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Ropelength criticality

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The *ropelength problem* asks for the minimum-length configuration of a knotted diameter-one tube embedded in Euclidean three-space. The core curve of such a tube is called a tight knot, and its length is a knot invariant measuring complexity. In terms of the core curve, the thickness constraint has two parts: an upper bound on curvature and a self-contact condition.

We give a set of necessary and sufficient conditions for criticality with respect to this constraint, based on a version of the Kuhn–Tucker theorem that we established in previous work. The key technical difficulty is to compute the derivative of thickness under a smooth perturbation. This is accomplished by writing thickness as the minimum of a C^1–compact family of smooth functions in order to apply a theorem of Clarke. We give a number of applications, including a classification of the "supercoiled helices" formed by critical curves with no self-contacts (constrained by curvature alone) and an explicit but surprisingly complicated description of the "clasp" junctions formed when one rope is pulled tight over another.

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Unlike the classical machine that is composed of well-defined parts that interact according to well-understood rules (gears and cogs), the sliding interaction of two ropes under tension is extraordinary and interactive, with tension, topology, and the system providing the form which finally results.
—Louis H Kauffman, *Knots and physics*, 1992

1 Introduction

Our goal in this paper is to investigate what shape a knot or link attains when it is tied in rope of a given diameter (or thickness) and then pulled tight. Ignoring elastic deformations within the rope, we formulate this as the *ropelength problem*: to minimize the length of a knot or link L in Euclidean space subject to the condition that it remains one unit thick. Although there are many equivalent formulations (see Cantarella, Kusner

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and Sullivan [5] and Gonzalez and Maddocks [14]) of this thickness constraint, perhaps the most elegant simply requires that the *reach* of L be at least $\frac{1}{2}$. Here, following Federer, the reach of L is the supremal $r \geq 0$ such that every point in space within distance r of L has a unique nearest point on L. Any curve of positive reach is $C^{1,1}$, that is, its unit tangent vector is a Lipschitz function of arc length.

In an earlier paper [4], we studied a simplified version, the Gehring link problem, in which the thickness constraint is replaced by the weaker requirement that the *link-thickness* — the minimal distance between different components of the link — is at least 1. Thinking of the components again as strands of rope of diameter 1, this means that different strands cannot overlap, but each strand can pass through itself. Our balance criterion [4] for the Gehring problem made precise the intuition that, in a critical configuration for a link L, the tension forces seeking to minimize length must be balanced by contact forces. More precisely, we defined a *strut* to be a pair of points on different components at distance exactly 1. The balance criterion says that L is critical if and only if there is a nonnegative measure on the set of struts, thought of as a system of compression forces, which balances the curvature vector field of L.

The strut measure should be thought of as giving Lagrange multipliers for the distance constraints; our proof was basically an infinite-dimensional Lagrange multipliers argument characterizing critical points of length constrained by the nonsmooth thickness functional. The general procedure for such a problem is to write the nonsmooth constraint as the minimum of a compact family of differentiable constraints. In the case of link-thickness, this is immediate: we just take the infinite family of pairwise distances between points on different components of the curve. Our proof was then based on two technical tools. First, Clarke's theorem [7] on the derivatives of "min-functions" (our Theorem 3.1) lets us compute the directional derivative of the link-thickness with respect to a smooth deformation of L. Second, we proved a new version of the Kuhn–Tucker theorem on extrema of functionals subject to convex constraints, similar in spirit to a version by Luenberger [15], but giving necessary and sufficient conditions for a strong form of criticality. This provided the required version of the Lagrange multipliers theorem.

In the present paper we adopt the same general approach to develop a criticality theory for the (technically much more difficult) ropelength problem. Again the main point is to express the thickness as the minimum of a compact family of smooth functions. For this, we recall some equivalent reformulations [5, Lemmas 1, 2] of thickness for a space curve. First, it is the infimal diameter of circles through three points on the curve, and this is always realized in a limit as at least two points approach each other. (This idea originates with [14] and leads to interesting work on approximating ropelength by smooth integral Menger curvature energies; see for instance Strzelecki, Szumańska and von der Mosel [21].) Second, the thickness is always either the minimum self-critical distance or twice the infimal radius of curvature, as illustrated in Figure 1.

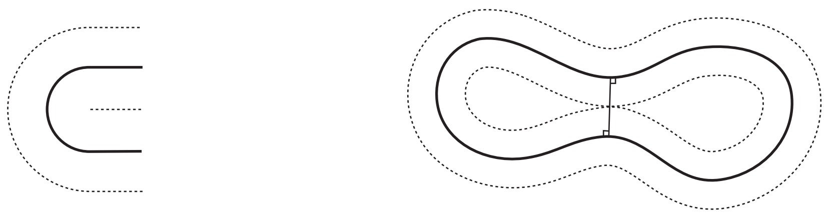

Figure 1: The diameter of an embedded tube around a curve is controlled by twice the radius of curvature (left) and by local minima of the self-distance function on the tube (right).

Guided by this last picture, we write thickness as the minimum of two compact subfamilies of smooth functions, controlling self-distance and curvature respectively.

The first subfamily is indexed by *all* pairs of points of the link L, but of course cannot simply be the distance, since this vanishes along the diagonal. Guided by the trigonometric factors that appear in the three-point diameter when two of the three points approach each other, we define a penalized distance between two points (depending also on the tangent direction at one of them) which equals distance for critical pairs and achieves its minimum only at such pairs (while blowing up along the diagonal). This yields a C^1–compact family of functions indexed by $L \times L$.

The second subfamily controls the curvature of L, but its construction is complicated by the fact that L need not be C^2. Nevertheless, since any thick curve is $C^{1,1}$ (meaning the tangent vector is Lipschitz continuous), L is twice differentiable — and thus admits an osculating circle — almost everywhere. It is now tempting to simply use the limit inferior to define a lower semicontinuous radius of curvature function along the curve. We can view this as a family indexed by the compact set L, but Clarke's theorem requires that the derivatives under any variation vector field also be lower semicontinuous, which is not the case here. (Knowing the derivative of curvature requires knowing the osculating plane, information which is lost in the lim inf.) Fixing this requires a genuinely new idea. We consider the closure $\overline{\operatorname{Osc} L}$ of the set of osculating circles in the space of all pointed circles in $\mathbb{R}^3$; the functions in our second subfamily simply measure the diameter of each circle.

Proceeding in this way, we formulate and prove our first main result — the general balance criterion of Theorem 3.17 — which gives a necessary and sufficient condition for a link to be (strongly) critical for length under the thickness constraint. As in the Gehring case, the condition requires the existence of a certain measure balancing the curvature of L, this time the sum of the strut measure and a *kink measure* on the space $\overline{\mathrm{Osc}\, L}$ of circles. In particular, in the case when there are no kinks, we recover the criticality criterion of Schuricht and von der Mosel [19], who discussed tight knots where the curvature constraint is nowhere active.

Our analysis also applies to the case where, in addition to the thickness constraint, the radius of curvature of the curve is constrained to be at least σ, a parameter giving the *stiffness* of the link. (Here we take $\sigma \geq \frac{1}{2}$, with $\sigma = \frac{1}{2}$ corresponding to the ordinary ropelength problem.)

The general balance criterion can be applied directly to curves without kinks; for example we classify curves with struts in one-to-one contact as double helices. The kink measure, on the other hand, is a bit arcane and can be difficult to work with: in general, L is no smoother than $C^{1,1}$, so the space $\overline{\mathrm{Osc}\, L}$ may be an unruly subspace of the normal bundle over L. For a C^2 link, of course, the kink measure reduces to a measure along L, but unfortunately, the only known example of a tight link which is C^2 is the round circle, the ropelength-minimizing unknot. On the other hand, all known explicit examples of tight links [5; 4] are piecewise C^2, indeed even piecewise analytic.

With a view towards the fact that other tight links (say, the tight trefoil knot) may not even be piecewise C^2, in Section 4, we impose the even milder smoothness assumption of *regulated kinks*. We conjecture that all critical links have regulated kinks, but an answer to this question seems far beyond our current understanding. For links with regulated kinks, we derive successively nicer forms of our balance criterion, concluding with Theorem 4.13, our second main result. It says the kink measure can be described by a scalar *kink tension function* — or equivalently, by a *virtual tangent* vector — along the curve. As an example, we use this theorem to classify all strut-free arcs in critical curves. We combine our results into Theorem 4.19, which will be the most useful form for most applications. Readers who are not interested in the underlying analytical machinery may wish to start there.

At the end of the paper, we apply our balance criterion to describe the ropelength-critical symmetric clasps. A curious feature of these clasps — whose analysis is based on the discussion in [4, Section 9] and whose form was independently derived by Starostin [20] — is the presence of a gap between the tips of the two components. In other words, there is a small cavity between two tight ropes of circular cross-section linked in this way.

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2 Curves, reach, curvature and thickness

We must begin this paper with the lengthy and somewhat intricate reformulation of thickness outlined in the introduction. Proposition 2.14 achieves the goal of writing thickness as the minimum of a compact family of functions; Corollary 2.16 extends this to a family of thicknesses modeling stiff ropes. This allows us to use Clarke's theorem (Theorem 3.1) to compute first variation of thickness in Section 3.1.

We consider generalized links, which may include arc components with constrained endpoints; our links are always C^1 but not necessarily C^2.

A C^1 *curve* L will mean a compact 1–dimensional C^1 submanifold with boundary embedded in $\mathbb{R}^3$. (For us, manifold will always mean manifold with boundary.) The curve L is thus a finite union of components, each a circle or an arc (compact interval). Our results are independent of orientation, but for convenience in taking derivatives we fix an orientation on each component. The Euclidean metric on $\mathbb{R}^3$ pulls back to give a Riemannian metric on L; we denote the positively oriented unit tangent vector at a point $x \in L$ by $T(x)$. The orientation induces a sign ± 1 on each endpoint $p \in \partial L$ such that $\pm T(p)$ is the outward tangent vector.

Each arc or circle component of length ℓ is of course isometric to $[0, \ell]$ or $\mathbb{R}/\ell\mathbb{Z}$, respectively. Writing M for the disjoint union of these intervals or circles, the isometry $\gamma\colon M \to L \subset \mathbb{R}^3$ is simply an arc length parametrization of L, and we use it to implicitly identify M with L.

All standard smoothness classes of functions on L are obtained via this identification. In particular, given a (vector-valued) function f on L, we write $f'(x)$ for the arc length derivative of f at any $x \in L$; for example $\gamma'(x) = T(x)$.

When we talk about the degree of smoothness of a Lipschitz curve L we mean the smoothness of the arc length parametrization; it is a standard and straightforward fact that no (immersive) reparametrization can be smoother. For any C^1 curve L, we let $E_L \subset L$ denote the set of points at which L (meaning its arc length parametrization) is twice differentiable. (At an endpoint $x \in \partial L$ we of course require only a one-sided second derivative.) No reparametrization has a second derivative at any point of $L \setminus E_L$. For $x \in E_L$, we write $\kappa(x) := T'(x) = \gamma''(x)$ for the curvature vector.

Suppose we have a C^2–smooth vector-valued function $f: \mathbb{R}^3 \to V$ on space. Its restriction to L is C^1 (with respect to arc length); indeed we have $f'(x) = D_x f(T(x))$. For $x \in E_L$, the second arc length derivative along L also exists and is given in terms of the spatial derivatives of f by

$$f''(x) = D_x^2 f(T(x), T(x)) + D_x f(\kappa(x)).$$

We say a sequence $L_1, L_2, \dots$ of C^1 curves *converges in the C^1 topology* to a C^1 curve L if there are C^1 immersions $\gamma_i: L \to \mathbb{R}^3$ with images $\gamma_i(L) = L_i$ such that the maps γ_i converge in C^1 to the inclusion map γ. Of course each γ_i has a reparametrization $\gamma_i \circ \phi_i$ with locally constant speed (that is, constant speed on each component). Since these also converge to γ, we usually assume each γ_i has locally constant speed.

2.1 Reach

To handle our generalized links, we need to reconsider the equivalence of the various formulations of reach or thickness mentioned in the introduction, that are by now standard for closed curves. Federer's definition [12] of reach can be rephrased as follows:

Definition 2.1 Given a link (or indeed any closed set) $L \subset \mathbb{R}^3$, its *medial axis* is the set of points $p \in \mathbb{R}^3$ for which the nearest point $x \in L$ is not unique. The *reach* of L, $\operatorname{reach}(L)$, is the distance from L to its medial axis.

Of course, a closed subset $L \subset \mathbb{R}^3$ has infinite reach if and only if it is convex. For curves, this means $\operatorname{reach}(L) = \infty$ if and only if L is a connected straight arc. We will often implicitly exclude this trivial case, for instance when discussing derivatives of reach.

To analyze the reach of a curve in more detail, we need to consider its tangent and normal cones. Let L be a C^1 curve in $\mathbb{R}^3$. At any interior point $x \in L$, the *tangent cone* $T_x L$ is the line through x tangent to L. At an endpoint $x \in \partial L$ of an arc component, $T_x L$ is the (inward) tangent ray. The *normal cone* $N_x L$ is

$$N_x L := \{p \in \mathbb{R}^3 \mid \langle p - x, q - x \rangle \le 0 \text{ for all } q \in T_x L\}.$$

At an interior point this is the normal plane, while at an endpoint $x \in \partial L$ it is a closed halfspace. (These cones are the translates by the base point x of the corresponding cones given by Federer [12] for general closed subsets of $\mathbb{R}^n$.)

The following alternate characterization of reach is then an immediate corollary of [12, Theorem 4.8].

Lemma 2.2 *If L is a C^1 curve in $\mathbb{R}^3$ then the reach of L equals the infimal $r > 0$ such that there exist $x \neq y \in L$ and $p \in N_x L$ with $|p-x| = r = |p-y|$.* □

If $p \notin N_x L$, then there are points near x in L which are closer to p. Thus if (x, y) is a local minimum for $|x-y|$ on $L \times L$ (away from the diagonal), then (x, y) is a critical pair in the following sense:

Definition 2.3 A pair of distinct points $x, y \in L$ is a *critical pair* if $x \in N_y L$ and $y \in N_x L$. We denote the set of all critical pairs by $\mathrm{Crit}(L)$.

We would now like to reformulate the lemma above in terms of the radii of circles tangent to the curve at one point and passing through another point.

Definition 2.4 For distinct points $x, y \in L$, let $C(x, y)$ denote the circle (or line) through y tangent to L at x. By plane geometry, its radius is

$$\frac{|x-y|}{2\cos\psi(x, y)} =: r(x, y),$$

where $\psi(x, y) \in [0, \pi/2]$ denotes the angle between the normal plane to L at x and the segment xy. (The notation we define here suppresses the dependence of C, r and ψ on L, in particular on $T_x L$.)

To properly handle endpoints of generalized links, we also need variants of these functions. So consider now circles in the plane of $T_x L$ and y, passing through x and y. Let $C^*(x, y)$ denote the smallest such circle whose center lies in $N_x L$. Then $C^*(x, y) = C(x, y)$ except when $x \in \partial L$ and $y \in N_x L$, in which case $C^*(x, y)$ is a circle with diameter xy. The radius of $C^*(x, y)$ is

$$\frac{|x-y|}{2\cos\psi^*(x, y)} =: r^*(x, y) \le r(x, y),$$

where $\psi^*(x, y) \in [0, \pi/2]$ denotes the angle at x between $N_x L$ and the segment xy. Thus $\psi^* = 0$ for $y \in N_x L$ and $\psi^* = \pi/2$ for $y \in T_x L$. Furthermore $\psi^*(x, y) = \psi(x, y)$ if x is an interior point of L.

Lemma 2.2 can now be rephrased as follows:

Corollary 2.5 *If L is a C^1 curve in $\mathbb{R}^3$ then*

$$\operatorname{reach}(L) = \inf_{x\neq y\in L} r^*(x,y) = \min\Bigl(\inf_{x\neq y\in L} r(x,y), \inf_{\substack{x\neq y\in L\\ x\in\partial L}} r^*(x,y)\Bigr).$$

Proof Any point $p \in N_x L$ as in Lemma 2.2 is the center of a circle through x and y; hence $|p-x| \geq r^*(x,y)$. Conversely, the center of any $C^*(x,y)$ is such a point p. This gives the first equality. The second follows from the fact that $r^*(x,y) \leq r(x,y)$ with equality unless $x \in \partial L$. □

(For closed curves, this was also the first statement in [5, Lemma 1]. The proof of the later parts of that lemma should have been more careful about the treatment of points where L is not twice differentiable.)

