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Nonnegatively curved 5–manifolds with almost maximal symmetry rank

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We show that a closed, simply connected, nonnegatively curved 5–manifold admitting an effective, isometric T^2 action is diffeomorphic to one of S^5, $S^3 \times S^2$, $S^3 \tilde{\times} S^2$ or the Wu manifold $SU(3)/SO(3)$.

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1 Introduction

The classification of Riemannian manifolds with positive and, more generally, nonnegative sectional curvature is a long-standing open problem in Riemannian geometry. As a step towards general classification results one may consider manifolds whose isometry group is large. This has been a fruitful avenue of research (see, for example, the surveys by Grove [20; 21], Wilking [46] and Ziller [48]). It is well known that the isometry group of a compact Riemannian manifold is a compact Lie group. In the context of this paper, the measure for the "size" of an isometry group is its rank. In particular, we are interested in manifolds with nonnegative curvature that have almost maximal symmetry rank, where the *symmetry rank* of a Riemannian manifold M is defined to be the rank of the isometry group of M.

Grove and Searle [22] showed that the symmetry rank of a closed, positively curved, Riemannian n–manifold is bounded above by $\lfloor (n+1)/2 \rfloor$ and classified closed, positively curved Riemannian manifolds with maximal symmetry rank up to diffeomorphism. For a closed, positively curved Riemannian n–manifold of almost maximal symmetry rank, that is, one whose isometry group has rank $\lfloor (n-1)/2 \rfloor$, Rong [39] found topological restrictions for all dimensions (distinguishing cases for even and odd ones) and showed that a closed, simply connected, positively curved Riemannian 5–manifold with almost maximal symmetry rank, that is, with an effective isometric T^2 action, must be homeomorphic to the 5–sphere (in fact, it will be diffeomorphic to it as a consequence of the Generalized Poincaré conjecture). Later, Wilking [45] improved these results significantly for closed, positively curved, simply connected n–manifolds of dimension $n \geq 10$, considering actions of rank approximately $n/4$.

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The maximal symmetry rank for closed, simply connected n–manifolds with nonnegative curvature and dimension $n \leq 9$ is $\lfloor 2n/3 \rfloor$ (see Galaz-Garcia and Searle [15]). Kleiner [25] and Searle and Yang [41] independently classified up to homeomorphism closed, simply connected 4–manifolds of nonnegative curvature with an effective isometric circle action, corresponding to the almost maximal symmetry rank case in dimension 4. In [15], the authors classified up to diffeomorphism closed, simply connected, nonnegatively curved Riemannian manifolds of dimensions 3, 4, 5 and 6 with maximal symmetry rank. In this paper we address the case of almost maximal symmetry rank for closed, simply connected, nonnegatively curved Riemannian manifolds in dimensions 3, 4 and 5. Our main result is the following.

Theorem A *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold. If T^2 acts isometrically and (almost) effectively on M^5, then M^5 is diffeomorphic to one of S^5, $S^3 \times S^2$, $S^3 \tilde{\times} S^2$ (the nontrivial S^3–bundle over S^2) or the Wu manifold $SU(3)/SO(3)$.*

We remark that the 5–manifolds listed in Theorem A are all the known examples of closed, simply connected 5–manifolds with nonnegative curvature and these manifolds are all the closed, simply connected 5–dimensional homogeneous spaces or biquotients of Lie groups (see DeVito [7] and Pavlov [33]). We also point out that the 5–manifolds listed in Theorem A coincide with the closed, simply connected 5–manifolds that are elliptic (see Paternain and Petean [32]). Further, each one of these 5–manifolds M is rationally elliptic, that is, $\dim \pi_*(M) \otimes \mathbb{Q} < \infty$, thus satisfying the Ellipticity conjecture, which states that all closed, simply connected manifolds of (almost) nonnegative curvature are rationally elliptic [20]. It is also worth noting that these are exactly the 5–dimensional topological manifolds M for which $\mathrm{cat}_{S^2}(M) = 2$, that is, M can be covered by two open subsets W_1, W_2 such that the inclusions $W_i \hookrightarrow M$ factor homotopically through maps $W_i \to S^2$ (see Gómez-Larrañaga, González-Acuña and Heil [18]).

This paper is divided into seven sections. The first two sections comprise the introduction and basic tools we will use throughout. In Section 3, using classification results for smooth circle actions on 3– and 4–manifolds, in combination with restrictions imposed by nonnegative curvature, we classify closed, orientable manifolds with nonnegative curvature and almost maximal symmetry rank in dimension 3 and recall the classification of closed, simply connected manifolds with nonnegative curvature and almost maximal symmetry rank in dimension 4. In Section 4 we consider the problem of almost maximal symmetry rank in dimension 5 from a purely topological perspective and in Section 5 we find restrictions imposed by nonnegative curvature. In Section 6 we classify closed, simply connected, nonnegatively curved 5–manifolds of almost maximal symmetry rank by applying the results of the previous three sections. Finally, in Section 7 we give examples of actions of almost maximal symmetry rank on some of the manifolds listed in Theorem A.

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2 Definitions and tools

In this section we gather several definitions and results that we will use in subsequent sections.

2.1 Transformation groups

Let G be a Lie group acting (on the left) on a smooth manifold M. We denote by $G_x = \{g \in G \mid gx = x\}$ the *isotropy group* at $x \in M$ and by $Gx = \{gx \mid g \in G\} \simeq G/G_x$ the *orbit* of x. The *ineffective kernel* of the action is the subgroup $K = \bigcap_{x\in M} G_x$. We say that G acts *effectively* on M if K is trivial. The action is called *almost effective* if K is finite. The action is *free* if every isotropy group is trivial and *almost free* if every isotropy group is finite. We will denote the *fixed point set* $M^G = \{x \in M \mid gx = x, g \in G\}$ of this action by $\mathrm{Fix}(M;G)$ and define its dimension as $\dim(\mathrm{Fix}(M;G)) = \max\{\dim(N) \mid N \text{ is a component of } \mathrm{Fix}(M;G)\}$. When convenient, we will denote the orbit space M/G by X. We will denote by $\overline{p}$ the image of a point $p \in M$ under the orbit projection map $\pi\colon M \to M/G$. Given a subset $A \subset M$, we will denote its image in X under the orbit projection map by A^* and when convenient, we shall also denote the orbit space M/G by M^*.

One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space M/G, also called the *cohomogeneity* of the action. This dimension is clearly constrained by the dimension of the fixed point set M^G of G in M. In fact, $\dim(M/G) \geq \dim(M^G)+1$ for any nontrivial action. In light of this, the *fixed-point cohomogeneity* of an action, denoted by $\mathrm{cohomfix}(M;G)$, is defined by

$$\mathrm{cohomfix}(M;G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed-point cohomogeneity 0 is also called a *fixed point homogeneous manifold*. We will use the latter term throughout this article. We observe that the fixed point set of a fixed point homogeneous action has codimension 1 in the orbit space.

Remark 2.1 Throughout the rest of the paper we will assume all manifolds to be smooth. We will only consider smooth (almost) effective actions and all homology and cohomology groups will have coefficients in $\mathbb{Z}$, unless otherwise stated.

2.2 Alexandrov geometry

Recall that a finite-dimensional length space (X,dist) is an *Alexandrov space* if it has curvature bounded from below (see Burago, Burago and Ivanov [3]). When M is a complete, connected Riemannian manifold and G is a compact Lie group acting on M by isometries, the orbit space $X = M/G$ is equipped with the orbital distance metric induced from M, that is, the distance between $\bar{p}$ and $\bar{q}$ in X is the distance between the orbits Gp and Gq as subsets of M. If, in addition, M has sectional curvature bounded below, that is, $\sec M \geq k$, then the orbit space X is an Alexandrov space with $\mathrm{curv}\, X \geq k$.

The *space of directions* of a general Alexandrov space at a point x is by definition the completion of the space of geodesic directions at x. In the case of orbit spaces $X = M/G$, the space of directions $\Sigma_{\bar{p}}X$ at a point $\bar{p} \in X$ consists of geodesic directions and is isometric to

$$S_p^{\perp}/G_p,$$

where $S_p^{\perp}$ is the unit normal sphere to the orbit Gp at $p \in M$.

We now state Kleiner's Isotropy lemma [25], which we will use to obtain information on the distribution of the isotropy groups along minimal geodesics joining two orbits and, in consequence, along minimal geodesics joining two points in the orbit space $X = M/G$.

Isotropy lemma 2.2 *Let $c\colon [0,d] \to M$ be a minimal geodesic between the orbits $Gc(0)$ and $Gc(d)$. Then, for any $t \in (0,d)$, $G_{c(t)} = G_c$ is a subgroup of $G_{c(0)}$ and of $G_{c(d)}$.*

Recall that the *q–extent* $\mathrm{xt}_q(X)$, $q \geq 2$, of a compact metric space (X,d) is the maximum average distance between q points in X:

$$\mathrm{xt}_q(X) = \binom{q}{2}^{-1} \max\left\{ \sum_{1\leq i<j\leq q} d(x_i,x_j) \,\middle|\, \{x_i\}_{i=1}^n \subset X \right\}$$

The Extent lemma (see Grove and Searle [23]) stated below provides an upper bound on the total number of isolated singular points in $X = M/G$.

Extent lemma 2.3 *Let $\bar{p}_0,\dots,\bar{p}_q$ be $q+1$ distinct points in $X=M/G$. If* $\operatorname{curv} X \geq 0$, *then*

$$\frac{1}{q+1}\sum_{i=0}^{q} \mathrm{xt}_q(\Sigma_{\bar{p}_i} X) \geq \pi/3.$$

We remark that in the case of strictly positive curvature, the inequality is also strict.

We will also use the following analogue for orbit spaces of the Cheeger–Gromoll Soul theorem to obtain information on the geometry of the orbit space $X = M/G$ (see Cheeger–Gromoll [23] and Perelman [37]).

Soul theorem 2.4 *Let $X = M/G$. If* $\operatorname{curv} X \geq 0$ *and $\partial X \neq \varnothing$, then there exists a totally convex compact subset $S \subset X$ with $\partial S = \varnothing$, which is a strong deformation retract of X. If* $\operatorname{curv} M/G > 0$, *then $S = \bar{s}$ is a point, and ∂X is homeomorphic to $\Sigma_{\bar{s}} X \simeq S_s^{\perp}/G_s$.*

When M is a nonnegatively curved, fixed point homogeneous Riemannian G–manifold, the orbit space X is a nonnegatively curved Alexandrov space and ∂X contains a component N of $\operatorname{Fix}(M;G)$. Let $C \subset X$ denote the set at maximal distance from $N \subset \partial X$ and let $B = \pi^{-1}(C)$. Soul theorem 2.4 implies that M can be written as the union of neighborhoods $D(N)$ and $D(B)$ along their common boundary E, that is,

$$M = D(N) \cup_E D(B).$$

In particular, when $G = T^1$ and C is another fixed point set component with maximal dimension, one has the following result from [41].

Double soul theorem 2.5 *Let M be a complete, nonnegatively curved Riemannian manifold admitting an isometric T^1 action. If* $\operatorname{Fix}(M;T^1)$ *contains two codimension-2 components N_1 and N_2, with one of them being compact, then N_1 is isometric to N_2,* $\operatorname{Fix}(M;T^1) = N_1 \cup N_2$ *and M is diffeomorphic to an S^2–bundle over N_1 with T^1 as its structure group. In other words, there is a principal T^1–bundle, P, over N_1 such that M is diffeomorphic to $P \times_{T^1} S^2$.*

2.3 Closed 3–manifolds with a smooth T^2 action

We recall the list of closed 3–manifolds with a smooth cohomogeneity one T^2 action (see Mostert [27] and Neumann [28]), as they will appear throughout the paper. They are S^3, $L_{p,q}$, $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$, T^3, $S^2 \tilde{\times} S^1$, $\mathrm{Kl} \times S^1$ and A. Here $L_{p,q}$ denotes a lens space, Kl the 2–dimensional Klein bottle and $S^2 \tilde{\times} S^1$ the nontrivial S^2–bundle over S^1. The manifold A is obtained by gluing $\mathrm{Mb} \times S^1$ and $S^1 \times \mathrm{Mb}$ along their boundary torus, where Mb denotes the Möbius band.

