Geometry & Topology 18 (2014) 141–177

Minimal surfaces with positive genus and finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$

FRANCISCO MARTÍN
RAFE MAZZEO
M MAGDALENA RODRÍGUEZ

We construct the first examples of complete, properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and positive genus. These are constructed by gluing copies of horizontal catenoids or other nondegenerate summands. We also establish that every horizontal catenoid is nondegenerate.

49Q05, 53A10, 53C42

1 Introduction

Amidst the great activity in the past several years concerning the existence and nature of complete minimal surfaces in homogeneous three-manifolds, the study of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ has witnessed particular success. The central problem is the solvability of the asymptotic Dirichlet problem, ie, the existence of complete surfaces asymptotic to a given embedded curve γ in the boundary of the compactification of this space $B^2 \times I$, where B^2 is the closure of the Poincaré ball model of $\mathbb{H}^2$ in $\mathbb{R}^2$ and the interval I is the stereographic compactification of $\mathbb{R}$.

There have been three main approaches to this problem. The first is based on the method of Anderson [1] for the analogous problem in $\mathbb{H}^3$: one defines a sequence of curves γ_R lying on the geodesic sphere of radius R around some point, solves the Plateau problem for each of these curves, then attempts to take a limit as $R \to \infty$. The main points are to show that the sequence of minimal surfaces with boundary does not drift off to infinity and that the limit has γ as its asymptotic boundary curve; both of these are accomplished using suitable barrier surfaces, the existence and nature of which depends upon the convexity of $\mathbb{H}^3$ at infinity. This approach has also been used successfully for the analogous asymptotic Plateau problem in higher dimensions and codimensions for various classes of nonpositively curved spaces. The second approach generalizes the classical method of Jenkins and Serrin [7] for minimal graphs in $\mathbb{R}^3$, and was developed in this setting by Nelli and Rosenberg [21], Collin and Rosenberg [2] and Mazet, Rosenberg, and the third author [11]. This involves finding a minimal

Published: 9 January 2014 DOI: 10.2140/gt.2014.18.141

graph over domains of $\mathbb{H}^2$ with prescribed boundary data, possibly $\pm\infty$. The third approach is by an analytic gluing construction, and this is the method we follow here.

Before describing our work, let us draw attention to the issue of obtaining surfaces with finite total curvature. (We recall that the total curvature of a surface is defined as the integral on the surface of its Gauss curvature.) It turns out to be far easier to obtain complete minimal surfaces of finite topology in $\mathbb{H}^2 \times \mathbb{R}$ with infinite total curvature, and we refer to some of the papers above for a good (but not yet definitive) existence theory. The simplest example is the slice $\mathbb{H}^2 \times \{0\}$, but more generally there exist minimal surfaces asymptotic to a vertical graph $\{(\theta, f(\theta)) : \theta \in \mathbb{S}^1\} \subset \partial B^2 \times \mathbb{R}$ for any $f \in \mathcal{C}^1(\mathbb{S}^1)$. Other examples include the one-parameter family of Costa–Hoffman–Meeks type surfaces, each asymptotic to three parallel horizontal copies of $\mathbb{H}^2$. These have positive genus and were constructed by Morabito [18] also using a gluing method. On the other hand, surfaces of finite total curvature have proved to be more elusive. The basic examples are the vertical plane $\gamma \times \mathbb{R}$, where γ is a complete geodesic in $\mathbb{H}^2$, and the Scherk minimal graphs over ideal polygons constructed by Nelli and Rosenberg [21] and Collin and Rosenberg [2]. There is also a family of *horizontal catenoids* K_η constructed in [19; 24] (called *2–noids* in [19]), each consisting of a catenoidal handle that is orthogonal to the vertical direction, and asymptotic to two disjoint vertical planes that are neither asymptotic nor too widely separated. The recent paper Hauswirth, Nelli, Sa Earp and Toubiana [5] shows that these are the unique proper minimal surfaces with finite total curvature and two ends asymptotic to vertical planes. A large number of other examples of genus zero have been constructed recently by Pyo [24], and Morabito and the third author [19], independently. Both papers use the conjugate surface method. The theory of conjugate minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ was elaborated by Daniel [3] and Hauswirth, Sa Earp and Toubiana [6]. The surfaces in [19; 24] are shown to have total curvature $-4\pi(k-1)$, where k is the number of ends, and each end is asymptotic to a vertical plane. The horizontal catenoids, which have total curvature -4π, are a special case.

Despite all this progress, it has remained open whether there exist complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and positive genus. The aim of this paper is to construct such surfaces, which we do by gluing together certain configurations of horizontal catenoids. There is a dichotomy in the types of configurations one may glue together. The ones for which the horizontal catenoid components have "necksize" bounded away from zero are simpler to handle, and the gluing construction in this case is quite elementary; the trade-off is that the minimal surfaces obtained using only this type of component have a very large number of ends relative to the genus. Alternatively, one may glue together horizontal catenoids with very small necksizes, which allows one to obtain viable configurations with relatively few ends for a given genus. Unfortunately this turns out to involve more analytic details because these the horizontal catenoids with very small necks are "nearly degenerate", and because of this we will address this second case in a sequel to this paper.

Our main result here is:

Theorem 1.1 *For each $g \geq 0$, there is a $k_0 = k_0(g)$ such that if $k \geq k_0$, then there exists a properly embedded minimal surface with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, with genus g and k ends, each asymptotic to a vertical plane.*

The proof involves gluing together component minimal surfaces that are nondegenerate in the sense that they have no decaying Jacobi fields. Unfortunately, every minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with each end asymptotic to a vertical plane is degenerate since vertical translation (ie, in the $\mathbb{R}$ direction) always generates such a Jacobi field. Because of this we shall work within the class of surfaces that are symmetric with respect to a fixed horizontal plane $\mathbb{H}^2 \times \{0\}$, and then it suffices to work with surfaces that are *horizontally nondegenerate* in the sense that they possess no decaying Jacobi fields that are even with respect to the reflection across this horizontal plane. The surfaces obtained in Theorem 1.1 are all even with respect to the vertical reflection, and all are horizontally nondegenerate as well.

This leads to the problem of showing that there are component minimal surfaces which satisfy this condition, and our second main result guarantees that many such surfaces exist.

Theorem 1.2 *Each horizontal catenoid K_η is horizontally nondegenerate.*

Our final result concerns the deformation theory of this class of surfaces.

Theorem 1.3 *Let $\mathcal{M}_k$ denote the space of all complete, properly embedded minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ with k ends, each asymptotic to an entire vertical plane. If $\Sigma \in \mathcal{M}_k$ is horizontally nondegenerate, then the component of this moduli space containing Σ is a real analytic space of dimension $2k$, and Σ is a smooth point in this moduli space. In any case, even without this nondegeneracy assumption, $\mathcal{M}_k$ is a real analytic space of virtual dimension $2k$.*

We make two remarks on this. First, this dimension count coincides with the dimension of the family constructed by our gluing methods, and also with the dimension of the family of genus 0 surfaces constructed in [19]. The fact that the dimension does not depend on the genus may be surprising at first, but this is also the case for the space of complete Alexandrov-embedded minimal or CMC surfaces of finite topology in $\mathbb{R}^3$; see Kusner, Mazzeo and Pollack [8] and Pérez and Ros [22]. As is the case in these other theories, it turns out to be very hard to construct surfaces which are actually degenerate, and we leave this as an interesting open problem here as well. The second remark is that it would also be quite interesting to know whether the vertical symmetry condition we are imposing is anything more than a technical convenience. More specifically, we ask whether there exist finite total curvature minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with vertical planar ends which do not have a horizontal plane of symmetry.

Our results show that the existence theory for these properly embedded minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ is in some ways opposite to that in $\mathbb{R}^3$. Indeed, Meeks, Pérez and Ros [16] have proved that there is an upper bound, depending only on the genus, for the number of ends of a properly embedded minimal surface of finite topology in $\mathbb{R}^3$. This is a significant step toward resolving the conjecture of Hoffman and Meeks that a connected minimal surface of finite topology, genus g and $k > 2$ ends can be properly minimally embedded in $\mathbb{R}^3$ if and only if $k \le g + 2$. By contrast, our result gives some indication that a connected surface of finite topology and finite total curvature can be properly minimally embedded in $\mathbb{H}^2 \times \mathbb{R}$ only if the number of ends k has a specific lower bound in terms of the genus g. Going out on a limb, we conjecture that the correct bound for the surfaces constructed by gluing horizontal catenoids with small necks is $k \ge 2g + 1$. Note also that our construction shows that if there exists a surface of this type of genus g and k ends, then we can construct such surfaces with genus g and any larger number of ends, so there definitely is no upper bound as in the Euclidean space to the number of ends that a surface of fixed genus may have.

The plan of this paper is as follows: In Section 2 we describe the horizontal catenoids in more detail, reviewing known properties and developing some new ones as well. This is where we prove Theorem 1.2. Next in Section 3 we describe the configurations of approximate minimal surfaces formed by patching together horizontal catenoids. The actual gluing, ie, the perturbation of these approximately minimal surfaces to actual minimal surfaces, which is possible when some parameter in the construction is sufficiently large, is carried out in Section 4. The analytic steps involve a parametrix construction that is perhaps not so well known in the minimal surface literature but fairly standard elsewhere; we refer to the recent paper Mazzeo and Sáez [14] that uses a similar method to construct multi-layer solutions of the Allen–Cahn equation in $\mathbb{H}^n$. In Section 5 the general construction is given for gluing together any two horizontally nondegenerate properly embedded minimal surfaces of finite total curvature; this is a simple variant of the proof of the main result. Finally, in Section 6, we study the deformation theory.

Acknowledgements

The second author is very grateful to the Department of Geometry and Topology at the University of Granada, where this work was initiated. R M is supported by the NSF grant DMS-1105050. F M and M M R are partially supported by MEC-FEDER Grant number MTM2011-22547 and a Regional J Andalucía Grant number P09-FQM-5088.

2 Horizontal catenoids

We now describe the fundamental building blocks in our gluing construction, which are the horizontal catenoids K_η in $\mathbb{H}^2 \times \mathbb{R}$, originally constructed by Morabito and the third author [19] and by Pyo [24]. Each K_η has genus zero and two ends asymptotic to vertical geodesic planes. The parameter η is the hyperbolic distance between these two planes; it varies in an open interval $(0, \eta_0)$, where the upper bound η_0 corresponds to the distance between two opposite sides of an ideal regular quadrilateral. These catenoids have total curvature -4π, and have "axes" orthogonal to the $\mathbb{R}$ direction, whence the moniker horizontal.

The horizontal catenoid as a vertical bigraph The initial construction of K_η in the papers above describes it as a bigraph over a region $\Omega_\eta \subset \mathbb{H}^2$ with a reflection symmetry across the central $\mathbb{H}^2 \times \{0\}$. This means the following: first, there is a nonnegative function u defined in Ω_η such that

$$K_\eta = \{(z, u(z)) : z \in \Omega_\eta\} \cup \{(z, -u(z)) : z \in \Omega_\eta\}.$$

The domain Ω_η is bounded by four smooth curves of infinite length that intersect only at infinity; two of these are hyperparallel geodesics, denoted γ_{-1} and γ_1, and the parameter η equals the hyperbolic distance between them; the other two curves, denoted C_{-1} and C_1, connect the adjacent pairs of endpoints of $\gamma_{\pm 1}$. The function u is strictly positive in the interior of K_η, vanishes and has infinite gradient on $C_{-1} \cup C_1$, and tends to $+\infty$ along $\gamma_{-1} \cup \gamma_1$. We also let C'_{-1} and C'_1 be the geodesic lines with the same endpoints as C_{-1} and C_1, respectively, and Ω'_η the ideal geodesic quadrilateral bounded by $\gamma_{-1} \cup \gamma_1 \cup C'_{-1} \cup C'_1$. Using vertical planes (which are minimal) as barriers, we see that C_{-1} and C_1 are strictly concave with respect to Ω_η. In particular, they lie in the interior of Ω'_η. For later reference, we identify a few other curves that will enter the discussion. First, let Γ denote the unique geodesic that is orthogonal to both $\gamma_{\pm 1}$; next, let γ_0 be the geodesic perpendicular to Γ and midway between $\gamma_{\pm 1}$; finally, denote by $\widetilde{\gamma}_{\pm 1}$ the two geodesics which connect the opposite ideal vertices of Ω_η. Observe that γ_0 is perpendicular to both $C'_{\pm 1}$; in addition the points of intersection $\gamma_0 \cap \Gamma$ and $\widetilde{\gamma}_{-1} \cap \widetilde{\gamma}_1$ are the same, and we denote this centerpoint by Q.

We finish this discussion by noting that the horizontal catenoid with ends asymptotic to the two vertical planes $\gamma_1 \times \mathbb{R}$ and $\gamma_{-1} \times \mathbb{R}$ is unique (when it exists). This follows from the fact that this surface is a bigraph across the plane $t = 0$ as well as in the two horizontal directions associated to the geodesics Γ and γ_0 (see Proposition 2.4).

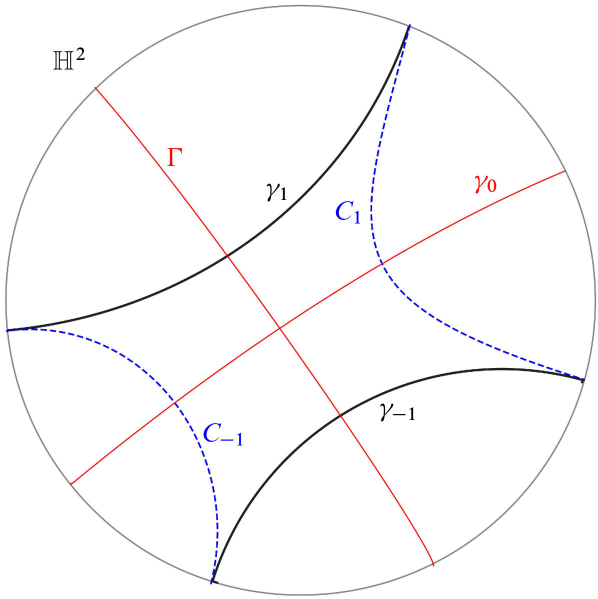

Figure 1: The boundary of the region Ω_η

The extremal surface The family of catenoids K_η exists only for $0 < \eta < \eta_0$. This critical value η_0 corresponds to the case where the pairs of geodesics $\widetilde{\gamma}_{\pm 1}$ intersect orthogonally at Q. The limiting domain Ω_{η_0} is the same as Ω'_{η_0} (so $C_{-1} = C'_{-1}$ and $C_1 = C'_1$ in this limit). Furthermore, as $\eta \nearrow \eta_0$, the value $u(Q)$ tends to $+\infty$. In fact, recentering K_η by translating down by $-u(Q)$, there is a limiting surface that is a graph over Ω'_{η_0} taking the boundary values $\pm\infty$ on alternate sides. It is planar of genus zero with one end. This surface is qualitatively similar to the classical Scherk surface of $\mathbb{R}^3$, and so we also call it the Scherk surface. As already mentioned in the introduction, this example was constructed in [2; 21].

