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A categorification of $U_T(\mathfrak{sl}(1|1))$ and its tensor product representations

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We define the Hopf superalgebra $U_T(\mathfrak{sl}(1|1))$, which is a variant of the quantum supergroup $U_q(\mathfrak{sl}(1|1))$, and its representations $V_1^{\otimes n}$ for $n > 0$. We construct families of DG algebras A, B and R_n, and consider the DG categories $\mathrm{DGP}(A)$, $\mathrm{DGP}(B)$ and $\mathrm{DGP}(R_n)$, which are full DG subcategories of the categories of DG A–, B– and R_n–modules generated by certain distinguished projective modules. Their 0^{th} homology categories $\mathrm{HP}(A)$, $\mathrm{HP}(B)$ and $\mathrm{HP}(R_n)$ are triangulated and give algebraic formulations of the contact categories of an annulus, a twice punctured disk and an n times punctured disk. Their Grothendieck groups are isomorphic to $U_T(\mathfrak{sl}(1|1))$, $U_T(\mathfrak{sl}(1|1)) \otimes_{\mathbb{Z}} U_T(\mathfrak{sl}(1|1))$ and $V_1^{\otimes n}$, respectively. We categorify the multiplication and comultiplication on $U_T(\mathfrak{sl}(1|1))$ to a bifunctor $\mathrm{HP}(A) \times \mathrm{HP}(A) \to \mathrm{HP}(A)$ and a functor $\mathrm{HP}(A) \to \mathrm{HP}(B)$, respectively. The $U_T(\mathfrak{sl}(1|1))$–action on $V_1^{\otimes n}$ is lifted to a bifunctor $\mathrm{HP}(A) \times \mathrm{HP}(R_n) \to \mathrm{HP}(R_n)$.

18D10; 16D20, 57M50

1 Introduction

1.1 Background

This paper is a sequel to [37], in which we categorified the algebra structure of an integral version of the quantum supergroup $U_q(\mathfrak{sl}(1|1))$. The goal of this paper is to present a categorification of a Hopf superalgebra $U_T(\mathfrak{sl}(1|1))$ (a variant of $U_q(\mathfrak{sl}(1|1))$) and its representations $V_1^{\otimes n}$ for $n > 0$, where V_1 is the two-dimensional fundamental representation.

In the late 1980s, Witten [41] and Reshetikhin and Turaev [31] established a connection between quantum groups and knot invariants. In particular, the Jones polynomial could be recovered as the Witten–Reshetikhin–Turaev invariant of the fundamental representation of $U_q(\mathfrak{sl}_2)$. For quantum supergroups, Kauffman and Saleur [15] developed an analogous representation-theoretic approach to the Alexander polynomial, by considering the fundamental representation V_0 of $U_q(\mathfrak{sl}(1|1))$. Rozansky and Saleur [33] gave a corresponding quantum field-theoretic description.

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The connection between quantum groups and knot invariants can be lifted to the categorical level. The existence of such a lifting process, called *categorification*, was conjectured by Crane and Frenkel in [5]. In the seminal paper [18], Khovanov defined a doubly graded homology, now called *Khovanov homology*, whose graded Euler characteristic agreed with the Jones polynomial. Chuang and Rouquier [4] categorified locally finite $\mathfrak{sl}_2$–representations, and more generally, Rouquier [32] constructed a 2–category associated with a Kac–Moody algebra. For the quantum groups themselves, Lauda [22] gave a diagrammatic categorification of $\boldsymbol{U}_q(\mathfrak{sl}_2)$ and general cases are given by Khovanov and Lauda [19; 20; 21]. The program of categorifying Witten–Reshetikhin–Turaev invariants was brought to fruition by Webster [38; 39] using the diagrammatic approach.

On the other hand, the Alexander polynomial is categorified by *knot Floer homology*, defined independently by Ozsváth and Szabó [28] and Rasmussen [30]. Although its initial definition was through Lagrangian Floer homology, knot Floer homology admits a completely combinatorial description by Manolescu, Ozsváth and Sarkar [25]. It is natural to ask whether there is a categorical program for $\boldsymbol{U}_q(\mathfrak{sl}(1|1))$ which is analogous to that of $\boldsymbol{U}_q(\mathfrak{sl}_2)$ and which recovers knot Floer homology.

This paper presents another step towards such a categorical program. We first define the Hopf superalgebra $\boldsymbol{U}_T(\mathfrak{sl}(1|1))$ as a variant of $\boldsymbol{U}_q(\mathfrak{sl}(1|1))$ and the representations $V_1^{\otimes n}$ of $\boldsymbol{U}_T(\mathfrak{sl}(1|1))$. Then we categorify the multiplication and comultiplication on $\boldsymbol{U}_T(\mathfrak{sl}(1|1))$, and its representations $V_1^{\otimes n}$. In a subsequent paper [36], we will categorify the action of the braid group on $V_1^{\otimes n}$ which is induced by the R–matrix structure of $\boldsymbol{U}_T(\mathfrak{sl}(1|1))$.

Our motivation is from the *contact category* introduced by Honda [8], which presents an algebraic way to study 3–dimensional contact topology. The contact category is closely related to *bordered Heegaard–Floer homology* defined by Lipshitz, Ozsváth and Thurston [23]; see Section 1.3 for more detail on the contact category. Motivated by the strands algebra in bordered Heegaard–Floer homology, Khovanov in [17] categorified the positive part of $\boldsymbol{U}_q(\mathfrak{gl}(1|2))$. A counterpart of our construction in Lie theory is developed by Sartori in [34] using subquotient categories of $\mathcal{O}(\mathfrak{gl}_n)$.

1.2 Main results

We define the Hopf superalgebra $\boldsymbol{U}_T(\mathfrak{sl}(1|1))$ as an associative $\mathbb{Z}$–algebra with unit I, generators E, F, T, T^{-1} and relations

$$EF + FE = I - T, \tag{1}$$

$$E^2 = F^2 = 0, \quad ET = TE, FT = TF, \quad TT^{-1} = T^{-1}T = I.$$

The comultiplication $\Delta\colon U_T(\mathfrak{sl}(1|1)) \to U_T(\mathfrak{sl}(1|1)) \otimes_{\mathbb{Z}} U_T(\mathfrak{sl}(1|1))$ is given by

$$\Delta(E) = E \otimes I + I \otimes E, \quad \Delta(F) = F \otimes T + I \otimes F, \quad \Delta(T) = T \otimes T.$$

Recall from [15] that the commutator relation of $U_q(\mathfrak{sl}(1|1))$ is

$$EF + FE = \frac{H - H^{-1}}{q - q^{-1}}. \tag{2}$$

To see the relation between $U_T(\mathfrak{sl}(1|1))$ and $U_q(\mathfrak{sl}(1|1))$, we compare their commutator relations (1) and (2) by setting $T = H^{-2}$. Then the right-hand side of (2) is equal to that of (1) multiplied by $H/(q-q^{-1})$.

Let U_T denote $U_T(\mathfrak{sl}(1|1))$ from now on. Let V_1 be a free $\mathbb{Z}[t^{\pm 1}]$–module spanned by $\mathcal{B}_1 = \{|0\rangle, |1\rangle\}$ which admits an action of U_T given by

$$E|0\rangle = 0, \quad F|0\rangle = |1\rangle,$$
$$E|1\rangle = (1-t)|0\rangle, \quad F|1\rangle = 0,$$
$$T|0\rangle = t|0\rangle, \quad T|1\rangle = t|1\rangle.$$

Consider the n^{th} tensor product representation $V_1^{\otimes n}$ induced by the iterated comultiplication of U_T. Note that $T \cdot v = t^n v$ for $v \in V_1^{\otimes n}$ since $\Delta(T) = T \otimes T$. Our categorification of $V_1^{\otimes n}$ is built on a distinguished basis $\mathcal{B}'_n = \mathcal{B}_1^{\times n} = \{\boldsymbol{a} = |a_1 \cdots a_n\rangle \mid a_i \in \{0,1\}\}$, where $|a_1 \cdots a_n\rangle$ is shorthand for $|a_1\rangle \otimes \cdots \otimes |a_n\rangle$. The following are the main results of this paper:

Theorem 1.1 (Categorification of the multiplication on U_T) *There exist a triangulated category* $\mathrm{HP}(A)$ *whose Grothendieck group is* U_T*, and an exact bifunctor*

$$\mathcal{M}\colon \mathrm{HP}(A) \times \mathrm{HP}(A) \to \mathrm{HP}(A)$$

whose induced map $K_0(\mathcal{M})\colon U_T \times U_T \to U_T$ *on the Grothendieck groups agrees with the multiplication on* U_T*.*

Theorem 1.2 (Categorification of the comultiplication on U_T) *There exist a triangulated category* $\mathrm{HP}(B)$ *whose Grothendieck group is* $U_T \otimes_{\mathbb{Z}} U_T$*, and an exact functor*

$$\delta\colon \mathrm{HP}(A) \to \mathrm{HP}(B)$$

whose induced map $K_0(\delta)\colon U_T \to U_T \otimes_{\mathbb{Z}} U_T$ *on the Grothendieck groups agrees with the comultiplication on* U_T*.*

Theorem 1.3 (Categorification of the $\boldsymbol{U_T}$–module $V_1^{\otimes n}$) *For each $n>0$, there exist a triangulated category $\mathrm{HP}(H(R_n))$ whose Grothendieck group is $V_1^{\otimes n}$, and an exact bifunctor*

$$\mathcal{M}_n : \mathrm{HP}(A)\times \mathrm{HP}(H(R_n)) \to \mathrm{HP}(H(R_n))$$

whose induced map $K_0(\mathcal{M}_n)$ on the Grothendieck groups agrees with the action $\boldsymbol{U_T}\times V_1^{\otimes n}\to V_1^{\otimes n}$.

The topological motivation for categorifying the multiplication and comultiplication is completely different. In the corresponding algebraic formulations, we use $\mathrm{HP}(A)\times\mathrm{HP}(A)$ for the multiplication and use $\mathrm{HP}(B)$ for the comultiplication; see Figures 2 and 4 for more detail about the topological motivation.

1.3 Motivation from contact topology

The contact category $\mathcal{C}(\Sigma,F)$ of (Σ,F) is an additive category associated to an oriented surface Σ and a finite subset F of $\partial\Sigma$. The objects of $\mathcal{C}(\Sigma,F)$ are formal direct sums of isotopy classes of *dividing sets* on Σ whose restrictions to $\partial\Sigma$ agree with F. A *dividing set* Γ on Σ is a properly embedded 1–manifold, possibly disconnected and possibly with boundary, which divides Σ into positive and negative regions. The morphism $\mathrm{Hom}_{\mathcal{C}(\Sigma,F)}(\Gamma_0,\Gamma_1)$ is an $\mathbb{F}_2$–vector space spanned by isotopy classes of *tight* contact structures on $\Sigma\times[0,1]$ with the dividing sets Γ_i on $\Sigma\times\{i\}$ for $i=0,1$. The composition is given by vertically stacking contact structures. Any dividing set with a contractible component is isomorphic to the zero object since there is no tight contact structure in a neighborhood of the dividing set by a criterion of Giroux [7]. As basic blocks of morphisms, *bypass attachments* introduced by Honda [9] locally change dividing sets as in Figure 1. Honda, Kazez and Matić [11] gave a criterion for the addition of a collection of disjoint bypasses to be tight.

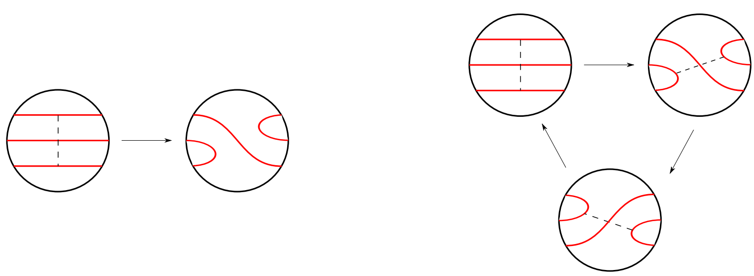

Figure 1: The picture on the left is a bypass attachment along the dashed arc; the one on the right is a distinguished triangle given by a triple of bypass attachments.

The connection between 3–dimensional contact topology and *Heegaard–Floer homology* was established by Ozsváth and Szabó [29] in the closed case. Honda, Kazez and Matić generalized it to the case of a contact 3–manifold with *convex* boundary in [13] and formulated it in the framework of TQFT in [12]. The combinatorial properties of this TQFT were studied by Mathews in the case of disks [26] and annuli [27]. The connection on the categorical level is observed by Zarev [43].

There is a refined version, called the *universal cover* $\widetilde{\mathcal{C}}(\Sigma, F)$, of the contact category $\mathcal{C}(\Sigma, F)$ given as follows. Choose a dividing set Γ_0 as a base point. The basic objects of $\widetilde{\mathcal{C}}(\Sigma, F)$ are pairs $(\Gamma, [\zeta])$, where Γ is an isotopy class of dividing sets on (Σ, F), and $[\zeta]$ is a homotopy class of a 2–plane field ζ on $\Sigma \times [0,1]$ which is contact near $\Sigma \times \{0,1\}$ with the dividing sets Γ_0 on $\Sigma \times \{0\}$ and Γ on $\Sigma \times \{1\}$. The morphism set $\mathrm{Hom}_{\widetilde{\mathcal{C}}(\Sigma,F)}((\Gamma_1, [\zeta_1]), (\Gamma_2, [\zeta_2]))$ is spanned by tight contact structures $\{\xi\}$ such that $[\zeta_2] = [\xi \cup \zeta_1]$, where $\xi \cup \zeta_1$ denotes a concatenation of the 2–plane fields ξ and ζ_1. In other words, the component $[\zeta]$ gives a grading $\mathrm{gr}(\Sigma)$ on the objects of $\widetilde{\mathcal{C}}(\Sigma, F)$ which takes values in homotopy classes of 2–plane fields. Equivalently, the grading $\mathrm{gr}(\Sigma)$ is given by a central extension by $\mathbb{Z}$ of the homology group $H_1(\Sigma)$, ie there is a short exact sequence $0 \to \mathbb{Z} \to \mathrm{gr}(\Sigma) \to H_1(\Sigma) \to 0$. Note that a similar grading appears in bordered Heegaard–Floer homology [23, Section 3.3]. The main feature of the universal cover $\widetilde{\mathcal{C}}(\Sigma, F)$ is the existence of distinguished triangles given by a triple of bypass attachments as in Figure 1. The subgroup $\mathbb{Z}$ of the grading $\mathrm{gr}(\Sigma)$ is related to the shift functor in a triangulated category. In particular, Huang [14] showed that a triple of bypass attachments changes the $\mathbb{Z}$ component by one.

This paper provides an algebraic formulation of the universal covers of the contact categories of an annulus, a twice punctured disk and an n times punctured disk. Let $\widetilde{\mathcal{C}_o}$ be the universal cover of $\mathcal{C}_o := \mathcal{C}(S_o, F_o)$, where S_o is an annulus and F_o consists of two points on each boundary component. Then $\widetilde{\mathcal{C}_o}$ is a monoidal category with a bifunctor $\mathcal{M} : \widetilde{\mathcal{C}_o} \times \widetilde{\mathcal{C}_o} \to \widetilde{\mathcal{C}_o}$ defined by stacking two dividing sets along their common boundaries of two annuli for objects and gluing two contact structures for morphisms; see Figure 2. The Grothendieck group $K_0(\widetilde{\mathcal{C}_o})$ is isomorphic to $\boldsymbol{U_T}$, where the multiplication on $\boldsymbol{U_T}$ is lifted to the monoidal functor $\mathcal{M}$. A $\mathbb{Z}[T^{\pm 1}]$–basis of $K_0(\widetilde{\mathcal{C}_o})$ is given by classes of dividing sets in $\mathcal{B} = \{I, E, F, EF\}$, where EF is the stacking of E and F under $\mathcal{M}$. The generator of $H_1(S_o)$ in the grading $\mathrm{gr}(S_o)$ corresponds to the central element $T \in \boldsymbol{U_T}$. The commutator relation (1) is lifted to two distinguished triangles in $\widetilde{\mathcal{C}_o}$: $I \to EF \to K^{-1}$ and $K^{-1} \to I \to FE$ as in Figure 3, where the homotopy gradings are ignored; see Section 2.3 for a computation of $\mathcal{C}_o$.

To categorify $\boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$ in the comultiplication, consider $\widetilde{\mathcal{C}_{oo}}$ as the universal cover of $\mathcal{C}(S_{oo}, F_{oo})$, where S_{oo} is a twice-punctured disk and F_{oo} consists of two points on

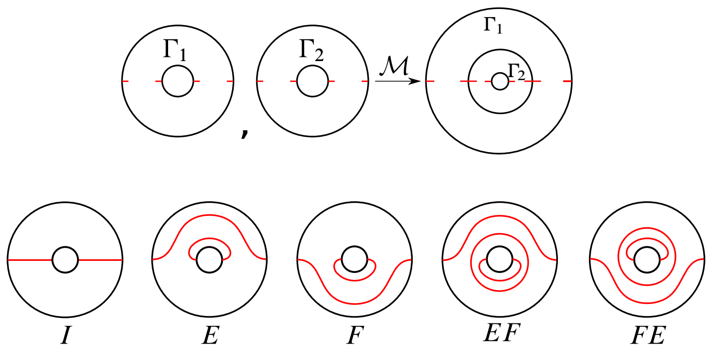

Figure 2: The upper picture describes the monoidal functor $\mathcal{M}$ on objects; the lower one consists of the distinguished basis of $K_0(\widetilde{\mathcal{C}}_o)$ and the dividing set FE.

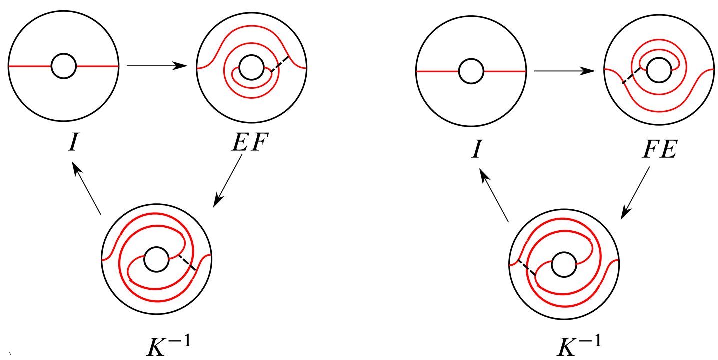

Figure 3: Two distinguished triangles lift the commutator relation (1).

each boundary component. A distinguished basis of the Grothendieck group $K_0(\widetilde{\mathcal{C}}_{oo})$ is given by classes of dividing sets in $\{\Gamma_1 \otimes \Gamma_2 \mid \Gamma_1, \Gamma_2 \in \mathcal{B}\}$ as in Figure 4. There are two generators $t_1, t_2 \in H_1(S_{oo})$ in the grading $\mathrm{gr}(S_{oo})$ given by the two loops. They correspond to the central elements $T \otimes I, I \otimes T \in \boldsymbol{U}_T \otimes_{\mathbb{Z}} \boldsymbol{U}_T$. Hence $K_0(\widetilde{\mathcal{C}}_{oo})$ is isomorphic to $\boldsymbol{U}_T \otimes_{\mathbb{Z}} \boldsymbol{U}_T$. To categorify the comultiplication $\Delta\colon \boldsymbol{U}_T \to \boldsymbol{U}_T \otimes_{\mathbb{Z}} \boldsymbol{U}_T$, define a functor $\delta\colon \widetilde{\mathcal{C}}_o \to \widetilde{\mathcal{C}}_{oo}$ on objects by stacking dividing sets $\Gamma \in \widetilde{\mathcal{C}}_o$ with the specific dividing set $I \otimes I \in \widetilde{\mathcal{C}}_{oo}$ along the outmost boundary of S_{oo}, on morphisms by gluing contact structures in $S_o \times [0,1]$ with the I*–invariant* contact structure of $I \otimes I$ in $S_{oo} \times [0,1]$. An I–invariant contact structure is the trivial contact structure which contains no nontrivial bypass attachments. Then the decategorification $K_0(\delta)\colon \boldsymbol{U}_T \to \boldsymbol{U}_T \otimes_{\mathbb{Z}} \boldsymbol{U}_T$

agrees with the comultiplication Δ. For instance, $\Delta(E) = E \otimes I + I \otimes E$ is lifted to a distinguished triangle: $I \otimes E \to \delta(E) \to E \otimes I$.

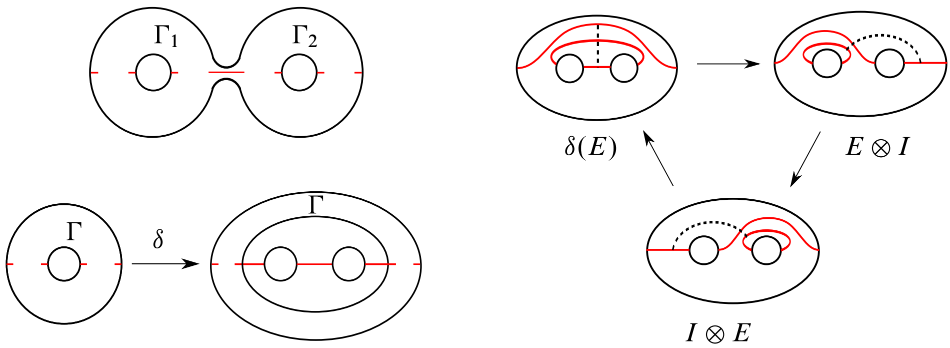

Figure 4: The upper picture on the left describes the basis $\{\Gamma_1 \otimes \Gamma_2\}$ of $K_0(\widetilde{\mathcal{C}}_{oo})$; the lower picture on the left gives the comultiplication δ; the picture on the right is the triangle $I \otimes E \to \delta(E) \to E \otimes I$ in $\widetilde{\mathcal{C}}_{oo}$.

To categorify $V_1^{\otimes n}$, consider $\widetilde{\mathcal{C}}_n$ as the universal cover of $\mathcal{C}(\Sigma_n, F_n)$, where Σ_n is an n times punctured disk and F_n contains two marked points on the outermost boundary and no points on the other boundary components.[1] The Grothendieck group $K_0(\widetilde{\mathcal{C}}_n)$ is a free module over $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$, where t_i is the generator in $H_1(\Sigma_n)$ corresponding to the i^{th} loop. A quotient of $K_0(\widetilde{\mathcal{C}}_n)$ by the relations $t_1 = t_2 = \cdots = t_n = t$ is isomorphic to the $\boldsymbol{U_T}$–module $V_1^{\otimes n}$. A distinguished collection of dividing sets in $\widetilde{\mathcal{C}}_n$ is obtained by lifting the basis $\mathcal{B}'_n$ of $V_1^{\otimes n}$; see Figure 5 for $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$. Note that $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_o$ have the same underlying surface but with different boundary conditions. The $\boldsymbol{U_T}$ action on $V_1^{\otimes n}$ is lifted to a functor $\mathcal{M}_n : \widetilde{\mathcal{C}}_o \times \widetilde{\mathcal{C}}_n \to \widetilde{\mathcal{C}}_n$ given by stacking dividing sets on the annulus S_o with those on Σ_n along the outermost boundary of Σ_n.

We give some morphism sets in $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ as follows. There is a unique tight contact structure $e_\varnothing \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_1}(|0\rangle, |0\rangle)$ which is I–invariant. In particular, $e_\varnothing$ is an idempotent under the composition: $e_\varnothing \cdot e_\varnothing = e_\varnothing$. There are exactly two tight contact structures $e_1, \rho_1 \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_1}(|1\rangle, |1\rangle)$, where e_1 is I–invariant and ρ_1 is nilpotent, ie the composition $\rho_1 \cdot \rho_1$ is not tight:

$$\mathrm{Hom}_{\widetilde{\mathcal{C}}_1}(|0\rangle, |0\rangle) = \langle e_\varnothing \rangle, \quad \mathrm{Hom}_{\widetilde{\mathcal{C}}_1}(|1\rangle, |1\rangle) = \langle e_1, \rho_1 \mid \rho_1^2 = 0 \rangle$$

[1] Since there is no marked point on interior boundary components $\partial \Sigma'_n$, the boundary restriction of contact structures in $\mathrm{Hom}_{\widetilde{\mathcal{C}}_n}(\Gamma_1, \Gamma_2)$ is a collection of dividing sets $(\partial\Sigma'_n \times \{1/2\} \cup \Gamma_1 \times \{0\} \cup \Gamma_2 \times \{1\}) \subset \partial(\Sigma \times [0,1])$.

The nonzero morphism sets in $\widetilde{\mathcal{C}}_2$ are

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|00\rangle, |00\rangle) &= \langle e_\varnothing \rangle, \\
\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|01\rangle, |01\rangle) &= \langle e_2, \rho_2 \mid \rho_2^2 = 0 \rangle, \\
\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|10\rangle, |10\rangle) &= \langle e_1, \rho_1 \mid \rho_1^2 = 0 \rangle, \\
\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|01\rangle, |10\rangle) &= \langle r, \rho_2 \cdot r, r \cdot \rho_1, \rho_2 \cdot r \cdot \rho_1 \rangle, \\
\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|11\rangle, |11\rangle) &= \langle e_{1,2}, \rho_1, \rho_2, \rho_1 \cdot \rho_2 \mid \rho_1^2 = \rho_2^2 = 0,\ \rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1 \rangle.
\end{aligned}$$

There is one nilpotent endomorphism of $\boldsymbol{x} \in \mathcal{B}_2'$ associated to each factor $|1\rangle$ in $\boldsymbol{x}$. The two nilpotent endomorphisms of $|11\rangle$ commute. There is one tight contact structure $r \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|01\rangle, |10\rangle)$ given by a single bypass attachment as in Figure 5. Note that $\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|10\rangle, |01\rangle) = 0$ which is related to the nondecreasing restriction on diagrams in strands algebras; see Definition 5.2 for more detail.

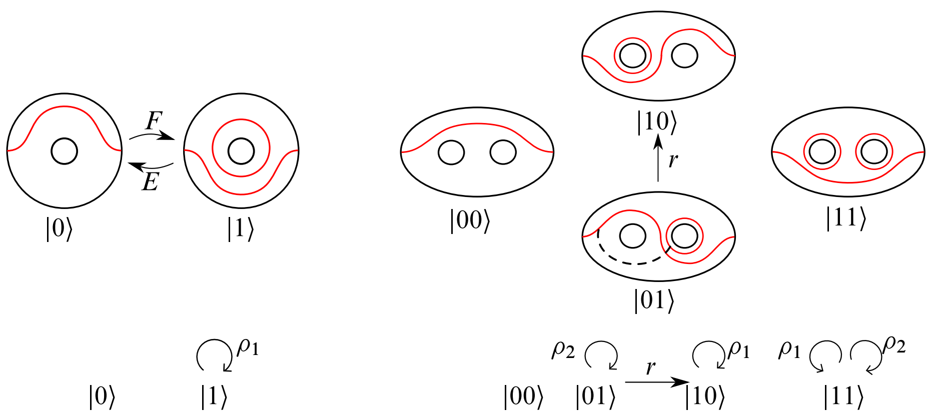

Figure 5: The upper picture on the left describes actions of E and F exchanging $|0\rangle$ and $|1\rangle$ in $\widetilde{\mathcal{C}}_1$. The upper picture on the right gives the distinguished collections of objects in $\widetilde{\mathcal{C}}_2$, where the arrow r is a bypass attachment along the dashed line in $\mathrm{Hom}_{\widetilde{\mathcal{C}}_2}(|01\rangle, |10\rangle)$. The quivers in the lower picture give morphisms in $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$.

1.4 Algebraic formulations

At this point we pass to algebra.[2] An algebraic formulation of $\widetilde{\mathcal{C}}_n$ is given as follows. We construct a quiver Γ_n whose vertex set $V(\Gamma_n)$ is the basis $\mathcal{B}_n'$ of $V_1^{\otimes n}$. The arrow set $A(\Gamma_n)$ is given by morphisms between the objects of $\widetilde{\mathcal{C}}_n$ in the distinguished

[2] In fact, the rest of this paper is just algebra motivated by the contact category.

collection which lifts the basis $\mathcal{B}'_n$. Consider the path algebra $\mathbb{F}_2\Gamma_n$ of the quiver Γ_n with an additional t–grading. Finally, we define a t–graded DG algebra R_n by adding a nontrivial differential on a quotient of $\mathbb{F}_2\Gamma_n$. We prove that R_n is formal, ie it is quasi-isomorphic to its cohomology $H(R_n)$. Similarly, we define t–graded DG algebras A and B from the contact categories $\widetilde{\mathcal{C}}_o$ and $\widetilde{\mathcal{C}}_{oo}$.

The DG algebra R_n is closely related to the strands algebra for an n times punctured disk. In general, the strands algebra of any surface with boundary was defined by Zarev [42]. Motivated by the *rook monoid* (see Solomon [35]) and its diagrammatic presentation *rook diagrams* (see Flath, Halverson and Herbig [6]), we describe R_n in terms of *decorated rook diagrams*. The rook diagram is used to study the Alexander and Jones polynomials by Bigelow, Ramos and Yi [3], and tensor representations of $\mathfrak{gl}(1|1)$ by Benkart and Moon [1].

Consider the DG category $\mathrm{DG}(R_n)$ of t–graded DG R_n–modules and its full subcategory $\mathrm{DGP}(R_n)$ generated by some distinguished projective DG R_n–modules. We model $\widetilde{\mathcal{C}}_n$ by the 0^{th} homology category of $\mathrm{DGP}(R_n)$ which is denoted by $\mathrm{HP}(R_n)$. Let $\mathrm{HP}(H(R_n))$ denote the 0^{th} homology category of $\mathrm{DGP}(H(R_n))$. Then $\mathrm{HP}(R_n)$ and $\mathrm{HP}(H(R_n))$ are equivalent as triangulated categories since R_n is formal. Their Grothendieck groups are isomorphic to free $\mathbb{Z}[t^{\pm 1}]$–modules over $\mathcal{B}'_n$: $K_0(\mathrm{HP}(R_n)) \cong K_0(\mathrm{HP}(H(R_n))) \cong \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}'_n \rangle \cong V_1^{\otimes n}$. Similarly, $\widetilde{\mathcal{C}}_o$ and $\widetilde{\mathcal{C}}_{oo}$ are modeled by triangulated categories $\mathrm{HP}(A)$ and $\mathrm{HP}(B)$ whose Grothendieck groups are isomorphic to $\boldsymbol{U_T}$ and $\boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$, respectively.

In order to categorify the $\boldsymbol{U_T}$–action on $V_1^{\otimes n}$, we define a DG algebra $A \boxtimes R_n$ by adding a differential to $A \otimes R_n$. Consider the 0^{th} homology category $\mathrm{HP}(A \boxtimes R_n)$ of $\mathrm{DGP}(A \boxtimes R_n)$ whose Grothendieck group is isomorphic to a quotient $\boldsymbol{U_T} \otimes_{\{T=t^n\}} V_1^{\otimes n}$ of $\boldsymbol{U_T} \times V_1^{\otimes n}$ by the relation $(T, v) = (I, t^n v)$. Motivated from stacking of dividing sets in the contact categories, we define a DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n which is the key construction in our categorification. A functor defined by tensoring with C_n over $A \boxtimes R_n$,

$$\mathrm{DGP}(A \boxtimes R_n) \xrightarrow{C_n \otimes -} \mathrm{DGP}(H(R_n)),$$

induces an exact functor η_n between their 0^{th} homology categories. The decategorification $K_0(\eta_n)$ on the Grothendieck groups agrees with the $\boldsymbol{U_T}$–action on $V_1^{\otimes n}$: $\boldsymbol{U_T} \otimes_{\{T=t^n\}} V_1^{\otimes n} \to V_1^{\otimes n}$. Similarly, we construct functors

$$\eta\colon \mathrm{HP}(A \otimes A) \xrightarrow{N \otimes -} \mathrm{HP}(A)$$

by tensoring with a DG $(A, A \otimes A)$–bimodule N, and

$$\delta\colon \mathrm{HP}(A) \xrightarrow{S \otimes -} \mathrm{HP}(B)$$

by tensoring with a DG (B, A)–bimodule S. We show that η and δ categorify the multiplication and comultiplication on $\boldsymbol{U_T}$, respectively:

$$\begin{array}{lccc} \eta_n: & \mathrm{HP}(A \boxtimes R_n) & \xrightarrow{C_n \otimes -} & \mathrm{HP}(H(R_n)) \\ & \downarrow K_0 & & \downarrow K_0 \\ K_0(\eta_n): & \boldsymbol{U_T} \otimes_{\{T=t^n\}} V_1^{\otimes n} & \longrightarrow & V_1^{\otimes n} \end{array}$$

The organization of the paper Since many algebraic constructions are quite technical, we will try to give the motivation from contact topology in remarks after the algebraic definitions.

- In Section 2 we define the Hopf superalgebra $\boldsymbol{U_T}$ and categorify its multiplication via the DG $(A, A \otimes A)$–bimodule N.
- In Section 3 we categorify the comultiplication on $\boldsymbol{U_T}$ via the DG (B, A)–bimodule S.
- In Section 4 we define the representations V_1 and $V_1^{\otimes n}$ of $\boldsymbol{U_T}$.
- In Section 5 we define the *decorated rook diagrams* and the t–graded DG algebras R_n, $A \boxtimes R_n$ and show that they are formal as DG algebras.
- In Section 6 we define the DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n.
- In Section 7 we give a categorification of the $\boldsymbol{U_T}$ action on $V_1^{\otimes n}$.