For any C^1 link L, the angles ψ and ψ^* extend continuously to the diagonal, since $\lim_{y\to x}\psi(x,y) = \pi/2 = \lim_{y\to x}\psi^*(x,y)$. But without additional smoothness of L, the functions r and r^* do not extend. For smooth curves, of course, it is a standard fact that as $y \to x$, the circles tangent at x through y approach the osculating circle at x. For completeness, we verify that the existence of a second derivative at x is sufficient for this:

Lemma 2.6 *Suppose L is a C^1 curve with curvature vector κ at a point $x \in E_L$. Then*

$$\lim_{y\to x} r(x,y) = \lim_{y\to x} r^*(x,y) = 1/|\kappa|.$$

Proof First note that for y sufficiently near x, we have $y \notin N_x L$ so $\psi^*(x,y) = \psi(x,y)$ and thus $r^*(x,y) = r(x,y)$. Assume $x = 0 \in \mathbb{R}^3$ and let γ be an arc length parametrization around x so

$$\gamma(0) = 0, \quad \gamma'(0) = T = T(x), \quad \gamma''(0) = \kappa.$$

Taylor's theorem implies that

$$\gamma(s) = sT + \tfrac{s^2}{2}\kappa + o(s^2).$$

For $y = \gamma(s)$, we can compute ψ from the equation $|T \times y| = |y| \cos\psi(x,y)$. We get

$$r(x,y) = \frac{|\gamma(s)|^2}{2|T\times\gamma(s)|} = \frac{s^2 + o(s^3)}{|\kappa|s^2 + o(s^2)} = 1/|\kappa| + o(1). \qquad \square$$

Lemma 2.7 *Suppose a C^1 curve L is twice differentiable at $x \in E_L$, and suppose $y \in L \smallsetminus N_x L$. Fix the orientation at x such that $\langle T(x), y-x\rangle > 0$. If $r(x,y) < \infty$, then the partial derivative $\partial r/\partial x$ exists, with*

$$\frac{\partial r}{\partial x}(x,y) \le (r(x,y)|\kappa(x)| - 1)\tan\psi(x,y).$$

Proof From plane geometry, the rotation speed of the vector $x-y$ is

$$\left|\frac{\partial}{\partial x}\left(\frac{x-y}{|x-y|}\right)\right| = \frac{1}{2r(x,y)}.$$

The normal plane $N_x L$ of course turns at rate $|\kappa(x)|$. Comparing these rates gives

$$-\frac{1}{2r(x,y)} - |\kappa(x)| \le \frac{\partial\psi(x,y)}{\partial x} \le -\frac{1}{2r(x,y)} + |\kappa(x)|.$$

On the other hand differentiating the definition of r gives

$$\frac{\partial r(x,y)}{\partial x} = -\tfrac{1}{2}\tan\psi + r\tan\psi\frac{\partial\psi}{\partial x}.$$

The desired inequality follows at once. □

2.2 Penalized distance

Recall that in order to apply Clarke's theorem (Theorem 3.1) to compute the derivative of reach(L) under a smooth deformation of L, we must express the reach as the minimum of a compact family of functions. For a closed C^2 curve L, we could simply extend r continuously to the diagonal $x=y$ by Lemma 2.6, getting a compact family parametrized by $L \times L$. Unfortunately, the examples of [5] show that even ropelength minimizers may fail to be C^2. (For the same reason, the three-point curvature defined off the diagonal in $L \times L \times L$ has no nice extension to the diagonal, and thus cannot be used in Clarke's theorem.)

On the other hand by [5, Lemma 4], the reach condition implies that L is $C^{1,1}$, meaning that T is a Lipschitz function of arc length. Recall that by Rademacher's theorem (cf Royden [18, Section 5.4]), a Lipschitz function is differentiable almost everywhere, so E_L has full measure if L is $C^{1,1}$. This turns out to be enough to make Clarke's theorem work using the more technical approach that we now describe.

The expression of thickness in terms of minimum self-distance and minimum radius of curvature is mirrored in the following dichotomy. First, if the infimal r is achieved, then it is achieved for a critical pair (x,y), where $r = |x-y|/2$. To avoid the problem that the infimal r might also be achieved at noncritical pairs, we next define a penalized distance function that achieves its minimum only on critical pairs. Second, if the infimal r is not achieved, then it is approached in the limit as $y \to x$. Intuitively, this should happen at a point of maximum curvature, but in fact L might not even be twice differentiable at the limit point. To handle this limiting behavior near the diagonal, in Section 2.3 we will look at the set of osculating circles (at points where L is twice differentiable) and compactify it within the space of all pointed circles in space.

Definition 2.8 Given a link L, the *penalized distance* between two distinct points $x, y \in L$ is

$$\mathrm{pd}(x, y) := |x - y| \sec^2 \psi(x, y) = 2r(x, y) \sec \psi(x, y).$$

For $y = x$, we set $\mathrm{pd}(x, x) = \infty$. When we want to emphasize the dependence on L, we will write $\mathrm{pd}^L(x, y)$. Similarly the *penalized endpoint distance* is

$$\mathrm{pd}^*(x, y) := |x - y| \sec^2 \psi^*(x, y) = 2r^*(x, y) \sec \psi^*(x, y) \le \mathrm{pd}(x, y).$$

For $y = x$, we set $\mathrm{pd}^*(x, x) = \infty$. Of course $\mathrm{pd}^*(x, y) = \mathrm{pd}(x, y)$ except when $x \in \partial L$.

Lemma 2.9 *Given a link L of positive reach, the penalized distance is a continuous function from $L \times L$ to $(0, \infty]$. Similarly, the penalized endpoint distance is continuous when restricted to $\partial L \times L$.*

Proof First, we note that the angle $\psi(x, y)$ (extended to be $\pi/2$ on the diagonal $x = y$) is continuous. The formula for $\mathrm{pd}(x, y)$ shows it shares this continuity away from the diagonal. But we also have continuity on the diagonal, since $r \ge \mathrm{reach}(L) > 0$, while ψ approaches $\pi/2$ as $(x, y) \to (z, z)$.

On the other hand the penalized endpoint distance $\mathrm{pd}^*(x, y)$ is merely lower semicontinuous, since it equals $\mathrm{pd}(x, y)$ away from endpoints $x \in \partial L$ but can jump down there. But the continuity claimed here is easy: for fixed $x \in \partial L$, the angle $\psi^*(x, y)$ is continuous in y, and the rest follows as above. □

Lemma 2.10 *Suppose $0 < \mathrm{reach}(L) < \infty$. We have $\mathrm{pd}^*(x, y) \ge 2\,\mathrm{reach}(L)$ for all $x, y \in L$; equality can hold only if x, y is a critical pair.*

Proof Clearly $\mathrm{pd}^*(x, y) \ge 2r^*(x, y)$, with equality only when $\psi^*(x, y) = 0$, that is, when $y \in N_x L$. Since $r^*(x, y) \ge \mathrm{reach}(L)$ by Corollary 2.5, it only remains to show that $x \in N_y L$ in the case $\mathrm{pd}^*(x, y) = 2\,\mathrm{reach}(L)$. If not, there is a tangent vector T to L at y such that $\langle x - y, T\rangle > 0$. The directional derivative of $|x - y|$ in the direction T is negative; since $\psi^*(x, y) = 0$, the directional derivative of $\mathrm{pd}^*(x, y)$ is the same negative value, contradicting the fact that $\mathrm{pd}^*(x, y) = \mathrm{reach}(L)$ is a minimum. □

2.3 Osculating circles

Capturing the curvature portion of the thickness information on a $C^{1,1}$ curve as a min-function will require a genuinely new idea. As mentioned in the introduction, one might be tempted to use lim inf to replace the radius of curvature defined on E_L by a lower semicontinuous function on L. But its time derivative under a variation of L would not be lower semicontinuous, so Clarke's theorem would not work.

Instead we recall that at each point in the dense set $E_L \subset L$ there is an osculating circle. Taking the closure of the set of these osculating circles inside the space of pointed circles in $\mathbb{R}^3$ gives the compact index set on which the radius function is C^1–continuous. This construction is the most important technical idea in this paper, and we note that a similar idea should be essential in extending our results to surfaces or higher-dimensional submanifolds.

Thus we consider the space Circ of all oriented pointed circles (including lines) in $\mathbb{R}^3$. We describe a circle through $p \in \mathbb{R}^3$ by its oriented unit tangent $T \in \mathbb{S}^2$ at p together with its curvature vector $\kappa \in T_T\mathbb{S}^2$ there. This identifies Circ with $\mathbb{R}^3 \times TS^2 \ni (p, T, \kappa)$. Here of course $\kappa = 0$ exactly when the circle degenerates to a line. Let $R(p, T, \kappa) := 1/|\kappa| \in (0, \infty]$ be the radius function on Circ and let Π denote the projection $\Pi\colon (p, T, \kappa) \mapsto p$.

Given a $C^{1,1}$ link L, the set E_L on which the second derivative exists has full measure. Note that the minimal Lipschitz constant $\operatorname{Lip}(T)$ for the tangent vector as a function of arc length is exactly $\sup_{E_L} |\kappa|$. We let $\operatorname{Osc} L \subset \operatorname{Circ}$ be the set of all osculating circles:

$$\operatorname{Osc} L := \{(x, T(x), \kappa(x)) \mid x \in E_L\} \subset \operatorname{Circ}.$$

Its closure $\overline{\operatorname{Osc} L}$ is a compact subset of Circ since $|\kappa|$ is bounded on E_L. Note that for any $(x, T, \kappa) \in \overline{\operatorname{Osc} L}$ we have $x \in L$ and $T = T(x)$, while of course $\kappa \perp T$ is some normal vector; thus we can view $\overline{\operatorname{Osc} L}$ as a subset of the normal bundle to L.

For $x \in L$, we set $\overline{\operatorname{Osc} L}_x := \overline{\operatorname{Osc} L} \cap \Pi^{-1}\{x\}$. Since $E_L \subset L$ is dense, it follows that $\overline{\operatorname{Osc} L}_x$ is nonempty for every point $x \in L$. Thus for $x \in L$ we may define

$$\rho(x) := \min_{\overline{\operatorname{Osc} L}_x} R = \Big(\overline{\lim_{E_L \ni y \to x}} |\kappa(y)| \Big)^{-1}.$$

Note that ρ is essentially a Clarke upper derivative of the tangent vector T. Clearly ρ is lower semicontinuous, so it attains its minimum along L, which we can view as a minimum radius of curvature. For $x \in E_L$ we have $\rho(x) \le 1/|\kappa(x)|$, but equality might not hold.

Lemma 2.11 *If L is a $C^{1,1}$ curve and $c \in \overline{\operatorname{Osc} L}$ then $R(c) \geq \operatorname{reach}(L)$.*

Proof By continuity of R, it is enough to prove this for osculating circles $c \in \operatorname{Osc} L$. There it follows immediately from Corollary 2.5 and Lemma 2.6. □

Lemma 2.12 *If $r(x, y) = \operatorname{reach}(L)$ with $y \notin N_x L$, then $\rho(x) = \operatorname{reach}(L)$.*

Proof If not, we have $r(x, y) < \rho(x)$, in which case by lower semicontinuity of ρ there is a neighborhood U of x in L such that $r(x', y) < \rho(x')$ for $x' \in U$. At any $x' \in E_L \cap U$ we have $r(x', y)|\kappa(x')| < 1$, so by Lemma 2.7 we get $\partial r/\partial x < 0$. Since L is $C^{1,1}$, the function r is Lipschitz (at least locally where it is finite), so its values near x can be computed by integrating this derivative. But this contradicts the fact that r is minimized at x. □

Remark In fact under the hypothesis of Lemma 2.12, x and y lie on the same component of L, and the arc of L from x to y (in the direction of the tangent T at x with $\langle T, y - x\rangle > 0$) must be an arc of a circle, but we will not need to invoke this stronger statement.

Lemma 2.13 *Suppose γ is a subarc of L joining x to y with length at most $\pi r(x, y)$. Then $\sup_{\gamma \cap E_L} |\kappa| \geq 1/r(x, y)$, so $\inf_\gamma \rho \leq r(x, y)$.*

Proof In the case $r(x, y) = \infty$ there is nothing to prove. Otherwise, for convenience we rescale so that $r(x, y) = 1$ and translate so that $C(x, y)$ is centered at the origin. Letting B denote the open unit ball, $C(x, y)$ is then a great circle on ∂B.

First suppose there is a subarc $\alpha \subset \gamma$ disjoint from B and with endpoints $a, b \in \partial B$. Then α has length at most π but at least that of the great circular arc from a to b. Let β denote the extension of this latter arc (within the same great circle) with one endpoint at a and having the same length as α. Since this is still less than a semicircle, the distance between the endpoints of β is at least $|a - b|$. Applying Schur's comparison theorem to α and β, we conclude that the curvature of α is somewhere at least that of β, that is, that $\sup_\alpha |\kappa| \geq 1$ as desired. (In [22], we show that the standard proof (see Chern [6]) of Schur's theorem for smooth curves actually applies to all $W^{1,\mathrm{BV}}$ curves, that is to all curves of finite total curvature. In particular, it applies to $C^{1,1}$ curves, with the curvature comparison being between the measures $|\kappa|\, ds$.)

If there is no such subarc, then $B \cap \gamma$ is dense in γ. In particular there is a sequence $x_i \in \gamma \cap B$ with $x_i \to x$. It now suffices to show $\overline{\lim}_{y \to x} |\kappa(y)| \geq 1$.

The function $f(p) := |p|^2 - 1$ is $C^{1,1}$ along L with $f(x) = 0 = f'(x)$. Since $f(x_i) < 0$ there is some y_i between x and x_i with $f'(y_i) < 0$, and thus some z_i

between x and y_i such that $f''(z_i) < 0$. In fact the set of such z_i has positive measure, so we may choose $z_i \in E_L$. Then by the chain rule,

$$f''(z_i) = 2(1 + \langle z_i, \kappa(z_i)\rangle) > 2(1 - |z_i||\kappa(z_i)|),$$

so we find that $|\kappa(z_i)||z_i| > 1$. Since $|z_i| \to 1$, we have $\overline{\lim}\,|\kappa| \geq 1$ as desired. □

2.4 Thickness and stiff ropes

We can now prepare for the application of Clarke's theorem by expressing the reach of L as the minimum of a family of functions parametrized by the disjoint union $(L \times L) \sqcup \overline{\operatorname{Osc} L}$:

Proposition 2.14 *For any $C^{1,1}$ curve L,*

$$\begin{aligned}\operatorname{reach}(L) &= \min\Big\{\tfrac{1}{2}\min_{x,y\in L} \operatorname{pd}^*(x,y), \min_L \rho\Big\}\\ &= \min\Big\{\tfrac{1}{2}\min_{x,y\in L} \operatorname{pd}^*(x,y), \min_{c\in\overline{\operatorname{Osc} L}} R(c)\Big\}.\end{aligned}$$

Proof The right-hand sides are equal and by Lemmas 2.10 and 2.11 they are at least $\operatorname{reach}(L)$. It remains to prove that either $2\operatorname{reach}(L) = \operatorname{pd}^*(x,y)$ for some $x, y \in L$, or $\operatorname{reach}(L) = R(c)$ for some $c \in \overline{\operatorname{Osc} L}$.

By Corollary 2.5, we can find a sequence (x_i, y_i) with $r^*(x_i, y_i) \to \operatorname{reach}(L)$. By compactness, a subsequence converges to some pair (x, y). We consider three cases.

First, if $x \neq y$ and $y \in N_x L$ then $\psi^*(x,y) = 0$. Therefore, $\operatorname{pd}^*(x,y) = 2r^*(x,y) = 2\operatorname{reach}(L)$.

Second, if $x \neq y$ and $y \notin N_x L$, then by Lemma 2.12 we have $\operatorname{reach}(L) = \rho(x)$, which is the radius of some circle in $\overline{\operatorname{Osc} L_x}$ by compactness.

Third, if $x = y$, then for large i the subarc γ_i from x_i to y_i satisfies the length bound of Lemma 2.13. Applying the lemma, we find a point $z_i \in \gamma_i \cap E_L$ with $1/|\kappa(z_i)| \leq r(x_i, y_i) + 1/i$. Since $z_i \to x$ while $r(x_i, y_i) \to \operatorname{reach}(L)$, we conclude as desired that $\rho(x) \leq \operatorname{reach}(L)$. □

Proposition 2.14 permits us also to model *stiff* ropes, which cannot bend as much as the reach constraint permits.

Definition 2.15 If L is a $C^{1,1}$ curve and $\sigma \geq \frac{1}{2}$, we define the *σ–thickness* of L as

$$\operatorname{Thi}_\sigma(L) := \min\Big\{2\operatorname{reach}(L), \frac{1}{\sigma}\min_L \rho\Big\}.$$

We note that a link with $\mathrm{Thi}_\sigma \geq 1$ cannot have an osculating circle with radius less than σ. We specify $\sigma \geq \frac{1}{2}$ because otherwise this formula would simply give twice the reach. (It is tempting to try to define a thickness for $\sigma < \frac{1}{2}$ by combining the curvature term with a minimum distance of critical pairs. But this is unphysical in the sense that it permits the thick rope to penetrate itself near points of large curvature; furthermore it is not amenable to our analysis since the reformulation in terms of penalized distance does not apply.)

As a corollary, we get the main result of this section; it writes thickness as a min-function, which will let us apply Clarke's theorem.