Further symmetries Unlike the Euclidean case, or even the case of vertical catenoids in $\mathbb{H}^2 \times \mathbb{R}$, the horizontal catenoid K_η has only a discrete isometry group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Each $\mathbb{Z}_2$ corresponds to a reflection. The first reflection, which we call $\mathcal{R}_t$, sends (z, t) to $(z, -t)$, and thus interchanges the top and bottom halves of

K_η. The second, $\mathcal{R}_s$, is the reflection across $\Gamma \times \mathbb{R}$; it interchanges the "left" and "right" sides of each asymptotic end. The final one, $\mathcal{R}_o$, is the reflection across $\gamma_0 \times \mathbb{R}$ and interchanges the two ends of K_η and has fixed point set a loop around the neck.

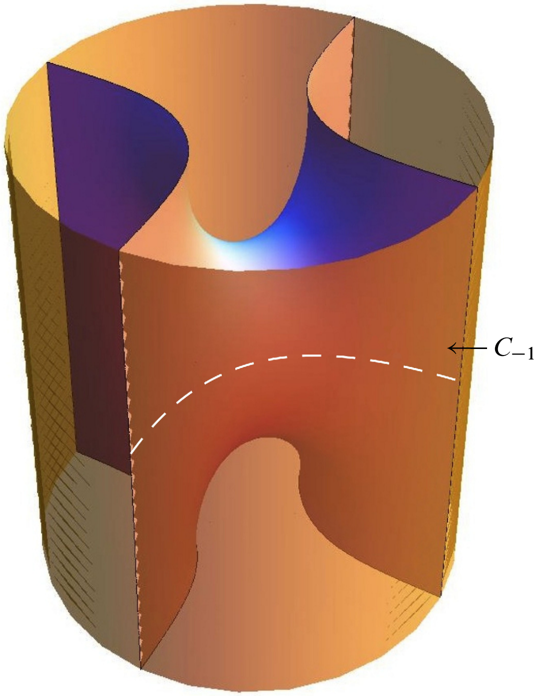

Figure 2: A horizontal catenoid K_η

Graphical representation of the ends of K_η Each end of K_η is asymptotic to one of the totally geodesic vertical planes $P_j = \gamma_j \times \mathbb{R}$, $j = \pm 1$. The intermediate vertical plane $\gamma_0 \times \mathbb{R}$ fixed by $\mathcal{R}_o$ bisects K_η, decomposing it into two pieces, $K_\eta^1 \cup K_\eta^{-1}$, which are interchanged by this reflection. Each K_η^j is a smooth manifold with compact boundary and one end, which is asymptotic to the vertical plane P_j. Outside of some compact set, K_η^j is a normal graph over P_j, with graph function v^j that is strictly positive and defined on an exterior region $E_\eta^j = P_j \setminus \mathcal{O}_\eta$.

The two ends are equivalent, so let us fix one and drop the sub- and superscripts j for the time being. Use parameters (s,t) on P, where t is the vertical coordinate and s is the signed distance function along the geodesic γ, as measured from $\gamma \cap \Gamma$. The restrictions of $\mathcal{R}_s$ and $\mathcal{R}_t$ to the plane P correspond to $(s,t) \mapsto (-s,t)$ and

$(s,t) \mapsto (s,-t)$, respectively. We assume that the domain E_η is invariant under both these reflections.

The parameter η, strictly speaking, measures the distance between the asymptotic vertical planes of K_η, but also measures the size of the neck of K_η, which we take, for example, as the length of the closed curve $K_\eta \cap (\gamma_0 \times \mathbb{R})$. This function, which we denote by $n(\eta)$, has $n(\eta) \to 0$ as $\eta \to 0$ and $n(\eta) \to \infty$ as $\eta \to \eta_0$. This can be thought of as the original parameter for this family used in [19; 24].

We now describe the asymptotic decay profile of the graphical representation of K_η over P. Introduce polar coordinates (r,θ) in the (s,t) plane, so $s = r\cos\theta$ and $t = r\sin\theta$, where the coordinates (s,t) have been defined above.

Proposition 2.1 *For each $\eta \in (0,\eta_0)$, as $r \to \infty$, the graph function v has the asymptotic expansion*

$$v(r,\theta) = A_\eta(\theta) r^{-\frac{1}{2}} e^{-r} + \mathcal{O}(r^{-\frac{3}{2}} e^{-r}), \tag{2-1}$$

where $A_\eta(\theta)$ is some strictly positive smooth function on $\mathbb{S}^1$.

This decay profile is essentially a linear phenomenon and corresponds to the known asymptotic properties of homogeneous solutions of the Jacobi operator on P. Recall that for any minimal surface Σ, its Jacobi operator (for the minimal surface equation) is the elliptic operator

$$L_\Sigma := \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}(N,N); \tag{2-2}$$

here Δ_Σ is the Laplacian on Σ, A_Σ the second fundamental form of the surface, N its unit normal, and Ric the Ricci tensor of the ambient space. When $\Sigma = P$ is a vertical plane, this simplifies substantially. Indeed, $A_P \equiv 0$ and N has no vertical component, so that $\mathrm{Ric}(N,N) \equiv -1$, hence

$$L_P = \Delta_{\mathbb{R}^2} - 1. \tag{2-3}$$

We now deduce Proposition 2.1 from a slightly more general result.

Proposition 2.2 *Let $E \subset P$ be an unbounded region with complement $P \setminus E$ smoothly bounded and compact. Let $K \subset \mathbb{H}^2 \times \mathbb{R}$ be a minimal surface that is a normal graph over E with compact boundary over ∂E, and denote by $v\colon E \to \mathbb{R}$ the graph function. Suppose that $v \to 0$ at infinity in P. Then there exists $A \in \mathcal{C}^\infty(\mathbb{S}^1)$, such that*

$$v(r,\theta) = A(\theta) r^{-\frac{1}{2}} e^{-r} + \mathcal{O}(r^{-\frac{3}{2}} e^{-r}). \tag{2-4}$$

Furthermore, if K lies on one side of P at infinity, then A is either strictly positive or strictly negative.

Proof The minimal surface equation for a horizontal graph over P is a quasilinear elliptic equation $\mathcal{N}(v, \nabla v, \nabla^2 v) = 0$, the linearization of which at $v = 0$ is just $L_{\boldsymbol{P}}$. Let p_j be any sequence of points in P tending to infinity, and consider the restriction of v to the unit ball $B_1(p_j)$ around p_j. Recenter this ball at the origin and write the translated function as v_j. We are assuming that $v_j \to 0$, and it follows from standard regularity theory for the minimal surface equation that

(2-5) $$\|v_j\|_{2,\mu;B_1(0)} \to 0 \quad \text{as } j \to \infty,$$

where $\|\cdot\|_{2,\mu;B_1(0)}$ denotes the norm on the Holder space $\mathcal{C}^{2,\mu}$ on the unit ball $B_1(0)$ (see Gilbarg and Trudinger [4]). This means that we can write

(2-6) $$\mathcal{N}(v, \nabla v, \nabla^2 v) = L_{\boldsymbol{P}} v + Q(v),$$

where Q is quadratic in v, ∇v and $\nabla^2 v$, and has the property that if $\|v\|_{2,\mu}$ is small, then

(2-7) $$\|Q(v)\|_{0,\mu} \le C\|v\|_{2,\mu}^2.$$

Now, applying the inverse $G_{\boldsymbol{P}} = (\Delta_{\mathbb{R}^2} - 1)^{-1}$ of the Jacobi operator (this $G_{\boldsymbol{P}}$ is also called the Green operator or Green function) to (2-6) shows that $\mathcal{N}(v, \nabla v, \nabla^2 v) = 0$ is equivalent to the equation

(2-8) $$v = G_{\boldsymbol{P}}(-Q(v)).$$

We assume initially only that $v \to 0$ at infinity in P, but without any particular rate. We first show that v decays at some exponential rate; this is done using the maximum principle. The second and final step is to obtain the asymptotic formula (2-4).

To begin, using (2-5) and (2-7), the following is true: There exists a constant $C_1 > 0$ such that, given any $\delta_0 > 0$ sufficiently small, there exists $R_0 \ge 1$ so that if $\delta < \delta_0$, $R > R_0$ and $|v| < \delta$ for all $r \ge R$ then $\sup|Q(v)| \le C_1\delta^2$.

Now, define $w = ae^{-r} + b$. This satisfies $L_{\boldsymbol{P}} w = -ar^{-1}e^{-r} - b$. Suppose that $\delta < \delta_0, 1$ and $R > R_0$ are such that $\sup_{r \ge R}|v| = \delta$ is attained at $r = R$, and choose the coefficients a and b so that $ae^{-R} + b \ge \delta$ and $b \ge C_1\delta^2$; to be specific, we take $b = C_1\delta^2$ and $a = \delta(1 - C_1\delta)e^{R}$. Then $v - w \le 0$ when $r = R$, and furthermore (taking $\delta \le 1/C_1$),

$$L_{\boldsymbol{P}}(v - w) = -Q(v) + ar^{-1}e^{-r} + b \ge -Q(v) + C_1\delta^2 \ge 0,$$

where we drop the middle term since $ar^{-1}e^{-r} > 0$. Thus $v - w$ is a subsolution of the equation which is non-positive at $r = R$ and is bounded as $r \ge R$, hence $v - w \le 0$

for all $r \geq R$. This implies that

$$v(R+1,\theta) \leq w(R+1) = \delta(1-C_1\delta)e^R e^{-R-1} + C_1\delta^2 = \delta((1-C_1\delta)e^{-1} + C_1\delta).$$

Since C_1 is independent of δ, we can choose δ so small that $(1-C_1\delta)e^{-1}+C_1\delta < \frac{1}{2}$, and hence $v(R+1,\theta) \leq \frac{1}{2}\delta$. In other words, we see that

$$\sup_{r=R+1} |v| \leq \tfrac{1}{2} \sup_{r=R} |v|,$$

for all $R \geq R_0$, or equivalently $|v(r,\theta)| \leq Ce^{-mr}$ for some $m > 0$. This completes the first step.

Now, by local a priori estimates, if $\mathcal{A}(\rho)$ is the annulus $\{\rho \leq r \leq \rho+1\}$, then $\|v\|_{2,\mu;\mathcal{A}(\rho)} \leq Ce^{-m\rho}$, and hence $|Q(v)| \leq C_2 e^{-2mr}$ for all $r \geq R_0$. Assuming that $m < 1$, we use the maximum principle again, this time with $w = e^{-\beta r}$ for some $\beta \in (m, \min\{1, 2m\})$. Since

$$L_P w = (\beta^2 - r^{-1}\beta - 1)e^{-\beta r} < (\beta^2 - 1)e^{-\beta r},$$

we obtain $L_P(v - C_3 w) \geq -Q(v) + C_3(1-\beta^2)e^{-\beta r} \geq 0$ for all $r \geq R_0$; in addition $v - C_3 w \leq 0$ along $r = \rho$ for C_3 sufficiently large, and $v - C_3 w \to 0$ as $r \to \infty$. We conclude that $v \leq C_3 e^{-\beta r}$ for $r \geq R_0$.

With this argument we have improved the exponent in the decay rate from $m < 1$ to any $\beta \in (m, \min\{1, 2m\})$. Iterating this a finite number of times shows that we can obtain a decay rate with exponent as close to 1 as we please. In other words, we conclude that $v \leq C_4 e^{-(1-\epsilon)r}$ for some very small $\epsilon > 0$, and hence $|Q(v)| \leq C_5 e^{-2(1-\epsilon)r}$, and then that $\|Q(v)\|_{0,\mu;\mathcal{A}(\rho)} \leq C_6 e^{-2(1-\epsilon)\rho}$ as well.

Now write $v = G_P(-Q(v))$ as in (2-8). Since L_P commutes with rotations and translations in P, the Green function $G_P((s,t),(s',t'))$ depends only on the (Euclidean) distance between (s,t) and (s',t'), and hence reduces to a function of one variable which satisfies a modified Bessel equation. We thus arrive at the well-known classical formula

$$G_P((s,t),(s',t')) = \frac{1}{4\pi} K_0\Big(\sqrt{|s-s'|^2 + |t-t'|^2}\Big). \tag{2-9}$$

Here $K_0(r)$ is the Bessel function of imaginary argument (see Lebedev [9]), which has the well-known asymptotics

$$\begin{aligned} &K_0(r) \sim \log r && \text{as } r \searrow 0,\\ &K_0(r) \sim r^{-\frac{1}{2}}e^{-r} + \mathcal{O}(r^{-\frac{3}{2}}e^{-r}) && \text{as } r \nearrow \infty \end{aligned} \tag{2-10}$$

(we are omitting the normalizing constant $(4\pi)^{-1}$ for simplicity). It is a straightforward exercise to check that if f is continuous and $|f| \le Ce^{-2(1-\epsilon)r}$, then

$$v = G_P f = \int_{\mathbb{R}^2} G_P((s,t),(s',t')) f(s',t')\, ds'dt' = A(\theta) r^{-1/2} e^{-r} + \mathcal{O}(r^{-3/2} e^{-r}),$$

and if $f \in \mathcal{C}^\infty$, then $A \in \mathcal{C}^\infty(\mathbb{S}^1)$. We refer to Melrose [17] for an explanation of the linear mapping $f(s,t) \mapsto A(\theta)$ (it is the adjoint of the Poisson operator and is closely related to the scattering operator for L_P).