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2 $\boldsymbol{U_T}(\mathfrak{sl}(1|1))$ and the categorification of its multiplication

In Section 2.1 we define the Hopf superalgebra $\boldsymbol{U_T}$. In Section 2.2 we define the t–graded DG algebra A and the triangulated category $\mathrm{HP}(A)$ whose Grothendieck group is isomorphic to $\boldsymbol{U_T}$. In Section 2.3 we define the triangulated category $\mathrm{HP}(A \otimes A)$ whose Grothendieck group is isomorphic to $\boldsymbol{U_T} \otimes_{\mathbb{Z}[T^{\pm 1}]} \boldsymbol{U_T}$. In Section 2.4 we construct the t–graded DG $(A, A \otimes A)$–bimodule N. In Section 2.5 we categorify the multiplication $\boldsymbol{U_T} \otimes_{\mathbb{Z}[T^{\pm 1}]} \boldsymbol{U_T} \to \boldsymbol{U_T}$ to the exact functor

$$\eta\colon \mathrm{HP}(A \otimes A) \xrightarrow{N \otimes_{A \otimes A} -} \mathrm{HP}(A).$$

2.1 The Hopf superalgebra U_T

Definition 2.1 Define the Hopf superalgebra $\{U_T, m, \mathrm{p}, \Delta, \epsilon, S\}$ over $\mathbb{Z}$ as follows.

(1) The multiplication m makes U_T into an associative $\mathbb{Z}$–algebra with unit I, generators E, F, T, T^{-1} and relations

$$E^2 = F^2 = 0, \quad EF + FE = I - T,$$
$$ET = TE, \quad FT = TF, \quad TT^{-1} = T^{-1}T = I.$$

(2) The parity p is a $\mathbb{Z}$–grading[3] defined by $\mathrm{p}(E) = -1, \mathrm{p}(F) = 1, \mathrm{p}(I) = \mathrm{p}(T) = 0$.

(3) The comultiplication $\Delta\colon U_T \to U_T \otimes_{\mathbb{Z}} U_T$ is an algebra map defined on the generators by

$$\Delta(E) = E \otimes I + I \otimes E, \quad \Delta(F) = F \otimes T + I \otimes F, \quad \Delta(T) = T \otimes T.$$

(4) The counit $\epsilon\colon U_T \to \mathbb{Z}$ is an algebra map defined by $\epsilon(E) = \epsilon(F) = 0, \epsilon(I) = \epsilon(T) = 1$.

(5) The antipode $S\colon U_T \to U_T$ is an antihomomorphism of superalgebras, ie $S(ab) = (-1)^{\mathrm{p}(a)\mathrm{p}(b)} S(b)S(a)$, defined by $S(T) = T^{-1}$, $S(E) = -E$ and $S(F) = -FT^{-1}$.

Remark 2.2 (1) Since T is a central element, U_T can be viewed as a free $\mathbb{Z}[T^{\pm}]$–module over the basis $\mathcal{B} = \{F; I, EF; E\}$.

(2) The parity p actually comes from the Euler number of a dividing set. Recall a dividing set divides the surface into positive and negative regions. Then the *Euler number* is the Euler characteristic of the positive region minus the Euler characteristic of the negative region.

(3) The multiplication on $U_T \otimes_{\mathbb{Z}} U_T$ is graded: $(a \otimes b) \cdot (c \otimes d) = (-1)^{\mathrm{p}(b)\mathrm{p}(c)} ac \otimes bd$.

(4) The counit corresponds to a functor $\widetilde{\mathcal{C}}_o \to \widetilde{\mathcal{C}}(S^2)$ between the contact categories of an annulus S_o and a sphere S^2 which is given by capping each component of ∂S_o off with a disk.

(5) The antipode corresponds to a functor $\widetilde{\mathcal{C}}_o \to \widetilde{\mathcal{C}}_o$ given by an inversion about the core of the annulus.

Lemma 2.3 *The definition above gives a Hopf superalgebra $\{U_T, m, \mathrm{p}, \Delta, \epsilon, S\}$:*

(1) *Δ is an algebra map*

(2) *Δ is coassociative: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*

(3) *S is an antipode: $m \circ (S \otimes \mathrm{id}) \circ \Delta(a) = m \circ (\mathrm{id} \otimes S) \circ \Delta(a) = \epsilon(a) I$ for all $a \in U_T$*

[3] In fact, this is a "categorical parity" rather than the usual parity taking values in $\mathbb{F}_2$.

Proof We verify (1) and leave (2) and (3) to the reader:

$$\begin{aligned}\Delta(E)\Delta(E) &= (E\otimes I + I\otimes E)(E\otimes I + I\otimes E)\\ &= E^2\otimes I + I\otimes E^2 + (E\otimes I)(I\otimes E) + (I\otimes E)(E\otimes I)\\ &= E\otimes E - E\otimes E = 0\end{aligned}$$

Similarly, $\Delta(F)\Delta(F)=0$. Finally,

$$\begin{aligned}&\Delta(E)\Delta(F)+\Delta(F)\Delta(E)\\ &\quad = (E\otimes I + I\otimes E)(F\otimes T + I\otimes F) + (F\otimes T + I\otimes F)(E\otimes I + I\otimes E)\\ &\quad = (EF\otimes T - F\otimes ET + E\otimes F + I\otimes EF) + (FE\otimes T + F\otimes TE - E\otimes F + I\otimes FE)\\ &\quad = (EF+FE)\otimes T + I\otimes(EF+FE)\\ &\quad = I\otimes I - T\otimes T = \Delta(I-T) = \Delta(EF+FE).\end{aligned}$$

This completes the proof. □

2.2 The t–graded DG algebra A

We refer to Bernstein and Lunts [2, Section 10] for an introduction to DG algebras, DG modules and *projective* DG modules, and to Keller [16] for an introduction to DG categories and their homology categories. A t–graded DG algebra R is a DG algebra with an additional t–grading. Let $\mathrm{DG}(R)$ denote the DG category of t–graded DG left R–modules. We refer to [37] for more detail.

Definition 2.4 Let A be a t–graded DG $\mathbb{F}_2$–algebra with idempotents $e(\Gamma)$ for $\Gamma\in\mathcal{B}=\{F; I, EF; E\}$, generators $\rho(I,EF)$, $\rho(EF,I)$ and relations

$$\begin{gathered} e(\Gamma)\cdot e(\Gamma') = \delta_{\Gamma,\Gamma'}e(\Gamma) \text{ for } \Gamma,\Gamma'\in\mathcal{B},\\ e(I)\cdot\rho(I,EF) = \rho(I,EF)\cdot e(EF) = \rho(I,EF),\\ e(EF)\cdot\rho(EF,I) = \rho(EF,I)\cdot e(I) = \rho(EF,I),\\ \rho(I,EF)\cdot\rho(EF,I) = 0.\end{gathered}$$

The differential on A is trivial. The grading $\deg = (\deg_h, \deg_t)$ is defined by

$$\deg(a) = \begin{cases}(1,1) & \text{if } a=\rho(EF,I),\\ (0,0) & \text{otherwise,}\end{cases}$$

where $\deg_h$ is the cohomological grading and $\deg_t$ is the t–grading.

Remark 2.5 (1) The algebra $A \cong \bigoplus_{\Gamma_1,\Gamma_2\in\mathcal{B}} \mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(\Gamma_1,\Gamma_2)$ describes *all* tight contact structures between the dividing sets in $\mathcal{B}$. Each of $e(\Gamma)$ is the I–invariant contact structure associated to $\Gamma \in \mathcal{B}$. Each of $\rho(I, EF) \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(I, EF)$ and $\rho(EF, I) \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(EF, I)$ is given by a single nontrivial bypass attachment; see Proposition 2.10 for more detail.

(2) The stacking $\rho(I, EF)\cdot\rho(EF, I) = 0 \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(I, I)$ is a nontight contact structure. On the other hand, the stacking $\rho(EF, I)\cdot\rho(I, EF) \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(EF, EF)$ is tight and nonzero. This nonzero product compared to $\rho(I, EF)\cdot\rho(EF, I) = 0$ reflects the differences between I and EF as dividing sets, where I is the identity in categorical actions but EF is not the identity.

(3) The algebra A is a quotient of the path algebra $\mathbb{F}_2 Q_A$ of a quiver Q_A:

$$F, \quad I \rightleftarrows EF, \quad E$$

There is a decomposition $A = A_1 \oplus A_0 \oplus A_{-1}$, where we have $A_1 = e(F)Ae(F)$, $A_{-1} = e(E)Ae(E)$ and $A_0 = e(I)Ae(I) \oplus e(EF)Ae(EF)$.

Definition 2.6 Define a *parity* $\mathrm{p}\colon A \to \mathbb{Z}$ on A by $\mathrm{p}(a) = i$ for $a \in A_i$. Note that p is not a grading with respect to the multiplication on A.

Consider a collection of projective DG A–modules $\{P(\Gamma) = A\cdot e(\Gamma) \mid \Gamma \in \mathcal{B}\}$. As a left A–module, $P(\Gamma)$ is generated by the idempotent $e(\Gamma)$. To distinguish from $e(\Gamma) \in A$, let $m(\Gamma)$ denote the generator of $P(\Gamma)$. The grading on $P(\Gamma)$ is inherited from A, ie $\deg(m(\Gamma)) = (0,0)$.

Definition 2.7 We define a *parity* $\mathrm{p}\colon \bigsqcup_{\Gamma\in\mathcal{B}} P(\Gamma) \to \mathbb{Z}$ by $\mathrm{p}(m) = \mathrm{p}(\Gamma)$ for all $m \neq 0 \in P(\Gamma)$ and $\Gamma \in \mathcal{B} \subset \boldsymbol{U_T}$, where $\mathrm{p}(\Gamma)$ is the parity of Γ in $\boldsymbol{U_T}$ from Definition 2.1.

For $a \in A$ and $m \in P(\Gamma)$, $\mathrm{p}(a\cdot m) = \mathrm{p}(m) = \mathrm{p}(a)$ if m and $a\cdot m$ are nonzero. In particular, p on $P(\Gamma)$ is not a grading with respect to the A action on $P(\Gamma)$. The two parities in Definitions 2.6 and 2.7 will be used to define twisted gradings on $A\otimes A$ in Definition 2.11 and on $P(\Gamma_1)\otimes P(\Gamma_2)$ in Definition 2.15.

Definition 2.8 Let $\mathrm{DGP}(A)$ be the smallest full subcategory of $\mathrm{DG}(A)$ which contains the projective DG A–modules $\{P(\Gamma) = A\cdot e(\Gamma) \mid \Gamma \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor [1], the t–grading shift functor $\{1\}$ and taking mapping cones.

The 0^{th} homology category $\mathrm{HP}(A)$ of $\mathrm{DGP}(A)$ is the homotopy category of t–graded DG projective A–modules generated by $\{P(\Gamma) \mid \Gamma \in \mathcal{B}\}$. It is a triangulated category and the Grothendieck group $K_0(\mathrm{HP}(A))$ has a $\mathbb{Z}[T^{\pm 1}]$–basis $\{[P(\Gamma)] \mid \Gamma \in \mathcal{B}\}$, where the multiplication by T is induced by the t–grading shift: $[M\{1\}] = T[M] \in K_0(\mathrm{HP}(A))$ for $M \in \mathrm{HP}(A)$.

Lemma 2.9 *There is an isomorphism $K_0(\mathrm{HP}(A)) \cong \boldsymbol{U}_T$ of free $\mathbb{Z}[T^{\pm 1}]$–modules.*

2.3 The connection to contact topology

This subsection is a digression to contact topology. We give a computation of the contact category $\mathcal{C}_o := \mathcal{C}(S_o, F_o)$, where S_o is an annulus and F_o consists of two points on each component of ∂S_o. In particular, we relate the numbers of tight contact structures on $S_o \times [0,1]$ to the dimensions of the corresponding subspaces of the algebra A.

Recall that an object Γ of $\mathcal{C}_o$ is a dividing set satisfying $\Gamma \cap \partial S_o = F_o$. The collection $\mathcal{B} = \{F; I, EF; E\}$ of distinguished dividing sets of $\mathcal{C}_o$ is given in Figure 2. Let $\overline{F}_o$ be a subset of ∂S_o consisting of two points on each component of ∂S_o such that the points of F_o and $\overline{F}_o$ are alternating along ∂S_o; see Figure 6. Given two dividing sets Γ_1, Γ_2, let $\mathrm{ER}(\Gamma_1, \Gamma_2)$ denote a dividing set on $\partial(S_o \times [0,1])$ which is given by connecting a union $(\Gamma_1 \times \{0\}) \cup (\Gamma_2 \times \{1\}) \cup (\overline{F}_o \times [0,1])$ along the corner $\partial S_o \times \{0,1\}$ via the *Edge-rounding Lemma*. We refer to [9, Lemma 3.11] for more detail about edge-rounding. An example of $\mathrm{ER}(I, I)$ is given in Figure 6.

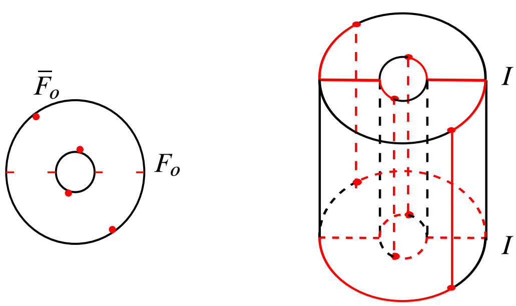


Figure 6: The alternating points of F_o and $\overline{F}_o$ are depicted on the left, where F_o consists of four red bars and $\overline{F}_o$ consists of four red dots; $\mathrm{ER}(I, I)$ consisting of two components is given on the right.

For objects Γ_1, Γ_2 of $\mathcal{C}_o$, let $\mathrm{Tight}(\Gamma_1, \Gamma_2) := \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathcal{C}_o}(\Gamma_1, \Gamma_2)$ denote the number of tight contact structures on $S_o \times [0,1]$ with the dividing set $\mathrm{ER}(\Gamma_1, \Gamma_2)$ on $\partial(S_o \times [0,1])$. Then we have the following connection between the algebra A and the contact category $\mathcal{C}_o$.

Proposition 2.10 *There exists a collection of dividing sets $\mathcal{B} = \{F; I, EF; E\}$ as the objects of $\mathcal{C}_o$ such that* $\mathrm{Tight}(\Gamma_1, \Gamma_2) = \dim_{\mathbb{F}_2}(e(\Gamma_1) \cdot A \cdot e(\Gamma_2))$ *for $\Gamma_1, \Gamma_2 \in \mathcal{B}$.*

Proof We compute $\mathrm{Tight}(\Gamma_1, \Gamma_2)$ depending on parities of Γ_1 and Γ_2.

Case I We have $\mathrm{Tight}(\Gamma_1, \Gamma_2) = 0$ if $\mathrm{p}(\Gamma_1) \neq \mathrm{p}(\Gamma_2)$. It is easy to see that $\mathrm{ER}(\Gamma_1, \Gamma_2)$ contains at least one contractible component if $\mathrm{p}(\Gamma_1) \neq \mathrm{p}(\Gamma_2)$. An example of $\mathrm{ER}(I, E)$ is given in Figure 7. Then $\mathrm{Tight}(\Gamma_1, \Gamma_2) = 0$ by a criterion of Giroux [7].

Case II We have $\mathrm{Tight}(\Gamma_1, \Gamma_2) = 1$ for

$$(\Gamma_1, \Gamma_2) \in \mathcal{D} = \{(E, E), (F, F), (I, EF), (EF, I), (I, I)\}.$$

It is easy to see that $\mathrm{ER}(\Gamma_1, \Gamma_2)$ consists of two components of longitude for $(\Gamma_1, \Gamma_2) \in \mathcal{D} \backslash (I, I)$ and $\mathrm{ER}(I, I)$ consists of two components with slope equal to -1 as in Figure 6. Examples of $\mathrm{ER}(E, E)$ and $\mathrm{ER}(I, EF)$ are given in Figure 7. Then $\mathrm{Tight}(\Gamma_1, \Gamma_2) = 1$ by a result of Makar and Limanov [24] on the classification of tight contact structures on solid torus.

Case III We have $\mathrm{Tight}(EF, EF) = 2$. It is easy to see that $\mathrm{ER}(EF, EF)$ consists of four components of longitude. Then $\mathrm{Tight}(EF, EF) = 2$ by a result of Honda [10] on gluing tight contact structures.

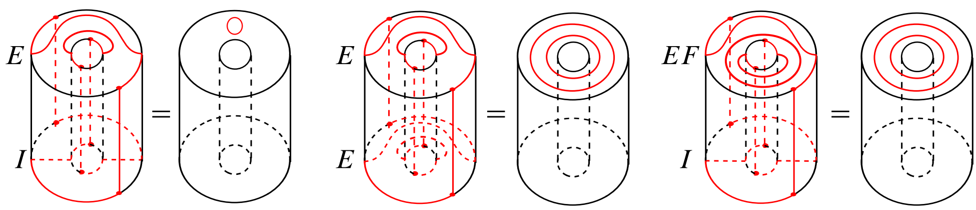


Figure 7: From the left to the right: $\mathrm{ER}(I, E), \mathrm{ER}(E, E)$ and $\mathrm{ER}(I, EF)$

Then it is straightforward to check $\mathrm{Tight}(\Gamma_1, \Gamma_2) = \dim_{\mathbb{F}_2}(e(\Gamma_1) \cdot A \cdot e(\Gamma_2))$ case by case. □

2.4 The t–graded DG algebra $A \otimes A$

Definition 2.11 Let $A \otimes A$ be the tensor product of two A's over $\mathbb{F}_2$ as an algebra. The differential is trivial. The grading $\deg = (\deg_h, \deg_t)$ is defined for generators $a, b \in A$ by

$$\deg_t(a \otimes b) = \deg_t(a) + \deg_t(b),$$
$$\deg_h(a \otimes b) = \deg_h(a) + \deg_h(b) + 2\deg_t(a)\mathrm{p}(b).$$

Remark 2.12 The cohomological grading $\deg_h$ of $A \otimes A$ is the sum of two $\deg_h$'s twisted by the t–grading $\deg_t$ and the parity p. The topological meaning of $\deg_h$ is the framing of links in $S_o \times [0,1]$. In general, $\deg_h(a \otimes b) \neq \deg_h(a) + \deg_h(b)$ when two links are nontrivially linked.

Lemma 2.13 *The grading* deg *is well defined:*

$$\deg(ac \otimes bd) = \deg(a \otimes b) + \deg(c \otimes d)$$

Proof If $ac \otimes bd \neq 0$ for generators $a, b, c, d \in A$, then $\mathrm{p}(b) = \mathrm{p}(d) = \mathrm{p}(bd)$ and $\deg_t(ac) = \deg_t(a) + \deg_t(c)$. By definition

$$\begin{aligned}
&\deg_h(ac \otimes bd)\\
&\qquad = \deg_h(ac) + \deg_h(bd) + 2\deg_t(ac)\mathrm{p}(bd)\\
&\qquad = \deg_h(a) + \deg_h(c) + \deg_h(b) + \deg_h(d) + 2\deg_t(a)\mathrm{p}(b) + 2\deg_t(c)\mathrm{p}(d)\\
&\qquad = \deg_h(a \otimes b) + \deg_h(c \otimes d).
\end{aligned}$$

The equation for the t–component is obvious. □

Definition 2.14 Let $\mathrm{DGP}(A \otimes A)$ be the smallest full subcategory of $\mathrm{DG}(A \otimes A)$ which contains the projective DG $A \otimes A$–modules $\{P(\Gamma, \Gamma') = (A \otimes A)\cdot(e(\Gamma) \otimes e(\Gamma')) \mid \Gamma, \Gamma' \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor $[1]$, the t–grading shift functor $\{1\}$ and taking mapping cones.

The 0^{th} homology category $\mathrm{HP}(A \otimes A)$ of $\mathrm{DGP}(A \otimes A)$ is the homotopy category of t–graded DG projective $A \otimes A$–modules generated by $\{P(\Gamma, \Gamma') \mid \Gamma, \Gamma' \in \mathcal{B}\}$.

Definition 2.15 Define a tensor product functor

$$\begin{aligned}
\chi\colon \mathrm{HP}(A) \times \mathrm{HP}(A) &\to \mathrm{HP}(A \otimes A),\\
M_1, M_2 &\mapsto M_1 \otimes_{\mathbb{F}_2} M_2,
\end{aligned}$$

where the grading on $M_1 \otimes M_2$ is given for $m_1 \in M_1$ and $m_2 \in M_2$ by

$$\begin{aligned}
\deg_t(m_1 \otimes m_2) &= \deg_t(m_1) + \deg_t(m_2),\\
\deg_h(m_1 \otimes m_2) &= \deg_h(m_1) + \deg_h(m_2) + 2\deg_t(m_1)\mathrm{p}(m_2).
\end{aligned}$$

Remark 2.16 The grading on $M_1 \otimes M_2$ is compatible with the grading on $A \otimes A$. We have

$$\chi(P(\Gamma)\{n\}, P(\Gamma')\{n'\}) = P(\Gamma, \Gamma')\{n + n'\}[2n\mathrm{p}(\Gamma')].$$

Note that the twisting $[2n\mathrm{p}(\Gamma')]$ cannot be seen on the level of Grothendieck groups.

Since $K_0(\mathrm{HP}(A \otimes A))$ has a $\mathbb{Z}[T^{\pm 1}]$–basis $\mathcal{B} \times \mathcal{B}$, we have the following:

Lemma 2.17 *There is an isomorphism $K_0(\mathrm{HP}(A \otimes A)) \cong \boldsymbol{U_T} \otimes_{\mathbb{Z}[T^{\pm}]} \boldsymbol{U_T}$ of free $\mathbb{Z}[T^{\pm 1}]$–modules. Additionally, the functor χ induces a tensor product on their Grothendieck groups:*

$$K_0(\chi)\colon \boldsymbol{U_T} \times \boldsymbol{U_T} \xrightarrow{\otimes_{\mathbb{Z}[T^{\pm}]}} \boldsymbol{U_T} \otimes_{\mathbb{Z}[T^{\pm}]} \boldsymbol{U_T}$$

2.5 The t–graded DG $(A, A \otimes A)$–bimodule N

To define a functor $\eta\colon \mathrm{DGP}(A \otimes A) \to \mathrm{DGP}(A)$ lifting the multiplication on $\boldsymbol{U_T}$, we construct a DG $(A, A \otimes A)$–bimodule N in two steps: a left DG A–module N in Section 2.5.1 and a right DG $A \otimes A$–module N in Section 2.5.2.

In practice, the functor η is obtained by "reverse-engineering": we have all the essential information about η from the contact topology and we construct the bimodule N to realize the functor. More precisely, we first figure out the behavior of η on the objects $P(\Gamma_1, \Gamma_2)$ and set

$$N = \bigoplus_{\Gamma_1, \Gamma_2 \in \mathcal{B}} N(\Gamma_1, \Gamma_2) = \bigoplus_{\Gamma_1, \Gamma_2 \in \mathcal{B}} \eta(P(\Gamma_1, \Gamma_2)) \in \mathrm{DGP}(A),$$

as left DG A–modules. We then determine the right $A \otimes A$–module structure on N by considering morphism sets $\mathrm{Hom}(P(\Gamma_1, \Gamma_2), P(\Gamma_1', \Gamma_2'))$ in $\mathrm{DGP}(A \otimes A)$. For instance, the right multiplication by $e(F) \otimes \rho(I, EF)$ in $A \otimes A$ defines a morphism

$$\begin{aligned} f\colon\ & P(F, I) \to P(F, EF), \\ & m \mapsto m \cdot (e(F) \otimes \rho(I, EF)). \end{aligned}$$

Then the right multiplication on N by $e(F) \otimes \rho(I, EF)$ is given by the morphism

$$\eta(f)\colon \eta(P(F, I)) \to \eta(P(F, EF))$$

in $\mathrm{DGP}(A)$, where $\eta(P(F, I))$ and $\eta(P(F, EF))$ are viewed as left A–submodules of N. This technique will be used to construct various bimodules in the paper.

2.5.1 The left DG A–module N

Definition 2.18 Define a left DG A–module

$$N = \bigoplus_{\Gamma_1, \Gamma_2 \in \mathcal{B}} N(\Gamma_1, \Gamma_2),$$

where $(N(\Gamma_1,\Gamma_2), d(\Gamma_1,\Gamma_2)) \in \mathrm{DGP}(A)$ is defined on a case by case basis as

$$N(E,E) = N(E,EF) = N(F,F) = N(EF,F) = 0,$$
$$N(I,\Gamma) = N(\Gamma,I) = P(\Gamma) \text{ for all } \Gamma \in \mathcal{B},$$
$$N(E,F) = P(EF),$$
$$N(F,EF) = P(F) \oplus P(F)\{1\}[1],$$
$$N(EF,E) = P(E) \oplus P(E)\{1\}[-1],$$
$$N(EF,EF) = P(EF) \oplus P(EF)\{1\}[1],$$
$$N(F,E) = P(I) \oplus P(EF)[-1] \oplus P(I)\{1\}[-1].$$

We have $d(\Gamma_1,\Gamma_2) = 0$ for all $(\Gamma_1,\Gamma_2) \neq (F,E)$ and $d(F,E)$ is a map of left A–modules defined on generators of $N(F,E)$ by

$$\begin{aligned} d(F,E)\colon N(F,E) &\to N(F,E), \\ m_{F,E}(I) &\mapsto \rho(I,EF) \cdot m_{F,E}(EF), \\ m_{F,E}(EF) &\mapsto \rho(EF,I) \cdot m'_{F,E}(I), \\ m'_{F,E}(I) &\mapsto 0, \end{aligned}$$

where $m_{F,E}(I) \in P(I), m_{F,E}(EF) \in P(EF)[-1]$ and $m'_{F,E}(I) \in P(I)\{1\}[-1]$.

Remark 2.19 (1) The left DG A–module $N(\Gamma_1,\Gamma_2)$ is supposed to be a categorical multiplication of DG A–modules $P(\Gamma_1)$ and $P(\Gamma_2)$. In particular, the class $[N(\Gamma_1,\Gamma_2)] \in \boldsymbol{U_T}$ is the multiplication $\Gamma_1 \cdot \Gamma_2 \in \boldsymbol{U_T}$ under the isomorphism in Lemma 2.9.

(2) In the contact category $\widetilde{\mathcal{C}}_0$, the stacking $EF \cdot E$ of dividing sets is the union of E and a pair of loops. The pair of loops corresponds to tensoring with $\mathbb{Z}^2$ up to grading. Correspondingly, the left A–module $N(EF,E)$ is a direct sum of two $P(E)$'s.

(3) The definition of $N(F,E)$ is motivated from the two distinguished triangles in $\widetilde{\mathcal{C}}_0$: $I \to EF \to K^{-1}$ and $K^{-1} \to I \to FE$ as in Figure 3, where the gradings are ignored. The isomorphisms $FE \cong (K^{-1} \to I)$ and $K^{-1} \cong (I \to EF)$ give the isomorphism $FE \cong (I \to EF \to I)$ as in the definition of $N(F,E)$.

Lemma 2.20 *We have $(N(F,E), d(F,E))$ is a t–graded DG A–module.*

Proof It suffices to prove that $d = d(F,E)$ is of degree $(1,0)$ such that $d^2 = 0$. We verify that

$$d^2(m_{F,E}(I)) = d(\rho(I,EF) \cdot m_{F,E}(EF)) = \rho(I,EF) \cdot \rho(EF,I) \cdot m'_{F,E}(I) = 0.$$

The degrees of the generators of $N(F,E)$ are

$$\deg(m_{F,E}(I)) = (0,0), \quad \deg(m_{F,E}(EF)) = (1,0), \quad \deg(m'_{F,E}(I)) = (1,-1).$$

Hence the differential d is of degree $(1,0)$, since

$$\begin{aligned}\deg(d(m_{F,E}(I))) &= \deg(\rho(I,EF)) + \deg(m_{F,E}(EF))\\ &= (1,0) = \deg(m_{F,E}(I)) + (1,0),\\ \deg(d(m_{F,E}(EF))) &= \deg(\rho(EF,I)) + \deg(m'_{F,E}(I))\\ &= (2,0) = \deg(m_{F,E}(EF)) + (1,0).\end{aligned}$$

This completes the proof. □

2.5.2 The right $A \otimes A$–module structure on N In this subsection we describe the right $A\otimes A$–module structure on N. Let $m \times (a \otimes b)$ denote the right multiplication for $m \in N, a\otimes b \in A \otimes A$ and let $m \cdot a$ denote the multiplication in A for $m \in P(\Gamma) \subset A, a \in A$.

We fix the notation for the generators of $N(\Gamma_1,\Gamma_2)$ as

$$\begin{gathered} m_{\Gamma,I}(\Gamma) \in P(\Gamma) = N(\Gamma, I) \text{ for all } \Gamma \in \mathcal{B},\\ m_{I,\Gamma}(\Gamma) \in P(\Gamma) = N(I, \Gamma) \text{ for all } \Gamma \in \mathcal{B},\\ m_{E,F}(EF) \in P(EF) = N(E,F),\\ m_{EF,E}(E) \in P(E), m'_{EF,E}(E) \in P(E)\{1\}[-1] \text{ in } N(EF,E),\\ m_{F,EF}(F) \in P(F), m'_{F,EF}(F) \in P(F)\{1\}[1] \text{ in } N(F,EF),\\ m_{EF,EF}(EF) \in P(EF), m'_{EF,EF}(EF) \in P(EF)\{1\}[1] \text{ in } N(EF,EF),\\ m_{F,E}(I) \in P(I), m_{F,E}(EF) \in P(EF)[-1], m'_{F,E}(I) \in P(I)\{1\}[-1] \text{ in } N(F,E).\end{gathered}$$

We define right multiplications by the generators of $A \otimes A$ on a case by case basis. Each right multiplication is a map of left A–modules defined on the generators of N as follows.

(1) For an idempotent $e(\Gamma_1)\otimes e(\Gamma_2)$, define

$$\begin{aligned}\times(e(\Gamma_1)\otimes e(\Gamma_2))\colon N(\Gamma'_1,\Gamma'_2) &\to N(\Gamma'_1,\Gamma'_2),\\ m &\mapsto \delta_{\Gamma_1,\Gamma'_1}\delta_{\Gamma_2,\Gamma'_2} m.\end{aligned}$$

(2) For generators $\rho(EF, I) \otimes e(E)$ and $\rho(I, EF) \otimes e(E)$, define

$$\begin{aligned}
\times(\rho(EF, I) \otimes e(E))\colon N(EF, E) &\to N(I, E),\\
m_{EF,E}(E) &\mapsto 0,\\
m'_{EF,E}(E) &\mapsto m_{I,E}(E),\\
\times(\rho(I, EF) \otimes e(E))\colon N(I, E) &\to N(EF, E),\\
m_{I,E}(E) &\mapsto m_{EF,E}(E).
\end{aligned}$$

(3) For generators $e(F) \otimes \rho(EF, I)$ and $e(F) \otimes \rho(I, EF)$, define

$$\begin{aligned}
\times(e(F) \otimes \rho(EF, I))\colon N(F, EF) &\to N(F, I),\\
m_{F,EF}(F) &\mapsto 0,\\
m'_{F,EF}(F) &\mapsto m_{F,I}(F),\\
\times(e(F) \otimes \rho(I, EF))\colon N(F, I) &\to N(F, EF),\\
m_{F,I}(F) &\mapsto m_{F,EF}(F).
\end{aligned}$$

(4) For generators $e(I) \otimes a$ and $a \otimes e(I)$, where $a \in \{\rho(I, EF), \rho(EF, I)\}$, define

$$\begin{aligned}
\times(\rho(EF, I) \otimes e(I))\colon N(EF, I) &\to N(I, I),\\
m_{EF,I}(EF) &\mapsto \rho(EF, I) \cdot m_{I,I}(I),\\
\times(\rho(I, EF) \otimes e(I))\colon N(I, I) &\to N(EF, I),\\
m_{I,I}(I) &\mapsto \rho(I, EF) \cdot m_{EF,I}(EF),\\
\times(e(I) \otimes \rho(EF, I))\colon N(I, EF) &\to N(I, I),\\
m_{I,EF}(EF) &\mapsto \rho(EF, I) \cdot m_{I,I}(I),\\
\times(e(I) \otimes \rho(I, EF))\colon N(I, I) &\to N(I, EF),\\
m_{I,I}(I) &\mapsto \rho(I, EF) \cdot m_{I,EF}(EF).
\end{aligned}$$

(5) For generators $e(EF) \otimes a$ and $a \otimes e(EF)$, where $a \in \{\rho(I, EF), \rho(EF, I)\}$, define

$$\begin{aligned}
\times(\rho(EF, I) \otimes e(EF))\colon N(EF, EF) &\to N(I, EF),\\
m_{EF,EF}(EF) &\mapsto 0,\\
m'_{EF,EF}(EF) &\mapsto m_{I,EF}(EF),\\
\times(e(EF) \otimes \rho(EF, I))\colon N(EF, EF) &\to N(EF, I),\\
m_{EF,EF}(EF) &\mapsto 0,\\
m'_{EF,EF}(EF) &\mapsto m_{EF,I}(EF),\\
\times(\rho(I, EF) \otimes e(EF))\colon N(I, EF) &\to N(EF, EF),\\
m_{I,EF}(EF) &\mapsto m_{EF,EF}(EF),
\end{aligned}$$

$$\times(e(EF)\otimes\rho(I,EF))\colon N(EF,I)\to N(EF,EF),$$
$$m_{EF,I}(EF)\mapsto m_{EF,EF}(EF).$$

(6) Define the right multiplication to be the zero map for generators $e(E)\otimes\rho(EF,I)$, $e(E)\otimes\rho(I,EF)$, $\rho(EF,I)\otimes e(F)$ and $\rho(I,EF)\otimes e(F)$, since the corresponding domains or ranges are trivial from Definition 2.18.