Corollary 2.16 *For any link L and any $\sigma \geq \frac{1}{2}$ we have*

$$\begin{aligned}\mathrm{Thi}_\sigma(L) &= \min\Big\{\min_{x,y\in L} \mathrm{pd}^*(x,y), \frac{1}{\sigma}\min_L \rho\Big\}\\ &= \min\Big\{\min_{x,y\in L} \mathrm{pd}(x,y), \min_{\substack{x\in\partial L\\ y\in L}} \mathrm{pd}^*(x,y), \frac{1}{\sigma}\min_L \rho\Big\}.\end{aligned}$$

Proof The first equality follows immediately from Proposition 2.14. The second follows from the fact that $\mathrm{pd}^*(x,y) \leq \mathrm{pd}(x,y)$ with equality unless $x \in \partial L$. □

Clearly for any σ we have $\mathrm{Thi}_\sigma(L) = \infty$ if and only if L is a connected straight arc, since this is true of $\mathrm{reach}(L)$. From Lemma 2.10 and the definition of σ–thickness we immediately get:

Corollary 2.17 *Suppose* $0 < \mathrm{Thi}_\sigma(L) < \infty$. *If $x, y \in L$ satisfy* $\mathrm{pd}^*(x,y) = \mathrm{Thi}_\sigma(L)$ *then* $\mathrm{Thi}_\sigma(L) = 2\,\mathrm{reach}(L)$, *so* $(x,y) \in \mathrm{Crit}(L)$. □

Definition 2.18 We refer to pairs $(x,y) \in \mathrm{Crit}$ with $\mathrm{pd}^*(x,y) = \mathrm{Thi}_\sigma(L)$ as *struts*; and to circles $c \in \overline{\mathrm{Osc}\,L}$ such that $R(c) = \sigma\,\mathrm{Thi}_\sigma(L)$ as *kinks*. We denote the sets of struts and kinks by

$$\mathrm{Strut} = \mathrm{Strut}(L) \subset \mathrm{Crit} \subset L \times L, \quad \mathrm{Kink} = \mathrm{Kink}(L) \subset \overline{\mathrm{Osc}\,L} \subset \mathrm{Circ}\,.$$

Thus the σ–thickness of L is realized exactly at the struts and kinks.

Every kink is a circle of the same radius σ, indeed it is a point in Circ of the form $(x, T(x), n/\sigma)$ with $|n| = 1$. Thus we identify it with (x,n), and we can and will view $\mathrm{Kink}(L)$ as a subset of the unit normal bundle to L. But without additional smoothness assumptions on L it is hard to say anything about the possible structure of this kink set.

The σ–ropelength problem is to minimize length subject to the condition $\mathrm{Thi}_\sigma \geq 1$. For a closed link L, we minimize over the usual link type $[L]$. When L includes arc components, we constrain each endpoint $p \in \partial L$ to lie in an affine subspace denoted H^0_p (of dimension 0, 1 or 2). Furthermore we allow for Neumann or first-order boundary constraints by specifying that the tangent vector $T(p)$ at each endpoint stay in a linear subspace H^1_p; we consider only the cases of clamped tangents ($\dim H^1_p = 1$) and free tangents ($\dim H^1_p = 3$). We define the *constrained link type* $[L]$ (as in [4, Section 8]) by requiring that each endpoint p stay on H^0_p, with tangent $T(p) \in H^1_p$, during any isotopy. (Of course it would be easy to allow more general constraint manifolds but we will not need this for our examples.)

To prevent isotopy classes from being too large, we could also include obstacles for the curve, as in [4]. The resulting wall struts in the criticality theory work just as in the Gehring problem considered there. However, in the examples we have in mind (like the simple clasp) the obstacles are never active constraints, so the wall struts are not needed. Thus we leave this extension of the theory as a straightforward exercise for the reader.

Definition 2.19 Suppose $\mathrm{Thi}_\sigma(L) \geq 1$. We say that L is a ropelength minimizer constrained by σ–thickness (or, for short, a Thi_σ*–constrained minimizer*) in its (possibly constrained) link type $[L]$ if it minimizes length among all curves in $[L]$ with $\mathrm{Thi}_\sigma \geq 1$. We say L is a *local minimizer* if it minimizes length among all curves with $\mathrm{Thi}_\sigma \geq 1$ in some C^1 neighborhood.

Proposition 2.20 *The thickness* Thi_σ *is upper semicontinuous with respect to the* C^1 *metric on the space of* $C^{1,1}$ *curves* L*.*

Proof By definition, Thi_σ is the minimum of $\mathrm{reach}(L)$ and a scaled radius-of-curvature term. Federer has shown [12, Theorem 4.13] that $\mathrm{reach}(L)$ is upper semicontinuous even with respect to the (coarser) topology induced by Hausdorff distance.

It only remains to check that $\min_L \rho$ is semicontinuous with respect to C^1 convergence of L. Since ρ is a local function, it suffices to consider a connected curve L. Suppose L_i are $C^{1,1}$ curves converging to L. As we have noted earlier, we may assume that the convergent C^1 maps $\gamma_i\colon L \to L_i$ each have constant speed v_i (with $v_i \to 1$ of course). Now, by the lower semicontinuity of Lipschitz constants, we have

$$\Big(\min_L \rho\Big)^{-1} = \sup_{x \in E_L} |\kappa(x)| = \mathrm{Lip}(T) \leq \underline{\lim}\, \mathrm{Lip}(\gamma_i') = \underline{\lim}\, v_i^2 \sup_{x \in E_{L_i}} |\kappa_i(x)|$$
$$= \lim(v_i^2)\, \underline{\lim}\Big(\min_{L_i} \rho_i\Big)^{-1} = \underline{\lim}\Big(\min_{L_i} \rho_i\Big)^{-1}$$

which yields the desired conclusion. □

We now prove the existence of thickness-constrained minimizers, under a mild technical hypothesis that prevents the length of any component from shrinking to zero. Since a circle component of thickness $\mathrm{Thi}_\sigma \geq 1$ necessarily has length at least π, we only have to worry here about arc components. An arc component with endpoints p and q clearly has length bounded away from 0 if the constraints H_p^0 and H_q^0 are disjoint.

Corollary 2.21 *Suppose the constrained link type* $[L]$ *contains at least one curve* L *with* $\mathrm{Thi}_\sigma(L) \geq 1$*, and suppose that, in at least one length-minimizing sequence* L_i *of such curves, the length of each component stays bounded away from zero. Then there exists a* σ*–thickness constrained minimizer in* $[L]$*.*

Proof We may assume the L_i are parametrized at locally constant speed on a common domain (say L_1). By Arzela–Ascoli we may extract a subsequence converging in C^1 to a limit curve L_0. (If the link L is split, we assume without loss of generality that the various pieces stay within a common ball while they shrink.) Because the convergence is in C^1, we have $\mathrm{Len}(L_i) \to \mathrm{Len}(L_0)$, and by Proposition 2.20 we know $\mathrm{Thi}_\sigma(L_0) \geq \overline{\lim}\, \mathrm{Thi}_\sigma(L_i) \geq 1$. That the endpoints of L still satisfy the given constraints is clear. Finally, by C^1 convergence, L_0 is isotopic to all but finitely many of the L_i and in particular, $L_0 \in [L]$. □

3 The general balance criterion

We give an analytic condition, Theorem 3.17, that is both necessary and sufficient for a general curve to be critical for σ–ropelength (subject to the ancillary condition of Thi_σ–regularity). The condition may be viewed as an equation of vector distributions on $\mathbb{R}^3$. The approach follows the one we used in [4]: using Clarke's Theorem 3.1 we compute the derivative of the thickness of a curve L under a variation induced by a smooth vector field ξ; then we apply the Kuhn–Tucker theorem.

3.1 The derivative of thickness

Here we give a formula for the first variation of the σ–thickness of L, which will be key to the technical definition of criticality for length subject to thickness constraints. The proof is an application of a theorem of Clarke [7] on the directional derivatives of a function g that may be expressed as the minimum of a C^1–compact family $\{g_u\}$ of C^1 functions. Essentially this theorem states that the directional derivative of g at a point x is the minimum of the directional derivatives of those g_u for which $g_u(x) = g(x)$. In our case, this will mean that the first variation of thickness in the direction of a deforming vector field is given (in Theorem 3.5) as the minimum of the derivatives of the strut lengths and kink radii.

We use Clarke's theorem [7] in the following special case:

Theorem 3.1 (Clarke) *Let U be a sequentially compact topological space. Suppose that for each $u \in U$ and some $\epsilon > 0$ there is a C^1 function $g_u\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that the functions $(t, u) \mapsto g_u(t)$ and $(t, u) \mapsto g'_u(t)$ are lower semicontinuous. Then, putting $g(t) := \min_{u \in U} g_u(t)$, the right derivative of g at $t = 0$ exists and is given by*

$$\left.\frac{dg}{dt^+}\right|_{t=0} = \min\{g'_u(0) \mid u \in U, g_u(0) = g(0)\}. \qquad \square$$

That the *minima* exist (in the definition of g and the formula for its derivative) as opposed to infima, is of course an immediate consequence of the compactness hypothesis. There is nothing special about $t = 0$; the min function g has both one-sided derivatives at each $t \in (-\epsilon, \epsilon)$.

We have previously expressed thickness as the minimum of penalized distances between pairs of points on our curve and scaled radii over the closure of the set of osculating circles to L. It will be easy to differentiate penalized distances as we vary our curve, but somewhat more complicated to differentiate radii of curvature. We now turn to the task of defining and computing these derivatives.

While the main technical difficulties we face in this work are due to the fact that our curves may fail to be C^2, when we consider derivatives, it suffices to consider only variations arising from C^2–smooth deformations of the ambient space $\mathbb{R}^3$: our balance criteria show that criticality with respect to such variations suffices to get balancing measures.

We start by noting that any C^2 diffeomorphism $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ induces a homeomorphism ϕ_* on the space Circ of pointed circles: If $c \subset \mathbb{R}^3$ is the circle $(x, T, \kappa) \in \mathrm{Circ}$, then $\phi_*(x, T, \kappa)$ is the osculating circle at $\phi(x)$ to the C^2–smooth curve $\phi(c)$. It is clear that ϕ maps the circle c to a curve with velocity $v := D_x\phi(T)$ and acceleration $a := D_x^2\phi(T, T) + D_x\phi(\kappa)$. Thus

$$\phi_*(x, T, \kappa) = \left(\phi(x), \frac{v}{|v|}, \frac{a}{|v|^2} - \frac{\langle a, v\rangle v}{|v|^4}\right).$$

Expressing the length of the new curvature vector in the usual way in terms of the vector cross product gives

$$R(\phi_*(x, T, \kappa)) = \frac{|v|^3}{|v \times a|} = \frac{|D_x\phi(T)|^3}{|D_x\phi(T) \times (D_x^2\phi(T, T) + D_x\phi(\kappa))|}.$$

The variations of a link that we consider are generated by a C^1–smooth family of C^2 diffeomorphisms ϕ^t with $\phi^0 = \mathrm{Id}$. The *initial velocity* $\frac{\partial}{\partial t}|_{t=0}\phi^t$ is thus a C^2 vector field ξ. (Conversely, any C^2 vector field ξ on $\mathbb{R}^3$ is the initial velocity of some such family ϕ^t, for instance its local autonomous flow, given by $\partial\phi^t/\partial t = \xi \circ \phi^t$.) The diffeomorphisms ϕ^t induce a C^1–smooth family ϕ^t_* of homeomorphisms of Circ, whose initial velocity is a continuous vector field ξ_* on Circ depending only on ξ. The formula we need expresses the derivative of the radius function R in the direction ξ_* in terms of the given vector field ξ and its spatial derivatives.

Lemma 3.2 *Given a C^1–smooth one-parameter family of C^2 diffeomorphisms ϕ^t with initial velocity ξ, the time derivative of the radius function R (where this is finite) is*

$$\delta_\xi R(x,T,\kappa) := D_{(x,T,\kappa)}R(\xi_*) = 2R\langle T, D_x\xi(T)\rangle - R^3\langle \kappa, D_x^2\xi(T,T) + D_x\xi(\kappa)\rangle.$$

Proof By smoothness, the time derivatives commute with spatial derivatives. From $\phi^0 = \mathrm{Id}$ we see $D_x\phi^0 = \mathrm{Id}$ and $D_x^2\phi^0 = 0$. Thus we can write $\delta_\xi R(x,T,\kappa)$ as

$$\begin{aligned}\frac{3\langle T, D_x\xi(T)\rangle}{|T\times\kappa|} &- \frac{\langle T\times\kappa, D_x\xi(T)\times\kappa + T\times(D_x^2\xi(T,T)+D_x\xi(\kappa))\rangle}{|T\times\kappa|^3}\\ &= 3R\langle T, D_x\xi(T)\rangle - R^3(\langle T, D_x\xi(T)\rangle\langle\kappa,\kappa\rangle + \langle\kappa, D_x^2\xi(T,T)+D_x\xi(\kappa)\rangle),\end{aligned}$$

using the facts that $|T| = 1$ and $|T\times\kappa| = 1/R$. Since $\langle\kappa,\kappa\rangle = R^{-2}$, this reduces to the formula given. □

Of course if (x,T,κ) is the osculating circle to L at a point $x \in E_L$, then the quantity $D_x^2\xi(T,T) + D_x\xi(\kappa)$ appearing here is simply the second derivative ξ'' of ξ along L.

Corollary 3.3 *Suppose L is a $C^{1,1}$ curve and ξ a C^2 vector field on space. At any point $x \in E_L$ with osculating circle $c = (x,T,\kappa)$, $\kappa \neq 0$, we have*

$$\delta_\xi R(c) = 2R\langle \xi', T\rangle - R^3\langle \xi'', \kappa\rangle.$$

Lemma 3.4 *Suppose $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a C^2 diffeomorphism and $L \subset \mathbb{R}^3$ is a C^1 curve. Then its image ϕL is a C^1 curve with $E_{\phi L} = \phi E_L$. Assuming L is $C^{1,1}$, we also have $\phi_*(\overline{\mathrm{Osc}\,L}) = \overline{\mathrm{Osc}\,\phi L}$.*

Proof If γ is the arc length parametrization of L, then $\phi\circ\gamma$ is an immersive parametrization of ϕL. Since its second derivative exists at all points of ϕE_L we have $\phi E_L \subset E_{\phi L}$. The reverse inclusion follows by considering L as the image of ϕL under ϕ^{-1}. For a $C^{1,1}$ curve, we now see $\phi_*(\mathrm{Osc}\,L) = \mathrm{Osc}\,\phi L$; since ϕ_* is a homeomorphism, it follows that $\phi_*(\overline{\mathrm{Osc}\,L}) = \overline{\mathrm{Osc}\,\phi L}$. □

We are now ready to apply Clarke's theorem to give our first main result, a formula for the first variation of thickness of a link.

Theorem 3.5 *Let ϕ^t for $t \in (-\epsilon, \epsilon)$ be a C^1–smooth family of C^2 diffeomorphisms of $\mathbb{R}^3$ with $\phi^0 = \mathrm{Id}$, and let ξ be the initial velocity vector field*

$$\xi_x := \frac{\partial \phi^t(x)}{\partial t}\Big|_{t=0}.$$

Let L be a $C^{1,1}$ curve with $\mathrm{reach}(L) < \infty$. Then the function $t \mapsto \mathrm{Thi}_\sigma(\phi^t(L))$ is differentiable from the right at $t = 0$, with right-hand derivative

$$\begin{aligned}\delta_\xi \,\mathrm{Thi}_\sigma(L) &:= \frac{d\,\mathrm{Thi}_\sigma(\phi^t(L))}{dt^+}\Big|_{t=0} \\ &= \min\Bigl(\min_{(x,y)\in\mathrm{Strut}(L)} \frac{1}{2}\Bigl\langle \frac{x-y}{|x-y|}, \xi_x - \xi_y \Bigr\rangle, \frac{1}{\sigma} \min_{c\in\mathrm{Kink}(L)} \delta_\xi R(c)\Bigr).\end{aligned}$$

Proof We will apply Clarke's Theorem 3.1 to a family of functions of t parametrized by the compact space $(L\times L)\sqcup\overline{\mathrm{Osc}\,L}$. The functions are the following: for $(x,y)\in L\times L$ we use $t \mapsto \mathrm{pd}^{\phi^t(L)}(\phi^t(x), \phi^t(y))$, and for $c \in \overline{\mathrm{Osc}\,L}$ we use $t \mapsto \frac{1}{\sigma} R(\phi^t_*(c))$. These functions and their derivatives depend continuously on the parameters; they form the family to which we will apply Clarke's theorem.

By the last lemma, $\phi^t_*(\overline{\mathrm{Osc}\,L}) = \overline{\mathrm{Osc}\,\phi^t L}$. Thus by Corollary 2.16 and the definition of Thi_σ, the minimum of our Clarke family is the thickness $\mathrm{Thi}_\sigma(\phi^t L)$. Clarke's theorem thus shows that thickness has a forward time derivative given by the minimum derivative of $\mathrm{pd}(x,y)$ or R/σ where these functions equal thickness.

By Corollary 2.17, struts are critical pairs: we have that $\mathrm{pd}(x,y) = \mathrm{Thi}_\sigma(L)$ only if $(x,y) \in \mathrm{Crit}$. Differentiating the formula defining $\mathrm{pd}(x,y)$, using the fact that $\psi(x,y) = 0$, we see that the derivative equals the derivative of $|x-y|/2$ given above.