To complete the argument, suppose that $v > 0$. Since $A(\theta)e^{-r}r^{-1/2}$ dominates the expansion for r large, clearly $A(\theta) > 0$. □

Asymptotics of Jacobi fields Let Σ be a complete properly embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with a finite number of ends, each one asymptotic to a vertical plane P_α, $\alpha \in A$. We could also let Σ be an exterior region in any such surface, ie, the discussion below incorporates the case where Σ has compact boundary. We now recall some facts about the asymptotic properties of solutions of the equation $L_\Sigma \psi = 0$, where L_Σ is the Jacobi operator (2-2). This operator has the particularly simple form (2-3) when Σ is a vertical plane P, and this provides the asymptotic model for L_Σ in our more general setting. When Σ is a horizontal catenoid K_η, we write the Jacobi operator as L_η.

There are many classical sources for the material in this section; we refer in particular to [17] since the treatment there is specifically geometric.

It is a classical fact in scattering theory that any solution of $L_P \psi = 0$ (defined either on all of P or just on the complement of a relatively compact domain) has a so-called far-field expansion as $r \to \infty$; this takes the form

$$\psi(r,\theta) \sim (F^+(\theta) r^{-\frac{1}{2}} + \mathcal{O}(r^{-\frac{3}{2}}))e^r + (F^-(\theta) r^{-\frac{1}{2}} + \mathcal{O}(r^{-\frac{3}{2}}))e^{-r}. \tag{2-11}$$

In the particular case where P is all of $\mathbb{R}^2$ and has no boundary, then $F^-(\theta) = F^+(-\theta)$, but in general the relationship is more complicated. The subtlety in such an expansion is that the coefficients $F^\pm(\theta)$ are allowed to be arbitrary distributions on $\mathbb{S}^1$, and if these coefficients are not smooth, then the expansion must be interpreted weakly, ie, as holding only after we pair with an arbitrary test function $\varphi(\theta)$. The simplest "plane wave" solution of this equation, e^s, exhibits an expansion with coefficients which are Dirac delta functions:

$$e^s \sim \delta(\theta) r^{-\frac{1}{2}} e^r + \delta(-\theta) r^{-\frac{1}{2}} e^{-r}.$$

One can interpret this as reflecting the obvious fact that this solution grows exponentially as $s \to \infty$ (which corresponds to $\theta = 0$) and decays exponentially as $s \to -\infty$ (which is $\theta = \pi$). On the other hand, the Green function for this operator with pole at 0, $G_P((s,t),(0,0)) = K_0(r)$, has

$$G_P \sim r^{-\frac{1}{2}} e^{-r} \quad \text{as } r \to \infty,$$

ie, $F^+ = 0$ and $F^- = 1$. Similarly, since ∂_s commutes with L_P, the function $\partial_s G_P$ is another Jacobi field, and it has

$$\partial_s G_P \sim r^{-\frac{1}{2}} \cos\theta \, e^{-r}.$$

Now return to the Jacobi operator on more general minimal surfaces with ends asymptotic to vertical planes.

Proposition 2.3 *Suppose that $L_\Sigma \psi = 0$. Let r_α denote the radial function on the asymptotic end P_α, and transfer this (via the horizontal graph description) to a function on Σ. Then ψ has the far-field expansion*

$$\psi \sim \sum_{\alpha \in A} (F_\alpha^+(\theta) r_\alpha^{-\frac{1}{2}} + \mathcal{O}(r_\alpha^{-\frac{3}{2}})) e^{r_\alpha} + (F_\alpha^-(\theta) r_\alpha^{-\frac{1}{2}} + \mathcal{O}(r_\alpha^{-\frac{3}{2}})) e^{-r_\alpha}.$$

The set of possible leading (distributional) coefficients $\{F_\alpha^+, F_\alpha^-\}$ that can occur is called the scattering relation for L_Σ. If Σ is preserved by the reflection $\mathcal{R}_t$ and we restrict to functions that are even with respect to $\mathcal{R}_t$, then any collection $\{F_\alpha^+\}$ uniquely determines a solution, and hence determines the other set of coefficients $\{F_\alpha^-\}$; the same is true on the complement of a finite dimensional subspace if we drop the evenness condition. The map $\{F_\alpha^+\} \mapsto \{F_\alpha^-\}$ is called the scattering operator.

Geometric Jacobi fields We now describe the special family of global Jacobi fields on the horizontal catenoid K_η generated by the "integrable", or geometric, deformations of K_η. In other words, these Jacobi fields are tangent at K_η to families of horizontal catenoids.

We have already described the space $\mathcal{C}_K$ of all horizontal catenoids that are symmetric about the plane $t = 0$. Indeed, there is a unique such catenoid associated to any two geodesics $\gamma_\pm$ in $\mathbb{H}^2$ with $0 < \operatorname{dist}(\gamma_+, \gamma_-) = \eta < \eta_0$. Thus $\mathcal{C}_K$ is identified with an open subset of the space of distinct four-tuples of points on $\mathbb{S}^1$: writing any such four-tuple in consecutive order around $\mathbb{S}^1$ as $(\zeta_{-,1}, \zeta_{-,2}, \zeta_{+,1}, \zeta_{+,2})$, then we let $\gamma_\pm$ be the unique geodesic connecting $\zeta_{\pm,1}$ to $\zeta_{\pm,2}$. Note that we do not allow arbitrary four-tuples simply because the distances between these geodesics must be less than η_0. In any case, $\dim \mathcal{C}_K = 4$.

There are various different ways to describe the complete family of horizontal catenoids (symmetric about $\{t = 0\}$). First we can vary the points $\zeta_{\pm,\ell}$ independently. Second, we can transform K_η using the three-dimensional space of isometries of $\mathbb{H}^2$, and then, to obtain the entire four-dimensional family, we augment this by the extra deformation corresponding to changing the parameter η, ie, moving the geodesics relative to one another.

Using the first parametrization of this family, let $\zeta(\epsilon)$ be a smooth curve in the space of (allowable) four-tuples where we vary only one end of one of the geodesics. The corresponding Jacobi field decays exponentially in all directions but one (this holds by Proposition 2.1 and the behavior of the hyperbolic metric at infinity). For example, if we vary only $\zeta_{+,2}$, then this Jacobi field decays exponentially in all directions at infinity on P_-, while on P_+, it decays exponentially as $s \to -\infty$ but grows exponentially as $s \to +\infty$ (we assume that s increases as we move along γ_+ from $\zeta_{+,1}$ to $\zeta_{+,2}$).

In computing the infinitesimal variations here, note that if $K_\eta(\epsilon)$ is a one-parameter family of horizontal catenoids as described here, with $K_\eta(0) = K_\eta$, then for $\epsilon \neq 0$ we can write $K_\eta(\epsilon)$ as a normal graph over some proper subset of K_η. However, as $\epsilon \to 0$, this proper subset fills out all of K_η, and hence the derivative of the normal graph function at $\epsilon = 0$ is defined on the entire surface.

Denote by $\Phi_{\pm,\ell}$ the Jacobi field generated by varying only the one point $\zeta_{\pm,\ell}$, and note that each $\Phi_{\pm,\ell} \sim e^s = e^{r\cos\theta}$. For any four real numbers $E_{\pm,\ell}$, $\ell = 1, 2$, we define

$$\Phi_E = \sum_{\pm,\ell} E_{\pm,\ell}\Phi_{\pm,\ell}.$$

K_η as a horizontal bigraph The geometric Jacobi fields can be used to show that K_η is a horizontal bigraph in two distinct directions: over the vertical plane $\gamma_0 \times \mathbb{R}$ and also over the vertical plane $\Gamma \times \mathbb{R}$. These two new graphical representations were also obtained in the recent paper [5] using an Alexandrov reflection argument. We present a separate argument using these Jacobi fields since it is somewhat less technical. Note that the assertion about horizontal graphicality must be clarified first since there are two geometrically natural ways of writing a surface with a vertical end in $\mathbb{H}^2 \times \mathbb{R}$ as a horizontal graph over a vertical plane. Indeed, let $\gamma(s)$ be an arclength parametrized geodesic in $\mathbb{H}^2$. We can then coordinatize $\mathbb{H}^2$ using Fermi coordinates off of γ, ie, $(s,\sigma) \mapsto \exp_{\gamma(s)}(\sigma\nu(s))$ (where ν is the unit normal), or else by $(s,\sigma) \mapsto D_\sigma(\gamma(s))$, where D_σ is the one-parameter family of isometries of $\mathbb{H}^2$ that are dilations along the geodesic $\gamma^\perp$ orthogonal to γ and meeting γ at $\gamma(0)$. We use the latter, and then say that a curve is a graph over γ in the direction of $\gamma^\perp$ if $\sigma = f(s)$. Hence $f \equiv$ const. corresponds to a geodesic γ' that is hyperparallel to γ and perpendicular to $\gamma^\perp$. This transfers immediately to the notion of a horizontal graph over $\gamma \times \mathbb{R}$ in the direction of $\gamma^\perp$ in $\mathbb{H}^2 \times \mathbb{R}$.

Now, recall the two orthogonal geodesics Γ and γ_0 (see Figure 1). The vertical plane $\Gamma \times \mathbb{R}$ (resp. $\gamma_0 \times \mathbb{R}$) fixed by $\mathcal{R}_s$ (resp. $\mathcal{R}_o$) bisects K_η, decomposing it into two pieces denoted by $K_\eta^{s,1}, K_\eta^{s,-1}$ (resp. $K_\eta^{o,1}, K_\eta^{o,-1}$), which are interchanged by this reflection. The result of Hauswirth, Nelli, Sa Earp and Toubiana [5, Lemmas 3.1 and 3.2] is the following:

Proposition 2.4 *For each $\eta \in (0, \eta_0)$, $K_\eta^{o,+}$ is a horizontal graph in the direction of Γ over some portion of the vertical plane $\gamma_0 \times \mathbb{R}$ while $K_\eta^{s,+}$ is a horizontal graph in the direction of γ_0 over some portion of the vertical plane $\Gamma \times \mathbb{R}$.*

As noted, we sketch an independent proof of this.

Proof First, notice that the first assertion in Proposition 2.4 is equivalent to the fact that the Jacobi field Φ_o generated by dilations along Γ is strictly positive on $K_\eta^{o,+}$ and vanishes along the fixed point set of $\mathcal{R}_o$; similarly, the second assertion is equivalent to claiming that the Jacobi field Φ_s generated by dilations along γ_0 is strictly positive on $K_\eta^{s,+}$ and vanishes along the fixed point set of $\mathcal{R}_s$.

The proof has two steps. We first show that these Jacobi fields have the required positivity property when η is very close to the upper limit η_0. We then show that as we vary η from η_0 down to 0, they maintain this positivity.

We begin by asserting that the limiting Scherk surface K_{η_0} is a horizontal bigraph over $\Gamma \times \mathbb{R}$ and also over $\gamma_0 \times \mathbb{R}$. In fact, this surface has a symmetry obtained by rotating by $\pi/2$ and flipping $t \mapsto -t$; this interchanges these two graphical representations. This can be proved by a simple Alexandrov reflection argument: Consider the family of geodesics γ_σ perpendicular to Γ and intersecting it at $\Gamma(\sigma)$ (where $\Gamma(0) = \Gamma \cap \gamma_0$). The plane $\gamma_\sigma \times \mathbb{R}$ only intersects K_{η_0} when $\sigma < \eta_0/2$, and for σ just slightly smaller, the reflection of the "smaller" portion of K_{η_0} across this vertical plane does not intersect the other component. Pushing σ lower, it is standard to see that these two half-surfaces do not intersect until $\sigma = 0$, in which case they coincide. These planes of reflection are the images of $\gamma_0 \times \mathbb{R}$ with respect to dilation along Γ, so we deduce that the vector field X generated by this dilation is everywhere transverse to the component $K_{\eta_0}^+$ of K_{η_0} on one side of this plane of symmetry. Note finally that the angle between X and $K_{\eta_0}^+$ is bounded below by a positive constant if we remain a bounded distance away from $\partial K_{\eta_0}^+$.

Now recall that an appropriate vertical translate of K_η converges locally uniformly in $\mathcal{C}^\infty$ to K_{η_0}, and indeed this convergence (of the translated K_η) is uniform in the half-plane $t \geq -C$ for any fixed C. It is then clear that the angle between X and

$K_\eta^{o,+} \cap \{t \geq -C\}$ is also positive everywhere when η is sufficiently close to η_0. Since K_η is invariant by $\mathcal{R}_t$, this finishes the first step.

For the second step, to be definite consider Φ_o, and let us study what happens as η varies in the interval $(0, \eta_0)$. We use that $L_\eta \Phi_o = 0$ and Φ_o is nonnegative on $K_\eta^{o,+}$ for η close to η_0, vanishing only on the boundary, and by the Hopf boundary point lemma, has strictly positive normal derivative there. As η decreases, Φ_o must remain strictly positive in the interior; the alternative would be that it develops some interior zeroes or else its normal derivative vanishes at the boundary while the function still remains nonnegative in the interior, and both contradict the maximum principle. Note that we are using two additional facts: first, we use the form of the maximum principle that states that a nonnegative solution of $(\Delta + V)u = 0$ cannot have an interior zero, regardless of the sign of V; we also use that because from the graphical representation of the ends, it is clear that Φ_o is bounded away from 0 outside a compact set. This proves that $\Phi_o > 0$ on $K_\eta^{o,+}$ for all $\eta \in (0, \eta_0)$, which shows that this half remains graphical.