3 Nonnegatively curved 3– and 4–manifolds with almost maximal symmetry rank

In this section we classify closed, orientable 3–manifolds and closed, simply connected 4–manifolds, assuming they have nonnegative curvature and admit an isometric action of a circle T^1.

3.1 Dimension 3

In the case of a T^1 action, we have the following result, which follows from the Orlik–Raymond–Seifert classification of 3–manifolds with a smooth T^1 action; see Orlik [29] and Orlik and Raymond [30].

Theorem 3.1 *Let T^1 act isometrically on M^3, a closed, orientable 3–manifold of nonnegative curvature. Then M^3 is equivariantly diffeomorphic to a spherical 3–manifold, $S^2 \times S^1$, $\mathbb{R}P^3 \# \mathbb{R}P^3$, T^3 or one of four T^2–bundles over S^1.*

Proof We break the proof into three cases: where the action is free, where the action is almost free and where the action has nontrivial fixed point set.

Case 1: T^1 acts freely In this case $X^2 = M^3/T^1$ is a closed, orientable 2–manifold of nonnegative curvature and thus $X^2 = S^2$ or $X^2 = T^2$ by the Gauss–Bonnet Theorem. Since the action is free, M^3 is a principal circle bundle over X^2 and therefore M^3 is diffeomorphic to one of S^3, $L_{p,q}$, $S^2 \times S^1$ or T^3.

Case 2: T^1 acts almost freely Here M^3 is a Seifert manifold supporting a smooth circle action. Since we have assumed that M^3 has nonnegative curvature, M^3 admits a geometric structure modeled on S^3, $S^2 \times \mathbb{R}$ or Euclidean space E^3 (see Scott [40]). Closed, orientable Seifert manifolds with S^3, E^3 or $S^2 \times \mathbb{R}$ geometry supporting a smooth T^1 action have been classified [40; 29]. When M^3 has S^3 geometry, M^3 must be diffeomorphic to a spherical 3–manifold, that is, a quotient of S^3 by a finite subgroup of $SO(4)$ acting freely on S^3. We denote these manifolds in the usual fashion by their 2–dimensional orbit spaces. The 3–sphere S^3 is denoted by S^2 and $L_{p,q}$ by $S^2(p)$ or $S^2(p,q)$. The remaining manifolds with S^3 geometry are denoted by $S^2(2,2,n)$, $S^2(2,3,3)$, $S^2(2,3,4)$ and $S^2(2,3,5)$, where $n \geq 2$ is an integer.

When M^3 has $S^2 \times \mathbb{R}$ geometry, M^3 must be $S^2 \times S^1$ and, when M^3 has E^3 geometry, it must be diffeomorphic to T^3 or to one of four of the remaining five possible orientable, closed flat manifolds covered by T^3. The fifth possibility is excluded immediately since it does not admit a circle action. These four flat manifolds covered by T^3 are T^2 bundles over S^1, described in [29]. Their orbit spaces are $S^2(2,2,2,2)$, $S^2(2,4,4)$, $S^2(2,3,6)$ and $S^2(3,3,3)$. Further, all of these closed, orientable 3–manifolds with E^3, S^3 or $S^2 \times \mathbb{R}$ geometry, with a Seifert fibration induced by an almost free circle action, do admit isometric circle actions inducing the given Seifert fibration [40; 29].

Case 3: T^1 has nontrivial fixed point set By definition, the action is fixed point homogeneous. Closed fixed point homogeneous manifolds 3–manifold with nonnegative curvature were classified by Galaz-Garcia in [13] and we recall their classification in the orientable case. Observe first that the fixed point set is 1–dimensional, with at most two components, and these components are circles. If $\mathrm{Fix}(M^3; S^1)$ contains two components, then by Double soul theorem 2.5 we see that M^3 is one of the two S^2 bundles over S^1 and since M^3 is assumed to be orientable, it must be $S^2 \times S^1$. If $\mathrm{Fix}(M^3; S^1)$ consists of a single component F^1, then $X^2 = M^3/S^1$ is a 2–dimensional Alexandrov space of nonnegative curvature with boundary $F^1 \cong S^1$. Thus X^2 is an orientable, nonnegatively curved topological manifold with boundary and the only possibilities are D^2 and $S^1 \times I$. We may exclude $S^1 \times I$ since M^3 is assumed to be orientable. Thus D^2 is the only possible orbit space. The nonnegative curvature hypothesis implies that the interior of D^2 has either no points with nontrivial finite isotropy, one point with finite isotropy $\mathbb{Z}_p$ or two points with finite isotropy $\mathbb{Z}_2$. These correspond, respectively, to S^3, a lens space $L_{p,q}$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$.

It follows from the three cases analyzed above that M^3 can only be S^3/Γ, where Γ is a finite subgroup of $SO(4)$, $S^2 \times S^1$, T^3, one of the four flat T^2–bundles over S^1 covered by T^3 or, finally, $\mathbb{R}P^3 \# \mathbb{R}P^3$. Each of these manifolds supports only one isometric T^1 action with nonnegative curvature yielding the possible orbit space structures (see Raymond [38]). □

3.2 Dimension 4

Given Perelman's work on the Poincaré conjecture [34; 35; 36], the classification of closed, simply connected, nonnegatively curved 4–manifolds admitting an isometric T^1 action follows from earlier classification results in a curvature-free setting and a restriction on the Euler characteristic, which is a simple consequence of the nonnegative curvature assumption. The first theorem is due to Fintushel [11; 12] in combination with work of Pao [31] and Perelman's proof of the Poincaré conjecture [34; 36; 35].

Theorem 3.2 *A closed, simply connected smooth* 4*–manifold with a* T^1 *action is equivariantly diffeomorphic to a connected sum of* S^4*,* $\pm\mathbb{C}P^2$ *and* $S^2 \times S^2$*.*

Let M^4 be a closed, simply connected, nonnegatively curved 4–manifold and let $\chi(M^4)$ be its Euler characteristic. It follows from work done independently by Kleiner [25] and Searle and Yang [41] that $2 \leq \chi(M^4) \leq 4$. Combining this with Theorem 3.2 yields the following result in the case of nonnegative curvature (see [13]).

Theorem 3.3 *A closed, simply connected, nonnegatively curved* 4*–manifold with an isometric* T^1 *action is diffeomorphic to* S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ *or* $\mathbb{C}P^2 \# \pm\mathbb{C}P^2$*.*

4 Cohomogeneity three torus actions on simply connected 5–manifolds

In this section we gather general facts about smooth cohomogeneity three torus actions $T^n \times M^{n+3} \to M^{n+3}$ on simply connected, smooth manifolds and then consider the specific case when M is 5–dimensional. The main goal of this section is to understand the structure of the singular sets, that is, the set of points in the orbit space M^* corresponding to orbits with nonprincipal isotropy groups.

4.1 General considerations

We begin with the following theorem from Bredon [2], which characterizes the orbit space of a cohomogeneity three action.

Theorem 4.1 *Let* G *be a compact Lie group acting by cohomogeneity three on* M*, a compact, simply connected smooth manifold. If all orbits are connected, then* M^* *is a simply connected topological* 3*–manifold with or without boundary.*

It follows from the resolution of the Poincaré conjecture [34; 35; 36] that M^* is homeomorphic to one of S^3, D^3, $S^2 \times I$ or, more generally, to S^3 with a finite number of disjoint open 3–balls removed. We will see in the next section that nonnegative curvature implies that M^* can only be one of the first three manifolds from this list.

We also recall the following general result of Bredon [2] about the fundamental group of the orbit space.

Theorem 4.2 *Let G be a compact Lie group acting on a topological space X. If either G is connected or G has a nonempty fixed point set, then the orbit projection map $\pi\colon X \to X/G$ induces an onto map on fundamental groups.*

The next theorem, again from Bredon [2], implies the absence of special exceptional orbits and, in particular, allows us to conclude that no fixed point set of finite $\mathbb{Z}_2$–isotropy has codimension one in M. This result will be used in the proof of Lemma 4.4.

Theorem 4.3 *Let M be a smooth, simply connected manifold admitting an action by a compact Lie group. If a principal orbit is connected (and hence all orbits are connected) then there are no special exceptional orbits, that is, the set of points belonging to exceptional orbits is of codimension greater than or equal to 2.*

Lemma 4.4 *Let T^n act on M^{n+3}, a closed, simply connected smooth manifold. Then some circle subgroup has nontrivial fixed point set.*

Proof If all circle subgroups were to act freely, this would imply a free circle action on a closed, simply connected 4–manifold $M^4 = M^{n+3}/T^{n-1}$, which is impossible. Likewise, if the action is almost free, then there are finitely many finite isotropy groups. Let Γ be the finite group generated by all these finite groups and consider the action of T^n/Γ on M^{n+3}/Γ. Note first that we may consider the successive quotients

$$M \to M/\Gamma_1 \to \cdots \to M/\Gamma_k = M/\Gamma,$$

where $\Gamma = \Gamma_k \supset \cdots \supset \Gamma_1$ is a filtration with prime-order quotients Γ_i/Γ_{i-1}. Such a filtration exists because $\Gamma \subset T^n$ is abelian. Then each quotient is a closed, simply connected topological space by Theorem 4.2 and hence M^{n+3}/Γ is as well. We claim that M^{n+3}/Γ must be a topological manifold. Note that the fixed point set of any subgroup of finite isotropy must be at least n–dimensional since it is invariant under the T^n action and it will be at most $(n+1)$–dimensional because there are no special exceptional orbits by Theorem 4.3. The space of directions normal to the projection of a codimension 2 fixed point set in M^{n+3}/Γ is a circle. In the codimension-3 case, the isotropy subgroup will be a finite subgroup of $SO(3) \cap T^n$, $n \geq 2$; hence it must be a cyclic group of rotations or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In both cases the quotient of the isotropy action on the normal 2–sphere will be again a topological 2–sphere. Hence M^{n+3}/Γ must be a closed, simply connected topological manifold. Now, T^n/Γ must act freely on M^{n+3}/Γ and we have just seen that this is impossible. Therefore T^n cannot act almost freely on M^{n+3} either. □

Let M^{n+3} be a closed, simply connected $(n+3)$–manifold with a cohomogeneity three T^n action. By the previous lemma, there is a circle subgroup $T^1 \subset T^n$ with nontrivial fixed point set. In the case where $M^* = D^3$, there is a unique codimension-2 fixed point set component. In general, when M^* is homeomorphic to S^3 with k disjoint open 3–balls removed, $k \geq 1$, the k boundary components correspond to the quotients of unions of fixed point sets of possibly different circles.

In the case where M^* is homeomorphic to S^3, the components of $\mathrm{Fix}(M^{n+3}; T^1)$ are of codimension greater than or equal to 4. In this case, the following proposition, generalizing a result of Rong in dimension 5 [39], shows there must be a minimum number of codimension-4 fixed point set components, corresponding to isolated singular orbits T^{n-1}.

Proposition 4.5 *Let T^n act on M^{n+3}, a closed, simply connected, smooth manifold. Suppose that M^* is homeomorphic to S^3 and that there are exactly two orbit types: principal orbits T^n and isolated singular orbits T^{n-1}, that is, the isotropy subgroups are either trivial or isomorphic to T^1. Then there are at least $n+1$ isolated singular orbits T^{n-1}.*

Proof Let M_0 denote the manifold with boundary obtained by removing a small tubular neighborhood around each isolated singular orbit T^{n-1}. Let M_0^* denote the quotient space M_0/T^2. By a standard transversality argument we know that

$$\pi_1(M_0) \cong \pi_1(M) = \{1\},$$
$$\pi_2(M_0) \cong \pi_2(M).$$

Since there is no isotropy group of finite order we obtain a fibration

$$T^n \to M_0 \to M_0^*,$$

and therefore a long exact sequence in homotopy

$$0 \to \pi_2(M_0) \to \pi_2(M_0^*) \to \pi_1(T^n) \to \pi_1(M_0) \to \pi_1(M_0^*) \to 0.$$

Since $\pi_1(M) \cong \pi_1(M_0) = 0$, it follows that $\pi_1(M_0^*) = 0$. Since M^* is a 3–sphere, by applying the Mayer–Vietoris sequence to the pair $(M_0^*, \mathrm{cl}(M^* \backslash M_0^*))$, noting that $\mathrm{cl}(M^* \backslash M_0^*)$ is a disjoint union of closed 3–discs, we obtain that $H_2(M_0^*) \cong \mathbb{Z}^r$, where $(r+1)$ is the number of isolated singular orbits. It follows from the Hurewicz isomorphism that $\pi_2(M_0^*) \cong H_2(M_0^*) \cong \mathbb{Z}^r$ and the above exact sequence in homotopy becomes

$$0 \to \pi_2(M_0) \to \mathbb{Z}^r \to \mathbb{Z}^n \to 0.$$

We conclude that $n \leq r$ and thus there are at least $n+1$ isolated singular orbits. □

Corollary 4.6 *Proposition 4.5 remains valid in the presence of finite isotropy.*

Proof Let Γ be the finite group generated by the finite isotropy groups of the action.