This concludes the definition of the right multiplications by the generators of $A\otimes A$. In general, define $m\times(r_1\cdot r_2):=(m\times r_1)\times r_2$ for $r_1,r_2\in A\otimes A$ and $m\in N$.

Remark 2.21 (1) The definition is motivated from studying tight contact structures on the gluing of two $\widetilde{\mathcal{C}}_o$'s in Figure 2. For instance, the right multiplication with $\rho(EF,I)\otimes e(E)$ is determined by the corresponding tight contact structure in $\mathrm{Hom}_{\widetilde{\mathcal{C}}_o}(EF\cdot E, I\cdot E)$.

(2) The right multiplication with r on $N(F,E)$ is nonzero only if $r=e(F)\otimes e(E)$. In that case the right multiplication is the identity from Case (1).

Lemma 2.22 *We have that N is a t–graded right $A\otimes A$–module:*

(1) $(m\times r_1)\times r_2=(m\times r_1')\times r_2'$, *for* $r_1\cdot r_2=r_1'\cdot r_2'\in A\otimes A$

(2) $\deg(m\times r)=\deg(m)+\deg(r)$

Proof We verify (1) for $r_1=\rho(I,EF)\otimes e(I)$, $r_2=\rho(EF,I)\otimes e(I)$ and $m=m_{I,I}(I)$ in Case (4). It suffices to show that $(m\times r_1)\times r_2=0$ since $r_1\cdot r_2=0$. We have

$$(m\times r_1)\times r_2=\rho(I,EF)\cdot m_{EF,I}(EF)\times r_2=\rho(I,EF)\cdot\rho(EF,I)\cdot m_{I,I}(I)=0,$$

since $\rho(I,EF)\cdot\rho(EF,I)=0\in A$.

The proofs for other cases are similar and we leave them to the reader.

For (2), the only nontrivial case is $m\times r=m'_{EF,E}(E)\times(\rho(EF,I)\otimes e(E))=m_{I,E}(E)$ in Case (2), where the gradings are given in

$$\begin{aligned}\deg(m'_{EF,E}(E))+\deg(\rho(EF,I)\otimes e(E))&=(1,-1)+(-1,1)\\&=(0,0)=\deg(m_{I,E}(E)).\end{aligned}$$

This completes the proof. □

Since the right multiplications are the maps of left A–modules, $a\cdot(m\times r)=(a\cdot m)\times r$, for $a\in A,r\in A\otimes A$ and $m\in N$. Hence N is a t–graded DG $(A,A\otimes A)$–bimodule.

2.6 The categorification of the multiplication on $\boldsymbol{U_T}$

In this section, we categorify the multiplication on $\boldsymbol{U_T}$, ie prove Theorem 1.1. Let

$$\eta\colon \mathrm{DGP}(A\otimes A) \xrightarrow{N\otimes_{A\otimes A}-} \mathrm{DGP}(A)$$

be a functor given by tensoring with the DG $(A, A\otimes A)$–bimodule N over $A\otimes A$.

Lemma 2.23 *We have that the functor η maps $P(\Gamma_1,\Gamma_2)$ to $N(\Gamma_1,\Gamma_2)\in \mathrm{DGP}(A)$ for all $\Gamma_1,\Gamma_2\in\mathcal{B}$.*

Proof Since $N=\bigoplus_{\Gamma_1',\Gamma_2'\in\mathcal{B}} N(\Gamma_1',\Gamma_2')$ as left DG A–modules, then we have that $N\otimes P(\Gamma_1,\Gamma_2)$ is the quotient of $\bigoplus_{\Gamma_1',\Gamma_2'\in\mathcal{B}}(N(\Gamma_1',\Gamma_2')\times P(\Gamma_1,\Gamma_2))$ by the relations

$$\{(m\times r, e(\Gamma_1)\otimes e(\Gamma_2)) = (m, r\cdot(e(\Gamma_1)\otimes e(\Gamma_2)))\mid m\in N(\Gamma_1',\Gamma_2'), r\in A\otimes A\}.$$

Since $\{(m, r\cdot(e(\Gamma_1)\otimes e(\Gamma_2)))\mid m\in N(\Gamma_1',\Gamma_2'), r\cdot e(\Gamma_1)\otimes e(\Gamma_2)\neq 0\}$ spans $N(\Gamma_1',\Gamma_2')\times P(\Gamma_1,\Gamma_2)$, $N\otimes P(\Gamma_1,\Gamma_2)$ is spanned by

$$\{(m\times r, e(\Gamma_1)\otimes e(\Gamma_2))\mid m\in N(\Gamma_1',\Gamma_2'), r\cdot(e(\Gamma_1)\otimes e(\Gamma_2))\neq 0\} \\ \cong N(\Gamma_1,\Gamma_2)\in \mathrm{DGP}(A).$$

This completes the proof. □

There is an induced exact functor $\eta\colon \mathrm{HP}(A\otimes A)\xrightarrow{N\otimes_{A\otimes A}-}\mathrm{HP}(A)$ between the 0^{th} homology categories. Let $\mathcal{M}=\eta\circ\chi$ be the composition

$$\mathcal{M}\colon \mathrm{HP}(A)\times\mathrm{HP}(A)\xrightarrow{\chi}\mathrm{HP}(A\otimes A)\xrightarrow{\eta}\mathrm{HP}(A).$$

Proof of Theorem 1.1 We compute the multiplication

$$K_0(\mathcal{M})\colon K_0(\mathrm{HP}(A))\times K_0(\mathrm{HP}(A))\to K_0(\mathrm{HP}(A)).$$

(1) By Lemma 2.23, $\mathcal{M}(P(\Gamma),P(\Gamma'))=\eta(P(\Gamma,\Gamma'))=N(\Gamma,\Gamma')$, for $\Gamma,\Gamma'\in\mathcal{B}$. Its class $[N(\Gamma,\Gamma')]$ agrees with $\Gamma\cdot\Gamma'\in\boldsymbol{U_T}$ by Remark 2.19.

(2) The class $[P(I)]$ is a unit of $K_0(\mathrm{HP}(A))$, since $P(I)$ is a unit under $\mathcal{M}$:

$$\mathcal{M}(P(\Gamma),P(I))=\eta(P(\Gamma,I))=N(\Gamma,I)=P(\Gamma)$$
$$\mathcal{M}(P(I),P(\Gamma))=\eta(P(I,\Gamma))=N(I,\Gamma)=P(\Gamma)$$

(3) By Remark 2.16, $\mathcal{M}(P(\Gamma), P(I)\{1\}) = \eta(P(\Gamma, I)\{1\}) = P(\Gamma)\{1\}$,

$$\mathcal{M}(P(I)\{1\}, P(\Gamma)) = \eta(P(I,\Gamma)\{1\}[2\mathrm{p}(\Gamma)]) = P(\Gamma)\{1\}[2\mathrm{p}(\Gamma)].$$

Although $\mathcal{M}(P(\Gamma), P(I)\{1\})$ and $\mathcal{M}(P(I)\{1\}, P(\Gamma))$ differ by $2\mathrm{p}(\Gamma)$ in their cohomological gradings, their classes agree in $K_0(\mathrm{HP}(A))$. Hence $[P(I)\{1\}]$ corresponds to the variable T in the $\mathbb{Z}[T^{\pm 1}]$–algebra $K_0(\mathrm{HP}(A))$.

Parts (1), (2) and (3) together imply that the following map is an isomorphism of $\mathbb{Z}[T^{\pm 1}]$–algebras:

$$\begin{aligned} \boldsymbol{U_T} &\to K_0(\mathrm{HP}(A)) \\ \Gamma &\mapsto [P(\Gamma)] \\ T &\mapsto [P(I)\{1\}] \end{aligned}$$

This completes the proof. □

Remark 2.24 It is natural to ask whether $\mathcal{M}$ is a monoidal functor, ie the following diagram commutes up to equivalence:

$$\begin{tikzcd} \mathrm{HP}(A) \times \mathrm{HP}(A) \times \mathrm{HP}(A) \arrow[r, "\mathrm{id} \times \mathcal{M}"] \arrow[d, "\mathcal{M}\times\mathrm{id}"] & \mathrm{HP}(A) \times \mathrm{HP}(A) \arrow[d, "\mathcal{M}"] \\ \mathrm{HP}(A) \times \mathrm{HP}(A) \arrow[r, "\mathcal{M}"] & \mathrm{HP}(A) \end{tikzcd}$$

We believe that the answer is positive and it could be done by verifying some associativity relation on various DG bimodules.

3 The categorification of the comultiplication on $\boldsymbol{U_T}(\mathfrak{sl}(1|1))$

To categorify the comultiplication $\Delta\colon \boldsymbol{U_T} \to \boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$, we define the (t_1, t_2)–graded DG algebra B and the triangulated category $\mathrm{HP}(B)$ whose Grothendieck group is isomorphic to $\boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$. Then we construct the (t_1, t_2)–graded DG (B, A)–bimodule S to give an exact functor

$$\delta\colon \mathrm{HP}(A) \xrightarrow{S \otimes_A -} \mathrm{HP}(B).$$

The decategorification $K_0(\delta)$ agrees with the comultiplication on $\boldsymbol{U_T}$.

3.1 The (t_1, t_2)–graded DG algebra B

We define the algebra B via a quiver Q_B.

Definition 3.1 (Quiver $Q_B = (V(Q_B), A(Q_B))$) Let $V(Q_B) = \mathcal{B} \times \mathcal{B}$ be the set of vertices. Let $\Gamma_1 \otimes \Gamma_2$ denote a vertex of Q_B for $\Gamma_1, \Gamma_2 \in \mathcal{B}$. Let $A(Q_B)$ be the set of arrows given as follows:

$$\begin{tikzcd}
E \otimes EF \arrow[d, shift left] & EF \otimes E \arrow[d, shift left] & EF \otimes F \arrow[d, shift left] & F \otimes EF \arrow[d, shift left] \\
E \otimes I \arrow[u, shift left] \arrow[r] & I \otimes E \arrow[u, shift left] & I \otimes F \arrow[u, shift left] \arrow[r] & F \otimes I \arrow[u, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
E \otimes F \arrow[r] & I \otimes EF \arrow[r, shift left] \arrow[d, shift left] & I \otimes I \arrow[l, shift left] \arrow[d, shift left] \\
& EF \otimes EF \arrow[u, shift left] \arrow[r, shift left] & EF \otimes I \arrow[l, shift left] \arrow[u, shift left] \arrow[d] \\
& & F \otimes E
\end{tikzcd}$$

Remark 3.2 (1) The quiver has 5 components, $Q_B = \bigsqcup_{i=-2}^{2} Q_{B,i}$, where a vertex $\Gamma_1 \otimes \Gamma_2$ is in $Q_{B,i}$ if $\mathrm{p}(\Gamma_1) + \mathrm{p}(\Gamma_2) = i$. The diagrams on the left are the components $Q_{B,-1}$ and $Q_{B,1}$. The diagram on the right is the component $Q_{B,0}$. There are no arrows in $Q_{B,-2}$ and $Q_{B,2}$.

(2) The arrows give all tight contact structures between the corresponding dividing sets in $S_{oo} \times I$. Most of the arrows are inherited from $Q_A \times Q_A$, where Q_A is the quiver for the algebra A in Remark 2.5 (3). The extra 4 arrows in red are given by some tight contact structures which do not exist on $(S_o \sqcup S_o) \times I$. For instance, see Figure 4 for the bypass attachment $E \otimes I \to I \otimes E$.

(3) Stackings of contact structures corresponding to compositions $E \otimes F \to I \otimes EF \to I \otimes I$ and $I \otimes I \to EF \otimes I \to F \otimes E$ are tight. On the other hand, stackings of contact structures corresponding to compositions $E \otimes F \to I \otimes EF \to EF \otimes EF$ and $EF \otimes EF \to EF \otimes I \to F \otimes E$ are not tight; see relation (iii-3) in Definition 3.3.

We define the (t_1, t_2)–graded algebra $B = \bigoplus_{i=-2}^{2} B_i$, where B_i is a quotient of the path algebra $\mathbb{F}_2 Q_{B,i}$ of the component $Q_{B,i}$.

Definition 3.3 The algebra B is an associative (t_1, t_2)–graded $\mathbb{F}_2$–algebra with a trivial differential and a grading $\deg = (\deg_h; \deg_{t_1}, \deg_{t_2}) \in \mathbb{Z}^3$.

(1) The algebra B has idempotents $e(\Gamma_1 \otimes \Gamma_2)$ for all vertices $\Gamma_1 \otimes \Gamma_2 \in \mathcal{B} \times \mathcal{B}$, generators $\rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2')$ for all arrows $\Gamma_1 \otimes \Gamma_2 \to \Gamma_1' \otimes \Gamma_2'$ in Q_B. The relations consist of the following 4 groups.

(i) Idempotents:

$$e(\Gamma_1 \otimes \Gamma_2) \cdot e(\Gamma_1' \otimes \Gamma_2') = \delta_{\Gamma_1, \Gamma_1'} \delta_{\Gamma_2, \Gamma_2'} e(\Gamma_1 \otimes \Gamma_2) \text{ for } \Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2' \in \mathcal{B}$$

$$\begin{aligned} e(\Gamma_1 \otimes \Gamma_2) \cdot \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2') &= \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2') \cdot e(\Gamma_1' \otimes \Gamma_2') \\ &= \rho(\Gamma_1 \otimes \Gamma_2, \Gamma_1' \otimes \Gamma_2') \end{aligned}$$

(ii) Relations in B_{-1} of 2–groups

(A) from the algebra A:

$$\rho(E \otimes I, E \otimes EF) \cdot \rho(E \otimes EF, E \otimes I) = 0$$
$$\rho(I \otimes E, EF \otimes E) \cdot \rho(EF \otimes E, I \otimes E) = 0$$

(B) for $\rho(E \otimes I, I \otimes E)$:

$$\rho(E \otimes EF, E \otimes I) \cdot \rho(E \otimes I, I \otimes E) = 0$$
$$\rho(E \otimes I, I \otimes E) \cdot \rho(I \otimes E, EF \otimes E) = 0$$

(iii) Relations in B_0 of 3–groups

(A) from the algebra A:

$$\rho(I \otimes I, I \otimes EF) \cdot \rho(I \otimes EF, I \otimes I) = 0$$
$$\rho(I \otimes I, EF \otimes I) \cdot \rho(EF \otimes I, I \otimes I) = 0$$
$$\rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EF, I \otimes EF) = 0$$
$$\rho(EF \otimes I, EF \otimes EF) \cdot \rho(EF \otimes EF, EF \otimes I) = 0$$

(B) commutativity relations:

$$\rho(I \otimes I, I \otimes EF) \cdot \rho(I \otimes EF, EF \otimes EF) = \rho(I \otimes I, EF \otimes I) \cdot \rho(EF \otimes I, EF \otimes EF)$$
$$\rho(I \otimes EF, I \otimes I) \cdot \rho(I \otimes I, EF \otimes I) = \rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EF, EF \otimes I)$$
$$\rho(EF \otimes I, I \otimes I) \cdot \rho(I \otimes I, I \otimes EF) = \rho(EF \otimes I, EF \otimes EF) \cdot \rho(EF \otimes EF, I \otimes EF)$$
$$\rho(EF \otimes EF, I \otimes EF) \cdot \rho(I \otimes EF, I \otimes I) = \rho(EF \otimes EF, EF \otimes I) \cdot \rho(EF \otimes I, I \otimes I)$$

(C) for $E \otimes F$ and $F \otimes E$:

$$\rho(E \otimes F, I \otimes EF) \cdot \rho(I \otimes EF, EF \otimes EF) = 0$$

$$\rho(EF \otimes EF, EF \otimes I) \cdot \rho(EF \otimes I, F \otimes E) = 0$$

(iv) Relations in B_1 of 2–groups

(A) from the algebra A:

$$\rho(I \otimes F, EF \otimes F) \cdot \rho(EF \otimes F, I \otimes F) = 0$$

$$\rho(F \otimes I, F \otimes EF) \cdot \rho(F \otimes EF, F \otimes I) = 0$$

(B) for $\rho(I \otimes F, F \otimes I)$:

$$\rho(EF \otimes F, I \otimes F) \cdot \rho(I \otimes F, F \otimes I) = 0$$

$$\rho(I \otimes F, F \otimes I) \cdot \rho(F \otimes I, F \otimes EF) = 0$$

(2) The grading $\deg = (\deg_h; \deg_{t_1}, \deg_{t_2})$ is defined on the generators by

$$\deg(a) = \begin{cases} (1;0,0) & \text{if } a = \rho(E \otimes I, I \otimes E), \rho(E \otimes F, I \otimes EF), \\ (1;1,0) & \text{if } a = \rho(EF \otimes \Gamma, I \otimes \Gamma) \text{ for all } \Gamma \in \mathcal{B}, \\ (1;0,1) & \text{if } a = \rho(I \otimes F, F \otimes I), \rho(\Gamma \otimes EF, \Gamma \otimes I) \text{ for all } \Gamma \in \mathcal{B}, \\ (0;0,0) & \text{otherwise,} \end{cases}$$

where $\deg_h$ is the cohomological grading and $(\deg_{t_1}, \deg_{t_2})$ is the (t_1,t_2)–grading.

Remark 3.4 (1) Relations (ii-A), (iii-A) and (iv-A) come from the fact $\rho(I, EF) \cdot \rho(EF, I) = 0$ in A. The relations in (iii-B) come from certain isotopies of tight contact structures. Other relations come from the fact that stackings of the corresponding contact structures are not tight.

(2) Generators $\rho(E \otimes I, I \otimes E)$, $\rho(I \otimes F, F \otimes I)$, $\rho(E \otimes F, I \otimes EF)$ and $\rho(EF \otimes I, F \otimes E)$ of B do not inherited from $A \otimes A$. Note that we have $\deg(\rho(EF \otimes I, F \otimes E)) = (0;0,0)$.

(3) The algebra B is actually the homology of the strands algebra for a specific handle decomposition of a twice punctured disk. We refer to Section 5.1 for more detail.

Definition 3.5 Let $\mathrm{DGP}(B)$ be the smallest full subcategory of $\mathrm{DG}(B)$ which contains the projective DG B–modules $\{P(\Gamma_1 \otimes \Gamma_2) = B \cdot e(\Gamma_1 \otimes \Gamma_2) \mid \Gamma_1, \Gamma_2 \in \mathcal{B}\}$ and is closed under the cohomological grading shift functor $[1]$, two (t_1,t_2)–grading shift functors $\{1,0\}$ and $\{0,1\}$, and taking mapping cones.

Let $m(\Gamma_1\otimes\Gamma_2)\in P(\Gamma_1\otimes\Gamma_2)$ denote the generator with $\deg(m(\Gamma_1\otimes\Gamma_2))=(0;0,0)$. The 0^{th} homology category $\mathrm{HP}(B)$ of $\mathrm{DGP}(B)$ is a triangulated category and the Grothendieck group $K_0(\mathrm{HP}(B))$ has a $\mathbb{Z}[T_1^{\pm1},T_2^{\pm1}]$–basis $\{P(\Gamma_1\otimes\Gamma_2)\mid \Gamma_1,\Gamma_2\in\mathcal{B}\}\cong\mathcal{B}\times\mathcal{B}$, where the multiplication by T_1 and T_2 are induced by the (t_1,t_2)–grading shifts:

$$[M\{1,0\}]=T_1[M],\quad [M\{0,1\}]=T_2[M]\in K_0(\mathrm{HP}(B)),$$

for $M\in\mathrm{HP}(B)$.

Lemma 3.6 *There is an isomorphism of free $\mathbb{Z}[T_1^{\pm1},T_2^{\pm1}]$–modules $K_0(\mathrm{HP}(B))\cong \boldsymbol{U_T}\otimes_{\mathbb{Z}}\boldsymbol{U_T}$, where T_1 and T_2 act on $\boldsymbol{U_T}\otimes_{\mathbb{Z}}\boldsymbol{U_T}$ by multiplying $T\otimes I$ and $I\otimes T$, respectively.*

3.2 The (t_1,t_2)–graded DG (B,A)–bimodule S

To define a functor δ: $\mathrm{DGP}(A)\to\mathrm{DGP}(B)$ lifting the comultiplication on $\boldsymbol{U_T}$, we construct a (t_1,t_2)–graded DG (B,A)–bimodule S in two steps: a left DG B–module S in Section 3.2.1 and a right A–module S in Section 3.2.2.

3.2.1 The left B–module S

Definition 3.7 Define a (t_1,t_2)–graded left DG B–module $S=\bigoplus_{\Gamma\in\mathcal{B}}S(\Gamma)$, where $(S(\Gamma),d(\Gamma))$ in $\mathrm{DGP}(B)$ is defined on a case by case basis as follows:

(1) $S(I)=P(I\otimes I); d(I)=0$

(2) $S(E)=P(E\otimes I)\oplus P(I\otimes E)$; $d(E)$ is a map of left B–modules defined on the generators by:

$$\begin{aligned}d(E)\colon S(E)&\to S(E)\\ m(E\otimes I)&\mapsto \rho(E\otimes I,I\otimes E)\cdot m(I\otimes E)\\ m(I\otimes E)&\mapsto 0\end{aligned}$$

(3) $S(F)=P(I\otimes F)\oplus P(F\otimes I)\{0,1\}$; $d(F)$ is a map of left B–modules defined by:

$$\begin{aligned}d(F)\colon S(F)&\to S(F)\\ m(I\otimes F)&\mapsto \rho(I\otimes F,F\otimes I)\cdot m(F\otimes I)\\ m(F\otimes I)&\mapsto 0\end{aligned}$$

(4) $S(EF) = P(E \otimes F) \oplus P(I \otimes EF) \oplus P(EF \otimes I)\{0,1\} \oplus P(F \otimes E)\{0,1\}[-1]$;
$d(EF)$ is a map of left B–modules defined by:

$$\begin{aligned}
d(EF)\colon S(EF) &\to S(EF)\\
m(E \otimes F) &\mapsto \rho(E \otimes F, I \otimes EF) \cdot m(I \otimes EF)\\
m(I \otimes EF) &\mapsto \rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EFEF \otimes I) \cdot m(EF \otimes I),\\
m(EF \otimes I) &\mapsto \rho(EF \otimes I, F \otimes E) \cdot m(F \otimes E)\\
m(F \otimes E) &\mapsto 0
\end{aligned}$$

Remark 3.8 (1) The DG B–modules $S(\Gamma)$ are supposed to be the categorical comultiplication of the DG A–modules $P(\Gamma)$, for all $\Gamma \in \mathcal{B}$. In particular, the classes $[S(\Gamma)] \in K_0(\mathrm{HP}(B))$ agree with the comultiplication $\Delta(\Gamma) \in \boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$ under the isomorphism in Lemma 3.6.

(2) The definition of $S(E)$ is motivated from an isomorphism $\delta(E) \cong (E \otimes I \to I \otimes E)$ in the contact category $\widetilde{\mathcal{C}}_{oo}$ as in Figure 4 in Section 1.3. The other definitions have similar motivations.

Lemma 3.9 *We have $(S(\Gamma), d(\Gamma))$ is a (t_1, t_2)–graded DG B–module.*

Proof It suffices to prove that $d(\Gamma)$ is of degree $(1;0,0)$ such that $d(\Gamma)^2 = 0$. We verify it for $\Gamma = EF$ and leave other cases to the reader:

$$\begin{aligned}
d^2(m(E \otimes F)) &= \rho(E \otimes F, I \otimes EF) \cdot \rho(I \otimes EF, EF \otimes EF)\\
&\qquad\qquad \cdot \rho(EF \otimes EF, EF \otimes I) \cdot m(EF \otimes I)\\
&= 0 \cdot \rho(EF \otimes EF, EF \otimes I) \cdot m(EF \otimes I) = 0,\\
d^2(m(I \otimes EF)) &= \rho(I \otimes EF, EF \otimes EF) \cdot \rho(EF \otimes EF, EF \otimes I)\\
&\qquad\qquad \cdot \rho(EF \otimes I, F \otimes E) \cdot m(F \otimes E)\\
&= \rho(I \otimes EF, EF \otimes EF) \cdot 0 \cdot m(F \otimes E) = 0,
\end{aligned}$$

from relation (iii-C) in Definition 3.7. That $d^2 = 0$ is obvious for the other two generators of $S(EF)$.

The degrees of the generators of $S(EF)$ are

$$\deg(m(E \otimes F)) = \deg(m(I \otimes EF)) = (0;0,0),$$
$$\deg(m(EF \otimes I)) = (0;0,-1), \quad \deg(m(F \otimes E)) = (1;0,-1).$$

Then d is of degree $(1;0,0)$ since

$$\begin{aligned}
\deg(d(m(E\otimes F))) &= \deg(\rho(E\otimes F, I\otimes EF))+\deg(m(I\otimes EF))\\
&= (1;0,0)=\deg(m(E\otimes F))+(1;0,0),\\
\deg(d(m(EF\otimes I))) &= \deg(\rho(EF\otimes I, F\otimes E))+\deg(m(F\otimes E))\\
&= (1;0,-1)=\deg(m(E\otimes F))+(1;0,0),\\
\deg(d(m(I\otimes EF))) &= \deg(\rho(I\otimes EF, EF\otimes EF))+\deg(\rho(EF\otimes EF, EF\otimes I))\\
&\qquad +\deg(m(EF\otimes I))\\
&= (1;0,0)=\deg(m(I\otimes EF))+(1;0,0).
\end{aligned}$$

This completes the proof. □

3.2.2 The right A–module structure on S In this subsection we describe the right A–module structure on S. Let $m\times a$ denote the right multiplication for $m\in S, a\in A$ and let $m\cdot b$ denote the multiplication in B for $m\in P(\Gamma_1\otimes\Gamma_2)\subset B, b\in B$.

Definition 3.10 For $m\in S$ and $a\in A$, define the grading of the right multiplication $m\times a$ by

$$\deg(m\times a)=\deg(m)+(\deg_h(a);\deg_t(a),\deg_t(a)),$$

where deg is the grading in B, $\deg_h$ and $\deg_t$ are the gradings in A.

Remark 3.11 This definition is related to the categorification of $\Delta(T)=T\otimes T$ in the proof of Theorem 1.2 in Section 3.3. Topologically, the definition comes from the fact that the generator $t\in H_1(S_o)$ is mapped to $t_1+t_2\in H_1(S_{oo})$ under δ as in Figure 4 in Section 1.3.

The right multiplication is a map of left DG B–modules defined on generators as follows.

(1) For an idempotent $e(\Gamma)$, define

$$\begin{aligned}\times e(\Gamma)\colon S(\Gamma')&\to S(\Gamma'),\\ m&\mapsto \delta_{\Gamma,\Gamma'}m.\end{aligned}$$

(2) For the generator $\rho(I,EF)$, define

$$\begin{aligned}\times\rho(I,EF)\colon S(I)&\to S(EF),\\ m(I\otimes I)&\mapsto \rho(I\otimes I, I\otimes EF)\cdot m(I\otimes EF).\end{aligned}$$

(3) For the generator $\rho(EF, I)$, define

$$\begin{aligned} \times\rho(EF, I)\colon S(EF) &\to S(I), \\ m(EF\otimes I) &\mapsto \rho(EF\otimes I, I\otimes I)\cdot m(I\otimes I), \\ m(E\otimes F) &\mapsto 0, \\ m(I\otimes EF) &\mapsto 0, \\ m(F\otimes E) &\mapsto 0. \end{aligned}$$

In general, define $m\times(a_1a_2) := (m\times a_1)\times a_2$ for $a_1, a_2\in A$ and $m\in S$.

Lemma 3.12 *The definition above gives a right DG A–module S:*

(1) $(m\times a_1)\times a_2 = (m\times a_1')\times a_2'$ *for* $a_1\cdot a_2 = a_1'\cdot a_2'\in A$ *and* $m\in S$

(2) $d(m\times a) = d(m)\times a$ *for* $a\in A$ *and* $m\in M$

(3) *The right multiplication is compatible with the grading in Definition 3.10*

Proof For (1), since the only nontrivial relation in A is $\rho(I, EF)\cdot\rho(EF, I) = 0$, it suffices to prove that $(m\times\rho(I, EF))\times\rho(EF, I) = 0$, which follows from the definition.

For (2), we verify the following from relations (iii-A) and (iii-B) in Definition 3.7:

$$\begin{aligned} d(m(I\otimes I)\times\rho(I, EF)) &= \rho(I\otimes I, I\otimes EF)\cdot d(m(I\otimes EF)) \\ &= \rho(I\otimes I, I\otimes EF)\cdot\rho(I\otimes EF, I\otimes I)\cdot\rho(I\otimes I, EF\otimes I) \\ &= 0 = d(m(I\otimes I))\times\rho(I, EF). \end{aligned}$$

Similarly, $d(m(I\otimes EF))\times\rho(EF, I) = 0 = d(m(I\otimes EF)\times\rho(EF, I))$.

For (3), we verify that

$$\begin{aligned} \deg(m(EF\otimes I)\times\rho(EF, I)) &= (0; 0, -1) + (1; 1, 1) \\ &= \deg(\rho(EF\otimes I, I\otimes I)\cdot m(I\otimes I)). \end{aligned}$$

Similarly, $\deg(m(I\otimes I)\times\rho(I, EF)) = \deg(\rho(I\otimes I, I\otimes EF)\cdot m(I\otimes EF))$. □

Since the right multiplication is a map of left A–modules, we have: $b\cdot(m\times a) = (b\cdot m)\times a$, for $a\in A, b\in B$ and $m\in S$. Hence S is a (t_1, t_2)–graded DG (B, A)–bimodule.

3.3 The categorification of the comultiplication on U_T

In this section, we use the bimodule S to categorify the comultiplication on U_T, ie prove Theorem 1.2.

Let $\delta\colon \mathrm{DGP}(A) \xrightarrow{S\otimes_A -} \mathrm{DGP}(B)$ given by tensoring with the DG (B,A)–bimodule S over A.

Lemma 3.13 *The functor δ maps $P(\Gamma)$ to $S(\Gamma) \in \mathrm{DGP}(B)$ for all $\Gamma \in \mathcal{B}$.*

Proof The proof is similar to that of Lemma 2.23. □

There is an induced exact functor $\delta\colon \mathrm{HP}(A) \xrightarrow{S\otimes_A -} \mathrm{HP}(B)$ between the homology categories.

Proof of Theorem 1.2 We compute the map on the Grothendieck groups

$$K_0(\delta)\colon K_0(\mathrm{HP}(A)) \to K_0(\mathrm{HP}(B)).$$

(1) By Lemma 3.13, $\delta(P(\Gamma)) = S(\Gamma)$, for $\Gamma \in \mathcal{B} = \{I, E, F, EF\}$. Hence by Remark 3.8,

$$K_0(\delta)[P(\Gamma)] = [S(\Gamma)] = \Delta(\Gamma) \in U_T \otimes_{\mathbb{Z}} U_T.$$

(2) By the grading in Definition 3.10, $\delta(P(\Gamma)\{n\}) = S(\Gamma)\{n,n\}$ for $n \in \mathbb{Z}$. Hence,

$$K_0(\delta)(T^n[P(\Gamma)]) = K_0(\delta)([P(\Gamma)\{n\}]) = [S(\Gamma)\{n,n\}] = T_1^n T_2^n [S(\Gamma)].$$

Parts (1) and (2) together imply that $K_0(\delta) = \Delta\colon U_T \to U_T \otimes_{\mathbb{Z}} U_T$ since the $\mathbb{Z}$–linear maps $K_0(\delta)$ and Δ agree on the $\mathbb{Z}$–basis $\{T^n\Gamma \mid \Gamma \in \mathcal{B}, n \in \mathbb{Z}\}$ of U_T. □

Remark 3.14 It is interesting to ask whether the properties of the comultiplication in Lemma 2.3, such as coassociativity, can be lifted to the categorical level. We believe that the answer is positive since on the topological side $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ are both categorified to a functor $\widetilde{\mathcal{C}}(S_o, F_o) \to \widetilde{\mathcal{C}}(S_{ooo}, F_{ooo})$, where S_{ooo} is a triple punctured disk with F_{ooo} consisting of two points on each boundary component of ∂S_{ooo}.

4 The linear action of U_T on $V_1^{\otimes n}$

In this section, we give a distinguished basis $\mathcal{B}_n$ of $V_1^{\otimes n}$ and express the action in terms of $\mathcal{B}_n$.