The case of the Jacobi field Φ_s is quite similar. Taking into account the asymptotic behavior of K_η, it is not hard to see that there exists a constant $T \gg 0$ so that $K_\eta \cap \{|t| > T\}$ is a horizontal graph over the vertical plane $\Gamma \times \mathbb{R}$, for all $\eta \in (0, \eta_0)$. We can then apply the same argument as in the previous paragraphs to $K_\eta \cap \{|t| \leq T\}$. □

Fluxes Closely related to the geometry in the last subsection is the computation of the flux homomorphism. We recall that if Σ is an oriented minimal surface in an ambient space (Z, g), then its flux is a linear mapping

$$\mathcal{F}\colon H_1(\Sigma) \times \mathcal{K}(Z, g) \longrightarrow \mathbb{R},$$

where $\mathcal{K}(Z, g)$ is the space of Killing vector fields on Z, ie, infinitesimal generators of one-parameter families of isometries. The definition is simple: if $c \in H_1(\Sigma)$ is a homology class represented by a smooth oriented closed curve γ and if $X \in \mathcal{K}(Z, g)$, then

$$\mathcal{F}(c, X) = \int_\gamma X \cdot \nu \, ds,$$

where ν is the unit normal to γ in Σ. This is only interesting when the ambient space Z admits Killing fields, but this is certainly the case in our setting. Indeed, $\mathcal{K}(\mathbb{H}^2 \times \mathbb{R})$ (with the product metric) is four-dimensional: there is one Killing field X_t generated by vertical translation, and a three-dimensional space of Jacobi fields on $\mathbb{H}^2$ that lift to the product to act trivially on the $\mathbb{R}$ factor. If K_η is a horizontal catenoid and if $o = \gamma_0 \cap \Gamma \in \mathbb{H}^2$ is its "center", then this three-dimensional space is generated by the infinitesimal rotation X_R around o, and the infinitesimal dilations X_{γ_0} and X_Γ along γ_0 and Γ, respectively.

The first homology (with real coefficients), $H_1(K_\eta)$, is one-dimensional and is generated by the loop $\gamma = (\gamma_0 \times \mathbb{R}) \cap K_\eta$. Thus it suffices to consider $\mathcal{F}([\gamma], X_j)$, where $X_j = X_t$, X_R, X_{γ_0} or X_Γ.

Proposition 2.5 *The quantity $\mathcal{F}([\gamma], X_j)$ vanishes when $X = X_t$, X_R or X_{γ_0}, and is nonzero when $X = X_\Gamma$.*

Proof The vector field X_t is odd with respect to the reflection $\mathcal{R}_t$; similarly, X_R and X_{γ_0} are odd with respect to one or more of the reflections $\mathcal{R}_o$, $\mathcal{R}_s$. Since the choice of generator γ for H_1 is invariant under all three reflections, it is easy to see that $\mathcal{F}([\gamma], X) = 0$ when X is any one of these three vector fields. However, X_Γ is a positive multiple of ν at every point of γ, so that $\mathcal{F}([\gamma], X_\Gamma) > 0$, as claimed.

We do not actually compute the value of this one nonvanishing flux. □

Unlike many other gluing constructions for minimal surfaces, these fluxes turn out to play no interesting role in the analysis below. This traces, ultimately, to the fact that we will be gluing together copies of horizontal catenoids and these are already "balanced". We explain this point further at the end of Section 5.

Spectrum of the Jacobi operator We now study the L^2 spectrum of the Jacobi operator L_η. By the general considerations described above,

$$\operatorname{spec}(-L_\eta) = \{\lambda_j(\eta)\}_{j=1}^N \cup [1, \infty).$$

The ray $[1, \infty)$ consists of absolutely continuous spectrum (this is because K_η is a decaying perturbation of the union of two planes outside a compact set, so that the essential spectrum of $-L_\eta$ coincides with that of $-L_P$), while the discrete spectrum lies entirely in $(-\infty, 1)$; note that, even counted according to multiplicity, the number of eigenvalues may depend on η.

Our main result is the following:

Proposition 2.6 *For each $\eta \in (0, \eta_0)$, the only one of the eigenvalues of $-L_\eta$ that is negative is $\lambda_0(\eta)$, and only $\lambda_1(\eta) = 0$. All the remaining eigenvalues are strictly positive. The ground-state eigenfunction $\phi_0 = \phi_0(\eta)$ is even with respect to all three reflections, $\mathcal{R}_t$, $\mathcal{R}_s$ and $\mathcal{R}_o$; the eigenfunction ϕ_1, which is the unique L^2 Jacobi field, is generated by vertical translations and is odd with respect to $\mathcal{R}_t$ but even with respect to $\mathcal{R}_s$ and $\mathcal{R}_o$. In particular, if we restrict $-L_\eta$ to functions that are even with respect to $\mathcal{R}_t$, then L_η is nondegenerate.*

Proof We can decompose the spectrum of $-L_\eta$ into the parts that are either even or odd with respect to each of the isometric reflections $\mathcal{R}_t$, $\mathcal{R}_s$ and $\mathcal{R}_o$. Indeed, for each such reflection, there is an even/odd decomposition

$$L^2(K_\eta) = L^2(K_\eta)_{j \text{ ev}} \oplus L^2(K_\eta)_{j \text{ odd}}, \quad j = t, s, o.$$

The reduction of $-L_\eta$ to the odd part of any one of these decompositions corresponds to this operator acting on functions on the appropriate half $K_\eta^{j,+}$ of K_η with Dirichlet boundary conditions.

Our first claim is that the restriction of $-L_\eta$ to $L^2(K_\eta)_{j \text{ odd}}$ with $j = s, o$ is strictly positive, and is nonnegative if $j = t$, with one-dimensional nullspace spanned by the Jacobi field Φ_t generated by vertical translations.

To prove this, note first that since $\Phi_t \in L^2(K_\eta)_{t \text{ odd}}$ and Φ_t is strictly positive on $K_\eta^{t,+}$, it must be the ground state eigenfunction for this reduction and is thus necessarily simple, with all the other eigenvalues strictly positive.

On the other hand, we have proved above that Φ_s and Φ_o are strictly positive solutions of this operator on the appropriate halves of K_η, vanishing on the boundary, but of course do not lie in L^2. We shall invoke the following lemma.

Lemma 2.7 *Consider the operator $-L = -\Delta + V$ on a Riemannian manifold M, where V is smooth and bounded. Assume either that M is complete, or else, if it has boundary, then we consider $-L$ with Dirichlet boundary conditions at ∂M. Suppose that there exists an L^2 solution u_0 of $Lu_0 = 0$ such that $u_0 > 0$, at least away from ∂M. If v is any other solution of $Lv = 0$ with $v > 0$ in M and $v = 0$ on ∂M, then $v = cu_0$ for some constant c.*

Remark 2.8 We can certainly relax the hypotheses on V. The proof below is from the paper of Murata [20]; the result appears in earlier work by Agmon, and is proved by different methods in Sullivan [26, Theorem 2.8] and Pinsky [23, Chapter 4, Theorem 3.4].

Proof It is technically simpler to work on a compact manifold with smooth boundary, so let Ω_j be a sequence of nested, compact smoothly bounded domains that exhaust M, and in the case where $\partial M \neq \varnothing$, assume that $\overline{\Omega}_j \cap \partial M = \varnothing$ for all j. The last condition is imposed since it is convenient to have that v is strictly positive on the closure of each Ω_j.

It is well-known that the lowest eigenvalue λ_0^j of $-L$ with Dirichlet boundary conditions on Ω_j converges to the lowest Dirichlet eigenvalue λ_0 of $-L$ on all of M (indeed, this follows from the Rayleigh quotient characterization of the lowest eigenvalue). We are assuming that $\lambda_0 = 0$, so by domain monotonicity, $\lambda_0^j \searrow 0$.

Now choose a nonnegative (and not identically vanishing) function $\psi \in \mathcal{C}_0^\infty(\Omega_0)$ and define $-L_k = -L - \frac{1}{k}\psi$ for any $k \in \mathbb{R}^+$. Denoting the lowest eigenvalue of this operator on Ω_j by $\lambda_0^{j,k}$, then by the same Rayleigh quotient characterization, we have that $\lambda_0^{j,k} \le \langle -L_k u, u\rangle$ for any fixed $u \in H_0^1(\Omega_j)$ with $\|u\|_{L^2} = 1$. In particular, inserting the ground state eigenfunction $\hat{u}_0^j$ for $-L$ on Ω_j, we obtain

$$\lambda_0^{j,k} \le \lambda_0^j - \frac{1}{k}\int_{\Omega_j} \psi |\hat{u}_0^j|^2 \, dV_g.$$

In particular, fixing $k > 0$, then since the first term on the right can be made arbitrarily close to 0 by assumption, we can choose j so that $\lambda_0^{j-1,k} > 0$ and $\lambda_0^{j,k} \le 0$. This is because the integral in the second term on the right is bounded away from zero, which holds because

$$\hat{u}_0^j \le \hat{u}_0^{j+1}$$

on the support of ψ (this can be proved using the maximum principle for $-L - \lambda_0^{j+1}$ to compare $\hat{u}_0^j$ and $\hat{u}_0^{j+1}$ on the smaller domain Ω_j). If we recall also that the eigenvalue $\lambda_0^{j,k}$ depends continuously (in fact, analytically) on k, then we can adjust the value of k slightly to a nearby value k_j so that $\lambda_0^{j,k_j} = 0$. Clearly $k_j \to \infty$. We have thus obtained a solution $u_0^j > 0$ of $-L_{k_j} u_0^j = 0$ on Ω_j with $u_0^j = 0$ on $\partial\Omega_j$.

Since the solution v is strictly positive, we have that $\Delta \log v = V - |\nabla \log v|^2$. Now, using that u_0^j vanishes on $\partial\Omega_j$, we compute that

$$\begin{aligned}
\int_{\Omega_j} |\nabla(u_0^j/v)|^2 v^2 \, dV_g &= \int_{\Omega_j} |\nabla u_0^j|^2 - \nabla(u_0^j)^2 \cdot \nabla \log v + (u_0^j)^2 |\nabla \log v|^2 \, dV_g \\
&= \int_{\Omega_j} |\nabla u_0^j|^2 + (u_0^j)^2 (V - |\nabla \log v|^2) + (u_0^j)^2 |\nabla \log v|^2 \, dV_g \\
&= \int_{\Omega_j} u_0^j(-\Delta + V) u_0^j \, dV_g = \frac{1}{k_j}\int_{\Omega_j} \psi (u_0^j)^2 \, dV_g.
\end{aligned}$$

Normalizing so that $\|u_0^j\|_{L^2} = 1$, then it is straightforward to show that $u_0^j \to u_0$ on any compact subdomain of M. Since the right hand side of this equation tends to 0, so does the left, hence in particular the integral of $|\nabla(u_0/v)|^2$ over any fixed $\Omega_{j'}$ vanishes, ie, $v = c u_0$ as claimed. □

This lemma implies that it is impossible for $-L_\eta$ to have lowest eigenvalue equal to 0 on either of the subspaces $L^2(K_\eta)_{j \text{ odd}}$, $j = s, o$, since if this were the case, then we could use the corresponding eigenfunction as u_0 in Lemma 2.7 and let $v = \Phi_j$ to get a contradiction since $\Phi_j \notin L^2$.

We shall justify below that when η is very close to its maximal value η_0, the lowest eigenvalue of $-L_\eta$ on $L^2(K_\eta)_{j \text{ odd}}$ is strictly positive.

Using the continuity of the ground state eigenvalue as η decreases combined with the argument above, we see that this lowest eigenvalue can never be negative on any one of these odd subspaces, and the only odd L^2 Jacobi field is Φ_t. This proves the claim.

We have finally reduced to studying the spectrum of $-L_\eta$ on $L^2(K_\eta)_{\text{ev}}$, ie, the subspace which is even with respect to all three reflections (we call this "totally even"). Because of the existence of an L^2 solution of $L_\eta u = 0$ that changes signs, namely $u = \Phi_t$, we know that the bottom of the spectrum of $-L_\eta$ is strictly negative, and we have proved above that the corresponding eigenfunction must live in the totally even subspace. (This is also obvious because of the simplicity of this eigenspace and the fact that the corresponding eigenfunction is everywhere positive.) Thus $\lambda_0(\eta) < 0$ as claimed.

Now suppose that the next eigenvalue $\lambda_1(\eta)$ lies in the interval $(\lambda_0(\eta), 0]$, and if $\lambda_1(\eta) = 0$, assume that there exists a corresponding eigenfunction that is totally even. Since this is the *second* eigenvalue, we know that the corresponding eigenfunction $\phi_1(\eta)$ has exactly two nodal domains. However, it is straightforward to see using the symmetries of K_η that if ϕ is any function on K_η that is totally even and changes sign, then it cannot have exactly two nodal domains. Indeed, if that were the case, then the nodal line $\{\phi = 0\}$ would have to either be a connected simple closed curve or else two arcs, and these would then necessarily be the fixed point set of one of the three reflections. This is clearly incompatible with ϕ being totally even.

We are almost finished. It remains finally to prove that the lowest eigenvalue of $-L_\eta$ on any one of the odd subspaces is nonnegative when η is sufficiently large.

As a first step, we first prove that $\lambda_0(\eta) \nearrow 0$ as $\eta \nearrow \eta_0$. Recall that in this limit, K_η converges (once we translate vertically by an appropriate distance) to the limiting Scherk surface K_{η_0}. Moreover, K_{η_0} is strictly stable because the Jacobi field Φ_t generated by vertical translation is strictly positive on it.

Now suppose that $\lambda_0(\eta) \leq -c < 0$. When η is sufficiently close to η_0, we can construct a cutoff $\widetilde{\phi}_0(\eta)$ of the corresponding eigenfunction $\phi_0(\eta)$ that is supported in the region $t > 0$ (we are still assuming that K_η is centered around $t = 0$); this function lies in L^2 and regarding it as a function on K_{η_0}, it is straightforward to show that

$$\frac{\int_{K_{\eta_0}} (-L_{\eta_0} \widetilde{\phi}_0) \widetilde{\phi}_0}{\int_{K_{\eta_0}} |\widetilde{\phi}_0|^2} \leq -c/2 < 0.$$

This contradicts the strict stability of K_{η_0}, and hence proves that $\lambda_0(\eta) \nearrow 0$.

