Geometry & Topology 18 (2014) 669–716

Classification of Hamiltonian torus actions with two-dimensional quotients

YAEL KARSHON
SUSAN TOLMAN

We construct all possible Hamiltonian torus actions for which all the nonempty reduced spaces are two-dimensional (and not single points) and the manifold is connected and compact, or, more generally, the moment map is proper as a map to a convex set. This construction completes the classification of tall complexity-one spaces.

53D35; 53D20, 57S15, 52B20, 14M25

1 Introduction

Fix a torus $T \cong (S^1)^{\dim T}$ with Lie algebra $\mathfrak{t}$ and dual space $\mathfrak{t}^*$. Let T act on a symplectic manifold (M, ω) with moment map $\Phi\colon M \to \mathfrak{t}^*$, so that

$$\iota(\zeta_M)\omega = -d\langle \Phi, \zeta\rangle \quad \text{for all } \zeta \in \mathfrak{t}, \tag{1.1}$$

where ζ_M is the vector field on M induced by ζ. Assume that the T–action is faithful (effective) on each connected component of M. We call (M, ω, Φ) a *Hamiltonian T–manifold*. An *isomorphism* between two Hamiltonian T–manifolds is an equivariant symplectomorphism that respects the moment maps. The *complexity* of (M, ω, Φ) is the difference $\frac{1}{2}\dim M - \dim T$; it is half the dimension of the reduced space $\Phi^{-1}(\alpha)/T$ at a regular value α in $\Phi(M)$. Assume that (M, ω, Φ) has complexity one; it is *tall* if every reduced space $\Phi^{-1}(\alpha)/T$ is a two-dimensional topological manifold. If M is connected, $\mathcal{T}$ is a convex open subset of $\mathfrak{t}^*$ containing $\Phi(M)$ and the map $\Phi\colon M \to \mathcal{T}$ is proper, then we call $(M, \omega, \Phi, \mathcal{T})$ a *complexity-one space*. For example, if M is compact and connected then $(M, \omega, \Phi, \mathfrak{t}^*)$ is a complexity-one space, which is tall exactly if the preimage of each vertex of the moment polytope $\Phi(M)$ is a fixed surface; see Corollary 2.6.

In this paper we complete our classification of tall complexity-one spaces of arbitrary dimension. More precisely, in a previous paper [21] we defined an invariant of a tall complexity-one space called the *painting*, which subsumes two other invariants:

Published: 20 March 2014 DOI: 10.2140/gt.2014.18.669

the *genus* and the *skeleton* (see the paragraph above Remark 1.3). We proved that these invariants, together with the Duistermaat–Heckman measure, determine the tall complexity-one space up to isomorphism. In this paper we give a necessary and sufficient condition for a measure and a painting to arise from a tall complexity-one space.

Symplectic toric manifolds (see the paragraph below Remark 1.3) serve as extremely important examples in symplectic geometry, illuminating many different aspects of the field. We hope that the existence theorems of this paper will enable complexity-one spaces to serve a similar role. These spaces are more flexible than symplectic toric manifolds. For example, in a paper in progress, the second author uses the methods of [46] to show that many complexity-one spaces do not admit equivariant Kähler structures. Additionally, she constructs an infinite family of symplectic forms in a fixed cohomology class which are equivariantly deformation equivalent but are not equivariantly isotopic.

Symplectic toric manifolds are classified by their moment images (by Delzant [10]); see the first author and Lerman [19] for the noncompact case. Compact connected nonabelian complexity-zero actions are determined by their moment image and the principal isotropy subgroup; this is the Delzant conjecture, recently proved by Knop [26] and Losev [33], following earlier work by Iglésias [17] and Woodward [49].

The simplest complexity-one spaces, compact connected symplectic 2–manifolds with no group action, are classified by their genus and total area; see Moser [37]. Four-dimensional compact connected complexity-one spaces are classified by the first author in [18] (also Ahara and Hattori [1] and Audin [6; 7]); see Example 1.7. Similar techniques apply to complexity-one nonabelian group actions on six-manifolds (see Chiang [9]) and to two-torus actions on five-dimensional K–contact manifolds (see Nozawa [39]). From the algebraic geometric point of view, complexity-one actions (of possibly nonabelian groups) have been studied by Kempf, Knudsen, Mumford and Saint-Donat [24, Chapter IV], Timashëv [44; 45], and Altmann and Hausen [2], Altmann, Hausen and Süss [3], Altmann and Petersen [4] and Vollmert [48]. Moreover, a complexity-one symplectic torus action on a compact symplectic manifold is Hamiltonian if and only if it has a fixed point; see Kim [25].

Work on Hamiltonian circle actions on six-manifolds, which have complexity two, appeared in Li [30; 31], McDuff [35], the second author [47], González [12] and Li and the second author [32]. Finally, classification in arbitrary complexity is feasible for "centered spaces;" see [10, Section 1], the authors [22, Section 2], and the first author and Ziltener [23].

We begin by recalling the invariants of complexity-one spaces.

Let (M,ω,Φ) be a $2n$–dimensional Hamiltonian T–manifold. Recall that the Liouville measure on M is given by integrating the volume form $\omega^n/n!$ with respect to the symplectic orientation and that the *Duistermaat–Heckman measure* is the pushforward of the Liouville measure by the moment map. The isotropy representation at x is the linear representation of the stabilizer $\{\lambda \in T \mid \lambda \cdot x = x\}$ on the tangent space $T_x M$. Points in the same orbit have the same stabilizer, and their isotropy representations are linearly symplectically isomorphic; this isomorphism class is the *isotropy representation* of the orbit.

Now assume that (M,ω,Φ) is a tall complexity-one Hamiltonian T–manifold. An orbit is *exceptional* if every nearby orbit in the same moment fiber $\Phi^{-1}(\alpha)$ has a strictly smaller stabilizer. The *skeleton* of M is the set $M_{\text{exc}} \subset M/T$ of exceptional orbits in M/T, with the subset topology, with each orbit labeled by its isotropy representation, and with the map $\bar{\Phi}\colon M_{\text{exc}} \to \mathfrak{t}^*$ that is induced from the moment map. Let M'_{exc} be the skeleton of another tall complexity-one space; an *isomorphism* from M_{exc} to M'_{exc} is a homeomorphism that respects the maps to $\mathfrak{t}^*$ and sends each orbit to an orbit with the same (stabilizer and) isotropy representation.

The next proposition is a slight modification of [21, Proposition 2.2]; see Remark 1.9.

1.2 Proposition *Let $(M,\omega,\Phi,\mathcal{T})$ be a tall complexity-one space. There exist a closed oriented surface Σ and a map $f\colon M/T \to \Sigma$ so that*

$$(\bar{\Phi}, f)\colon M/T \to (\text{image}\,\Phi) \times \Sigma$$

is a homeomorphism and the restriction $f\colon \Phi^{-1}(\alpha)/T \to \Sigma$ is orientation preserving for each $\alpha \in \text{image}\,\Phi$. Here, $\bar{\Phi}$ is induced by the moment map. Given two such maps f and f', there exists an orientation preserving homeomorphism $\xi\colon \Sigma' \to \Sigma$ so that f is homotopic to $\xi \circ f'$ through maps which induce homeomorphisms $M/T \to (\text{image}\,\Phi) \times \Sigma$.

The *genus* of a tall complexity-one space $(M,\omega,\Phi,\mathcal{T})$ is the genus of the surface $\Phi^{-1}(\alpha)/T$ for $\alpha \in \text{image}\,\Phi$. By Proposition 1.2, it is well defined. A *painting* is a map f from M_{exc} to a closed oriented surface Σ such that the map

$$(\bar{\Phi}, f)\colon M_{\text{exc}} \to \mathcal{T} \times \Sigma$$

is one-to-one, where M_{exc} is the set of exceptional orbits. Two paintings, $f\colon M_{\text{exc}} \to \Sigma$ and $f'\colon M'_{\text{exc}} \to \Sigma'$, are *equivalent* if there exists an isomorphism $i\colon M'_{\text{exc}} \to M_{\text{exc}}$ and an orientation preserving homeomorphism $\xi\colon \Sigma' \to \Sigma$ such that $f \circ i\colon M'_{\text{exc}} \to \Sigma$ and $\xi \circ f'\colon M'_{\text{exc}} \to \Sigma$ are homotopic *through paintings*. A painting $f\colon M_{\text{exc}} \to \Sigma$ is *trivial* if it is equivalent to a painting that is constant on each component of M_{exc}.

Proposition 1.2 implies that there is a well-defined equivalence class of paintings associated to every tall complexity-one space; just restrict f to M_{exc}. By a slight abuse of terminology, we refer to any painting in this equivalence class as *the* painting associated to the complexity-one space.

1.3 Remark The notion of a painting is simplest when $\bar{\Phi}\colon M_{\mathrm{exc}} \to \mathfrak{t}^*$ is one-to-one, as in Examples 1.11 and 1.12. In this case, every map from M_{exc} to a closed oriented surface Σ is a painting, and two paintings $f\colon M_{\mathrm{exc}} \to \Sigma$ and $f'\colon M_{\mathrm{exc}} \to \Sigma'$ are equivalent exactly if there exists an orientation preserving homeomorphism $\xi\colon \Sigma' \to \Sigma$ such that f and $\xi \circ f'$ are homotopic.

In the next two examples, we construct complexity-one spaces out of *symplectic toric manifolds*, ie, compact connected complexity-zero Hamiltonian $(S^1)^n$–manifolds. The moment image of a symplectic toric manifold is a Delzant polytope; see Remark 1.18. In fact, every Delzant polytope occurs as the moment image of a symplectic toric manifold, and this manifold is unique up to equivariant symplectomorphism [10].

1.4 Example Let (M, ω, ψ) be a symplectic toric manifold with moment image $\Delta = \psi(M) \subset \mathbb{R}^n$. The moment map induces a homeomorphism $\bar{\psi}\colon M/(S^1)^n \to \Delta$. Moreover, given $x \in \Delta$, let F_x be the smallest face of Δ containing x. The stabilizer of the preimage $\psi^{-1}(x)$ is the connected subgroup $H_x \subset (S^1)^n$ with Lie algebra

$$\mathfrak{h}_x = \{\zeta \in \mathbb{R}^n \mid \langle \zeta, y - z\rangle = 0 \text{ for all } y, z \in F_x\}.$$

Let $\Phi\colon M \to \mathbb{R}^{n-1}$ be the composition of the moment map ψ with the projection $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then $(M, \omega, \Phi, \mathbb{R}^{n-1})$ is a complexity-one space for the subtorus $(S^1)^{n-1} \subset (S^1)^n$. It is tall exactly if

$$\Delta_{\mathrm{ceiling}} \cap \Delta_{\mathrm{floor}} = \varnothing, \tag{1.5}$$

where

$$\Delta_{\mathrm{ceiling}} = \{x \in \Delta \mid x_n \geq x'_n \text{ for all } x' \in \pi^{-1}(\pi(x))\},$$
$$\Delta_{\mathrm{floor}} = \{x \in \Delta \mid x_n \leq x'_n \text{ for all } x' \in \pi^{-1}(\pi(x))\}.$$

Assume that (1.5) holds.

For $x \in \Delta$ such that $\pi(x)$ is in the interior of $\pi(\Delta)$, the preimage $\psi^{-1}(x)$ is exceptional exactly if its $(S^1)^{n-1}$ stabilizer, $H_x \cap (S^1)^{n-1}$, is nontrivial. (In particular, $\psi^{-1}(x)$ is always exceptional if $\dim F_x < n-1$, and it is never exceptional if x is not in $\Delta_{\text{ceiling}} \cup \Delta_{\text{floor}}$.)[1] The skeleton M_{exc} is the closure of the set of such orbits. The Duistermaat–Heckman measure of (M, ω, Φ) is the pushforward to $\mathbb{R}^{n-1}$ of Lebesgue measure on Δ, and the genus of (M, ω, Φ) is zero. Finally, let $M_{\text{ceiling}} = \psi^{-1}(\Delta_{\text{ceiling}})/T$ and $M_{\text{floor}} = \psi^{-1}(\Delta_{\text{floor}})/T$. Since $\bar{\Phi}\colon M_{\text{ceiling}} \to \mathcal{T}$ and $\bar{\Phi}\colon M_{\text{floor}} \to \mathcal{T}$ are homeomorphisms onto $\pi(\Delta)$, and since the painting $f\colon M_{\text{exc}} \to S^2$ associated to M can be extended to a map $\tilde{f}\colon M_{\text{ceiling}} \cup M_{\text{floor}} \to S^2$ such that $(\bar{\Phi}, \tilde{f})\colon M_{\text{ceiling}} \cup M_{\text{floor}} \to \mathcal{T} \times S^2$ is one-to-one, the painting f is equivalent to a painting with at most two values.

Thus, if a compact tall complexity-one space $(M, \omega, \Phi, \mathcal{T})$ can be obtained from a symplectic toric manifold in this way, then we have that its genus is zero and its painting $f\colon M_{\text{exc}} \to \Sigma = S^2$ can be chosen to take at most two values. When $\dim M = 4$, the converse is also true; this follows from [18, Proposition 5.16].

1.6 Example Let $P \to \Sigma$ be a principal $(S^1)^n$ bundle over a closed oriented surface Σ of genus g with first Chern class $c_1(P) \in H^2(\Sigma, \mathbb{Z}^n)$, and let N be a symplectic toric manifold with moment map $\psi\colon N \to \mathbb{R}^n$. There exists a T–invariant symplectic form ω on $M := P \times_T N$ whose restriction to the fibers is the symplectic form on N; the moment map $\Phi\colon M \to \mathbb{R}^n$ is given by $\Phi([p, n]) = \psi(n)$; see Guillemin, Lerman and Sternberg [13]. In this case, $(M, \omega, \Phi, \mathbb{R}^n)$ is a tall complexity space of genus g, $M_{\text{exc}} = \varnothing$, and the Duistermaat–Heckman measure of (M, ω, Φ) is Lebesgue measure on Δ times an affine function with slope $c_1(P)$.

1.7 Example In [18], the first author showed that a Hamiltonian circle action on a compact, connected symplectic four-manifold M is determined up to isomorphism by the following labelled graph: the vertices correspond to connected components of the fixed point set; each vertex is labelled by its moment map value, and – if the corresponding component is a two-dimensional fixed surface – the genus and area of that surface. The edges correspond to two spheres in M that the circle rotates at speed $k > 1$; such an edge is labelled by the integer k.

The space M is tall exactly if the minimum and maximum of the moment map are attained along two-dimensional fixed surfaces; see Corollary 2.6. It is fairly straightforward to check that in this case the invariants that we describe in this paper

[1] More generally, $H_x \cap (S^1)^{n-1}$ is trivial exactly if $\pi(\mathbb{Z}^n \cap TF_x) = \mathbb{Z}^{n-1}$, where $TF_x = \mathfrak{h}_x^\circ = \{\lambda(y - z) \mid \lambda \in \mathbb{R} \text{ and } y, z \in F_x\}$. To see this, note that $\pi(\mathbb{Z}^n \cap TF_x) = \mathbb{Z}^{n-1}$ exactly if every character of $(S^1)^{n-1}$ is the restriction of a character of $(S^1)^n$ that vanishes on H_x. If $H_x \cap (S^1)^{n-1}$ is not trivial, then there are characters of $(S^1)^{n-1}$ that do not vanish on $H_x \cap (S^1)^{n-1}$, and these can not be the restrictions of characters that vanish on H_x. On the other hand, if $H_x \cap (S^1)^{n-1} = \{1\}$, then either $H_x = \{1\}$ or $(S^1)^n \simeq H_x \times (S^1)^{n-1}$. In either case, every character of $(S^1)^{n-1}$ is the restriction of a character of $(S^1)^n$ that vanishes on H_x.

determine, and are determined by, the labelled graph described above. In particular, the moment map identifies each component of the skeleton with an interval, and so every painting is trivial.

By [21, Theorem 1], the invariants that we have defined completely determine the tall complexity-one space.

1.8 Theorem (Global uniqueness) *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be tall complexity-one spaces. They are isomorphic if and only if they have the same moment image[2] and Duistermaat–Heckman measure, the same genus and equivalent paintings.*

1.9 Remark In our definition of "equivalent paintings," we require the homeomorphism ξ to be orientation preserving. This requirement, which is necessary for Theorem 1.8 to be true, was mistakenly omitted from [21, page 29]. Similarly, Definition 1.16 of the current paper is the correction to [21, Definition 18.1]. Finally, both [21, Proposition 2.2] and its smooth analogue, [21, Lemma 18.4], should state that $f|_{\Phi^{-1}(\alpha)/T}$ and ξ are orientation preserving. The maps that are obtained in the proofs of these propositions in [21] do satisfy this additional requirement.

Before stating our most general existence theorem, Theorem 3, we give two existence theorems – Theorems 1 and 2 – that are easier to state and simpler to apply but are sufficient for constructing interesting examples. These two theorems are actually consequences of the most general theorem; all three are proved in Section 10; cf Remark 1.10.

The simplest existence theorem

We now state our first existence theorem, which shows that – given a tall complexity-one space – we can find another tall complexity-one space with an isomorphic skeleton (and the same Duistermaat–Heckman measure) but a different genus and painting.

Theorem 1 *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity-one space. Let Σ be a closed oriented surface, and let $f\colon M_{\text{exc}} \to \Sigma$ be any painting. Then there exists a tall complexity-one space $(M', \omega', \Phi', \mathcal{T})$ with the same moment image and Duistermaat–Heckman measure whose painting is equivalent to f.*

[2] Since the moment image is the support of the Duistermaat–Heckman measure, we could omit the condition that the spaces have the same moment image. Nevertheless, we will sometimes include this condition for emphasis.

1.10 Remark The special case of Theorem 1 where the genus of Σ is equal to the genus of M is easier to prove than the general case; see Theorem 6.1.