4.1 The representations V_1 and V_2

Let V_1 be a free $\mathbb{Z}[t^{\pm 1}]$–module with a basis $\mathcal{B}_1 = \{|0\rangle, |1\rangle\}$. A *parity* p is a $\mathbb{Z}$–grading on V_1 given by $\mathrm{p}(|0\rangle) = 0, \mathrm{p}(|1\rangle) = 1$. Define an action of $\boldsymbol{U_T}$ on V_1 by

$$E|0\rangle = 0, \quad F|0\rangle = |1\rangle,$$
$$E|1\rangle = (1-t)|0\rangle, \quad F|1\rangle = 0,$$
$$T|0\rangle = t|0\rangle, \quad T|1\rangle = t(-1)^2|1\rangle.$$

Note that the factor $(-1)^2$ in $T|1\rangle$ is the shadow of a cohomological grading shift by 2 on the categorical level. The parities on $\boldsymbol{U_T}$ and V_1 are compatible with respect to the action. More precisely, the operators E and F change the parity by -1 and 1, respectively:

$$\mathrm{p}(F|0\rangle) = \mathrm{p}(|1\rangle) = 1 = \mathrm{p}(F) + \mathrm{p}(|0\rangle), \quad \mathrm{p}(E|1\rangle) = \mathrm{p}(|0\rangle) = 0 = \mathrm{p}(E) + \mathrm{p}(|1\rangle)$$

Let $V_2 = V_1 \otimes_{\mathbb{Z}[t^{\pm 1}]} V_1$ be a free $\mathbb{Z}[t^{\pm 1}]$–module with a basis $\mathcal{B}_2' = \mathcal{B}_1 \times \mathcal{B}_1 = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The action of $\boldsymbol{U_T}$ on V_2 is induced by the comultiplication Δ: $a\cdot(v\otimes w) = \Delta(a)(v\otimes w)$, for $a \in \boldsymbol{U_T}$, $v, w \in V_1$. Note that the action of $\boldsymbol{U_T} \otimes_{\mathbb{Z}} \boldsymbol{U_T}$ on $V_1 \otimes V_1$ is the graded tensor product: $(a_1 \otimes a_2)(v \otimes w) = (-1)^{\mathrm{p}(a_2)\mathrm{p}(v)} a_1 v \otimes a_2 w$.

Lemma 4.1 *The action of $\boldsymbol{U_T}$ on V_2 is given in the basis $\mathcal{B}_2$ as*

$$E|00\rangle = 0, \quad F|00\rangle = |01\rangle + t|10\rangle,$$
$$E|01\rangle = (1-t)|00\rangle, \quad F|01\rangle = t|11\rangle,$$
$$E|10\rangle = (1-t)|00\rangle, \quad F|10\rangle = -|11\rangle,$$
$$E|11\rangle = (1-t)|01\rangle - (1-t)|10\rangle, \quad F|11\rangle = 0,$$
$$T(v) = t^2 v \text{ for all } v = v_1 \otimes v_2 \in \mathcal{B}_2, \text{ where } v_1, v_2 \in \mathcal{B}_1.$$

Proof We verify some of the formulas and leave others to the reader:

$$T(v) = \Delta(T)(v_1 \otimes v_2) = (T \otimes T)(v_1 \otimes v_2) = T(v_1) \otimes T(v_2) = t^2 v$$
$$F|00\rangle = \Delta(F)|00\rangle = (1 \otimes F + F \otimes T)|00\rangle = |01\rangle + t|10\rangle$$
$$F|10\rangle = \Delta(F)|10\rangle = (1 \otimes F + F \otimes T)|10\rangle = (1 \otimes F)|10\rangle = -|11\rangle$$

This completes the proof. □

4.2 The representations $V_1^{\otimes n} = V_1^{\otimes n}$

There is an action of U_T on the n^{th} tensor product $V_1^{\otimes n} = V_1^{\otimes n}$ induced by iterated comultiplication. Consider a $\mathbb{Z}[t^{\pm 1}]$–basis $\mathcal{B}'_n$ of $V_1^{\otimes n}$:

$$\mathcal{B}'_n = \mathcal{B}_1^{\times n} = \{\boldsymbol{a} = |a_1 \cdots a_n\rangle \mid a_i \in \{0,1\}\}$$

We call $\mathcal{B}'_n$ the tensor product basis of $V_1^{\otimes n}$. Consider another presentation of the basis:

$$\mathcal{B}_n = \{\boldsymbol{x} = (x_1, \dots, x_k) \mid 1 \le x_1 < \cdots < x_k \le n, 1 \le k \le n\} \sqcup \{\varnothing\}$$

There is a one-to-one correspondence between $\mathcal{B}_n$ and $\mathcal{B}'_n$,

$$\begin{aligned} \mathcal{B}_n &\to \mathcal{B}'_n, \\ \varnothing &\mapsto a = |0 \cdots 0\rangle, \\ \boldsymbol{x} = (x_1, \dots, x_k) &\mapsto a = |a_1 \cdots a_n\rangle, \end{aligned}$$

where

$$a_i = \begin{cases} 1 & \text{if } i = x_l, \text{ for some } 1 \le l \le k, \\ 0 & \text{otherwise.} \end{cases}$$

There is a partition $\mathcal{B}_n = \bigsqcup_{k=0}^{n} \mathcal{B}_{n,k}$, where $\mathcal{B}_{n,k} = \{\boldsymbol{x} = (x_1, \dots, x_k) \mid 1 \le x_1 < \cdots < x_k \le n\}$ for $1 \le k \le n$ and $\mathcal{B}_{n,0} = \{\varnothing\}$. Let $V_1^{\otimes n} = \bigoplus_{k=0}^{n} V_{n,k}$ be the corresponding decomposition of $V_1^{\otimes n}$, where $V_{n,k}$ is spanned by the basis $\mathcal{B}_{n,k}$ for $0 \le k \le n$.

In the U_T–action, F converts a state from $|0\rangle$ to $|1\rangle$ for one factor of the state in $\mathcal{B}'_n$. In particular, F increases the number of $|1\rangle$ states by 1; similarly, E decreases the number of $|1\rangle$ states by 1:

$$F\colon V_{n,k} \to V_{n,k+1}, \quad E\colon V_{n,k} \to V_{n,k-1}$$

We introduce some notation in order to describe the action in terms of $\mathcal{B}_n$. For $\boldsymbol{x} = (x_1, \dots, x_k) \in \mathcal{B}_{n,k}$, let $\bar{\boldsymbol{x}} = (\bar{x}_1, \dots, \bar{x}_{n-k}) \in \mathcal{B}_{n,n-k}$ be the increasing sequence consisting of the complement $\{1, \dots, n\} \backslash \{x_1, \dots, x_k\}$ of $\boldsymbol{x}$ in $\{1, \dots, n\}$. Define

$$\beta(\boldsymbol{x}, \bar{x}_j) = |\{l \in \{1, \dots, k\} \mid x_l < \bar{x}_j\}| + 2|\{l \in \{1, \dots, k\} \mid x_l > \bar{x}_j\}|.$$

Let $\boldsymbol{x} \sqcup \{\bar{x}_j\}$ be an increasing sequence obtained by adjoining $\bar{x}_j$ to $\boldsymbol{x}$ and $\boldsymbol{x} \backslash \{x_i\}$ be an increasing sequence obtained by removing x_i from $\boldsymbol{x}$.

Lemma 4.2 *The $\boldsymbol{U}_T$–action on $V_1^{\otimes n}$ is given for $\boldsymbol{x} \in \mathcal{B}_{n,k}$ by*

$$I(\boldsymbol{x}) = \boldsymbol{x}, \quad T(\boldsymbol{x}) = t^n(-1)^{2n}\boldsymbol{x},$$

$$F(\boldsymbol{x}) = \sum_{j=1}^{n-k} t^{n-\overline{x}_j}(-1)^{\beta(\boldsymbol{x},\overline{x}_j)}\boldsymbol{x} \sqcup \{\overline{x}_j\},$$

$$E(\boldsymbol{x}) = \sum_{i=1}^{k}((-1)^{1-i}\boldsymbol{x}\backslash\{x_i\} + t(-1)^{2-i}\boldsymbol{x}\backslash\{x_i\}).$$

Proof We only check the action of F:

$$\begin{aligned} F(\boldsymbol{x}) = \Delta^n(F)(\boldsymbol{x}) &= \sum_{j=1}^{n}(1 \otimes \cdots 1 \otimes \underset{j^{\text{th}}}{F} \otimes T \cdots \otimes T)(\boldsymbol{x}) \\ &= \sum_{j=1}^{n-k} t^{n-\overline{x}_j}(-1)^{\beta(\boldsymbol{x},\overline{x}_j)}\boldsymbol{x} \sqcup \{\overline{x}_j\}, \end{aligned}$$

where the exponent of t comes from $(n-\overline{x}_j)$'s T in the j^{th} term of $\Delta^n(F)$; the exponent of -1 comes from the graded tensor product and the action of T on the state $|1\rangle$: $T|1\rangle = t(-1)^2|1\rangle$. □

The exponents of (-1) in the expressions including $\beta(\boldsymbol{x},\overline{x}_j)$ in $F(\boldsymbol{x})$ and $2n$ in $T(\boldsymbol{x})$ will be used as cohomological grading shifts in the bimodule C_n in Section 6.1 which categorifies the $\boldsymbol{U}_T$–action.

5 The t–graded DG algebra R_n through the quiver Q_n

We define a family of t–graded DG algebras R_n which are closely related to the strands algebras associated to an n times punctured disk in bordered Heegaard–Floer homology. In Section 5.1 we briefly review the definition of the strands algebras and introduce the *decorated rook diagrams* as a variant of *rook diagrams*. In Section 5.2 we construct the quivers Q_n whose arrows are given by the decorated rook diagrams. In Section 5.3 we define R_n as a quotient of the path algebra $\mathbb{F}_2 Q_n$ whose differential is given by resolutions of crossings and markings for the decorated rook diagrams. We show that R_n is formal and categorify $V_1^{\otimes n}$ through $\mathrm{DGP}(R_n)$ generated by some projective DG R_n–modules. In Section 5.4 we define a variant $A \boxtimes R_n$ of $A \otimes R_n$ which will be used in the construction of the $(H(R_n), A \boxtimes R_n)$–bimodule C_n in Section 6.

5.1 Background on the strands algebras and the rook monoid

5.1.1 The strands algebra Lipshitz, Ozsváth and Thurston in [23] introduced the *strands algebra* associated to a connected surface with one closed boundary component. Later on, the strands algebra of any surface with boundary was defined by Zarev [42]. We refer to [42, Section 2] for more detail on the strands algebras associated to an *arc diagram*.

Definition 5.1 An *arc diagram* $\mathcal{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$ is a triple consisting of a collection $\boldsymbol{Z} = \{Z_1, \ldots, Z_l\}$ of line segments, a collection $\boldsymbol{a} = \{a_1, \ldots, a_{2k}\}$ of distinct $2k$ points in $\boldsymbol{Z}$, and a matching M of $\boldsymbol{a}$ into k pairs of points.

A surface $F(\mathcal{Z})$ can be constructed from an arc diagram $\mathcal{Z}$ by starting with a collection of rectangles $[0,1] \times Z_j$ for $j = 1, \ldots, l$, and attaching a 1–handle with endpoints on $M^{-1}(i) \times \{0\}$ for each $i = 1, \ldots, k$. An n times punctured disk Σ_n can be parametrized by $\mathcal{Z}(2n) = (\boldsymbol{Z}, \boldsymbol{a}, M)$:

- $\boldsymbol{Z} = \{Z\}$ is a single vertical line segment.
- $\boldsymbol{a} = \{1, \ldots, 2n\}$ is a collection of $2n$ points in Z ordered from bottom to top.
- M matches $\boldsymbol{a}$ into n pairs of adjacent points $\{2i-1, 2i\}$ for $i = 1, \ldots, n$.

We fix the arc diagram $\mathcal{Z}_n = \mathcal{Z}(2n) = (\boldsymbol{Z}, \boldsymbol{a}, M)$ throughout this paper; see the diagram on the left of Figure 8. The associated strands algebra is generated by *strands diagrams*.

Definition 5.2 Given the arc diagram $\mathcal{Z}_n$, a *strands diagram with k strands* is a triple (S, T, ϕ), where S, T are k–element subsets of $\boldsymbol{a}$ and $\phi\colon S \to T$ is a bijection with $i \le \phi(i)$ for all $i \in S$.

Geometrically, a strands diagram (S, T, ϕ) with k–strands is an isotopy class of a set of k strands with a minimal number of crossings which connect the k points in S as a subset of the $2n$ points on the left to the k points in T as a subset of the $2n$ points on the right. The restriction that ϕ is nondecreasing means that strands stay horizontal or move up when read from left to right.

The associated *strands algebra* $\mathcal{A}(\mathcal{Z}_n) = \bigoplus_{k=0}^{n} \mathcal{A}(\mathcal{Z}_n, k)$, where $\mathcal{A}(\mathcal{Z}_n, k)$ is $\mathbb{F}_2$–vector space generated by strands diagrams with k strands with the following two constraints. The first constraint on a strands diagram (S, T, ϕ) is $|S \cap \{2i-1, 2i\}| \le 1$ and $|T \cap \{2i-1, 2i\}| \le 1$ for $i = 1, \ldots, n$, ie the number of intersection points of the strands diagram with any pair $\{2i-1, 2i\}$ is at most 1. We call it the 1*–handle*

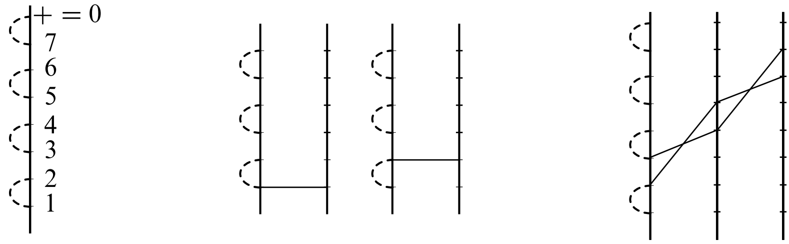


Figure 8: The diagram on the left is the arc diagram $\mathcal{Z}_4$; the diagram in the middle gives a primitive idempotent in $\mathcal{A}(\mathcal{Z}_3, 1)$ according to the idempotent constraint; the diagram on the right describes a double crossing.

constraint. The second constraint is about horizontal strands: if any generator $r \in \mathcal{A}(\mathcal{Z}_n)$ contains (S_1, T_1, ϕ_1) as a summand where $\phi_1(i) = i$ for some i, then r must contain another summand (S_2, T_2, ϕ_2), where $S_2 = S_1 \backslash \{i\} \cup \{j\}, T_2 = T_1 \backslash \{i\} \cup \{j\}$ and $\phi_2|_{S_2 \backslash \{j\}} = \phi_1|_{S_1 \backslash \{i\}}$, $\phi_2(j) = j$ for $j = i - (-1)^i$. Note that i and j form a pair in $\mathcal{Z}_n$ so that $j \notin S_1, T_1$ according to the 1–handle constraint. As in the middle diagram in Figure 8, a primitive idempotent is a sum of two horizontal strands: $(S_1, T_1, \phi_1) + (S_2, T_2, \phi_2)$, where $S_1 = T_1 = \{1\}, S_2 = T_2 = \{2\}$ and ϕ_1, ϕ_2 are the identities. In particular, each summand (S_1, T_1, ϕ_1) or (S_2, T_2, ϕ_2) is not a generator of $\mathcal{A}(\mathcal{Z}_3, 1)$. Since horizontal strands represent idempotents in the algebra, we call it the *idempotent constraint*.

The product $a \cdot b$ of two strands diagrams is set to be zero if the right side of a does not match the left side of b; otherwise, the product is the horizontal juxtaposition of a and b. If two strands cross each other twice in the juxtaposition, the product is set to be zero. It is called the *double crossing relation*; see the right diagram in Figure 8.

The differential of a strand diagram is the sum over all ways of resolving one crossing of the diagram; see Figure 9 for an example.

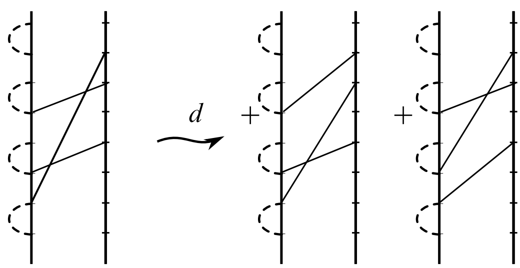


Figure 9: Differential given by resolving crossings

5.1.2 The generalized rook diagrams Since the matching in the arc diagram $\mathcal{Z}_n$ simply identifies $2i-1$ and $2i$ in each pair, we introduce a generalization of *rook diagrams* to describe the strands algebra $\mathcal{A}(\mathcal{Z}_n)$. We first recall the definition of *rook monoid* from [35].

Definition 5.3 Let n be a positive integer and $\boldsymbol{n}=\{1,\dots,n\}$. The *rook monoid* $\mathcal{R}_n$ is the set of all one-to-one maps σ with domain $I(\sigma)\subset\boldsymbol{n}$ and range $J(\sigma)\subset\boldsymbol{n}$. The multiplication on $\mathcal{R}_n$ is given by composition of maps.

There is a diagrammatic presentation of the rook monoid, called *rook diagrams*, given in [6]. A rook diagram associated to an element $\sigma\in\mathcal{R}_n$ is a graph on two rows of n vertices such that vertex i in the bottom row is connected to vertex j in the top row if and only if $\sigma(i)=j$. The multiplication is given by vertical concatenation of two rook diagrams.

We define the *generalized rook diagrams* by adding a new type of diagrams with loops attached at vertices to the rook diagrams. The strand diagrams in $\mathcal{A}(\mathcal{Z}_n)$ corresponding to the loops and the multiplication rule on the loops will be given in Section 5.1.3.

5.1.3 From strands diagrams to generalized rook diagrams We describe the translation from the strands diagrams in $\mathcal{A}(\mathcal{Z}_n)$ to the generalized rook diagrams on some generators of $\mathcal{A}(\mathcal{Z}_n)$.

In the left diagram in Figure 10, an idempotent in $\mathcal{A}(\mathcal{Z}_n)$ as a sum of two horizontal strands is translated to a single vertical rook diagram id from the state $|001\rangle$ to itself. The translation consists of three steps: rotate a strand diagram counterclockwise by $\pi/2$; replace the identified points $\{2i-1,2i\}$ of i^{th} pair in the strand diagram by the i^{th} vertex from the right in a rook diagram; combine two horizontal strands in the strand diagram into a single vertical rook diagram.

In the middle diagram in Figure 10, an upward strand connecting two points in a pair of $\mathcal{Z}_n$ is translated to a loop ρ attached at the corresponding state $|1\rangle$. Note that the loop ρ is nilpotent in $\mathcal{A}(\mathcal{Z}_n)$: $\rho^2=0$. Correspondingly, the square of any loop is defined as zero. In the right diagram in Figure 10, a strand connecting two points in different pairs is translated to a left-veering rook diagram. In general, a strand diagram with k strands is translated to a generalized rook diagram as a superposition of the corresponding k generalized rook diagrams.

Since the strands diagrams stay horizontal or move up, the corresponding rook diagrams always have negative or infinity slopes, ie they stay vertical or move to the left when read from bottom to top. We call these generalized rook diagrams as *left-veering rook diagrams*.

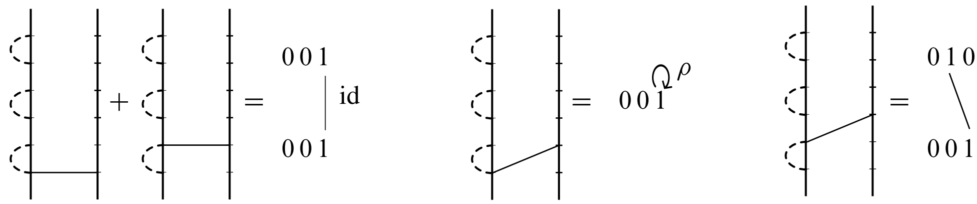


Figure 10: The translation for $\mathcal{A}(\mathcal{Z}_3, 1)$: the left one is the idempotent id; the middle one is the nilpotent element ρ; the right one is a left-veering rook diagram.

The differential on $\mathcal{A}(\mathcal{Z}_n)$ induces a differential d_1 on the left-veering rook diagrams. As in Figure 11 the resolution of such a crossing contains two left-veering rook diagrams with loops, since a vertical strand in a left-veering rook diagram corresponds to a sum of two terms in $\mathcal{A}(\mathcal{Z}_n)$.

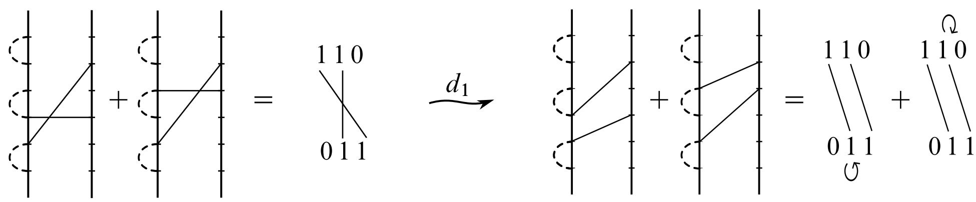


Figure 11: The translation for a differential of a crossing

5.1.4 New ingredients We define the *decorated rook diagrams* as the left-veering rook diagrams, possibly added with some *markings*.

The motivation of introducing diagrams with markings is given as follows. There is a relation in $\mathcal{A}(\mathcal{Z}_n)$: $(S, T, \phi) \cdot (S', T', \phi') = 0$, if $T \cap \{2i-1, 2i\} = \{2i-1\}$ and $S' \cap \{2i-1, 2i\} = \{2i\}$ for some i since their endpoints do not match; see the left diagram in Figure 12. But the endpoints of the corresponding decorated rook diagrams do match. We introduce a new rook diagram with a marking at the position corresponding to the pair $\{2i-1, 2i\}$. We deform the relation in $\mathcal{A}(\mathcal{Z}_n)$ to be a differential d_0 of this new diagram with the marking. In general, the differential d_0 of a strand diagram is the sum over all ways of resolving one marking of the diagram.

We define a differential $d = d_0 + d_1$ on the decorated rook diagrams. In other words, d is a combination of the resolutions of crossings and those of markings as in the right part of Figure 12. For decorated rook diagrams with no crossings and markings, the differential d is defined to be zero.

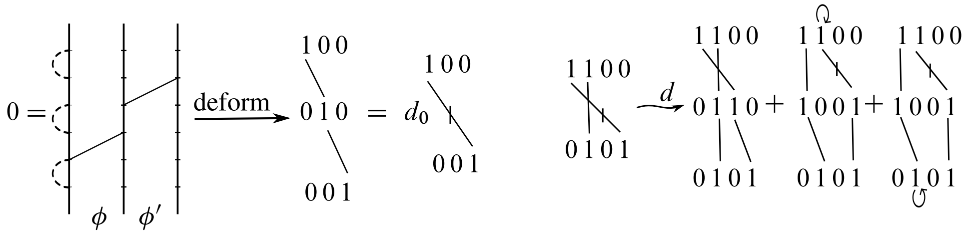


Figure 12: The deformation of a relation in $\mathcal{A}(\mathcal{Z}_n)$ is on the left; a differential of a decorated rook diagram with one marking and one crossing is on the right.

Remark 5.4 There are two types of resolutions d_0, d_1 for the decorated rook diagrams. On the one hand, d_1 is inherited from the strands algebras via studying moduli spaces of holomorphic curves. On the other hand, the author does not know a moduli space or contact topological interpretation of d_0. The definition is only used for the algebraic construction of the bimodule C_n in Section 6.

We introduce the notion of *elementary decorated rook diagrams* which cannot be decomposed as a concatenation of two nontrivial pieces. The elementary decorated rook diagrams will give generators of the algebra R_n in Definition 5.8. Notice that n–tuples of $\{|0\rangle, |1\rangle\}$ are elements in the basis $\mathcal{B}_n$ of the representation $V_1^{\otimes n}$. Hence, a decorated rook diagram can be viewed as a map from one element of $\mathcal{B}_n$ in the bottom row to the other element of $\mathcal{B}_n$ in the top row.

Definition 5.5 *Elementary decorated rook diagrams* consist of two types:

(1) A loop $\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}$ attached at x_i for $i = 1, \dots, k$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$.

(2) A decorated rook diagram $\boldsymbol{x} \xrightarrow{i,s_1} \boldsymbol{y}$ with s_1 crossings and $\sum_{j=i}^{i+s_1}(x_j - y_j - 1)$ markings associated to a matching $\sigma\colon \boldsymbol{x} = (x_1, \dots, x_k) \to \boldsymbol{y} = (y_1, \dots, y_k)$, where $i \in \{1, \dots, k\}$, $s_1 \geq 0$ such that

$$\begin{aligned}
\sigma(x_{i+s_1}) &= y_i, \\
\sigma(x_j) &= y_{j+1} = x_j \quad \text{for } j \in \{i, \dots, i+s_1-1\}, \\
\sigma(x_j) &= y_j = x_j \quad \text{for } j \notin \{i, i+1, \dots, i+s_1\}.
\end{aligned}$$

The algebraic definition above is technical while the corresponding rook diagrams are easier to follow as in Figure 13. Given an elementary decorated rook diagram

$$\boldsymbol{x} \xrightarrow{i,s_1} \boldsymbol{y},$$

define

$$v = (x_i - y_i - 1, \dots, x_{i+s_1} - y_{i+s_1} - 1) \in \mathbb{N}^{s_1+1}.$$

The vector v counts the numbers of $|0\rangle$ states between the j^{th} $|1\rangle$ states $\{x_j\}$ in x and $\{y_j\}$ in y for $j = i, \dots, i+s_1$. Let $s_0(v) = \sum_{l=0}^{s_1} v_l$ denote the total number of $|0\rangle$ states between x_{i+s_1} and y_i. Then x and y only differ at two positions x_{i+s_1} and y_i which are connected by a left-veering strand. On this strand, there are s_1 crossings with vertical strands and $s_0(v)$ markings, ie all possible crossings and markings must be on the strand.

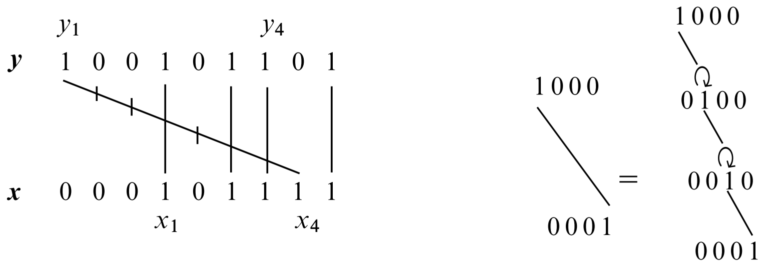


Figure 13: The left-hand diagram is an elementary rook diagram $x \xrightarrow{i,s_1,v} y$, where $i = 1, s_1 = 3, v = (2,1,0,0)$; in the right-hand diagram, we use the left-hand side to denote the composition of elementary diagrams on the right-hand side.

Although v is determined by

$$x \xrightarrow{i,s_1} y,$$

we will still write the decorated rook diagram as

$$x \xrightarrow{i,s_1,v} y.$$

Relations for concatenation of decorated rook diagrams are quite different from those for the strands diagrams as in Figure 14:

- The double crossing in decorated rook diagrams is not zero.
- An isotopy of a crossing does not give the same decorated rook diagram.

For more detail about the relations, refer to Definition 5.8 of the algebra R_n.

5.2 The quiver Q_n

In this section, we construct the quiver $Q_n = \bigsqcup_{k=0}^{n} Q_{n,k}$ for $n > 0$.

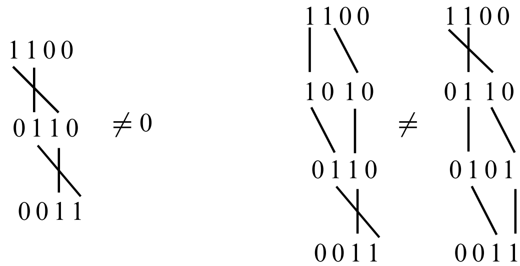


Figure 14: Some relations for decorated rook diagrams

Definition 5.6 (Quiver $Q_{n,k} = (V(Q_{n,k}), A(Q_{n,k}))$) (1) Let $V(Q_{n,k}) = \mathcal{B}_{n,k}$ be the set of vertices.

(2) Let $A(Q_{n,k})$ be the set of arrows consisting of two types:

$$\text{loops:} \quad \{\boldsymbol{x} \xrightarrow{i} \boldsymbol{x} \mid i = 1, \dots, k; \boldsymbol{x} \in \mathcal{B}_{n,k}\}$$

$$\text{arrows:} \quad \{\text{elementary decorated rook diagrams } \boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}\}$$

Example 5.7 (Quiver $Q_{4,2}$) We have

$$V(Q_{4,2}) = \{(3,4), (2,4), (1,4), (2,3), (1,3), (1,2)\}.$$

For $\boldsymbol{x} = (x_1, x_2)$, there exist two loops, one for each x_i. There are 6 arrows without crossings or markings,

$$\{(3,4) \xrightarrow{1,0,(0)} (2,4), \quad (2,4) \xrightarrow{1,0,(0)} (1,4), \quad (2,4) \xrightarrow{2,0,(0)} (2,3),$$
$$(1,4) \xrightarrow{2,0,(0)} (1,3), \quad (2,3) \xrightarrow{1,0,(0)} (1,3), \quad (1,3) \xrightarrow{2,0,(0)} (1,2)\},$$

and 6 arrows with crossings or markings,

$$\{(3,4) \xrightarrow{1,0,(1)} (1,4), \quad (3,4) \xrightarrow{1,1,(0,0)} (2,3), \quad (3,4) \xrightarrow{1,1,(1,0)} (1,3),$$
$$(1,4) \xrightarrow{2,0,(1)} (1,2), \quad (2,3) \xrightarrow{1,1,(0,0)} (1,2), \quad (2,4) \xrightarrow{1,1,(0,1)} (1,2)\}.$$

5.3 The t–graded DG algebra R_n

We define the t–graded DG algebra $R_n = \bigoplus_{k=0}^{n} R_{n,k}$, where $R_{n,k} = \mathbb{F}_2 Q_{n,k}/\sim$ is a quotient of the path algebra $\mathbb{F}_2 Q_{n,k}$ of the quiver $Q_{n,k}$ with a differential.

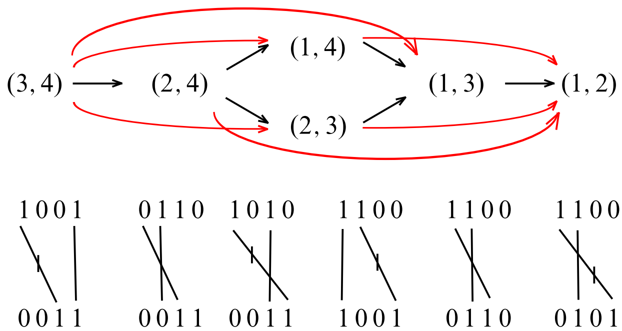


Figure 15: The top diagram describes the quiver $Q_{4,2}$, where black lines denote arrows without crossings or markings and red lines denote arrows with crossings or markings which are represented in the bottom diagram.

Definition 5.8 (t–graded DG algebra R_n) The algebra R_n is an associative t–graded $\mathbb{F}_2$–algebra with a differential d and a grading $\deg = (\deg_h, \deg_t) \in \mathbb{Z}^2$.

(A) The algebra R_n has idempotents $e(\boldsymbol{x})$ for each vertex $\boldsymbol{x}$ in Q_n, generators

$$\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \text{ for each loop } \boldsymbol{x} \xrightarrow{i} \boldsymbol{x},$$
$$r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \text{ for each arrow } \boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y},$$

in Q_n. The relations consist of 4 groups:

(i) Idempotents

$$e(\boldsymbol{x}) \cdot e(\boldsymbol{y}) = \delta_{\boldsymbol{x},\boldsymbol{y}} \cdot e(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y}$$
$$e(\boldsymbol{x}) \cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \cdot e(\boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \quad \text{for all } \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$$
$$e(\boldsymbol{x}) \cdot r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \cdot e(\boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \quad \text{for all } r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$$

(ii) nilpotent loops

$$\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = 0 \quad \text{for all } \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$$

(iii) commutativity for disjoint diagrams:

$$\rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x}) \cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \cdot \rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x}) \quad \text{if } i' \neq i$$
$$\rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x}) \cdot r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \cdot \rho(\boldsymbol{y} \xrightarrow{i'} \boldsymbol{y}) \quad \text{if } i' \notin \{i, \dots, i+s_1\}$$
$$r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \cdot r(\boldsymbol{y} \xrightarrow{i',s_1',\boldsymbol{v}'} \boldsymbol{z}) = r(\boldsymbol{x} \xrightarrow{i',s_1',\boldsymbol{v}'} \boldsymbol{w}) \cdot r(\boldsymbol{w} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{z}) \quad \text{if } x_{i+s_1} < z_{i'}$$

(iv) sliding over a crossing:

$$\rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x}) \cdot r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) \cdot \rho(\boldsymbol{y} \xrightarrow{i'+1} \boldsymbol{y})$$

if $i' \in \{i, \dots, i+s_1-1\}, s_1 > 0$

(B) The differential $d = d_0 + d_1$ is defined on the generators by the resolutions of crossings and markings on the corresponding decorated rook diagrams. In general, the differential is extended by Leibniz's rule: $d(r_1 \cdot r_2) = dr_1 \cdot r_2 + r_1 \cdot dr_2$ for $r_1, r_2 \in R_n$.

