if $\mathbb{R} \times M$ is diffeomorphic to $\mathbb{R} \times M'$, then M and M' are h–cobordant.

Remark 4.4 (1) Given a closed contact manifold (M, ξ) of dimension greater than or equal to 5, we have associated to any h–cobordism from M a contact manifold (M', ξ') such that $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are exact symplectomorphic. So by the s–cobordism theorem, this produces as many contact manifolds as the cardinality of $\mathrm{Wh}(\pi_1 M)$. Of course, this is only interesting when $\mathrm{Wh}(\pi_1 M) \neq 0$. Note that the example given in the introduction together with s–cobordism theorem shows that $\mathrm{Wh}(\mathbb{Z}/7\mathbb{Z}) \neq 0$ (see Cohen [3, pages 42–45] for more examples of nontrivial Whitehead groups).

(2) A contact invariant which is functorial with respect to Liouville cobordisms (such as SFT invariants, see [5]) cannot distinguish (M, ξ) and (M', ξ') though they are not diffeomorphic.

4.2 Contact manifolds at infinity of Weinstein and Stein manifolds

A Weinstein structure on an open manifold V is a triple (ω, X, ϕ), where ω is a symplectic form, X is a complete vector field such that $X.\omega = \omega$, ϕ is a Morse function on V (proper and bounded from below) for which X is a pseudogradient vector field. Notice that the region between two regular values of ϕ is a Weinstein cobordism in the sense of Section 3. We call (V, ω, X, ϕ) of *finite type* if there is $c > 0$ such that $\phi^{-1}([c, +\infty[)$ does not contain any critical point. In this case, the level sets of ϕ above c are all contactomorphic by flowing along the Liouville vector field X, we call it the *contact manifold at infinity* of (V, ω, X, ϕ). This depends only on (ω, X) and we may think that it is actually independent of X (see [2, pages 238–239]). As a corollary of the proof of Theorem 4.3, we show that this is not the case.

We will need the following notion of homotopy for an open Weinstein manifold (see [2, page 246]). A *Weinstein homotopy* on V is a smooth path $(\omega_s, X_s, \phi_s)_{s \in [0,1]}$ of Weinstein structures such that (X_s, ϕ_s) is a generic path (it encounters only birth-death type singularities), there is a subdivision $0 = a_0 < a_1 < \cdots < a_p = 1$, and for each $i \in \{0, \ldots, p-1\}$ an unbounded increasing sequence (c_k^i) of regular values of ϕ_s for all $s \in [a_i, a_{i+1}]$. This definition prevents critical points to escape at infinity during a Weinstein homotopy.

Corollary 4.5 *Let (V, ω, X, ϕ) be a finite type Weinstein manifold of dimension greater than or equal to 6 with contact manifold at infinity contactomorphic to (M, ξ). For any h–cobordism $(W; M, M')$ there is a Weinstein homotopy $(\omega_s, X_s, \phi_s)_{s\in[0,1]}$ such that $(\omega_0, X_0, \phi_0) = (\omega, X, \phi)$ and $(W, \omega_1, X_1, \phi_1)$ is a finite type Weinstein manifold with contact manifold at infinity diffeomorphic to M'.*

Proof Let c be sufficiently close to $+\infty$ so that ϕ has no critical points in $\{\phi \geq c\}$. Then $\phi^{-1}(c)$ is contactomorphic to (M, ξ) and the flow of X identifies $\{\phi \geq c\}$ with $[0, +\infty[\times M$. The proof of Theorem 4.3 shows that there is a Weinstein homotopy (ω_s, X_s, ϕ_s) on $[0, +\infty[\times M$ such that

- $(\omega_0, X_0, \phi_0) = (\omega, X, \phi)$,
- (ω_s, X_s, ϕ_s) is fixed near $\{0\} \times M$,
- for $c' > 0$ sufficiently large, $\{\phi_1 \geq c'\}$ contains no critical points of ϕ_1 and $\phi_1^{-1}(c')$ is diffeomorphic to M'.

We extend the Weinstein homotopy by a constant homotopy on $\{\phi \leq c\} = \{\phi_s \leq c\}$ to get the result. □

Remark 4.6 (1) If M and M' are not diffeomorphic, critical points have to appear out of every compact set during the Weinstein homotopy in Corollary 4.5 because otherwise the topology of the contact manifold at infinity would not change.

(2) The Weinstein homotopy can be made fixed on an arbitrary large compact set of V: in some sense, it only moves things at infinity.

(3) According to the proof of Theorem 4.3, we can find an isotopy Ψ_s of V such that $\Psi_s^*\lambda_s = \lambda_0 + \mathrm{d} f_s$. In particular, we get a Weinstein homotopy $(\omega_0, \Psi_s^* X_s, \Psi_s^*\phi_s)$ with fixed symplectic form during which the topology of the contact manifold at infinity changes.

(4) Since the homotopy in Corollary 4.5 only concerns the cylindrical end $[0, +\infty[\times M$, the result also holds for any symplectic manifold with cylindrical end, not necessarily Weinstein.

And finally using the Weinstein–Stein correspondence from [2], we can give a corollary concerning the complex geometry of Stein manifolds.

Corollary 4.7 *Let (V, J, ϕ) be a finite type Stein manifold of dimension greater than or equal to 6 with contact manifold at infinity contactomorphic to (M, ξ). For any h–cobordism $(W; M, M')$, there is a Stein homotopy $(J, \phi_s)_{s\in[0,1]}$ such that $\phi_0 = \phi$ and (V, J, ϕ_1) is a finite type Stein manifold with contact manifold at infinity diffeomorphic to M'.*

Proof In the spirit of [2], the proof goes from Stein to Weinstein and back. Let $(\omega = -\mathrm{dd}^c\phi, X = \nabla_\phi \phi, \phi)$ be the Weinstein structure which is associated to (V, J, ϕ); see [2, pages 244–245]. By Corollary 4.5, there is a Weinstein homotopy (ω_s, X_s, ϕ_s) such that $(\omega_0, X_0, \phi_0) = (\omega, X, \phi)$ and level sets of ϕ_1 at infinity are diffeomorphic to M'. Now by [2, Theorem 15.3], there is an isotopy Ψ_s of V and an isotopy g_s of $\mathbb{R}$ such that $(J, g_s \circ \phi_s \circ \Psi_s^{-1})$ is a Stein homotopy. The level sets at infinity of $g_1 \circ \phi_1 \circ \Psi_1^{-1}$ are then diffeomorphic to M'. □

5 Questions

We now state a few questions that remain open.

(1) Does there exist contact structures ξ and ξ' on a closed manifold M that are not contactomorphic but whose symplectizations $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M$ are (exact) symplectomorphic? There are many examples of closed manifolds M of dimension greater than or equal to 5 for which there are nontrivial h–cobordisms from M to itself (see Hatcher and Lawson [6]). A flexible Weinstein structure on such a cobordism produces two contact structures on M whose symplectizations are exact symplectomorphic according to Theorem 4.3 but we do not know if they are contactomorphic or not.

(2) What about contact 3–manifolds? On one hand, it follows from the combination of the work of Perelman and Turaev that if M and M' are closed 3–manifolds such that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic, then M and M' are diffeomorphic (this was pointed out to me by Vladimir Chernov). On the other hand, the existence of a smooth 4–dimensional h–cobordism which is not diffeomorphic to a product cobordism is an open question (see the discussion by Chen in [1]). So the method used in this paper will hardly adapt to the 3–dimensional case.

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