As we saw earlier in the proof of Lemma 4.4, M^{n+3}/Γ is a closed, topological manifold. Moreover, M^{n+3}/Γ is simply connected. Finally, observe that T^n/Γ acts without finite isotropy on M^{n+3}/Γ and the isolated $T^{n-1}/(T^{n-1}\cap\Gamma)$ orbits in M^{n+3}/Γ correspond to isolated T^{n-1} orbits in M^{n+3}. □

Remark 4.7 We observe that in the case where we have a T^2 action on M^5, Proposition 4.5 implies that when $M^* = S^3$, there are at least three isolated circle orbits.

4.2 Possible isotropy groups

In this subsection, we use the isotropy representation of the possible isotropy groups to better understand fixed point components of finite isotropy and their corresponding images in the orbit space M^*.

By a theorem of Chang and Skjelbred [5], components of $\mathrm{Fix}(M;\mathbb{Z}_k)$ are smooth submanifolds. When $k \neq 2$ these components are orientable and of even codimension. If $k = 2$, components of $\mathrm{Fix}(M;\mathbb{Z}_2)$ may also be nonorientable and by Theorem 4.3, of codimension at least 2. In the case of a smooth T^2 action on a closed, simply connected smooth 5–manifold M^5, components of $\mathrm{Fix}(M^5;\mathbb{Z}_k)$ must be at least 2–dimensional, as we saw in the proof of Lemma 4.4. An analysis of the isotropy representations will show that for all cases the components of $\mathrm{Fix}(M^5;\mathbb{Z}_k)$ must be 3–dimensional.

Proposition 4.8 *Let T^2 act smoothly on M^5, a closed, simply connected smooth 5–manifold. If $M^* = S^3$, then the following hold.*

(1) *The singular orbits of the action are T^1 and $T^1/\mathbb{Z}_k$, $k \in \mathbb{Z}^+$.*
(2) *The exceptional orbits are $T^2/\mathbb{Z}_k$, $k \geq 2$ and $T^2/(\mathbb{Z}_2\times\mathbb{Z}_2)$.*
(3) *In all cases where there is finite cyclic isotropy, the corresponding fixed point set of finite isotropy is of dimension 3.*

Proof Since we have assumed that M^* is homeomorphic to S^3, there are no points with T^2 isotropy. Observe that the normal sphere at any point of an exceptional orbit will be of dimension two. Thus the finite isotropy group of an exceptional orbit must be a subgroup of $SO(3)$ and of T^2. Hence the only possible finite isotropy groups are $\mathbb{Z}_k$, $k \geq 2$, and $\mathbb{Z}_2\times\mathbb{Z}_2$. This proves parts (1) and (2).

Now we prove part (3). We first consider the singular orbits, observing that if we have a singular orbit of the form $T^1/\mathbb{Z}_k$, then we have a $T^1\times\mathbb{Z}_k$ action on the normal 3–sphere to any point of the orbit. In particular, there will be a finite cyclic subgroup of order k in $T^1\times\mathbb{Z}_k$ fixing circles in this normal 3–sphere and therefore this orbit is contained in a fixed point set of finite isotropy of dimension 3. If the singular orbit is T^1, then the action of the circle on the normal S^3 is either free or almost free. In the latter case, a finite cyclic subgroup fixes a 3–dimensional submanifold which contains the singular orbit.

We now consider the exceptional orbits. For a $T^2/\mathbb{Z}_k$ orbit, $k \neq 2$, the $\mathbb{Z}_k$ action on S^2 is never free and thus this exceptional orbit will be contained in a 3–dimensional submanifold fixed by $\mathbb{Z}_k$, $k \neq 2$. It remains to show that for the exceptional orbit $T^2/\mathbb{Z}_2$, the $\mathbb{Z}_2$ isotropy group also does not act freely on its normal S^2. This follows from the fact that the antipodal map, which reverses orientation, generates the only free $\mathbb{Z}_2$ action on S^2 and it is not a subgroup of $SO(3)$.

Finally, we consider the exceptional orbit $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The action of the isotropy subgroup, $\mathbb{Z}_2 \times \mathbb{Z}_2$, on the normal S^2 produces a quotient space equal to the double right-angled spherical triangle with three vertices, each of which is fixed by a different $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each fixed vertex corresponds to a 3–dimensional submanifold fixed by the corresponding $\mathbb{Z}_2$ subgroup. For each $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit we will have exactly three such fixed point sets intersecting in this orbit. Thus, we conclude that the fixed point set of a finite cyclic group is always of dimension 3. □

4.3 The singular sets in $M^* = S^3$

We now determine the structure of the singular sets in the orbit space in the particular case when M is 5–dimensional and $M^* = S^3$.

Proposition 4.9 *Let T^2 act smoothly on M^5, a closed, simply connected smooth 5–manifold. If $M^* = S^3$, then the set of points in M^* with nontrivial isotropy corresponds to a graph and the following hold.*

(1) *The vertices of the graph correspond to isolated singular orbits or to isolated exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

(2) *The graph must contain at least three vertices corresponding to isolated singular orbits.*

(3) *The vertices corresponding to isolated singular orbits have degree 0, 1 or 2.*

(4) *The vertices corresponding to isolated exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$ have degree 3.*

(5) *The edges of the graph correspond to points with nontrivial, finite, cyclic isotropy.*

(6) *Every edge must meet two different vertices.*

(7) *The points in the edges meeting an isolated exceptional orbit with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$ have isotropy $\mathbb{Z}_2$.*

(8) *The preimage of the closure of an edge corresponds to a 3–dimensional manifold fixed by a nontrivial finite cyclic group admitting a T^2 action of cohomogeneity one.*

Proof Parts (1), (3), (4), (5) and (7) follow from the proof of Proposition 4.8 by looking at the isotropy representation at the corresponding orbits. Part (2) follows from Proposition 4.5 and Corollary 4.6. To prove (6), observe first that the existence of simple closed curves in M^* whose points correspond to exceptional orbits with nontrivial finite cyclic isotropy $\mathbb{Z}_k$ is ruled out by work of Montgomery and Yang [26, Lemma 2.3]. By Kleiner's isotropy lemma [25], the isotropy type on a cycle with one vertex and one edge must be constant, ruling out this configuration. Therefore, there cannot be cycles of (graph-theoretic) length 1 and, in particular, any edge must connect two different vertices. □

We will denote by *arc* the closure of an edge with finite cyclic isotropy in the set of orbits with nontrivial isotropy in M^*. Since the graphs corresponding to the singular set in M^* carry isotropy information, we will refer to them as *weighted graphs*. We further note that in the figures we will use the following scheme to distinguish the possible weighted graphs:

- *Black vertices* will correspond to singular orbits and have degree 0, 1 or 2.
- *White vertices* will correspond to exceptional orbits with $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropy and have degree 3.
- *Edges* will correspond to nonisolated exceptional orbits with nontrivial, finite, cyclic isotropy.

We now begin the process of determining what 3–manifolds may actually occur as fixed point set components of a finite cyclic isotropy group. Since these components admit an (almost) effective T^2 action, they must be one of the manifolds listed in Section 2.3. We will eventually show, in Section 6, that the only such 3–manifolds that can occur are S^3, $L_{p,q}$, $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$.

We first observe that we may immediately rule out T^3, since its orbit space would correspond to a simple closed curve in M^* with finite cyclic isotropy and, as mentioned above, simple closed curves with finite cyclic isotropy will not occur.

Of the possible 3–manifolds on the list, the nonorientable ones are $\mathbb{R}P^2 \times S^1$, $S^2 \tilde{\times} S^1$, $\mathrm{Kl} \times S^1$ and A, and as such, they may only be fixed point set components of $\mathbb{Z}_2$ isotropy. All have at least one exceptional orbit and correspond to the possible preimages of arcs containing a vertex of degree three.

If the singular set in M^* contains a vertex of degree three, then it may contain different types of trees as subgraphs. Two types of trees may occur. The first type occurs if either $\mathbb{R}P^2 \times S^1$ or $S^2 \tilde{\times} S^1$ is the preimage of an arc of $\mathbb{Z}_2$ isotropy, in which case, the singular set contains a tree with one vertex of degree three joined to three vertices of degree one or two only. The second type occurs if $\mathrm{Kl}\times S^1$ or A is the preimage of an arc of $\mathbb{Z}_2$ isotropy, in which case the singular set contains a tree with an edge terminating in two vertices of degree three, each of which is joined to two more vertices of degree one or two. We will see that when we take into consideration the lower curvature bound this second type of tree cannot occur, allowing us to exclude $\mathrm{Kl}\times S^1$ and A as possible fixed point set components of $\mathbb{Z}_2$ isotropy.

The first type of tree is the bipartite graph $K_{1,3}$, commonly known as a *claw* (see Diestel [8] and Gross and Yellen [19]). Since vertices and edges carry isotropy information, we shall refer to this configuration as a *weighted claw* (see Figure 1). An example of the second possible tree appears in Figure 2. We will refer to such graphs as *weighted trees*. These graphs will appear in our analysis of the finite isotropy case in Section 6.2.

Finally, we point out that the weighted graph could also contain a cycle. Moreover, this cycle could potentially be knotted in $M^* = S^3$. We will see in Section 5.4 that when the orbit space is nonnegatively curved the cycle cannot be knotted.

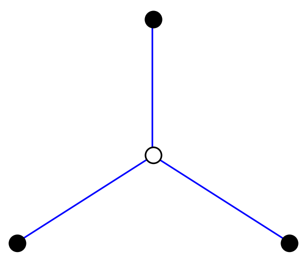

Figure 1: Weighted claw: the central vertex has isotropy $\mathbb{Z}_2\times\mathbb{Z}_2$, the external vertices have isotropy conjugate to S^1 and the edges have isotropy $\mathbb{Z}_2$.

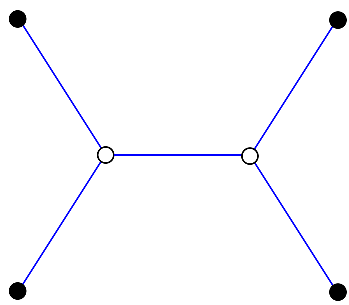

Figure 2: Weighted tree: the two central vertices have isotropy $\mathbb{Z}_2\times\mathbb{Z}_2$, the external vertices have isotropy conjugate to S^1 and the edges have isotropy $\mathbb{Z}_2$.

5 Restrictions on the orbit space imposed by nonnegative curvature

In this section we will see how nonnegative curvature restricts the structure of the orbit space of an isometric T^2 action on a closed, simply connected 5–manifold. Throughout this section we will let M^5 be a closed, simply connected 5–manifold of nonnegative curvature with an isometric T^2 action.

5.1 Topology of orbit spaces with nonnegative curvature

As we noted earlier, the quotient space of a smooth T^2 action on a closed, simply connected smooth 5–manifold is homeomorphic to one of S^3 or S^3 with a finite number of disjoint open 3–balls removed. For every open 3–ball we remove we obtain an S^2 boundary component. In the presence of nonnegative curvature we have the following proposition.