1.11 Example Let (M,ω,ψ) be a six-dimensional symplectic toric manifold with moment image

$$\Delta = \{(x,y,z) \in [-3,3]\times[-2,2]\times[1,4] \mid |x| \le z, |y| \le z\}.$$

We let $\Phi\colon M \to \mathbb{R}^2$ be the composition of the map $\psi\colon M \to \mathbb{R}^3$ with the projection $(x,y,z) \mapsto (x,y)$. Then $(M,\omega,\Phi,\mathbb{R}^2)$ is a tall complexity-one space, and Φ induces a homeomorphism from the skeleton M_{exc} to the set

$$\Phi(M_{\mathrm{exc}}) = \{(x,y) \in \mathbb{R}^2 \mid |x| \le |y| = 1, \text{ or } |y| \le |x| = 1 \text{ or } 1 \le |x| = |y| \le 2\};$$

thus, M_{exc} is homotopy equivalent to S^1; see Example 1.4.

Fix a closed oriented surface Σ. Let $[S^1,\Sigma]$ denote the set of homotopy classes of loops in Σ. By Remark 1.3, there is a one-to-one correspondence between equivalence classes of paintings $M_{\mathrm{exc}} \to \Sigma$ and the quotient of $[S^1,\Sigma]$ by the action of the group $\mathrm{Aut}(\Sigma)$ of orientation preserving homeomorphisms of Σ. If Σ has genus zero, then since Σ is simply connected any two paintings are equivalent. In contrast, if Σ has positive genus, then the quotient of $[S^1,\Sigma]$ by $\mathrm{Aut}(\Sigma)$ is infinite. For example, if Σ has genus one, then this quotient is naturally isomorphic to the set of nonnegative integers.

Therefore, if $g=0$ then Theorem 1.8 implies that every tall complexity-one space of genus g whose skeleton is isomorphic to M_{exc} and whose Duistermaat–Heckman measure is equal to that of (M,ω,Φ) is isomorphic to (M,ω,Φ). In contrast, if $g>0$ then Theorems 1.8 and 1 imply that there exist infinitely many nonisomorphic tall complexity-one spaces with these properties.

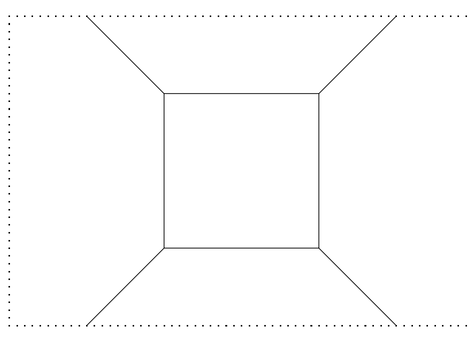

Figure 1: Moment image and skeleton for Example 1.11

1.12 Example Fix an integer $n > 1$. Let (M, ω, ψ) be a $(2n+4)$–dimensional symplectic toric manifold with moment image

$$\Delta' = \{(x, y_1, \ldots, y_n, z) \in [-3,3] \times [-2,2]^n \times [1,4] \\ \mid |x| \le z \text{ and } |y_i| \le z \text{ for all } i = 1, \ldots, n\}.$$

Composing the moment map ψ with the projection $(x, y_1, \ldots, y_n, z) \mapsto (x, y_1, \ldots, y_n)$, we obtain a tall complexity-one space $(M, \omega, \Phi, \mathbb{R}^{n+1})$ such that M_{exc} is homotopy equivalent to S^n and $\overline{\Phi}\colon M_{\mathrm{exc}} \to \mathbb{R}^{n+1}$ is one-to-one. Moreover, the group of orientation preserving homeomorphisms acts trivially on the set $[S^n, \Sigma]$ of homotopy classes of maps from S^n to Σ if Σ is an oriented surface of genus 0, while $[S^n, \Sigma]$ itself is trivial if Σ has positive genus. Therefore, if $g > 0$ then Theorem 1.8 implies that every complexity-one space of genus g whose skeleton is isomorphic to M_{exc} and whose Duistermaat–Heckman measure is equal to that of (M, ω, Φ) is isomorphic to (M, ω, Φ). In contrast, if $g = 0$ then Theorems 1.8 and 1 give a bijection between the set of isomorphism classes of tall complexity-one spaces with these properties and the set $[S^n, S^2]$. Thus, there are infinitely many nonisomorphic such spaces if $n = 2$ or $n = 3$.

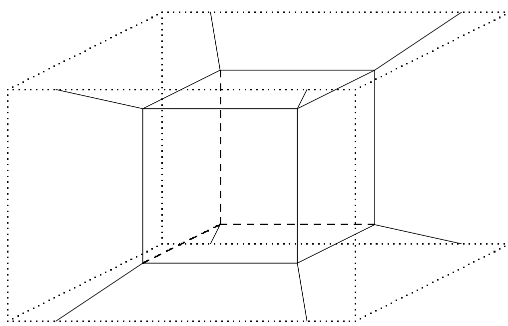

Figure 2: Moment image and skeleton for Example 1.12

The intermediate existence theorem

Our second existence theorem, Theorem 2, allows us to construct complexity-one spaces with prescribed painting and moment image, even when the skeleton does not a priori come from a complexity-one space. To state this theorem, we need an abstract notion of "skeleton." To apply this theorem, one must check that the skeleton and moment image satisfy certain conditions. These conditions are automatically satisfied whenever the skeleton and moment image can be obtained from complexity-one spaces over sufficiently small open sets in $\mathfrak{t}^*$; see Lemma 7.3. This allows us to construct new examples by performing surgery that attaches pieces of *different* complexity-one manifolds. Such surgeries were carried out in [46] and Morton [36]; this theorem gives a systematic way to perform such surgeries.

1.13 Definition A *tall skeleton* over an open subset $\mathcal{T}$ of $\mathfrak{t}^*$ is a topological space S whose points are labeled by (equivalence classes of) representations of subgroups of T, together with a proper map $\pi\colon S \to \mathcal{T}$. This data must be locally modeled on the set of exceptional orbits of a tall complexity-one space in the following sense. For each point $s \in S$, there exists a tall complexity-one Hamiltonian T–manifold (M, ω, Φ) with exceptional orbits $M_{\text{exc}} \subset M/T$, and a homeomorphism Ψ from a neighbourhood of s to an open subset of M_{exc} that respects the labels and such that $\overline{\Phi} \circ \Psi = \pi$, where $\overline{\Phi}\colon M_{\text{exc}} \to \mathfrak{t}^*$ is induced from the moment map. An *isomorphism* between tall skeletons (S', π') and (S, π) is a homeomorphism $i\colon S' \to S$ that sends each point to a point with the same isotropy representation and such that $\pi' = \pi \circ i$; cf [21, page 72].

1.14 Remark In [21, Definition 16.1] we called this notion "skeleton." Here we added the adjective "tall" in order to later allow for skeletons that are not tall.

1.15 Example The skeleton of a tall complexity-one space, as was defined before Proposition 1.2, is a tall skeleton in the sense of Definition 1.13. This follows immediately from the fact that the map $\overline{\Phi}\colon M_{\text{exc}} \to \mathcal{T}$ is proper; see Lemma 7.1.

1.16 Definition Let (S, π) be a tall skeleton over an open set $\mathcal{T} \subset \mathfrak{t}^*$ and let Σ be a closed oriented surface. A *painting* is a map $f\colon S \to \Sigma$ such that the map $(\pi, f)\colon S \to \mathcal{T} \times \Sigma$ is one-to-one. Paintings $f\colon S \to \Sigma$ and $f'\colon S' \to \Sigma'$ are *equivalent* if there exists an isomorphism $i\colon S' \to S$ and an orientation preserving homeomorphism $\xi\colon \Sigma' \to \Sigma$ such that $f \circ i\colon S' \to \Sigma$ and $\xi \circ f'\colon S' \to \Sigma$ are homotopic through paintings.

The notions of painting and of equivalence of paintings given in Definition 1.16 are consistent with our earlier definitions, which only applied to the special case $(S, \pi) = (M_{\text{exc}}, \overline{\Phi})$.

Let ℓ denote the integral lattice in $\mathfrak{t}$ and ℓ^* the weight lattice in $\mathfrak{t}^*$. Thus, we have $\ell = \ker(\exp\colon \mathfrak{t} \to T)$ and $\ell^* \cong \operatorname{Hom}(T, S^1)$. Here, the Lie algebra of S^1 is identified with $\mathbb{R}$ by setting the exponential map $\mathbb{R} \to S^1$ to be $t \mapsto e^{2\pi i t}$. Let $\mathbb{R}_+$ denote the set of nonnegative numbers.

1.17 Definition A subset $C \subset \mathfrak{t}^*$ is a *Delzant cone*[3] at $\alpha \in \mathfrak{t}^*$ if there exists an integer $0 \le k \le n$ and a linear isomorphism $A\colon \mathbb{R}^n \to \mathfrak{t}^*$ that sends $\mathbb{Z}^n$ onto the weight lattice ℓ^*, such that

$$C = \alpha + A(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}).$$

Let $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$. A subset $\Delta \subset \mathcal{T}$ is a *Delzant subset* if it is closed in $\mathcal{T}$ and if for every point $\alpha \in \Delta$ there exists a neighbourhood $U \subset \mathcal{T}$ and a Delzant cone C at α such that $\Delta \cap U = C \cap U$.

1.18 Remark A compact convex set $\Delta \subset \mathfrak{t}^*$ is a Delzant subset exactly if it is a *Delzant polytope*, ie, a convex polytope such that at each vertex the edge vectors are generated by a basis to the lattice.

1.19 Remark If Δ is a Delzant subset of a *convex* open subset $\mathcal{T} \subset \mathfrak{t}^*$ then, by the Tietze–Nakajima theorem [43; 38], Δ is convex exactly if it is connected; see Bjorndahl and the first author [8].

1.20 Definition The *moment cone* corresponding to a point s in a tall skeleton (S, π) is the cone

$$C_s := \pi(s) + (i_H^*)^{-1}(\text{image}\,\Phi_s) \text{ in } \mathfrak{t}^*,$$

where the label associated to s is a linear symplectic representation of the subgroup H of T with quadratic moment map Φ_s, and where $i_H^*\colon \mathfrak{t}^* \to \mathfrak{h}^*$ is the natural projection map. It is straightforward to check that C_s is the moment image of the complexity-one model corresponding to s; see Definition 1.23.

1.21 Definition Let $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$. A Delzant subset Δ of $\mathcal{T}$ and a tall skeleton (S, π) over $\mathcal{T}$ are *compatible* if for every point $s \in S$ there exists a neighbourhood U of $\pi(s)$ in $\mathcal{T}$ such that $U \cap \Delta = U \cap C_s$, where C_s is the moment cone corresponding to s.

Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity-one space. Then its moment map image is a convex Delzant subset of $\mathcal{T}$ that is compatible with the skeleton $(M_{\text{exc}}, \overline{\Phi})$; see Lemma 7.3. Our next theorem shows that this compatibility condition is also sufficient for a subset of $\mathcal{T}$ and a painting to arise from a complexity-one space.

Theorem 2 *Let (S, π) be a tall skeleton over a convex open subset $\mathcal{T} \subset \mathfrak{t}^*$. Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π). Let Σ be a closed oriented surface, and let $f\colon S \to \Sigma$ be a painting. Then there exists a tall complexity-one space $(M, \omega, \Phi, \mathcal{T})$ with moment map image Δ whose associated painting is equivalent to f.*

[3] Such a set is also called a "unimodular cone."

The most general existence theorem

Our final existence theorem, Theorem 3, provides a complete list of all the possible values of the invariants of tall complexity-one spaces. Together with Theorem 1.8, this gives a complete classification of tall complexity-one spaces.

The *Duistermaat–Heckman function* of a Hamiltonian T–manifold is a real-valued function on the moment image whose product with Lebesgue measure is equal to the Duistermaat–Heckman measure. If such a function exists, then it is almost unique; any two such functions are equal almost everywhere. When we say that the Duistermaat–Heckman function of a Hamiltonian T–manifold has some property (eg, continuity), we mean that this holds after possibly changing the function on a set of measure zero. Here, we normalize Lebesgue measure on $\mathfrak{t}^*$ such that the volume of the quotient $\mathfrak{t}^*/\ell^*$ is one.

1.22 Remark Some authors define the Duistermaat–Heckman function at $\alpha \in \mathfrak{t}^*$ to be the symplectic volume of the reduced space $\Phi^{-1}(\alpha)/T$.

A function $\rho\colon \mathfrak{t}^* \to \mathbb{R}$ is *integral affine* if it has the form

$$\rho(x) = \langle x, A\rangle + B,$$

where A is an element of the integral lattice $\ell \subset \mathfrak{t}$, where $B \in \mathbb{R}$, and where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. The Duistermaat–Heckman theorem implies that the Duistermaat–Heckman function of a complexity-one space with no exceptional orbits is integral affine.

Once and for all, fix an inner product on $\mathfrak{t}$. Let a closed subgroup $H \subset T$ act on $\mathbb{C}^n$ as a subgroup of $(S^1)^n$ with quadratic moment map $\Phi_H\colon \mathbb{C}^n \to \mathfrak{h}^*$. Let $\mathfrak{h}^0 \subset \mathfrak{t}^*$ be the annihilator of the Lie algebra $\mathfrak{h}$, and consider the model

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0,$$

where $[ta, z, \nu] = [t, az, \nu]$ for all $(t, z, \nu) \in T \times \mathbb{C}^n \times \mathfrak{h}^0$ and $a \in H$. There exists a T invariant symplectic form on Y with moment map

$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu,$$

where $\alpha \in \mathfrak{t}^*$ and where we use the inner product to embed $\mathfrak{h}^*$ in $\mathfrak{t}^*$. (Explicitly, this symplectic form is obtained by identifying Y with the symplectic quotient of $T^*(T) \times \mathbb{C}^n$ by H.) The isotropy representation of the *central orbit* $T \cdot [1, 0, 0] = \{[t, 0, 0] \mid t \in T\}$ determines the model up to permutation of the coordinates in $\mathbb{C}^n$. If $\dim T = \frac{1}{2} \dim Y - 1$, or, equivalently, $n = h + 1$ where $h = \dim H$, we call the space Y a *complexity-one model*.

1.23 Definition Given a point s in a tall skeleton S, the *corresponding model* is the model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ such that s is labeled by the isotropy representation of $T \cdot [1,0,0]$ in Y.

Such a model exists and is unique up to permutation of the coordinates in $\mathbb{C}^{h+1}$. Moreover, by Corollary 2.4, the corresponding model is always tall.

Let Y be a tall complexity-one model. In Section 8 we define the *Duistermaat–Heckman functions for truncations of the model*. (In fact, such functions are the Duistermaat–Heckman functions of compact spaces that are obtained from Y by extending the action to a toric action, choosing a subcircle that is complementary to the original action, and taking a symplectic cut with respect to this circle.)

1.24 Definition Let (S, π) be a tall skeleton over an open subset $\mathcal{T}$ of $\mathfrak{t}^*$. Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π). Fix a point $\alpha \in \Delta$. A function $\rho\colon \Delta \to \mathbb{R}_{>0}$ is *compatible with the skeleton* (S, π) *at a point* $\alpha \in \Delta$ if there exists for each $s \in \pi^{-1}(\alpha)$ a Duistermaat–Heckman function ρ_s for a truncation of the tall complexity-one model associated to s such that the difference

$$\rho - \sum_{s \in \pi^{-1}(\alpha)} \rho_s \tag{1.25}$$

is integral affine on some neighbourhood of α in Δ. (In particular, if $\pi^{-1}(\alpha)$ is empty, then the condition is that ρ itself be integral affine near α.) The function ρ is *compatible with the skeleton* (S, π) if it is compatible with (S, π) at every $\alpha \in \Delta$.

1.26 Remark The above notion of "compatible" is in fact well defined; moreover, the difference between any two compatible functions is integral affine near α. To see this, let (S, π) be a tall skeleton over $\mathcal{T}$; fix $\alpha \in \mathcal{T}$. The preimage $\pi^{-1}(\alpha) \subset S$ is finite; see Corollary 2.5. Thus, the summation in (1.25) is finite. By Corollary 8.23, for each $s \in \pi^{-1}(\alpha)$, there exists a Duistermaat–Heckman function ρ_s for a truncation of the tall complexity-one model Y_s associated to s; moreover, ρ_s is defined on a neighborhood of α in image Φ_{Y_s}. By Definitions 1.20 and 1.21, the moment cone $C_s = \text{image}\, \Phi_{Y_s}$ coincides with Δ near α for all $s \in \pi^{-1}(\alpha)$. Thus, the function in (1.25) is defined on a neighbourhood of α in Δ. Finally, if both ρ_s and ρ_s' are Duistermaat–Heckman functions for truncations of the model Y_s, then by Corollary 8.24 there exists a neighbourhood of α in C_s, hence in Δ, on which the difference $\rho_s - \rho_s'$ coincides with an integral affine function.

The Duistermaat–Heckman function of a tall complexity-one space is compatible with the skeleton; see Proposition 9.2. Our final theorem shows that this compatibility condition is also sufficient for a function, a subset of $\mathcal{T}$, and a painting to arise from a complexity-one space.