Now suppose that there is some sequence $\eta^\ell \nearrow \eta_0$ and a corresponding sequence of eigenvalues $\lambda^\ell \in (\lambda_0(\eta^\ell), 0)$ and eigenfunctions $\phi^\ell \in L^2(K_\eta)_{j \text{ odd}}$, $j = s, o$. We know that $\lambda^\ell \nearrow 0$. Suppose that the maximum of $|\phi^\ell|$ is attained at some point $p^\ell \in K_{\eta^\ell}$. Normalize by setting $\widehat{\phi}^\ell = \phi^\ell / \sup|\phi^\ell|$ and take the limit as $\ell \to \infty$. Depending on the limiting location of p^ℓ, we obtain a bounded solution of the limiting equation on the pointed Gromov–Hausdorff limit of the sequence (K_{η^ℓ}, p^ℓ). There are, up to isometries, only two possible such limits: either the limiting Scherk surface K_{η_0} or else a vertical plane $P = \gamma \times \mathbb{R}$. In the latter case, the limiting function $\widehat{\phi}$ satisfies $L_P\widehat{\phi} = 0$. However, $L_P = \Delta_P - 1$ and it follows by an easy argument using the Fourier transform on P that there are no bounded solutions of $L_P\widehat{\phi} = 0$ on all of P, so this case cannot occur. Therefore, we have obtained a function $\widehat{\phi}$ on K_{η_0} that is a solution of the Jacobi equation there and that is bounded. We now invoke [10, Theorem 2.1], which is a result very similar to Lemma 2.7, but instead of assuming that v is positive, we assume instead that v is bounded, and then conclude that $v = cu_0$ where u_0 is the positive L^2 solution. The proof proceeds by a somewhat more intricate cutoff argument than the one above. In any case, this proves that $\widehat{\phi}$ must equal the unique *positive* L^2 Jacobi field on K_{η_0}, but this is impossible because of the oddness of ϕ^ℓ with respect to either $\mathcal{R}_s$ or $\mathcal{R}_o$.

This completes the proof of the main proposition. □

3 Families of nearly minimal surfaces

We now describe a collection of families of "nearly minimal" surfaces, exhibiting a wide variety of topological types. In the next section we prove that these can be deformed to actual minimal surfaces, at least when certain parameters in the family are sufficiently large. The geometry of each such configuration is encoded by a finite network of geodesic lines and arcs in $\mathbb{H}^2$. Each complete geodesic γ in this network corresponds to a vertical plane $P = P_\gamma = \gamma \times \mathbb{R}$. The geodesic segments connecting these geodesic lines correspond to catenoidal necks connecting the associated vertical planes. The approximate minimal surfaces themselves are constructed by gluing together horizontal catenoids. Thus we take advantage of the existence of these components, the existence of which already incorporates some of the nonlinearities of the problem; this is in lieu of working with the more primitive component set comprised of vertical planes and catenoidal necks. The parameter that measures the "strength" of the interaction between these pieces is the distance between the finite geodesic segments. Once this distance is sufficiently large, we expect that the approximately minimal surface can be perturbed to be exactly minimal. We prove this here under one extra hypothesis, that the catenoidal necksizes remain bounded away from zero. The more general case will be handled in a subsequent paper. The joint requirement that the distances between geodesic "connector" arcs be large and that the necksizes are bounded away from zero imposes restrictions, which we describe below.

We now describe all of this more carefully.

Geodesic networks An admissible geodesic network $\mathcal{F}$ (see Figure 3) consists of a finite set of (complete) geodesic lines $\Gamma = \{\gamma_\alpha\}_{\alpha \in A}$ and geodesic segments $\mathcal{T} = \{\tau_{\alpha\beta}\}_{(\alpha,\beta)\in A'}$ connecting various pairs of elements in Γ. Here A is some finite index set and A' is a subset of $A \times A \setminus \operatorname{diag}$ that indexes all "contiguous" geodesics, γ_α and γ_β that are joined by some $\tau_{\alpha\beta}$. We now make various assumptions on these data and set notation:

(i) If $\alpha \neq \beta$, then $\operatorname{dist}(\gamma_\alpha, \gamma_\beta) := \eta_{\alpha\beta} \in (0, \eta_0)$, where η_0 is the maximal separation between vertical planes that support a horizontal catenoid.

(ii) The segment $\tau_{\alpha\beta}$ realizes the distance $\eta_{\alpha\beta}$ between γ_α and γ_β, and hence is perpendicular to both these geodesic lines.

(iii) Set $p_\alpha(\beta) = \tau_{\alpha\beta} \cap \gamma_\alpha$ and $p_\beta(\alpha) = \tau_{\alpha\beta} \cap \gamma_\beta$, and then define

$$D_\alpha = \min_{(\alpha,\beta),(\alpha,\beta')\in A'} \{\operatorname{dist}(p_\alpha(\beta), p_\alpha(\beta'))\} \quad \text{and} \quad D = \min_\alpha D_\alpha.$$

This number D is called the minimal neck separation of the configuration $\mathcal{F}$.

(iv) We also write $\eta := \sup \eta_{\alpha\beta}$, and call it the maximal neck parameter.

We shall be considering sequences of geodesic networks $\mathcal{F}_j$ for which the minimal neck separation D_j tends to infinity. Such sequences have two distinct types of behaviour: either all of the $(\eta_{\alpha\beta})_j \geq c > 0$, or else at least some of the $(\eta_{\alpha\beta})_j \to 0$. The main analytic construction below turns out to be fairly straightforward for the first type, but unfortunately the simplest geometries (a relatively small number of ends for a given genus) can only happen in the second setting.

Proposition 3.1 *Let $\mathcal{F}_j$ be a sequence of configurations with $D_j \to \infty$, and suppose that no $\mathcal{F}_j$ is contractible. If the cardinalities of the index sets $A(\mathcal{F}_j)$ and $A'(\mathcal{F}_j)$ (ie, the number of geodesics and geodesic segments) remain bounded independently of j, then at least some of the necksizes $(\eta_{\alpha\beta})_j$ must tend to 0.*

Proof By hypothesis, for each j the configuration $\mathcal{F}_j$ contains a cycle c_j. Referring to the geometry of each $\mathcal{F}_j$, it is clear that each side of every c_j is a geodesic segment,

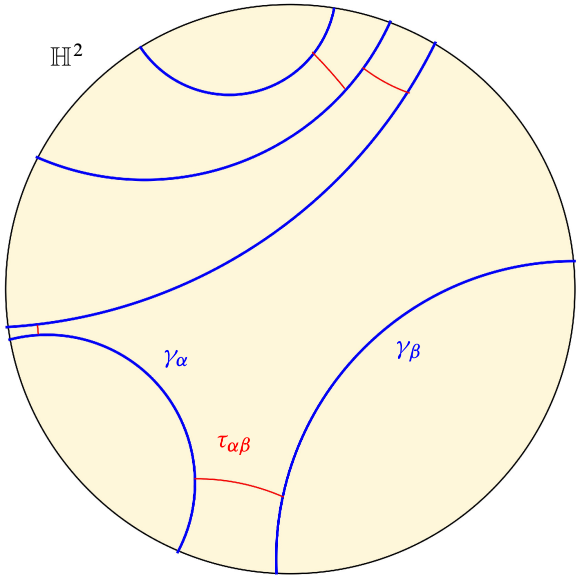

Figure 3: The geodesic network $\mathcal{F}$

and moreover, each c_j is a convex hyperbolic polygon whose sides meet at right angles. By hypothesis then we have a sequence of such polygons where the number of sides remains bounded, so we may as well assume that each c_j is a k–gon for some fixed k. Suppose that all $(\eta_{\alpha\beta})_j \geq c > 0$. Then by hypothesis, the successive adjacent sides of c_j are geodesic segments of length at least D_j and geodesic segments of length lying in the interval $[c, \eta_0]$. However, it is a standard fact in hyperbolic geometry that a geodesic polygon with every other side lying in such an interval must have all sidelengths uniformly controlled, which is a contradiction. □

In summary, for any sequence of configurations with fixed nontrivial topology and a fixed number of geodesic lines, at least some of the $\eta_{\alpha\beta}$ must converge to 0.

From geodesic networks to nearly minimal surfaces To each geodesic network $\mathcal{F}$ satisfying the properties above we now associate an approximately minimal surface Σ. The idea is straightforward: each geodesic line γ_α is replaced by the vertical plane $P_\alpha = \gamma_\alpha \times \mathbb{R}$, and each geodesic segment $\tau_{\alpha\beta}$ corresponds to a catenoidal neck connecting P_α and P_β at the points $p_\alpha(\beta)$ and $p_\beta(\alpha)$. The resulting surface is denoted $\Sigma_{\mathcal{F}}$.

The arguments used below to deform $\Sigma_{\mathcal{F}}$ to an actual minimal surface are perturbative, so we must construct sequences of nearly minimal surfaces for which the error, which is a quantitative measure of how far $\Sigma_{\mathcal{F}}$ is from being minimal, tends to zero. To make the error term small, it is necessary to consider a sequence of networks $\mathcal{F}_j$ where the minimum neck separation D_j tends to infinity. As proved above, if the necksizes stay bounded away from zero, the number of component pieces must grow with j. Because the proof is much cleaner in this case, we assume that $(\eta_{\alpha\beta})_j \geq c > 0$ in all the rest of this paper. The more general case can be treated using techniques similar to those in [12], but we shall address this in a separate paper.

The surface $\Sigma_{\mathcal{F}}$ will be constructed by assigning to each $\tau_{\alpha\beta}$ a vertical strip in the catenoid $K_{\eta_{\alpha\beta}}$ (see below) that contains a very wide neighbourhood around the neck region. Using that none of the necksizes tend to zero, we will prove that the Jacobi operator has a uniformly bounded inverse, acting between certain weighted Hölder spaces.

For each $\mathcal{F}$, we now show how to construct $\Sigma_{\mathcal{F}}$. Fix a line γ_α in $\mathcal{F}$, and enumerate the points $p_\alpha(\beta)$ along this line consecutively as $p_{\alpha,1}, \dots, p_{\alpha,N}$. (The number of such points, $N = N_\alpha$, depends on α, but for the sake of simplicity, the notation does not record this.) Let $q_{\alpha,j}$ be the midpoint of the geodesic segment from $p_{\alpha,j}$ to $p_{\alpha,j+1}$, $j = 1, \dots, N-1$, and denote the length of such a segment by $d_{\alpha,j}$. Hence,

$$\operatorname{dist}(p_{\alpha,j}, q_{\alpha,j}) = \tfrac{1}{2} d_{\alpha,j}, \quad \operatorname{dist}(p_{\alpha,j}, q_{\alpha,j-1}) = \tfrac{1}{2} d_{\alpha,j-1}.$$

Note that each $d_{\alpha,j} \geq D_\alpha > D$. Finally, let $S_{\alpha,j}$ denote the vertical strip in P_α bounded by the two lines $q_{\alpha,j} \times \mathbb{R}$ and $q_{\alpha,j+1} \times \mathbb{R}$. For the extreme values $j = 0$ and N, let $S_{\alpha,j}$ be the half-plane in P_α bounded by $q_{\alpha,1}$ (on the right) and $q_{\alpha,N-1}$ (on the left), respectively.

Now, consider a geodesic segment $\tau_{\alpha\beta} \in \mathcal{F}$, and write its two endpoints as $p_{\alpha,j}$ and $p_{\beta,k}$. Let $K_{\alpha\beta}$ be the horizontal catenoid with vertical ends $P_\alpha \sqcup P_\beta$ and parameter $\eta_{\alpha\beta}$. Writing this catenoid as a horizontal normal graph over the relevant portions of the planes P_α and P_β (ie, away from the neck regions), we let $K^c_{\alpha\beta}$ denote the portion of the catenoid that includes the neck region and that lies over the strips $S_{\alpha,j}$ and $S_{\beta,k}$ (this is possible when D is large enough).

This ensemble is not quite in final form since the edges of the different truncated horizontal catenoids do not quite match up. Write the corresponding portion of $K_{\alpha\beta}$ over the strip $S_{\alpha,j}$ as a normal graph with graph function $v_{\alpha,j}$. Choose a smooth cutoff function $\chi_{\alpha,j} \geq 0$ that equals 1 in the interior of $S_{\alpha,j}$ at all points that are a distance at least 2 from the boundaries, and that vanishes at all points which are distance at most 1 from these boundaries, and such that $|\nabla \chi_{\alpha,j}| + |\nabla^2 \chi_{\alpha,j}| \leq 2$ (again this is possible for D large enough). We then let $K^0_{\alpha\beta}$ be the slightly modified surface that agrees with $K^c_{\alpha\beta}$ near the neck region and is the graph of $\chi_{\alpha,j} v_{\alpha,j}$ over the rest of

$S_{\alpha,j}$. Of course, this is no longer quite minimal where the modifications have been made.