Proposition 5.1 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold. If T^2 acts isometrically on M^5, then M^* is homeomorphic to one of the following:*

(1) S^3, *if for any* $T^1 \subset T^2$ *for which* $\mathrm{Fix}(M^5; T^1) \neq \varnothing$, $\dim(\mathrm{Fix}(M^5; T^1)) = 1$

(2) D^3 *or* $S^2 \times I$, *if* $\dim(\mathrm{Fix}(M^5; T^1)) = 3$ *for some* $T^1 \subset T^2$

Proof Part (1) follows easily since only points belonging to a codimension-two fixed point set of a circle will correspond to boundary points in the orbit space M^*. Note that part (1) is independent of the curvature assumption. Part (2) follows from Double soul theorem 2.5. $\square$

5.2 Upper bound on the number of isolated circle orbits in M^5

In the previous section, in Proposition 4.5, we found a lower bound of three for the number of isolated circle orbits in M^5 for the case where $M^* = S^3$. We now propose to determine an upper bound on the number of isolated circle orbits when M^5 is nonnegatively curved. Theorem 5.2 below will show that there can be at most four such orbits.

A simple application of the Extent lemma tells us that in $M^* = M/G$, where G acts isometrically on M, a closed manifold of positive curvature, there are at most 3 singular points with space of directions isometric to $S^2(\frac{1}{2})$ or a "thin" $S^2(\frac{1}{2})$, that is, the quotient of $S^3(1)$ by an almost free S^1 action. If M is nonnegatively curved, the

Extent lemma tells us that there will be at most 5 such singular points. A closer analysis of the geometry will allow us to show that in the case where M is 5–dimensional, nonnegatively curved and admits an isometric T^2 action, there will be at most 4 isolated circle orbits.

This upper bound follows from a triangle comparison argument in the orbit space, generalizing an argument used in Kleiner's thesis [25] showing that an isometric circle action on a closed, simply connected, nonnegatively curved 4–manifold has at most four isolated fixed points. These bounds, in turn, are a particular instance of the more general fact that a three-dimensional nonnegatively curved Alexandrov space can have at most four points whose space of directions is not larger than $S^2(\frac{1}{2})$ (see Grove and Wilking [24]). We remark that the same result as in Kleiner's thesis was obtained in [41], but the argument used to prove the result was specific to dimension 4 and does not generalize to higher dimensions. The key observation that allows us to apply the techniques in [25] to our situation is that the normal sphere at a point to each one of the circles fixed by some $S^1 \subset T^2$ is 3–dimensional. We include the proof of the theorem here for the sake of completeness since Kleiner's result was never published.

Theorem 5.2 *Let M^5 be a closed nonnegatively curved 5–manifold with an isometric T^2 action. Then there are at most 4 isolated circle orbits of the T^2 action.*

The proof of Theorem 5.2 will occupy the remainder of this subsection. We begin by fixing some notation and recasting several lemmas from [25] to meet our needs.

Let $\{p_i\}_{i=1}^4$ be four distinct points in M^5 and let $\{\bar{p}_i\}_{i=1}^4$ be their corresponding projections in the orbit space $X^3 = M^5/T^2$, which is a nonnegatively curved Alexandrov space and a topological manifold. Given two distinct points $\bar{p}_i$, $\bar{p}_j$, $1 \leq i, j \leq 4$, let $\bar{\gamma}_{ij}$ be a minimizing geodesic from $\bar{p}_i$ to $\bar{p}_j$. For each triple of distinct points $\bar{p}_i$, $\bar{p}_j$, $\bar{p}_k$, $1 \leq i, j, k \leq 4$, and a pair of minimizing geodesics $\bar{\gamma}_{ij}$, $\bar{\gamma}_{ik}$, let

$$\alpha_{ijk} = \angle(\bar{\gamma}_{ij}, \bar{\gamma}_{ik}).$$

This angle is the distance in $\Sigma_{\bar{p}_i} = S^3/G_{p_i}$ between the directions of the minimizing geodesics $\bar{\gamma}_{ij}, \bar{\gamma}_{ik}$. Finally, let T_{ijkl} denote the (possibly degenerate) tetrahedron determined by the four points $\bar{p}_i, \bar{p}_j, \bar{p}_k, \bar{p}_l$ and minimal geodesics between these.

Before proceeding with the proof of Theorem 5.2, we recall the following fact (cf [41]).

Lemma 5.3 *Suppose S^1 acts isometrically and fixed point freely on $S^3(1)$. Then S^3/S^1 is smaller than $S^2(\frac{1}{2}) = S^3/S^1_{\mathrm{Hopf}}$. That is, there is a surjective 1–Lipschitz map $S^2(\frac{1}{2}) \to S^3/S^1$.*

We have the following lemma.

Lemma 5.4 *If there are 4 isolated circle orbits $\{N_i\}_{i=1}^4$, then, for distinct points $p_i \in N_i$, $1 \le i \le 4$, and every quadruple of distinct integers $1 \le i, j, k, l \le 4$, a tetrahedron T_{ijkl} in the orbit space with vertices $\bar{p}_i$, $1 \le i \le 4$, and edges corresponding to minimal geodesic between the vertices, is rigid in the sense that*

$$\alpha_{ijk} + \alpha_{ijl} + \alpha_{ikl} = \pi, \tag{1}$$

$$\alpha_{ijk} + \alpha_{jki} + \alpha_{kij} = \pi, \tag{2}$$

that is, the sum of angles at each vertex and the sum of angles of each face of T_{ijkl} are both π.

Proof In the orbit space $X^3 = M^5/T^2$, the 4 circles $\{N_i\}_{i=1}^4$ correspond to 4 points $\{\bar{p}_i\}_{i=1}^4$. By Toponogov's Theorem for Alexandrov spaces (see Burago, Gromov and Perelman [4]), we know that the sum of the angles of a geodesic triangle in X^3 will be greater than or equal to π. Connecting each pair of distinct points in $\{\bar{p}_i\}_{i=1}^4$ by a minimal geodesic we obtain a configuration of four triangles and the total sum of the angles in this configuration will be greater than or equal to 4π.

For each one of the four points $\{p_i\}_{i=1}^4$ the corresponding isotropy group acts freely or almost freely on the normal space $T_{p_i}N_i^{\perp}$ and the quotient of the unit normal sphere $S^3 \subset T_{p_i}N_i^{\perp}$ is $S^2(\frac{1}{2})$, the round sphere of radius $\frac{1}{2}$ in the first case or a "thin" $S^2(\frac{1}{2})$ in the second case. Hence

$$\mathrm{xt}_q(\Sigma_{\bar{p}_i} X^3) \le \mathrm{xt}_q(S^2(\tfrac{1}{2}))$$

for any $q \ge 2$.

Using the fact that $\mathrm{xt}_3(S^2(\frac{1}{2})) = \pi/3$, it is easily seen that for any triple of distinct points $x_j, x_k, x_l \in S^2(\frac{1}{2})$, we have

$$\mathrm{dist}(x_j, x_k) + \mathrm{dist}(x_j, x_l) + \mathrm{dist}(x_k, x_l) \le \pi.$$

Thus summing over all the triangles formed by the points $\{\bar{p}_i\}_{i=1}^4$ we find that the sum of their angles should be less than or equal to 4π. Therefore this sum of angles must be exactly 4π. □

Lemma 5.5 *If there are 5 isolated circle orbits $\{N_i\}_{i=1}^5$ then, for fixed $1 \le i \le 5$ and points $p_j \in N_j$, $1 \le j \le 5$, $j \ne i$, the following hold.*

1. *For each i and each $p_i \in N_i$, we have $G_{p_i} = S^1$ and its slice representation is the Hopf action.*
2. *The directions in $\Sigma_{\bar{p}_i} = S^3/G_{p_i}$ corresponding to minimal geodesics from $\bar{p}_i$ to $\bar{p}_j$, $j \ne i$, come in mutually orthogonal pairs, that is, given i, for each set of distinct j, k, l, m, up to reordering, we can assume that*

$$\alpha_{ijk} = \alpha_{ilm} = \pi/2.$$

Proof For convenience, let $i = 5$. For $s = j,k,l,m$, let $v_s \in \Sigma_{\bar{p}_5}$ be the initial direction of the minimizing geodesic $\bar{\gamma}_{5s}$ from $\bar{p}_5$ to $\bar{p}_s$. By Lemma 5.4, we have

$$\alpha_{5jk} + \alpha_{5kl} + \alpha_{5lj} = \pi,$$

for the 4 points $\bar{p}_j, \bar{p}_k, \bar{p}_l, \bar{p}_5$, with $j,k,l \neq 5$.

We have already seen in the proof of Lemma 5.4 that the sum of the distances between any set of three distinct points in $\Sigma_{\bar{p}_i}$ is equal to π for any $i \in \{1,2,3,4,5\}$. Consider now the three points v_j, v_k, v_l in $\Sigma_{\bar{p}_5}$. In the case where $\Sigma_{\bar{p}_5} = S^2(\frac{1}{2})$ then either two of them are antipodal or all three of them lie on a great circle. Note that in this last case the three points cannot lie in half of the great circle. In the second case, where $\Sigma_{\bar{p}_5}$ is a "thin" $S^2(\frac{1}{2})$, two of the points must be at distance $\pi/2$. Since this is true for any choice of three of the four possible directions v_s, one may conclude that $\Sigma_{\bar{p}_5}$ cannot be smaller than $S^2(\frac{1}{2})$. In particular, this implies that the isotropy group of each isolated circle orbit is S^1 and that the action is the Hopf action. This proves part (1) of the lemma. Finally, one can conclude that the four directions v_s, $s = j,k,l,m$ in the space of directions must lie on a great circle and consist of two pairs of antipodal points, thus proving part (2) of the lemma. □

Proof of Theorem 5.2 Suppose that there are 5 singular points $\bar{p}_i$, $1 \leq i \leq 5$, in the orbit space X^3 corresponding to isolated circle orbits. Lemmas 5.4 and 5.5 imply that every triangle determined by any three such points has exactly one angle $\pi/2$. It follows from the discussion of the equality case in the proof of Toponogov's Theorem (see Cheeger and Ebin [6]) that these triangles must be flat. Assume, after relabeling if necessary, that $\mathrm{dist}(\bar{p}_1, \bar{p}_2)$ is the minimum of the distances $\mathrm{dist}(\bar{p}_i, \bar{p}_j)$ between distinct points $\bar{p}_i, \bar{p}_j$, $1 \leq i, j \leq 5$. That is, assume that the geodesic edge from $\bar{p}_1$ to $\bar{p}_2$ is the shortest in the configuration with vertices $\bar{p}_i$, $1 \leq i \leq 5$, and edges corresponding to minimal geodesics $\bar{\gamma}_{ij}$ between distinct points $\bar{p}_i$, $\bar{p}_j$. Now choose $\bar{p}_i$ in $\{\bar{p}_3, \bar{p}_4, \bar{p}_5\}$ such that neither one of the angles α_{12i}, α_{21i} is equal to $\pi/2$. This choice implies that the minimal geodesic $\bar{\gamma}_{12}$ is the hypotenuse of the triangle determined by $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_i$. On the other hand, $\mathrm{dist}(\bar{p}_1, \bar{p}_2)^2 = \mathrm{dist}(\bar{p}_1, \bar{p}_i)^2 + \mathrm{dist}(\bar{p}_2, \bar{p}_i)^2$, which contradicts the choice of $\bar{p}_1$ and $\bar{p}_2$ as determining the shortest geodesic edge of the configuration determined in the orbit space by the five isolated circle orbits. □

Corollary 5.6 *Let M^{n+3} be a closed, nonnegatively curved manifold with an isometric T^n action. Suppose that $M^* = S^3$ and that there are isolated T^{n-1} orbits. Then there are at most four such isolated T^{n-1} orbits. In particular, if $n \geq 7$ then there are none.*

Proof The first result follows directly from the proof of Theorem 5.2. The second result follows by Proposition 4.5. □

5.3 Possible components with finite isotropy

The following lemma, easily generalized from Rong [39], allows us to calculate the Betti numbers with $\mathbb{Z}_p$ coefficients of M^5.