Note that our functions sometimes take the value $+\infty$. This is not really an obstacle to applying Clarke's theorem: we simply choose a smooth increasing map $h\colon \mathbb{R} \to \mathbb{R}$ that is bounded above but satisfies $h(x) = x$ for $x \le \mathrm{Thi}_\sigma(L) + 1$. Composing each function in our family with h gives a family to which Clarke's theorem as stated applies. Since h is the identity near all points where its value matters, it drops out of the formula for the derivative. □

Since superlinear functions may be characterized as infima of families of linear functions, we immediately get:

Corollary 3.6 *Suppose L is a $C^{1,1}$ curve with* $\operatorname{reach}(L) < \infty$*. Then the functional $\xi \mapsto \delta_\xi \operatorname{Thi}_\sigma(L)$ is superlinear for $\xi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$. That is, for $a \geq 0$ and vector fields ξ and η, we have*

$$\delta_{a\xi} \operatorname{Thi}_\sigma(L) = a\delta_\xi \operatorname{Thi}_\sigma(L), \quad \delta_{\xi+\eta} \operatorname{Thi}_\sigma(L) \geq \delta_\xi \operatorname{Thi}_\sigma(L) + \delta_\eta \operatorname{Thi}_\sigma(L).$$

3.2 The balance criterion

Having computed the derivative of the function $\operatorname{Thi}_\sigma$ representing the one-sided constraint, we can now start to formulate our balance criterion. Recall that in a constrained link type, at each endpoint $p \in \partial L$ we have constraints given by the subspaces H_p^0 and H_p^1.

Definition 3.7 Let L be a $C^{1,1}$ curve in the constrained link type $[L]$. A vector field $\eta \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ is *compatible* with $[L]$ at L if $\eta(p)$ is tangent to H_p^0 and $\eta'(p) = D_p\eta(T) \in H_p^1$ at each endpoint $p \in \partial L$.

These conditions of course mean that the vector field η preserves the endpoint constraints to first order. While the autonomous flow of η might violate these constraints to second order, we next show how to modify it locally near the endpoints to fix this.

Lemma 3.8 *Suppose L is a constrained link and η is a compatible vector field. Then there exists a C^1 family of C^2 diffeomorphisms ϕ^t with initial velocity η such that $\phi^t(L)$ satisfies the endpoint constraints for all small t.*

Proof Let $\widetilde{\phi}^t$ be the autonomous flow of η, satisfying $\partial\widetilde{\phi}^t/\partial t = \eta \circ \widetilde{\phi}^t$. We will make local modifications in a ball $B_r(p)$ around each endpoint, choosing the radius $r > 0$ small enough that these balls are disjoint. We focus on a single endpoint $p \in H_p^0$, where the tangent vector to L is some $v^0 \in H_p^1$. After flowing by time t, the link $\widetilde{\phi}^t(L)$ has endpoint $p^t = \phi^t(p)$ and velocity $v^t = D_p\widetilde{\phi}^t(v^0)$ there. These are close to H_p^0 and H_p^1, respectively, and there is a unique "smallest" Euclidean rigid motion ρ^t restoring these constraints exactly: first we rotate around p^t until v^t lies in H_p^1 and then we translate p^t to its orthogonal projection in H_p^0. This motion depends smoothly on p^t and v^t and thus is a C^1 function of t. The compatibility of η with the endpoint conditions means exactly that $(\partial\rho^t/\partial t)|_{t=0} = 0$, since only second-order corrections are necessary.

Now fix a smooth bump function ψ supported on $B_r(p)$ and with $\psi \equiv 1$ on some smaller neighborhood of p. Then define ϕ^t as the linear combination

$$\phi^t(x) := \psi(x)\rho^t(\widetilde{\phi}^t(x)) + (1 - \psi(x))\widetilde{\phi}^t(x).$$

In a small neighborhood of p, only the first term is active, so $\phi^t(L)$ satisfies the endpoint constraints. But because $d\rho^t/dt = 0$, the initial velocity of ϕ^t is still η. □

Definition 3.9 Assuming $\operatorname{reach}(L) < \infty$, we say that L is Thi_σ*–regular* if it has a *thickening field*, meaning a compatible C^2 vector field η on $\mathbb{R}^3$ with $\delta_\eta \mathrm{Thi}_\sigma(L) > 0$.

Regularity is a form of constraint qualification; we will use it for instance to show that minimizers are critical points. Note that for a classical link type (with all components closed curves), any L with $\mathrm{Thi}_\sigma > 0$ is Thi_σ–regular: the Euler vector field $\eta_p := p$ generating homotheties is a thickening field. Regularity also holds for many examples of constrained links.

A link is critical for the ropelength problem if its length cannot be decreased without also decreasing thickness. For technical reasons we will also need a strong version of criticality.

Definition 3.10 Suppose $\mathrm{Thi}_\sigma(L) = 1$. We say L is σ*–critical* if

$$\delta_\xi \operatorname{Len}(L) < 0 \Longrightarrow \delta_\xi \mathrm{Thi}_\sigma(L) < 0$$

for every compatible $\xi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$. We say L is *strongly* σ*–critical* if there exists $\epsilon > 0$ such that

$$\delta_\xi \operatorname{Len}(L) = -1 \Longrightarrow \delta_\xi \mathrm{Thi}_\sigma(L) \le -\epsilon$$

for every compatible $\xi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$.

Clearly strong criticality implies criticality. Under the assumption of Thi_σ–regularity they are in fact equivalent.

Lemma 3.11 *If L is Thi_σ–regular and σ–critical, then L is in fact strongly σ–critical.*

Proof Let η be a thickening field for L. Scaling η if necessary, we may assume that $\delta_\eta \operatorname{Len}(L) \le \frac{1}{2}$. Thus for ξ as in the definition of strong criticality, $\delta_{\xi+\eta} \operatorname{Len}(L) \le -\frac{1}{2}$. Using the superlinearity of Corollary 3.6, and the criticality of L, we get

$$0 > \delta_{\xi+\eta} \mathrm{Thi}_\sigma(L) \ge \delta_\xi \mathrm{Thi}_\sigma(L) + \delta_\eta \mathrm{Thi}_\sigma(L).$$

Thus we may take $\epsilon := \delta_\eta \mathrm{Thi}_\sigma(L)$. □

The next two lemmas characterize Thi_σ–constrained local minimizers L. In the trivial case when $\mathrm{Thi}_\sigma(L) > 1$, the thickness constraint is not active; if $\mathrm{Thi}_\sigma(L) = 1$ and L is Thi_σ–regular, then it is critical.

Lemma 3.12 *If L is a* Thi_σ*–constrained local minimizer with* $\mathrm{Thi}_\sigma(L) > 1$*, then each component of L is a straight arc.*

Proof Since the constraint $\mathrm{Thi}_\sigma \geq 1$ is not active at L, the curve is a local length minimizer without constraints. Thus $\delta_\xi \mathrm{Len}(L) = 0$ for all compatible ξ, so L has zero curvature everywhere. □

Lemma 3.13 *If L is a* Thi_σ*–constrained local minimizer with* $\mathrm{Thi}_\sigma(L) = 1$*, and L is* Thi_σ*–regular, then L is (strongly) σ–critical.*

Proof Suppose ξ is a compatible vector field such that $\delta_\xi \mathrm{Len}(L) < 0$, but $\delta_\xi \mathrm{Thi}_\sigma \geq 0$. Let η be a thickening field, and choose $c > 0$ small enough that $\delta_{\xi+c\eta} \mathrm{Len} < 0$. By Corollary 3.6, we see $\delta_{\xi+c\eta} \mathrm{Thi}_\sigma > 0$. Using Lemma 3.8, we can flow to get nearby curves in the same constrained link type with $\mathrm{Thi}_\sigma > 1$ but smaller length than L, which is a contradiction. □

The rest of our results deal with strongly σ–critical curves L with $\mathrm{Thi}_\sigma(L) = 1$, and thus apply to Thi_σ–regular local minimizers (ignoring the trivial case of minimizers with $\mathrm{Thi}_\sigma(L) > 1$, classified above). Our main theorem, the general balance criterion, says that a link is strongly critical if and only if its curvature is balanced by certain measures on the kinks and struts.

Definition 3.14 Let L be a $C^{1,1}$ link. A *kink measure* for L is a nonnegative Radon measure on $\mathrm{Kink}(L)$. A *strut measure* for L is a nonnegative Radon measure on $\mathrm{Strut}(L) \subset L \times L$ that is invariant under $(x, y) \mapsto (y, x)$. Given a strut measure μ on $\mathrm{Strut}(L)$ we define the associated *strut force measure* Ω on L to be the vector-valued measure obtained by projecting the vector-valued Radon measure $2(x - y)\mu(x, y)$ to L via $(x, y) \mapsto x$. Thus

$$\int_{\mathrm{Strut}(L)} \langle x - y, \xi_x - \xi_y \rangle \, d\mu(x, y) = \int_L \langle \xi, d\Omega \rangle.$$

Physically one should think of a strut measure as a system of compressions on the points of self-contact of the embedded tube around L, or alternatively on certain compression-bearing elements of length 1 connecting critical pairs of L. The strut force measure then gives the resultant force along L itself. The physical interpretation of the kink measure is more elusive in general.

Definition 3.15 A $C^{1,1}$ link L with $\mathrm{Thi}_\sigma(L) = 1$ is *σ–balanced* if there exist a strut measure μ (with strut force measure Ω) and a kink measure ν for L such that for any compatible vector field ξ we have

$$\delta_\xi \mathrm{Len}(L) = \int_L \langle \xi, d\Omega \rangle + \int_{\mathrm{Kink}(L)} \delta_\xi R(c) \, d\nu(c).$$

We refer to this as the *balance equation*. Note that it may be viewed as an equation of distributions acting on vector fields $\xi\colon \mathbb{R}^3 \to \mathbb{R}^3$. The kink term has distributional order 2 by Lemma 3.2, while the other terms have order 0: in particular the variation of length can be written as

$$\delta_\xi \operatorname{Len}(L) = \int_L \langle \xi', T\rangle \, ds = -\int_L \langle \xi, \kappa\rangle \, ds + \sum_{p\in\partial L} \langle \xi, \pm T\rangle,$$

pairing ξ with a vector-valued Radon measure which is absolutely continuous on the interior and has outward-pointing atoms at each endpoint.

The general balance criterion is an application of the following version of the Kuhn–Tucker theorem from linear programming, which we proved in [4] following ideas of [15]. As usual $C(Y)$ denotes the space of continuous functions on a space Y.

Theorem 3.16 *Let X be any vector space and Y be a compact topological space. For any linear functional f on X and any linear map $A\colon X \to C(Y)$, the following are equivalent.*

(a) *There exists $\epsilon > 0$ such that for each $\xi \in X$ with $f(\xi) = -1$ there exists $y \in Y$ with $(A\xi)(y) \le -\epsilon$.*

(b) *There exists a nonnegative Radon measure μ on Y such that $f(\xi) = \int_Y A(\xi) d\mu$ for all $\xi \in X$.* □

Theorem 3.17 (General balance criterion) *A link L with $\operatorname{Thi}_\sigma(L) = 1$ is strongly σ–critical (Definition 3.10) if and only if it is σ–balanced (Definition 3.15).*

Proof We apply Theorem 3.16 with X being the space of compatible vector fields ξ and f the linear functional $f(\xi) := \delta_\xi \operatorname{Len}(L)$. The idea is to capture the derivative $\delta_\xi \operatorname{Thi}_\sigma(L)$ as the minimum value of a continuous function $A(\xi)$. Thus following Theorem 3.5 we take $Y := \mathrm{Strut} \sqcup \mathrm{Kink}$ and define $A\colon X \to C(Y)$ via

$$A(\xi) := \begin{cases} \frac{1}{2}\langle x - y, \xi_x - \xi_y\rangle & (x, y) \in \mathrm{Strut}, \\ \sigma^{-1}\delta_\xi R(c) & c \in \mathrm{Kink}. \end{cases}$$

The conclusion of Theorem 3.16 is then exactly that L is strongly critical if and only if it is balanced. □

The special case of a critical knot with no kinks was analyzed by Schuricht and von der Mosel [19]. Of course in this case our balance criterion reduces to theirs, involving only the strut measure. We next consider other links that can be balanced by strut measure alone.

Proposition 3.18 *Suppose L is a critical link for the Gehring problem of minimizing length subject to maintaining distance 1 between components. Then L is also σ–critical for any σ for which $\mathrm{Thi}_\sigma(L) \geq 1$.*

Proof The main theorem of [4] gives a strut measure on the set of Gehring struts (connecting points at distance 1 on distinct components). Under the assumption that $\mathrm{Thi}_\sigma(L) \geq 1$, these Gehring struts are also struts in our sense. Even if there are kinks or further struts (between points on a single component) the Gehring strut measure alone balances the link, so by the general balance criterion it is σ–critical. □

Consider for instance, the known ropelength-minimizing links from [5], where each component is a convex planar curve built from straight segments and arcs of unit circles. They have $\mathrm{Thi}_\sigma = 1$ for any $\sigma \in [\frac{1}{2}, 1]$ and thus are global minimizers also for these more restrictive problems. By Lemma 3.13 they are then strongly σ–critical. The same strut measure that balances them for the Gehring problem [4] also shows they are σ–balanced, again for any $\sigma \leq 1$. (For $\sigma = 1$ the curved sections are kinks and balance can be achieved in other ways as well.)

The Gehring τ–clasp of [4, Section 9] has maximum curvature $1/\sqrt{1-\tau^2}$ at the tip. Since neither component approaches itself closely, for $\sigma \leq \sqrt{1-\tau^2}$ we have $\mathrm{Thi}_\sigma = 1$. For these values of σ, the strut measure used for the Gehring problem shows the clasp is also σ–balanced. Below in Section 7 we explore what happens for larger stiffnesses, when the clasps include kinks.

Similarly, we described in [4, Section 10] a Gehring-critical configuration B_0 of the Borromean rings. It curvature is bounded by 1.52802 (and no component approaches itself closely), so the same strut measure shows it is σ–balanced for any $\sigma < 0.65444$. We also described a nearby configuration B_2 (with length less than 1% more than that of B_0) where each component is made of arcs of unit circles centered on the other components. For $\sigma = 1$ these arcs are kinks, and it is not hard to show (using Lemma 4.18 below) that B_2 is 1–balanced. We have computed σ–balanced configurations also for intermediate stiffnesses and plan to report on these separately.

3.3 Kink-free arcs with special strut patterns

The kink term in the general balance criterion is a bit arcane; in Section 4 we will give nicer versions under certain minimal smoothness assumptions. But of course the kink term is irrelevant along kink-free arcs (or even kinked arcs over which the kink measure vanishes), so we can apply the general balance criterion directly.

Lemma 3.19 *Suppose L is σ–balanced and A is an open subarc over which the kink measure vanishes. Then along A the strut force measure is absolutely continuous, given by $\Omega = -\kappa\, ds$.*

Proof For any vector field ξ vanishing on $L \smallsetminus A$ the kink term in the balance equation vanishes, so we get

$$\int_A \langle \xi, d\Omega \rangle = \delta_\xi \operatorname{Len}(L) = \int_A \langle \xi', T \rangle \, ds.$$

Integrating by parts gives the desired result. □

As a first application, we can easily analyze "free" sections of a critical curve, with no struts or kinks. (This result was first discussed — in the case of a C^2 knot — by Gonzalez and Maddocks [14].)

Proposition 3.20 *If L is σ–balanced and A is a subarc with zero strut force measure and zero kink measure, then A is a line segment.*

Proof By the lemma, $\kappa\, ds = -\Omega = 0$ along the subarc. □

We now consider the case of two subarcs in "one-to-one contact".

Proposition 3.21 *Let L be σ–balanced. Suppose A and B are two subarcs with zero kink measure and suppose they are in one-to-one contact, meaning there is a homeomorphism $\phi\colon A \to B$ such that there is a strut from a to $\phi(a)$ for each $a \in A$ but no other struts touching $A \cup B$. Then $A \cup B$ forms a piece of a standard symmetric double helix of pitch at least 1 (or of a circle).*

Remark We could start with the weaker assumption of a (weakly) monotonic family of struts, where a single point $a \in A$ might touch a whole subarc $B' \subset B$ or vice versa. In fact this cannot happen, since B' is a subarc of the unit normal circle to A at a, so the tangent vector has nonzero change along B'; this would imply an atom of strut force measure at a which is impossible since Ω is absolutely continuous on a kink-free arc.

Proof Change the orientation on B if necessary to assume that ϕ is orientation-preserving. Since the kink measure vanishes on $A \cup B$, the lemma applies, giving $\Omega = -T'$. For any subarc $aa' \subset A$, by the symmetry of Ω we get

$$T(a) - T(a') = \Omega(aa') = -\Omega(\phi(aa')) = T(\phi(a')) - T(\phi(a)).$$

This means that $W := T(a) + T(\phi(a))$ is a constant vector along A.

Now define the continuous vector field $N(a) := \phi(a) - a$ along A. Since struts have unit length and $\phi(a) \in N_a L$, this is a unit normal field. Since Ω acts in the direction $-N$ of the single strut, we deduce that $T' = |\kappa| N$ almost everywhere. That is, N is the Frenet principal normal.

Reversing the roles of A and B, we see equally well that $N(a) \perp T(\phi(a))$. (Indeed the principal normal at $\phi(a) \in B$ is $-N(a)$.) It follows that $N(a) \perp W$, which in turn implies that $\langle W, T(a) \rangle$ is constant along A. But from the definition of W, we have

$$\langle W, T(a) \rangle = 1 + \langle T(a), T(\phi(a)) \rangle = \langle W, T(\phi(a)) \rangle,$$

so $\langle W, T \rangle$ is the same constant along B.