(C) The grading $\deg = (\deg_h, \deg_t)$ is defined on generators by

$$\deg(e(\boldsymbol{x})) = (0,0),$$
$$\deg(\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})) = (-1,-1),$$
$$\deg(r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})) = (1-s_1, 1+s_0).$$

Remark 5.9 The relations in (iii) are from isotopies of stackings of disjoint decorated rook diagrams.

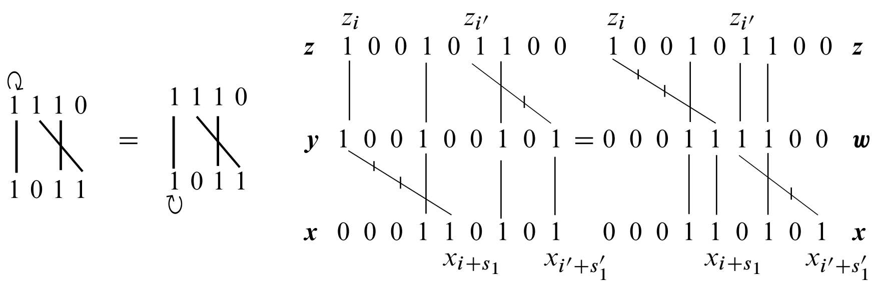


Figure 16: The diagrams for the second and third relations in (iii) on the left and right, respectively

Lemma 5.10 *We have that d is well defined and is a differential on R_n.*

Proof We use the geometric description of d in terms of resolving crossings and markings.

We show that d is well defined under the relations of R_n. Since the commutativity relations correspond to isotopies of stackings of disjoint rook diagrams, their resolutions commute as well. The pictorial proof of the invariance of the differential under the sliding relation (relation (iv)) is given in Figure 17. Resolutions of both sides are obviously the same under the sliding relation except for the resolution of the crossing over which the loop slides.

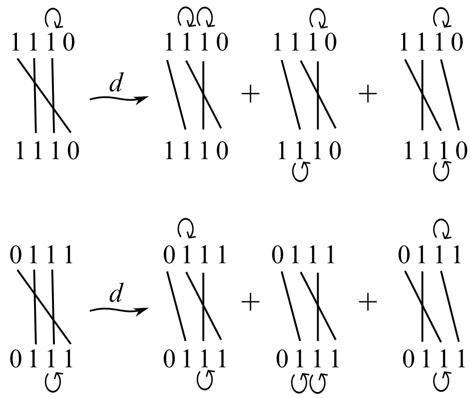


Figure 17: Differentials of both sides in the sliding relation

Next we verify that d is a differential. It is easy to check that d is of degree $(1,0)$. We have to show that $d^2(r)=0$ for any generator $r\in R_n$. In the expansion of $d^2(r)$, any term comes from a resolution of two of crossings and markings. If r has at least two crossings or markings, then the coefficient of each term is even since there are two ways to resolve them depending on the different orders of resolutions. Hence, $d^2=0$ since we are working in $\mathbb{F}_2$. If r has only one crossing or marking, then dr has no crossings or markings and $d(dr)=0$. □

Lemma 5.11 *The cohomology $H(R_n)$ is generated by idempotents $e(\boldsymbol{x})$, loops*

$$\rho(\boldsymbol{x}\xrightarrow{i}\boldsymbol{x})$$

and arrows

$$r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y})$$

without crossings or markings, ie $s_1=s_0(\boldsymbol{v})=0$.

Proof It is easy to see that $R_n=\bigoplus_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{B}_n}R_n(\boldsymbol{x},\boldsymbol{y})$, where $R_n(\boldsymbol{x},\boldsymbol{y})$ is the subspace of R_n generated by all the arrows from $\boldsymbol{x}$ to $\boldsymbol{y}$. It suffices to prove the lemma for $R_n(\boldsymbol{x},\boldsymbol{y})$. Since the differential $d=d_0+d_1$ can be decomposed into two differentials, we have a double complex $C=\bigoplus_{p,q}R_n(\boldsymbol{x},\boldsymbol{y})_{p,q}$, where $R_n(\boldsymbol{x},\boldsymbol{y})_{p,q}$ is the subspace of $R_n(\boldsymbol{x},\boldsymbol{y})$ generated by all

$$r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y})$$

with $s_1 = p, s_0 = q$, and the horizontal and vertical differentials are d_1 and d_0, respectively:

$$\begin{tikzcd}
R_n(\boldsymbol{x},\boldsymbol{y})_{0,2} \arrow[d, "d_0"] & R_n(\boldsymbol{x},\boldsymbol{y})_{1,2} \arrow[l, "d_1"'] \arrow[d, "d_0"] & \cdots \arrow[l, "d_1"'] \arrow[d, "d_0"] \\
R_n(\boldsymbol{x},\boldsymbol{y})_{0,1} \arrow[d, "d_0"] & R_n(\boldsymbol{x},\boldsymbol{y})_{1,1} \arrow[l, "d_1"'] \arrow[d, "d_0"] & \cdots \arrow[l, "d_1"'] \arrow[d, "d_0"] \\
R_n(\boldsymbol{x},\boldsymbol{y})_{0,0} & R_n(\boldsymbol{x},\boldsymbol{y})_{1,0} \arrow[l, "d_1"'] & \cdots \arrow[l, "d_1"']
\end{tikzcd}$$

Then the differential in the total complex $\mathrm{Tot}(C)$ is $d = d_0 + d_1$ in R_n. Note that the double complex C is finite since there are at most n crossings and markings in any arrow. Consider the two spectral sequences of C from the two filtrations which converge to the homology of $\mathrm{Tot}(C)$; see Weibel [40, Section 5.6]. Let $E^1_{p,q} = H^v_q(C_{p,*})$ and ${}'E^1_{p,q} = H^h_p(C_{*,q})$ be the first pages by taking the homology of the vertical differential d_0 and the horizontal differential d_1, respectively. We will show that

$$E^1_{p,q} = 0 \quad \text{for } q > 0, \quad {}'E^1_{p,q} = 0 \quad \text{for } p > 0.$$

Therefore, $H_{p+q}\,\mathrm{Tot}(C) = 0$ for $p+q > 0$, ie $H(R_n)$ is generated by loops and arrows without crossings or markings.

For E^1_{pq} with $q > 0$, suppose $d_0 r = 0$, where $r = \sum_i r_i \in C_{p,q}$ of a finite sum and each r_i is a product of elementary decorated rook diagrams given by a path ϕ_i in Q_n:

$$\boldsymbol{x} \to z^{i_1} \to \cdots \to z^{i_{j(i)}} \to \boldsymbol{y}$$

Assume further that r is primitive, ie any nontrivial partial sum of $\sum_i r_i$ is nonclosed. A path consisting of elementary decorated rook diagrams is called a *composition path*.

The key observation is that d_0 only locally changes a decorated rook diagram by replacing the marking α with the gap β, as shown in Figure 18. For each $r_i(s)$ in $d_0 r_i = \sum_s r_i(s)$, the corresponding path $\phi_{i(s)}$ is given by inserting one vertex of Q_n to ϕ_i. A composition path ϕ is called a *quotient path* of ϕ' if the set of vertices $V(\phi)$ in ϕ is a proper subset of $V(\phi')$. In particular, ϕ_i is a quotient path of $\phi_{i(s)}$ for all s. The goal is to find a universal path ϕ_r for r which is a quotient path of ϕ_i for all i.

We start from $r_1(1)$ which must be equal to $r_j(s)$ as a summand of $d_0 r_j$ for some $j \neq 1$ since r is d_0–closed. Without loss of generality, assume $r_1(1) = r_2(1)$. The composition path for any element in R_n is uniquely determined up to isotopies of disjoint diagrams and slides of loops over crossings. We choose a composition path $\phi_{1(1)}$ for $r_1(1)$ which in turn determines the composition paths ϕ_1 for r_1 and $\phi_{1(s)}$ for $r_1(s)$

for $s > 1$. Since $r_2(1) = r_1(1)$, we choose $\phi_{2(1)} = \phi_{1(1)}$ as the composition path for $r_2(1)$ which determines the composition paths ϕ_2 for r_2 and $\phi_{2(s)}$ for $r_2(s)$ for $s > 1$. Then both ϕ_1 and ϕ_2 are quotient paths of $\phi_{1(1)} = \phi_{2(1)}$. Let $\phi_{1,2}$ be the unique composition path such that $V(\phi_{1,2}) = V(\phi_1) \cap V(\phi_2)$.

We repeat the same procedure for $r_1(2)$. If $r_1(2) = r_2(s)$ for some s, then we move on to $r_1(3)$. Otherwise, assume $r_1(2) = r_3(1)$. We choose $\phi_{1(2)}$ as the composition path for $r_3(1)$. Then ϕ_3 and $\phi_{1,2}$ are both quotient paths of $\phi_{1(2)}$ since ϕ_1 is a quotient path of $\phi_{1(2)}$. Let $\phi_{1,2,3}$ be the unique composition path such that $V(\phi_{1,2,3}) = V(\phi_{1,2}) \cap V(\phi_3)$. Since we assume r is primitive, by iterating this procedure we can finally find a universal path ϕ_r for r such that ϕ_r is a quotient path of ϕ_i for all i. More precisely, there exists a decorated rook diagram $\Gamma(r)$ corresponding to ϕ_r such that each r_i is represented by a diagram $\Gamma(r_i)$ which is obtained from $\Gamma(r)$ by changing some of the markings to gaps.

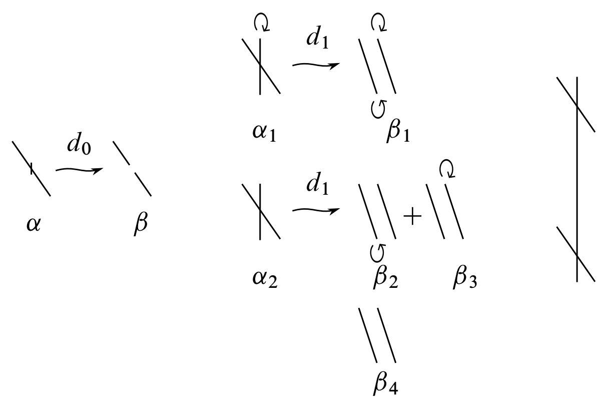


Figure 18: The left picture is W_0; the middle is W_1; the right is $W_1 \otimes_O W_1$.

We construct a *chain complex of markings* $C_0(r)$ for $r \in R_n$ such that $d_0(r) = 0$ as follows. Let

$$W_0 = \langle \alpha \rangle \xrightarrow{d_0} \langle \beta \rangle$$

be a chain complex of $\mathbb{F}_2$–vector spaces generated by a marking α and a gap β, where $\langle \alpha \rangle$ and $\langle \beta \rangle$ are in degree 1 and 0, respectively. The differential d_0 resolves a marking α and yields a gap β. The homology of (W_0, d_0) is zero. Note that (W_0, d_0) is the local model for one marking in $\Gamma(r)$. Let $\mathcal{I}$ be a finite set indexed by all markings in $\Gamma(r)$. Define the chain complex of markings $C_0(r)$ by $|\mathcal{I}|^{\text{th}}$ tensor product $W_0^{\otimes |\mathcal{I}|}$ of W_0 over $\mathbb{F}_2$. In other words, $C_0(r)$ encodes the information of all markings in $\Gamma(r)$. Then each $r_i \in R_n$ corresponds to a generator in the chain complex. The differential d_0 in R_n corresponds to the differential in $W_0^{\otimes |\mathcal{I}|}$.

We compute the homology of $W_0^{\otimes|\mathcal{I}|}$ in the following. Recall the Künneth formula from [40, Theorem 3.6.3]: If P and Q are right and left complexes of R–modules such that P_n and $d(P_n)$ are flat for each n, then there is an exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(P), H_q(Q)) \to 0.$$

The homology of $(W_0^{\otimes|\mathcal{I}|}, d_0)$ is zero by taking $R = \mathbb{F}_2$ and $P = Q = W_0$. It is easy to see that $r = \sum_i r_i$ corresponds a closed element in $W^{\otimes|\mathcal{I}|}$. Then there exists another element $w \in W^{\otimes|\mathcal{I}|}$ such that $d_0(w) = r$ since the homology of $(W_0^{\otimes|\mathcal{I}|}, d_0)$ is zero. Hence, there exists a corresponding element w in R_n such that $d_0(w) = r \in R_n$.

For ${}'E^1_{pq}$, the proof is similar to that for E^1_{pq}. A key difference is that the collection of local diagrams consists of 6 patterns $\{\alpha_i; \beta_j \mid i = 1, 2; j = 1, 2, 3, 4\}$ as in Figure 18. Let (W_1, d_1) be the chain complex of $\mathbb{F}_2$–vector spaces given by locally resolving a crossing,

$$\begin{aligned} \langle \alpha_1, \alpha_2 \rangle &\xrightarrow{d_1} \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle, \\ \alpha_1 &\mapsto \beta_1, \\ \alpha_2 &\mapsto \beta_2 + \beta_3, \end{aligned}$$

where α_i's are in degree 1 and β_j's are in degree 0. Then (W_1, d_1) is the local model for one crossing. Notice that (W_1, d_1) can be viewed as a chain complex of O–bimodules, where the ring $O = \langle 1, \rho \mid \rho^2 = 0 \rangle$ acts on W_1 by attaching a loop ρ to the decorated rook diagrams along the middle column:

$$\rho\alpha_2 = \alpha_2\rho = \alpha_1, \quad \rho\beta_4 = \beta_2, \quad \beta_4\rho = \beta_3, \quad \rho\beta_4\rho = \beta_1$$

Similarly, we construct a *chain complex of crossings* $C_1(r')$ as a tensor product of W_1's for any element $r' \in R_n$ such that $d_1(r') = 0$. The chain complex $C_1(r')$ is supposed to encode the information of all crossings in Γ' associated to r'. There are two types of tensor products in $C_1(r')$. The first one is a tensor product of two W_1's over O if the corresponding two crossings can be connected by a vertical strand as in Figure 18 since the loop ρ could slide over a crossing and along a vertical strand. Otherwise, we use a tensor product over $\mathbb{F}_2$.

It is easy to verify the conditions for $W_1 \otimes_O W_1$ and $W_1 \otimes_{\mathbb{F}_2} W_1$ in the Künneth formula. The homology $H_1(W_1)$ is zero at degree 1 and $H_0(W_1)$ is isomorphic to the ring O. Hence, $H_0(W_1)$ is free as left and right O modules and the Tor group in the Künneth formula vanishes. We have

$$H_n(W_1 \otimes_R W_1) \cong \bigoplus_{p+q=n} H_p(W_1) \otimes_R H_q(W_1),$$

where R is either O or $\mathbb{F}_2$ depending on the type of the tensor product. It follows that the homology of $C_1(r')$ is zero at degree greater than 0. Hence, ${}'E^1_{pq} = 0$ for $p > 0$ and we conclude the proof. $\square$

Remark 5.12 The first page E^1_{pq} given by the differential d_0 for resolving markings is very close to the strands algebra $\mathcal{A}(2n)$. They only differ at the relation of a double crossing which is set to be zero in the strands algebra.

Let $r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})$ denote the class $[r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})]$ with $s_1 = s_0(\boldsymbol{v}) = 0$ in the cohomology $H(R_n)$.

Proposition 5.13 *The cohomology $H(R_n)$ is an associative t–graded DG algebra with a trivial differential. It has idempotents $e(\boldsymbol{x})$ for each vertex $\boldsymbol{x}$ in Q_n, generators*

$$\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \quad \text{for each loop } \boldsymbol{x} \xrightarrow{i} \boldsymbol{x},$$
$$r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y}) \quad \text{for each arrow } \boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y},$$

with $s_1 + s_0(\boldsymbol{v}) = 0$ in Q_n. The relations consist of 4 groups:

(i) *idempotents*

$$e(\boldsymbol{x})\cdot e(\boldsymbol{y}) = \delta_{\boldsymbol{x},\boldsymbol{y}}\cdot e(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x},\boldsymbol{y}$$
$$e(\boldsymbol{x})\cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})\cdot e(\boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \quad \text{for all } \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$$
$$e(\boldsymbol{x})\cdot r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot e(\boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y}) \quad \text{for all } r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})$$

(ii) *unstackability relations* (R1)

$$\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})\cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = 0 \tag{R1-1}$$
$$r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot r(\boldsymbol{y} \xrightarrow{i} \boldsymbol{z}) = 0 \tag{R1-2}$$

(iii) *commutativity relations* (R2)

$$\rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x})\cdot \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) = \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})\cdot \rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x}) \quad \text{if } i' \neq i \tag{R2-1}$$
$$\rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x})\cdot r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y}) = r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot \rho(\boldsymbol{y} \xrightarrow{i'} \boldsymbol{y}) \quad \text{if } i' \neq i \tag{R2-2}$$
$$r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot r(\boldsymbol{y} \xrightarrow{i'} \boldsymbol{z}) = r(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{w})\cdot r(\boldsymbol{w} \xrightarrow{i} \boldsymbol{z}) \quad \text{if } x_i < z_{i'} \tag{R2-3}$$

(iv) *relation* (R3) *from the differential of a crossing:*

$$\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})\cdot r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot r(\boldsymbol{y} \xrightarrow{i+1} \boldsymbol{z}) = r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y})\cdot r(\boldsymbol{y} \xrightarrow{i+1} \boldsymbol{z})\cdot \rho(\boldsymbol{z} \xrightarrow{i+1} \boldsymbol{z}) \tag{R3}$$

if $z_{i+1} = x_i$

The t–graded DG algebra R_n is formal since its cohomology $H(R_n)$ is concentrated along the line $\deg_h - \deg_t = 0$ by Lemma 5.11.

Lemma 5.14 *The t–graded DG algebra R_n is quasi-isomorphic to its cohomology $H(R_n)$.*

Proof A quasi-isomorphism is given by

$$\begin{aligned} g_n\colon R_n &\to H(R_n), \\ e(\boldsymbol{x}) &\mapsto e(\boldsymbol{x}), \\ \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) &\mapsto \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}), \\ r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) &\mapsto r(\boldsymbol{x} \xrightarrow{i} \boldsymbol{y}) \qquad \text{if } s_1 = s_0(\boldsymbol{v}) = 0, \\ r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}) &\mapsto 0 \qquad\qquad\quad\;\; \text{otherwise.} \end{aligned}$$

□

Definition 5.15 (1) Let $\mathrm{DGP}(R_n)$ be the smallest full subcategory of $\mathrm{DG}(R_n)$ which contains the projective DG R_n–modules $\{P(\boldsymbol{x}) = R_n \cdot e(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor [1], the t–grading shift functor $\{1\}$ and taking mapping cones.

(2) Let $\mathrm{DGP}(H(R_n))$ be the smallest full subcategory of $\mathrm{DG}(H(R_n))$ which contains the projective DG $H(R_n)$–modules $\{PH(\boldsymbol{x}) = H(R_n) \cdot e(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor [1], the t–grading shift functor $\{1\}$ and taking mapping cones.

Let $\mathrm{HP}(R_n), \mathrm{HP}(H(R_n))$ denote 0^{th} homology categories of $\mathrm{DGP}(R_n), \mathrm{DGP}(H(R_n))$, respectively. Then $\mathrm{HP}(R_n), \mathrm{HP}(H(R_n))$ are triangulated categories. Since R_n is formal, it is easy to see the following equivalence of triangulated categories.

Lemma 5.16 *The triangulated categories $\mathrm{HP}(R_n)$ and $\mathrm{HP}(H(R_n))$ are equivalent. Thus there are isomorphisms of $\mathbb{Z}[t^{\pm 1}]$–modules: $K_0(\mathrm{HP}(R_n)) \cong K_0(\mathrm{HP}(H(R_n))) \cong \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}_n \rangle \cong V_1^{\otimes n}$.*

5.4 The t–graded DG algebra $A \boxtimes R_n$

Ideally, we want to use $\mathrm{DGP}(A \otimes R_n)$ of DG projective $A \otimes R_n$–modules to categorify $\boldsymbol{U}_T \otimes V_1^{\otimes n}$ and construct a $(R_n, A \otimes R_n)$–bimodule to categorify the $\boldsymbol{U}_T$–action. But this ideal approach does not work. We have to modify $A \otimes R_n$ to a DG algebra $A \boxtimes R_n$ by adding an extra differential which enables us to construct the DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n in Section 6. We show that $A \boxtimes R_n$ is formal, hence it is quasi-isomorphic to $A \otimes H(R_n)$. The definition of $A \boxtimes R_n$ is rather technical and the reader can pretend it is $A \otimes R_n$ at a first reading.

Definition 5.17 The algebra $A \boxtimes R_n$ is an associative t–graded DG $\mathbb{F}_2$–algebra with a differential d and a grading $\deg = (\deg_h, \deg_t) \in \mathbb{Z}^2$.

(A) The algebra $A \boxtimes R_n$ has generators of 3 types:

(1) $e(\Gamma) \boxtimes r$ and $a \boxtimes e(\boldsymbol{x})$ for $\Gamma \in \mathcal{B}, r \in R_n, a \in A, \boldsymbol{x} \in \mathcal{B}_n$

(2) $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$ for $1 \le i \le k < n$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ such that $x_i = n-k+i$

(3) $\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$ for $1 \le j \le k < n$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$

(B) The relations consist of 5 groups:

(1) relations from A and R_n: for $\Gamma \in \mathcal{B}, r_1, r_2 \in R_n, a_1, a_2 \in A, \boldsymbol{x} \in \mathcal{B}_n$,

$$\begin{aligned} e(\Gamma) \boxtimes (r_1 + r_2) &= e(\Gamma) \boxtimes r_1 + e(\Gamma) \boxtimes r_2 \\ e(\Gamma) \boxtimes (r_1 r_2) &= (e(\Gamma) \boxtimes r_1) \cdot (e(\Gamma) \boxtimes r_2) \\ (a_1 + a_2) \boxtimes e(\boldsymbol{x}) &= a_1 \boxtimes e(\boldsymbol{x}) + a_2 \boxtimes e(\boldsymbol{x}) \\ (a_1 a_2) \boxtimes e(\boldsymbol{x}) &= (a_1 \boxtimes e(\boldsymbol{x})) \cdot (a_2 \boxtimes e(\boldsymbol{x})) \end{aligned}$$

(2) commutativity relation from $A \otimes R_n$: $(a \boxtimes e(\boldsymbol{x})) \cdot (e(\Gamma_2) \boxtimes r) = (e(\Gamma_1) \boxtimes r) \cdot (a \boxtimes e(\boldsymbol{y}))$ for $e(\Gamma_1) \cdot a \cdot e(\Gamma_2) = a \in A, e(\boldsymbol{x}) \cdot r \cdot e(\boldsymbol{y}) = r \in R_n$, except when

$$(a, r) = \begin{cases} (\rho(I, EF), \rho(\boldsymbol{x} \xrightarrow{k} \boldsymbol{x})) & \text{if } x_k = n \\ (\rho(EF, I), \rho(\boldsymbol{x} \xrightarrow{1} \boldsymbol{x})) & \text{if } x_1 = 1 \end{cases} \tag{$**$}$$

(3) relations for $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$:

$$\begin{aligned} (e(I) \boxtimes e(\boldsymbol{x})) \cdot \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) &= \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \cdot (e(EF) \boxtimes e(\boldsymbol{x})) \\ &= \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \end{aligned}$$

$$\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \cdot (e(EF) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x})) = (e(I) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i'} \boldsymbol{x})) \cdot \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \tag{$*$}$$

if $i \neq i'+1$ or $x_{i'} \neq n-k+i'$

$$\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \cdot (e(EF) \boxtimes r(\boldsymbol{x} \xrightarrow{i', s_1, \boldsymbol{v}} \boldsymbol{y})) = (e(I) \boxtimes r(\boldsymbol{x} \xrightarrow{i', s_1, \boldsymbol{v}} \boldsymbol{y})) \cdot \rho(I\boldsymbol{y} \xrightarrow{i} EF\boldsymbol{y})$$

if $i' + s_1 < i$ and

$$\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \cdot (\rho(EF, I) \boxtimes e(\boldsymbol{x})) = 0$$

(4) relations for $\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$:

$$(e(EF)\boxtimes e(\boldsymbol{x}))\cdot\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x}) = \rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})\cdot(e(I)\boxtimes e(\boldsymbol{x})) = \rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$$

$$\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})\cdot(e(I)\boxtimes \rho(\boldsymbol{x} \xrightarrow{j'} \boldsymbol{x})) = (e(EF)\boxtimes \rho(\boldsymbol{x} \xrightarrow{j'} \boldsymbol{x}))\cdot\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$$

if $j \neq j'-1$ or $x_{j'} \neq j'$

$$\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})\cdot(e(I)\boxtimes r(\boldsymbol{x} \xrightarrow{j',s_1,\boldsymbol{v}} \boldsymbol{y})) = (e(EF)\boxtimes r(\boldsymbol{x} \xrightarrow{j',s_1,\boldsymbol{v}} \boldsymbol{y}))\cdot\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$$

if $j < j'$ and

$$(\rho(I,EF)\boxtimes e(\boldsymbol{x}))\cdot\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x}) = 0$$

(5) relations for $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$ and $\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$:

$$\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})\cdot\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x}) = 0$$

(C) The differential is defined on generators in the following and extended by the Leibniz rule:

(1) $$d(a\boxtimes e(\boldsymbol{x})) = 0 \text{ for } a\in A, \boldsymbol{x}\in\mathcal{B}_n$$

(2) $$d(e(\Gamma)\boxtimes r) = e(\Gamma)\boxtimes d(r) \quad \text{for } \Gamma\in\mathcal{B}, r\in R_n$$

(3) $$d(\rho(I\boldsymbol{x} \xrightarrow{k} EF\boldsymbol{x})) = (\rho(I,EF)\boxtimes e(\boldsymbol{x}))\cdot(e(EF)\boxtimes\rho(\boldsymbol{x} \xrightarrow{k} \boldsymbol{x})) + (e(I)\boxtimes\rho(\boldsymbol{x} \xrightarrow{k} \boldsymbol{x}))\cdot(\rho(I,EF)\boxtimes e(\boldsymbol{x}))$$

(4) $$d(\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})) = \rho(I\boldsymbol{x} \xrightarrow{i+1} EF\boldsymbol{x})\cdot(e(EF)\boxtimes\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})) + (e(I)\boxtimes\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}))\cdot(\rho(I\boldsymbol{x} \xrightarrow{i+1} EF\boldsymbol{x})) \quad \text{for } i<k$$

(5) $$d(\rho(EF\boldsymbol{x} \xrightarrow{1} I\boldsymbol{x})) = (\rho(EF,I)\boxtimes e(\boldsymbol{x}))\cdot(e(I)\boxtimes\rho(\boldsymbol{x} \xrightarrow{1} \boldsymbol{x})) + (e(EF)\boxtimes\rho(\boldsymbol{x} \xrightarrow{1} \boldsymbol{x}))\cdot(\rho(EF,I)\boxtimes e(\boldsymbol{x}))$$

(6) $$d(\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})) = \rho(EF\boldsymbol{x} \xrightarrow{j-1} I\boldsymbol{x})\cdot(e(I)\boxtimes\rho(\boldsymbol{x} \xrightarrow{j} \boldsymbol{x})) + (e(EF)\boxtimes\rho(\boldsymbol{x} \xrightarrow{j} \boldsymbol{x}))\cdot\rho(EF\boldsymbol{x} \xrightarrow{j-1} I\boldsymbol{x}) \quad \text{for } j>1$$

(D) The grading $\deg = (\deg_h, \deg_t)$ is defined for $a \in A, r \in R_{n,k}$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ by

$$\begin{aligned}\deg_h(a \boxtimes r) &= \deg_h(a) + \deg_h(r) + 2k \deg_t(a),\\ \deg_t(a \boxtimes r) &= n \deg_t(a) + \deg_t(r),\\ \deg(\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})) &= (-2(k-i+1), -(k-i+1)),\\ \deg(\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})) &= (2k+1-2j, n-j).\end{aligned}$$

Remark 5.18 (1) Equations (1) and (2) in part (C) are inherited from the differential on $A \otimes R_n$. Equations (3) and (5) correspond to the conditions in $(**)$ in part (B-2) where the commutativity relation fails. Equations (4) and (6) are higher homotopies.

(2) The condition $x_i = n-k+i$ for $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$, where $\boldsymbol{x} \in \mathcal{B}_{n,k}$, implies that $x_j = n-k+j$ for all $i \le j \le k$. In other words, all states between the $(n-k+i)^{\text{th}}$ state and the last state are $|1\rangle$. Similarly, the condition $x_j = j$ for

$$\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$$

implies that all states between the first state and j^{th} state are $|1\rangle$; see the proof of Lemma 5.20 for an example.

(3) The new ingredients $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}), \rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$ are only used in the construction of the $(H(R_n), A \boxtimes R_n)$–bimodules C_n in Section 6.4. A contact topological interpretation of them is still missing.

Lemma 5.19 *We have that d is well defined: d preserves the relations $d^2 = 0$,* $\deg(d) = (1,0)$.

Proof There is a decomposition of the differential $d = d_1 + d_2$, where d_1 is defined by Equations (1), (2) and extended to the other cases by zero, and d_2 is defined by Equations (3)–(6) and extended to the other cases by zero. It is easy to see that $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$. Since d_1 is inherited from the differential on R_n, it suffices to prove the lemma for d_2.

We verify that d_2 preserves $(*)$ in relation (B-3). Fixing $\boldsymbol{x} \in \mathcal{B}_{n,k}$, we set

$$\overline{\rho}_{k+1} := \rho(I, EF) \boxtimes e(\boldsymbol{x}), \quad \overline{\rho}_i := \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) \quad \text{if } x_i = n-k+i,$$
$$\rho(\Gamma, i) := e(\Gamma) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \quad \text{for } \Gamma = I, EF.$$

Then the differential is given by $d_2(\overline{\rho}_i) = \overline{\rho}_{i+1}\rho(EF, i) + \rho(I, i)\overline{\rho}_{i+1}$ for $i \le k$. Equation $(*)$ reads as $\overline{\rho}_i \rho(EF, i') = \rho(I, i')\overline{\rho}_i$ if $i \ne i'+1$. We apply d_2 to both sides and get

$$d_2(\overline{\rho}_i\rho(EF,i')) = \overline{\rho}_{i+1}\rho(EF,i)\rho(EF,i') + \rho(I,i)\overline{\rho}_{i+1}\rho(EF,i'),$$
$$d_2(\rho(I,i')\overline{\rho}_i) = \rho(I,i')\overline{\rho}_{i+1}\rho(EF,i) + \rho(I,i')\rho(I,i)\overline{\rho}_{i+1}.$$

Case 1 If $i \neq i'$, then $\overline{\rho}_{i+1}\rho(EF,i') = \rho(I,i')\overline{\rho}_{i+1}$ from $(*)$. We have

$$d_2(\overline{\rho}_i\rho(EF,i')) = \overline{\rho}_{i+1}\rho(EF,i)\rho(EF,i') + \rho(I,i)\rho(I,i')\overline{\rho}_{i+1} = d_2(\rho(I,i')\overline{\rho}_i).$$

Case 2 If $i = i'$, then $\rho(EF,i)\rho(EF,i') = \rho(I,i')\rho(I,i) = 0$ from relation (B-1). We have

$$d_2(\overline{\rho}_i\rho(EF,i')) = \rho(I,i)\overline{\rho}_{i+1}\rho(EF,i) = d_2(\rho(I,i')\overline{\rho}_i).$$

In either case, we proved that d_2 preserves $(*)$.

The proofs for other relations are similar and we leave it to the reader.

It suffices to show that $d_2^2(r) = 0$ for any generator $r \in A \boxtimes R_n$. The computation is similar to that above.

By definition, $\deg(e(\Gamma) \boxtimes \rho(\boldsymbol{x} \xrightarrow{j} \boldsymbol{x})) = (-1,-1)$ for $\Gamma = I, EF$ from Definition 5.8. For $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$, $i < k$ we have

$$\begin{aligned}\deg(d) &= (-1,-1) + \deg(\rho(I\boldsymbol{x} \xrightarrow{i+1} EF\boldsymbol{x})) - \deg(\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})) \\ &= (-1,-1) + (2,1) = (1,0).\end{aligned}$$

We have similar computations for other generators. □

We compute the cohomology $H(A \boxtimes R_n)$ and show that $A \boxtimes R_n$ is formal.

Lemma 5.20 *The t–graded DG algebra $A \boxtimes R_n$ is quasi-isomorphic to its cohomology $A \otimes H(R_n)$.*

Proof We first compute $H(A \boxtimes R_n)$. Note that $A \boxtimes R_n$ is a finite double complex with respect to $d = d_1 + d_2$ in previous lemma. Since the cohomology $H_{d_1}(A \otimes R_n)$ with respect to d_1 is $A \otimes H(R_n)$, the following claim implies that the cohomology $H_d(A \boxtimes R_n)$ is $A \otimes H(R_n)$.

Claim The cohomology $H_{d_2}(A \boxtimes R_n)$ with respect to d_2 is $A \otimes R_n$.