Theorem 3 (Global existence) *Let (S,π) be a tall skeleton over a convex open subset $\mathcal{T}\subset\mathfrak{t}^*$, let $\Delta\subset\mathcal{T}$ be a convex Delzant subset that is compatible with (S,π), and let $\rho\colon\Delta\to\mathbb{R}_{>0}$ be a function that is compatible with (S,π). Let Σ be a closed oriented surface, and let $f\colon S\to\Sigma$ be a painting. Then there exists a tall complexity-one space over $\mathcal{T}$ with moment image Δ and Duistermaat–Heckman function ρ whose painting is equivalent to f.*

Section 2 contains some general facts about complexity-one spaces. The remainder of the paper is divided into two parts. Sections 3–6 constitute Part I of the paper and lead to Theorem 6.1. This is a *reconstruction theorem* in the sense that we take a tall complexity-one space, break it into pieces, and glue the pieces together so as to obtain a new complexity-one space. In Section 3, we prove some facts about the cohomology of spaces that are locally modeled on the quotients of complexity-one spaces. In Section 4, we glue local pieces of complexity-one spaces as T–manifolds. In Section 5, we show how to arrange that the symplectic forms on these local pieces will agree on their overlaps. In Section 6, we use the technology developed so far and a crucial technical result from our previous paper [21, Proposition 20.1] to prove Theorem 6.1. Sections 7–10 constitute Part II of the paper. In Section 7 we show that the moment map image and skeleton of a tall complexity-one space satisfy our compatibility conditions. In Section 9 we use technical results from Section 8 to show that the Duistermaat–Heckman measure of a complexity-one space is compatible with its skeleton, and we give a local existence theorem: any compatible data *locally* comes from a complexity-one space. Finally, in Section 10, we combine these results with a variant of the reconstruction theorem from Section 6 to prove the main existence theorems: Theorems 1, 2 and 3.

2 Basic properties of complexity-one spaces

In this section we recall the local normal form theorem and the convexity package, and analyze some of their basic consequences for complexity-one Hamiltonian torus actions.

Local normal form theorem

For every orbit x in a Hamiltonian T–manifold M there is a corresponding model $Y=T\times_H\mathbb{C}^n\times\mathfrak{h}^0$ such that the isotropy representation of the central orbit $T\cdot[1,0,0]$

is the same as that of x. The *local normal form for Hamiltonian torus actions* asserts that there exists an equivariant symplectomorphism from an invariant neighbourhood of x in M to an invariant open subset of Y that carries x to $T\cdot[1,0,0]$; see Guillemin and Sternberg [15] and Marle [34].

Convexity package

Let (M,ω,Φ) be a connected Hamiltonian T–manifold. Suppose that there exists a convex open subset $\mathcal{T}$ of $\mathfrak{t}^*$ that contains $\Phi(M)$ and such that $\Phi\colon M\to\mathcal{T}$ is proper. Then we have the following *convexity package*.

Convexity The moment map image, $\Phi(M)$, is convex.

Connectedness The moment fiber, $\Phi^{-1}(\alpha)$, is connected for all $\alpha\in\mathcal{T}$.

Stability As a map to $\Phi(M)$, the moment map is open.

These three properties also hold for the moment map of a local model. Note that together the three properties imply that the moment map preimage of every convex set is connected. Moreover, by convexity, stability and the local normal form theorem, $\Delta := \Phi(M)$ is a convex polyhedral subset of $\mathcal{T}$ whose faces have rational slopes.

For the compact case, we refer the reader to Atiyah [5], Guillemin and Sternberg [14] and Sjamaar [42, Theorem 6.5]; also see Lerman and the second author [29]. For convexity and connectedness in the case of proper moment maps to open convex sets, see Lerman, Meinrenken, the second author and Woodward [28]. Stability then follows from the local normal form theorem and stability for local models; see [42, Theorem 5.4 and Example 5.5]. Also see [8, Section 7].

Some consequences

In order to apply these theorems to complexity-one spaces, we now analyze complexity-one models.

2.1 Lemma *Let $Y = T\times_H \mathbb{C}^{h+1}\times\mathfrak{h}^0$ be a complexity-one model in which the central orbit $T\cdot[1,0,0]$ is a nonexceptional orbit. Then, after possibly permuting the coordinates, $Y = T\times_H\mathbb{C}^h\times\mathbb{C}\times\mathfrak{h}^0$ and H acts on $\mathbb{C}^h$ through an isomorphism with $(S^1)^h$. Consequently, every orbit in Y is nonexceptional, and the image of Y is a Delzant cone.*

Proof Inside the model, the set of points that have stabilizer H and that lie in the same moment fiber as $T\cdot[1,0,0]$ is $T\times_H(\mathbb{C}^{h+1})^H\times\{0\}$, where $(\mathbb{C}^{h+1})^H$ is the subspace fixed by H. Since $T\cdot[1,0,0]$ is not an exceptional orbit, this subspace is not trivial. The result then follows from a dimension count. □

2.2 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a complexity-one model with moment map $\Phi_Y([t,z,\nu]) = \alpha + \Phi_H(z) + \nu$. Then one of the following conditions occurs.*

- *Let F be the smallest face of the convex polyhedral cone* image Φ_Y, *which contains α. Then, for every $\beta \in F$, the reduced space $\Phi_Y^{-1}(\beta)/T$ consists of a single orbit.*
- *There exists a homeomorphism*

$$Y/T \to (\text{image } \Phi_Y) \times \mathbb{C} \tag{2.3}$$

whose first component is induced from the moment map and whose second component takes the set of exceptional orbits to zero.

Proof Suppose first the map $\Phi_Y \colon Y \to \mathfrak{t}^*$ is proper. Then the map $\Phi_H \colon \mathbb{C}^{h+1} \to \mathfrak{h}^*$ is also proper. Recall that $\Phi_H(z) = \sum_{j=0}^{h} \pi |z_j|^2 \eta_j$, where the $\eta_j \in \mathfrak{h}^*$ are the weights for the H action. Hence, properness implies that $\Phi_H^{-1}(0) = \{0\}$. We claim that the weights η_j generate a strictly convex cone in $\mathfrak{h}^*$. Indeed, otherwise there exists a nonzero element β of $\mathfrak{h}^*$ such that both β and $-\beta$ are nonnegative linear combinations of the weights, and their sum gives the origin $0 \in \mathfrak{h}^*$ as a nontrivial nonnegative linear combination of the weights. Finally, because image Φ_H is a strictly convex polyhedral cone, the smallest face of image Φ_Y is $F = \alpha + \mathfrak{h}^0$, and because $\Phi_H^{-1}(0) = \{0\}$, for every $\beta \in F$ the preimage $\Phi_Y^{-1}(\beta)$ is a single T orbit.

Finally, if the map Φ_Y is not proper, then there exists a homeomorphism (2.3) with the required properties; see the authors [20, Lemma 6.2], and also [20, Definition 8.2] and the sentences that follow it. □

2.4 Corollary *In a tall complexity-one Hamiltonian T–manifold, the corresponding local models are tall.*

Proof By Lemma 2.2, a complexity-one model Y is tall exactly if there exists a neighbourhood of the central orbit $T \cdot [1,0,0]$ in Y that is tall. Hence, the claim follows immediately from the local normal form theorem. □

2.5 Corollary *Let (S,π) be a tall skeleton over an open subset $\mathcal{T}$ of $\mathfrak{t}^*$. Then $\pi^{-1}(\alpha)$ is finite for every $\alpha \in \mathcal{T}$.*

Proof Let s be a point in S and let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be the corresponding complexity-one model, which is tall by Corollary 2.4. By the local normal form theorem and Lemma 2.2 there exists a neighbourhood of s in S whose intersection with $\pi^{-1}(\alpha)$ consists of the single element set $\{s\}$. The result then follows from the properness of π. □

For a complexity-one space $(M,\omega,\Phi,\mathcal{T})$ with image $\Phi = \Delta$, define

$$\Delta_{\text{tall}} := \{\alpha \in \Delta \mid \Phi^{-1}(\alpha)/T \text{ is a two-dimensional topological manifold}\},$$
$$\Delta_{\text{short}} := \{\alpha \in \Delta \mid \Phi^{-1}(\alpha)/T \text{ consists of a single orbit}\}.$$

2.6 Corollary (Short/tall dichotomy) *Let $(M,\omega,\Phi,\mathcal{T})$ be a complexity-one space with moment image $\Delta = \operatorname{image}\Phi$. Then:*

- $\Delta = \Delta_{\text{tall}} \cup \Delta_{\text{short}}$*; moreover, $\Phi^{-1}(\alpha)/T$ is connected and oriented for all $\alpha \in \Delta_{\text{tall}}$.*
- Δ_{short} *is a union of (closed) faces of Δ.*

Proof By Lemma 2.2 and the local normal form theorem there exists an open set $U \subset M/T$ such that, for each $\alpha \in \Delta$, the intersection $\Phi^{-1}(\alpha)/T \cap U$ is a (possibly empty) two-dimensional topological manifold, and its complement in $\Phi^{-1}(\alpha)/T$ is discrete. Hence, by the connectedness of the level sets, either $\alpha \in \Delta_{\text{short}}$ or $\Phi^{-1}(\alpha)/T$ is a connected two-dimensional topological manifold. In the latter case, the reduced space $\Phi^{-1}(\alpha)/T$ has a natural symplectic orientation on the complement of the set of exceptional orbits by Lemma 2.1; moreover, the set of exceptional orbits is discrete by Corollary 2.5.

The set Δ_{short} is a closed subset of Δ, because it is the image of the closed subset $M/T \smallsetminus U$ under the proper map $\overline{\Phi}\colon M/T \to \Delta$. By the stability of the moment map Φ, the local normal form theorem, and Lemma 2.2, the intersection of Δ_{short} with the relative interior $\mathring{F}$ of any face F of Δ is open in the relative topology of F. These two facts imply that Δ_{short} is a union of (closed) faces of Δ. □

Part I: Reconstruction

3 Topology of complexity-one quotients

In this section, we prove two results about the topology of complexity-one quotients which we will need in order to prove the main propositions in Sections 4 and 5. For future reference, whenever possible we will allow complexity-one spaces that are not tall.

For a topological space X and a presheaf $\mathcal{S}$ of abelian groups on X, we let $\check{H}^i(X,\mathcal{S})$ denote the Čech cohomology of $\mathcal{S}$. If X is paracompact,[4] this agrees with the Čech

[4] We adopt the convention that, by definition, every paracompact space is Hausdorff.

cohomology $\check{H}^i(X, \mathcal{S}^+)$ of the sheafification $\mathcal{S}^+$ of $\mathcal{S}$ and with the sheaf cohomology $H^i(X, \mathcal{S}^+)$ of $\mathcal{S}^+$ that is defined through derived functors; see Theorem 5.10.1 and the corollary to Theorem 5.10.2 of Godement [11, Chapter II].

Consider a continuous map of topological spaces, $\bar{\Phi}\colon Q \to B$. Given an abelian group A and a nonnegative integer i, define a presheaf $\mathcal{H}^i_A$ on B by

$$\mathcal{H}^i_A(U) = \check{H}^i(\bar{\Phi}^{-1}(U); A) \quad \text{for each open set } U \subset B.$$

Note that $\mathcal{H}^i_A(\varnothing) = \{0\}$. This presheaf is the pushforward by $\bar{\Phi}\colon Q \to B$ of the presheaf on Q that associates to each open set $W \subset Q$ the group $\check{H}^i(W; A)$. In general, neither presheaf is a sheaf.

3.1 Proposition *Let Q be a topological space, $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$, and $\bar{\Phi}\colon Q \to \mathcal{T}$ be a continuous map such that $\Delta = \operatorname{image} \bar{\Phi}$ is convex. Assume that for every point in $\mathcal{T}$ there exists a convex neighbourhood U in $\mathcal{T}$, a complexity-one space $(M_U, \omega_U, \Phi_U, U)$, and a homeomorphism from $\bar{\Phi}^{-1}(U)$ to M_U/T that carries $\bar{\Phi}|_{\bar{\Phi}^{-1}(U)}$ to the map $\bar{\Phi}_U\colon M_U/T \to U$ induced by Φ_U. Then for any abelian group A,*

$$\check{H}^i(\mathcal{T}, \mathcal{H}^0_A) = \check{H}^i(\mathcal{T}, \mathcal{H}^1_A) = 0 \quad \text{for all } i > 0.$$

Moreover, if at least one of the spaces M_U is not tall, then

$$\check{H}^0(\mathcal{T}, \mathcal{H}^2_A) = 0.$$

Proof We first show that Q is paracompact. Let $\mathfrak{W}$ be an arbitrary open covering of Q. There exists a locally finite covering ν of $\mathcal{T}$ by open balls such that every $B \in \nu$, the preimage $\bar{\Phi}^{-1}(\bar{B})$ of the closure $\bar{B}$ of B is compact. For each $B \in \nu$, let $\mathfrak{W}_B \subset \mathfrak{W}$ be a finite subset that covers $\bar{\Phi}^{-1}(\bar{B})$; then

$$\bigcup_{B \in \nu} \{W \cap f^{-1}(B) \mid W \in \mathfrak{W}_B\}$$

is a locally finite open refinement of $\mathfrak{W}$ that covers Q.

The map $\bar{\Phi}\colon Q \to \Delta \subset \mathcal{T}$ induces presheaves $\mathcal{H}^j_A$ on Δ and $\mathcal{T}$. Moreover, since $\mathcal{H}^j_A(U) = \mathcal{H}^j_A(U \cap \Delta)$ for all open $U \subset \mathcal{T}$, we have

$$\check{H}^i(\mathcal{T}, \mathcal{H}^j_A) = \check{H}^i(\Delta, \mathcal{H}^j_A) \quad \text{for all } i \text{ and } j.$$

Let $(\mathcal{H}^j_A)^+$ denote the sheafification of the presheaf $\mathcal{H}^j_A$ on Δ. Because the Čech cohomology of a presheaf on a paracompact space is equal to that of its sheafification, it is enough to prove that

$$\check{H}^i(\Delta, (\mathcal{H}^0_A)^+) = \check{H}^i(\Delta, (\mathcal{H}^1_A)^+) = 0 \quad \text{for all } i > 0$$

and that, if at least one of the spaces M_U is not tall, then

$$\check{H}^0(\Delta, (\mathcal{H}_A^2)^+) = 0.$$

Assume first that all of the complexity-one spaces M_U are tall. By Proposition 1.2, this implies that for every point in $\mathcal{T}$ there exists a convex neighbourhood U in $\mathcal{T}$, a surface Σ, and a function $f\colon \bar{\Phi}^{-1}(U) \to \Sigma$, such that

$$(\bar{\Phi}, f)\colon \bar{\Phi}^{-1}(U) \to (\Delta \cap U) \times \Sigma$$

is a homeomorphism. Hence, we have $(\mathcal{H}_A^0)^+$ is a constant sheaf and $(\mathcal{H}_A^j)^+$ is a locally constant sheaf for all $j > 0$. Since Δ is convex, it is contractible; thus $\check{H}^i(\Delta, (\mathcal{H}_A^j)^+) = \{0\}$ for all j and all $i > 0$.

Next, assume that at least one of the complexity-one spaces M_U is not tall. As before, let Δ_{tall} denote the set of $\alpha \in \Delta$ such that $\bar{\Phi}^{-1}(\alpha)$ is a two-dimensional topological manifold, and let Δ_{short} denote the set of $\alpha \in \Delta$ such that $\bar{\Phi}^{-1}(\alpha)$ is a single point. Corollary 2.6 implies that $\Delta_{\text{short}} = \Delta \smallsetminus \Delta_{\text{tall}}$, that Δ_{tall} is open in Δ, and that $\bar{\Phi}^{-1}(\alpha)$ is connected and oriented for all $\alpha \in \Delta_{\text{tall}}$.

By assumption, for every point in $\mathcal{T}$ there exists a convex neighbourhood U, a complexity-one space $(M_U, \omega_U, \Phi_U, U)$, and a homeomorphism from $\bar{\Phi}^{-1}(U)$ to M_U/T that carries $\bar{\Phi}|_{\bar{\Phi}^{-1}(U)}$ to the map $\bar{\Phi}_U\colon M_U/T \to U$ induced by Φ_U. In fact, the convexity package implies that the preimage $\Phi_U^{-1}(V)$ is connected for any convex subset $V \subset U$; see Section 2. Hence, the neighbourhood U can be chosen to be arbitrarily small.

In particular, every $\alpha \in \Delta$ has arbitrarily small neighbourhoods whose preimages in Q are connected. Hence, $(\mathcal{H}_A^0)^+$ is a constant sheaf. Since Δ is convex, this implies that $\check{H}^i(\Delta, (\mathcal{H}_A^0)^+) = 0$ for all $i > 0$.

The following result is proved in [20, Lemma 5.7]:

(3.2) Let (M, ω, Φ, U) be a complexity-one space. Suppose that $\Phi^{-1}(\alpha)$ consists of a single orbit. Then every neighbourhood of α contains a smaller neighbourhood V such that the quotient $\Phi^{-1}(V)/T$ is contractible. Moreover, every regular nonempty symplectic quotient $\Phi^{-1}(y)/T$ in $\Phi^{-1}(V)/T$ is homeomorphic to a 2–sphere.

By Proposition 1.2, the genus of the reduced space is locally constant on Δ_{tall}. Hence, since regular values are dense, (3.2) implies that this genus is zero for all two-dimensional reduced spaces over a neighbourhood of Δ_{short}. Since Δ is connected and Δ_{short} is not empty, this implies that every two-dimensional reduced space has genus zero. Hence, by Proposition 1.2 and (3.2), $(\mathcal{H}_A^1)^+$ is the zero sheaf. Therefore, $\check{H}^i(\Delta, (\mathcal{H}_A^1)^+) = 0$ for all $i > 0$.

Finally, consider a global section $\gamma \in \check{H}^0(\Delta, (\mathcal{H}_A^2)^+)$. By (3.2), the support of γ is a subset of Δ_{tall}. Therefore, since the restriction of $(\mathcal{H}_A^2)^+$ to Δ_{tall} is a locally constant sheaf by Proposition 1.2, the support of γ is an open and closed subset of Δ_{tall}. Since Δ is connected and Δ_{short} is nonempty, this implies that $\gamma = 0$. Thus, $\check{H}^0(\Delta, (\mathcal{H}_A^2)^+) = 0$. □

3.3 Proposition *Let* $(M, \omega, \Phi, \mathcal{T})$ *be a complexity-one space. The restriction map* $H^2(M/T; \mathbb{Z}) \to H^2(\Phi^{-1}(y)/T; \mathbb{Z})$ *is one-to-one for each* $y \in \text{image}\,\Phi$.