Consider first the degenerate case where $W = 0$, meaning $T(\phi(a)) = -T(a)$. The arcs A and B stay in the plane of $T(a)$ and $N(a)$, and indeed are centrally symmetric around the midpoint of any strut. Since a and $\phi(a)$ are always at unit distance, it follows that A and B are antipodal arcs of a circle of diameter 1, a degenerate double helix of pitch zero.

Clearly this case only arises when $\sigma = \frac{1}{2}$. Since points near $\phi(a)$ are at distance less than 1 from a, it follows that A and B belong to the same component of L. Furthermore, by the remark after Lemma 2.12, this component is the full circle of diameter 1. Since this circle is kinked, balance could alternatively be obtained through a kink measure instead of the strut measure.

For the general case $W \neq 0$, think of W as a vertical vector. Since $N \perp W$, each strut connects points at equal height. Since $\langle W, T \rangle$ is the same constant along each curve, the homeomorphism ϕ is actually an isometry. Consider now the midpoints $M(a) := (a + \phi(a))/2$ of the struts. Since ϕ is an isometry, differentiating gives $M' = W/2$, meaning these midpoints move at constant speed in direction W. Since T makes a constant angle with W, the strut vectors $N(a)$ also rotate at constant speed in the plane perpendicular to W. The arcs A and B, given as $M \mp N/2$, thus form a symmetric double helix as claimed.

(In the degenerate case where $|W| = 2$, we have $T(\phi(a)) = T(a) \equiv W/2$. That is, both A and B are straight segments, giving a degenerate double helix of infinite pitch. The strut measure vanishes on the struts connecting A and B.)

Consider the squared distance function from a fixed point $(-1/2, 0, 0) \in A$ to the other strand $B = \{(\cos\theta, \sin\theta, k\theta)/2\}$ of a helix of pitch k. Since its second derivative is $(k^2 - \cos\theta)/2$, we see that it is convex (with a single minimum at the claimed strut) for $k \geq 1$. For smaller pitch, the distance has a local maximum at $\theta = 0$, so the thickness of the double helix is less than 1 and the curves are not in one-to-one contact. □

This agrees with the result of Maddocks and Keller [16] which states (under different hypotheses) that two intertwined ropes in equilibrium with one-to-one contact should form a double helix where the radii of the helices depend on the tension in the ropes. Schuricht and von der Mosel [19] show in this situation that the curvature vectors of A and B must point along the common strut, without carrying the analysis through to prove that the curves form a double helix.

4 Balance with regulated kinks

The general balance criterion can be hard to apply without some control on the kink set. In the balance equation, as we have already noted, the second-order kink term is equated to strut and length terms which are distributions of order zero in the variation vector field ξ. If we knew that kinked arcs were C^2, then there would be at most one kink over each point of L and furthermore, Corollary 3.3 would give the kink term in terms of the second arc length derivative of ξ. In this case, standard distributional calculus (cf Duistermaat and Kolk [10]) then says this second-order term can be integrated by parts. This would give us a simpler form of the balance criterion as an equality of measures in which the variational vector field does not appear.

Our goal is to carry out as much of this program as possible for less smooth links, like those in our examples. Over a junction point along a piecewise C^2 curve, for instance, there may be two kinks. Our first theorem below says that we can essentially ignore such points: the kink measure is nonatomic even after projection down to L, so even any countable subset of L can be ignored.

In the later parts of this section we discuss the balance criterion under certain mild regularity assumptions about the kinked arcs of L; these suffice first to guarantee a single kink over all but a countable subset of L, then to transfer the balance equations to distributions along L, and thus to apply the calculus of distributions. We end up with friendlier versions of the balance criterion, and can bootstrap to greater smoothness of the critical link L.

4.1 The projection of the kink measure is nonatomic

The kink measure ν for a balanced link L is supported on $\mathrm{Kink}(L)$, which we view as a subset of the unit normal bundle $N_1(L)$ via $(x,n) \longleftrightarrow (x, T(x), n/\sigma)$. Thus we think of ν as a measure on this circle bundle with support on Kink. We recall the projection Π: Circ $\to \mathbb{R}^3$, in particular $\Pi\colon N_1(L) \to L$. If ν is a kink measure for L, then we write $\overline{\nu}$ for the projection of $\sigma\nu$ to L, which of course is supported on $\Pi\,\mathrm{Kink}(L)$. (The factor of σ here simplifies several formulas later.)

Using Lemma 3.2 we can write the kink term in the balance equation as

$$\int_{\text{Kink}} \delta_\xi R(x,n)\, d\nu(x,n) = 2\int_L \langle \xi', T\rangle\, d\overline{\nu}(x) - \sigma^2 \int_{\text{Kink}} \langle D_x^2\xi(T,T), n\rangle\, d\nu(x,n) \\ - \sigma \int_{\text{Kink}} \langle D_x\xi(n), n\rangle\, d\nu(x,n).$$

We note the linear and quadratic dependence on n in the last two terms; these could also be written as integrals over L, now with respect to projected vector- and tensor-valued measures. Thus it is really only the projections to L of the three measures ν, $n\nu$ and $(n\otimes n)\nu$ which enter into the balance equation. (What this essentially means is that if we Fourier-decompose the measure ν on each normal circle, then it is only the components of order 0, 1 and 2 which matter.)

Our first result shows that no single normal circle has positive mass. This will later allow us to ignore countably many points along L.

Theorem 4.1 *If L is σ–balanced, then the projection $\overline{\nu}$ of the kink measure ν to L is nonatomic.*

Proof Fix a point on L, which by translation we assume is at the origin. We must show that $\nu(\Pi^{-1}\{0\}) = 0$. We will obtain this equation as the limit of the balance equation applied to a family of variation fields ξ^ϵ.

Let f denote a smooth nonnegative bump function supported on the unit ball, with $f \equiv 1$ in a small neighborhood of 0. Given any vector $v \in \mathbb{R}^3$ we write $v^\perp := v - \langle v, T_0\rangle T_0$ for its part perpendicular to the tangent vector $T_0 := T(0)$ at the origin. Then we define

$$\xi^\epsilon(x) := f(x/\epsilon)x^\perp.$$

Since ξ^ϵ is supported on the ϵ–ball its L^∞ norm is $O(\epsilon)$. Thus in the limit $\epsilon \to 0$ the order 0 (strut and $\delta\,\mathrm{Len}$) terms in the balance equation approach 0 (even though the strut force measure might have an atom at the origin). Therefore the kink term approaches 0 as well.

We easily calculate the derivatives

$$D_x\xi^\epsilon(v) = D_{x/\epsilon} f(v)x^\perp/\epsilon + f(x/\epsilon)v^\perp,$$
$$D_x^2\xi^\epsilon(v,v) = 2D_{x/\epsilon} f(v)v^\perp/\epsilon + D_{x/\epsilon}^2 f(v,v)x^\perp/\epsilon^2.$$

Note that $D\xi^\epsilon$ is $O(1)$ while $D^2\xi^\epsilon$ is $O(1/\epsilon)$. At the origin (independent of ϵ) we have $D_0\xi^\epsilon(v) = v^\perp$, while the second derivatives vanish.

Note that ξ^ϵ is supported on the ϵ–ball; since $\mathrm{reach}(L) \geq \mathrm{Thi}_\sigma(L) = 1$ we know (from Denne, Diao and Sullivan [9, Lemma 3.1]) that for small ϵ this ball contains a single arc α^ϵ of L whose length is at most $2\arcsin\epsilon$. Now suppose $x \in \alpha^\epsilon$ is at arc length $s = O(\epsilon)$ from 0. Using the curvature bound and the fact that $\sigma \leq \frac{1}{2}$, we get $|T(x) - T_0| \leq |s|/\sigma \leq 2|s|$ and thus $|x - sT_0| \leq s^2$. In particular, $|T^\perp| = O(\epsilon)$ and $|x^\perp| = O(\epsilon^2)$ along the whole arc α^ϵ.

The integrand in the kink term is

$$\delta_{\xi^\epsilon} R(x,n) = 2\sigma\langle T, D_x\xi^\epsilon(T)\rangle - \sigma\langle n, D_x\xi^\epsilon(n)\rangle - \sigma^2\langle n, D_x^2\xi^\epsilon(T,T)\rangle.$$

First we show that this integrand is uniformly bounded as $\epsilon \to 0$. Clearly the first two terms are $O(1)$. Writing

$$\langle n, D_x^2\xi^\epsilon(T,T)\rangle = 2D_{x/\epsilon} f(T)\langle n, T^\perp\rangle/\epsilon + D_{x/\epsilon}^2 f(T,T)\langle n, x^\perp\rangle/\epsilon^2$$

shows — using our estimates on $T^\perp$ and $x^\perp$ — that the third term is also $O(1)$. We also note that at $x = 0$ the integrand reduces to

$$\delta_{\xi^\epsilon} R(0,n) = 0 - \sigma\langle n, n\rangle - 0 = -\sigma,$$

independent of ϵ.

Now as $\epsilon \to 0$ the arcs α^ϵ shrink to the single point $\{0\}$, so since the kink integrand is uniformly bounded, the kink integral $\int_{\Pi^{-1}(\alpha^\epsilon)} \delta_{\xi^\epsilon} R(x,n)\, d\nu$ approaches the integral over $\Pi^{-1}\{0\}$, which as noted is $-\sigma\nu(\Pi^{-1}\{0\})$, independent of ϵ. Thus this measure is zero, as desired. □

4.2 Regularly balanced links

To reformulate the balance criterion in a nicer way it will be important to consider curves with regulated second derivative. While regulated functions are usually defined (as in Bourbaki [2, Chapter 2.1]) on an interval in $\mathbb{R}$, it is equivalent to define them on Riemannian 1–manifolds; in our context we speak of submanifolds M of a C^1 curve L. (Any 1–manifold is a countable union of components, each a circle or an open, half-open or compact interval.) Note that a submanifold $M \subset L$ with empty boundary is exactly an open subset $U \subset L \smallsetminus \partial L$.

Let $M \subset L$ be a submanifold of a C^1 curve. A *regulated function* on M is a function $f\colon D \to \mathbb{R}^n$ defined on a dense subset $D \subset M$ whose one-sided limits exist at every $x \in M$. An interior point $x \in M \smallsetminus \partial M$ is called a *jump point* of f if $f(x-) \neq f(x+)$. For $\epsilon > 0$ we let J_ϵ denote the set on which the jump is large:

$$J_\epsilon(f) := \{x \in M \smallsetminus \partial M \mid |f(x-) - f(x+)| \geq \epsilon\}.$$

If M is compact then J_ϵ is finite; for any M it follows that J_ϵ is countable and closed in M (though not necessarily in L). The union $J = J(f) := \bigcup J_\epsilon(f) \subset M$ is the countable set of all jump points (which may of course be dense). Let $\overline{f}\colon M \to \mathbb{R}^n$ denote any function such that $\overline{f}(x) \in \{f(x-), f(x+)\}$ for each x. (Note that $\overline{f} = f$ at all but countably many points of D, a statement which is vacuous if D is countable.) Then $\overline{f}$ is continuous on $M \setminus J$ but has a jump discontinuity at each $x \in J$. The following lemma is then immediate:

Lemma 4.2 *Let f be a regulated function on M. Consider the smoothings $f_\epsilon := \overline{f} * \phi_\epsilon$ obtained by convolution with a sequence of mollifiers (cf [10, Chapter 1]). Here f_ϵ is defined away from an ϵ–neighborhood of ∂M. For any $x \in M \setminus (\partial M \cup J)$, the continuity of $\overline{f}$ at x implies that $f_\epsilon(x) \to \overline{f}(x)$. In particular we have this pointwise convergence at all but countably many points of M.* □

We will say that an absolutely continuous function $g\colon M \to \mathbb{R}^n$ has *regulated derivative* if its arc length derivative g' (which is defined almost everywhere) is regulated. Note that in this case the mean value theorem implies that $g'(x\pm)$ are the one-sided derivatives of g, so these exist everywhere, and g is differentiable exactly at those x where $g'(x+) = g'(x-)$.

Lemma 4.3 *Let $f\colon (a,b) \to (c,d)$ be a $C^{1,1}$ diffeomorphism with $\frac{1}{2} \le f' \le 1$. Its inverse g is also $C^{1,1}$ with $1 \le g' \le 2$. Furthermore f has regulated second derivative if and only if g does.*

Proof The chain rule gives $g'(f(x)) = 1/f'(x)$; therefore if f' is L–Lipschitz then g' is $8L$–Lipschitz. The second derivative g'' exists almost everywhere and from the formula $g''(f(x)) = -f''(x)/f'(x)^3$ we see that it has a one-sided limit at $f(x)$ if and only if f'' has a one-sided limit at x. □

Definition 4.4 Suppose a link L is σ–balanced (Definition 3.15) by strut measure μ and kink measure ν. We say L is *regularly balanced* if there is an open subset $U \subset L$ such that $\overline{\nu}(L \setminus U) = 0$ and the unit tangent T has regulated derivative κ on U.

We conjecture that every σ–balanced link is regularly balanced, but this seems difficult to prove. But there is a condition on L which will ensure this.

Definition 4.5 We say a $C^{1,1}$ curve L has *regulated kinks* if Π Kink is contained in a submanifold $M \subset L$ on which T has regulated derivative. (As above, this means M is a countable union of circles and intervals.)

With this in hand, we prove the following.

Lemma 4.6 *Suppose that L has regulated kinks. Then L is regularly balanced (see Definition 4.4) if and only if L is σ–balanced (see Definition 3.15). (By Theorem 3.17, this holds if and only if L is strongly σ–critical.)*

Proof It only remains to show that if L is σ–balanced then it is regularly balanced. Let M be the submanifold on which T has regulated derivative and set $U := M \smallsetminus \partial M$. We know $\overline{\nu}$ is supported on $\Pi\,\mathrm{Kink} \subset M$. Since ∂M is countable and $\overline{\nu}$ is nonatomic, we have $\overline{\nu}(L \smallsetminus U) = 0$. □

In the rest of this section we analyze regularly balanced links to get several equivalent conditions that are easier to apply. First we show that we can reformulate the balance equation to involve distributions along L instead of on $\mathbb{R}^3$; then we integrate by parts twice, ending with a balance equation that can be stated as an equality of measures with no explicit variation vector field. This is the condition we use later to show our examples are (regularly) balanced.

Suppose L is regularly balanced. We let J denote the jump set of κ on U; since J is countable and $\overline{\nu}$ is nonatomic, $\overline{\nu}(J) = 0$. Over each point of $U \smallsetminus J$ there is at most one kink; a kink exists only when $|\kappa| = 1/\sigma$. (Over each point in J there are at most two kinks, but we may ignore these with regards to the kink measure.)

Now we claim that we may replace U (in the definition of regularly balanced) by an open subset on which $|\kappa|$ is bounded away from zero. Writing $c := 1/2\sigma$ for notational convenience, remove from U the set J_c where κ jumps by at least c. We may do this because J_c is closed in U and, being countable, has measure zero with respect to the nonatomic $\overline{\nu}$. Now let A be the closure — in this new U — of $\{x \in U \mid \kappa(x) < c\}$. At any point in A, some one-sided limit of κ is at most c, while on $\Pi\,\mathrm{Kink}$ some one-sided limit of κ is $2c = 1/\sigma$. Since all jumps on U are by less than c, we see A is disjoint from $\Pi\,\mathrm{Kink}$, so $\overline{\nu}(A) = 0$. Thus we may remove A from U, proving the claim.

From now on we assume we have adjusted U in this way. It follows that the unit principal normal vector $N := \kappa/|\kappa|$ is well defined as a regulated function on U (with jumps only on J). We can rewrite the kink term in the balance equation in terms of this normal vector, using Corollary 3.3:

Lemma 4.7 *On a regularly balanced link L, the kink measure ν is uniquely determined by its projection $\overline{\nu}$, and the kink term in the balance equation becomes*

$$\int_{\mathrm{Kink}} \delta_\xi R(x,n)\, d\nu(x,n) = \int_U (2\langle \xi', T\rangle - \sigma\langle \xi'', N\rangle)\, d\overline{\nu}. \qquad \square$$

Here we note that in the last term, both N and ξ'' are regulated functions (with jumps only on J). Since their product is also regulated and $\overline{\nu}$ is nonatomic, the integral is well-defined.

By this lemma, the balance equation for a regularly balanced L can be expressed entirely in terms of derivatives of the vector field ξ along the curve L. Of course, ξ here is still a C^2 vector field in space, and the balance equation is an equation of distributions on such vector fields. Our next result shows, however, that we can translate it into an equation of distributions on C^2 vector fields along L. (We recall that the C^2 structure on L comes not directly from the embedding in $\mathbb{R}^3$ but instead from the local identification with $\mathbb{R}$ given by an arc length parametrization.) This sets us up to use the standard calculus of distributions: by examining the highest-order term, we can integrate by parts and bootstrap to higher smoothness.

Theorem 4.8 *Let L be a link with $\mathrm{Thi}_\sigma(L) = 1$. Then L is regularly balanced (Definition 4.4) by strut force measure Ω and kink measure ν if and only if*

$$\int_L \langle \eta', T\rangle\, ds - \int_L \langle \eta, d\Omega\rangle = \int_U (2\langle \eta', T\rangle - \sigma\langle \eta'', N\rangle)\, d\overline{\nu}$$

for all compatible C^2 vector fields $\eta \in C^2(L, \mathbb{R}^3)$ along L.