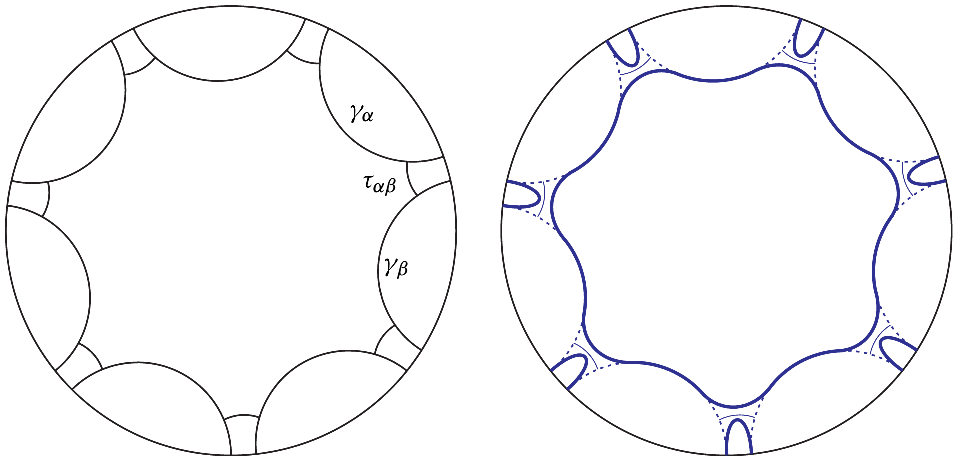

Figure 4: An example of $\Sigma_{\mathcal{F}}$ and the corresponding (approximately) minimal surface

Our final definition of the approximately minimal surface $\Sigma_{\mathcal{F}}$ in this case, where all neck parameters are bounded below by c, is

$$\Sigma_{\mathcal{F}} = \bigsqcup_{(\alpha\beta)\in A'} K^0_{\alpha\beta}. \tag{3-1}$$

Proposition 3.2 *Let $\mathcal{F}$ be a geodesic network that satisfies the properties* (i), (ii) *and* (iii), *and let $\Sigma = \Sigma_{\mathcal{F}}$ be the associated surface in $\mathbb{H}^2 \times \mathbb{R}$ just constructed. Then Σ is smooth and has $H \equiv 0$ except in the vertical strips $Q_{\alpha,j}$ of width 2 around the lines $q_{\alpha,j} \times \mathbb{R}$. In the vertical strip $Q_{\alpha,j}$,*

$$\sup_{B_t} \|H\|_{0,\mu;B_t} \le C r^{-\frac{1}{2}} e^{-r}.$$

Here

$$r = \min\left\{\sqrt{(d_{\alpha,j})^2 + t^2}, \sqrt{(d_{\alpha,j+1})^2 + t^2}\right\},$$

and B_t is the square of width 2 and height 2 centered at $(q_{\alpha,j}, t) \in Q_{\alpha,j}$. The constant C is independent of all parameters in the construction provided $D = \min D_\alpha$ is sufficiently large.

The only point that needs to be checked is the decay of the local Hölder norm of the mean curvature. However, this follows directly from the corresponding estimate for the decay of the horizontal graph functions $v_{\alpha,j}$; see (2-4).

Proposition 3.3 *Let $\mathcal{F}$ be a geodesic network satisfying* (i), (ii) *and* (iii), *and write $\Sigma = \Sigma_{\mathcal{F}}$. If D is sufficiently large, then the Jacobi operator L_Σ is non-degenerate when restricted to functions which are even respect to $\mathcal{R}_t$.*

Proof We proceed by contradiction. Assume there exists a sequence of networks $\mathcal{F}_j$ with $D_j = D(\mathcal{F}_j) \nearrow +\infty$ such that each $\Sigma_j = \Sigma(\mathcal{F}_j)$ admits a nonvanishing, even, L^2 Jacobi field ϕ_j. Let $p_j \in \Sigma_j$ be a point where $|\phi_j|$ attains its maximum, and set $a_j := |\phi_j(p_j)|$. If T_j is an isometry of $\mathbb{H}^2 \times \mathbb{R}$ with $T_j(p_j) = (0,0)$, and $S_j = T_j(\Sigma_j)$, then

$$\psi_j = \Big(\frac{1}{a_j}\phi_j\Big) \circ T_j^{-1}$$

is a Jacobi field on S_j with $\sup|\psi_j| = 1$ attained at $(0,0)$. Passing to a subsequence, S_j converges to a surface S_∞, which is clearly either a vertical plane or a horizontal catenoid K_η, for some $\eta \in (0, \eta_0)$, and ψ_j converges to a nontrivial, bounded Jacobi field ψ_∞ on S_∞. However, it is clear that no such Jacobi field exists on a vertical plane. Furthermore, the expansion (2-11) holds also on ends of K_η and shows that a bounded Jacobi field must, in fact, lie in L^2 (this could also be proved more directly using a variant of the proof of Proposition 2.2). However, ψ_∞ is the limit of functions invariant with respect to $\mathcal{R}_t$, hence also has this property. This contradicts the vertical nondegeneracy of horizontal catenoids as proved in Proposition 2.6, and completes the proof. □

Examples It is possible to construct nearly minimal surfaces as sketched above, assuming that all necksizes $\eta_{\alpha\beta}$ are bounded away from 0, with arbitrary genus, though possibly a large number of ends. Since each plane P_α is diffeomorphic to a once-punctured sphere, we see that $\Sigma_{\mathcal{F}}$ is a connected sum of such spheres, with one connection corresponding to each geodesic segment $\tau_{\alpha\beta}$.

To carry out the perturbation analysis, we must consider networks $\mathcal{F}$ with $D(\mathcal{F})$ sufficiently large. As already explained, this imposes various restrictions. For example, to find a sequence of networks $\mathcal{F}_j$ with precisely one loop, and with $D(\mathcal{F}_j) \to \infty$ and all $\eta_{\alpha\beta} \geq c > 0$, standard formulas from hyperbolic trigonometry show that the number of edges must grow with j. One construction is to take a hyperideal polygon in $\mathbb{H}^2$ that is invariant with respect to rotation by $2\pi/j$, by which we mean a collection of j disjoint geodesics with a cyclic ordering and such that the minimal distance between any pair of adjacent geodesics is some fixed number η. If η does not tend to zero, then the only way to have the minimal neck separation tend to infinity is if $j \to \infty$ (see Proposition 3.1). By contrast, we can find sequences of such networks with $j = 3$, for example, if we let $\eta \to 0$.

4 Perturbation of $\Sigma_{\mathcal{F}}$ to a minimal surface

We now complete the perturbation analysis to show how to pass from the nearly minimal surfaces $\Sigma_{\mathcal{F}}$ to actual minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ when $\mathcal{F}$ is a geodesic network with minimal neck separation D sufficiently large, and with a uniform lower bound $\eta_{\alpha\beta} \geq c > 0$ on the neck parameters.

Fixing $\mathcal{F}$, let $\Sigma = \Sigma_{\mathcal{F}}$ and let ν be the unit normal on Σ with respect to a fixed orientation. For any $u \in \mathcal{C}^{2,\mu}(\Sigma)$, consider the normal graph over Σ with graph function u,

$$\Sigma(u) = \{\exp_q(u(q)\nu(q)),\ q \in \Sigma\}.$$

Assuming that all $\eta_{\alpha\beta} \geq c > 0$, then there exists $C = C(c) > 0$ such that if $\|u\|_{2,\mu} < C$, then $\Sigma(u)$ is embedded.

The surface $\Sigma(u)$ is minimal if and only if u satisfies a certain quasilinear elliptic partial differential equation, $\mathcal{N}(u) = 0$, which calculates the mean curvature of $\Sigma(u)$. (A similar argument was considered in the proof of Proposition 2.2 considering the vertical plane P instead of Σ.) We do not need to know much about $\mathcal{N}$ except the following. If we write $\mathcal{N}(u) = \mathcal{N}(0) + D\mathcal{N}\big|_0(u) + Q(u)$, then

(i) $\mathcal{N}(0) = H_\Sigma$,

(ii) the linearization at $u = 0$ is the Jacobi operator of Σ,

$$D\mathcal{N}\big|_0 = L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}(\nu, \nu),$$

(iii) if ϵ is sufficiently small and $\|u\|_{2,\mu} < \epsilon$, then

$$\|\mathcal{N}(u)\|_{0,\mu} \leq C\epsilon \quad \text{and} \quad \|Q(u)\|_{0,\mu} \leq C\epsilon^2.$$

The equation $\mathcal{N}(u) = 0$ is equivalent to

$$L_\Sigma u = -H_\Sigma - Q(u). \tag{4-1}$$

The strategy is now a standard one: we shall define certain weighted Hölder spaces X and Y, and first prove that $L_\Sigma\colon X \to Y$ is Fredholm. A more careful analysis will show that, at least when the minimal neck separation D is sufficiently large, this map is invertible and moreover its inverse $G_\Sigma\colon Y \to X$ has norm that is uniformly bounded by a constant depending only on the lower bounds D for the minimal neck separation and c for the maximal neck parameter (see Proposition 4.2 and Corollary 4.3). Given these facts, we then rewrite (4-1) as

$$u = -G_\Sigma(H_\Sigma + Q(u)), \tag{4-2}$$

and solve this equation by a standard contraction mapping argument.

Somewhat remarkably, in this instance, this argument works almost exactly as stated. The only subtlety is that we must restrict to functions that are even with respect to the vertical reflection $t \mapsto -t$, since this subspace avoids the exponentially decaying element of the nullspace of the Jacobi operator on Σ.

The basic function spaces are standard Hölder spaces $\mathcal{C}^{k,\mu}(\Sigma)$ defined using the seminorm

$$[u]_{0,\mu} = \sup_{\substack{z \neq z' \\ \mathrm{dist}\,(z,z') \leq 1}} \frac{|u(z) - u(z')|}{\mathrm{dist}\,(z,z')^{\mu}}.$$

Although the result could be proved using these spaces alone, we can obtain finer results by including a weight factor, which involves the exponential of a piecewise radial function R. On each strip $S_{\alpha,j}$, define a radial function $r_{\alpha,j} = \sqrt{s^2 + t^2}$, where s is the arclength parameter along γ_α and $s = 0$ corresponds to the point $p_{\alpha,j}$. The functions $r_{\alpha,j}$ and $r_{\alpha,j+1}$ match up continuously at $S_{\alpha,j} \cap S_{\alpha,j+1}$. Now define a function R on Σ as follows: on each neck region of every horizontal catenoid set $R \equiv 1$; on the portion of Σ that is a graph over $S_{\alpha,j} \setminus \mathcal{O}_{\alpha,j}$ (where $\mathcal{O}_{\alpha,j}$ is some ball that is larger than the projection of the neck region), set $R = r_{\alpha,j}$. It is convenient to replace this function with a slightly mollified version which is smooth everywhere, and which has the property that $|\nabla R| + |\nabla^2 R| \leq 2$, so we assume this is the case. Finally, define

$$e^{\kappa R}\mathcal{C}^{k,\mu}(\Sigma) = \{u = e^{\kappa R} v : v \in \mathcal{C}^{k,\mu}(\Sigma)\}.$$

Given $u \in e^{\kappa R}\mathcal{C}^{k,\mu}(\Sigma)$, we consider the following norm:

$$\|u\|_{k,\mu,\kappa} = \|e^{-\kappa R}\, u\|_{k,\mu}.$$

Proposition 4.1 *Fix any $\kappa \in (-1,1)$ and $\mu \in (0,1)$. If Σ is any nearly minimal surface, as constructed above, then*

$$L_\Sigma \colon e^{\kappa R}\mathcal{C}^{2,\mu}(\Sigma) \longrightarrow e^{\kappa R}\mathcal{C}^{0,\mu}(\Sigma)$$

is Fredholm.

Proof If the elliptic operator L_Σ has local parametrices with compact remainder on each end of Σ, then we can patch together these local parametrices to obtain a parametrix on all of Σ with similarly good properties. Recall that a local parametrix on a bounded open set $\mathcal{U}$ in Σ is a continuous linear operator $\widetilde{G}_{\mathcal{U}} \colon \mathcal{E}'(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$, between the spaces of compactly supported and all distributions on $\mathcal{U}$, satisfying

$$L\widetilde{G}_{\mathcal{U}} = \mathrm{Id} - \mathfrak{R}, \quad \widetilde{G}_{\mathcal{U}} L = \mathrm{Id} - \mathfrak{R}', \tag{4-3}$$

where $\mathfrak{R}$ and $\mathfrak{R}'$ are smoothing of infinite order, and such that

$$\widetilde{G}_{\mathcal{U}}\colon \mathcal{C}^{0,\mu} \cap \mathcal{E}'(\mathcal{U}) \longrightarrow \mathcal{C}^{2,\mu}(\mathcal{U}).$$

Similarly, if E is any infinite end of Σ, then a local parametrix on E is a linear operator $\widetilde{G}_E$ that is bounded as a map $e^{\kappa R}\mathcal{C}^{0,\mu}(E) \to e^{\kappa R}\mathcal{C}^{2,\mu}(E)$, and satisfies the analogue of (4-3), where $\mathfrak{R}$ and $\mathfrak{R}'$ are again infinite order smoothing operators that have range in a space of (smooth) functions which have a fixed rate of decay at infinity. It follows directly from the Arzelà–Ascoli theorem and these mapping properties that $\mathfrak{R}$ and $\mathfrak{R}'$ are compact operators on these weighted Hölder spaces. Thus, once we produce this parametrix, which is an inverse to L modulo compact remainder terms, a standard argument from functional analysis then shows that L is Fredholm.

Since L_Σ is uniformly elliptic, the existence of local parametrices on bounded open sets is one of the basic theorems of microlocal analysis; see Shubin [25]. The construction of parametrices on the ends of Σ uses more, namely that L_Σ is "fully elliptic" near infinity, which means that it is strongly invertible there in a sense we make precise below.

Each end of Σ has the form $P \setminus \mathcal{O}$ where P is a vertical plane and $\mathcal{O}$ is a large ball of finite radius. The restriction of L to each end is a decaying perturbation of the basic operator $\Delta_{\mathbb{R}^2} - 1$. The restriction to the complement of $\mathcal{O}$ of this operator has an inverse, the Schwartz kernel of which, also known as the Green function, is expressed in terms of the modified Bessel function

$$G_{\mathbb{R}^2}(z,z') = cK_0(|z-z'|) \sim c'|z-z'|^{-\frac{1}{2}}e^{-|z-z'|} \quad \text{as } |z-z'| \to \infty.$$

It is not hard to check (see Mazzeo and Vasy [15]) that if $|\kappa| < 1$ and $r = |z|$, then

$$G_{\mathbb{R}^2}\colon e^{\kappa r}\mathcal{C}^{0,\mu}(\mathbb{R}^2) \longrightarrow e^{\kappa r}\mathcal{C}^{2,\mu}(\mathbb{R}^2).$$

(The fact that this operator increases regularity by 2 orders is classical; the slightly more subtle point is that it also preserves the growth or decay rate $e^{\kappa r}$ when $|\kappa| < 1$.) Now write $L_\Sigma = \Delta_{\mathbb{R}^2} - 1 + F$ where F is a second order operator with smooth coefficients that decay like e^{-r}. From this we deduce that

$$L_\Sigma G_{\mathbb{R}^2} - \mathrm{Id} = \mathfrak{R}\colon e^{\kappa r}\mathcal{C}^{0,\mu}(\mathbb{R}^2 \setminus \mathcal{O}) \longrightarrow e^{(\kappa-1)r}\mathcal{C}^{0,\mu}(\mathbb{R}^2 \setminus \mathcal{O}).$$

This does not yet compactly include into $e^{\kappa r}\mathcal{C}^{0,\mu}$ since there is no gain of regularity so we cannot apply the Arzelà–Ascoli Theorem. There are two effective ways to overcome this: first, restricting to the complement of an even larger ball, we can make the norm of this remainder term as small as desired, hence $\mathrm{Id} + \mathfrak{R}$ can be inverted using a Neumann series. Equivalently, we can use a standard elliptic parametrix construction to modify $G_{\mathbb{R}^2}$ by an asymptotic series so that the new modified parametrix satisfies

$LG = \mathrm{Id} - \mathfrak{R}$, where $\mathfrak{R}$ maps into $e^{(\kappa-1)r}\mathcal{C}^\infty(\mathbb{R}^2 \setminus \mathcal{O})$. Either of these methods produces a global parametrix for L_Σ with compact remainder on each end of Σ.