There is a decomposition

$$\begin{aligned}A \boxtimes R_n &= \bigoplus_{\Gamma_1,\Gamma_2 \in \mathcal{B}, \boldsymbol{x},\boldsymbol{y} \in \mathcal{B}_n} L(\Gamma_1,\Gamma_2;\boldsymbol{x},\boldsymbol{y}) \\ &= \bigoplus_{\Gamma_1,\Gamma_2 \in \mathcal{B}, \boldsymbol{x},\boldsymbol{y} \in \mathcal{B}_n} (e(\Gamma_1) \boxtimes e(\boldsymbol{x})) \cdot A \boxtimes R_n \cdot (e(\Gamma_2) \boxtimes e(\boldsymbol{y})).\end{aligned}$$

Since d_2 is nontrivial only on

$$\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}), \quad \rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x}),$$

it suffices to prove the claim for summands $L(\Gamma_1, \Gamma_2; \boldsymbol{x}, \boldsymbol{y})$ where we have that $(\Gamma_1, \Gamma_2) = (I, EF), (EF, I)$ and $\boldsymbol{x} = \boldsymbol{y}$.

We compute $H_{d_2}(L)$, where we have $L = L(\Gamma_1, \Gamma_2; \boldsymbol{x}, \boldsymbol{y})$ for $(\Gamma_1, \Gamma_2) = (EF, I)$, $\boldsymbol{x} = \boldsymbol{y} = |1101\rangle$. By definition there exist

$$\overline{\rho}_2 := \rho(EF|1101\rangle \xrightarrow{2} I|1101\rangle), \quad \overline{\rho}_1 := \rho(EF|1101\rangle \xrightarrow{1} I|1101\rangle).$$

Let

$$\overline{\rho}_0 := \rho(EF, I) \boxtimes e(|1101\rangle), \quad \rho(\Gamma, i) := e(\Gamma) \boxtimes \rho(|1101\rangle \xrightarrow{i} |1101\rangle)$$

for $\Gamma = I, EF$ and $i = 1, 2, 3$. The nontrivial differential on the generators is given by

$$d_2(\overline{\rho}_2) = \rho(EF, 2)\overline{\rho}_1 + \overline{\rho}_1\rho(I, 2), \quad d_2(\overline{\rho}_1) = \rho(EF, 1)\overline{\rho}_0 + \overline{\rho}_0\rho(I, 1).$$

From relations in (B-4), we have

$$\begin{aligned}
\rho(EF, i)\overline{\rho}_2 &= \overline{\rho}_2\rho(I, i) \quad \text{for } i = 1, 2, 3,\\
\rho(EF, i)\overline{\rho}_1 &= \overline{\rho}_1\rho(I, i) \quad \text{for } i = 1, 3,\\
\rho(EF, i)\overline{\rho}_0 &= \overline{\rho}_0\rho(I, i) \quad \text{for } i = 2, 3.
\end{aligned}$$

Then L is a complex $L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_2} L_0$, where $L_i = (A \boxtimes R_n)\overline{\rho}_i(A \boxtimes R_n)$. A direct computation shows that L_2 is a 8–dimensional $\mathbb{F}_2$–vector space, L_1 and L_0 are 16–dimensional $\mathbb{F}_2$–vector spaces. Moreover, the complex is exact except at L_0 and $H_{d_2}(L)$ is isomorphic to $\rho(EF, I) \otimes e(\boldsymbol{x})Ae(\boldsymbol{x})$ in $A \otimes R_n$ for $\boldsymbol{x} = |1101\rangle$.

The proof of the claim in general is similar and we leave it to the reader.

It is easy to see that the following map gives a quasi-isomorphism:

$$\begin{aligned}
A \boxtimes R_n &\to A \otimes R_n\\
e(\Gamma) \boxtimes r &\mapsto e(\Gamma) \otimes r\\
a \boxtimes e(\boldsymbol{x}) &\mapsto a \otimes e(\boldsymbol{x})\\
\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}) &\mapsto 0\\
\rho(EF\boldsymbol{x} \xrightarrow{i} I\boldsymbol{x}) &\mapsto 0
\end{aligned}$$

Then $A \boxtimes R_n$ is quasi-isomorphic to $A \otimes H(R_n)$ since R_n is formal. □

Consider projective DG $A \boxtimes R_n$–modules given by $\{P(\Gamma, x) = (A \boxtimes R_n)(e(\Gamma) \boxtimes e(x)) \mid \Gamma \in \mathcal{B}, x \in \mathcal{B}_n\}$, and projective DG $A \otimes H(R_n)$–modules given by $\{PH(\Gamma, x) = (A \otimes H(R_n))(e(\Gamma) \otimes e(x)) \mid \Gamma \in \mathcal{B}, x \in \mathcal{B}_n\}$.

Definition 5.21 (1) Let $\mathrm{DGP}(A \boxtimes R_n)$ denote the smallest full subcategory of $\mathrm{DG}(A \boxtimes R_n)$ which contains the projective DG $A \boxtimes R_n$–modules $\{P(\Gamma, x) \mid \Gamma \in \mathcal{B}, x \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor [1], the t–grading shift functor $\{1\}$ and taking mapping cones.

(2) Let $\mathrm{DGP}(A \otimes H(R_n))$ denote the smallest full subcategory of $\mathrm{DG}(A \otimes H(R_n))$ containing the projective DG $A \otimes H(R_n)$–modules $\{PH(\Gamma, x) \mid \Gamma \in \mathcal{B}, x \in \mathcal{B}_n\}$ and is closed under the cohomological grading shift functor [1], the t–grading shift functor $\{1\}$ and taking mapping cones.

Let $\mathrm{HP}(A \boxtimes R_n)$ and $\mathrm{HP}(A \otimes H(R_n))$ denote 0^{th} homology categories of $\mathrm{DGP}(A \boxtimes R_n)$ and $\mathrm{DGP}(A \otimes H(R_n))$, respectively. Then $\mathrm{HP}(A \boxtimes R_n)$ and $\mathrm{HP}(A \otimes H(R_n))$ are triangulated categories. Since $A \boxtimes R_n$ is formal, we have the following equivalence of triangulated categories.

Lemma 5.22 *The triangulated categories* $\mathrm{HP}(A \boxtimes R_n)$ *and* $\mathrm{HP}(A \otimes H(R_n))$ *are equivalent. Hence, there are isomorphisms of $\mathbb{Z}[t^{\pm 1}]$–modules:*

$$K_0(\mathrm{HP}(A \boxtimes R_n)) \cong K_0(\mathrm{HP}(A \otimes H(R_n))) \cong U_T \otimes_{\{T = t^n\}} V_1^{\otimes n}$$

Proof It is easy to see that $K_0(\mathrm{HP}(A \boxtimes R_n))$ is isomorphic to a quotient of

$$\mathbb{Z}[T^{\pm 1}]\langle \mathcal{B} \rangle \times \mathbb{Z}[t^{\pm 1}]\langle \mathcal{B}_n \rangle$$

by the relation $(\Gamma \cdot T, x) = (\Gamma, t^n x)$ for $\Gamma \in \mathcal{B}$ and $x \in \mathcal{B}_n$ from the t–grading in $A \boxtimes R_n$:

$$\deg_t(a \boxtimes r) = n \deg_t(a) + \deg_t(r)$$ □

Definition 5.23 Define a tensor product functor

$$\chi_n \colon \mathrm{HP}(A) \times \mathrm{HP}(H(R_n)) \to \mathrm{HP}(A \otimes H(R_n)),$$
$$M, M' \mapsto M \otimes M',$$

where the grading of $M \otimes M'$ is given by

$$\deg_t(m \otimes m') = n \deg_t(m) + \deg_t(m'),$$
$$\deg_h(m \otimes m') = \deg_h(m) + \deg_h(m') + 2k \deg_t(m)$$

for $m \in M$ and $m' \in M'$ in $\mathrm{HP}(H(R_{n,k}))$.

Remark 5.24 The grading on $M \otimes M'$ makes it into a t–graded DG $A \otimes H(R_n)$–module.

6 The t–graded DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n

In order to define a functor $\mathrm{DGP}(A \boxtimes R_n) \to \mathrm{DGP}(H(R_n))$, we construct the t–graded DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n in four steps.

(1) We define the first part of the left $H(R_n)$–module C_n corresponding to the categorical action of I, E, F on the objects of $\mathrm{DGP}(R_n)$ in Section 6.1.

(2) We define the first part of the right $A \boxtimes R_n$–module structure on C_n corresponding to the categorical action of I, E, F on the morphisms of $\mathrm{DGP}(R_n)$ in Section 6.2.

(3) We finish the construction of the left $H(R_n)$–module C_n corresponding to the action of EF in Section 6.3.

(4) We finish the definition of the right $A \boxtimes R_n$–module structure on C_n corresponding to the action of EF in Section 6.4.

The algebraic construction is quite technical, but the geometric presentation in terms of decorated rook diagrams is easy to follow.

6.1 The left DG $H(R_n)$–module C_n, Part I

As a left DG $H(R_n)$–module,

$$C_n = \bigoplus_{\Gamma \in \mathcal{B}, \boldsymbol{x} \in \mathcal{B}_n} C_n(\Gamma, \boldsymbol{x}).$$

In this subsection we define $C_n(\Gamma, \boldsymbol{x})$ for $\Gamma \in \{I, E, F\}$ and $\boldsymbol{x} \in \mathcal{B}_n$. We fix some $n > 0$ throughout this section and omit the subscript n.

6.1.1 The case $\Gamma = I$ Define $C(I, \boldsymbol{x}) = PH(\boldsymbol{x}) \in \mathrm{DGP}(H(R_n))$ for all $\boldsymbol{x} \in \mathcal{B}_n$.

6.1.2 The case $\Gamma = F$ For $\boldsymbol{x} \in \mathcal{B}_{n,k}$, recall the linear action $F(\boldsymbol{x})$ from Lemma 4.2. Define

$$C(F, \boldsymbol{x}) = \bigoplus_{j=1}^{n-k} C_j(F, \boldsymbol{x}) = \bigoplus_{j=1}^{n-k} PH((F\boldsymbol{x})_j)\{n - \overline{x}_j\}[\beta(\boldsymbol{x}, \overline{x}_j)],$$

where $(F\boldsymbol{x})_j$ denotes $\boldsymbol{x} \sqcup \{\overline{x}_j\}$ in $\mathcal{B}_{n,k+1}$. Define a differential

$$d(F,\boldsymbol{x}) = \sum_{j=2}^{n-k} d_j(F,\boldsymbol{x}),$$

where $d_j(F,\boldsymbol{x})\colon C_j(F,\boldsymbol{x}) \xrightarrow{\cdot r_F(\boldsymbol{x};j)} C_{j-1}(F,\boldsymbol{x})$ is defined below for $2 \le j \le n-k$.

Let $q_j = |\{l \in \{1,\dots,k\} \mid x_l < \overline{x}_j\}|$, for $1 \le j \le n-k$. Then the number of states $|1\rangle$ between $\overline{x}_{j-1}$ and $\overline{x}_j$ is $q_j - q_{j-1}$. Recall that $\boldsymbol{x} \xrightarrow[i]{} \boldsymbol{y}$ is the shorthand for the arrow $\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}$ with $s_1 = s_0(\boldsymbol{v}) = 0$ in the quiver Q_n. Then there exists a path

$$(F\boldsymbol{x})_j \xrightarrow{q_{j-1}+1} \boldsymbol{z}^1 \xrightarrow{q_{j-1}+2} \cdots \xrightarrow{q_j} \boldsymbol{z}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\boldsymbol{x})_{j-1}.$$

Define $r_F(\boldsymbol{x};j) \in H(R_{n,k+1})$ as the product of the corresponding $q_j - q_{j-1} + 1$ generators:

$$r_F(\boldsymbol{x};j) = r((F\boldsymbol{x})_j \xrightarrow{q_{j-1}+1} \boldsymbol{z}^1) \cdots r(\boldsymbol{z}^{q_j-q_{j-1}} \xrightarrow{q_j+1} (F\boldsymbol{x})_{j-1});$$

see Figure 19 for an example.

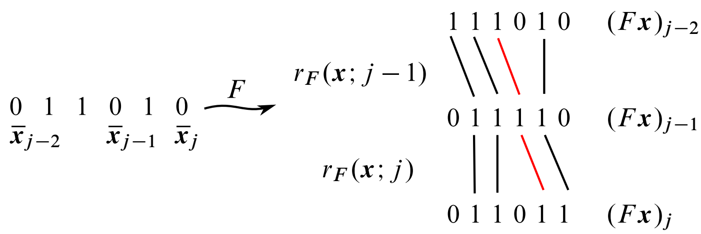


Figure 19: A local diagram of $r_F(\boldsymbol{x};j)\cdot r_F(\boldsymbol{x};j-1)$ equals 0 since a product of two red strands is zero in $H(R_n)$.

Definition 6.1 The differential $d_j(F,\boldsymbol{x})\colon C_j(F,\boldsymbol{x}) \xrightarrow{\cdot r_F(\boldsymbol{x};j)} C_{j-1}(F,\boldsymbol{x})$ is a map of left $H(R_n)$–modules defined by right multiplication with $r_F(\boldsymbol{x};j)$:

$$d_j(F,\boldsymbol{x})(m((F\boldsymbol{x})_j)) = r_F(\boldsymbol{x};j)\cdot m((F\boldsymbol{x})_{j-1})$$

Here $m((F\boldsymbol{x})_j) \in C_j(F,\boldsymbol{x}) = PH((F\boldsymbol{x})_j)\{n-\overline{x}_j\}[\beta(\boldsymbol{x},\overline{x}_j)]$ is the generator of the left projective $H(R_n)$–module for $1 \le j \le n-k$.

Remark 6.2 The definition of the left $H(R_n)$–module $C(F,\boldsymbol{x})$ comes from a projective resolution of the left $H(R_n)$–module which corresponds to the dividing set $F\cdot\boldsymbol{x} \in \widetilde{\mathcal{C}}_n$; see Figures 2, 4 and 5 in Section 1.3 for the dividing sets. For instance, there is a distinguished triangle $F|010\rangle \to |011\rangle \to |110\rangle$ in $\widetilde{\mathcal{C}}_3$ as in Figure 20 which gives an isomorphism $F|010\rangle \cong (|011\rangle \to |110\rangle)$.

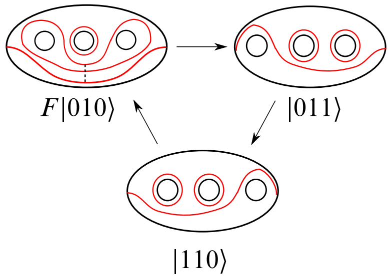


Figure 20

Lemma 6.3 *We have $d_j(F,\boldsymbol{x})$ is a map of degree $(1,0)$ and $d_{j-1}\circ d_j=0$.*

Proof The degrees of the generators are as follows:

$$\deg(m((F\boldsymbol{x})_j)) = (-\beta(\boldsymbol{x},\bar{x}_j),\bar{x}_j-n)$$
$$\deg(m((F\boldsymbol{x})_{j-1})) = (-\beta(\boldsymbol{x},\bar{x}_{j-1}),\bar{x}_{j-1}-n)$$
$$\deg(r_F(\boldsymbol{x};j)) = (q_j-q_{j-1}+1,q_j-q_{j-1}+1)$$

Since $\beta(\boldsymbol{x},\bar{x}_{j-1})-\beta(\boldsymbol{x},\bar{x}_j)=q_j-q_{j-1}$ and $(n-\bar{x}_{j-1})-(n-\bar{x}_j)=q_j-q_{j-1}+1$,

$$\deg(m((F\boldsymbol{x})_j))-\deg(m((F\boldsymbol{x})_{j-1}))=\deg(r_F(\boldsymbol{x};j))-(1,0),$$

which implies that $d_j(F,\boldsymbol{x})$ is a map of degree $(1,0)$.

For a diagrammatic proof of $d_{j-1}\circ d_j=0$, see Figure 19. The composition $d_{j-1}\circ d_j$ is right multiplication by $r_F(\boldsymbol{x};j)\cdot r_F(\boldsymbol{x};j-1)$ and is induced by the path

$$(F\boldsymbol{x})_j\xrightarrow{q_{j-1}+1}\boldsymbol{z}^1\to\cdots\to\boldsymbol{z}^{q_j-q_{j-1}}\xrightarrow{q_j+1}(F\boldsymbol{x})_{j-1}\xrightarrow{q_{j-2}+1}\boldsymbol{w}^1\to\cdots$$
$$\to\boldsymbol{w}^{q_{j-1}-q_{j-2}}\xrightarrow{q_{j-1}+1}(F\boldsymbol{x})_{j-2}.$$

By using the commutation relation Equation (R2-3) to rearrange the arrows, the path above can be written as

$$(F\boldsymbol{x})_j\xrightarrow{q_{j-2}+1}\boldsymbol{u}^1\to\cdots\to\boldsymbol{u}^{q_{j-1}-q_{j-2}}\xrightarrow{q_{j-1}+1}\boldsymbol{v}^0\xrightarrow{q_{j-1}+1}\boldsymbol{v}^1\to\cdots$$
$$\to\boldsymbol{v}^{q_j-q_{j-1}}\xrightarrow{q_j+1}(F\boldsymbol{x})_{j-2}.$$

Hence, $r_F(\boldsymbol{x};j)\cdot r_F(\boldsymbol{x};j-1)=\cdots r(\boldsymbol{u}^{q_{j-1}-q_{j-2}}\xrightarrow{q_{j-1}+1}\boldsymbol{v}^0)\cdot r(\boldsymbol{v}^0\xrightarrow{q_{j-1}+1}\boldsymbol{v}^1)\cdots=0$ by (R1-2). Here the product

$$r(\boldsymbol{u}^{q_{j-1}-q_{j-2}}\xrightarrow{q_{j-1}+1}\boldsymbol{v}^0)\cdot r(\boldsymbol{v}^0\xrightarrow{q_{j-1}+1}\boldsymbol{v}^1)=0$$

is given by a product of two red strands in Figure 19. Then we have $d_{j-1}\circ d_j=0$. □

6.1.3 The case $\Gamma = E$ For $x \in \mathcal{B}_{n,k}$, recall the linear action $E(x)$ from Lemma 4.2. Define

$$\begin{aligned} C(E,x) &= \bigoplus_{i=1}^{k} (C^i(E,x) \oplus C^i(E,x)') \\ &= \bigoplus_{i=1}^{k} (PH((Ex)^i)[1-i] \oplus PH((Ex)^i)\{1\}[2-i]), \end{aligned}$$

where $(Ex)^i$ denotes $x \backslash \{x_i\}$ in $\mathcal{B}_{n,k-1}$. Define a differential

$$d(E,x) = \sum_{i=1}^{k-1} d^i(E,x),$$

where $d^i(E,x)$: $C^i(E,x) \oplus C^i(E,x)' \to C^{i+1}(E,x) \oplus C^{i+1}(E,x)'$ is defined below.

Consider a path $(Ex)^i \xrightarrow{i} z^1 \xrightarrow{i} \cdots \xrightarrow{i} z^{x_{i+1}-x_i-1} \xrightarrow{i} (Ex)^{i+1}$ in $Q_{n,k-1}$. Define $r_E(x;i) \in H(R_{n,k-1})$ as the product of the generators corresponding to the $x_{i+1}-x_i$ arrows in the path and the $x_{i+1}-x_i-1$ loops attached at the vertices z^s:

$$\begin{aligned} r_E(x;i) = r((Ex)^i \xrightarrow{i} z^1) \cdot \rho(z^1 \xrightarrow{i} z^1) \cdot r(z^1 \xrightarrow{i} z^2) \cdots \\ \cdot r(z^{x_{i+1}-x_i-2} \xrightarrow{i} z^{x_{i+1}-x_i-1}) \cdot \rho(z^{x_{i+1}-x_i-1} \xrightarrow{i} z^{x_{i+1}-x_i-1}) \\ \cdot r(z^{x_{i+1}-x_i-1} \xrightarrow{i} (Ex)^{i+1}). \end{aligned}$$

Define loops

$$\theta(x;i) = \rho((Ex)^i \xrightarrow{i} (Ex)^i), \quad \sigma(x;i) = \rho((Ex)^{i+1} \xrightarrow{i} (Ex)^{i+1});$$

see Figure 21 for an example. Let $m((Ex)^i) \in C^i(E,x) = PH(x\backslash\{x_i\})[1-i]$ and $m'((Ex)^i) \in C^i(E,x)' = PH(x\backslash\{x_i\})\{1\}[2-i]$ be the generators of the left $H(R_n)$ modules for $1 \le i \le k$.

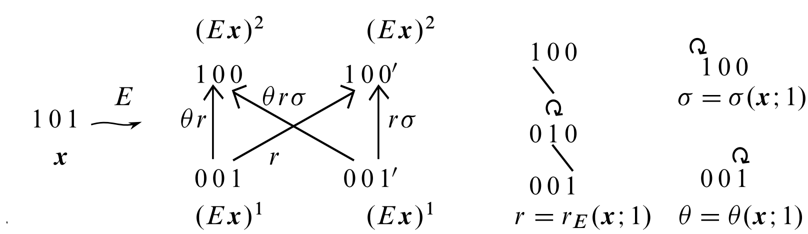


Figure 21: We have $C(E,x)$ for $x = |101\rangle$, where 4 upward arrows denote the differential.

Definition 6.4 The differential $d^i(E,\boldsymbol{x})\colon C^i(E,\boldsymbol{x})\oplus C^i(E,\boldsymbol{x})' \to C^{i+1}(E,\boldsymbol{x})\oplus C^{i+1}(E,\boldsymbol{x})'$ is a map of left $H(R_n)$–modules defined on the generators by

$$\begin{aligned} d^i(E,\boldsymbol{x})(m((E\boldsymbol{x})^i)) &= \theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot m((E\boldsymbol{x})^{i+1}) + r_E(\boldsymbol{x};i)\cdot m'((E\boldsymbol{x})^{i+1}),\\ d^i(E,\boldsymbol{x})(m'((E\boldsymbol{x})^i)) &= \theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot \sigma(\boldsymbol{x};i)\cdot m((E\boldsymbol{x})^{i+1})\\ &\qquad + r_E(\boldsymbol{x};i)\cdot\sigma(\boldsymbol{x};i)\cdot m'((E\boldsymbol{x})^{i+1}). \end{aligned}$$

Lemma 6.5 *We have $d(E,\boldsymbol{x})$ is a map of degree $(1,0)$ and $d^{i+1}\circ d^i = 0$.*

Proof It is easy to verify that $d(E,\boldsymbol{x})$ is a map of degree $(1,0)$ since

$$\deg(r_E(\boldsymbol{x};i)) = (1,1),\quad \deg(\theta(\boldsymbol{x};i)) = (-1,-1),\quad \deg(\sigma(\boldsymbol{x};i)) = (-1,-1).$$

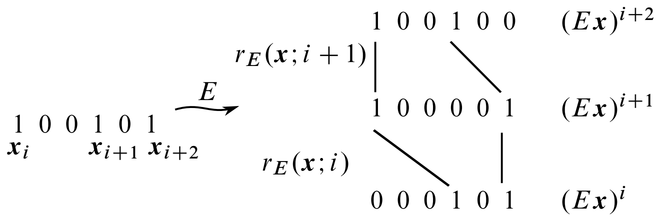
We show that $d^{i+1}(d^i(m((E\boldsymbol{x})^i))) = 0$ and leave the case of $m'((E\boldsymbol{x})^i))$ to the reader:

$$\begin{aligned} &d^{i+1}(d^i(m((E\boldsymbol{x})^i)))\\ &\quad= d^{i+1}(\theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot m((E\boldsymbol{x})^{i+1}) + r_E(\boldsymbol{x};i)\cdot m'((E\boldsymbol{x})^{i+1}))\\ &\quad= \theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot d^{i+1}(m((E\boldsymbol{x})^{i+1})) + r_E(\boldsymbol{x};i)\cdot d^{i+1}(m'((E\boldsymbol{x})^{i+1}))\\ &\quad= (\theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i+1) + r_E(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i+1)\cdot\sigma(\boldsymbol{x};i+1))\\ &\qquad\qquad \cdot m'((E\boldsymbol{x})^{i+2}) + \theta(\boldsymbol{x};i)\cdot r_E(\boldsymbol{x};i)\cdot\theta(\boldsymbol{x};i+1)\cdot r_E(\boldsymbol{x};i+1)\cdot m((E\boldsymbol{x})^{i+2})\\ &\qquad\qquad\qquad + r_E(\boldsymbol{x};i)\cdot\theta(\boldsymbol{x};i+1)\cdot r_E(\boldsymbol{x};i+1)\cdot\sigma(\boldsymbol{x};i+1)\cdot m((E\boldsymbol{x})^{i+2}) \end{aligned}$$

We compute the coefficient of $m'((E\boldsymbol{x})^{i+2})$ in Figure 22. The coefficient is zero by (R3). Similarly, we can prove that the coefficient of $m((E\boldsymbol{x})^{i+2})$ is zero by (R2-1), (R2-2) and (R3). Hence $d^{i+1}(d^i(m((E\boldsymbol{x})^i))) = 0$. □

6.2 The right $A\boxtimes R_n$–module C_n, Part I

In this subsection we define the right multiplication with the idempotents $e(\Gamma)\boxtimes e(\boldsymbol{x})$ and generators $e(\Gamma)\boxtimes\rho(\boldsymbol{x}\xrightarrow{i}\boldsymbol{x}), e(\Gamma)\boxtimes r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y})$ of $A\boxtimes R_n$ for $\Gamma\in\{I,E,F\}$ and $\boldsymbol{x}\in\mathcal{B}_n$. Let $m\times r$ denote the right multiplication for $m\in C$ and generators $r\in A\boxtimes R_n$. The definition of right multiplication in general is extended by the associativity: $m\times(r_1\cdot r_2) = (m\times r_1)\times r_2$. We will check this is well defined in Proposition 6.25. The case by case definition will be labeled by (M$*$). Let $m\cdot r'$ denote the multiplication in $H(R_n)$ for $m\in PH(\boldsymbol{x})\subset H(R_n)$ and $r'\in H(R_n)$. Let $j(\boldsymbol{x},i)$ be the number in $\{0,1,\ldots,n-k\}$ such that $\overline{x}_{j(\boldsymbol{x},i)} < x_i < \overline{x}_{j(\boldsymbol{x},i)+1}$ for $\boldsymbol{x}\in\mathcal{B}_{n,k}$. Let j_0 denote $j(\boldsymbol{x},i)$ when $\boldsymbol{x}$ and i are understood.



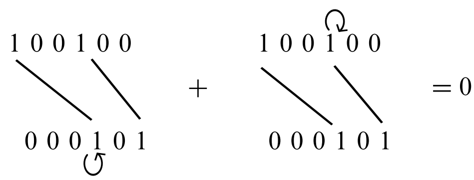


Figure 22: The top diagram describes $(Ex)^i$ and the bottom diagram represents the coefficient $\theta(x;i)\cdot r_E(x;i)\cdot r_E(x;i+1)+r_E(x;i)\cdot r_E(x;i+1)\cdot \sigma(x;i+1)$. Here, the diagrams representing $r_E(x;i)$'s are defined in Figure 13 in Section 5.1.4.

6.2.1 Idempotents Let $a\boxtimes r = e(\Gamma)\boxtimes e(x)$ be an idempotent. Then define for $m\in C(\Gamma', x')$

(M1) $$m\times(e(\Gamma)\boxtimes e(x)) = \delta_{\Gamma,\Gamma'}\delta_{x,x'}m.$$

6.2.2 The case $a = e(I)$ For $a\boxtimes r = e(I)\boxtimes r$, where $r\in\{\rho(x\xrightarrow{i}x), r(x\xrightarrow{i,s_1,v}y)\}$, define

(M2) $$m\times(e(I)\boxtimes r) = \begin{cases} m\cdot\rho(x\xrightarrow{i}x) & \text{if } r=\rho(x\xrightarrow{i}x),\\ m\cdot r(x\xrightarrow{i}y) & \text{if } r = r(x\xrightarrow{i,s_1,v}y), s_1 = s_0(v)=0,\\ 0 & \text{otherwise,}\end{cases}$$

for $m\in C(I,x)=PH(x)$. We view $\bigoplus_{x\in\mathcal{B}_n} C(I,x) = H(R_n)$ as an $(H(R_n),R_n)$–bimodule, where R_n acts from right via the quasi-isomorphism $g_n: R_n\to H(R_n)$ in Lemma 5.14.

6.2.3 The case $a = e(F)$ Let $a\boxtimes r = e(F)\boxtimes\rho(x\xrightarrow{i}x)$. The right multiplication is a map of left $H(R_n)$–modules $C(F,x)\to C(F,x)$ defined on the generators by

(M3-1) $$m((Fx)_j)\times(e(F)\boxtimes\rho(x\xrightarrow{i}x)) = \begin{cases}\rho((Fx)_j\xrightarrow{i}(Fx)_j)\cdot m((Fx)_j) & \text{if } j>j_0,\\ \rho((Fx)_j\xrightarrow{i+1}(Fx)_j)\cdot m((Fx)_j) & \text{if } j\le j_0.\end{cases}$$

Remark 6.6 The morphism

$$e(F) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}) \in \mathrm{Hom}_{\widetilde{\mathcal{C}}_n}(F \cdot \boldsymbol{x}, F \cdot \boldsymbol{x})$$

represents a tight contact structure from the dividing curve $F \cdot \boldsymbol{x}$ to itself. Recall from Remark 6.2 that $(C(F,\boldsymbol{x}), d(F,\boldsymbol{x}))$ is the "projective resolution" of $F \cdot \boldsymbol{x}$. Then the right multiplication $C(F,\boldsymbol{x}) \to C(F,\boldsymbol{x})$ defined above is the corresponding morphism between their projective resolutions.

Lemma 6.7 *The right multiplication with $u = e(F) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$ is compatible with the relation in $A \boxtimes R_n$: $(m \times u) \times u = 0 = m \times (u \cdot u) = 0$.*

Proof It follows from (M3-1) and $\rho((F\boldsymbol{x})_j \xrightarrow{i} (F\boldsymbol{x})_j) \cdot \rho((F\boldsymbol{x})_j \xrightarrow{i} (F\boldsymbol{x})_j) = 0$ for all i and j. □

Lemma 6.8 *The right multiplication by $e(F) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$ commutes with the differential.*

Proof The commutativity for each square which is not in the diagram below follows from the commutativity relation (R2-2) since the corresponding decorated rook diagrams are disjoint:

$$\begin{tikzcd}
\cdots \arrow[r] & PH((F\boldsymbol{x})_{j_0+1}) \arrow[r, "d"] & PH((F\boldsymbol{x})_{j_0}) \arrow[r] & \cdots \\
\cdots \arrow[r] & PH((F\boldsymbol{x})_{j_0+1}) \arrow[r, "d"] \arrow[u, "\cdot\rho((F\boldsymbol{x})_{j_0+1} \xrightarrow{i} (F\boldsymbol{x})_{j_0+1})"] & PH((F\boldsymbol{x})_{j_0}) \arrow[r] \arrow[u, "\cdot\rho((F\boldsymbol{x})_{j_0} \xrightarrow{i+1} (F\boldsymbol{x})_{j_0})"'] & \cdots
\end{tikzcd}$$

The commutativity for the square follows from (R3) since the sum of maps is a resolution of a crossing in R_n, hence zero in $H(R_n)$; see Figure 23. □

Remark 6.9 The right multiplication with $e(F) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$ as in the left-hand diagram in Figure 23 can be viewed as the functor F applying to the morphism $\rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$ in $\mathrm{Hom}(\boldsymbol{x}, \boldsymbol{x})$. From now on, we will omit labels $\boldsymbol{x}, (F\boldsymbol{x})_j, r_F(\boldsymbol{x}; i)$'s in this type of diagram to express right multiplications.

Let $a \boxtimes r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_1 = s_0(\boldsymbol{v}) = 0$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_{n,k}$. Note that

$$\overline{y}_{j_0} = \overline{x}_{j_0} + 1, \quad \overline{y}_j = \overline{x}_j \quad \text{for } j \neq j_0.$$

Then we have $(F\boldsymbol{x})_{j_0} = (F\boldsymbol{y})_{j_0} \in \mathcal{B}_{n,k+1}$ and there exist arrows in $\mathcal{Q}_{n,k+1}$,

$$(F\boldsymbol{x})_j \xrightarrow{i} (F\boldsymbol{y})_j \text{ for } j > j_0, \quad (F\boldsymbol{x})_j \xrightarrow{i+1} (F\boldsymbol{y})_j \text{ for } j < j_0.$$

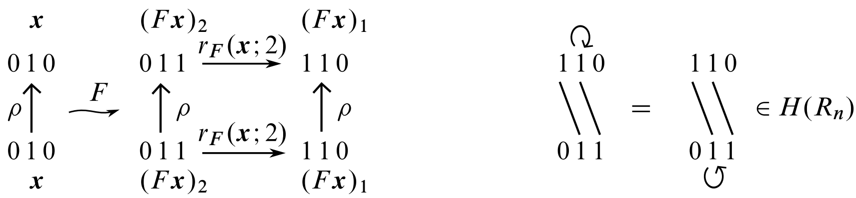


Figure 23: In the left-hand diagram, the upward arrow on the left is $\rho(x \xrightarrow{1} x)$ for $x = |010\rangle$. The horizontal arrows in $|011\rangle \to |110\rangle$ are the differential in $C(F, |010\rangle)$ given by the right multiplication with $r_F(x, 2)$. Two upward arrows between $|011\rangle \to |110\rangle$'s are the right multiplication with $e(F) \boxtimes \rho(x \xrightarrow{i} x)$. The right-hand diagram shows that the right multiplication and differential commute.