Lemma 5.7 *Suppose T^2 acts isometrically on M^5, a closed, simply connected 5–manifold. If there are exactly 3 isolated circle orbits, then $H_2(M^5)$ has trivial free rank. If there are exactly 4 isolated circle orbits, then $H_2(M^5)$ has free rank equal to 1.*

We will now show that a weighted graph containing a tree with a vertex of degree three, that is, a weighted graph containing a weighted claw or a weighted tree, may occur only when there are exactly 3 isolated circle orbits. With this result, we may then conclude that neither $\mathrm{Kl} \times S^1$ nor A can never occur as the fixed point set of a finite group.

Proposition 5.8 *Let T^2 act isometrically on M^5, a closed, simply connected, nonnegatively curved 5–manifold. If $M^* = S^3$ and there exists a nonorientable 3–manifold F^3 fixed by a $\mathbb{Z}_2$ subgroup, then the projection of F^3 in M^* must belong to a weighted claw and there can be no other singular points in M^* corresponding to an isolated circle orbit, besides the three external vertices of the claw.*

Proof Let W be the weighted graph corresponding to the set of orbits with nontrivial isotropy in M^*. There are two cases we must exclude. The first case is where W contains a weighted claw as a subgraph and a vertex of degree 0, 1 or 2 (see, for example, Figure 3). The second case is when W contains a weighted tree as a subgraph (see, for example, Figure 4).

We begin with the first case. Let $\bar{p}_1$ denote the center point in M^* of the weighted claw, that is, whose space of directions $\Sigma_{\bar{p}_1}$ is the double right-angled spherical triangle $S^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, and let $\bar{p}_i$, $i = 2, 3, 4$, denote the points in M^* corresponding to the vertices of the weighted claw, each of which corresponds to an orbit with T^1 or $T^1 \times \mathbb{Z}_2$ isotropy. We note that the space of directions for each of these external vertices is either an $S^2(\frac{1}{2})/\mathbb{Z}_2$, that is a "thin" 2–sphere of diameter $\pi/2$ or a possibly thinner 2–sphere of diameter $\pi/2$. If there is a fourth singular point $\bar{p}_5$ corresponding to an isolated circle orbit in M^5, then $\Sigma_{\bar{p}_5}$ is either an $S^2(\frac{1}{2})$ or a "thin" $S^2(\frac{1}{2})$. Since $S^2(\frac{1}{2})$ is the "largest" of these spaces of directions (cf Lemma 5.3), we will assume that $\Sigma_{\bar{p}_5} = S^2(\frac{1}{2})$.

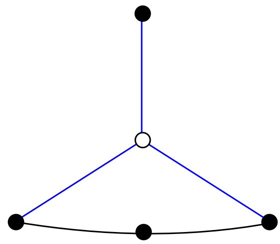

Figure 3: Weighted graph containing a claw: the solid vertices correspond to isolated circle orbits; the vertex of degree 3 corresponds to an exceptional orbit with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$.

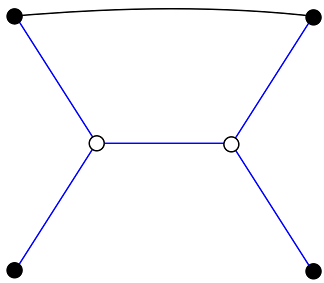

Figure 4: Weighted graph containing a tree: the solid vertices correspond to isolated circle orbits; the two vertices of degree 3 correspond to exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$

It is clear that in $\Sigma_{\overline{p}_1}$ the vertices of the spherical triangle correspond to the geodesic directions to the points $\overline{p}_2, \overline{p}_3$ and $\overline{p}_4$, and consequently $\alpha_{123} = \alpha_{124} = \alpha_{134} = \pi/2$. Without loss of generality, we will assume that $\alpha_{135} + \alpha_{145} = \alpha + \beta = \pi/2$, in which case it follows that $\alpha_{125} = \pi/2$.

Now, by Lemma 5.4, the tetrahedron T_{2345} is rigid, in the sense that the angles of every face sum to π and the angles at every vertex sum to π. In particular, because of this rigidity and because each of the points $\overline{p}_2, \overline{p}_3$ and $\overline{p}_4$ has space of directions a thin $S^2(\frac{1}{2})$, it follows that at every one of the vertices $\overline{p}_2, \overline{p}_3$ and $\overline{p}_4$ of T_{2345} there will be an angle of $\pi/2$. Further, the maximal configuration for the spaces of directions of the points $\overline{p}_2, \overline{p}_3$ and $\overline{p}_4$ will be where the remaining angles at each vertex in T_{2345} are all $\pi/4$, that is, $\alpha_{j1k} = \pi/4$ for all $j \in \{2,3,4\}$ and $k \in \{2,3,4,5\}$, where $j \neq k$, whereas, α_{51j} will be equal to $\pi/2$ for one value of $j \in \{2,3,4\}$ and for the remaining values it will be equal to $\pi/4$. Without loss of generality we may choose specific values for all angles of the form α_{2jk}, $j,k \in \{1,3,4,5\}$. Once these choices are determined, the rigidity of T_{2345} will determine the remaining angles.

It now follows by Toponogov's Theorem that the angle sum of any triangle in any tetrahedron formed by these five singular points must be greater than or equal to π. When we consider the tetrahedron T_{1345}, we see that when we substitute all the known values for the angles the lower bound on the sum of the angles for any triangle forces the inequalities

$$\alpha_{513} + \alpha_{315} + \alpha \geq \pi, \quad \alpha_{514} + \alpha_{415} + \beta \geq \pi.$$

As we saw previously, $\alpha_{415} = \alpha_{315} = \pi/4$ and one of α_{513} or α_{514} is equal to $\pi/2$ and the other is equal to $\pi/4$. In particular this tell us that one of the angles α or β is greater than or equal to $\pi/2$ and the other is greater than or equal to $\pi/4$. Since $\alpha + \beta = \pi/2$ this immediately gives us a contradiction and thus this case cannot occur.

For the second case, where the weighted graph contains a weighted tree, we observe that the addition of the singular point $\overline{p}_6$, corresponding to a $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit will produce an analogous contradiction and thus this case cannot occur either. $\square$

We summarize the results of this subsection in the following theorem.

Theorem 5.9 *Let T^2 act isometrically on M^5, a closed, simply connected, nonnegatively curved 5–manifold. If $M^* = S^3$, then the fixed point set components of finite cyclic isotropy (if they exist) are:*

(1) S^3, $L_{p,q}$, $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$ or $S^2 \tilde{\times} S^1$ *when there are three isolated circle orbits*

(2) S^3, $L_{p,q}$ or $S^2 \times S^1$ *when there are four isolated circle orbits*

We recall the following theorem of Bredon [2].

Theorem 5.10 *Suppose that p is a prime and that $G = \mathbb{Z}_p$ acts on the finite-dimensional space X with $B \subset X$ closed and invariant. Suppose that G acts trivially on $H^*(X, B; \mathbb{Z})$ and let $F = \mathrm{Fix}(X; \mathbb{Z}_p)$. Then, for any $k \geq 0$, we have*

$$\sum_{i \geq 0} \mathrm{rk}\, H^{k+2i}(F, F \cap B; \mathbb{Z}_p) \leq \sum_{i \geq 0} \mathrm{rk}\, H^{k+2i}(X, B; \mathbb{Z}_p).$$

We observe that any diffeomorphism in T^2 is homotopic to the identity, since it is contained in a torus. Thus we may apply this theorem to the situation at hand to obtain the following corollary.

Corollary 5.11 *Let T^2 act isometrically on M^5, a closed, simply connected, nonnegatively curved 5–manifold. If $M^* = S^3$ and the orbit space contains a weighted claw, then M^5 is not S^5.*

Proof This follows directly by applying the inequality in Theorem 5.10, observing that if either $S^2 \tilde{\times} S^1$ or $\mathbb{R}P^2 \times S^1$ is contained in $\mathrm{Fix}(M^5; \mathbb{Z}_2)$, then $H_2(M^5) \neq 0$. $\square$

5.4 Unknottedness of cycles

We will now analyze the special case where the singular set in the orbit space contains a cycle. Following work of Grove and Wilking [24; 47], we will show that this cycle is unknotted in $M^* = S^3$. Recall that the arcs in a cycle correspond to the projection of fixed point sets of finite isotropy. We have the following result.

Theorem 5.12 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold with an isometric T^2 action and orbit space $M^* \simeq S^3$. If the singular set in the orbit space M^* contains a cycle K^1, then the following hold.*

(1) *The cycle K^1 is the only cycle in the singular set in M^*.*

(2) *If there are four isolated circle orbits, then K^1 comprises all of the singular set, that is, $M^* \setminus K^1$ is smooth.*

(3) *Suppose there are exactly three isolated circle orbits, the cycle K^1 contains only two of them, and there are no exceptional orbits of isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then the finite isotropy fixing one of the 3–manifolds corresponding to one of the arcs of the cycle must be $\mathbb{Z}_2$.*

(4) *The cycle K^1 is unknotted in M^*.*

Proof We will first prove parts (1) and (2) of the theorem. Note first that the weighted graph $W \subset M^*$ corresponding to the set of singular orbits can contain two cycles only if W has four vertices, since a cycle must contain at least two isolated singular orbits [25; 26]. In particular, this shows part (1) when there are only three isolated singular orbits. To then obtain parts (1) and (2) when there are 4 isolated singular orbits we will employ the following strategy. For any given weighted graph containing a cycle K^1 in M^* we will construct a branched double cover $\kappa\colon B \to M^*$ with branching set K^1 and show that B is a simply connected Alexandrov space of nonnegative curvature; see Lemmas 5.13, 5.14 and 5.15 below. By Lemma 5.14 below and the proof of Theorem 5.2, the cover B can contain at most four points projecting down to isolated circle orbits in M^*. This then shows us that if we have a cycle containing fewer than 4 isolated singular orbits in M^*, then in B we will have more than 4 isolated singular orbits, a contradiction.

To construct the branched cover, first observe that a generator of $H_1(M^* \setminus K^1; \mathbb{Z}) = \mathbb{Z}$ is given by a normal circle to K^1. Recall that index two subgroups of $H_1(M^* \setminus K^1; \mathbb{Z}) = \mathbb{Z}$ are in one-to-one correspondence with two-fold covers of $M^* \setminus K^1$. Hence there is a unique two-fold cover B' of $M^* \setminus K^1$. Let $B_r(K^1)$ be a tubular neighborhood of K^1 in $M^* \simeq S^3$. Observe that $B_r(K^1)$ is a solid torus and thus $H_1(B_r(K^1) \setminus K^1; \mathbb{Z}) = \mathbb{Z}^2$. It follows that $B_r(K^1) \setminus K^1$ also has a two-fold cover. Now let $B = B' \cup K^1$ so that $\kappa\colon B \to M^*$ is a two-fold branched cover, with branching set K^1. Further, B admits a $\mathbb{Z}_2$ action, which is isometric with respect to the metric induced by the orbital distance metric from M^*, with fixed point set K^1.

Lemma 5.13 *The space B is a nonnegatively curved Alexandrov space.*

Proof of Lemma 5.13 Observe that B is locally isometric to M^* outside of the branching set K^1. Let C_2 be the union of arcs in K^1 with isotropy $\mathbb{Z}_k$, $k \neq 2$. We have two cases: case one, where the cycle K^1 contains all the singular points corresponding to isolated circle orbits, and case two, when there are exactly three isolated circle orbits and of the corresponding singular points only two belong to a cycle.

For case one, proceeding as in the proof of Lemma 2.3 in [24], one verifies that the set $B \setminus C_2$ is convex in B. The conclusion follows after observing that any geodesic triangle in B is the limit of geodesic triangles in $B \setminus C_2$.