Now, cover Σ by open sets of the form $P_\alpha \setminus \mathcal{O}_\alpha$ and one relatively compact open set $\mathcal{U}$. Using the standard elliptic parametrix construction on this bounded set and the parametrices constructed above on each P_α, we may form a global parametrix as follows. Choose a partition of unity for this open cover, $\{\chi_0, \chi_\alpha\}_{\alpha \in A}$, and for each open set here choose another smooth function $\tilde{\chi}_i$ with support in $\mathcal{U}$ for $i = 0$ and in $P_\alpha \setminus \mathcal{O}_\alpha$ for $i = \alpha$, such that $\tilde{\chi}_i = 1$ on the support of χ_i. Now define

$$\tilde{G}_\Sigma = \tilde{\chi}_0 G_0 \chi_0 + \sum_{\alpha \in A} \tilde{\chi}_\alpha G_\alpha \chi_\alpha.$$

We calculate that

$$\begin{aligned} L_\Sigma \tilde{G}_\Sigma &= \tilde{\chi}_0 L_\Sigma G_0 \chi_0 + \sum_\alpha \tilde{\chi}_\alpha L_\Sigma G_\alpha \chi_\alpha + [L_\Sigma, \tilde{\chi}_0] G_0 \chi_0 + \sum_\alpha [L_\Sigma, \tilde{\chi}_\alpha] G_\alpha \chi_\alpha \\ &= \mathrm{Id} + \mathfrak{R}_\Sigma . \end{aligned}$$

We use here that $L_\Sigma G_i = \mathrm{Id}$ on the support of χ_i so the first set of terms on the right is equal to $\sum \tilde{\chi}_i \mathrm{Id} \chi_i = \sum \chi_i \mathrm{Id} = \mathrm{Id}$. The remainder $\mathfrak{R}_\Sigma$ is a pseudodifferential operator of order -1 with image lying in the union of the supports of the $\nabla \tilde{\chi}_i$, which is a compact set. Hence, using the well-known mapping properties of such operators, $\mathfrak{R}_\Sigma \colon e^{\kappa R}\mathcal{C}^{0,\mu} \to e^{\kappa R}\mathcal{C}^{0,\mu}$ is a compact operator. A similar calculation shows that $\tilde{G}_\Sigma L_\Sigma = \mathrm{Id} + \mathfrak{R}''_\Sigma$ is also compact.

We have now produced an approximate inverse modulo compact remainders, and as explained at the beginning of the proof, this suffices to prove that L is Fredholm. □

The next step is to show that L_Σ is invertible provided the minimal neck separation D is sufficiently large. This fails of course if L_Σ acts on the entire space $e^{\kappa R}\mathcal{C}^{2,\mu}(\Sigma)$ because of the exponentially decaying Jacobi field generated by vertical translations. To circumvent this issue, we restrict L_Σ to the subspace $e^{\kappa R}\mathcal{C}^{k,\mu}_{\mathrm{ev}}(\Sigma)$ of even functions with respect to the reflection $t \mapsto -t$. (Note that we can assume that the radial function R is even.)

Proposition 4.2 *Let Σ be a nearly minimal surface associated to the geodesic network $\mathcal{F}$. There exists a $D_0 > 0$ such that if the minimal neck separation D is greater than D_0, then*

$$L_\Sigma \colon e^{\kappa R}\mathcal{C}^{2,\mu}_{\mathrm{ev}}(\Sigma) \longrightarrow e^{\kappa R}\mathcal{C}^{0,\mu}_{\mathrm{ev}}(\Sigma)$$

is invertible.

Proof We have already proved that the mapping L_Σ is Fredholm on the entire weighted Hölder space, and it is clear that this remains true when restricting to the subspace of even functions. Let G_Σ denote the generalized inverse of L_Σ. Recall that, by definition, this means that $L_\Sigma G_\Sigma - \mathrm{Id} = \mathfrak{R}_\Sigma$ is a projector onto the complement of the range of L_Σ and $G_\Sigma L_\Sigma - \mathrm{Id} = \mathfrak{R}'_\Sigma$ is a projector onto the nullspace of L_Σ. In particular, these projectors both have finite rank. Since the index of L_Σ vanishes, $\mathrm{Tr}\,\mathfrak{R}'_\Sigma - \mathrm{Tr}\,\mathfrak{R}_\Sigma = \mathrm{Ind}(L_\Sigma) = 0$.

To proceed, we sketch a slightly different version of the parametrix construction. Recall that Σ is a union of truncated (and slightly perturbed) horizontal catenoids $K^0_{\alpha\beta}$. These catenoids are joined along the lines $\{q_{\alpha,j}\} \times \mathbb{R}$, which are at distance at least $D/2$ away from each neck region. In fact, when $R > D/2$, Proposition 3.2 shows that

$$\sup e^{-\kappa R}|H_\Sigma| \le C \sup e^{-(\kappa+1)R} R^{-1/2} \le C e^{-(\kappa+1)D/2} D^{-1/2}.$$

On the other hand, this inequality trivially holds when $R \le D/2$.

Now choose an open cover comprised by slightly larger truncations of these catenoids, a partition of unity $\chi_{\alpha\beta}$ associated to this open cover, and smooth cutoff functions $\widetilde{\chi}_{\alpha\beta}$ that are supported in these same open sets and that equal 1 on the support of $\chi_{\alpha\beta}$. Then set

$$\widehat{G}_\Sigma = \sum_{(\alpha\beta)\in A'} \widetilde{\chi}_{\alpha\beta} G_{\alpha\beta} \chi_{\alpha\beta}.$$

Exactly the same computation as above shows that $L_\Sigma \widehat{G}_\Sigma - \mathrm{Id} = -\widehat{\mathfrak{R}}_\Sigma$ is compact on $e^{\kappa R}\mathcal{C}^{0,\mu}_{\mathrm{even}}(\Sigma)$, but furthermore has norm $\|\widehat{\mathfrak{R}}_\Sigma\| \le C e^{-(\kappa+1)D/2}$, where C is independent of D.

Finally, choosing D sufficiently large, then $\mathrm{Id} - \widehat{\mathfrak{R}}_\Sigma$ is invertible on $e^{\kappa R}\mathcal{C}^{0,\mu}_{\mathrm{even}}$, and hence $L_\Sigma G_\Sigma = \mathrm{Id}$, where

$$G_\Sigma = \widehat{G}_\Sigma \circ (\mathrm{Id} - \widehat{\mathfrak{R}}_\Sigma)^{-1}.$$

This shows that L_Σ is surjective. The proof of Proposition 3.3 applies equally well here and implies that L_Σ is injective as well, provided D is large enough. (Alternately, the index of L_Σ is zero, hence injectivity follows directly from surjectivity.) This concludes the proof. □

It is clear from the local nature of the Hölder norms and the definition of this parametrix that the operator norm of $\widehat{G}_\Sigma$ is uniformly bounded as $D \to \infty$. Its modification by $(\mathrm{Id} - \widehat{\mathfrak{R}})^{-1}$ does not change this, so we obtain:

Corollary 4.3 *If Σ satisfies all the assumptions of the previous proposition, then the norm of the inverse G_Σ on $e^{\kappa R}\mathcal{C}^{0,\mu}_{\mathrm{even}}$ is uniformly bounded as $D \to \infty$.*

The slightly surprising fact is that these estimates are independent of the topology or number of ends of $\mathcal{F}$ and Σ, but this is due to the character of the function spaces being used.

Theorem 4.4 *Let $\mathcal{F}$ be a geodesic network in $\mathbb{H}^2$ and $\Sigma_{\mathcal{F}}$ the nearly minimal surface constructed from it. Also, fix $\kappa \in (-1,0)$. If the minimal neck separation D is sufficiently large, then there exists a function $u \in e^{\kappa R}\mathcal{C}^{2,\mu}_{\text{even}}(\Sigma_{\mathcal{F}})$ with $\|u\|_{2,\mu,\kappa} \le Ce^{-(\kappa+1)D/2}D^{-1/2}$ such that $\Sigma_{\mathcal{F}}(u)$ is an embedded minimal surface that is a small normal graph over $\Sigma_{\mathcal{F}}$.*

Proof We solve $\mathcal{N}(u) = 0$ in the function space $e^{\kappa R}\mathcal{C}^{2,\mu}_{\text{even}}$ by rewriting this equation as in (4-2). As $\kappa \in (-1,0)$, then

$$\|Q(u)\|_{0,\mu,\kappa} \le C_1\|u\|^2_{2,\mu,\kappa},$$

and hence if $\|H_\Sigma\|_{0,\mu,\kappa} \le A$, then

$$\|G_\Sigma(H_\Sigma + Q(u))\|_{2,\mu,\kappa} \le C(A + C_1\|u\|^2_{2,\mu,\kappa}).$$

If $\|u\|_{2,\mu,\kappa} \le \beta$, then the right hand side here is bounded by $C(A + C_1\beta^2)$, and $C(A + C_1\beta^2) \le \beta$ provided we choose $\beta = \lambda A$ for some large λ and then let A be very small. With these choices, if we write (4-2) as $u = \mathcal{T}(u)$, then $\mathcal{T}$ maps the ball of radius β in $e^{\kappa R}\mathcal{C}^{2,\mu}_{\text{even}}$ to itself. A similar analysis shows that $\mathcal{T}$ is a contraction on this ball.

This proves that there is a unique solution to $\mathcal{N}(u) = 0$, and that $\|u\|_{2,\mu,\kappa} \le \beta$. Finally, since $\kappa < 0$, we have $|u| \le \beta e^{\kappa R} \le \beta$; the derivatives of u are similarly small, which implies that $\Sigma_{\mathcal{F}}(u)$ is embedded. □

Proposition 4.5 *Let $\mathcal{F}_j$ be a sequence of geodesic networks as in Theorem 4.4 (in particular the minimal neck separation $D(\mathcal{F}_j) \to \infty$) and Σ_j the corresponding minimal surfaces. Suppose that the necksizes $(\eta_{\alpha\beta})_j$ in the constituent horizontal catenoids all lie in a fixed interval $[c_1, c_2] \subset (0, \eta_0)$. Then for j sufficiently large, Σ_j is horizontally nondegenerate.*

Proof Suppose that this is not the case, so that there exists some subsequence $\Sigma_{j'}$, which we immediately relabel as Σ_j and a function $\varphi_j \in L^2(\Sigma_j)$ that is even with respect to $\mathcal{R}_t$ and that lies in the nullspace of the Jacobi operator L_j on Σ_j. Renormalize φ_j to have supremum equal to 1, and suppose that this supremum is attained at a point $p_j \in \Sigma_j$.

At this point we can reason as in the proof of Proposition 3.3 to get a contradiction with the nondegeneracy of the vertical plane and the horizontal nondegeneracy of the horizontal catenoid. □

5 Gluing nondegenerate surfaces

A construction that is closely related to the one in the last section is as follows. Let Σ_1 and Σ_2 be two minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with a finite number of vertical ends, each one symmetric with respect to the reflection $\mathcal{R}_t$, and each one horizontally nondegenerate. Fix a vertical planar end $E_\ell \subset \Sigma_\ell$, and choose a sequence of isometries $\phi_{\ell,j}$ (of the form $\varphi_{\ell,j} \times \mathrm{id}$ where each $\varphi_{\ell,j}$ is an isometry of $\mathbb{H}^2$) such that the surface $\Sigma_{\ell,j} := \phi_{\ell,j}(\Sigma_\ell)$ converges to a fixed vertical plane $P = \gamma \times \mathbb{R}$. Parametrizing γ as $\gamma(s)$, then we suppose that a half-plane in the end E_1 in $\Sigma_{1,j}$ is a horizontal graph over $(-B_{1,j}, \infty) \times \mathbb{R}$ with $B_{1,j} \to \infty$ and with graph function $v_{1,j}$, while a half-plane E_2 in $\Sigma_{2,j}$ is a horizontal graph over $(-\infty, B_{2,j}) \times \mathbb{R}$ with $B_{2,j} \to \infty$ and with graph function $v_{2,j}$. We assume finally that both $v_{\ell,j}$ converge to 0 as $j \to \infty$.

Now let $\widetilde{\Sigma}_{1,j}$ be the surface that agrees with $\Sigma_{\ell,j}$ away from the half-plane $(-1, \infty) \times \mathbb{R}$, and where the graph function is altered to $\chi_1(s) v_{1,j}$; here $\chi_1(s)$ is a smooth monotone decreasing function that equals 1 for $s \le -1$ and vanishes for $s \ge 0$. We let $\widetilde{\Sigma}_{2,j}$ be a similar alteration of $\Sigma_{2,j}$. Finally, let

$$\Sigma(j) = \big(\widetilde{\Sigma}_{1,j} \setminus ((0,\infty) \times \mathbb{R})\big) \sqcup \big(\widetilde{\Sigma}_{2,j} \setminus ((-\infty, 0) \times \mathbb{R})\big).$$

It is clear that $\Sigma(j)$ is exactly minimal outside of the vertical strip $(-1,1) \times \mathbb{R}$.