Note that this is the same balance equation we already have for C^2 fields on space; the only difference is that it is now supposed to hold for C^2 fields along L. For such fields η, *compatible* means again that at each endpoint $p \in \partial L$ we have $\eta(p)$ tangent to H^0_p and $\eta'(p) \in H^1_p$.

Proof First suppose this balance equation holds for all compatible $\eta \in C^2(L, \mathbb{R}^3)$. Given a compatible C^2 vector field ξ on space, to check the balance equation for ξ it suffices to find a sequence of compatible smooth fields η_i along L with uniformly bounded C^2 norms such that $|\eta_i - \xi|_{C^1(L)} \to 0$ and $\eta_i'' \to \xi''$ pointwise on $U \smallsetminus J$. For then each term in the balance equation for η_i approaches the corresponding term for ξ (in Lemma 4.7). In particular, to handle the second-order term $\int_{U \smallsetminus J} \langle N, \eta_i''\rangle\, d\overline{\nu}$ we use the dominated convergence theorem. But the construction of the η_i is easy: we simply start with the restriction of ξ to L and smooth it by convolving with a sequence of mollifiers. (Small modifications near the endpoints suffice to maintain the compatibility conditions.) Since ξ'' is regulated on U with jumps only on J, the desired pointwise convergence follows from Lemma 4.2.

Conversely, if L is regularly balanced, then given any compatible C^2 field η along L it suffices to find a sequence of smooth ξ_i on $\mathbb{R}^3$ that have uniformly bounded C^2 norms, that converge to η in $C^1(L)$ and whose second derivatives converge pointwise on $U \setminus J$. Indeed it suffices to construct the ξ_i locally in a neighborhood of any given point $p \in L$; these pieces can be patched together with a partition of unity. By translation we assume $p = 0$ and let T_0 be the tangent there. The idea is to extend η to $\overline{\eta}$ on a neighborhood of $0 \in \mathbb{R}^3$ by making $\overline{\eta}$ constant on each plane perpendicular to T_0, and then smooth this in space.

More precisely, consider the function $f\colon x \mapsto \langle T_0, x\rangle$. Restricted to L, it is $C^{1,1}$ and has regulated second derivative on U. On some neighborhood $V \subset L$ of p we have $\frac{1}{2} < f' \le 1$, so in particular $f|_V$ is a C^1 diffeomorphism onto its image $(a,b) \subset \mathbb{R}$. Lemma 4.3 applies to show the inverse function $g\colon (a,b) \to V$ is a $C^{1,1}$ parametrization with speed in $[1,2)$, and has regulated second derivative on the subset $f(U \cap V)$. Thus if we set $\overline{\eta} := \eta \circ g$ then $\overline{\eta}$ is also $C^{1,1}$ with regulated second derivative on $f(U \cap V)$. To get the ξ_i, we simply smooth $\overline{\eta}$ by convolving it with a sequence of mollifiers:

$$\xi_i := (\overline{\eta} * \phi_i) \circ f.$$

The desired properties again follow immediately using Lemma 4.2. □

On a regularly balanced link L, we have discussed the principal normal N as a regulated function on U. For convenience we extend it arbitrarily outside of U. (Of course for points $x \in E_L$ with $\kappa \neq 0$ we are free to pick $N = \kappa/|\kappa|$ but this will be irrelevant.) In the balance equation of Theorem 4.8, since $\overline{\nu}$ vanishes outside U, we can thus equally well write the integral over U as an integral over all of L.

For our further analysis, it will be important to make use of the space $\mathrm{BV}(M, \mathbb{R}^n)$ of functions of bounded (essential) variation, again on a submanifold $M \subset L$ of a C^1 curve. For $k \ge 1$ we write $W^{k,\mathrm{BV}}(M, \mathbb{R}^n)$ for the Sobolev space of functions whose k^{th} (distributional) derivatives (with respect to arc length) lie in $\mathrm{BV}(M, \mathbb{R}^n)$. We write $\mathrm{BV}_{\mathrm{loc}}(M, \mathbb{R}^n)$ for the space of functions with locally bounded variation in M, and similarly for $W^{k,\mathrm{BV}}_{\mathrm{loc}}(M, \mathbb{R}^n)$. We recall a few facts about BV functions. (Compare the discussion in [22, Section 1] and the references there.)

- Any $f \in \mathrm{BV}_{\mathrm{loc}}(M, \mathbb{R}^n)$ (after modification on a set of measure zero) is regulated, that is, has only jump discontinuities. (On the other hand, of course not even every continuous function is in $\mathrm{BV}_{\mathrm{loc}}$.)
- We have $f \in \mathrm{BV}_{\mathrm{loc}}(M, \mathbb{R}^n)$ if and only if its distributional derivative is a vector-valued Radon measure (with atoms at the jumps of f).
- Any function $g \in W^{1,\mathrm{BV}}_{\mathrm{loc}}(M, \mathbb{R}^n)$ is continuous and locally Lipschitz. (A continuous curve is in $W^{1,\mathrm{BV}}$ if and only if it has finite total curvature.)

Lemma 4.9 *Suppose L is regularly balanced. Then the projected kink measure $\overline{\nu}$ is absolutely continuous with respect to ds and indeed there exists $\Phi \in W^{1,\mathrm{BV}}(L,\mathbb{R}^3)$ such that $N\overline{\nu} = \Phi\, ds$ and $\Phi(p) \perp H^1_p$ at each endpoint $p \in \partial L$. The balance equation for L can then be written as*

$$\int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', T - 2|\Phi| T - \sigma \Phi' \rangle\, ds.$$

Proof The balance equation from Theorem 4.8 equates $\int_L \langle \eta'', N\, d\overline{\nu} \rangle$ with terms of order at most one in η, so this term is also order one. Thus we can write $N\overline{\nu} = \Phi\, ds$ with $\Phi \in \mathrm{BV}(L,\mathbb{R}^3)$. Since $\overline{\nu}$ is nonnegative, it follows that $\Phi = |\Phi| N$; of course $|\Phi| \in \mathrm{BV}(L)$ is nonnegative and vanishes (a.e.) outside U. Now we may integrate by parts to obtain

$$-\int_L \langle \eta'', N \rangle\, d\overline{\nu} = -\int_L \langle \eta'', \Phi \rangle\, ds = \int_L \langle \eta', \Phi'\, ds \rangle - \sum_{p \in \partial L} \langle \pm\eta', \Phi \rangle,$$

where $\pm\eta'$ is the derivative of η in the outward direction $\pm T$. Note that the value $\Phi(p)$ of a BV function at an endpoint is well defined as the one-sided limit.

Thus we may write the balance equation from Theorem 4.8 as

$$\int_L \langle \eta', T \rangle\, ds - \int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', 2|\Phi| T + \sigma \Phi' \rangle\, ds - \sigma \sum_{p \in \partial L} \langle \pm\eta', \Phi \rangle.$$

Since the left-hand side has order 0, so does the right-hand side. Our first conclusion is that the atomic terms $\langle \eta', \Phi \rangle$ vanish at each endpoint. Since a compatible vector field η can have an arbitrary value $\eta'(p) \in H^1_p$ at $p \in \partial L$, this simply means that $\Phi(p) \perp H^1_p$. The balance equation then reduces to the form given in the lemma.

Our second conclusion is that the integrand $2|\Phi| T + \sigma \Phi'$ (which gets paired with η') is a BV function. Since T and $|\Phi|$ are both BV, so is their product and we conclude that $\Phi' \in \mathrm{BV}$, that is, that $\Phi \in W^{1,\mathrm{BV}}(L,\mathbb{R}^3)$, as desired. In particular Φ is continuous. □

A few comments on the boundary conditions are in order. Let $p \in \partial L$ be an endpoint. By continuity it is clear that $\Phi(p)$ is a normal vector. Thus if $\dim H^1_p = 1$ (that is, if the tangent vector at p is fixed) then the condition $\Phi \perp H^1_p$ is automatic. If on the other hand $\dim H^1_p = 3$ (that is, if the tangent vector is free) then of course $\Phi \perp H^1_p$ means $\Phi(p) = 0$.

Corollary 4.10 *If L is regularly balanced then the vector field Φ of Lemma 4.9 vanishes on the jump set $J \subset U$ of κ.*

Proof Suppose $x \in J$ is a jump point of κ. If at least one one-sided limit has $|\kappa|(x\pm) < 1/\sigma$, then there are no kinks in some one-sided neighborhood of x. Thus $\overline{\nu}$ vanishes on that neighborhood and so does Φ, so $\Phi(p) = 0$ by continuity. Otherwise, the jump in κ reflects a jump between kinks in different normal directions, that is, N also has a jump at x. But the continuity of Φ implies that $N = \Phi/|\Phi|$ is continuous at any point where $\Phi \neq 0$. Thus again we conclude $\Phi(p) = 0$. □

Definition 4.11 Suppose L has $\mathrm{Thi}_\sigma = 1$. A *kink tension function* for L is a nonnegative $\phi \in W^{1,\mathrm{BV}}(L)$, vanishing at any endpoint $p \in \partial L$ with free tangent vector, such that on the open set $U := \{p \in L \mid \phi(p) > 0\}$ the link L is C^2 with constant curvature $|\kappa| \equiv 1/\sigma$. We call the BV vector field

$$V := (1 - 2\phi)T - \sigma(\phi N)'$$

the *virtual tangent* associated to ϕ, noting that it agrees with T outside U.

We are now ready to give our final reformulation of the balance criterion.

Definition 4.12 Suppose L has $\mathrm{Thi}_\sigma = 1$. We say L is *nicely balanced* if it has a strut measure μ (with strut force measure Ω) and a kink tension function ϕ (with virtual tangent V) such that $\Omega + V' = 0$ as measures on the interior of L, while at each endpoint $p \in \partial L$, we have $\Omega\{p\} \mp V(p) \perp H_p^0$.

Note that this nice form $\Omega = -V'$ of the balance equation generalizes the equation $\Omega = -T'$ for kink-free arcs (where of course $V = T$) from Lemma 3.19. Physically, of course, for a (nonkinked) curve under tension (minimizing its length), the tangent vector T at a point p can be thought of as the force exerted by the arc of the curve after p on the arc before p. Along a kinked arc, this force is instead V, due to the fact that the curvature bound is active. The kink tension function ϕ can be thought of as giving the Lagrange multipliers for the curvature bounds at each point along the curve. Physically one could imagine a "triple strut" acting like an archer's bow to transmit force between a point q and points some tiny arc length ϵ before and after it along L, through bars attached to each other at the center of the osculating circle. Then $\phi(q)$ gives the relative strength to which this triple strut is used, in a limit as $\epsilon \to 0$. The formula above for $V(p)$ then follows as the net transmitted force between the arcs before and after p.

The next theorem is our final main technical result.

Theorem 4.13 *A link L is regularly balanced (Definition 4.4) if and only if it is nicely balanced (Definition 4.12).*

Proof Suppose first that L is regularly balanced. In view of Lemma 4.9 we set $\phi := |\Phi|$. Since this is continuous, $\{\phi > 0\}$ is open, and we may replace the original U (in the definition of regularly balanced) by this open subset. Since ϕ vanishes on J by Corollary 4.10, we know that L is C^2 on U. In terms of the virtual tangent $V = (1-2\phi)T - \sigma\Phi'$, the balance equation of the lemma is $\int_L \langle \eta, d\Omega \rangle = \int_L \langle \eta', V \rangle \, ds$. Integrating by parts gives $\Omega + V' = 0$ on the interior and $\langle \eta, \Omega\{p\} \mp V(p) \rangle$ at each endpoint $p \in \partial L$. Recalling that a compatible vector field η can have any value parallel to H^0_p at p, we obtain $\Omega\{p\} \mp V(p) \perp H^0_p$.

Conversely, if L is nicely balanced with strut measure μ and kink tension function ϕ, we define $\overline{\nu} := \phi \, ds$. Since L is C^2 along $U = \{\phi > 0\}$ there is a unique kink measure ν projecting to this $\overline{\nu}$. Retracing our steps in the integrations by parts, we see that L is regularly balanced by μ and this ν. □

We note that it would be possible to do the analysis of this section for a single subarc $A \subset L$. If A has regulated kinks, then the kink measure over A can be expressed in terms of a kink tension function and virtual tangent. If A abuts other kinked arcs, the boundary conditions of course get more complicated. We have not carried this out in detail even though it would allow a slight strengthening of the results below on strut-free kinked arcs; we would only need to assume regulated kinks along the arc in question rather than on the whole link.

Given Theorem 4.13, we can rephrase the conjecture mentioned above as follows:

Conjecture 4.14 *Every σ–balanced link is nicely balanced. In particular, the kink measure is supported over piecewise C^2 arcs of the link.*

We gain some hope that this conjecture is true from the analysis above: we have seen, for instance, that if an arc A has regulated kinks but the jump set J of κ is dense in A, then the kink measure vanishes over A. The effect of the kink measure, as seen in the kink tension function, grows only in the interior of C^2 pieces of the link.

Corollary 4.15 *Suppose L is nicely balanced with kink tension ϕ. Then along U we have $L \in W^{3,\mathrm{BV}}_{\mathrm{loc}}(U, \mathbb{R}^3)$. The normal N and thus also the binormal $B := T \times N$ are in $W^{1,\mathrm{BV}}_{\mathrm{loc}}(U)$, so the torsion $\tau := \langle N', B \rangle$ is locally* BV *on U.*

Proof Recall that $\phi \in W^{1,\mathrm{BV}}(L)$ and $\phi > 0$ on U. Since $(1/\phi)' = -\phi'/\phi^2$ we see that $1/\phi \in W^{1,\mathrm{BV}}_{\mathrm{loc}}(U)$. Since $\phi N = \Phi \in W^{1,\mathrm{BV}}(L, \mathbb{R}^3)$ we conclude $N \in W^{1,\mathrm{BV}}_{\mathrm{loc}}(U, \mathbb{R}^3)$. But on U, we have $N = \sigma\kappa$, so this means that $L \in W^{3,\mathrm{BV}}_{\mathrm{loc}}(U, \mathbb{R}^3)$, as claimed. The Leibniz product rule for BV functions shows $B := T \times N \in W^{1,\mathrm{BV}}_{\mathrm{loc}}(U, \mathbb{R}^3)$, and then $\tau := \langle N', B \rangle \in \mathrm{BV}_{\mathrm{loc}}(U)$. □

It follows that along U we have the usual Frenet equations

$$T' = N/\sigma, \quad N' = -T/\sigma + \tau B, \quad B' = -\tau N.$$

We can thus write

$$V = (1-\phi)T - \sigma\phi' N - \sigma\tau\phi B,$$
$$V' = ((1-\phi)/\sigma - \sigma\phi'' + \sigma\tau^2\phi)N - \sigma(\tau'\phi + 2\phi'\tau)B.$$

Along U we may decompose the restricted strut force measure $\Omega|_U$ into two signed Radon measures

$$\Omega|_U = \Omega_N N + \Omega_B B, \quad \Omega_N := \langle \Omega, N\rangle, \quad \Omega_B := \langle \Omega, B\rangle.$$

We now rewrite the balance equation $\Omega = -V'$ in terms of these measures.

Corollary 4.16 *If L is nicely balanced, then we have the following equalities of signed Radon measures on U:*

$$\sigma^2\phi'' + (1-\sigma^2\tau^2)\phi = 1 + \sigma\Omega_N,$$
$$\sigma(\phi^2\tau)' = \phi\Omega_B.$$

Further smoothness results would depend on better understanding how the geometry of the rest of the curve affects the struts converging on a given arc. Of course we know that outside the closure of U, the strut force measure $\Omega = -T'$ is absolutely continuous. On this closure, however, Ω can even have atoms. The next result describes their effect on τ and ϕ.

Corollary 4.17 *At a point $p \in U$, an atom of Ω_N corresponds to a jump in ϕ', while an atom of Ω_B corresponds to a jump in τ. If $\Omega\{p\} = 0$ at a limit point p of $L \smallsetminus U$, then $\phi'(p) = 0$. If $\Omega\{p\} = 0$ at an isolated point p of $L \smallsetminus U$, then $\phi'_+(p) + \phi'_-(p) = 0$ and if these are nonzero then N changes sign at p.*

Proof From the equation $\Omega = -V'$ and the fact that $(1-2\phi)T$ is continuous, we see that

$$\text{atom of } \Omega \longleftrightarrow \text{jump in } V \longleftrightarrow \text{jump in } (\phi N)'.$$

Thus on U, an atom of Ω_N corresponds to a jump in ϕ' while an atom of Ω_B corresponds to a jump in $\phi^2\tau$, that is, to a jump in τ.

Now recall that $\phi \equiv 0$ on $L \smallsetminus U$. Thus if p is a limit point, at least one of the one-sided derivatives $\phi'_\pm(p)$ vanishes. If Ω has no atom at p, the derivative $\phi'(p)$ exists, hence is 0.

Finally, suppose p is an isolated point of $L \smallsetminus U$. If Ω has no atom there, then $\phi' N$ is continuous at p, which yields the desired conclusion. □

As an example, we consider a planar kinked arc, that is, a circular arc, say of total turning angle 2α.