For case two, there are only two graphs that correspond to this case: graph (4) of Figure 5 and graph (2) of Figure 6. For graph (4) of Figure 5, we may proceed as in case one. Here we verify that the set $B \setminus \{C_2 \cup \{p_1, p_2\}\}$ is convex in B, where p_1 and p_2 are the lifts of the point p corresponding to the isolated circle orbit in M^* that does not belong to the cycle K^1. For graph (2) of Figure 6, let A_1 be the arc in K^1 with isotropy $\mathbb{Z}_k$, $k \neq 2$, and let A_2 be the lift in B of the arc in the claw with isotropy $\mathbb{Z}_2$ not contained in K^1. Observe that A_2 is a minimal geodesic between the lifts of the isolated circle orbit not contained in the cycle K^1. As above, one verifies that the set $B \setminus (A_1 \cup A_2)$ is convex in B and the conclusion follows after observing that geodesics triangles in B are limits of geodesic triangles in $B \setminus (A_1 \cup A_2)$. □

Lemma 5.14 *Let $\overline{p} \in K^1 \subset M^*$ be a point corresponding to an isolated circle orbit in M^5 and consider $\overline{p}$ as a point in B. Then $\Sigma_{\overline{p}} B$ is smaller than or equal to $S^2(\frac{1}{2})$.*

Proof of Lemma 5.14 There is a two-fold branched cover $\Sigma_{\overline{p}} B \to \Sigma_{\overline{p}} M^*$. We know that the space of directions $\Sigma_{\overline{p}} M^*$ is a "thin" $S^2(\frac{1}{2})$. We will denote this space by $X_{k,l}$. Observe that $\Sigma_{\overline{p}} M^* = X_{k,l}$ can be written as the join of a circle, $S^1/\mathbb{Z}_{kl}$, of diameter π/kl and S^0, of diameter $\pi/2$, where the former is the normal space of directions to K^1 at the point $\overline{p}$ and the latter corresponds to the tangent space of directions at $\overline{p}$ of K^1 in M^*. Since B is a two-fold branched cover of M^* with branching locus K^1, the corresponding space of normal directions in B will be of twice the diameter as the space of normal directions in M^*. In particular, this means that $\Sigma_{\overline{p}} B$ corresponds to the orbifold $X_{2k,l}$ or $X_{k,2l}$. Since at least one of k, l is greater than 2, it follows that this orbifold is smaller than or equal to $S^2(\frac{1}{2})$, as we saw earlier (see Lemma 5.3). □

Lemma 5.15 *The space B is simply connected.*

Proof of Lemma 5.15 We will prove this by contradiction, so assume that $\pi_1(B)$ contains at least two elements. Observe that $\widetilde{B}$, the universal cover of B, is a compact Alexandrov space of nonnegative curvature. There are at least three singular points p_i in B, corresponding to isolated circle orbits in M^*. Therefore, in $\widetilde{B}$ there will be at least six points $\tilde{p}_k$ covering these points M^*. By Lemma 5.14, the spaces of directions $\Sigma_{\tilde{p}_k}\widetilde{B}$ are smaller than or equal to $S^2(\frac{1}{2})$. On the other hand, the Extent lemma implies there can be at most five such points in $\widetilde{B}$, yielding a contradiction. □

Now we prove part (3). Let $\mathbb{Z}_k$ and $\mathbb{Z}_l$ be the finite isotropy groups fixing the two 3–manifolds corresponding to the two arcs of the cycle K^1. Without loss of generality, we may assume that $2 \le k < l$. In this case we may take a k–fold branched cover of M^* with branching locus K^1. It follows from the proof of Theorem 5.2 that $k = 2$, since otherwise the branched cover would have more than four singular points.

Finally, we prove part (4). Observe that B is a topological 3–manifold and, by Lemma 5.15, B is simply connected. Hence, by the resolution of the Poincaré conjecture, B must be homeomorphic to S^3. Recall that we have an isometric $\mathbb{Z}_2$ action on B fixing K^1. By the equivariant version of the Poincaré conjecture (see Dinkelbach and Leeb [9]), it follows that this action is equivalent to a linear action on a standard $S^3(1)$. Therefore $\mathbb{Z}_2 \subset SO(4)$. Note that $\mathbb{Z}_2$ is not equivalent to the action of $-\mathrm{Id}$, since the branching locus is a unique circle fixed by the $\mathbb{Z}_2$ action. Thus, without loss of generality,

$$\mathbb{Z}_2 = \left\{ \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cup \mathrm{Id} \right\}.$$

Thus, $K^1 \subset M^* = B/\mathbb{Z}_2$ is not knotted. This concludes the proof of Theorem 5.12. □

6 Nonnegatively curved 5–manifolds with almost maximal symmetry rank

We can now classify closed, simply connected, nonnegatively curved 5–manifolds with an isometric T^2 action, corresponding to the almost maximal symmetry rank case in dimension 5. We summarize our results in the following theorem.

Theorem 6.1 *Let M^5 be a closed, simply connected 5–manifold with nonnegative curvature admitting an isometric T^2 action. Then M^5 is diffeomorphic to S^5, $S^3 \times S^2$, $S^3 \tilde{\times} S^2$ or $SU(3)/SO(3)$.*

By Lemma 4.4, the T^2 action is neither free nor almost free. In particular, this tells us that there is always some circle subgroup with nonempty fixed point set. We then have two cases to consider: case A, where some circle subgroup acts fixed point homogeneously and therefore M^* is D^3 or $S^2 \times I$, and case B, where no circle subgroup acts fixed point homogeneously and hence $M^* = S^3$. Throughout, our main goal will be to compute $H_2(M^5)$. The conclusions of Theorem 6.1 will then follow by the Barden–Smale classification of simply connected 5–manifolds [1; 42]. We remark that it is only in case B, where $M^* = S^3$ and we have finite isotropy, that the Wu manifold, $SU(3)/SO(3)$, may occur. Observe also that if one circle subgroup acts freely, then M^5 fibers over one of the four manifolds $\mathbb{C}P^2, S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ (see Theorem 3.3). The corresponding total space is diffeomorphic to S^5, $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$ (see Duan and Liang [10]). It follows that in the case where we obtain the Wu manifold, no circle subgroup may act freely.

6.1 Case A: ∂M^* is nonempty

Let M be a nonnegatively curved manifold with a fixed point homogeneous T^1 action. By definition, this means that there is a component F of $\mathrm{Fix}(M; T^1)$ of codimension 2. We begin with the following proposition.

Proposition 6.2 *Let M^n be a closed, simply connected, nonnegatively curved manifold of dimension $n \geq 4$ with an isometric T^1 action and suppose that* $\mathrm{Fix}(M^n; T^1)$ *contains an $(n-2)$–dimensional component F^{n-2}. Let C^k be the set at maximal distance from F^{n-2} in the orbit space $X^{n-1} = M^n/T^1$.*

(1) *If $\dim(F) = \dim(C) = n-2$ and we further suppose that $\pi^{-1}(C) = B$ is a topological manifold without boundary, then B is fixed by the T^1 action, is isometric to F^{n-2} and $F^{n-2} \cong C^{n-2}$ is simply connected.*

(2) *If C^k has dimension $k \leq n-4$, then F^{n-2} is simply connected.*

Proof First we prove (1). If we suppose that C is not fixed by the S^1 action, then $B = \pi^{-1}(C)$ has dimension $n-1$. We may decompose M as a union of neighborhoods of N^{n-2} and B, which we will denote V and U, respectively. Their common boundary is a circle bundle over N^{n-2} which we denote by ∂V. Observe that both N and ∂V are orientable, but that C is not (this follows from the Mayer–Vietoris sequence of the triple (M, V, U)). Since M^n is simply connected it follows by duality that $H_{n-1}(M) = 0$ and that the torsion subgroup of $H_{n-2}(M)$ is trivial. Further, since ∂V is a compact, orientable manifold of dimension $n-1$, the torsion subgroup of $H_{n-2}(\partial V)$ is also trivial. Likewise, since V deformation retracts onto N^{n-2}, $H_{n-2}(V) = \mathbb{Z}$. Since B

is nonorientable, the torsion subgroup of $H_{n-2}(U)$ is equal to $\mathbb{Z}_2$. If we write down the first few elements of the Mayer–Vietoris sequence of the triple (M,V,U) we have

$$\begin{aligned}0 \to H_n(M) \to H_{n-1}(\partial V) \to H_{n-1}(U) \oplus H_{n-1}(V) \to H_{n-1}(M)\\ \to H_{n-2}(\partial V) \to H_{n-2}(U) \oplus H_{n-2}(V) \to H_{n-2}(M).\end{aligned}$$

Substituting known values we obtain

$$\begin{aligned}0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \oplus 0 \to 0\\ \to \mathrm{Fr}(H_{n-2}(\partial V)) \to (\mathrm{Fr}(H_{n-2}(U)) \oplus \mathbb{Z}_2) \oplus \mathbb{Z} \to \mathrm{Fr}(H_{n-2}(M)).\end{aligned}$$

The sequence is not exact and thus this case cannot occur. This in turn implies that the inverse image of C^{n-2} in M must be exactly C^{n-2} and thus C^{n-2} is a component of $\mathrm{Fix}(M;T^1)$. It then follows from Double soul theorem 2.5 that M is an S^2 bundle over F^{n-2} and hence N^{n-2} must be simply connected.

To prove (2), let γ be a loop in $F^{n-2} \subset M^n$. Since M^n is simply connected, γ bounds a 2–disc D^2. Let $B^{k'} = \pi^{-1}(C^k)$ and observe that $k' \le n-3$. By transversality, we can perturb D^2 so as to lie in the complement of $D(B^{k'})$, a tubular neighborhood of $B^{k'}$, while keeping $\gamma = \partial D^2$ in F^{n-2}. The conclusion follows after observing that D^2 is now contained in $D(F^{n-2})$, which deformation retracts onto F^{n-2}. □

Remark 6.3 Proposition 6.2 holds trivially in dimension 2, since in this case the fixed point set components of codimension 2 are isolated points. In dimension 3, however, the conditions of Parts 1 and 2 of Proposition 6.2 cannot occur, since a 1–dimensional fixed point set component, being a closed submanifold of M, must be a circle and thus has infinite fundamental group.

Simply connected 5–manifolds with nonnegative curvature and a fixed point homogeneous isometric circle action were classified by Galaz-Garcia and Spindeler [16; 17]. To obtain the classification, it suffices to show that there is some convex subset of the set C at maximal distance from a fixed point set component $F \subset \partial M^5/S^1$ whose inverse image in M^5 is a smooth manifold H without boundary. In particular, one shows that the dimension of H is always either 1 or 3 and M can be decomposed as a union of disc bundles. One may then conclude that $H_2(M^5;\mathbb{Z})$ is either trivial or $\mathbb{Z}$ and using the Barden–Smale classification [1; 42], the result is obtained.

The case where $\dim(C) = \dim(F)$ is simplified somewhat by the use of Proposition 6.2; indeed, when C has no boundary, it is the soul of M/S^1 and the orbit type on C must be constant, so that $\pi^{-1}(C) = H$ is a smooth manifold. The cases where $\partial C \neq \varnothing$ are then considered individually and in those cases where H has boundary, one can easily produce a convex subset $C' \subset C$, where $\dim(C') < \dim(C)$ and whose inverse image in M is a smooth manifold (see [16] or Spindeler [43, Chapter 5] for details).

The following theorem summarizes the result.

Theorem 6.4 *Let S^1 act isometrically and fixed point homogeneously on M^5, a closed, nonnegatively curved, simply connected 5–manifold. Then M^5 is diffeomorphic to S^5, $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$.*

6.2 Case B: $M^* = S^3$

We consider the case where no circle acts fixed point homogeneously, that is, $M^* = S^3$, and there are either three or four isolated circle orbits. We prove the following theorem.

Theorem 6.5 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold admitting an isometric T^2 action. If $M^* = S^3$, then M^5 is diffeomorphic to S^5, $SU(3)/SO(3)$, $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$.*

We first consider the case where there is no nontrivial finite isotropy.

Proposition 6.6 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold admitting an isometric T^2 action. Suppose $M^* = S^3$ and that there is no nontrivial finite isotropy.*

(1) *If there are exactly three isolated circle orbits, then M^5 is diffeomorphic to S^5.*

(2) *If there are four isolated circle orbits, then M^5 is diffeomorphic to $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$.*

Proof This follows directly from the proof of Proposition 4.5 and the Barden–Smale classification of closed, simply connected smooth 5–manifolds [1; 42]. □

We now consider the case where the T^2 action admits nontrivial finite isotropy groups. We will devote the rest of this section to the proof of the following proposition.