Proof If the complexity-one space is tall, this proposition is an immediate consequence of Proposition 1.2. So assume that it is not tall. Let $\bar{\Phi}\colon M/T \to \mathcal{T}$ be the map induced by Φ. Then there is the Leray spectral sequence converging to $H^*(M/T; \mathbb{Z})$ with

$$E_2^{i,j} = \check{H}^i(\mathcal{T}, \mathcal{H}_{\mathbb{Z}}^j);$$

see [11, Chapter II, Theorem 4.17.1]. By Proposition 3.1, $E_2^{i,j} = 0$ for all i and j such that $i + j = 2$. Consequently, $H^2(M/T; \mathbb{Z}) = 0$. □

4 Lifting from the quotient

An important step in gluing together local pieces of complexity-one spaces is to glue them together as T–manifolds. To carry this out, which we will do in this section, we need a notion of diffeomorphisms of quotient spaces.

Let a compact torus T act on a manifold N. The quotient N/T can be given a natural differential structure, consisting of the sheaf of real-valued functions whose pullbacks to N are smooth.[5] We say that a map $h\colon N/T \to N'/T$ is *smooth* if it pulls back smooth functions to smooth functions; it is a *diffeomorphism* if it is smooth and has a smooth inverse. If N and N' are oriented, the choice of an orientation on T determines orientations on the smooth part of N/T and N'/T. Whether or not a diffeomorphism $f\colon N/T \to N'/T$ preserves orientation is independent of this choice.

We now recall several definitions from [20].

[5] This notion of a differential structure on quotient spaces was used by Schwarz [40]. An axiomatization of "differential structure" appeared in Sikorski [41].

4.1 Definition Let a torus T act on oriented manifolds M and M' with T–invariant maps $\Phi\colon M \to \mathfrak{t}^*$ and $\Phi'\colon M' \to \mathfrak{t}^*$. A *$\Phi$–$T$–diffeomorphism* from (M, Φ) to (M', Φ') is an orientation preserving equivariant diffeomorphism $f\colon M \to M'$ that satisfies $\Phi' \circ f = \Phi$.

4.2 Definition Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be complexity-one Hamiltonian T–manifolds. A *Φ–diffeomorphism* from M/T to M'/T is an orientation preserving diffeomorphism $f\colon M/T \to M'/T$ such that $\overline{\Phi}' \circ f = \overline{\Phi}$, and such that f and f^{-1} lift to Φ–T–diffeomorphisms in a neighbourhood of each exceptional orbit. Here, $\overline{\Phi}$ and $\overline{\Phi}'$ are induced by the moment maps.

We now state the main result of this section.

4.3 Proposition *Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open subset, $\Delta \subset \mathcal{T}$ a convex subset, and $\rho\colon \Delta \to \mathbb{R}_{>0}$ a function. Let $\mathfrak{U}$ be a cover of $\mathcal{T}$ by convex open sets. For each $U \in \mathfrak{U}$, let (M_U, ω_U, Φ_U) be a complexity-one space over U with* image $\Phi_U = U \cap \Delta$ *and Duistermaat–Heckman function $\rho|_U$. For each U and V in $\mathfrak{U}$, let*

$$f_{UV}\colon M_V|_{U\cap V}/T \to M_U|_{U\cap V}/T$$

be a Φ–diffeomorphism, such that $f_{UV} \circ f_{VW} = f_{UW}$ on $M_W|_{U\cap V\cap W}/T$ for all $U, V, W \in \mathfrak{U}$. Then, after possibly passing to a refinement of the cover, there exist Φ–T–diffeomorphisms $g_{UV}\colon M_V|_{U\cap V} \to M_U|_{U\cap V}$ that lift f_{UV} and such that $g_{UV} \circ g_{VW} = g_{UW}$ on $M_W|_{U\cap V\cap W}$ for all $U, V, W \in \mathfrak{U}$.

Under the assumptions of Proposition 4.3, let Q denote the topological space obtained from the disjoint union $\bigsqcup_{U\in\mathfrak{U}} M_U/T$ by identifying x with $f_{UV}(x)$ for all U and V in $\mathfrak{U}$ and all $x \in M_V|_{U\cap V}/T$. Let

$$\overline{\Phi}\colon Q \to \mathcal{T}$$

denote the map induced by the moment maps. As in the proof of Proposition 3.1, Q is paracompact.

We define a differential structure on Q by declaring a real-valued function to be smooth if it lifts to a smooth function on each M_U; notice that this is well defined. We can use smooth partitions of unity on the spaces M_U and $\mathcal{T}$ to construct smooth partitions of unity on Q.

For any abelian Lie group A, let A^∞ denote the sheaf of smooth functions to A. Let $\mathcal{H}^i_{A^\infty}$ denote the presheaf on $\mathcal{T}$ which associates the group $\check{H}^i(\overline{\Phi}^{-1}(U); A^\infty)$ to each open set $U \subset \mathcal{T}$. We will need the following lemma.

4.4 Lemma *In the above situation,*

$$\check{H}^2(\mathcal{T}, \mathcal{H}^0_{T^\infty}) = 0.$$

Proof Every short exact sequence of sheaves on Q gives rise to a long exact sequence in Čech cohomology. Therefore, the short exact sequence of sheaves on Q,

$$0 \to \ell \to \mathfrak{t}^\infty \to T^\infty \to 1,$$

gives rise to a long exact sequence of presheaves on $\mathcal{T}$

$$\textbf{(4.5)} \qquad 0 \to \mathcal{H}^0_\ell \to \mathcal{H}^0_{\mathfrak{t}^\infty} \to \mathcal{H}^0_{T^\infty} \to \mathcal{H}^1_\ell \to \mathcal{H}^1_{\mathfrak{t}^\infty} \to \cdots .$$

Because the sheaf $\mathfrak{t}^\infty$ is fine, $\check{H}^1(W, \mathfrak{t}^\infty) = 0$ for all open sets $W \subset Q$. Hence, $\mathcal{H}^1_{\mathfrak{t}^\infty}$ is the zero presheaf. Thus (4.5) breaks up into two short exact sequences of presheaves,

$$0 \to \mathcal{H}^0_\ell \to \mathcal{H}^0_{\mathfrak{t}^\infty} \to \kappa \to 0, \quad 0 \to \kappa \to \mathcal{H}^0_{T^\infty} \to \mathcal{H}^1_\ell \to 0,$$

where κ denotes the kernel of the homomorphism $\mathcal{H}^0_{T^\infty} \to \mathcal{H}^1_\ell$. From these short exact sequences we get long exact sequences

$$\textbf{(4.6)} \quad \cdots \to \check{H}^i(\mathcal{T}, \mathcal{H}^0_{\mathfrak{t}^\infty}) \to \check{H}^i(\mathcal{T}, \kappa) \to \check{H}^{i+1}(\mathcal{T}, \mathcal{H}^0_\ell) \to \check{H}^{i+1}(\mathcal{T}, \mathcal{H}^0_{\mathfrak{t}^\infty}) \to \cdots ,$$

$$\textbf{(4.7)} \qquad \cdots \to \check{H}^i(\mathcal{T}, \kappa) \to \check{H}^i(\mathcal{T}, \mathcal{H}^0_{T^\infty}) \to \check{H}^i(\mathcal{T}, \mathcal{H}^1_\ell) \to \cdots .$$

Because $\mathcal{H}^0_{\mathfrak{t}^\infty}(U) = \mathfrak{t}^\infty(\bar{\Phi}^{-1}(U))$ for all $U \subset \mathcal{T}$, the sheaf $\mathcal{H}^0_{\mathfrak{t}^\infty}$ is a fine sheaf, and so

$$\check{H}^i(\mathcal{T}, \mathcal{H}^0_{\mathfrak{t}^\infty}) = 0 \quad \text{for all } i > 0.$$

Hence, (4.6) implies that $\check{H}^i(\mathcal{T}, \kappa) = \check{H}^{i+1}(\mathcal{T}, \mathcal{H}^0_\ell)$ for all $i > 0$. Thus, (4.7) becomes

$$\cdots \to \check{H}^{i+1}(\mathcal{T}, \mathcal{H}^0_\ell) \to \check{H}^i(\mathcal{T}, \mathcal{H}^0_{T^\infty}) \to \check{H}^i(\mathcal{T}, \mathcal{H}^1_\ell) \to \cdots$$

for all $i > 0$. The claim now follows immediately from Proposition 3.1. □

The proof of Proposition 4.3 will use the following result.

4.8 Lemma *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity-one spaces that have the same Duistermaat–Heckman function. Then, every Φ–diffeomorphism $f\colon M/T \to M'/T$ lifts to a Φ–T–diffeomorphism from M to M'.*

Proof Let $f\colon M/T \to M'/T$ be a Φ–diffeomorphism. By Lemma 2.1, a local model for a nonexceptional orbit is determined (up to permutation of the coordinates) by its momentum image. This, the local normal form theorem, and stability of the moment map, imply that the local model of a nonexceptional orbit x in M/T is the same (up to permutation of the coordinates) as the local model of its image $f(x)$ in M'/T. Moreover, [20, Lemma 4.10] reads as follows.

> Let Y be a local model for a nonexceptional orbit with a moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. Let W and W' be invariant open subsets of Y. Let $g\colon W/T \to W'/T$ be a diffeomorphism which preserves the moment map. Then g lifts to an equivariant diffeomorphism from W to W'.

When combined with Definition 4.2, this shows that every orbit in M/T has a neighbourhood on which f lifts to a Φ–T–diffeomorphism.

Condition (3.2) of [20] reads as follows:

$(*)$ The restriction map $H^2(M/T;\mathbb{Z}) \to H^2(\Phi^{-1}(y)/T;\mathbb{Z})$ is one-to-one for some regular value y of Φ.

Lemma 4.11 of [20] reads as follows.

> Let (M,ω,Φ,U) and (M',ω',Φ',U) be complexity-one spaces that satisfy Condition $(*)$ and have the same Duistermaat–Heckman measure. Then every homeomorphism from M/T to M'/T that locally lifts to a Φ–T–diffeomorphism also lifts globally to a Φ–T–diffeomorphism from M to M'.

The lemma follows from this and Proposition 3.3. □

We will also need the following result from Haefliger and Salem [16].

4.9 Theorem [16] *Let a torus T act on a manifold M. Let $h\colon M \to M$ be an equivariant diffeomorphism that sends each orbit to itself. Then there exists a smooth invariant function $f\colon M \to T$ such that $h(m) = f(m)\cdot m$ for all $m \in M$.*

Proof of Proposition 4.3 Fix any U and V in $\mathfrak{U}$. Since U and V, and hence $U\cap V$, are convex, we can apply Lemma 4.8 to the spaces $M_U|_{U\cap V}$ and $M_V|_{U\cap V}$. Thus, there exists a Φ–T–diffeomorphism

$$F_{UV}\colon M_V|_{U\cap V} \to M_U|_{U\cap V}$$

that lifts f_{UV}.

For every $U,V,W\in\mathfrak{U}$, by Theorem 4.9, $F_{UV}\circ F_{VW}\circ F_{UW}^{-1}$ is given by acting by a smooth T–invariant function $M_U|_{U\cap V\cap W} \to T$. This function is the pullback of a smooth function

$$h_{UVW}\colon Q|_{U\cap V\cap W} \to T. \tag{4.10}$$

On quadruple intersections, we have

$$(h_{UVW})(h_{UVX})^{-1}(h_{UWX})(h_{VWX})^{-1} = 1. \tag{4.11}$$

This is a cocycle condition; hence, the h_{UVW} represent a cohomology class in $H^2(\mathfrak{U}, \mathcal{H}^0_{T^\infty})$. By Lemma 4.4, after possibly passing to a refinement of the cover $\mathfrak{U}$, there exist smooth T–invariant functions

$$B_{UV}\colon Q|_{U\cap V} \to T$$

such that

$$B_{UV}\, B_{VW}\, B_{UW}^{-1} = h_{UVW} \tag{4.12}$$

on triple intersections.

Then

$$g_{UV}(x) := (B_{UV}(x))^{-1} \cdot F_{UV}(x)$$

are liftings of the f_{UV}'s that satisfy the required compatibility condition. □

5 Gluing symplectic forms

The last "local to global" step is to modify the symplectic forms on the local pieces so that they agree on overlaps.

5.1 Proposition *We let an $(n-1)$–dimensional torus T act on an oriented $2n$–dimensional manifold M, and let $\Phi\colon M \to \mathcal{T}$ be an invariant proper map to an open subset $\mathcal{T} \subset \mathfrak{t}^*$. Assume that $\Delta = \operatorname{image} \Phi$ is convex. Fix a function $\rho\colon \Delta \to \mathbb{R}_{>0}$ and an open cover $\mathfrak{U}$ of $\mathcal{T}$.*

Assume that, for all $U \in \mathfrak{U}$, there exists an invariant symplectic form ω_U on $\Phi^{-1}(U)$ with moment map $\Phi|_U$ and Duistermaat–Heckman function $\rho|_U$ such that ω_U is compatible with the given orientation. Then there exists an invariant symplectic form ω' on M with moment map Φ and Duistermaat–Heckman function ρ such that ω' is compatible with the given orientation.

Let a compact Lie group G act on a manifold M, and let $\{\zeta_M\}_{\zeta\in\mathfrak{g}}$ be the vector fields that generate this action. A differential form β on M is *basic* if it is G invariant and horizontal, that is, $\iota_{\zeta_M}\beta = 0$ for all $\zeta \in \mathfrak{g}$. The basic differential forms on M constitute a differential complex $\Omega^*_{\text{basic}}(M)$ whose cohomology coincides with the Čech cohomology of the topological quotient M/G; see Koszul [27].

We will need the following technical lemma; cf [20, Lemma 3.6].

5.2 Lemma *Let an $(n-1)$–dimensional abelian group T act faithfully on a $2n$–dimensional manifold M. Let $\Phi\colon M \to \mathfrak{t}^*$ be a smooth invariant map. Let ω_0 and ω_1 be invariant symplectic forms on M with moment map Φ that induce the same orientation on M. Let α be a basic two-form on M such that $\alpha(\eta,\eta')=0$ for all $\eta,\eta' \in \ker d\Phi$. Let λ_0 and λ_1 be nonnegative functions on M such that $\lambda_0+\lambda_1=1$. Then*

$$\lambda_0\omega_0 + \lambda_1\omega_1 + \alpha$$

is nondegenerate and induces the same orientation as ω_0 and ω_1.

Proof Let $x \in M$ be a point with stabilizer H; let h be the dimension of H. By the local normal form theorem, a neighbourhood of the orbit of x with the symplectic form ω_0 is equivariantly symplectomorphic to a neighbourhood of the central orbit $T\cdot[1,0,0]$ in the model $T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$. Then, the tangent space at x splits as $\mathfrak{t}/\mathfrak{h} \oplus \mathfrak{h}^0 \oplus \mathbb{C}^{h+1}$, where $\mathfrak{t}/\mathfrak{h}$ is the tangent space to the orbit. By the definition of the moment map, the forms $\omega_0|_x$ and $\omega_1|_x$ are given by block matrices of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \widetilde{\omega}_0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \widetilde{\omega}_1 \end{pmatrix},$$

where I is the natural pairing between the vector space $\mathfrak{t}/\mathfrak{h}$ and its dual, $\mathfrak{h}^0$, and where $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ are linear symplectic forms on $\mathbb{C}^{h+1}$ with the same moment map and the same orientation. By our assumptions, $\alpha|_x$ is given by a block matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{pmatrix}.$$

Hence, $(\lambda_0\omega_0 + \lambda_1\omega_1 + \alpha)|_x$ is given by a block matrix of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \widetilde{\omega} \end{pmatrix},$$

where

$$\widetilde{\omega} = \lambda_0(x)\widetilde{\omega}_0 + \lambda_1(x)\widetilde{\omega}_1.$$

It suffices to show $\widetilde{\omega}$ is nondegenerate and induces the same orientation as $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$.

Case 1 Suppose that the stabilizer of x is trivial. Then $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ are nonzero two-forms on $\mathbb{C}$ that induce the same orientation, and so $\widetilde{\omega}$ is nondegenerate and induces the same orientation.

Case 2 Suppose the stabilizer of x is nontrivial. Viewing $\widetilde{\omega}$ as a translation invariant differential two-form on $\mathbb{C}^{h+1}$, it is enough to find some $v \in \mathbb{C}^{h+1}$ such that $\widetilde{\omega}|_v$ is nondegenerate and induces the same orientation as $\widetilde{\omega}_0|_v$ and $\widetilde{\omega}_1|_v$. We choose $v \in \mathbb{C}^{h+1}$ whose stabilizer is trivial and apply Case 1 to the H action on $\mathbb{C}^{h+1}$. $\square$

Proof of Proposition 5.1 Given $j \in \mathbb{N}$, define a sheaf $\widetilde{\Omega}^j_{\text{basic}}$ on $\mathcal{T}$ by

$$\widetilde{\Omega}^j_{\text{basic}}(U) = \Omega^j_{\text{basic}}(\Phi^{-1}(U)) \quad \text{for all open } U \subset \mathcal{T}.$$

Consider the double complex

$$K^{i,j} = \check{C}^i(\mathfrak{U}, \widetilde{\Omega}^j_{\text{basic}}).$$

Let $d\colon K^{i,j} \to K^{i,j+1}$ denote the de Rham differential, and let $\delta\colon K^{i,j} \to K^{i+1,j}$ denote the Čech differential.