Lemma 4.18 *Suppose γ is a kinked circular arc of turning angle 2α, joined at each end to straight segments. Suppose further that γ bears no strut force except for a single atom. Then γ is balanced if and only if this atom acts at the midpoint p of the arc, in the principal normal direction $-N(p)$ with mass $2\sin\alpha$. The kink tension function is $\phi = 1 - \cos(\alpha - \sigma|s|)$, where s denotes the arc length from p.*

Proof Let T_0 and T_1 be the tangent vectors to the straight segments. Since $V = T$ on these segments, the jump in V is exactly $T_1 - T_0 = 2\sin\alpha N(p)$. This jump must cancel the atom of strut force measure. Since the strut force always acts in the normal plane and $N(p)$ is normal to the curve only at p, we see the atom is at p as claimed.

In the planar case of $\tau = 0$, the equations of Corollary 4.16 reduce on a strut-free arc to $\sigma^2\phi'' + \phi = 1$. Since ϕ vanishes at the ends of the arc, we solve to get $\phi = 1 - \cos(\alpha - \sigma|s|)$ as claimed. This solution for ϕ illustrates that ϕ' vanishes at the endpoints, but jumps by $-2\sin\alpha$ where the strut force is applied. □

Remark It is also interesting to consider where (along the unit normal circle around p) the atom of strut force can come from. For $\sigma \geq 1$ there could be a single strut in the plane of γ, but for small stiffnesses the strut force has to come from struts acting almost normal to the plane of γ. Thinking of γ in a vertical plane with p at the bottom, we know there must be struts acting downwards on p. But the points they come from cannot be higher than the center of the circle γ, that is, cannot be more than σ above p, because higher points would be closer to the rest of γ than to p. That means the downward-acting struts are all within angle $\arcsin\sigma$ of horizontal, on one side or the other of the plane of γ. In our critical clasps (Section 7) the kink near the tip of one component is balanced by pairs of such unit circle arcs (of angle less than $\arcsin\sigma$) along the other component; we refer to these as shoulders.

We have now proved our main theoretical results; the rest of the paper applies them to study various interesting examples. We can summarize our main theorems as follows:

$$\text{nicely balanced} \overset{\text{Theorem 4.13}}{\Longleftrightarrow} \text{regularly balanced} \overset{\text{Definition 4.4}}{\Longrightarrow} \sigma\text{–balanced},$$

$$\sigma\text{–balanced} \overset{\text{Theorem 3.17}}{\Longleftrightarrow} \text{strongly } \sigma\text{–critical} \overset{\text{Definition 3.10}}{\Longrightarrow} \sigma\text{–critical}.$$

We also have the following partial converses: a σ–balanced link with regulated kinks is nicely balanced (Lemma 4.6); a σ–critical link that is Thi_σ–regular is strongly σ–critical (Lemma 3.11). We recall that every closed link — with only circle components — is regular. We can assemble these ideas into the following form, which will be most useful in applications:

Theorem 4.19 *Let L be a link with* regulated kinks *(Definition 4.5). Then L is σ–critical for ropelength (Definition 3.10) if there is a kink tension function ϕ and a strut measure μ (with strut force measure Ω decomposed into normal and binormal parts Ω_N and Ω_B) so that L is $W^{3,\mathrm{BV}}_{\mathrm{loc}}$ on the support of ϕ and, as measures,*

$$\sigma^2\phi'' + (1-\sigma^2\tau^2)\phi = 1 + \sigma\Omega_N,$$
$$\sigma(\phi^2\tau)' = \phi\Omega_B.$$

If L is Thi_σ*–regular — in particular if it is closed — then these sufficient conditions for criticality are also necessary.*

5 Length-critical curves with an upper bound on curvature

If we restrict our attention to critical curves that are balanced by kink measure alone, we replace our original problem with a more classical one from differential geometry: to find critical curves for minimizing length subject to an upper bound on curvature. It is not immediately obvious from this formulation that nontrivial solutions exist; after all, the curves that minimize length absolutely are straight lines, which have curvature zero.

To develop some intuition, consider the one-parameter family of helices

$$h_r(t) := (r\cos t, r\sin t, t)$$

with curvature $r/(1+r^2)$ and torsion $1/(1+r^2)$. The curve-shortening flow decreases $r > 0$ while staying in this family. Thus it increases curvature for $r > 1$ (that is, for $|\tau| < \kappa$) but decreases curvature for $r < 1$. As this suggests, helices with $|\tau| < \kappa$ turn out to be critical for our problem of minimizing length subject to an upper bound on curvature, while those with $|\tau| > \kappa$ cannot be.

We now proceed to use our balance criterion to determine exactly which curves — including the helices just mentioned — are critical for this problem. We consider arcs of critical curves that are balanced by kink measure alone. In the absence of strut force, it is convenient to ignore struts completely and to rescale such that kinks have curvature 1. Essentially, we take a limit of the constraints $\sigma\,\mathrm{Thi}_\sigma(L) \geq 1$ as $\sigma \to \infty$, and are left with the curvature constraint

$$\mathrm{Thi}_\infty(L) := \min_L \rho \geq 1.$$

It should be clear that the derivative of Thi_∞ is like that of Thi_σ but sees only the kink terms, and that our general balance theorem adapts to this situation to say L is *strongly ∞–critical* if and only if it is *balanced by kink measure alone*. In case L has regulated kinks, it is of course regularly and indeed nicely balanced as before.

Lemma 5.1 *Suppose L is σ–balanced, and A is a compact subcurve such that the strut force measure Ω vanishes along the interior of A. (In particular this is the case if there are no struts with endpoints in the interior of A.) Then the rescaled curve A/σ has* $\mathrm{Thi}_\infty \geq 1$*. Considered as a curve with fixed endpoints and fixed tangent directions there, A/σ is balanced by kink measure alone, and is thus strongly ∞–critical. Conversely, if A is strongly ∞–critical, then for any $\sigma \geq 1/\mathrm{reach}(A)$ we find that σA is σ–balanced.*

Proof For the first direction, note that even if some struts to A carry strut measure necessary to balance other parts of the curve, they have by assumption no net effect on A and thus can be ignored when balancing A. The endpoint constraints on A ensure there is no restriction on the kink measure there.

For the converse, note first that $\mathrm{Thi}_\sigma(\sigma A) \geq 1$. In the case $\sigma = 1/\mathrm{reach}(A)$, the curve σA may have some struts, but even then it can be balanced with $\mu = 0$. □

Remark For this problem of minimizing length subject only to the curvature constraint $\mathrm{Thi}_\infty \geq 1$, we can treat each component of a link separately. As in Figure 3 (right), the curves do not necessarily stay embedded: we may have nonembedded critical configurations. Thus we should generalize our setup to allow nonembedded $C^{1,1}$ curves.

We proceed to classify connected, strongly ∞–critical curves, under the assumption that they have regulated kinks. That is, we classify connected curves which are nicely balanced by kink measure alone. Of course each such curve has positive reach if it is embedded, and is thus σ–critical for large enough σ, but we do not compute the reach for our individual examples. By the lemma above, any strut-free arc of a nicely balanced link will be one of the curves in our list.

To get started, suppose L is a connected curve, nicely balanced by kink measure alone. Note that although we are considering Thi_∞, we have rescaled to get curvature 1, so we should take $\sigma = 1$ in the formulas from the last section. For instance, the virtual tangent vector becomes

$$V = (1-2\phi)T - (\phi N)' = (1-\phi)T - \phi' N - \phi\tau B.$$

Since $V' = -\Omega = 0$, we see that V is constant along L. Indeed, this "force" V should be viewed as the conserved quantity along L corresponding to the translational symmetry of our problem.

With $\sigma = 1$ and $\Omega = 0$, the equations from Corollary 4.16 for the kink tension function ϕ along $U := \{\phi > 0\}$ become

$$\phi'' + (1-\tau^2)\phi = 1, \quad (\phi^2\tau)' = 0. \tag{1}$$

Thus on each component $C \subset U$ we see that $\phi^2\tau$ is some constant c. On C we can then express (1) as the semilinear ODE

$$\phi'' = 1 - \phi + \frac{c^2}{\phi^3} \tag{2}$$

for ϕ and we get

$$V = (1-\phi)T - \phi' N - \frac{c}{\phi} B. \tag{3}$$

In particular, along C we have

$$|V|^2 = (\phi - 1)^2 + \phi'^2 + \frac{c^2}{\phi^2}$$

and since V is constant, this is a conserved quantity for the ODE. For $c \neq 0$ consider as phase space the $\phi > 0$ half of the (ϕ, ϕ')–plane. (For $c = 0$ we take for now the whole (ϕ, ϕ')–plane and impose the requirement $\phi \geq 0$ later.) On this phase space, the above expression for $|V|^2$ is clearly a proper, strictly convex function. Thus it has a single minimum — at some fixed point $(\phi_0, 0)$ for the flow — and its other level sets are closed loops encircling this minimum. It follows that all solutions to (2) are periodic; each is determined by the parameters c and $|V|$. This discussion makes it clear that the cases $c \neq 0$ and $c = 0$ should be considered separately; we treat them in the next two subsections.

5.1 Supercoiled helices

Proposition 5.2 *Suppose a connected curve L is nicely balanced by kink measure alone and suppose at some point $p \in L$ we have $\phi^2\tau \neq 0$. Then $\phi^2\tau = c$ is constant along all of L, and $\phi > 0$ satisfies (2). The kink tension function ϕ on such L is uniquely determined.*

Proof As above, let C be the component of $\{\phi > 0\}$ containing p, and set $c := \phi(p)^2\tau(p) \neq 0$. On C we know ϕ satisfies (2) for some $c \neq 0$. The level set of $|V|^2$ is a closed loop in the half-plane $\phi > 0$, meaning the solution extends with nonvanishing ϕ to the whole curve L. For the final statement, note first that ϕ is uniquely determined up to a constant factor by the fact that $\phi^2\tau$ is constant; the constant is then determined by (1). □

To understand these solutions better, let us first consider helices again. A helix of constant curvature $\kappa \equiv 1$ and torsion $\tau \equiv m$ also has pitch m and lies on a cylinder of radius $1/(1+m^2)$; in appropriate coordinates it is parametrized as $(\cos t, \sin t, mt)/(1+m^2)$. If it is balanced then by (1), we see $\phi \geq 0$ is a constant $\phi \equiv \phi_0 = 1/(1-m^2)$. Clearly this works exactly when $|m| < 1$, that is, when $|\tau| < \kappa$. (We saw before that helices with $|\tau| > \kappa$ are not critical as they can be shortened while decreasing curvature.) We compute $c = m/(1-m^2)^2$ and

$$|V|^2 = cm(1+m^2) = \frac{m^2(1+m^2)}{(1-m^2)^2}.$$

Using (3), we see that the virtual tangent vector V points along the axis of the helix, but in the *opposite direction* from T, as $\langle V, T\rangle = 1-\phi < 0$. (Physically, the endpoint constraints are holding a kinked helix under compression, rather than tension as for a straight arc.)

To consider general solutions, we start again with any value of $m \in (-1, 1)$ and define c by $c := m/(1-m^2)^2$. A direct computation shows that the minimum value of $|V|^2$ on the (ϕ, ϕ')–plane is then $cm(1+m^2)$, occurring at $(1/(1-m^2), 0)$, and every solution to (2) then corresponds to a choice of $|V| \geq \sqrt{cm(1+m^2)}$. Equality gives the helix described above with $\tau \equiv m$ and $\phi \equiv 1/(1-m^2)$, while greater values of $|V|$ lead to solutions where τ and ϕ oscillate above and below these values. Each solution can also be described by the maximum value of ϕ along its orbit in the (ϕ, ϕ')–plane, which will be $k/(1-m^2)$ for some $k \geq 1$. This k determines $|V|$ by

$$|V|^2 = \frac{(k-1+m^2)^2 + m^2/k^2}{(1-m^2)^2}. \tag{4}$$

Even if these general solutions cannot be expressed in closed form, it is easy to integrate the ODE numerically for different values of m and $|V|$. Given their shapes (seen in Figure 2), we call these curves *supercoiled helices*. We can restate Proposition 5.2 as follows: Suppose a connected curve L has nonzero torsion somewhere and is nicely balanced by kink measure alone. Then L is a subarc of some supercoiled helix.

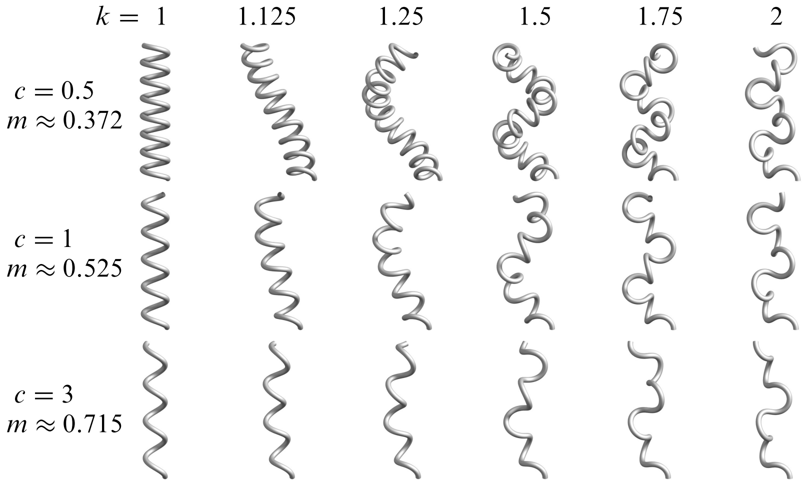

Figure 2: The picture shows σ–critical curves obtained by solving (2) with various values for $c = m/(1-m^2)^2$ and various initial conditions. For any c, there is one solution with constant $\phi \equiv 1/(1-m^2)$: a helix with torsion m. The solutions shown have initial conditions $\phi'(0) = 0$ and $\phi(0) = k/(1-m^2)$, for various $k \geq 1$. The shape of the curves explains why we call them supercoiled helices; they become progressively more twisted as k increases. The virtual tangent V is vertical in all of these pictures, and we can see that each curve is invariant under a screw motion along V, as guaranteed by Proposition 5.5.

This same family of curves was discovered by Hector Sussmann, who called them "helicoidal arcs". Sussmann gives a fascinating control-theoretic derivation of the family in his research abstract [23]. He considers the same problem of minimizing length subject to the curvature bound $\mathrm{Thi}_\infty \geq 1$ for arcs with fixed endpoints and fixed tangents there. He shows the absolute length minimizer (for any given boundary conditions) is either a helicoidal arc or a concatenation of at most three circular arcs and straight segments (as in our case $c = 0$ below). Our results are somewhat weaker than Sussmann's in that he has fewer regularity assumptions, but are stronger in that we classify all *critical* curves, rather than just minimizers. (Sussmann also claims to have a proof that any supercoiled helix is a local strict minimizer for length in the sense that each subarc of length less than some $\delta > 0$ is the unique length minimizer for its endpoints, but the promised paper with details does not seem to have appeared even as a preprint.)

As is clear from the pictures, each supercoiled helix is invariant with respect to some screw motion (perhaps degenerating to a translation) along the direction of V, which we call vertical. To prove this, we analyze the vertical and horizontal components separately.

Lemma 5.3 *Suppose an arc from p to q is nicely balanced by kink measure alone, with $\phi > 0$ and virtual tangent V. Then*

$$\langle q-p, V\rangle = \phi'(q) - \phi'(p) - c^2 \int_p^q \phi^{-3}\, ds.$$

Proof From (3) and (2) we have

$$\langle q-p, V\rangle = \int_p^q \langle T, V\rangle\, ds = \int_p^q (1-\phi)\, ds = \int_p^q \phi''\, ds - c^2 \int_p^q \phi^{-3}\, ds. \qquad \square$$

We conjecture that each supercoiled helix is embedded; while we do not attempt to prove this, the last lemma suffices to show that the curve does not close after any full number of periods:

Corollary 5.4 *Each period of a supercoiled helix makes negative progress in the direction of V. In particular, for $c \neq 0$ no solution to* (2) *gives a closed curve.*

Proof For $c \neq 0$ the lemma means that each period of the curve makes the same negative progress $-c^2 \int_L \phi^{-3}\, ds$ in the V direction. Thus we cannot close up after any number of periods. $\square$

Now we turn to analyzing the horizontal part of the supercoiled helix L. For this, consider the curve $V \times L$, a rotated and scaled version of the horizontal projection. Differentiating gives

$$(V \times L)' = V \times T = (-\phi' N - \phi\tau B) \times T = -\phi\tau N + \phi' B = (\phi B)'.$$

But this means that $V \times L - \phi B \equiv: W$ is a constant. Since ϕB is bounded, we immediately see (for $V \neq 0$) that L is contained in a cylinder around an axis parallel to V. Just as V can be viewed as a conserved force, the (pseudo)vector W is the conserved torque corresponding to the rotational invariance of our problem. This torque W of course depends on a choice of origin; by translating L we can change its horizontal component (perpendicular to V). In particular, we will translate to make W vertical, a scalar multiple of V. This minimizes $|W|$ and centers the bounding cylinder for L around the origin.

With this choice of origin, $V \times W = 0$. Thus, writing $L^\perp$ for the horizontal component of L, we have

$$L^\perp := -\frac{V \times (V \times L)}{|V|^2} = -\frac{V \times \phi B}{|V|^2}. \tag{5}$$

Since $\langle V, \phi B\rangle \equiv -c$, we get

$$|V \times \phi B|^2 = |V|^2|\phi B|^2 - \langle V, \phi B\rangle^2 = \phi^2|V|^2 - c^2. \tag{6}$$

Combining (5) and (6) gives

$$|L^\perp| = \sqrt{\frac{\phi^2}{|V|^2} - \frac{c^2}{|V|^4}}. \tag{7}$$

Since c and V are constant, it is clear that the radius $|L^\perp|$ from the cylinder axis is a monotone function of ϕ.