Proposition 6.7 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold with an isometric T^2 action. Suppose that $M^* = S^3$ and there is nontrivial finite isotropy.*

(1) *If there are exactly three isolated circle orbits, then M^5 is diffeomorphic to S^5 or the Wu manifold, $SU(3)/SO(3)$.*

(2) *If there are four isolated circle orbits, then M^5 is diffeomorphic to $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$.*

Before we proceed with the proof of Proposition 6.7 we summarize in the proposition below the properties of the weighted graphs we need to consider. Recall from Section 4.3 that black vertices correspond to isolated singular orbits, white vertices correspond to exceptional orbits with $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropy, and edges correspond to nonisolated exceptional orbits with nontrivial, finite, cyclic isotropy.

Proposition 6.8 *Let T^2 act isometrically on M^5, a closed, simply connected, nonnegatively curved 5–manifold. If $M^* = S^3$ and there is nontrivial finite isotropy, then the graphs corresponding to the possible singular sets are characterized by the following properties.*

(1) *The number of black vertices is either three or four.*

(2) *Black vertices have degree 0, 1 or 2 and white vertices have degree 3.*

(3) *The graph contains at least one edge.*

(4) *Every edge must connect two different vertices.*

(5) *The isotropy associated with each edge meeting a white vertex is $\mathbb{Z}_2$.*

(6) *Every edge connecting a white vertex and a black vertex belongs to a weighted claw.*

(7) *If a weighted graph contains a claw, then the graph has exactly 3 black vertices.*

(8) *The graph contains at most one cycle. Moreover, if a cycle exists in a graph with 4 black vertices, then the cycle contains every vertex and edge in the graph.*

Proof Part (1) follows from Proposition 4.9 and Theorem 5.2. Parts (2), (4) and (5) follow from Proposition 4.9. Part (3) follows from Proposition 4.9 and the fact that we are assuming the action has nontrivial finite isotropy. Parts (6) and (7) follow from Proposition 5.8. Finally, part (8) follows by Theorem 5.12. □

We may now make a complete list of all the graphs that can occur in the case where we have three isolated circle orbits and in the case where we have four isolated circle orbits. The graphs are listed in Figures 5, 6 and 7.

Recall that, by Theorem 5.12, if the weighted graph contains a cycle, then this cycle must be unknotted in $M^* = S^3$. We will now show in all cases where we have a cycle that we may decompose the manifold as a union of disc bundles, where at least one of the disc bundles is over one arc of the cycle. We have the following proposition.

Proposition 6.9 *Let M^5 be a closed, simply connected, nonnegatively curved 5–manifold with an isometric T^2 action. Suppose that $M^* = S^3$ and there is nontrivial finite isotropy.*

Suppose, in addition, that the weighted graph, W, corresponding to the singular set of the action contains a cycle K^1, corresponding to graphs (3) and (4) in Figure 5, graph (2) in Figure 6, and graph (5) in Figure 7. Then the following are true.

(1) *If W is graph (3) in Figure 5, then M^5 decomposes as the union of a disc bundle over a 3–dimensional submanifold $N^3 \subset M^5$ fixed by nontrivial finite cyclic isotropy, corresponding to the preimage of an arc in K^1, and a disc bundle over the remaining circle orbit not contained in N^3.*

(2) *If W is graph (4) in Figure 5, then M^5 decomposes as a union of disc bundles over two 3–dimensional submanifolds. One of these 3–manifolds corresponds to the fixed point set of $\mathbb{Z}_k$ isotropy, $k > 2$, and the other corresponds to the preimage of the arc between the remaining isolated circle orbit and an exceptional orbit $T^2/\mathbb{Z}_2$, which projects to an interior point of the arc of $\mathbb{Z}_2$ isotropy.*

(3) *If W is graph (2) in Figure 6, then M^5 decomposes as the union of disc bundles over two 3–dimensional submanifolds. One of these 3–manifolds corresponds to the preimage of the arc with $\mathbb{Z}_k$ isotropy, $k > 2$, and the other to the preimage of the arc with $\mathbb{Z}_2$ isotropy containing the remaining isolated circle orbit.*

(4) *If W is graph (5) in Figure 7, then M^5 decomposes as the union of two disc bundles over two disjoint 3–dimensional submanifolds fixed by nontrivial finite isotropy (although not necessarily the same group).*

Proof We will first prove parts (1) and (4) corresponding, respectively, to graph (3) in Figure 5 and graph (5) in Figure 7. In both cases the weighted graph is a cycle K^1.

Fix an arc A_1^* in K^1 corresponding to a fixed point set component of nontrivial finite cyclic isotropy $\mathbb{Z}_k$. Note that whether we have three or four isolated circle orbits, the corresponding edges of the weighted cycle K^1 in M^* form the angle $\pi/2$. Thus, at isolated circle orbits corresponding to the endpoints of the arc A_1^*, the normal space to the 3–dimensional submanifold $N^3_{\mathbb{Z}_k} = \pi^{-1}(A_1^*)$, fixed by $\mathbb{Z}_k$, will be the tangent space to the 3–dimensional submanifold fixed by nontrivial finite cyclic isotropy, corresponding to the lift of arcs adjacent to A_1^*. In graph (3) in Figure 5 the cycle K^1 contains three edges, and there are three 3–dimensional submanifolds fixed by nontrivial finite isotropy, each one corresponding to the lift of an arc in K^1.

Consider $N^3_{\mathbb{Z}_k} = \pi^{-1}(A_1^*)$ and let C_1 be the remaining isolated circle orbit which is not contained in $N^3_{\mathbb{Z}_k}$, so that C_1 projects to the vertex of K^1 not contained in A_1^*.

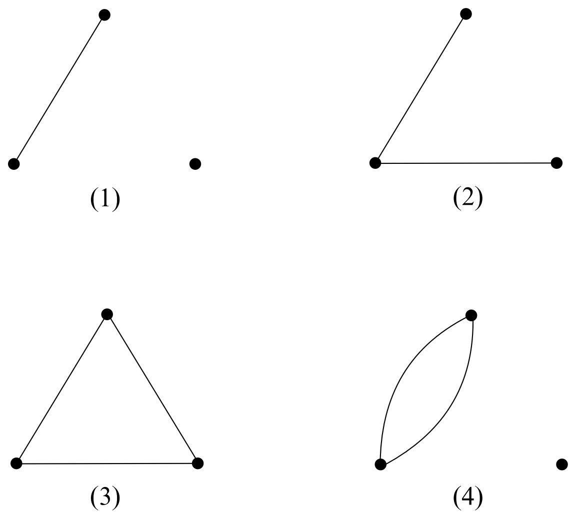

Figure 5: Possible weighted graphs when there are exactly three isolated circle orbits and only nontrivial finite cyclic isotropy

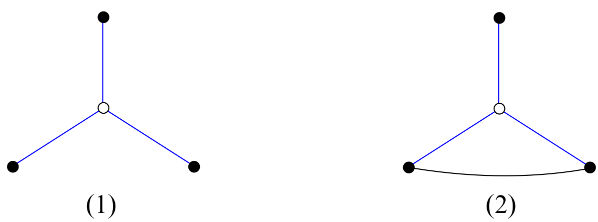

Figure 6: Possible weighted graphs when there are exactly three isolated circle orbits and an isolated exceptional orbit

The decomposition of M as a union of disc bundles over C_1 and over $N^3_{\mathbb{Z}_k}$ follows from an argument analogous to the one given by Grove and Wilking in [24, Section 3], where it is applied to obtain a double disc bundle decomposition of closed, simply connected 4–manifolds of nonnegative curvature with an isometric circle action. Our situation is analogous to the one in [24] and their argument, which we now recall, carries over to our case. Let U be a small ε–neighborhood of the preimage of K^1. In U one may construct a smooth T^2 invariant horizontal vector field V that is normally radial near $N^3_{\mathbb{Z}_k}$ and C_1 and which is tangential to the inverse image of the remaining two edges of the cycle K^1 in M^*. This vector field can be taken to be the horizontal lift of a smooth (in the orbifold sense) vector field V^* on U^* which is normal near the image of the boundaries of sufficiently small tubular neighborhoods of $N^3_{\mathbb{Z}_k}$ and C_1 and for which the remaining two edges of K^1 are integral curves.

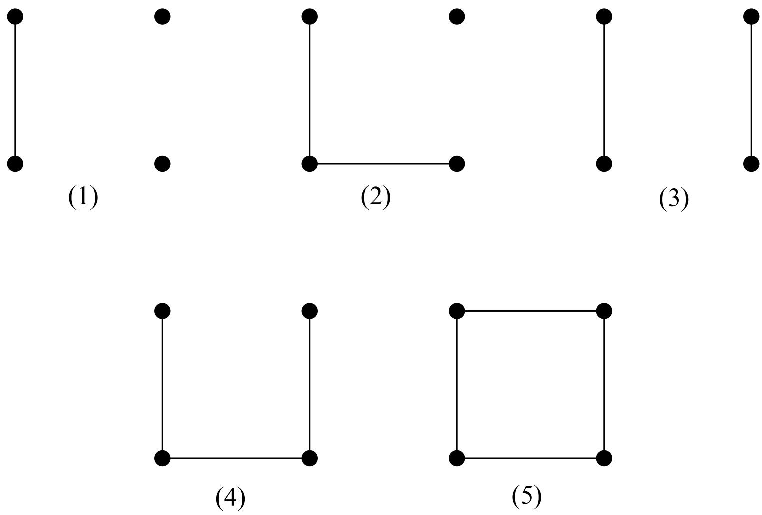

Figure 7: Possible weighted graphs when there are exactly four isolated circle orbits and nontrivial finite cyclic isotropy

Since the ε–neighborhoods of the images of $N^3_{\mathbb{Z}_k}$ and C_1 are 3–balls, as well as their complements, and K^1 is unknotted, V^* can be extended to a smooth nonvanishing vector field on the complement of U^* respecting the ball decomposition of M^* This extension uniquely lifts to an invariant extension of V, thus yielding the desired decomposition of M.

The same argument in the preceding paragraph works for graph (5) in Figure 7, corresponding to the case of four isolated circle orbits. Here we will isotope the boundary of a tubular neighborhood around N^3_1, corresponding to the preimage of an arc $A^*_1 \subset M^*$ in the cycle K^1, to the boundary of a tubular neighborhood around N^3_2, corresponding to the preimage of the arc opposite to A^*_1.

We now prove part (2), corresponding to graph (4) in Figure 5. Recall that in this case one of the edges in K^1 corresponds to orbits with isotropy $\mathbb{Z}_2$, while the other one corresponds to orbits with isotropy $\mathbb{Z}_k$, $k > 2$. We will denote the arc in K^1 corresponding to a fixed point set component of isotropy $\mathbb{Z}_2$ by A^*_0, and we will let A^*_1 be the arc in K^1 corresponding to the fixed point set component with finite isotropy $\mathbb{Z}_k$. We now form an arc A^*_2 in M^* by joining the vertex not contained in K^1 to A^*_1 via a shortest geodesic in M^*. The interior of this arc consists of principal orbits and the preimage of this arc is a cohomogeneity one 3–manifold, N^3_2. Proceeding as in cases (1) and (4), we may decompose M^5 as a union of disc bundles over $N^3_1 = \pi^{-1}(A^*_1)$ and N^3_2.

To prove part (3), we let A_1^* be the arc not contained in the weighted claw, that is, the arc containing two isolated circle orbits and corresponding to a fixed point set component of isotropy $\mathbb{Z}_k$, $k > 2$. We let A_2^* be the arc in the claw containing the isolated circle orbit not contained in A_1^*. Proceeding as above, we may decompose M^5 as a union of disc bundles over $N_1^3 = \pi^{-1}(A_1^*)$ and $N_2^3 = \pi^{-1}(A_2^*)$. □

We are now ready to prove Proposition 6.7. Our strategy will be to analyze the weighted graphs grouped into three separate cases, where the first two cases correspond to part (1) of Proposition 6.7 and the third case corresponds to part (2) of Proposition 6.7. The first case will be all the graphs in Figure 5, with the exception of graph (4). The second case will consist of graph (4) of Figure 5 and graphs (1) and (2) of Figure 6. The third case will be the graphs of Figure 7.