Furthermore, it is clear that if Σ_1 and Σ_2 carry radial functions R_1 and R_2 as in the previous section, then we can form a radial function $R(j)$ on $\Sigma(j)$, and define weighted Hölder spaces $e^{\kappa R(j)} \mathcal{C}^{k,\mu}(\Sigma(j))$. In terms of these, the mean curvature $H(j)$ of $\Sigma(j)$ tends to zero.

A straightforward modification of the arguments in the preceding section yields a proof of this:

Theorem 5.1 *Let $\Sigma(j)$ be a sequence of nearly minimal surfaces, constructed as above. Assume (as stated earlier) that both Σ_1 and Σ_2 are horizontally nondegenerate. Then for j sufficiently large, there exists a function $u \in e^{\kappa R(j)} \mathcal{C}^{2,\mu}(\Sigma(j))$ such that the surface $\Sigma(j,u)$, which is the normal graph over $\Sigma(j)$ with graph function u, is an embedded, horizontally nondegenerate minimal surface.*

One must check first that $\Sigma(j)$ itself is nondegenerate for j large, and then that the norm of the inverse of its Jacobi operator on these weighted Hölder spaces remains uniformly bounded as $j \to \infty$. These facts are both proved by contradiction, and the details of the proofs are very similar to what we have done above. The final step, using a contraction mapping to produce the function u whose graph is minimal, is again done as before.

Notice that if the genera of Σ_1 and Σ_2 are g_1 and g_2, respectively, then $\Sigma(j)$ and hence the minimal surface $\Sigma(j,u)$ has genus $g_1 + g_2$.

Corollary 5.2 *The construction in Theorem 5.1 can be continued indefinitely. In other words, let Σ_ℓ be an infinite sequence of minimal, horizontally nondegenerate surfaces, each with finite genus and finite number of planar ends, and let P_j be one of the planar ends of Σ_j. Suppose that we have constructed a sequence of minimal, horizontally nondegenerate surfaces $\Sigma^{(N)}$ inductively by gluing Σ_N to $\Sigma^{(N-1)}$ with the end P_N attached to the end corresponding to P_{N-1} in $\Sigma^{(N-1)}$. Then one can arrange the gluing parameters so that $\Sigma^{(N)}$ converges to a minimal surface with an infinite number of vertical planar ends.*

Indeed, each of the gluings here is given by Theorem 5.1, so it remains only to show that one can pass to the limit. For this, construct a sequence of properly embedded minimal surfaces $\{S_N\}$, and two sequences of positive real numbers $R_N \nearrow +\infty$ and $\varepsilon_N \searrow 0$ such that:

(a) S_N is obtained by gluing $\Sigma^{(N-1)}$ and $\Sigma^{(N)}$.

(b) If S_N is a normal graph of a function u_N, then $\|u_N\|_{2,\mu} \leq 2^{-N}$.

(c) $S_N \setminus B(p_0, R_N)$ consists of (disjoint) neighborhoods of the ends of S_N, where p_0 is a fixed point in $\mathbb{H}^2 \times \mathbb{R}$.

(d) For all $m \geq N$, we have that $S_m \cap B(p_0, R_N)$ lies on a ε_N–neighborhood of S_N and can be written as a normal graph over S_N.

The construction of such sequences is possible since we can choose the neck separation parameter D_N at the N^{th} stage sufficiently large. Thus it is clear (item (b)) that we can ensure that the sequence of normal graph functions u_N converge locally uniformly in $\mathcal{C}^\infty$ to a function that is uniformly small, so that embeddedness is maintained (items (c) and (d)), and that decays exponentially along all ends.

This corollary shows that there exist complete, properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with vertical planar ends, with either finite or infinite genus and with an infinite number of ends.

We conclude this section with a brief remark concerning why the fluxes of horizontal catenoids, or of the more general constituent pieces considered in this section, play no role in this gluing construction. The reason is that we glue along vertical lines orthogonal to the axis of the catenoid and positioned very far from it. Although these lines are not closed, they are limits of a sequence of closed curves, namely rectangles lying over regions $\{S_1 \leq s \leq S_2; |t| \leq T\}$ where $S_2, T \nearrow \infty$. These rectangles are homologically trivial, so the flux over them vanishes, and hence the same is true over the vertical lines. Because of this, there is no need to balance the fluxes of the summands in this construction against one another.

6 Deformation theory

We conclude this paper with a brief analysis of the moduli space of even, properly embedded complete minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. Let $\mathcal{M}_k$ denote the space of all such surfaces with k ends, each asymptotic to a vertical plane, and which are symmetric with respect to the reflection $\mathcal{R}_t$.

Theorem 6.1 *The space $\mathcal{M}_k$ is a real analytic set with formal dimension equal to $2k$. There is a stratum of $\mathcal{M}_k$ consisting of horizontally nondegenerate elements that has dimension exactly equal to $2k$.*

Remark 6.2 This dimension count agrees with our construction: indeed, $2k$ is precisely the dimension of the space of admissible geodesic networks with k geodesic lines, regardless of the number of "cross-piece" geodesic segments, since in a given network $\mathcal{F}$, each geodesic line γ_α has a two-dimensional deformation space, and any small perturbation of the geodesics uniquely determines the corresponding deformations of the geodesic segments $\tau_{\alpha\beta}$. Note, however, that we are *not* demanding here that the minimal surfaces be ones that we have constructed. For example, it is conceivable that there exist surfaces whose necks are not centered on the plane of symmetry. This analysis of the deformation space is insensitive to this.

We do not factor out by the 3–dimensional space of "horizontal" isometries of $\mathbb{H}^2 \times \mathbb{R}$. But if we do this, then the dimension count $2k-3$ agrees with the dimension of the family of minimal surfaces in [19].

Proof The proof is very similar to the ones in Mazzeo, Pollack and Uhlenbeck [13], and Kusner, Mazzeo and Pollack [8] (and in several places since then), so we shall be brief. A different approach to the moduli space theory – for minimal surfaces with finite total curvature and parallel ends – appears in [22], but that relies on a Weierstrass representation that is not available here.

Fix $\Sigma \in \mathcal{M}_k$ and enumerate its vertical planar ends as $\{P_\alpha\}_{\alpha\in A}$, so each $P_\alpha = \gamma_\alpha \times \mathbb{R}$. For any sufficiently small $\epsilon_{\alpha,j} \in \mathbb{R}$, $j = 1, 2$, we can deform γ_α, and hence P_α, by displacing the two endpoints of γ_α by these amounts, respectively (relative to a fixed metric on $\mathbb{S}^1$). Thus small deformations of the entire ensemble of vertical planes are in correspondence with $2k$–tuples $\epsilon = (\epsilon_{\alpha,1}, \epsilon_{\alpha,2})_{\alpha\in A}$ with $|\epsilon| \ll 1$.

For each such ϵ, let $\Sigma(\epsilon)$ denote a small deformation $\Sigma(\epsilon)$ of the surface $\Sigma = \Sigma(0)$, constructed as follows. For each α, write the end E_α of Σ as a normal graph over some exterior region $P_\alpha \setminus O_\alpha$ with graph function v_α defined in polar coordinates for $r \geq R_0$. Rotate P_α by the parameters ϵ_α to obtain a new vertical plane $P_\alpha(\epsilon)$. Using the same graph function v_α, now defined on an exterior region in $P_\alpha(\epsilon)$, we obtain the deformed end $E_\alpha(\epsilon)$; this is quite close to the original end E_α over the annulus $\{R_0+1 \leq r \leq R_0+2\}$, so we can write $E_\alpha(\epsilon)$ as the graph of a function $v_{\alpha,\epsilon}$ defined on this annulus in the *original* plane P_α. Finally, use a fixed cutoff function χ_α to define $\widetilde{v}_{\alpha,\epsilon} = \chi_\alpha v_\alpha + (1-\chi_\alpha)v_{\alpha,\epsilon}$ so that the graph of this new function agrees with the original surface Σ for $r \leq R_0 + 1$ and matches up smoothly with $E_\alpha(\epsilon)$ outside this annulus. This defines $\Sigma(\epsilon)$. Denoting its mean curvature function by $H(\epsilon)$, then clearly $H(\epsilon)$ vanishes outside the union of these annuli, hence $H(\epsilon) \to 0$ in $e^{\kappa R}\mathcal{C}^{2,\mu}(\Sigma(\epsilon))$ as $|\epsilon| \to 0$.

The remainder of the proof follows the corresponding arguments in [13] and [8] essentially verbatim. When Σ is horizontally nondegenerate, the implicit function theorem produces an analytic function $\epsilon \mapsto u_\epsilon$ such that the normal graph of u_ϵ over $\Sigma(\epsilon)$ is minimal. This is a real analytic coordinate chart in $\mathcal{M}_k$ around Σ. If Σ is horizontally degenerate, then we can apply a Lyapunov–Schmidt reduction argument to show that there exists a neighbourhood $\mathcal{U}$ of Σ in some fixed finite dimensional real analytic submanifold Y in the space of all surfaces (with a fixed weighted Hölder regularity) and a real analytic function $F\colon \mathcal{U} \to \mathbb{R}$ such that $\mathcal{M}_k \cap \mathcal{U} = F^{-1}(0) \cap \mathcal{U}$. □

References

[1] **M T Anderson**, *Complete minimal varieties in hyperbolic space*, Invent. Math. 69 (1982) 477–494 MR679768

[2] **P Collin**, **H Rosenberg**, *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math. 172 (2010) 1879–1906 MR2726102

[3] **B Daniel**, *Isometric immersions into* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$ *and applications to minimal surfaces*, Trans. Amer. Math. Soc. 361 (2009) 6255–6282 MR2538594

[4] **D Gilbarg**, **N S Trudinger**, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, Berlin (2001) MR1814364

[5] **L Hauswirth**, **B Nelli**, **R Sa Earp**, **E Toubiana**, *Minimal ends in* $\mathbb{H}^2 \times \mathbb{R}$ *with finite total curvature and a Schoen type theorem* (2012) arXiv:1111.0851v2

[6] **L Hauswirth**, **R Sa Earp**, **E Toubiana**, *Associate and conjugate minimal immersions in* $M \times \mathbf{R}$, Tohoku Math. J. 60 (2008) 267–286 MR2428864

[7] **H Jenkins**, **J Serrin**, *The Dirichlet problem for the minimal surface equation, with infinite data*, Bull. Amer. Math. Soc. 72 (1966) 102–106 MR0185522

[8] **R Kusner**, **R Mazzeo**, **D Pollack**, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. 6 (1996) 120–137 MR1371233

[9] **N N Lebedev**, *Special functions and their applications*, Dover Publications, New York (1972) MR0350075

[10] **J M Manzano**, **J Pérez**, **M M Rodríguez**, *Parabolic stable surfaces with constant mean curvature*, Calc. Var. Partial Differential Equations 42 (2011) 137–152 MR2819632

[11] **L Mazet**, **M M Rodríguez**, **H Rosenberg**, *The Dirichlet problem for the minimal surface equation, with possible infinite boundary data, over domains in a Riemannian surface*, Proc. Lond. Math. Soc. 102 (2011) 985–1023 MR2806098

[12] **R Mazzeo**, **F Pacard**, **D Pollack**, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math. 536 (2001) 115–165 MR1837428

[13] **R Mazzeo**, **D Pollack**, **K Uhlenbeck**, *Moduli spaces of singular Yamabe metrics*, J. Amer. Math. Soc. 9 (1996) 303–344 MR1356375

[14] **R Mazzeo**, **M Sáez**, *Multiple-layer solutions to the Allen–Cahn equation on hyperbolic space*, to appear in Proc. Amer. Math. Soc.

[15] **R Mazzeo**, **A Vasy**, *Resolvents and Martin boundaries of product spaces*, Geom. Funct. Anal. 12 (2002) 1018–1079 MR1937834

[16] **W H Meeks III**, **J Pérez**, **A Ros**, *Bounds on the topology and index of classical minimal surfaces* Available at `http://www.ugr.es/local/jperez/papers/papers.htm`

[17] **R B Melrose**, *Geometric scattering theory*, Stanford Lectures, Cambridge Univ. Press (1995) MR1350074

[18] **F Morabito**, *A Costa–Hoffman–Meeks type surface in $\mathbb{H}^2 \times \mathbb{R}$*, Trans. Amer. Math. Soc. 363 (2011) 1–36 MR2719669

[19] **F Morabito**, **M M Rodríguez**, *Saddle towers and minimal k–noids in $\mathbb{H}^2 \times \mathbb{R}$*, J. Inst. Math. Jussieu 11 (2012) 333–349 MR2905307

[20] **M Murata**, *Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbf{R}^n$*, Duke Math. J. 53 (1986) 869–943 MR874676

[21] **B Nelli**, **H Rosenberg**, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. Braz. Math. Soc. 33 (2002) 263–292 MR1940353

[22] **J Pérez**, **A Ros**, *The space of complete minimal surfaces with finite total curvature as Lagrangian submanifold*, Trans. Amer. Math. Soc. 351 (1999) 3935–3952 MR1487630

[23] **M A Pinsky**, *Large deviations for diffusion processes*, from: "Stochastic analysis", (A Friedman, M A Pinsky, editors), Academic Press, New York (1978) 271–283 MR517246

[24] **J Pyo**, *New complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Ann. Global Anal. Geom. 40 (2011) 167–176 MR2811623

[25] **M A Shubin**, *Pseudodifferential operators and spectral theory*, 2nd edition, Springer, Berlin (2001) MR1852334

[26] **D Sullivan**, *Related aspects of positivity in Riemannian geometry*, J. Differential Geom. 25 (1987) 327–351 MR882827

FM, MMR: *Departamento de Geometría y Topología*
Universidad de Granada, 18071 Granada, Spain

RM: *Department of Mathematics, Stanford University*
Stanford, CA 94305, USA

fmartin@ugr.es, mazzeo@math.stanford.edu, magdarp@ugr.es

http://www.ugr.es/local/fmartin, http://math.stanford.edu/~mazzeo/,
http://www.ugr.es/local/magdarp

Proposed: Tobias H Colding Received: 30 August 2012
Seconded: David Gabai, Ronald Stern Revised: 6 May 2013