The right multiplication is a map of left $H(R_n)$–modules: $C(F,x) \to C(F,y)$ defined on the generators by

$$\text{(M3-2)}\quad m((Fx)_j) \times (e(F) \boxtimes r(x \xrightarrow{i,s_1,v} y)) = \begin{cases} r((Fx)_j \xrightarrow{i} (Fy)_j) \cdot m((Fy)_j) & \text{if } j > j_0, \\ m((Fy)_j) & \text{if } j = j_0, \\ r((Fx)_j \xrightarrow{i+1} (Fy)_j) \cdot m((Fy)_j) & \text{if } j < j_0; \end{cases}$$

see the left-hand diagram in Figure 24 for an example.

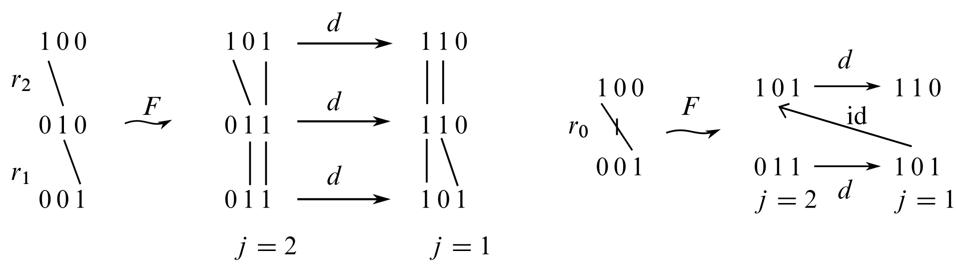


Figure 24: The right multiplications with $e(F) \boxtimes r_i$ for $i = 1, 2$ on the left, and for $i = 0$ on the right.

Lemma 6.10 *The right multiplication with $e(F) \boxtimes r(x \xrightarrow{i,s_1,v} y)$ such that $s_1 = s_0(v) = 0$ commutes with the differential.*

Proof We have the following diagram:

$$\begin{tikzcd}
PH((F\boldsymbol{y})_{j_0+1}) \arrow[r, "d"] & PH((F\boldsymbol{y})_{j_0}) \arrow[r, "d"] & PH((F\boldsymbol{y})_{j_0-1}) \\
PH((F\boldsymbol{x})_{j_0+1}) \arrow[r, "d"] \arrow[u, "\cdot r((F\boldsymbol{x})_{j_0+1}\xrightarrow{i}(F\boldsymbol{y})_{j_0+1})"] & PH((F\boldsymbol{x})_{j_0}) \arrow[r, "d"] \arrow[u, "\mathrm{id}"'] & PH((F\boldsymbol{x})_{j_0-1}) \arrow[u, "\cdot r((F\boldsymbol{x})_{j_0-1}\xrightarrow{i+1}(F\boldsymbol{y})_{j_0-1})"']
\end{tikzcd}$$

The commutativity follows from the commutativity relation (R2-3). □

Let $a \boxtimes r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_1 = 0, s_0(\boldsymbol{v}) > 0$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_{n,k}$. Let s_0 denote $s_0(\boldsymbol{v})$ for simplicity. Note that $(F\boldsymbol{x})_{j_0-s_0} = (F\boldsymbol{y})_{j_0} \in \mathcal{B}_{n,k+1}$. Then the right multiplication is a map of left $H(R_n)$–modules defined on the generators by

$$\text{(M3-3)} \quad m((F\boldsymbol{x})_j) \times (e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})) = \begin{cases} m((F\boldsymbol{y})_{j+s_0}) & \text{if } j = j_0 - s_0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $m((F\boldsymbol{y})_{j+s_0}) = m((F\boldsymbol{y})_{j_0}) = (F\boldsymbol{x})_j$ if $j = j_0 - s_0$; see the right-hand diagram in Figure 24 for an example when $s_0 = 1$.

We verify that the definition is compatible with the DG structure on $A \boxtimes R_n$.

Lemma 6.11 *We have $d(m \times r) = dm \times r + m \times dr$ holds for $r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,0,\boldsymbol{v}} \boldsymbol{y})$.*

Proof The case for $s_0 = 0$ is proved in Lemma 6.10.

If $s_0 = 1$, there is only one marking in $r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,0,(1)} \boldsymbol{y})$. Let

$$dr = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,0,(0)} \boldsymbol{z}) \cdot e(F) \boxtimes r(\boldsymbol{z} \xrightarrow{i,0,(0)} \boldsymbol{y})$$

for some $\boldsymbol{z}$.

We first discuss $m \in C_j(F, \boldsymbol{x})$ for $j < j_0 - 1$ or $j > j_0$, where the j^{th} vertical strand is disjoint from the marking. We have $m \times r = dm \times r = 0$ from (M3-3). The lemma follows from

$$m \times dr = \begin{cases} m \cdot r((F\boldsymbol{x})_j \xrightarrow{i} (F\boldsymbol{z})_j) \cdot r((F\boldsymbol{z})_j \xrightarrow{i} (F\boldsymbol{y})_j) \\ \qquad\qquad = m \cdot 0 = 0 \in H(R_n) \quad \text{if } j > j_0, \\ m \cdot r((F\boldsymbol{x})_j \xrightarrow{i+1} (F\boldsymbol{z})_j) \cdot r((F\boldsymbol{z})_j \xrightarrow{i+1} (F\boldsymbol{y})_j) \\ \qquad\qquad = 0 \in H(R_n) \quad \text{if } j < j_0 - 1. \end{cases}$$

From the above we can reduce to the local model: $j = j_0 - 1, j_0$ where the marking lives. A diagrammatic proof for

$$r_0 = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{1,0,(1)} \boldsymbol{y})$$

is given in Figure 24, where $\boldsymbol{x} = (3) = |001\rangle$, $\boldsymbol{y} = (1) = |100\rangle \in \mathcal{B}_{3,1}$. Recall that

$$d(r_0) = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{1,0,(0)} \boldsymbol{z}) \cdot e(F) \boxtimes r(\boldsymbol{z} \xrightarrow{1,0,(0)} \boldsymbol{y}) = r_1 \cdot r_2,$$

where $\boldsymbol{z} = (2) = |010\rangle$. In Figure 24, the right multiplications by r_1, r_2 and r_0 are given in the left-hand and right-hand diagrams, respectively.

We verify the equation for $m = m(|011\rangle) \in C(F, |001\rangle)$ by chasing the diagrams and leave other cases to the reader. The right-hand side of the equation is zero since

$$\begin{aligned} d(m(|011\rangle)) \times r_0 &= r(|011\rangle \xrightarrow{1} |101\rangle) \cdot (m(|101\rangle) \times r_0) \\ &= r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(F, |100\rangle), \\ m(|011\rangle) \times d(r_0) &= (m(|011\rangle) \times r_1) \times r_2 = m(|011\rangle) \times r_2 \\ &= r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(F, |100\rangle). \end{aligned}$$

The left-hand side is obviously zero since $m(|011\rangle) \times r_0 = 0$. Hence we proved

$$d(m(|011\rangle) \times r_0) = m(|011\rangle) \times dr_0 + dm(|011\rangle) \times r_0.$$

If $s_0 > 1$, for $m \in C_j(F, \boldsymbol{x})$, $j < j_0 - s_0$ or $j > j_0$ we have $m \times r = dm \times r = 0$ and $m \times dr = 0$ from (M3-3) since dr contains at least one marking.

Hence we reduce to the local model: $j_0 - s_0 \le j \le j_0$, ie $r_0 = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{1,0,(s_0)} \boldsymbol{y})$, where $\boldsymbol{x} = (s_0+2) = |0\cdots01\rangle$, $\boldsymbol{y} = (1) = |10\cdots0\rangle \in \mathcal{B}_{s_0+2,1}$. A diagrammatic proof for $s_0 = 2$ is given in Figure 25 by chasing the diagram. The proof for the case $s_0 > 1$ in general is similar and we leave it to the reader.

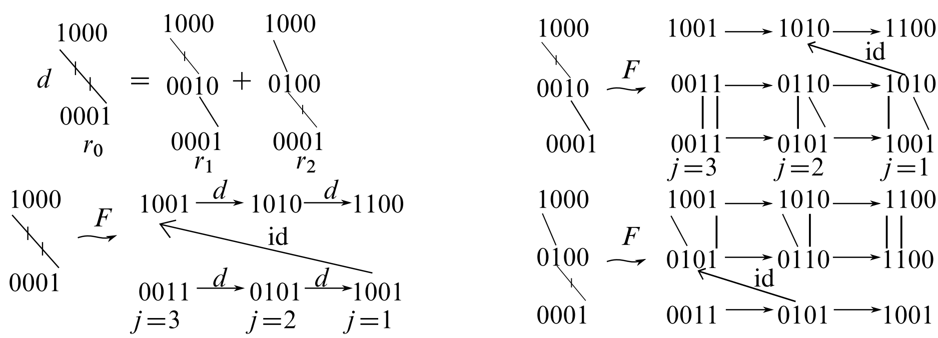


Figure 25: The right multiplications with $e(F) \boxtimes r_i$ for $i = 0$ on the left, and for $i = 1, 2$ on the right

Let $a \boxtimes r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_1 > 0$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_{n,k}$. The right multiplication is defined as the zero map

$$m \times (e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})) = 0. \tag{M3-4}$$

This completes the proof. □

6.2.4 The case $a = e(E)$ Let $a \boxtimes r = e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})$. The right multiplication is a map of left $H(R_n)$–modules: $C(E, \boldsymbol{x}) \to C(E, \boldsymbol{x})$ defined on the generators by

$$\text{(M4-1)} \qquad m((E\boldsymbol{x})^i) \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})) = \begin{cases} \rho((E\boldsymbol{x})^i \xrightarrow{i_0-1} (E\boldsymbol{x})^i) \cdot m((E\boldsymbol{x})^i) & \text{if } i < i_0, \\ m'((E\boldsymbol{x})^i) & \text{if } i = i_0, \\ \rho((E\boldsymbol{x})^i \xrightarrow{i_0} (E\boldsymbol{x})^i) \cdot m((E\boldsymbol{x})^i) & \text{if } i > i_0; \end{cases}$$

$$\text{(M4-1)} \qquad m'((E\boldsymbol{x})^i) \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})) = \begin{cases} \rho((E\boldsymbol{x})^i \xrightarrow{i_0-1} (E\boldsymbol{x})^i) \cdot m'((E\boldsymbol{x})^i) & \text{if } i < i_0, \\ 0 & \text{if } i = i_0, \\ \rho((E\boldsymbol{x})^i \xrightarrow{i_0} (E\boldsymbol{x})^i) \cdot m'((E\boldsymbol{x})^i) & \text{if } i > i_0. \end{cases}$$

An example is given in Figure 26; $|011\rangle$ denotes the first summand $PH((E|111\rangle)^1)$ and $|011\rangle'$ denotes the second summand $PH((E|111\rangle)^1)\{1\}[1]$ in $C(E, |111\rangle)$.

$$\begin{tikzcd}
111 & & 011 & \oplus & 011' \arrow[r, "d"] & 101 & \oplus & 101' \arrow[r, "d"] & 110 & \oplus & 110' \\
{} \arrow[r, squiggly, "E"] & {} & & & & & & & & & \\
111 \arrow[uu, "\rho"] & & 011 \arrow[uu, "\rho"] & \oplus & 011' \arrow[uu, "\rho"] \arrow[r, "d"] & 101 \arrow[uurr, "\mathrm{id}"] & \oplus & 101' \arrow[r, "d"] & 110 \arrow[uu, "\rho"'] & \oplus & 110' \arrow[uu, "\rho"'] \\
& & & i=1 & & & i=2 & & & i=3 &
\end{tikzcd}$$

Figure 26: The 5 upward arrows on the right are the right multiplication with $e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{2} \boldsymbol{x})$ for $\boldsymbol{x} = |111\rangle$.

Lemma 6.12 *The right multiplication with $u = e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})$ is compatible with the relation in $A \boxtimes R_n$: $(m \times u) \times u = 0 = m \times (u \cdot u)$.*

Proof For $m \in C^i(E, \boldsymbol{x})$, it follows from $\rho((E\boldsymbol{x})_i \xrightarrow{i'} (E\boldsymbol{x})_i) \cdot \rho((E\boldsymbol{x})_i \xrightarrow{i'} (E\boldsymbol{x})_i) = 0$ and (M4-1) if $i \neq i_0$. If $i = i_0$, it follows from the local model as in Figure 26. □

Lemma 6.13 *The right multiplication by $e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})$ commutes with the differential.*

Proof It suffices to prove that

$$d(m \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x}))) = d(m) \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x})),$$

for $m = m((E\boldsymbol{x})^i), m'((E\boldsymbol{x})^i)$ for $1 \le i \le k$. The equation is obviously true for $i \ne i_0 - 1, i_0$ from the commutativity relations (R2-1) and (R2-2) since the decorated rook diagrams in the differential and right multiplication are disjoint.

We verify the equation for $m = m((E\boldsymbol{x})^{i_0})$:

$$\begin{aligned}
&d(m((E\boldsymbol{x})^{i_0}) \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x}))) \\
&\quad = d(m'((E\boldsymbol{x})^{i_0})) \\
&\quad = \theta(\boldsymbol{x}; i_0) \cdot r_E(\boldsymbol{x}; i_0) \cdot \sigma(\boldsymbol{x}; i_0) \cdot m((E\boldsymbol{x})^{i_0+1}) + r_E(\boldsymbol{x}; i_0) \cdot \sigma(\boldsymbol{x}; i_0) \cdot m'((E\boldsymbol{x})^{i_0+1}) \\
&\quad = d(m((E\boldsymbol{x})^{i_0})) \cdot \rho((E\boldsymbol{x})^{i_0+1} \xrightarrow{i_0} (E\boldsymbol{x})^{i_0+1}) \\
&\quad = d(m((E\boldsymbol{x})^{i_0})) \times (e(E) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i_0} \boldsymbol{x}))
\end{aligned}$$

The proof for other cases is similar and we leave it to the reader. □

Let $a \boxtimes r = e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i_0, s_1, \boldsymbol{v}} \boldsymbol{y})$ with $s_1 = s_0(\boldsymbol{v}) = 0$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_{n,k}$. Note that

$$y_{i_0} = x_{i_0} - 1, \quad y_i = x_i \quad \text{for } i \ne i_0.$$

We have $(E\boldsymbol{x})^{i_0} = (E\boldsymbol{y})^{i_0} \in \mathcal{B}_{n,k-1}$ and there exist arrows

$$(E\boldsymbol{x})^i \xrightarrow{i_0 - 1} (E\boldsymbol{y})^i \quad \text{for } i < i_0, \quad (E\boldsymbol{x})^i \xrightarrow{i_0} (E\boldsymbol{y})^i \quad \text{for } i > i_0.$$

Then the right multiplication is a map of left $H(R_n)$–modules defined on the generators by

(M4-2) $$m((E\boldsymbol{x})^i) \times (e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i_0, s_1, \boldsymbol{v}} \boldsymbol{y})) = \begin{cases} r((E\boldsymbol{x})^i \xrightarrow{i_0 - 1} (E\boldsymbol{y})^i) \cdot m((E\boldsymbol{y})^i) & \text{if } i < i_0, \\ 0 & \text{if } i = i_0, \\ r((E\boldsymbol{x})^i \xrightarrow{i_0} (E\boldsymbol{y})^i) \cdot m((E\boldsymbol{y})^i) & \text{if } i > i_0, \end{cases}$$

(M4-2) $$m'((E\boldsymbol{x})^i) \times (e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i_0, s_1, \boldsymbol{v}} \boldsymbol{y})) = \begin{cases} r((E\boldsymbol{x})^i \xrightarrow{i_0 - 1} (E\boldsymbol{y})^i) \cdot m'((E\boldsymbol{y})^i) & \text{if } i < i_0, \\ m((E\boldsymbol{y})^i) & \text{if } i = i_0, \\ r((E\boldsymbol{x})^i \xrightarrow{i_0} (E\boldsymbol{y})^i) \cdot m'((E\boldsymbol{y})^i) & \text{if } i > i_0. \end{cases}$$

An example is given in Figure 27.

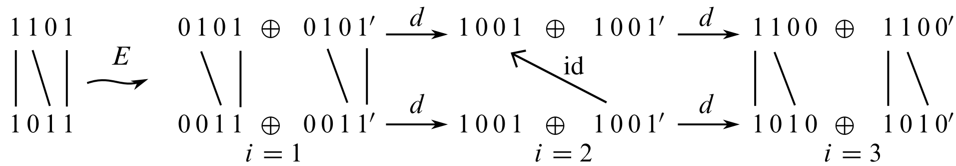


Figure 27: The 5 diagrams on the right represent the right multiplication with $e(E)\boxtimes r(|1011\rangle \xrightarrow{2,0,(0)} |1101\rangle)$.

Lemma 6.14 *The right multiplication by $e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y})$ such that $s_1 = s_0(\boldsymbol{v}) = 0$ commutes with the differential.*

Proof It suffices to prove that

$$d(m\times(e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y}))) = d(m)\times(e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y})),$$

where $m = m((E\boldsymbol{x})^i), m'((E\boldsymbol{x})^i)$ for $1\le i\le k$. The equation is obviously true for $i\neq i_0-1, i_0$ from the commutativity relations (R2-2) and (R2-3) since the decorated rook diagrams in the differential and right multiplication are disjoint.

We verify the equation for $m = m'((E\boldsymbol{x})^{i_0})$:

$$\begin{aligned}
d(m'((E\boldsymbol{x})^{i_0})\times(e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y})))&\\
&= d(m((E\boldsymbol{y})^{i_0}))\\
&= \theta(\boldsymbol{y};i_0)\cdot r_E(\boldsymbol{y};i_0)\cdot m((E\boldsymbol{y})^{i_0+1}) + r_E(\boldsymbol{y};i_0)\cdot m'((E\boldsymbol{y})^{i_0+1})\\
&= d(m'((E\boldsymbol{x})^{i_0}))\cdot r((E\boldsymbol{x})^{i_0+1} \xrightarrow{i_0} (E\boldsymbol{y})^{i_0+1})\\
&= d(m'((E\boldsymbol{x})^{i_0}))\times(e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y}))
\end{aligned}$$

The proof for other cases is similar and we leave it to the reader. □

Let $a\boxtimes r = e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_1>0, s_0(\boldsymbol{v})=0$ for $\boldsymbol{x},\boldsymbol{y}\in\mathcal{B}_{n,k}$. Note that $(E\boldsymbol{x})^{i_0+s_1} = (E\boldsymbol{y})^{i_0}\in\mathcal{B}_{n,k-1}$. The right multiplication is a map of left $H(R_n)$–modules defined on the generators by

(M4-3) $$m\times(e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})) = \begin{cases} m((E\boldsymbol{y})^{i-s_1}) & \text{if } m = m'((E\boldsymbol{x})^{i_0+s_1}),\\ 0 & \text{otherwise;}\end{cases}$$

see the right-hand diagram in Figure 28 for an example.

We verify that the definition is compatible with the DG structure on $A\boxtimes R_n$.

Lemma 6.15 *We have $d(m\times r) = dm\times r + m\times dr$ holds for $r = e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{i_0,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_0(\boldsymbol{v}) = 0$.*

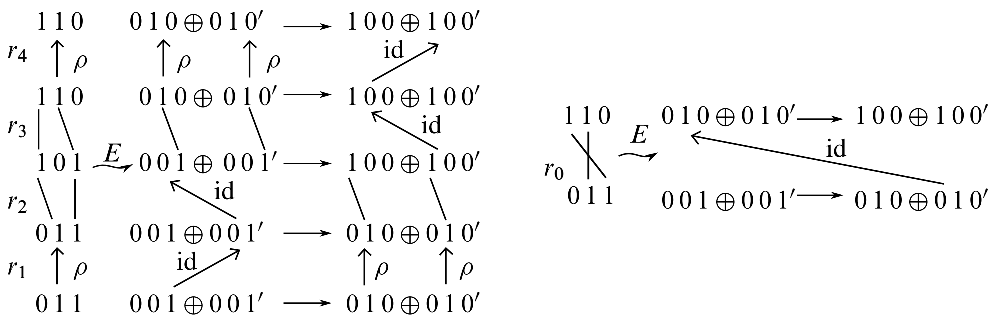


Figure 28: The right multiplication with $e(E)\boxtimes r_i$ for $i=1,2,3,4$ on the left and for $i=0$ on the right, where $dr_0 = r_1r_2r_3 + r_2r_3r_4$.

Proof The case for $s_1 = 0$ is proved in Lemma 6.14.

Suppose $s_1 > 0$. For $m \in C^i(E,\boldsymbol{x})$ or $C^i(E,\boldsymbol{x})'$, $m \times r = dm \times r = m \times dr = 0$ if $i < i_0 - s_1$ or $i > i_0$ from (M4-2) and (M4-3).

If $i_0 - s_1 \le i \le i_0$, we reduce to the local model: $r = e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{1,s_1,\boldsymbol{v}(s_1)} \boldsymbol{y})$, where $\boldsymbol{x} = (2,\dots,s_1+2) = |01\cdots 1\rangle$, $\boldsymbol{y} = (1,\dots,s_1+1) = |1\cdots 10\rangle \in \mathcal{B}_{s_1+2,s_1+1}$, and $\boldsymbol{v}(s_1) = (0,\dots,0) \in \mathbb{N}^{s_1+1}$. A diagrammatic proof for $s_1 = 1$ is given in Figure 28. Recall that $d(r_0) = r_1\cdot r_2\cdot r_3 + r_2\cdot r_3\cdot r_4$, where

$$r_1 = e(E)\boxtimes \rho(\boldsymbol{x} \xrightarrow{1} \boldsymbol{x}), \quad r_2 = e(E)\boxtimes r(\boldsymbol{x} \xrightarrow{1,0,(0)} \boldsymbol{z}),$$
$$r_4 = e(E)\boxtimes \rho(\boldsymbol{y} \xrightarrow{2} \boldsymbol{y}), \quad r_3 = e(E)\boxtimes r(\boldsymbol{z} \xrightarrow{2,0,(0)} \boldsymbol{y}),$$

and $\boldsymbol{z} = (1,3) = |101\rangle$. In Figure 28, the right multiplications by r_1, r_2, r_3, r_4 and r_0 are given in the left-hand and right-hand diagrams, respectively.

We verify the equation for $m = m(|001\rangle) \in C(E,|011\rangle)$ by chasing the diagrams and leave other cases to the reader. The right-hand side of the equation is zero since

$$\begin{aligned}
&d(m(|001\rangle)) \times r_0\\
&\quad= (\rho(|001\rangle \xrightarrow{1} |001\rangle)\cdot r(|001\rangle \xrightarrow{1} |010\rangle)\cdot m(|010\rangle) + r(|001\rangle \xrightarrow{1} |010\rangle)\cdot m'(|010\rangle)) \times r_0\\
&\quad= r(|001\rangle \xrightarrow{1} |010\rangle)\cdot (m'(|010\rangle) \times r_0)\\
&\quad= r(|001\rangle \xrightarrow{1} |010\rangle)\cdot m(|010\rangle) \in C(E,|110\rangle)
\end{aligned}$$

is the same as

$$\begin{aligned}
m(|001\rangle)\times d(r_0) &= m(|001\rangle)\times(r_1\cdot r_2\cdot r_3 + r_2\cdot r_3\cdot r_4) = ((m(|001\rangle)\times r_1)\times r_2)\times r_3\\
&= r(|001\rangle \xrightarrow{1} |010\rangle)\cdot m(|010\rangle) \in C(E,|110\rangle).
\end{aligned}$$

The left-hand side is obviously zero since $m(|001\rangle) \times r_0 = 0$.

The proof for the case $s_1 > 1$ is similar. □

Let $a \boxtimes r = e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_0(\boldsymbol{v}) > 0$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}_{n,k}$. The right multiplication is defined as the zero map

$$m \times (e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})) = 0. \tag{M4-4}$$

6.3 The left DG $H(R_n)$–module C_n, Part II

We finish the definition of the left $H(R_n)$–module structure on $C(\Gamma, \boldsymbol{x})$ for $\Gamma = EF$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$. The module $C(EF, \boldsymbol{x})$ is constructed through the action of E on the module $C(F, \boldsymbol{x})$. Let $(EF\boldsymbol{x})^i_j$ denote $(E(F\boldsymbol{x})_j)^i \in \mathcal{B}_{n,k}$, ie

$$(EF\boldsymbol{x})^i_j = \begin{cases} \boldsymbol{x} \sqcup \{\overline{x}_j\} \backslash \{x_i\} & \text{if } i < q_j + 1, \\ \boldsymbol{x} & \text{if } i = q_j + 1, \\ \boldsymbol{x} \sqcup \{\overline{x}_j\} \backslash \{x_{i-1}\} & \text{if } i > q_j + 1. \end{cases}$$

Define

$$\begin{aligned} C(EF, \boldsymbol{x}) &= \bigoplus_{j=1}^{n-k} C(E, (F\boldsymbol{x})_j)\{n - \overline{x}_j\}[\beta(\boldsymbol{x}, \overline{x}_j)] \\ &= \bigoplus_{j=1}^{n-k} \bigoplus_{i=1}^{k+1} (PH((EF\boldsymbol{x})^i_j)\{n - \overline{x}_j\}[\beta(\boldsymbol{x}, \overline{x}_j) + 1 - i] \\ &\qquad\qquad \oplus PH((EF\boldsymbol{x})^i_j)\{n - \overline{x}_j + 1\}[\beta(\boldsymbol{x}, \overline{x}_j) + 2 - i]). \end{aligned}$$

Recall that $r_F(\boldsymbol{x}; j) \in H(R_{n,k+1})$ is given by the path from $(F\boldsymbol{x})_j$ to $(F\boldsymbol{x})_{j-1}$. It can also be viewed as an element in $R_{n,k+1}$ which is still denoted by $r_F(\boldsymbol{x}; j)$. The right multiplication with $e(E) \boxtimes r_F(\boldsymbol{x}; j)$ defines a chain map

$$\times (e(E) \boxtimes r_F(\boldsymbol{x}; j))\colon C(E, (F\boldsymbol{x})_j) \to C(E, (F\boldsymbol{x})_{j-1}).$$

We view $C(EF, \boldsymbol{x})$ as a double complex with $(i, j)^{\text{th}}$ entry $C^i_j(EF, \boldsymbol{x})$ equal to

$$PH((EF\boldsymbol{x})^i_j)\{n - \overline{x}_j\}[\beta(\boldsymbol{x}, \overline{x}_j) + 1 - i] \oplus PH((EF\boldsymbol{x})^i_j)\{n - \overline{x}_j + 1\}[\beta(\boldsymbol{x}, \overline{x}_j) + 2 - i].$$

Let $m((EF\boldsymbol{x})^i_j)$ and $m'((EF\boldsymbol{x})^i_j)$ be the generators of the first and the second summand of $C^i_j(EF, \boldsymbol{x})$.

Definition 6.16 The differential $d(EF,\boldsymbol{x})$ is defined by

$$d(EF,\boldsymbol{x}) = \sum_{j=1}^{n-k}\sum_{i=1}^{k+1} d^i_j(EF,\boldsymbol{x}) = \sum_{j=1}^{n-k}\sum_{i=1}^{k+1}(d^i_j|_{\mathrm{ver}}(EF,\boldsymbol{x}) + d^i_j|_{\mathrm{hor}}(EF,\boldsymbol{x})),$$

where

$$d^i_j|_{\mathrm{ver}}(EF,\boldsymbol{x}) = d^i(E,(F\boldsymbol{x})_j)\colon C^i_j(EF,\boldsymbol{x}) \to C^{i+1}_j(EF,\boldsymbol{x}),$$
$$d^i_j|_{\mathrm{hor}}(EF,\boldsymbol{x}) = (\times e(E)\boxtimes r_F(\boldsymbol{x};j))\colon C^i_j(EF,\boldsymbol{x}) \to C^i_{j-1}(EF,\boldsymbol{x}).$$

We have the following double complex $(C(EF,\boldsymbol{x}), d(EF,\boldsymbol{x}))$:

$$\begin{tikzcd}
C^{k+1}_{n-k}(EF,\boldsymbol{x}) \arrow[r] & C^{k+1}_{n-k-1}(EF,\boldsymbol{x}) \arrow[r] & \cdots \arrow[r] & C^{k+1}_1(EF,\boldsymbol{x}) \\
\vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[u] \\
C^1_{n-k}(EF,\boldsymbol{x}) \arrow[r] \arrow[u, "d^1(E{,}(F\boldsymbol{x})_{n-k})"] & C^1_{n-k-1}(EF,\boldsymbol{x}) \arrow[r] \arrow[u, "d^1(E{,}(F\boldsymbol{x})_{n-k-1})"] & \cdots \arrow[r] \arrow[u] & C^1_1(EF,\boldsymbol{x}) \arrow[u, "d^1(E{,}(F\boldsymbol{x})_1)"]
\end{tikzcd} \tag{1}$$

Lemma 6.17 *The differential $d(EF,\boldsymbol{x})$ is well defined.*

Proof Since $r_F(\boldsymbol{x};j)$ is a product of the generators which satisfy $s_1 = s_0(\boldsymbol{v}) = 0$, the horizontal differential $d^i_j|_{\mathrm{hor}}(EF,\boldsymbol{x})$, ie the right multiplication by $e(E)\boxtimes r_F(\boldsymbol{x};j)$, commutes with the vertical differential $d^i_j|_{\mathrm{ver}}(EF,\boldsymbol{x})$ by Lemma 6.14. Therefore, $d(EF,\boldsymbol{x})\circ d(EF,\boldsymbol{x}) = 0$. □

This concludes the construction of the left $H(R_n)$–module C_n.

The following lemma is immediate:

Lemma 6.18 *If $\Gamma\in\mathcal{B}$ and $\boldsymbol{x}\in\mathcal{B}_n$, then $[C(\Gamma,\boldsymbol{x})] = \Gamma(\boldsymbol{x})$ when viewed as elements in*

$$K_0(\mathrm{HP}(H(R_n))) \cong V_1^{\otimes n}.$$

6.4 The right $A\boxtimes R_n$–module C_n, Part II

We finish the definition of the right multiplication with generators in

$$\{e(EF)\boxtimes\rho(\boldsymbol{x}\xrightarrow{i}\boldsymbol{x}), e(EF)\boxtimes r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y})\},$$

$$\{\rho(I,EF)\boxtimes e(\boldsymbol{x}), \rho(EF,I)\boxtimes e(\boldsymbol{x})\},$$

$$\{\rho(I\boldsymbol{x}\xrightarrow{i}EF\boldsymbol{x}), \rho(EF\boldsymbol{x}\xrightarrow{j}I\boldsymbol{x})\}.$$

6.4.1 The right multiplication by $e(EF)\boxtimes r$ The right multiplication by $e(EF)\boxtimes r$ corresponds to applying the functor EF to r as a morphism in DGP(R_n). It can be decomposed into two steps: first applying F to r to obtain a morphism, then applying E to the resulting morphism.

Let $a\boxtimes r = e(EF)\boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$. The right multiplication is a map $C(EF,\boldsymbol{x}) \to C(EF,\boldsymbol{x})$ of left $H(R_n)$–modules. Recall that

$$C(EF,\boldsymbol{x}) = \bigoplus_{j=1}^{n-k} C(E,(F\boldsymbol{x})_j)\{n-\overline{x}_j\}[\beta(\boldsymbol{x},\overline{x}_j)],$$

and the right multiplication

$$\times(e(F)\boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}))\colon PH((F\boldsymbol{x})_j) \to PH((F\boldsymbol{x})_j)$$

is given in (M3-1). Then the right multiplication by $e(EF)\boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})$ is defined by

(M5-1) $$m\times(e(EF)\boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x})) = \begin{cases} m\times(e(E)\boxtimes \rho((F\boldsymbol{x})_j \xrightarrow{i} (F\boldsymbol{x})_j)) & \text{if } j > j_0, \\ m\times(e(E)\boxtimes \rho((F\boldsymbol{x})_j \xrightarrow{i+1} (F\boldsymbol{x})_j)) & \text{if } j \le j_0, \end{cases}$$

where $m\in C(E,(F\boldsymbol{x})_j)\{n-\overline{x}_j\}[\beta(\boldsymbol{x},\overline{x}_j)] \subset C(EF,\boldsymbol{x})$.

Now let $a\boxtimes r = e(EF)\boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$. Recall that the right multiplication

$$\times(e(F)\boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y}))\colon PH((F\boldsymbol{x})_j) \to PH((F\boldsymbol{y})_{j+s_0})$$

is given in (M3-2), (M3-3) and (M3-4).