To prove the Proposition, it will be enough to find $\beta \in C^1(\mathfrak{U}, \widetilde{\Omega}^1_{\text{basic}})$ such that $\delta\beta = 0$ and $d\beta_{VW} = \omega_V - \omega_W$ for all V and W in $\mathfrak{U}$. To see this, let $\{\lambda_U\}_{U\in\mathfrak{U}}$ be the pull back to M of a smooth partition of unity on $\mathcal{T}$ subordinate to $\mathfrak{U}$. Define

$$\omega'_V := \sum_{U\in\mathfrak{U}} \lambda_U\omega_U + \sum_{U\in\mathfrak{U}} d\lambda_U \wedge \beta_{UV} \in \Omega^2(\Phi^{-1}(V)) \quad \text{for all } V \in \mathfrak{U}.$$

Since $\delta\beta = 0$ and $\sum_{U\in\mathfrak{U}} \lambda_U = 1$,

$$\omega'_V - \omega'_W = \sum_{U\in\mathfrak{U}} d\lambda_U \wedge (\beta_{UV} - \beta_{UW}) = d\Big(\sum_{U\in\mathfrak{U}} \lambda_U\Big) \wedge \beta_{WV} = 0.$$

Therefore, the ω'_V glue together to give a global form $\omega' \in \Omega^2(M)$. Since each ω_U is an invariant symplectic form, and since $d\lambda_U(\eta) = 0$ for all $U \in \mathfrak{U}$ and all $\eta \in \ker d\Phi$, by repeated application of Lemma 5.2 ω' is nondegenerate and is compatible with the given orientation. Moreover,

$$\begin{aligned}\omega_V + \sum_{U\in\mathfrak{U}} d(\lambda_U\beta_{UV}) &= \omega_V + \sum_{U\in\mathfrak{U}} (d\lambda_U \wedge \beta_{UV} + \lambda_U(\omega_U - \omega_V)) \\ &= \omega_V - \sum_{U\in\mathfrak{U}} \lambda_U\omega_V + \omega'_V = \omega'_V.\end{aligned}$$

Thus, each ω_V and ω'_V differ by the exterior derivative of a basic one-form. This implies that ω' is closed, and so it is a symplectic form compatible with the given orientation. It also implies that ω' is invariant, has the same moment map Φ as ω_V, and has the same Duistermaat–Heckman function ρ as ω_V.

As a first step towards finding the required cochain, we will show that we may assume that there exists $\beta \in C^1(\mathfrak{U}, \widetilde{\Omega}^1_{\text{basic}})$ such that $d\beta_{VW} = \omega_V - \omega_W$ for all V and W in $\mathfrak{U}$. After possibly passing to a refinement, we may assume that every $U \in \mathfrak{U}$ is convex. As we mentioned earlier, [20, Condition (3.2)] reads as follows:

$(*)$ The restriction map $H^2(M/T;\mathbb{Z}) \to H^2(\Phi^{-1}(y)/T;\mathbb{Z})$ is one-to-one for some regular value y of Φ.

Moreover, [20, Lemma 3.5] reads as follows.

> Let (M,ω,Φ,U) and (M',ω',Φ',U) be complexity-one spaces that satisfy Condition $(*)$ and have the same Duistermaat–Heckman measure. Then for every Φ–T–diffeomorphism $g\colon M \to M'$ there exists a basic one-form β on M such that $\mathrm{d}\beta = g^*\omega' - \omega$.

Hence, the claim follows immediately from Proposition 3.3.

Next, we will show we may assume there exists $\gamma \in \check{C}^2(\mathfrak{U}, \widetilde{\Omega}^0_{\text{basic}})$ such that $\delta\gamma = 0$ and $\delta\beta = d\gamma$. For all $j \in \mathbb{N}$, define a presheaf $\mathcal{H}^j_{\mathbb{R}}$ on $\mathcal{T}$ by $\mathcal{H}^j_{\mathbb{R}}(U) = \check{H}^j(\Phi^{-1}(U)/T;\mathbb{R})$ for all open $U \subset \mathcal{T}$. Recall the Čech cohomology of $\Phi^{-1}(U)/T$ coincides with the cohomology of $(\Omega^*_{\text{basic}}(\Phi^{-1}(U)), d)$. Since $\delta^2\beta = 0$ and $d\delta\beta = 0$, the cochain $\delta\beta$ represents a cohomology class in $\check{H}^2(\mathfrak{U};\mathcal{H}^1_{\mathbb{R}})$. By Proposition 3.1, $\check{H}^2(\mathcal{T},\mathcal{H}^1_{\mathbb{R}}) = 0$. Hence, after passing to a refinement, there exists $\beta' \in \check{C}^1(\mathfrak{U}, \widetilde{\Omega}^1_{\text{basic}})$ such that $d\beta' = 0$ and such that $\delta\beta$ and $\delta\beta'$ agree as elements of $\check{C}^2(\mathfrak{U},\mathcal{H}^1_{\mathbb{R}})$, ie, there exists $\gamma \in \check{C}^2(\mathfrak{U}, \widetilde{\Omega}^0_{\text{basic}})$ such that $\delta\beta - \delta\beta' = d\gamma$. By replacing β by $\beta - \beta'$, we may assume that $\delta\beta = d\gamma$, as required. Since $\check{H}^3(\mathcal{T},\mathcal{H}^0_{\mathbb{R}}) = 0$ by Proposition 3.1, we may assume that $\delta\gamma = 0$ by a similar argument.

Finally, we will use the fact that $\widetilde{\Omega}_{\text{basic}}$ is a fine sheaf to show that we may assume that $\delta\beta = 0$, as required. Define $\eta \in \check{C}^1(\mathfrak{U}, \widetilde{\Omega}^0_{\text{basic}})$ by

$$\eta_{VW} = \sum_{U\in\mathfrak{U}} \lambda_U \gamma_{UVW} \quad \text{for all } V, W \in \mathfrak{U}.$$

Since $\delta\gamma = 0$, we have $\delta\eta = \gamma$, and so $\delta d\eta = d\gamma = \delta\beta$. Hence, we may replace β by $\beta - d\eta$. □

6 Reconstruction

By breaking a space into the moment map preimages of small open subsets of $\mathfrak{t}^*$, and then gluing them back together, we obtain a special case of Theorem 1. This theorem is easier to prove than our other existence theorems, in that it does not require the "local existence" results proved in Sections 7–9, ie, it does not require us to determine which spaces can occur as preimages of small open subsets of $\mathfrak{t}^*$.

6.1 Theorem *Let $(M,\omega,\Phi,\mathcal{T})$ be a tall complexity-one space of genus g. Let Σ be a closed oriented surface of genus g, and let $f\colon M_{\mathrm{exc}} \to \Sigma$ be any painting. Then there exists a tall complexity-one space $(M',\omega',\Phi',\mathcal{T})$ with the same moment image and Duistermaat–Heckman function as M whose painting is equivalent to f.*

Since every tall complexity-one space $(M,\omega,\Phi,\mathcal{T})$ has a convex moment image $\Delta = \Phi(M)$, a positive Duistermaat–Heckman function ρ, and a skeleton $S = M_{\mathrm{exc}}$, Theorem 6.1 is simply the special case of Proposition 6.2 below with $\mathfrak{U} = \{\mathcal{T}\}$. (Proposition 6.2 is also a key ingredient in the proofs of Theorems 1, 2 and 3; see Section 10.)

6.2 Proposition *Let $\mathcal{T}$ be a convex open subset of $\mathfrak{t}^*$, $\Delta \subset \mathcal{T}$ a convex subset, and $\rho\colon \Delta \to \mathbb{R}_{>0}$ a positive function. Let (S,π) be a skeleton over $\mathcal{T}$, Σ a closed oriented surface of genus g, and $f\colon S \to \Sigma$ a painting. Finally, let $\mathfrak{U}$ be a cover of $\mathcal{T}$ by convex open sets.*

Suppose that for each $U \in \mathfrak{U}$ there exists a complexity-one space (M_U,ω_U,Φ_U) of genus g over U with moment image $\Delta\cap U$ and Duistermaat–Heckman function $\rho|_{\Delta\cap U}$ whose skeleton is isomorphic to $S \cap \pi^{-1}(U)$. Then there exists a complexity-one space $(M,\omega,\Phi,\mathcal{T})$ with moment image Δ and Duistermaat–Heckman function ρ whose painting is equivalent to f.

Proof By Proposition 20.1 from the elephant [21] (see Proposition 6.3 below), after (possibly) passing to a refinement of $\mathfrak{U}$, there exists Φ–diffeomorphisms

$$h_{VU}\colon M_U/T|_{U\cap V} \to M_V/T|_{U\cap V}$$

such that $h_{WV} \circ h_{VU} = h_{WU}$ on triple intersections and the following property holds.

> If $(M,\omega,\Phi,\mathcal{T})$ is a complexity-one space such that for every $U \in \mathfrak{U}$ there exists a Φ–T–diffeomorphism $\lambda_U\colon M|_U \to M_U$ so that h_{VU} is the map induced by the composition $\lambda_V \circ (\lambda_U)^{-1}$, then the painting associated to M is equivalent to f.

By Proposition 4.3, after passing to a refinement of $\mathfrak{U}$, there are Φ–T–diffeomorphisms $g_{VU}\colon M_U|_{U\cap V} \to M_V|_{U\cap V}$ that lift h_{VU} and such that $g_{WV} \circ g_{VU} = g_{WU}$ on every triple intersection.

We use these Φ–T–diffeomorphisms to glue together the manifolds M_U. This gives an oriented $2n$–dimensional manifold M with a T action and a T–invariant proper map $\Phi\colon M \to \mathcal{T}$. Moreover, there exists a Φ–T–diffeomorphism $\lambda_U\colon M|_U \to M_U$ for each $U \in \mathfrak{U}$ such that $g_{VU} = \lambda_V \circ (\lambda_U)^{-1}$ on each double intersection.

For each $U \in \mathfrak{U}$, the pullback $\omega'_U := \lambda_U^* \omega_U$ is a T–invariant symplectic form on $\Phi^{-1}(U)$ with moment map $\Phi|_U$ and Duistermaat–Heckman function $\rho|_U$ and such that ω'_U is compatible with the given orientation. Therefore, by Proposition 5.1, there exists a T–invariant symplectic form ω on M with moment map Φ and Duistermaat–Heckman function ρ and such that ω is compatible with the given orientation. Finally, by the property above, the painting associated to $(M, \omega, \Phi, \mathcal{T})$ is equivalent to f. □

For the reader's convenience, we now reformulate [21, Proposition 20.1].

6.3 Proposition *Let $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$ and $\Delta \subset \mathcal{T}$ a convex closed subset. Let $f\colon S \to \Sigma$ be a painting, where Σ is a closed oriented surface of genus g and (S, π) is a skeleton over $\mathcal{T}$. Let $\mathfrak{U}$ be a cover of $\mathcal{T}$ by convex open sets. For each $U \in \mathfrak{U}$, let $(M_U, \omega_U, \Phi_U, U)$ be a tall complexity-one space of genus g over U, so that* image $\Phi_U = U \cap \Delta$ *and so that the set of exceptional orbits $(M_U)_{\text{exc}}$ is isomorphic to the restriction $S|_U := S \cap \pi^{-1}(U)$.*

Then, after possibly refining the open cover, one can associate to each U and V in $\mathfrak{U}$ a Φ–diffeomorphism $h_{VU}\colon M_U/T|_{U\cap V} \to M_V/T|_{U \cap V}$ such that $h_{WV} \circ h_{VU} = h_{WU}$, and such that the following holds.

If $(M, \omega, \Phi, \mathcal{T})$ is a tall complexity-one space such that for every $U \in \mathcal{U}$ there exists a Φ–T–diffeomorphism $\lambda_U\colon M|_U \to M_U$ so that h_{VU} is the map induced by the composition $\lambda_V \circ (\lambda_U)^{-1}$, then the painting associated to M is equivalent to f.

Part II: Classification

7 Compatibility of skeleton

Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity-one space. The purpose of this section is to show that the set M_{exc} of exceptional orbits is a tall skeleton over $\mathcal{T}$, the moment image $\Phi(M)$ is a convex Delzant subset of $\mathcal{T}$, and the set $\Phi(M)$ and skeleton M_{exc} are compatible. See Definitions 1.13, 1.17 and 1.21.

7.1 Lemma *Let $(M,\omega,\Phi,\mathcal{T})$ be a tall complexity-one space. The set M_{exc} of exceptional orbits, labelled by the isotropy representations in M and equipped with the map $\bar{\Phi}\colon M_{\text{exc}} \to \mathfrak{t}^*$ induced by the moment map, is a tall skeleton over $\mathcal{T}$.*

Proof By construction, M_{exc} satisfies the local requirement in the definition of a tall skeleton. By the local normal form theorem and Lemma 2.1, the set of nonexceptional orbits is open, and hence M_{exc} is closed in M/T. So the restriction $\bar{\Phi}|_{M_{\text{exc}}}\colon M_{\text{exc}} \to \mathcal{T}$ is proper. □

7.2 Lemma *The moment image of a tall complexity-one model is a Delzant cone.*

Proof Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map Φ_Y, and let $\alpha = \Phi_Y([1,0,0])$ be the moment map value at the central orbit. By Lemma 2.2, there exists a nonexceptional orbit x in $\Phi_Y^{-1}(\alpha)$. Let Y_x be the corresponding complexity-one model. By Lemma 2.1, image Φ_{Y_x} is a Delzant cone at α. By the local normal form theorem and the stability of the moment map for Y_x and for Y (see Section 2), there exists a neighbourhood U of α in $\mathcal{T}$ such that $U \cap \text{image}\,\Phi_Y = U \cap \text{image}\,\Phi_{Y_x}$. Because image Φ_Y and image Φ_{Y_x} are invariant under dilations about α, this implies that they are equal. □

7.3 Lemma *Let $(M,\omega,\Phi,\mathcal{T})$ be a tall complexity-one space. Then the moment image $\Phi(M)$ is a convex Delzant subset of $\mathcal{T}$ that is compatible with the tall skeleton M_{exc}.*

Proof Because $\mathcal{T}$ is convex, M is connected, and $\Phi\colon M \to \mathcal{T}$ is proper, $\Phi(M)$ is a convex closed subset of $\mathcal{T}$; see Section 2. Let x be a T–orbit in M, let Y be the corresponding model, and let $\alpha = \bar{\Phi}(x)$. By the local normal form theorem, stability for the moment map on Y, and stability of the moment map on M, there exists a neighbourhood U of α such that $\Phi(M) \cap U = \text{image}\,\Phi_Y \cap U$. The claim now follows from Corollary 2.4 and Lemma 7.2. □

8 Duistermaat–Heckman functions for tall complexity-one models

The purpose of this section is to define the Duistermaat–Heckman functions for truncations of a tall complexity-one model (Definition 8.7) and to prove their basic properties (Corollaries 8.23 and 8.24). These functions were used in the introduction to define *compatibility* of a (Duistermaat–Heckman) function and a tall skeleton; see Definition 1.24.

Let $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ be a tall complexity-one model. The action of T on itself by left multiplication, and the action of $(S^1)^n$ on $\mathbb{C}^n$ by coordinatewise multiplication, induce a Hamiltonian action of $T \times (S^1)^n$ on $T^*(T) \times \mathbb{C}^n$. Under the reduction by H that produces the model Y, this descends to a faithful toric (complexity-zero) Hamiltonian action of the torus

$$G = T \times_H (S^1)^n$$

on the model Y.

8.1 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model, let $\chi\colon H \to (S^1)^{h+1}$ be the embedding through which H acts on $\mathbb{C}^{h+1}$, let*

$$G := T \times_H (S^1)^{h+1},$$

and let $i_T\colon T \to G$ denote the inclusion map $t \mapsto [t,1]$. Then there exists a unique $(h+1)$–tuple of nonnegative integers $(\xi_0, \dots, \xi_h)$ such that the following sequence is well defined and exact:

$$\{1\} \longrightarrow T \xrightarrow{i_T} G \xrightarrow{P} S^1 \longrightarrow \{1\}, \tag{8.2}$$

where $P([\lambda, a]) = a^\xi := \prod_{k=0}^{h} a_k^{\xi_k}$. The sequence

$$\{1\} \longrightarrow H \xrightarrow{\chi} (S^1)^{h+1} \xrightarrow{a \mapsto \prod_{k=0}^{h} a_k^{\xi_k}} S^1 \longrightarrow \{1\} \tag{8.3}$$

is also exact.

Proof Lemma 8.1 follows from [20, Lemmas 5.2, 5.3, and 5.8]. For completeness, we give a direct argument.

For any integers $\xi_0, \dots, \xi_h$, the sequence (8.2) is well defined and exact if and only if the sequence (8.3) is exact.

Because the quotient $(S^1)^{h+1}/\chi(H)$ is a one-dimensional compact connected Lie group, there exist integers $\xi_0, \dots, \xi_h$ such that (8.3) is exact; these integers are unique up to replacing $(\xi_0, \dots, \xi_h)$ by $(-\xi_0, \dots, -\xi_h)$.

Let $\eta_0, \dots, \eta_h$ be the weights for the H action on $\mathbb{C}^{h+1}$. Differentiating the relation $\chi(h)^\xi = 1$ gives $\sum_{k=0}^{h} \xi_k \eta_k = 0$.

The quadratic moment map for the H action on $\mathbb{C}^{h+1}$ is given by the equation $\Phi_H(z) = \sum_{k=0}^{h} \pi |z_k|^2 \eta_k$. Because Y is tall, the level set $\Phi_H^{-1}(0)$ contains more than one orbit. Hence, there exist complex numbers $z_0, \dots, z_h$, not all zero, such that $\sum_{k=0}^{h} \pi |z_k|^2 \eta_k = 0$.

Because the action is effective, the space of solutions $(x_0, \dots, x_h)$ of the equation $\sum x_k \eta_k = 0$ is one-dimensional. Hence, the previous two paragraphs imply that the vectors $(\xi_0, \dots, \xi_h)$ and $(|z_0|^2, \dots, |z_h|^2)$ are proportional. So, after possibly replacing $(\xi_0, \dots, \xi_h)$ by $(-\xi_0, \dots, -\xi_h)$, the integers $\xi_0, \dots, \xi_h$ are all nonnegative. □

8.4 Definition We call the map $P\colon G \to S^1$ described above the *defining monomial*; cf [20, Definition 5.12]. A *complementary circle* to T in G is a homomorphism $J\colon S^1 \to G$ such that $P \circ J = \mathrm{id}_{S^1}$.