Proof of Proposition 6.7(1) The weighted graph corresponding to the singular set of the action is one of those listed in Figure 5 or Figure 6. It follows from the discussion in Section 5.3 that if the weighted graph is one of those in Figure 5, then a fixed point set component of nontrivial finite isotropy can only be one of S^3, $L_{p,q}$ or $S^2 \times S^1$; if the weighted graph is one of those in Figure 6, then the fixed point set components of nontrivial finite isotropy corresponding to arcs in the claw can only be $S^2 \tilde{\times} S^1$ or $\mathbb{R}P^2 \times S^1$ and the corresponding isotropy subgroup is a $\mathbb{Z}_2$ subgroup of T^2 in each case.

As mentioned above, we have divided the proof of Proposition 6.7(1) into two cases: the case where the graphs are all those from Figure 5 with the exception of graph (4), and the case corresponding to graphs (4) from Figure 5 and graphs (1) and (2) from Figure 6.

For the first case, we have the following lemma.

Lemma 6.10 *Let T^2 act isometrically on M^5, a closed, simply connected 5–manifold of nonnegative curvature and suppose that $M^* = S^3$. If there are exactly 3 isolated circle orbits and the weighted graph of the action is one of graphs (1), (2) or (3) from Figure 5, then neither $L_{p,q}$ or $S^2 \times S^1$ may be a component of a fixed point set of nontrivial finite isotropy.*

Proof If the weighted graph is one of graphs (1) or (2), which do not contain a cycle, then we may complete it to a cycle by adding edges corresponding to curves consisting of regular points in the orbit space, so that each vertex in the graph has degree 2 (see Figure 8). We choose these curves so that they are geodesics near the vertices and any two edges meet at the maximal angle $\pi/2$. We may then decompose M^5 as the

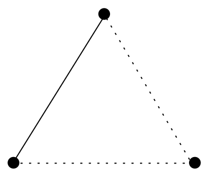

Figure 8: Completing a weighted graph with three vertices to form a cycle: the solid edge corresponds to orbits with nontrivial finite cyclic isotropy, while the dotted edges correspond to principal orbits

union of a disc bundle over a fixed point set component of nontrivial finite isotropy and the remaining isolated circle orbit. A tubular neighborhood around the isolated circle orbit will be a D^4–bundle over S^1 with boundary an S^3 bundle over S^1. A tubular neighborhood around the fixed point set component of nontrivial finite isotropy will be a D^2–bundle over $L_{p,q}$ or $S^2 \times S^1$ and therefore the boundary of both tubular neighborhoods must be $S^3 \times S^1$. When we consider the Mayer–Vietoris sequence of this decomposition we immediately obtain a contradiction and therefore neither of these two manifolds may occur as a fixed point set component of nontrivial finite isotropy. □

For the second case, we have the following lemma.

Lemma 6.11 *Let T^2 act isometrically on M^5, a closed, simply connected 5–manifold of nonnegative curvature and suppose that $M^* = S^3$, there are exactly three isolated circle orbits and the singular set corresponds to graph (4) of Figure 5 or graph (2) of Figure 6. Then a fixed point set component of finite isotropy $\mathbb{Z}_k$, $k > 2$, can only be one of S^3 or $\mathbb{R}P^3$.*

Proof We must rule out $L_{p,q}$, where $(p,q) \neq (2,1)$, and $S^2 \times S^1$ as fixed point set components of isotropy $\mathbb{Z}_k$, $k > 2$. Recall that, for both graphs under consideration, M^5 decomposes as a union of disc bundles over one of $L_{p,q}$ or $S^2 \times S^1$, and over one of $S^2 \tilde{\times} S^1$ or $\mathbb{R}P^2 \times S^1$. It follows from the Mayer–Vietoris sequence of this decomposition that only two possibilities do not give rise to a contradiction: namely M^5 may be the union of disc bundles over $\mathbb{R}P^3$ and $S^2 \tilde{\times} S^1$ or over S^3 and $S^2 \tilde{\times} S^1$. □

With these two lemmas we may now complete the proof of part (1) of Proposition 6.7. From Lemma 6.10 above we conclude that the only possible fixed point set components for graphs (1), (2), and (3) of Figure 5 are S^3. In this case, it follows from the Mayer–Vietoris sequence that $H_2(M^5) = 0$ and therefore M^5 is diffeomorphic to S^5 by work of Smale and Barden [42; 1].

From Lemma 6.11 we note for graph (4) of Figure 5 or graph (2) of Figure 6 that $H_2(M^5)=0$ when $\mathbb{R}P^3$ is the fixed point set component of isotropy $\mathbb{Z}_k, k>2$, and $H_2(M^5)=\mathbb{Z}_2$ when S^3 is. Both graphs (1) and (2) of Figure 6 contain a weighted claw and in the case of graph (1), we may complete the graph, joining two disjoint arcs via edges corresponding to shortest geodesics consisting of regular points in the orbit space. We may then decompose the manifold as a union of disc bundles over the preimage of the arc joining two edges of $\mathbb{Z}_2$ isotropy and the remaining edge of $\mathbb{Z}_2$ isotropy.

We further note that in these last two cases it follows from Corollary 5.11 that $H_2(M^5)=\mathbb{Z}_2$ and from Theorem 5.10 that the arcs corresponding to 3–manifolds of $\mathbb{Z}_2$ isotropy must be $S^2\tilde{\times} S^1$. It now follows by [42; 1] that for graph (4) of Figure 5 M^5 is diffeomorphic to S^5 or the Wu manifold and for graphs (1) and (2) of Figure 6 M^5 is diffeomorphic to the Wu manifold.

We have now completed the proof of part (1) of Proposition 6.7. □

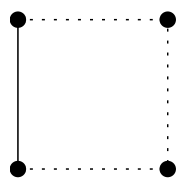

Figure 9: Completing a weighted graph with edges corresponding to principal orbits to obtain a cycle: the solid edge corresponds to orbits with finite cyclic isotropy, while the dotted edges correspond to principal orbits

It remains to prove part (2) of Proposition 6.7. To do this, it suffices to show that $H_2(M^5)\cong\mathbb{Z}$ for every possible fixed point set of nontrivial finite isotropy and thus, by [42; 1], M^5 is diffeomorphic to one of the two S^3 bundles over S^2.

Proof of Proposition 6.7(2) In this case the possible weighted graphs are shown in Figure 7. For graphs (1) through (4), we may complete the weighted graph by joining disjoint isolated circle orbits or arcs via edges corresponding to curves consisting of regular points in the orbit space. As before, we choose these curves so that they are geodesics near the vertices and any two edges meet at the maximal angle $\pi/2$. In this way we obtain a graph that is an unknotted cycle (see Figure 9) and now for all the possible graphs we may decompose M^5 as the union of two disc bundles over the 3–dimensional manifolds that correspond to opposite arcs of the cycle. In this particular case, the 3–dimensional manifold may be one of S^3, $L_{p,q}$ or $S^2\times S^1$. In all cases and for all possible combinations, we see that $H_2(M^5)=\mathbb{Z}$ and the result follows. □

7 Examples of isometric T^2 actions on simply connected, nonnegatively curved 5–manifolds

7.1 Examples of actions with codimension-2 fixed point set

It is easy to find examples of such actions and we list a few here.

Example 7.1 Given $(\theta_1, \theta_2) \in T^2$ and $(z_1, z_2, z_3) \in S^5 \subset \mathbb{C}^3$, let

$$\big((\theta_1, \theta_2), (z_1, z_2, z_3)\big) \longmapsto (e^{2\pi i\theta_1} z_1, e^{2\pi i\theta_2} z_2, z_3).$$

Here both circles θ_1 and θ_2 fix a 3–sphere. The corresponding singular set in the orbit space consists of 3 isolated singular points.

Example 7.2 Given $(\theta_1, \theta_2) \in T^2$ and $(z_1, z_2, x_1, x_2, x_3) \in S^3 \times S^2 \subset \mathbb{C}^2 \times \mathbb{R}^3$, let

$$\big((\theta_1, \theta_2), (z_1, z_2, x_1, x_2, x_3)\big) \longmapsto (e^{2\pi i\theta_1} z_1, e^{2\pi i\theta_2} z_2, x_1, x_2, x_3).$$

Here both circles θ_1 and θ_2 fix an $S^2 \times S^1$ and the action is the product of the cohomogeneity one action on S^3 combined with the trivial action on S^2. The corresponding singular set in the orbit space consists of 4 isolated singular points.

7.2 Examples of actions with finite isotropy

We give examples of actions on S^5 and on $S^3 \times S^2$ with finite isotropy and with 3 and 4 isolated circle orbits, respectively. The action on S^5 was given by Rong [39] and we include it here for the sake of completeness.

Example 7.3 Given $(\theta_1, \theta_2) \in T^2$ and $(z_1, z_2, z_3) \in S^5 \subset \mathbb{C}^3$, let

$$\big((\theta_1, \theta_2), (z_1, z_2, z_3)\big) \longmapsto (e^{2\pi i(\theta_1 + p\theta_2)} z_1, e^{2\pi i(\theta_1 + q\theta_2)} z_2, e^{2\pi i(\theta_1 + r\theta_2)} z_3).$$

Here there are 3 isolated circle orbits. If p, q, r are pairwise relatively prime and the differences $(p-q)$, $(p-r)$ and $(q-r)$ are also pairwise relatively prime, then the singular set of the action is a cycle in the orbit space and the closure of each edge corresponds to an S^3 fixed by finite isotropy.

Example 7.4 Given $(\theta_1, \theta_2) \in T^2$ and $v = (z_1, z_2, x_1, x_2, x_3) \in S^3 \times S^2 \subset \mathbb{C}^2 \times \mathbb{R}^3$, we let (θ_1, θ_2) act on v by

$$\big((\theta_1, \theta_2), v\big) \mapsto A(\theta_1, \theta_2)v,$$

where $A(\theta_1, \theta_2)$ is the matrix

$$\begin{pmatrix} e^{2\pi i(\theta_1 + p\theta_2)} & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi i(\theta_1 + q\theta_2)} & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta_1 + r\theta_2) & \sin(\theta_1 + r\theta_2) & 0 \\ 0 & 0 & -\sin(\theta_1 + r\theta_2) & \cos(\theta_1 + r\theta_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

p, q, r are pairwise relatively prime integers, as are the differences $(p-q)$, $(p-r)$ and $(q-r)$, and, without loss of generality, $p > q > r$. Here there are 4 isolated circle orbits and the finite groups $\mathbb{Z}_{p-r}, \mathbb{Z}_{q-r}$ each fix a distinct $S^2 \times S^1$ that has empty intersection with the other whereas the finite group $\mathbb{Z}_{p-q}$ fixes two disjoint copies of S^3, intersecting each of the fixed $S^2 \times S^1$ in an isolated circle orbit. The corresponding singular set in the orbit space is a quadrangle with vertices corresponding to isolated circle orbits and edges corresponding to arcs with finite isotropy.

Example 7.5 Let $T^2 \subset SU(3)$ act canonically on $SU(3)/SO(3)$. There are three involutions given by the diagonal matrices with entries $(-1,-1,1), (-1,1,-1)$ and $(1,-1,-1)$. Each of these involutions will fix an $S(U(2)\times U(1))/S(O(2)\times O(1)) = S^2 \tilde{\times} S^1$, each of which intersects in a $S(U(1)\times U(1)\times U(1))/S(O(1)\times O(1)\times O(1)) = T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The corresponding singular set in the orbit space is a weighted claw.

One can generate further examples by observing that $S^3 \times S^2$ and $S^3 \tilde{\times} S^2$ can be described as normal biquotients of $S^3 \times S^3$ by the free action of a circle (see [7], Galaz-Garcia and Kerin [14], [33] and Totaro [44]). The standard effective, isometric action of T^4 on the Lie group $S^3 \times S^3$ induces a maximal rank, effective, isometric torus action on each of these biquotients. This in turn induces effective, isometric T^2 actions on these manifolds.

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