If $s_1+s_0(\boldsymbol{v})=0$, the right multiplication by $e(EF)\boxtimes r(\boldsymbol{x} \xrightarrow{i,0,(0)} \boldsymbol{y})$ is defined by

(M5-2) $$m\times(e(EF)\boxtimes r(\boldsymbol{x} \xrightarrow{i,0,(0)} \boldsymbol{y})) = \begin{cases} m\times(e(E)\boxtimes r((F\boldsymbol{x})_j \xrightarrow{i,0,(0)} (F\boldsymbol{y})_j)) & \text{if } j > j_0, \\ m\times(e(E)\boxtimes e((F\boldsymbol{y})_j)) & \text{if } j = j_0, \\ m\times(e(E)\boxtimes r((F\boldsymbol{x})_j \xrightarrow{i+1,0,(0)} (F\boldsymbol{y})_j)) & \text{if } j < j_0, \end{cases}$$

where $m\in C(E,(F\boldsymbol{x})_j)\{n-\overline{x}_j\}[\beta(\boldsymbol{x},\overline{x}_j)] \subset C(EF,\boldsymbol{x})$.

If $s_1=0$ and $s_0(\boldsymbol{v})>0$, the right multiplication by $e(EF)\boxtimes r(\boldsymbol{x} \xrightarrow{i,0,(s_0)} \boldsymbol{y})$ is defined by

(M5-3) $\quad m\times(e(EF)\boxtimes r(\boldsymbol{x}\xrightarrow{i,0,(s_0)}\boldsymbol{y}))$

$$= \begin{cases} m\times(e(E)\boxtimes e((F\boldsymbol{y})_{j+s_0})) & \text{if } j=j_0-s_0,\\ 0 & \text{otherwise,}\end{cases}$$

where $m\in C(E,(F\boldsymbol{x})_j)\{n-\overline{x}_j\}[\beta(\boldsymbol{x},\overline{x}_j)]\subset C(EF,\boldsymbol{x})$.

If $s_1>0$, the right multiplication by $e(EF)\boxtimes r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y})$ is defined as the zero map

(M5-4) $$m\times(e(EF)\boxtimes r(\boldsymbol{x}\xrightarrow{i,s_1,\boldsymbol{v}}\boldsymbol{y}))=0.$$

6.4.2 The right multiplication by $\rho(I,EF)\boxtimes e(\boldsymbol{x})$

We discuss $(EF\boldsymbol{x})^{k+1}_{n-k}\in\mathcal{B}_{n,k}$ shown in the top left corner $C^{k+1}_{n-k}(EF,\boldsymbol{x})$ of the double complex (1) depending on $\overline{x}_{n-k}=n$ or $\overline{x}_{n-k}<n$.

Case 1 Suppose $\overline{x}_{n-k}=n$, ie the last state is $|0\rangle$. Then we have $(EF\boldsymbol{x})^{k+1}_{n-k}=\boldsymbol{x}$, $\beta(\boldsymbol{x},\overline{x}_{n-k})=k$ and $C^{k+1}_{n-k}(EF,\boldsymbol{x})=PH(\boldsymbol{x})\oplus PH(\boldsymbol{x})\{1\}[1]$. The right multiplication

(M6-1-1) $$\times(\rho(I,EF)\boxtimes e(\boldsymbol{x}))\colon C(I,\boldsymbol{x})\to C(EF,\boldsymbol{x})$$

is defined by the identity map from $PH(\boldsymbol{x})=C(I,\boldsymbol{x})$ to $PH(\boldsymbol{x})\subset C^{k+1}_{n-k}(EF,\boldsymbol{x})$. An example is given in Figure 29.

$$\begin{tikzcd}
 & C(I,010)=010 \arrow[dr, "\mathrm{id}"] & & & \\
 & & \underline{0\,1\,0}\oplus 0\,1\,0' \arrow[r, "d|_{\mathrm{hor}}"] & 1\,0\,0\oplus 1\,0\,0' & i=2 \\
C(EF,010)=C(E,011) \arrow[r] & C(E,110)= & & & \\
 & & 0\,0\,1\oplus 0\,0\,1' \arrow[r, "d|_{\mathrm{hor}}"] \arrow[uu, "d|_{\mathrm{ver}}"'] & 0\,1\,0\oplus \underline{0\,1\,0'} \arrow[uu, "d|_{\mathrm{ver}}"'] \arrow[ddr, "\mathrm{id}"] & i=1 \\
j=2 & j=1 & j=2 & j=1 & \\
 & & & & C(I,010)=010
\end{tikzcd}$$

Figure 29: The identity map from $C(I,010)$ to the top left corner is the right multiplication by $\rho(I,EF)\boxtimes e(\boldsymbol{x})$ when $\overline{x}_{n-k}=n$. The identity map from the bottom right corner to $C(I,010)$ is the right multiplication by $\rho(EF,I)\boxtimes e(\boldsymbol{x})$ when $\overline{x}_1=1$.

Lemma 6.19 *The right multiplication by $\rho(I,EF)\boxtimes e(\boldsymbol{x})$ in Case 1 commutes with d.*

Proof Since the differential on $C(I,\boldsymbol{x})$ is trivial, and

$$d(m(I\boldsymbol{x})\times(\rho(I,EF)\boxtimes e(\boldsymbol{x})))=d^{k+1}_{n-k}|_{\mathrm{hor}}(EF,\boldsymbol{x})(m((EF\boldsymbol{x})^{k+1}_{n-k}))$$
$$=m((EF\boldsymbol{x})^{k+1}_{n-k})\times(e(E)\boxtimes r_F(\boldsymbol{x};n-k)),$$

it suffices to prove that $m((EF\boldsymbol{x})^{k+1}_{n-k})\times(e(E)\boxtimes r_F(\boldsymbol{x};n-k))=0$.

Recall from Section 6.1.2 that $r_F(\boldsymbol{x};n-k)=r_0\cdot r(\boldsymbol{z}\xrightarrow{q_{n-k}+1}(F\boldsymbol{x})_{n-k-1})$ for some $r_0\in R_{n,k+1}$ and $\boldsymbol{z}\in\mathcal{B}_{n,k+1}$, where $q_{n-k}=|l\in\{1,\dots,k\}\mid x_l<\overline{x}_{n-k}=n\}|=k$. Moreover, $(E\boldsymbol{z})^{k+1}=(E(F\boldsymbol{x})_{n-k-1})^{k+1}\in\mathcal{B}_{n,k}$. Hence, there exists $r_1\in R_{n,k}$ such that

$$\begin{aligned}
&m((EF\boldsymbol{x})^{k+1}_{n-k})\times(e(E)\boxtimes r_F(\boldsymbol{x};n-k))\\
&\qquad=(m((EF\boldsymbol{x})^{k+1}_{n-k})\times(e(E)\boxtimes r_0))\times(e(E)\boxtimes r(\boldsymbol{z}\xrightarrow{k+1}(F\boldsymbol{x})_{n-k-1}))\\
&\qquad=(r_1\cdot m((E\boldsymbol{z})^{k+1}))\times(e(E)\boxtimes r(\boldsymbol{z}\xrightarrow{k+1}(F\boldsymbol{x})_{n-k-1}))\\
&\qquad=r_1\cdot(m((E\boldsymbol{z})^{k+1})\times(e(E)\boxtimes r(\boldsymbol{z}\xrightarrow{k+1}(F\boldsymbol{x})_{n-k-1})))\\
&\qquad=r_1\cdot 0=0,
\end{aligned}$$

where in the last step, $m((E\boldsymbol{z})^{k+1})\times(e(E)\boxtimes r(\boldsymbol{z}\xrightarrow{k+1}(F\boldsymbol{x})_{n-k-1}))=0$ is from (M4-2). □

Case 2 Suppose $\overline{x}_{n-k}<n$. Then $\beta(\boldsymbol{x},\overline{x}_{n-k})=k+n-\overline{x}_{n-k}$ and $C^{k+1}_{n-k}(EF,\boldsymbol{x})$ is $PH((EF\boldsymbol{x})^{k+1}_{n-k})\{n-\overline{x}_{n-k}\}[n-\overline{x}_{n-k}]\oplus PH((EF\boldsymbol{x})^{k+1}_{n-k})\{n-\overline{x}_{n-k}+1\}[n-\overline{x}_{n-k}+1]$.

Note that $(EF\boldsymbol{x})^{q_{n-k}+1}_{n-k}=\boldsymbol{x}$ and there is a path from $\boldsymbol{x}$ to $(EF\boldsymbol{x})^{k+1}_{n-k}$ in $Q_{n,k}$:

$$\boldsymbol{x}=(EF\boldsymbol{x})^{q_{n-k}+1}_{n-k}\xrightarrow{q_{n-k}+1}(EF\boldsymbol{x})^{q_{n-k}+2}_{n-k}\xrightarrow{q_{n-k}+2}\cdots\xrightarrow{k}(EF\boldsymbol{x})^{k+1}_{n-k}$$

Let $r_{I,EF}(\boldsymbol{x})$ be a product of the corresponding $n-\overline{x}_{n-k}=k-q_{n-k}$ generators in $H(R_{n,k})$. The right multiplication is a map $C(I,\boldsymbol{x})\to C(EF,\boldsymbol{x})$ of left $H(R_n)$–modules defined on the generators by

(M6-1-2) $$m(I\boldsymbol{x})\times(\rho(I,EF)\boxtimes e(\boldsymbol{x}))=r_{I,EF}(\boldsymbol{x})\cdot m((EF\boldsymbol{x})^{k+1}_{n-k}).$$

An example is given in Figure 30.

Lemma 6.20 *The right multiplication by $\rho(I,EF)\boxtimes e(\boldsymbol{x})$ in Case 2 commutes with d.*

Proof Let $r_{EF}(\boldsymbol{x};n-k,k+1)$ be a product of $q_{n-k}-q_{n-k-1}+1$ generators in $H(R_{n,k})$ induced by the path in $Q_{n,k}$:

$$(EF\boldsymbol{x})^{k+1}_{n-k}\xrightarrow{q_{n-k-1}+1}\cdots\xrightarrow{q_{n-k}+1}(EF\boldsymbol{x})^{k+1}_{n-k-1}$$

$$C(I,1001)=1001$$

$$C(EF,1001)=\underset{j=2}{C(E,1001)}\longrightarrow \underset{j=1}{C(E,1101)}=\left(\begin{array}{ccc} \underline{1\,0\,1\,0}\oplus 1\,0\,1\,0' & \longrightarrow & 1\,1\,0\,0\oplus 1\,1\,0\,0' \\ \uparrow & & \uparrow \\ 1\,0\,0\,1\oplus 1\,0\,0\,1' & \longrightarrow & 1\,0\,0\,1\oplus 1\,0\,0\,1' \\ \uparrow & & \uparrow \\ 0\,0\,1\,1\oplus 0\,0\,1\,1' & \longrightarrow & 0\,1\,0\,1\oplus \underline{0\,1\,0\,1'} \end{array}\right)\begin{array}{c} i=3 \\ \\ i=2 \\ \\ i=1 \end{array}$$

$$\begin{array}{ccc} j=2 & & j=1 \end{array}$$

$$C(I,1001)=1001$$

Figure 30: The map from $C(I,1010)$ to the top left corner is the right multiplication by $\rho(I,EF)\boxtimes e(\boldsymbol{x})$ when $\overline{x}_{n-k}<n$. The map from the bottom right corner to $C(I,1010)$ is the right multiplication by $\rho(EF,I)\boxtimes e(\boldsymbol{x})$ when $\overline{x}_1>1$.

Since the differential on $C(I,\boldsymbol{x})$ is zero, and

$$\begin{aligned} d^{k+1}_{n-k}|_{\mathrm{hor}}&(EF,\boldsymbol{x})(m(I\boldsymbol{x})\times(\rho(I,EF)\boxtimes e(\boldsymbol{x}))) \\ &=(r_{I,EF}(\boldsymbol{x})\cdot m((EF\boldsymbol{x})^{k+1}_{n-k}))\times(e(E)\boxtimes r_F(\boldsymbol{x};n-k)) \\ &=r_{I,EF}(\boldsymbol{x})\cdot r_{EF}(\boldsymbol{x};n-k,k+1), \end{aligned}$$

it suffices to prove that $r_{I,EF}(\boldsymbol{x})\cdot r_{EF}(\boldsymbol{x};n-k,k+1)=0$.

The product $r_{I,EF}(\boldsymbol{x})\cdot r_{EF}(\boldsymbol{x};n-k,k+1)$ is induced by the concatenation of the two paths

$$\boldsymbol{x}\xrightarrow{q_{n-k}+1}\cdots\xrightarrow{k}(EF\boldsymbol{x})^{k+1}_{n-k}\xrightarrow{q_{n-k-1}+1}\cdots\xrightarrow{q_{n-k}+1}(EF\boldsymbol{x})^{k+1}_{n-k-1}.$$

By using the commutation relations to rearrange the arrows, the path above can be written as

$$\boldsymbol{x}\xrightarrow{q_{n-k-1}+1}\cdots\xrightarrow{q_{n-k}}\boldsymbol{z}^0\xrightarrow{q_{n-k}+1}\boldsymbol{z}^1\xrightarrow{q_{n-k}+1}\boldsymbol{z}^2\xrightarrow{q_{n-k}+2}\cdots\xrightarrow{k}(EF\boldsymbol{x})^{k+1}_{n-k-1}.$$

Hence, $r_{I,EF}(\boldsymbol{x})\cdot r_{EF}(\boldsymbol{x};n-k,k+1)=\cdots r(\boldsymbol{z}^0\xrightarrow{q_{n-k}+1}\boldsymbol{z}^1)\cdot r(\boldsymbol{z}^1\xrightarrow{q_{n-k}+1}\boldsymbol{z}^2)\cdots=0$ from (R1-2). □

6.4.3 The right multiplication by $\rho(I\boldsymbol{x}\xrightarrow{i}EF\boldsymbol{x})$

Because the right multiplication with

$$(\rho(I,EF)\boxtimes e(\boldsymbol{x}))\cdot(e(EF)\boxtimes\rho(\boldsymbol{x}\xrightarrow{k}\boldsymbol{x}))+(e(I)\boxtimes\rho(\boldsymbol{x}\xrightarrow{k}\boldsymbol{x}))\cdot(\rho(I,EF)\boxtimes e(\boldsymbol{x}))$$

is possibly nonzero, we represent the above as the differential of $\rho(I\boldsymbol{x}\xrightarrow{k}EF\boldsymbol{x})$ for $\boldsymbol{x}\in\mathcal{B}_{n,k}$ with $x_k=n$ in Definition 5.17; see Figure 32 for an example.

Recall from Definition 5.17 that $\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})$ exists if and only if $1 \le i \le k < n$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ such that $x_i = n-k+i$. Note that the condition $x_i = n-k+i$ implies that $x_l = n-k+l$ for all $i \le l \le k$, ie each of the last $k-i+1$ states in $\boldsymbol{x}$ is $|1\rangle$. We are interested in $(i, n-k)^{\text{th}}$ entry $C^i_{n-k}(EF, \boldsymbol{x})$ of the double complex (1):

$$PH((EF\boldsymbol{x})^i_{n-k})\{n-\overline{x}_{n-k}\}[n-\overline{x}_{n-k}+k-i+1] \\ \oplus PH((EF\boldsymbol{x})^i_{n-k})\{n-\overline{x}_{n-k}+1\}[n-\overline{x}_{n-k}+k-i+2]$$

Note that $(EF\boldsymbol{x})^{q_{n-k}+1}_{n-k} = \boldsymbol{x}$ and $q_{n-k}+1 \le i$. Hence, there is a path from $\boldsymbol{x}$ to $(EF\boldsymbol{x})^{k+1}_{n-k}$ through $(EF\boldsymbol{x})^i_{n-k}$ in $Q_{n,k}$:

$$\boldsymbol{x} = (EF\boldsymbol{x})^{q_{n-k}+1}_{n-k} \xrightarrow{q_{n-k}+1} \cdots \xrightarrow{i-1} (EF\boldsymbol{x})^i_{n-k} \xrightarrow{i} \cdots \xrightarrow{k} (EF\boldsymbol{x})^{k+1}_{n-k}$$

Let $r_{I,EF}(\boldsymbol{x}; i)$ be the product of $i-q_{n,k}-1$ generators in $H(R_{n,k})$ corresponding to the path from $\boldsymbol{x}$ to $(EF\boldsymbol{x})^i_{n-k}$. Then the right multiplication is a map $C(I,\boldsymbol{x}) \to C(EF,\boldsymbol{x})$ of left $H(R_n)$–modules defined on the generators by

$$\text{(M6-2)} \qquad m(I\boldsymbol{x}) \times (\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x})) = r_{I,EF}(\boldsymbol{x}; i) \cdot m((EF\boldsymbol{x})^i_{n-k}),$$

where $m((EF\boldsymbol{x})^i_{n-k}) \in C^i_{n-k}(EF, \boldsymbol{x})$.

Example 6.21 Let $\boldsymbol{x} = (2,3) = |011\rangle \in \mathcal{B}_{3,2}$, then $x_i = n-k+i$ for $i = 1,2$ and $n = 3, k = 2$. Hence there exist

$$r_1 = \rho(I|011\rangle \xrightarrow{1} EF|011\rangle), \quad r_2 = \rho(I|011\rangle \xrightarrow{2} EF|011\rangle), \quad r_3 = \rho(I, EF) \boxtimes e(|011\rangle).$$

The right multiplications by r_1, r_2 and r_3 are described in Figure 31. More precisely,

$$m(I|011\rangle) \times r_3 = r(|011\rangle \xrightarrow{1} |101\rangle) \cdot r(|101\rangle \xrightarrow{2} |110\rangle) \cdot m(|110\rangle) \in C(EF, |011\rangle),$$

$$m(I|011\rangle) \times r_2 = r(|011\rangle \xrightarrow{1} |101\rangle) \cdot m(|101\rangle) \in C(EF, |011\rangle),$$

$$m(I|011\rangle) \times r_1 = m(|011\rangle) \in C(EF, |011\rangle).$$

We verify that the definitions (M6-1-1), (M6-1-2) and (M6-2) are compatible with the DG structure on $A \boxtimes R_n$.

Lemma 6.22 *For $1 \le i \le k$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ with $x_i = n-k+i$,*

$$d(m(I\boldsymbol{x}) \times (\rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}))) = m(I\boldsymbol{x}) \times d(\rho(I\boldsymbol{x} \xrightarrow{k} EF\boldsymbol{x})).$$

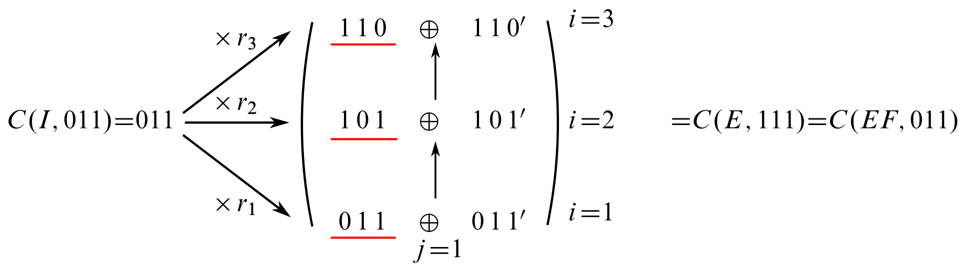


Figure 31: The right multiplication with r_i from $C(I,|011\rangle)$ to the summands of $C(EF,|011\rangle)$ with red lines underlying

Proof For $i=k$, we can reduce the case in general to the local model. We give a diagrammatic proof for the local model

$$r_1 = \rho(I|01\rangle \xrightarrow{1} EF|01\rangle)$$

in Figure 32. Let $\rho = \rho(|01\rangle \xrightarrow{1} |01\rangle)$ and $r_0 = r(|01\rangle \xrightarrow{1} |10\rangle)$. Recall that $d(r_1) = r_2 \cdot r_3 + r_4 \cdot r_2$, here

$$r_2 = \rho(I, EF) \boxtimes e(|01\rangle), \quad r_3 = e(EF) \boxtimes \rho, \quad r_4 = e(I) \boxtimes \rho.$$

In Figure 32, the right multiplications by r_1, r_2 and r_3, r_4 are given in the left-hand and right-hand diagrams, respectively.

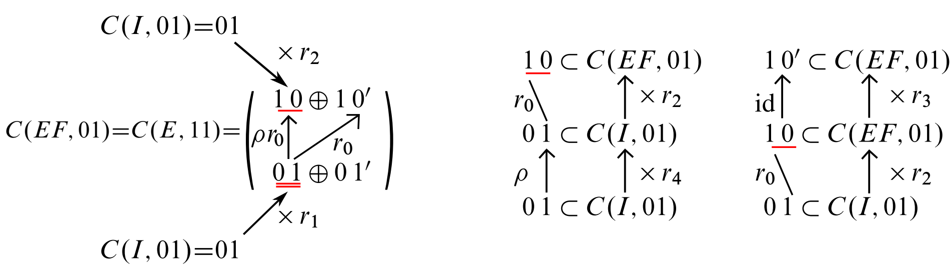


Figure 32: The map from $C(I,01)$ to the bottom left corner is the right multiplication by $\rho(I|01\rangle \xrightarrow{1} EF|01\rangle)$. The right-hand diagram gives the right multiplications by $d(\rho(I|01\rangle \xrightarrow{1} EF|01\rangle))$.

The right-hand side of the equation is

$$\begin{aligned} m(I|01\rangle) \times d(r_1) &= (m(I|01\rangle) \times r_4) \times r_2 + (m(I|01\rangle) \times r_2) \times r_3 \\ &= \rho \cdot r_0 \cdot m(|10\rangle) + r_0 \cdot m'(|10\rangle) \in C(EF,|01\rangle), \end{aligned}$$

which agrees with the left-hand side: $d(m(I|01\rangle) \times r_1) = d(m(|01\rangle)) \in C(EF,|01\rangle)$.

The proof for the case $i<k$ is similar. $\square$

6.4.4 The right multiplication by $\rho(EF, I) \boxtimes e(x)$ The construction in this subsection is dual to that in Section 6.4.2. We discuss $(EF\boldsymbol{x})_1^1 \in \mathcal{B}_{n,k}$ shown in the bottom right corner $C_1^1(EF, \boldsymbol{x})$ of the double complex (1) depending on $\overline{x}_1 = 1$ or $\overline{x}_1 > 1$.

Case 1 Suppose $\overline{x}_1 = 1$, ie the first state in the tensor product presentation is $|0\rangle$. Then we have $(EF\boldsymbol{x})_1^1 = \boldsymbol{x}$, $\beta(\boldsymbol{x}, \overline{x}_1) = 2k$ and $C_1^1(EF, \boldsymbol{x}) = PH(\boldsymbol{x})\{n-1\}[2k] \oplus PH(\boldsymbol{x})\{n\}[2k+1]$. The right multiplication

(M7-1-1) $$\times(\rho(EF, I) \boxtimes e(\boldsymbol{x}))\colon C(EF, \boldsymbol{x}) \to C(I, \boldsymbol{x})$$

is defined by the identity map from $PH(\boldsymbol{x})\{n\}[2k+1] \subset C_1^1(EF, \boldsymbol{x})$ to $PH(\boldsymbol{x}) = C(I, \boldsymbol{x})$; see Figure 29 for an example.

Case 2 Suppose $\overline{x}_1 > 1$, then $\beta(\boldsymbol{x}, \overline{x}_1) = 2k - \overline{x}_1 + 1$ and $C_1^1(EF, \boldsymbol{x})$ is

$$PH((EF\boldsymbol{x})_1^1)\{n-\overline{x}_1\}[2k-\overline{x}_1+1] \oplus PH((EF\boldsymbol{x})_1^1)\{n-\overline{x}_1+1\}[2k-\overline{x}_1+2].$$

Note that $(EF\boldsymbol{x})_1^{q_1+1} = \boldsymbol{x}$ and there is a path from $(EF\boldsymbol{x})_1^1$ to $\boldsymbol{x}$ in $Q_{n,k}$:

$$(EF\boldsymbol{x})_1^1 \xrightarrow{1} (EF\boldsymbol{x})_1^2 \xrightarrow{2} \cdots \xrightarrow{q_1} \boldsymbol{x}$$

Let $r_{EF,I}(\boldsymbol{x})$ be a product of the corresponding $q_1 = \overline{x}_1 - 1$ generators in $H(R_{n,k})$. The right multiplication is a map of left $H(R_n)$–modules: $C(EF, \boldsymbol{x}) \to C(I, \boldsymbol{x})$ defined on the generators by

(M7-1-2) $$m \times (\rho(EF, I) \boxtimes e(\boldsymbol{x})) = \begin{cases} r_{EF,I}(\boldsymbol{x}) \cdot m(I\boldsymbol{x}) & \text{if } m = m'((EF\boldsymbol{x})_1^1), \\ 0 & \text{otherwise.} \end{cases}$$

An example is given in Figure 30.

Lemma 6.23 *In both cases the right multiplication by $\rho(EF, I) \boxtimes e(\boldsymbol{x})$ commutes with d.*

Proof The proof is similar to those of Lemmas 6.19 and 6.20. □

6.4.5 The right multiplication by $\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$ The construction in this subsection is dual to that in Section 6.4.3. Recall from Definition 5.17 that

$$\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$$

exists if and only if $1 \le j \le k < n$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$. Note that the condition $x_j = j$ implies that $x_l = l$ for all $1 \le l \le j$, ie each of the first j states in $\boldsymbol{x}$ is $|1\rangle$. We are interested in the $(j+1, 1)^{\text{th}}$ entry $C_1^{j+1}(EF, \boldsymbol{x})$ of the double complex (1):

$$PH((EF\boldsymbol{x})_1^{j+1})\{n-\overline{x}_1\}[2k-\overline{x}_1+j-1] \oplus PH((EF\boldsymbol{x})_1^{j+1})\{n-\overline{x}_1+1\}[2k-\overline{x}_1+j]$$

Note that $(EF\boldsymbol{x})_1^{q_1+1} = \boldsymbol{x}$. Hence, there is a path from $(EF\boldsymbol{x})_1^1$ to $\boldsymbol{x}$ through $(EF\boldsymbol{x})_1^{j+1}$ in $Q_{n,k}$:

$$(EF\boldsymbol{x})_1^1 \xrightarrow{1} \cdots \xrightarrow{j} (EF\boldsymbol{x})_1^{j+1} \xrightarrow{j+1} \cdots \xrightarrow{q_1} (EF\boldsymbol{x})_1^{q_1+1} = \boldsymbol{x}$$

Let $r_{EF,I}(\boldsymbol{x}; j+1)$ be a product of $q_1 - j$ generators in $H(R_{n,k})$ corresponding to the path from $(EF\boldsymbol{x})_1^{j+1}$ to $\boldsymbol{x}$. Then the right multiplication is a map $C(EF,\boldsymbol{x}) \to C(I,\boldsymbol{x})$ of left $H(R_n)$–modules defined on the generators by

(M7-2) $\quad m \times (\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x}))$

$$= \begin{cases} r_{EF,I}(\boldsymbol{x}; j+1) \cdot m(I\boldsymbol{x}) & \text{if } m = m'((EF\boldsymbol{x})_1^{j+1}), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.24 *The right multiplication is compatible with the DG structure on $A \boxtimes R_n$:*

$$d(m'((EF\boldsymbol{x})_1^{j+1})) \times (\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})) = m'((EF\boldsymbol{x})_1^{j+1}) \times d((\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})))$$

for $1 \le j \le k < n$ and $\boldsymbol{x} \in \mathcal{B}_{n,k}$ such that $x_j = j$.

Proof The proof is similar to that of Lemma 6.22. □

This concludes the definition of the right $A \boxtimes R_n$–module structure on C_n.

Proposition 6.25 *The definitions of the right multiplications by $A \boxtimes R_n$ are well defined.*

Proof For the relations in $A \boxtimes R_n$, we need to verify that

$$(m \times r_1) \times r_2 = (m \times r_1') \times r_2'$$

if $r_1 \cdot r_2 = r_1' \cdot r_2' \in A \boxtimes R_n$ for $m \in C$ and generators $r_1, r_2, r_1', r_2' \in A \boxtimes R_n$. We checked the relations

$$(e(\Gamma) \boxtimes \rho(\boldsymbol{x} \xrightarrow{i} \boldsymbol{x}))^2 = 0$$

for $\Gamma = E, F$ in Lemmas 6.7 and 6.12. The commutation relations which come from isotopies of stackings of disjoint rook diagrams are easily verified since the definition of the right multiplication only depends on local properties of the rook diagrams.

For the DG structure, we need to verify that

$$d(m \times r) = dm \times r + m \times dr$$

for $m \in C$ and any generator $r \in A \boxtimes R_n$. We proved it when we showed that

- $d(r) = 0$ in Lemmas 6.8, 6.10, 6.13, 6.14, 6.19, 6.20 and 6.23,
- $r = e(F) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_1 = 0$ in Lemma 6.11,
- $r = e(E) \boxtimes r(\boldsymbol{x} \xrightarrow{i,s_1,\boldsymbol{v}} \boldsymbol{y})$ with $s_0(\boldsymbol{v}) = 0$ in Lemma 6.15,
- $r = \rho(I\boldsymbol{x} \xrightarrow{i} EF\boldsymbol{x}))$ and $\rho(EF\boldsymbol{x} \xrightarrow{j} I\boldsymbol{x})$ in Lemmas 6.22 and 6.24, respectively.

The proofs for other cases are similar and we leave them to the reader. □

Since the right multiplications are defined as maps of left $H(R_n)$–modules, we have

$$a \cdot (m \times r) = (a \cdot m) \times r,$$

for $a \in H(R_n)$, $r \in A \boxtimes R_n$ and $m \in C_n$. Hence C_n is a t–graded DG $(H(R_n), A \boxtimes R_n)$–bimodule.

7 The categorical action of HP(A) on HP($H(R_n)$)

In this section, we use the $(H(R_n), A \boxtimes R_n)$–bimodule C_n to categorify the action of $\boldsymbol{U}_T$ on $V_1^{\otimes n}$. Let

$$\eta_n\colon \mathrm{DGP}(A \boxtimes R_n) \xrightarrow{C_n \otimes_{A \boxtimes R_n} -} \mathrm{DGP}(H(R_n))$$

be a functor of tensoring with the DG $(H(R_n), A \boxtimes R_n)$–bimodule C_n over $A \boxtimes R_n$.

Lemma 7.1 *For all $\Gamma \in \mathcal{B}$ and $\boldsymbol{x} \in \mathcal{B}_n$, $\eta_n(P(\Gamma, \boldsymbol{x})) = C_n(\Gamma, \boldsymbol{x}) \in \mathrm{DGP}(H(R_n))$.*

Proof The proof is similar to that of Lemma 2.23. □

There is an induced exact functor

$$\eta_n\colon \mathrm{HP}(A \boxtimes R_n) \xrightarrow{C_n \otimes_{A \boxtimes R_n} -} \mathrm{HP}(H(R_n))$$

between the 0^{th} homology categories. From Lemma 5.22, we choose an equivalence $\mathcal{F}_n\colon \mathrm{HP}(A \otimes H(R_n)) \to \mathrm{HP}(A \boxtimes R_n)$ of triangulated categories. Let $\mathcal{M}_n = \eta_n \circ \mathcal{F}_n \circ \chi_n$ be the composition

$$\mathrm{HP}(A) \times \mathrm{HP}(H(R_n)) \xrightarrow{\chi_n} \mathrm{HP}(A \otimes H(R_n)) \xrightarrow{\mathcal{F}_n} \mathrm{HP}(A \boxtimes R_n) \xrightarrow{\eta_n} \mathrm{HP}(H(R_n)),$$

where χ_n is given in Definition 5.23, inducing the tensor product on the Grothendieck groups.

Proof of Theorem 1.3 We use $\{[P(\Gamma, \boldsymbol{x})]\}$ as a basis of $K_0(\mathrm{HP}(A \boxtimes R_n))$ to compute $K_0(\eta_n)$. By Lemma 6.18,

$$K_0(\eta_n)(\Gamma \otimes \boldsymbol{x}) = K_0(\eta_n)([P(\Gamma, \boldsymbol{x})]) = [C \otimes P(\Gamma, \boldsymbol{x})] = [C(\Gamma, \boldsymbol{x})] = \Gamma(\boldsymbol{x}) \in V_1^{\otimes n}.$$

Hence the $\mathbb{Z}[t^{\pm 1}]$–linear map

$$K_0(\mathcal{M}_n)\colon K_0(\mathrm{HP}(A)) \times K_0(\mathrm{HP}(H(R_n))) \to K_0(\mathrm{HP}(H(R_n)))$$

agrees with the action of $\boldsymbol{U_T}$ on $V_1^{\otimes n}$: $\boldsymbol{U_T} \times V_1^{\otimes n} \to V_1^{\otimes n}$. $\square$

Remark 7.2 It is natural to ask whether the categorical action is associative up to equivalence:

$$\begin{tikzcd}
\mathrm{HP}(A) \times \mathrm{HP}(A) \times \mathrm{HP}(H(R_n)) \arrow[r, "\mathrm{id} \times \mathcal{M}_n"] \arrow[d, "\mathcal{M} \times \mathrm{id}"] & \mathrm{HP}(A) \times \mathrm{HP}(H(R_n)) \arrow[d, "\mathcal{M}_n"] \\
\mathrm{HP}(A) \times \mathrm{HP}(H(R_n)) \arrow[r, "\mathcal{M}_n"] & \mathrm{HP}(H(R_n))
\end{tikzcd}$$

The question is equivalent to verifying some associativity relation on various DG bimodules. The computation is quite technical and we leave it to future work.

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