Let $\mathfrak{g}_{\mathbb{Z}}$ denote the integral lattice in $\mathfrak{g}$ and $\mathfrak{g}^*_{\mathbb{Z}}$ the weight lattice in $\mathfrak{g}^*$. Thus,

$$\mathfrak{g}_{\mathbb{Z}} \cong \mathrm{Hom}(S^1, G), \quad \mathfrak{g}^*_{\mathbb{Z}} \cong \mathrm{Hom}(G, S^1).$$

8.5 Remark Complementary circles always exist. To see this, note that the short exact sequence (8.2) gives rise to a short exact sequence of lattices, $\{0\} \to \ell \to \mathfrak{g}_{\mathbb{Z}} \to \mathbb{Z} \to \{0\}$. Any splitting of this sequence determines a complementary circle $J\colon S^1 \to G$ to T in G.

8.6 Definition Let (M, ω, Φ) be a Hamiltonian T–manifold, and let $A \subset M$ be a measurable subset. The *Duistermaat–Heckman measure for the restriction of* Φ *to* A is the pushforward by the moment map of the restriction to A of the Liouville measure; a real-valued function on $\Phi(A)$ is the *Duistermaat–Heckman function* for this restriction if its product with Lebesgue measure is the Duistermaat–Heckman measure. As before, it is almost unique; see the discussion above Remark 1.22.

8.7 Definition Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. A real-valued function ρ on a subset of $\mathfrak{t}^*$ is the *Duistermaat–Heckman function for a truncation of the model* if there exists a complementary circle J to T in G and a positive number κ such that ρ is the Duistermaat–Heckman function for the restriction of Φ_Y to the subset

$$Y_{J,\kappa} := \varphi_J^{-1}((-\infty, \kappa]) \tag{8.8}$$

of Y, where $\varphi_J\colon Y \to \mathbb{R}$ is the moment map for the resulting circle action on Y, normalized by $\varphi_J([1,0,0]) = 0$.

Let Y be a tall complexity-one space, and let $\alpha = \Phi_Y([1,0,0])$ be the moment map value at the central orbit. We will show in Corollaries 8.23 and 8.24 that there exist Duistermaat–Heckman functions for truncations of the model Y, that they are well defined and continuous on a neighbourhood of α in image Φ_Y, and that the difference of every two such functions is equal to an integral affine function on some neighbourhood of α in image Φ_Y.

8.9 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model, and let $\widetilde{\Phi}_Y \colon Y \to \mathfrak{g}^*$ be a moment map for the action of $G := T \times_H (S^1)^{h+1}$. There exists a linear isomorphism*

$$\mathfrak{g}^* \longrightarrow \mathfrak{h}^0 \times \mathbb{R}^{h+1} \tag{8.10}$$

with the following properties.

(1) *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{(8.10)} & \mathfrak{h}^0 \times \mathbb{R}^{h+1} \\ {\scriptstyle i_T^*}\downarrow & & \downarrow{\scriptstyle (\nu,s)\mapsto(\chi^*(s),\nu)} \\ \mathfrak{t}^* & \xrightarrow{\cong} & \mathfrak{h}^* \times \mathfrak{h}^0, \end{array}$$

where the bottom isomorphism is induced by the inner product on $\mathfrak{t}$ that we have chosen, $i_T \colon T \to G$ is the inclusion map, and $\chi \colon H \to (S^1)^{h+1}$ is the embedding through which H acts on $\mathbb{C}^{h+1}$.

(2) *The composition*

$$Y \xrightarrow{\widetilde{\Phi}_Y} \mathfrak{g}^* \xrightarrow{(8.10)} \mathfrak{h}^0 \times \mathbb{R}^{h+1}$$

has the form

$$[\lambda, z, \nu] \mapsto (\nu, \pi|z_0|^2, \dots, \pi|z_h|^2) + \textit{constant}. \tag{8.11}$$

(3) *Let ξ be the element of $\mathfrak{g}^*_{\mathbb{Z}}$ which corresponds to the defining monomial $P \in \mathrm{Hom}(G, S^1)$, and let $\xi_0, \dots, \xi_h$ be the exponents of the defining monomial. Then the isomorphism* (8.10) *carries ξ to $(0, (\xi_0, \dots, \xi_h))$.*

Proof Let $i_H \colon H \to T$ denote the inclusion map. The torus G is the quotient of $T \times (S^1)^{h+1}$ by the image of the H under embedding $a \mapsto (i_H(a)^{-1}, \chi(a))$. Hence,

$$\mathfrak{g}^* = \{(\gamma, s) \in \mathfrak{t}^* \times \mathbb{R}^{h+1} \mid i_H^*(\gamma) = \chi^*(s)\}. \tag{8.12}$$

Under the identification of $\mathfrak{t}^*$ with $\mathfrak{h}^* \times \mathfrak{h}^0$, the space $\mathfrak{g}^*$ becomes further identified with

$$\{(\beta, \nu, s) \in \mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1} \mid \beta = \chi^*(s)\}. \tag{8.13}$$

Now consider the composition

$$\mathfrak{g}^* \xrightarrow{\text{inclusion}} \mathfrak{t}^* \times \mathbb{R}^{h+1} \xrightarrow{\cong} \mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1} \xrightarrow{\text{projection}} \mathfrak{h}^0 \times \mathbb{R}^{h+1}. \tag{8.14}$$

The projection $(\beta, \nu, s) \mapsto (\nu, s)$ is a linear isomorphism from the space (8.13) – which is the image of $\mathfrak{g}^*$ in $\mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1}$ – to $\mathfrak{h}^0 \times \mathbb{R}^{h+1}$. This proves that the composition (8.14) is a linear isomorphism.

Moreover, if (β, ν, s) is in (8.13) then $(\beta, \nu) = (\chi^*(s), \nu)$. This gives (1).

We have that the moment map for the T action on Y, as a map to $\mathfrak{h}^* \times \mathfrak{h}^0$, has the form $[\lambda, z, \nu] \mapsto (\Phi_H(z), \nu) + \text{constant}$. The moment map for the $(S^1)^{h+1}$ action on Y has the form

$$[\lambda, z, \nu] \mapsto (\pi|z_0|^2, \dots, \pi|z_h|^2) + \text{constant}.$$

Therefore, the moment map for the G action on Y, as a map to $\mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1}$, has the form

$$[\lambda, z, \nu] \mapsto (\Phi_H(z), \nu, \pi|z_0|^2, \dots, \pi|z_h|^2) + \text{constant}.$$

This implies (2).

Since the restriction of $P \in \operatorname{Hom}(G, S^1)$ to the subtorus T of G is trivial, and the restriction of P to the subtorus $(S^1)^{h+1}$ of G is the homomorphism $a \mapsto \prod_{k=0}^{h} a_k^{\xi_k}$, the natural embedding of $\mathfrak{g}^*$ into $\mathfrak{t}^* \times \mathbb{R}^{h+1}$ carries ξ to $(0, (\xi_0, \dots, \xi_h))$. Projecting to $\mathfrak{h}^0 \times \mathbb{R}^{h+1}$, we get (3). □

8.15 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. Let $J \in \operatorname{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$, and let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$. Let $\widetilde{\Phi}_Y\colon Y \to \mathfrak{g}^*$ be the unique G moment map that satisfies $i_T^* \circ \widetilde{\Phi}_Y = \Phi_Y$ and $\langle \widetilde{\Phi}_Y([1,0,0]), j \rangle = 0$. There exists a unique continuous map*

$$\sigma\colon \text{image}\, \Phi_Y \to \mathfrak{g}^*$$

with the following properties:

(1) $i_T^*(\sigma(\beta) + t\xi) = \beta$ *for all* $\beta \in \text{image}\, \Phi_Y$ *and* $t \in \mathbb{R}$

(2) $\text{image}\, \widetilde{\Phi}_Y = \{\sigma(\beta) + t\xi \mid \beta \in \text{image}\, \Phi_Y \text{ and } t \geq 0\}$

(3) $\sigma(\Phi_Y([1,0,0])) = \widetilde{\Phi}_Y([1,0,0])$*; hence, if* $\alpha = \Phi_Y([1,0,0])$ *then* $\langle \sigma(\alpha), j \rangle = 0$

Here, $i_T\colon T \to G$ is the inclusion map, and $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$ corresponds to the defining monomial $P \in \operatorname{Hom}(G, S^1)$.

Proof Let $\chi\colon H \to (S^1)^{h+1}$ be the embedding through which H acts on $\mathbb{C}^{h+1}$. Let $\xi_0, \dots, \xi_h$ be the (nonnegative) exponents of the defining monomial. By (8.3), the level sets of the projection $\chi^*\colon \mathbb{R}^{h+1} \to \mathfrak{h}^*$ are the lines $s + \mathbb{R}(\xi_0, \dots, \xi_h)$.

After possibly reordering the coordinates, we may assume that

$$\textbf{(8.16)} \qquad \xi_k > 0 \text{ for } 0 \le k \le h', \quad \xi_k = 0 \text{ for } h' < k \le h;$$

let $h'' = h - h'$. Consider the subset $\partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ of $\mathbb{R}_+^{h+1}$, consisting of $(h+1)$–tuples of nonnegative numbers in which at least one of the first $h'+1$ entries is equal to zero. Consider the map

$$\textbf{(8.17)} \qquad \partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''} \to \chi^*(\mathbb{R}_+^{h+1}), \quad s \mapsto \chi^*(s).$$

This map is a bijection because the line $s + \mathbb{R}(\xi_0, \dots, \xi_h)$ meets $\partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ exactly once for each $s \in \mathbb{R}_+^{h+1}$, by (8.16). Restricted to each (closed) facet, this map coincides with a linear isomorphism, and hence is open as a map to its image. It follows that the map (8.17) is open.

Since the map (8.17) is a homeomorphism, it has a continuous inverse

$$\sigma_H\colon \chi^*(\mathbb{R}_+^{h+1}) \to \mathbb{R}_+^{h+1}.$$

We claim that σ_H has the following properties:

(1′) $\chi^*(\sigma_H(\beta) + t(\xi_0, \dots, \xi_h)) = \beta$ for all $\beta \in \chi^*(\mathbb{R}_+^{h+1})$ and $t \in \mathbb{R}$

(2′) $\mathbb{R}_+^{h+1} = \{\sigma_H(\beta) + t(\xi_0, \dots, \xi_h) \mid \beta \in \chi^*(\mathbb{R}_+^{h+1}) \text{ and } t \ge 0\}$

(3′) $\sigma_H(0) = 0$

Properties (1′) and (3′) follow immediately from the definition of σ_H. To prove (2′), consider $\beta \in \chi^*(\mathbb{R}_+^{h+1})$ and $t \in \mathbb{R}$. By (8.16), the fact that $\sigma_h(\beta) \in \partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ implies that $\sigma_H(\beta) + t(\xi_0, \dots, \xi_h)$ lies in $\mathbb{R}_+^{h+1}$ exactly if $t \ge 0$.

We may assume without loss of generality that $\Phi_Y([1,0,0]) = 0$. Then the identification of $\mathfrak{t}^*$ with $\mathfrak{h}^* \times \mathfrak{h}^0$ carries image Φ_Y onto $\chi^*(\mathbb{R}_+^{h+1}) \times \mathfrak{h}^0$. Define σ: image $\Phi_Y \to \mathfrak{g}^*$ so that the following diagram commutes:

$$\begin{array}{ccc}
\text{image } \Phi_Y & \xrightarrow{\ \cong\ } & \chi^*(\mathbb{R}_+^{h+1}) \times \mathfrak{h}^0 \\
\sigma \downarrow & & \downarrow (\beta,\nu)\mapsto(\nu,\sigma_H(\beta)) \\
\mathfrak{g}^* & \xrightarrow{\ (8.10)\ } & \mathfrak{h}^0 \times \mathbb{R}^{h+1},
\end{array}$$

where (8.10) is the isomorphism defined in Lemma 8.9. By part (2) of that lemma, (8.10) carries image $\widetilde{\Phi}_Y$ onto $\mathfrak{h}^0 \times \mathbb{R}_+^{h+1}$. Claims (1), (2) and (3) then follow from (1′), (2′) and (3′), respectively, by parts (1) and (3) of Lemma 8.9. □

8.18 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. Let $J \in \mathrm{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$; let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$; and let $\varphi\colon Y \to \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([1,0,0]) = 0$. Then the $T \times S^1$ moment map $(\Phi_Y, \varphi)\colon Y \to \mathfrak{t}^* \times \mathbb{R}$ is proper, and each fiber contains at most one $T \times S^1$ orbit. Moreover, if $\sigma\colon \mathrm{image}\,\Phi_Y \to \mathfrak{g}^*$ is the map given in Lemma 8.15, then*

$$\mathrm{image}(\Phi_Y, \varphi) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \mathrm{image}\,\Phi_Y \text{ and } s \geq \langle \sigma(\beta), j \rangle\}. \tag{8.19}$$

Proof Let $i_T\colon T \hookrightarrow G$ be the inclusion map. Let $\widetilde{\Phi}_Y\colon Y \to \mathfrak{g}^*$ be a G moment map, normalized so that $(i_T^*, j) \circ \widetilde{\Phi}_Y = (\Phi_Y, \varphi)$. Note that, in particular, $\langle \widetilde{\Phi}_Y([1,0,0], j\rangle = \varphi([1,0,0]) = 0$; cf Lemma 8.15. By Lemma 8.9(2), $\widetilde{\Phi}_Y$ is proper and each fiber contains at most one G orbit.

Let $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$ correspond to the defining monomial $P \in \mathrm{Hom}(G, S^1)$. Because $P \circ J$ is the identity map, $\langle \xi, j \rangle = 1$. Moreover, since the homomorphism J splits the short exact sequence (8.2), the map $(i_T, J)\colon T \times S^1 \to G$ is an isomorphism of groups. Hence, the induced map $(i_T^*, j)\colon \mathfrak{g}^* \to \mathfrak{t}^* \times \mathbb{R}$ is a linear isomorphism. Therefore, by the first paragraph, (Φ_Y, φ) is proper and each fiber contains at most one $T \times S^1$ orbit.

Finally, (8.19) follows easily from properties (1) and (2) of Lemma 8.15. □

8.20 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. Let $J \in \mathrm{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$; let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$; and let $\varphi\colon Y \to \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([1,0,0]) = 0$. Given $\kappa \in \mathbb{R}$, define*

$$Y_{J,\kappa} = \varphi^{-1}((-\infty, \kappa]) \subset Y.$$

(1) *If $\sigma\colon \mathrm{image}\,\Phi_Y \to \mathfrak{g}^*$ is the map given in Lemma 8.15, then the function*

$$\beta \mapsto \kappa - \langle \sigma(\beta), j \rangle \tag{8.21}$$

from $\Phi_Y(Y_{J,\kappa})$ to $\mathbb{R}$ is a Duistermaat–Heckman function for the restriction of Φ_Y to $Y_{J,\kappa}$; see Definition 8.6.

(2) *If $\kappa > 0$, then $\Phi_Y(Y_{J,\kappa})$ contains a neighbourhood of $\alpha = \Phi_Y([1,0,0])$ in $\Phi_Y(Y)$.*

(3) *The restriction of Φ_Y to $Y_{J,\kappa}$ is proper.*

Proof Since by Lemma 8.18 each level set of (Φ_Y, φ) contains at most a single $T\times S^1$ orbit, the Duistermaat–Heckman measure for the restriction of (Φ_Y, φ) to $Y_{J,\kappa}$ is Lebesgue measure on the set $(\Phi_Y, \varphi)(Y_{J,\kappa})$. The Duistermaat–Heckman measure for the restriction of Φ_Y to $Y_{J,\kappa}$ is the pushforward of this measure under the projection map from $\mathfrak{t}^* \times \mathbb{R}$ to $\mathfrak{t}^*$. Finally, by (8.19),

$$(\Phi_Y, \varphi)(Y_{J,\kappa}) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \operatorname{image} \Phi_Y \text{ and } \kappa \geq s \geq \langle \sigma(\beta), j\rangle\}. \tag{8.22}$$

Claim (1) follows immediately.

By Lemma 8.15, the function σ is continuous and $\langle\sigma(\alpha), j\rangle = \langle \widetilde{\Phi}_Y([1,0,0], j\rangle = \varphi([1,0,0]) = 0$. Therefore, if $\kappa > 0$, then (8.22) implies that there is a neighbourhood U of α in $\mathfrak{t}^*$ such that $\Phi_Y(Y) \cap U = \Phi_Y(Y_{J,\kappa}) \cap U$. This gives claim (2).

Since σ is continuous, (8.22) implies that the intersection $(K \times \mathbb{R}) \cap (\Phi_Y, \varphi)(Y_{J,\kappa})$ is compact for any compact set K. Moreover, by Lemma 8.18, (Φ_Y, φ) is proper. Claim (3) follows immediately. □

8.23 Corollary *For every tall complexity-one model Y, there exist Duistermaat–Heckman functions for truncations of the model. Each such function is well defined and continuous on a neighbourhood of α in* image Φ_Y.

Proof By Remark 8.5, there exist complementary circles to T in G. The result then follows from Lemma 8.20 with any $\kappa > 0$. □

8.24 Corollary *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model. Let ρ and ρ' be Duistermaat–Heckman functions for truncations of Y. Then the difference $\rho - \rho'$ is equal to an integral affine function on some neighbourhood of $\Phi_Y([1,0,0])$ in* image Φ_Y.

Proof Let J and J' be complementary circles to T in $G := T \times_H (S^1)^{h+1}$, and let j and j' be the corresponding elements of $\mathfrak{g}_{\mathbb{Z}}$. Let φ and φ' be the associated moment maps, normalized by $\varphi([1,0,0]) = \varphi'([1,0,0]) = 0$. Let $P\colon G \to S^1$ be the defining monomial. Because $P \circ J = P \circ J' = \mathrm{Id}_{S^1}$, there exists $\Delta j \in \mathfrak{t}_{\mathbb{Z}}$ such that

$$j - j' = i_T(\Delta j).$$

Let σ and σ' be the maps from image Φ_Y to $\mathfrak{g}^*$ that are associated to J and J', respectively, in Lemma 8.15. Let $\widetilde{\Phi}_Y$ and $\widetilde{\Phi}'_Y$ be the G moment maps, normalized as in Lemma 8.15. Because $i_T^* \circ \widetilde{\Phi}_Y = i_T^* \circ \widetilde{\Phi}'_Y$, there exists a real number c such that $\widetilde{\Phi}_Y - \widetilde{\Phi}'_Y = c\xi$, where $\xi \in \mathfrak{g}^*_{\mathbb{Z}}$ corresponds to $P \in \mathrm{Hom}(P, S^1)$.

Therefore, parts (1) and (2) of Lemma 8.15 imply that

$$\sigma - \sigma' = c\xi.$$

Let κ and κ' be positive numbers. Let $\rho_{J,\kappa}$ and $\rho_{J',\kappa'}$ be the Duistermaat–Heckman functions for the restrictions of Φ_Y to $\varphi^{-1}((-\infty,\kappa])$ and ${\varphi'}^{-1}((-\infty,\kappa'])$, respectively. By Lemma 8.20, there exists neighbourhood of $\Phi_Y([1,0,0])$ in image Φ_Y where

$$\begin{aligned}
\rho_{J,\kappa}(\beta) - \rho_{J',\kappa'}(\beta) &= \kappa - \langle \sigma(\beta), j\rangle - \kappa' + \langle \sigma'(\beta), j'\rangle \\
&= \kappa - \langle \sigma(\beta), j\rangle - \kappa' + \langle \sigma(\beta) - c\xi, j - i_T(\Delta j)\rangle \\
&= \kappa - \kappa' - c - \langle \iota_T^*(\sigma(\beta)), \Delta j\rangle \\
&= \kappa - \kappa' - c - \langle \beta, \Delta j\rangle.
\end{aligned}$$

Here, the penultimate equality uses the fact that $\langle \xi, j\rangle = 1$, and the last equality follows from Lemma 8.15. □

9 Compatibility of the Duistermaat–Heckman function and local existence

This section achieves two goals. In Proposition 9.2 we prove that a Duistermaat–Heckman function of a complexity-one space is compatible with its skeleton. In Proposition 9.10 we prove "local existence:" for any compatible values of our invariants, over sufficiently small open subsets of $\mathfrak{t}^*$ there exists a tall complexity-one space whose invariants take these values. The proofs of both propositions rely on a surgery that removes or adds exceptional orbits. In order to perform this surgery we identify a punctured neighbourhood of an exceptional orbit with a punctured neighbourhood of a nonexceptional orbit. This identification is done in Lemma 9.1.

We begin by setting up the relevant notation. Let C be a Delzant cone in $\mathfrak{t}^*$, and let (M_C, ω_C, Φ_C) be a symplectic toric manifold whose moment image is C. We recall how to obtain such a manifold. By Definition 1.17 there exists an integer $0 \le k \le n$ and a linear isomorphism $A\colon \mathbb{R}^n \to \mathfrak{t}^*$ that sends $\mathbb{Z}^n$ onto the weight lattice ℓ^* such that $C = \alpha + A(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$. We may take M_C to be the manifold

$$M_C := \mathbb{C}^k \times (T^*S^1)^{n-k},$$

with the standard symplectic structure; with the T–action given by the isomorphism $T \to (S^1)^k \times (S^1)^{n-k}$ induced by $A^*\colon \mathfrak{t} \to \mathbb{R}^n$; and with the moment map $\Phi_C(z,a,\eta) = \alpha + A(\pi|z_1|^2, \dots, \pi|z_k|^2, \eta_1, \dots, \eta_{n-k})$, where we have $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $(a,\eta) \in (S^1)^{n-k} \times \mathbb{R}^{n-k} \cong (T^*S^1)^{n-k}$.

We also consider the manifold $M_C \times \mathbb{C}$ with the product symplectic structure. This manifold admits a T action on the first factor with moment map $(m,z) \mapsto \Phi_C(m)$ for all $m \in M_C$ and $z \in \mathbb{C}$. It also admits a toric action of $T \times S^1$ with moment map $(m,z) \mapsto (\Phi_C(m), \kappa + \pi|z|^2)$, for any $\kappa \in \mathbb{R}$.

Given $\epsilon > 0$, let D_ϵ be the disk

$$D_\epsilon = \{z \in \mathbb{C} \mid \pi|z|^2 < \epsilon\}.$$

9.1 Lemma *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity-one model with moment map $\Phi_Y\colon Y \to \mathfrak{t}^*$. Let J be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$ and let $\varphi\colon Y \to \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([1,0,0]) = 0$. Let $\alpha = \Phi_Y([1,0,0])$.*

(1) *Let V be a neighbourhood of the central orbit $T \cdot [1,0,0]$ in Y. Then there exists a neighbourhood U of α in $\mathfrak{t}^*$ and a positive number κ' such that the preimage $(\Phi_Y, \varphi)^{-1}(U \times (-\infty, \kappa'))$ is contained in V.*

(2) *Let (M_C, ω_C, Φ_C) be a symplectic toric manifold whose moment image is*[6] *$C :=$ image Φ_Y. For every positive number κ, there exists $\epsilon > 0$, a neighbourhood U of $\alpha \in \mathfrak{t}^*$, and a T–equivariant symplectomorphism between*

$$\Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa, \kappa+\epsilon)) \subset Y \quad \text{and} \quad \Phi_C^{-1}(U) \times (D_\epsilon \smallsetminus \{0\}) \subset M_C \times \mathbb{C}$$

that intertwines $\varphi\colon Y \to \mathbb{R}$ and the map $(m,z) \mapsto \kappa + \pi|z|^2$. Here, T acts only on the first factor of $M_C \times \mathbb{C}$.

Proof Let j be the element of $\mathbb{Z}_G \subset \mathfrak{g}$ that corresponds to $J \in \mathrm{Hom}(S^1, G)$. Let $\sigma\colon$ image $\Phi_Y \to \mathfrak{g}^*$ be given as in Lemma 8.15. Then by Lemma 8.15 and Lemma 8.18, we have the following results:

(i) The map σ is continuous and $\langle \sigma(\alpha), j \rangle = 0$.

(ii) $(\Phi_Y, \varphi)\colon Y \to \mathfrak{t}^* \times \mathbb{R}$ is proper and each fiber contains at most one orbit.

(iii) $\mathrm{image}(\Phi_Y, \varphi) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \Phi_Y(Y) \text{ and } s \geq \langle \sigma(\beta), j \rangle\}$.

Let V be a neighbourhood of $T \cdot [1,0,0]$ in Y. By (ii), there exists a neighbourhood U of α in $\mathfrak{t}^*$ and a positive number κ' such that the preimage $(\Phi_Y, \varphi)^{-1}(U \times (-\kappa', \kappa'))$ is contained in V. Therefore, by (i) and (iii) above, after possibly shrinking U, the preimage $(\Phi_Y, \varphi)^{-1}(U \times (-\infty, -\kappa'])$ is empty; this proves claim (1).

Moreover, if κ is a positive number, then by (i) and (iii) above, there exists a neighbourhood W of $(\alpha, \kappa) \in \mathfrak{t}^* \times \mathbb{R}$ such that $\mathrm{image}(\Phi_Y, \varphi) \cap W = (C \times \mathbb{R}) \cap W$.

[6] The moment cone is a Delzant cone by Lemma 7.2.

Consider Y and $M_C \times S^1 \times \mathbb{R}$ as symplectic manifolds with toric $T \times S^1$ actions and with moment maps (Φ_Y, φ) and $(\Phi_C, \kappa + \mathrm{proj}_{\mathbb{R}})$, respectively. By the argument above, their moment images coincide with $C \times \mathbb{R}$, hence with each other, on a neighbourhood of (α, κ). Because (ii) holds, the local normal form for toric actions (see [10]) implies that, after possibly shrinking this neighbourhood, its preimages in Y and in $M_C \times S^1 \times \mathbb{R}$ are isomorphic. Thus, after possibly shrinking U, for sufficiently small ϵ the subset $\Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa - \epsilon, \kappa + \epsilon))$ of Y is isomorphic to the subset $\Phi_C^{-1}(U) \times S^1 \times (\kappa - \epsilon, \kappa + \epsilon)$ of $M_C \times S^1 \times \mathbb{R}$. Restricting to the preimage of $U \times (\kappa, \kappa + \epsilon)$, and composing further with the map $re^{i\theta} \mapsto (e^{i\theta}, \kappa + \pi r^2)$, which is an S^1–equivariant symplectomorphism from $D_\epsilon \smallsetminus \{0\}$ onto $S^1 \times (\kappa, \kappa + \epsilon)$ that carries the map $z \mapsto \kappa + \pi |z|^2$ to the map $(\lambda, s) \mapsto s$, we get claim (2). □

We are now ready to prove the first main result of this section.

9.2 Proposition *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity-one space. The skeleton and Duistermaat–Heckman function of M are compatible.*

Proof Fix a point α in the moment image $\Delta = \Phi(M)$. By Lemma 7.1 and Corollary 2.5, there are only finitely many exceptional orbits x in $\Phi^{-1}(\alpha)$. For each such x, let $Y_x = T \times_{H_x} \times \mathbb{C}^{h_x + 1} \times \mathfrak{h}_x^0$ be the tall complexity-one model with moment map $\Phi_x \colon Y_x \to \mathfrak{t}^*$ corresponding to x.

By applying surgery to neighbourhoods of the exceptional orbits, we will construct a complexity-one space (M', ω', Φ', U) with moment image $\Delta \cap U$ that has no exceptional orbits, where $U \subset \mathcal{T}$ is a convex open neighbourhood of α. Additionally, the Duistermaat–Heckman function $\rho' \colon \Delta \cap U \to \mathbb{R}_{>0}$ of M' will satisfy

$$\rho' = \rho - \sum_x \rho_x, \tag{9.3}$$

where ρ_x is the Duistermaat–Heckman function for a truncation of the model Y_x for each exceptional orbit x in $\Phi^{-1}(\alpha)$, and the sum is over all such orbits. Because M' is a complexity-one space with no exceptional orbits, ρ' is integral affine on $\Delta \cap U$. The proposition will follow immediately.

By Lemma 7.3 there exists a Delzant cone C at α and a convex open neighbourhood U of α in $\mathfrak{t}^*$ such that $\Delta \cap U = C \cap U$. Let (M_C, ω_C, Φ_C) be a toric manifold with moment image C.

By the local normal form theorem, for each exceptional orbit x in $\Phi^{-1}(\alpha)$ there exists an isomorphism Ψ_x from an invariant neighbourhood V_x of the central orbit $T \cdot [1, 0, 0]$

in Y_x to an invariant open subset of M that carries $T\cdot[1,0,0]$ to x. Moreover, we may assume that the closures in M of the open subsets $\Psi_x(V_x)$ are disjoint.

Given an exceptional orbit x in $\Phi^{-1}(\alpha)$, let J_x be a complementary circle to T in $T\times_{H_x}(S^1)^{h_x+1}$; see Remark 8.5. Let φ_x be a moment map for the resulting circle action, normalized by $\varphi_x([1,0,0])=0$. By Lemma 7.3, image $\Phi_x = C$. By Lemma 9.1, for sufficiently small $\epsilon>0$, after possibly shrinking U, there exists $\kappa_x>0$ such that $\Phi_x^{-1}(U)\cap\varphi_x^{-1}((-\infty,\kappa_x+\epsilon))$ is contained in V_x; moreover, there exists an isomorphism between

$$\Phi^{-1}(U)\cap\Psi_x(\varphi_x^{-1}((\kappa_x,\kappa_x+\epsilon)))\subset M \quad \text{and} \tag{9.4}$$

$$\Phi_C^{-1}(U)\times(D_\epsilon\smallsetminus\{0\})\subset M_C\times\mathbb{C} \tag{9.5}$$

that intertwines $\varphi_x\circ\Psi_x^{-1}\colon \Psi_x(V_x)\to\mathbb{R}$ and the map $(m,z)\mapsto\kappa_x+\pi|z|^2$.

We construct M' by gluing together the spaces

$$\Phi^{-1}(U)\smallsetminus\bigsqcup_x\Psi_x(\varphi_x^{-1}(-\infty,\kappa_x])\subset M \quad \text{and} \tag{9.6}$$

$$\bigsqcup_x\Phi_C^{-1}(U)\times D_\epsilon\subset M_C\times\mathbb{C} \tag{9.7}$$

by identifying their isomorphic open subsets (9.4) and (9.5) for every exceptional orbit x in $\Phi^{-1}(\alpha)$.

The space M' is the union of two *closed* subspaces: the images in M' of

$$\Phi^{-1}(U)\smallsetminus\bigsqcup_x\Psi_x(\varphi_x^{-1}(-\infty,\kappa_x+\tfrac{1}{2}\epsilon))\subset M \quad \text{and} \tag{9.8}$$

$$\bigsqcup_x\Phi_C^{-1}(U)\times\bar{D}_{(1/2)\epsilon}\subset M_C\times\mathbb{C}, \tag{9.9}$$

where $\bar{D}_{(1/2)\epsilon}\subset\mathbb{C}$ is the closed disk. Because each of these is Hausdorff, M' is Hausdorff.

By construction, M' is a manifold with a T action, a symplectic form ω', and a moment map Φ'. Moreover, as maps to U, the restriction of Φ to (9.8) and the restriction of Φ_C to (9.9) are both proper, and so Φ' is proper as well.

This yields a complexity-one space (M',ω',Φ',U) with moment image $\Delta\cap U$ that has no exceptional orbits in $\Phi'^{-1}(\alpha)$. By Lemma 7.1, the restriction of Φ' to the set of exceptional orbits is proper. Therefore, after possibly shrinking U further, we may assume that M' has no exceptional orbits.

By Lemma 8.20, after possibly shrinking U, there is a well-defined Duistermaat–Heckman function $\rho_x\colon \Delta \cap U \to \mathbb{R}_{>0}$ for the restriction of Φ_x to $\varphi_x^{-1}((-\infty, \kappa_x])$, and thus for the restriction of Φ to $\Psi_x(\varphi_x^{-1}((-\infty, \kappa_x]))$. Since M' and (9.6) differ by a set of measure zero, and since ρ_x is a Duistermaat–Heckman function for a truncation of the model Y_x, the Duistermaat–Heckman function ρ' for (M', ω', Φ') satisfies (9.3), as required. □

We proceed to the second main result of this section.

9.10 Proposition (Local existence) *Let (S, π) be a tall skeleton over an open subset $\mathcal{T} \subset \mathfrak{t}^*$, let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π), and let g be a nonnegative integer.*

1. *For any $\alpha \in \mathcal{T}$ there exists a convex open neighbourhood $U \subset \mathcal{T}$ of α and a tall complexity-one space of genus g over U with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$.*
2. *Let $\rho\colon \Delta \to \mathbb{R}_{>0}$ be a function that is compatible with (S, π). Then for any $\alpha \in \mathcal{T}$ there exists a convex open neighbourhood $U \subset \mathcal{T}$ of α and a tall complexity-one space of genus g over U with moment image $\Delta \cap U$, whose skeleton is isomorphic to $S|_U$ and whose Duistermaat–Heckman function is $\rho|_{\Delta \cap U}$.*

Proof of part (1) Let α be a point in $\mathcal{T}$ and fix an arbitrary positive number b. By Corollary 2.5, the level set $\pi^{-1}(\alpha)$ in S is finite. We will choose a convex open neighbourhood $U \subset \mathcal{T}$ of α and construct a complexity-one space (M', ω', Φ', U) of genus g with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$. Additionally, the Duistermaat–Heckman function ρ' of M' will be smaller than b near α; moreover, it will satisfy

$$\rho' = c + \sum_{s \in \pi^{-1}(\alpha)} \rho_s, \tag{9.11}$$

where c is a positive real number and where each ρ_s is the Duistermaat–Heckman function for a truncation of the model Y_s associated to $s \in S$.

Choose a closed symplectic 2–manifold (Σ, η) of genus g and a positive number κ_s for each $s \in \pi^{-1}(\alpha)$ such that

$$\int_\Sigma \eta + \sum_{s \in \pi^{-1}(\alpha)} \kappa_s < b. \tag{9.12}$$

Since Δ is a Delzant subset, there exists a convex open neighbourhood U of α and a Delzant cone C at α such that $\Delta \cap U = C \cap U$. Let (M_C, ω_C, Φ_C) be a toric manifold with moment image C.

By the Darboux theorem, if $\epsilon > 0$ is sufficiently small, for each $x \in \pi^{-1}(\alpha)$ we can choose an open set W_s in Σ and a symplectomorphism $\Psi_s\colon D_\epsilon \to W_s$; let $x_s = \Psi_s(0)$. Furthermore, we may assume that the closures of the sets W_s are disjoint. Clearly, for each $s \in \pi^{-1}(\alpha)$, there is an isomorphism between

$$\Phi_C^{-1}(U) \times (D_\epsilon \smallsetminus \{0\}) \subset M_C \times \mathbb{C} \quad \text{and} \tag{9.13}$$

$$\Phi_C^{-1}(U) \times (W_s \smallsetminus \{x_s\}) \subset M_C \times \Sigma, \tag{9.14}$$

given by $(m, z) \mapsto (m, \Psi_s(z))$.

Let $Y_s = T \times_{H_s} \mathbb{C}^{h_s+1} \times \mathfrak{h}_s^0$ be the tall complexity-one model with moment map $\Phi_s\colon Y_s \to \mathfrak{t}^*$ corresponding to $s \in \pi^{-1}(\alpha)$. Let J_s be a complementary circle to T in $T \times_{H_s} (S^1)^{h_s+1}$; see Remark 8.5. Let φ_s be a moment map for the resulting circle action, normalized by $\varphi_s([1,0,0]) = 0$. Since (S, π) and Δ are compatible, image $\Phi_{Y_s} = C$. By part (2) of Lemma 9.1, after possibly shrinking U and ϵ, there exists an isomorphism between

$$\Phi_s^{-1}(U) \cap \varphi_s^{-1}((\kappa_s, \kappa_s + \epsilon)) \subset Y_s \quad \text{and} \tag{9.15}$$

$$\Phi_C^{-1}(U) \times (D_\epsilon \smallsetminus \{0\}) \subset M_C \times \mathbb{C} \tag{9.16}$$

that intertwines $\varphi_s\colon Y_s \to \mathbb{R}$ and the map $(m, z) \mapsto \kappa_s + \pi |z|^2$.

Because the sets (9.14) and (9.15) are both isomorphic to the same set, there exists an isomorphism between them that intertwines the map $(m, \Psi_s(z)) \mapsto \kappa_s + \pi |z|^2$ and the map φ_s. We construct M' by gluing together the spaces

$$\Phi_C^{-1}(U) \times (\Sigma \smallsetminus \{x_s\}_{s \in \pi^{-1}(\alpha)}) \subset M_C \times \Sigma \quad \text{and} \tag{9.17}$$

$$\bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s + \epsilon)) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s \tag{9.18}$$

by identifying their isomorphic open subsets (9.14) and (9.15) for every $s \in \pi^{-1}(\alpha)$.

The space M' is the union of two *closed* subspaces: the images in M' of

$$\Phi_C^{-1}(U) \times \Big(\Sigma \smallsetminus \bigsqcup_{s \in \pi^{-1}(\alpha)} \Psi_s(D_{\epsilon/2})\Big) \subset M_C \times \Sigma \quad \text{and} \tag{9.19}$$

$$\bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s + \epsilon/2]) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s. \tag{9.20}$$

Because each of these is Hausdorff, M' is Hausdorff.

By construction, M' is a manifold with a T action, a symplectic form ω', and a moment map Φ'. Moreover, as a map to U, the restriction of Φ_C to (9.19) is proper, and the restriction of each Φ_s to (9.20) is proper by part (3) of Lemma 8.20. Therefore, Φ' is proper as a map to U.

This yields a complexity-one space (M', ω', Φ', U) with moment image $\Delta \cap U$, and an isomorphism from M'_{exc} onto a neighbourhood of $\pi^{-1}(\alpha)$ in S. Because π is proper, after possibly shrinking U further, we may assume that the image of M'_{exc} in S is $S|_U$. Since it is straightforward to check that M' has genus g, it remains to show that the Duistermaat–Heckman function ρ' of M' is smaller than b near α.

We can write M' as the disjoint union of (9.17) and

$$\bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s]) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s.$$

The function that takes the constant value $\int_\Sigma \eta$ on $C \cap U$ is a Duistermaat–Heckman function for the set (9.17). Moreover, by Lemma 8.20, after possibly shrinking U, for each $s \in \pi^{-1}(\alpha)$, the function $\rho_s \colon \Delta \cap U \to \mathbb{R}$ given by $\rho_s(\alpha) = \kappa_s - \langle \sigma(\alpha), j \rangle$ is a Duistermaat–Heckman function for the restriction of Φ_s to $\Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s])$. In particular, ρ_s is a Duistermaat–Heckman function for a truncation of the model Y_s. Thus, the Duistermaat–Heckman function ρ' for M' satisfies (9.11), with $c = \int_\Sigma \eta$. Since, moreover, $\rho_s(\alpha) = \kappa_s - \langle \sigma(\alpha), j \rangle = \kappa_s$ (by Lemma 8.15), Equation (9.12) implies that $\rho'(\alpha) < b$. By continuity, ρ' is smaller than b near α. □

Proof of part (2) By part (1) there exists a complexity-one space (M', ω', Φ', U) of genus g with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$ such that the Duistermaat–Heckman function ρ' of M' satisfies $\rho'(\alpha) < \rho(\alpha)$ and (9.11). Thus, since ρ is compatible with (S, π), Corollary 8.24 implies that $\rho - \rho'$ is integral affine on a neighbourhood of α in Δ. (Alternatively, this follows from Proposition 9.2.) Thus, since we have $\rho' < \rho$ near α, there exist $\kappa > 0$ and $\zeta \in \ell \subset \mathfrak{t}$ such that $\rho(\beta) - \rho'(\beta) = \kappa - \langle \beta - \alpha, \zeta \rangle$.

The tall complexity-one Hamiltonian T–manifold $Y = M_C \times \mathbb{C}$ with moment map $\Phi_Y(m, z) = \Phi_C(m)$ is isomorphic to the tall complexity-one model $T \times_H (\mathbb{C}^k \times \mathbb{C}) \times \mathfrak{h}^0$, where $T \cong (S^1)^n$, where $H \cong \{1\}^{n-k} \times (S^1)^k$, and where $\mathfrak{h}^0 \cong \mathbb{R}^{n-k}$. So we can apply Lemma 8.20 to it. We identify the corresponding torus $G = T \times_H (S^1)^{k+1}$ with $T \times S^1$, and we identify $[1, 0, 0]$ with the point $(m_0, 0)$ such that $\Phi_C(m_0) = \alpha$.

Define $J \colon S^1 \to T \times S^1$ by $J(\lambda) = (\lambda^\zeta, \lambda)$. Let $\varphi \colon Y \to \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi(m_0, 0) = 0$. The $T \times S^1$ moment map $\widetilde{\Phi}_Y \colon Y \to \mathfrak{t}^* \times \mathbb{R}$ given by $\widetilde{\Phi}_Y(m, z) = (\Phi_C(m), \pi |z|^2 - \langle \alpha, \zeta \rangle)$ is normalized as in Lemma 8.15. Its moment image is $\widetilde{\Phi}_Y(Y) = C \times [-\langle \alpha, \zeta \rangle, \infty)$. Hence, the map $\sigma\colon C \to \mathfrak{t}^* \times \mathbb{R}$ described in Lemma 8.15 is $\sigma(\beta) = (\beta, -\langle \alpha, \zeta \rangle)$. Therefore, by Lemma 8.20, the restriction of Φ_Y to $\varphi^{-1}((-\infty, \kappa])$ is proper and the Duistermaat–Heckman function for this restriction is $\rho_{J,\kappa}(\beta) = \kappa - \langle \beta - \alpha, \zeta \rangle$.

By part (2) of Lemma 9.1, there exists a T equivariant symplectomorphism from $\Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa, \kappa + \epsilon))$ to $\Phi_C^{-1}(U) \times (D_\epsilon \smallsetminus \{0\})$ that carries the map φ to the map $(m, z) \mapsto \kappa + \pi |z|^2$. If we glue this local model into M' following the same procedure as explained in the above proof of part (1), we get a new complexity-one space, which satisfies all our requirements. □

10 Proof of the existence theorems

We are now ready to prove the existence theorems that we stated in Section 1.

Proof of Theorem 3 By part (2) of Proposition 9.10 for each point in $\mathcal{T}$ there exists a convex open subset $U \subset \mathcal{T}$ containing the point and there exists a complexity-one space of genus $g := \operatorname{genus}(\Sigma)$ over U whose moment image is $\Delta \cap U$, whose skeleton is isomorphic to $S|_U$, and whose Duistermaat–Heckman function is $\rho|_U$. The result now follows from Proposition 6.2. □

Proof of Theorem 1 This theorem is an immediate consequence of Lemma 7.3, Proposition 9.2, and Theorem 3. □

Proof of Theorem 2 This theorem is an immediate consequence of Lemma 10.1 below and Theorem 3. □

10.1 Lemma *Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open subset. Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset. Let (S, π) be a skeleton over $\mathcal{T}$. If Δ and (S, π) are compatible, then there exists a function $\rho\colon \Delta \to \mathbb{R}_{>0}$ that is compatible with (S, π).*

Proof By part (1) of Proposition 9.10, there exists a cover $\mathfrak{U}$ of $\mathcal{T}$ by convex sets and, for each $U \in \mathfrak{U}$, a complexity-one space $(M_U, \omega_U, \Phi_U, U)$ whose moment image is $U \cap \Delta$ and whose skeleton is $S|_U$. Let

$$\rho_U\colon U \cap \Delta \to \mathbb{R}$$

be its Duistermaat–Heckman function. By Proposition 9.2, ρ_U is compatible with the moment image $\Delta \cap U$ and the skeleton $(S|_U, \pi_U)$. Hence, by Definition 1.24, and Corollary 8.24, on every intersection the difference

$$\rho_U|_{U \cap V} - \rho_V|_{U \cap V}$$

is locally an integral affine function. Hence this difference is given by a locally constant function

$$h_{UV}\colon U \cap V \to \ell \oplus \mathbb{R}.$$

Let $\mathcal{A}$ denote the sheaf of locally constant functions to $\ell \oplus \mathbb{R}$. Because Δ is convex, the Čech cohomology $H^1(\Delta, \mathcal{A})$ is trivial. Hence, after possibly passing to a refinement of the cover, there exist locally constant functions

$$h_U\colon U \to \ell \oplus \mathbb{R} \tag{10.2}$$

such that $h_U - h_V = h_{UV}$ on $U \cap V$. Therefore, we can define $\rho\colon \Delta \to \mathbb{R}$ by $\rho|_U = \rho_U - h_U$ for all $U \in \mathfrak{U}$, where h_U also denotes the integral affine function given by (10.2). □

Acknowledgements Yael Karshon is partially supported by an NSERC Discovery Grant. Susan Tolman was partially supported by NSF Grant DMS 0707122 and is partially supported by NSF Grant DMS 1206365. Finally, we would like to thank the referee for his or her careful reading of the paper.

References

[1] **K Ahara**, **A Hattori**, *4–dimensional symplectic S^1–manifolds admitting moment map*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 38 (1991) 251–298 MR1127083

[2] **K Altmann**, **J Hausen**, *Polyhedral divisors and algebraic torus actions*, Math. Ann. 334 (2006) 557–607 MR2207875

[3] **K Altmann**, **J Hausen**, **H Süss**, *Gluing affine torus actions via divisorial fans*, Transform. Groups 13 (2008) 215–242 MR2426131

[4] **K Altmann**, **L Petersen**, *Cox rings of rational complexity-one T–varieties*, J. Pure Appl. Algebra 216 (2012) 1146–1159 MR2875333

[5] **M F Atiyah**, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14 (1982) 1–15 MR642416

[6] **M Audin**, *Hamiltoniens périodiques sur les variétés symplectiques compactes de dimension* 4, from: "Géométrie symplectique et mécanique", (C Albert, editor), Lecture Notes in Math. 1416, Springer, Berlin (1990) 1–25 MR1047474

[7] **M Audin**, *The topology of torus actions on symplectic manifolds*, Progress in Mathematics 93, Birkhäuser, Basel (1991) MR1106194

[8] **C Bjorndahl**, **Y Karshon**, *Revisiting Tietze–Nakajima: Local and global convexity for maps*, Canad. J. Math. 62 (2010) 975–993 MR2730351

[9] **R Chiang**, *Complexity one Hamiltonian* SU(2) *and* SO(3) *actions*, Amer. J. Math. 127 (2005) 129–168 MR2115663

[10] **T Delzant**, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988) 315–339 MR984900

[11] **R Godement**, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris (1973) MR0345092

[12] **E González**, *Classifying semi-free Hamiltonian S^1–manifolds*, Int. Math. Res. Not. 2011 (2011) 387–418 MR2764868

[13] **V Guillemin**, **E Lerman**, **S Sternberg**, *Symplectic fibrations and multiplicity diagrams*, Cambridge Univ. Press (1996) MR1414677

[14] **V Guillemin**, **S Sternberg**, *Convexity properties of the moment mapping*, Invent. Math. 67 (1982) 491–513 MR664117

[15] **V Guillemin**, **S Sternberg**, *A normal form for the moment map*, from: "Differential geometric methods in mathematical physics", (S Sternberg, editor), Math. Phys. Stud. 6, Reidel, Dordrecht (1984) 161–175 MR767835

[16] **A Haefliger**, **É Salem**, *Actions of tori on orbifolds*, Ann. Global Anal. Geom. 9 (1991) 37–59 MR1116630

[17] **P Iglésias**, *Les* $\mathrm{SO}(3)$*–variétés symplectiques et leur classification en dimension* 4, Bull. Soc. Math. France 119 (1991) 371–396 MR1125672

[18] **Y Karshon**, *Periodic Hamiltonian flows on four-dimensional manifolds*, Mem. Amer. Math. Soc. 672, AMS (1999) MR1612833

[19] **Y Karshon**, **E Lerman**, *Non-compact symplectic toric manifolds* `arXiv:0907.2891`

[20] **Y Karshon**, **S Tolman**, *Centered complexity one Hamiltonian torus actions*, Trans. Amer. Math. Soc. 353 (2001) 4831–4861 MR1852084

[21] **Y Karshon**, **S Tolman**, *Complete invariants for Hamiltonian torus actions with two dimensional quotients*, J. Symplectic Geom. 2 (2003) 25–82 MR2128388

[22] **Y Karshon**, **S Tolman**, *The Gromov width of complex Grassmannians*, Algebr. Geom. Topol. 5 (2005) 911–922 MR2171798

[23] **Y Karshon**, **F Ziltener**, *Hamiltonian group actions on exact symplectic manifolds with proper momentum maps are standard*, in preparation

[24] **G Kempf**, **F F Knudsen**, **D Mumford**, **B Saint-Donat**, *Toroidal embeddings, I*, Lecture Notes in Mathematics 339, Springer, Berlin (1973) MR0335518

[25] **M K Kim**, *Frankel's theorem in the symplectic category*, Trans. Amer. Math. Soc. 358 (2006) 4367–4377 MR2231381

[26] **F Knop**, *Automorphisms of multiplicity free Hamiltonian manifolds*, J. Amer. Math. Soc. 24 (2011) 567–601 MR2748401

[27] **J L Koszul**, *Sur certains groupes de transformations de Lie*, from: "Géométrie différentielle: Colloques Int. CNRS", CNRS, Paris (1953) 137–141 MR0059919

[28] **E Lerman**, **E Meinrenken**, **S Tolman**, **C Woodward**, *Nonabelian convexity by symplectic cuts*, Topology 37 (1998) 245–259 MR1489203

[29] **E Lerman**, **S Tolman**, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. 349 (1997) 4201–4230 MR1401525

[30] **H Li**, *Semi-free Hamiltonian circle actions on* 6*–dimensional symplectic manifolds*, Trans. Amer. Math. Soc. 355 (2003) 4543–4568 MR1990761

[31] **H Li**, *On the construction of certain* 6*–dimensional symplectic manifolds with Hamiltonian circle actions*, Trans. Amer. Math. Soc. 357 (2005) 983–998 MR2110428

[32] **H Li**, **S Tolman**, *Hamiltonian circle actions with minimal fixed sets*, Internat. J. Math. 23 (2012) 1250071, 36 MR2949214

[33] **I V Losev**, *Proof of the Knop conjecture*, Ann. Inst. Fourier (Grenoble) 59 (2009) 1105–1134 MR2543664

[34] **C-M Marle**, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*, Rend. Sem. Mat. Univ. Politec. Torino 43 (1985) 227–251 MR859857

[35] **D McDuff**, *Some* 6*–dimensional Hamiltonian* S^1*–manifolds*, J. Topol. 2 (2009) 589–623 MR2546587

[36] **D Morton**, *GKM manifolds with low Betti numbers*, PhD thesis, University of Illinois at Urbana-Champaign (2011) MR3029681 Available at `http://search.proquest.com//docview/1010586168`

[37] **J Moser**, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120 (1965) 286–294 MR0182927

[38] **S Nakajima**, *Über konvexe Kurven and Flächen*, Tohoku Math. Journ. 29 (1928) 227–230

[39] **H Nozawa**, *Five dimensional* K*–contact manifolds of rank* 2 `arXiv:0907.0208`

[40] **G W Schwarz**, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Études Sci. Publ. Math. (1980) 37–135 MR573821

[41] **R Sikorski**, *Wstęp do geometrii różniczkowej*, Państwowe Wydawnictwo Naukowe, Warsaw (1972) MR0467544

[42] **R Sjamaar**, *Convexity properties of the moment mapping re-examined*, Adv. Math. 138 (1998) 46–91 MR1645052

[43] **H Tietze**, *Über Konvexheit im kleinen und im großen und über gewisse den Punkten einer Menge zugeordnete Dimensionszahlen*, Math. Z. 28 (1928) 697–707 MR1544985

[44] **D A Timashëv**, G*–manifolds of complexity* 1, Uspekhi Mat. Nauk 51 (1996) 213–214 MR1406071 In Russian; translated in *Russian Math. Surveys* **51**: (1996), 567–568.

[45] **D A Timashëv**, *Classification of* G*–manifolds of complexity* 1, Izv. Ross. Akad. Nauk Ser. Mat. 61 (1997) 127–162 MR1470147 In Russian; translated in *Izv. Math.* **61**: (1997), 363–397.

[46] **S Tolman**, *Examples of non-Kähler Hamiltonian torus actions*, Invent. Math. 131 (1998) 299–310 MR1608575

[47] **S Tolman**, *On a symplectic generalization of Petrie's conjecture*, Trans. Amer. Math. Soc. 362 (2010) 3963–3996 MR2638879

[48] **R Vollmert**, *Toroidal embeddings and polyhedral divisors*, Int. J. Algebra 4 (2010) 383–388 MR2652252

[49] **C Woodward**, *The classification of transversal multiplicity-free group actions*, Ann. Global Anal. Geom. 14 (1996) 3–42 MR1375064

Department of Mathematics, University of Toronto
40st George Street, 6th floor, Toronto, ON M5S 2E4, Canada

Department of Mathematics, University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801, USA

karshon@math.toronto.edu, stolman@math.uiuc.edu

http://www.math.toronto.edu/karshon, http://www.math.uiuc.edu/~stolman

Proposed: Frances Kirwan Received: 9 May 2012
Seconded: Leonid Polterovich, Richard Thomas Revised: 23 February 2013