Proposition 5.5 *For $c \neq 0$ every solution to (2) — that is, every supercoiled helix — is invariant under some screw motion (or perhaps a translation) in the direction of the virtual tangent V. For the supercoiled helix with $c = m/(1-m^2)^2$ and ϕ maximized at $k/(1-m^2)$, the curve is (tightly) contained in a cylinder of radius*

$$\frac{k(k-1+m^2)}{(k-1+m^2)^2+m^2/k^2} = \frac{k(k-1+m^2)c}{m|V|^2}.$$

Proof Any solution to (2) is periodic with some period P. Thus the torsion (and of course curvature) of the supercoiled helix L are P–periodic in arc length. Thus L is invariant under some rigid motion ρ of space in the sense that $L(s + P) = \rho L(s)$ for all s. But this motion must preserve the vertical direction of the constant virtual tangent V. That is, ρ is a screw motion along an axis parallel to V, perhaps degenerating to a translation or a rotation; the case of a rotation is ruled out by Corollary 5.4. Since we have translated to make $W \parallel V$, the screw axis passes through the origin. The cylinder radius is the maximum value of $|L^\perp|$, calculated from (7) at the maximum $\phi = k/(1-m^2)$. □

5.2 Planar critical curves

Now we turn to the case $c = 0$. Based on what we have already proved about the case $c \neq 0$, we see that if $c = 0$ on one component C of $U \subset L$, then we must have $c = 0$ on all of U. Thus $\tau \equiv 0$ on U, so each component of U is an arc of a unit circle (if not the whole circle). Thus L is made up of (potentially infinitely many) circular arcs (the components of U) possibly joined by straight segments ($L \smallsetminus U$). We will use Corollary 4.17 to analyze the possible junctions.

First we examine the possible kink tension functions ϕ on a circular arc, noting that for $c = 0$ equations (2), (3) become

$$\phi'' = 1 - \phi, \quad V = (1-\phi)T - \phi' N.$$

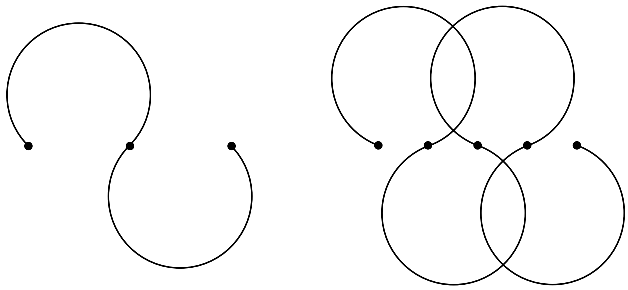

Figure 3: A wave is the planar C^1 concatenation of circular arcs of the same turning angle $\theta > \pi$. On the left, we see such an example. Since the straight line joining these endpoints is also critical, this shows that there are many σ–critical curves joining the same pair of fixed endpoints. If we allow nonembedded curves, there are infinitely many such critical configurations, like the one on the right.

Now suppose that L is a unit circle. Given any vector V in the plane of L, we define $\phi := 1 - \langle T, V \rangle$ on L. Clearly $\phi \geq 0$ on L if and only if $|V| \leq 1$. That is, the various possible kink measures balancing L correspond to the virtual tangent vectors V in the closed unit disk. For $V = 0$ we have $\phi \equiv 1$ (and it is interesting to think of L as a degenerate helix with $m = 0$ in the context of the discussion after Proposition 5.2). For $|V| < 1$ we have $\phi > 0$ on L. For $|V| = 1$ we have $\phi > 0$ except at a single point $p \in L$ where $\phi(p) = 0 = \phi'(p)$.

For $|V| > 1$, we cannot use this ϕ to balance the whole circle, but we do have $\phi > 0$ on an arc of more than half the circle, centered at the point where T points in the direction $-V$; at its endpoints $\phi = 0$ but $\phi' \neq 0$. Congruent such arcs can be joined end-to-end in a C^1 fashion such that V remains constant at each junction point while N flips sign; see Figure 3. We call an infinite such concatenation a *wave*. A wave is embedded if and only if the turning angle of each piece is less than $5\pi/3$, that is, if and only if $|V| > 2/\sqrt{3}$. (The borderline case corresponds to two rows of the hexagonal circle packing.)

Theorem 5.6 *Suppose L is an embedded connected curve, nicely balanced by kink measure alone (for fixed endpoints with fixed tangents). If L has any point of nonzero torsion, then as we have seen, it is a subarc of some supercoiled helix (for instance a helix of torsion less than 1). Otherwise L is either a straight segment (possibly joined to circular arcs at each end), a circle (or arc thereof), or a subarc of some wave.*

Proof We have already treated the case of nonzero torsion, so we may assume $c = 0$. Thus the curve is made up of straight segments and unit circular arcs. At any junction between two pieces we have $\phi = 0$, and by Corollary 4.17 we have $\phi' = 0$ unless N flips sign.

Our classification now proceeds according to $|V|$. Along any straight segment we have $V = T$, so $|V| = 1$; if the segment is joined to a circular arc at either end, this V uniquely determines the kink tension function on that arc. In particular the embeddedness of L means each arc is less than a full circle, so we never have $\phi = 0$ again along either arc and there are no further junctions. (If the segment degenerates to a point, the two circular arcs form part of a wave, and L can also be balanced as below with $|V| > 1$.)

If $|V| < 1$ on a circular arc then $\phi > 0$ so there are no junctions and L is a circle, or some subarc. (Here V is not uniquely determined. Since L is embedded we do not go more than once around the circle.)

Finally if $|V| > 1$ on a circular arc, then if the arc extends to where $\phi = 0$ we have $\phi' \neq 0$ there, so if there is a junction it is exactly the kind seen in a wave. Extending, there can be further junctions, but the whole curve is a subarc of the wave specified by V. (If there is no junction, we are really in the previous case of a circular arc. If there is a single junction, we can consider L as a subarc of many different waves, with any small enough $|V| > 1$, or balance it with $|V| = 1$. As long as there are at least two junctions, V and the wave containing L are uniquely determined.) □

Remark If we did allow nonembedded curves, then there would be additional examples as follows: at any point $p \in L$ where $\phi = 0 = \phi'$ (for instance any point along a straight segment of L), we can splice in a "hoop," a full circle tangent to L at p. Indeed we could traverse many different hoops at p before continuing further along the initial curve L. Comparing where we used embeddedness in the proof above, we see these (along with circles traversed more than once) are the only new examples.

Corollary 5.7 *Suppose L is an embedded connected curve, nicely balanced by kink measure alone (for fixed endpoints with* free *tangents). Then L is either a straight segment, a circle, or the subarc of a wave between some two junction points; that is, a planar C^1 concatenation of circular arcs with equal turning angle $\theta > \pi$ (and $\theta < 5\pi/3$ if there are more than two arcs).*

Proof Since the tangent vectors at the endpoints are free, we must have $\phi = 0$ there. That means we are looking for those examples from the theorem that satisfy this boundary condition. (Recall that on almost all examples, ϕ was uniquely determined.) Supercoiled helices are clearly excluded. In the other three examples, the endpoints are restricted to the special cases listed. □

Remark Analogous to the remark about curve-shortening flow on helices, we can give the following intuition for the condition that each piece in a wave has turning angle greater than π. Consider the one-parameter family of circular arcs through two fixed points in a plane. The curvature is maximized at the semicircle. The arcs of less than a semicircle can thus be shortened while decreasing curvature — even staying within our family — while the arcs of more than a semicircle cannot.

Durumeric [11] used Sussmann's work to prove that every closed $C^{1,1}$ curve which is a local minimum for ropelength has at least one strut. In our language, such curves are $\frac{1}{2}$–minimizing. We now prove a similar result which again is weaker in that it requires regulated kinks but stronger in that it applies to all critical curves, not just to minimizers.

Corollary 5.8 *Every closed $\frac{1}{2}$–critical curve with regulated kinks has at least one strut.*

Proof If the curve has nonzero strut force measure, it must have struts. If not, the curve is a circle of unit diameter by Theorem 5.6, and it again has struts. □

It is also interesting to see how two arcs of the type we have been considering can join at a point p where there is an atom of strut force measure. At p the virtual tangent V jumps by exactly $\Omega\{p\}$, while of course ϕ is continuous. If $\phi(p) = 0$ we are talking about a junction between circular arcs (or perhaps one straight segment); here the atom of Ω allows us to change the plane of the circle (and to change ϕ').

If on the other hand $\phi(p) > 0$, the Frenet frame is well defined, and we now consider atoms in Ω_N and in Ω_B separately using Corollary 4.17. At an atom of Ω_B we have a jump in $c = \phi^2\tau$ but ϕ' (like ϕ) is continuous. That is, we might change from one supercoiled helix to another, or might jump to or from the case $c = 0$. At an atom of Ω_N, on the other hand, c stays constant but ϕ' jumps. For $c \neq 0$ this means a vertical jump in the phase space, generally to a different supercoiled helix with the same c, but if ϕ' merely changes sign then $|V|$ is unchanged and we have merely jumped to a different point on the same supercoiled helix. For $c = 0$ we don't see any effect on the curve at p — it remains a circular arc — but the jump in ϕ' affects where ϕ vanishes to either side along this arc (as we saw in Lemma 4.18).

6 Noncompact curves

Sometimes it is interesting to consider noncompact (but still metrically complete) curves L. Since a complete curve L with positive reach is properly embedded, for any compact $K \subset \mathbb{R}^3$, the intersection $L \cap K$ is compact. Typically (for instance, by Sard's theorem for almost every closed ball K) this intersection is actually a compact subcurve of L.

Of course the length of L is infinite, but if we restrict our attention to variations ξ supported on some compact $K \subset \mathbb{R}^3$ then $\delta_\xi \operatorname{Len}(L)$ and $\delta_\xi \operatorname{Thi}_\sigma(L)$ are given by the same formulas as before, noting that only those struts and kinks touching $K \cap L$ — a compact subfamily — matter here.

Fix now a compact K and a complete curve L with $\operatorname{Thi}_\sigma(L) = 1$. We say that L is strongly σ–critical for variations supported on K if there exists $\epsilon > 0$ (depending on K) such that the condition in the earlier definition of strong criticality holds for all ξ supported on K. We say that L is σ–balanced for variations supported on K if there exist strut and kink measures (depending on K) such that the balance equation holds for all ξ supported on K.

It is straightforward to extend the general balance criterion (for each K) to say that L is strongly critical for variations supported on K if and only if it is σ–balanced for variations supported on K. Indeed, in the typical case when $K \cap L$ is a compact subcurve A, this statement is only slightly different from the general balance criterion for A (considered with any new endpoints and their tangents fixed): Essentially the parts of L at distance at most 1 from K act as obstacles for A.

Now suppose for a complete curve L with $\operatorname{Thi}_\sigma(L) = 1$ we can find a single strut measure μ and a single kink measure ν (typically given by a kink tension function $\phi \in W^{1,\mathrm{BV}}_{\mathrm{loc}}(L)$ vanishing outside C^2 arcs) such that the balance equation holds for all compactly supported ξ. It follows for each K that L is strongly critical for variations supported on K. In particular, L is critical; any compactly supported variation that decreases length must also decrease thickness.

In previous sections, we have implicitly seen several examples like this already.

- A straight line is balanced by $\mu = 0$ and $\nu = 0$.
- An infinite double helix of pitch at least 1 is balanced by the single family of struts in one-to-one contact.
- Any supercoiled helix is balanced by the $\phi > 0$ used to define it; in particular any infinite single helix with $\tau < |\kappa|$ is balanced by a constant ϕ.
- Any infinite wave (with each piece having turning angle more than π) is balanced by its ϕ, which vanishes at every junction.

With appropriate regularity and smoothness assumptions, one can show these are the only complete critical curves with the kink/strut patterns we considered before, that is, kink-free with controlled strut pattern as in Section 3.3, or strut-free as in Section 5.

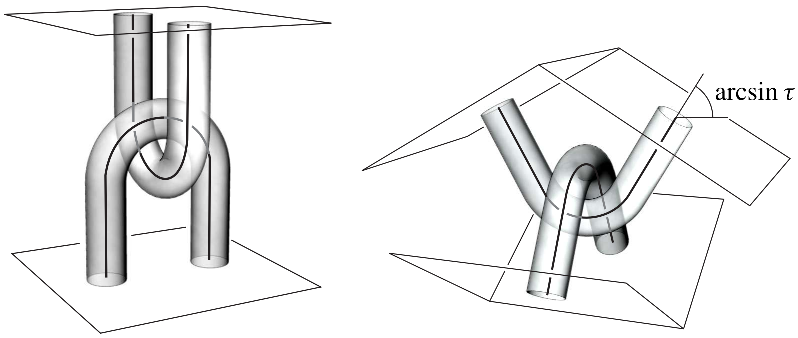

Figure 4: The clasp is the simplest configuration of two interlooped arcs. On the left, we see the basic clasp where the endpoints are constrained to lie in parallel planes. On the right, we have the angled clasp where the four ends of the rope make an angle of $\arcsin\tau$ with the horizontal. We will study σ–critical clasp configurations for varying values of τ and σ.

In the clasps we discuss next, the ends of each arc — attached the boundary planes — are straight segments. Clearly we could extend these to be infinite rays and talk about a complete clasp. It would be balanced by the same compactly supported strut and kink measures used for the compact clasp.

7 The tight clasp

Our next example is a variation on the "simple clasp" which we considered previously in [4, Section 9]. This clasp is a system of two interlooped ropes as in Figure 4 (left), one anchored to the floor and one to the ceiling. We studied the problem of minimizing the total length subject to the Gehring condition that the two strands are everywhere separated by at least unit distance, that is, that the link-thickness is at least 1.

In fact, we considered the entire family of "τ–clasp" problems, $0 \le \tau \le 1$, in which the four ends of the two ropes are no longer vertical but make an angle of $\arcsin\tau$ with the horizontal. (Thus the case $\tau = 1$ is the basic clasp described above.) In each case we described in detail a critical configuration (a "Gehring clasp") that we conjectured to be minimizing. Surprisingly, for $\tau = 1$ the Gehring clasp is a C^1 curve with unbounded curvature (that is, not $C^{1,1}$).

Here we consider the analogous problem in the more physically realistic setting of the present paper where the constraint is $\mathrm{Thi}_\sigma \ge 1$. Where the Gehring τ–clasp would have curvature greater than $1/\sigma$, our σ–critical τ–clasp now has a kinked arc. Note that the struts in these critical clasps always connect one component to the other. Thus (by an argument like Proposition 3.18) they are equally well critical for a Gehring problem with stiffness in which, in addition to the constraint on link-thickness, we insist that the curvature of each strand never exceed $1/\sigma$. For this problem we may permit the stiffness to assume the full range of values $0 \le \sigma < \infty$. The criticality theory for this problem is a straightforward combination of our work here with that in [4], and we refrain from developing it explicitly. In the remainder of this section, we will allow arbitrary values of σ; when $\sigma < \frac{1}{2}$ we implicitly then mean link-thickness with a curvature constraint instead of Thi_σ.

Definition 7.1 Consider a large tetrahedron with two edges forming an orthogonal frame with the line connecting their midpoints, where the dihedral angles along these edges are $2\arcsin\tau \in [0,\pi]$ as in Figure 4 (right). Suppose that the endpoints of two arcs are constrained to lie on the faces of this tetrahedra, and the arcs are linked as shown (giving a Hopf link if each component is closed with segments in its own boundary faces). The (τ,σ)*–clasp problem* is the problem of minimizing the length of this configuration subject to the constraint that $\mathrm{Thi}_\sigma(L) \ge 1$.

In this section we construct *critical curves* for the various (τ,σ)–clasp problems. These curves have the same symmetry (with the two components being congruent convex planar arcs in perpendicular planes) as our Gehring clasps. We believe these solutions are the length minimizers, but we do not see how to prove this. (Our arguments below might perhaps extend to show the curves we describe are the unique critical curves with the given symmetry, but it seems hard to show this symmetry is not broken in a minimizer.)

The maximum curvature of the Gehring τ–clasp is $1/\sqrt{1-\tau^2}$ at its tip. Thus for $0 \le \sigma \le \sqrt{1-\tau^2}$, the critical (τ,σ)–clasp is identical to the Gehring clasp, a curve explicitly described in terms of elliptic integrals. On the other hand, for larger σ, the curvature bound is active, and it is not surprising that our critical clasps include not only "Gehring arcs" (subarcs of the Gehring clasp), but also "kinks" (circular arcs of curvature $1/\sigma$) at the tips.

The curves that we obtain fall into four regimes, depending on the values of the parameters τ and σ, as shown in the phase diagram of Figure 5.

In each case they consist of two congruent arcs lying in orthogonal planes. Both components are symmetric with respect to the line of intersection of the two planes, which we take to be the z–axis. We describe the component lying in the xz–plane, which we take to be the one with endpoints attached to the ceiling, as in [4]. In the discussion below, we will refer to a circular arc of maximal curvature $1/\sigma$ as a *kink*.

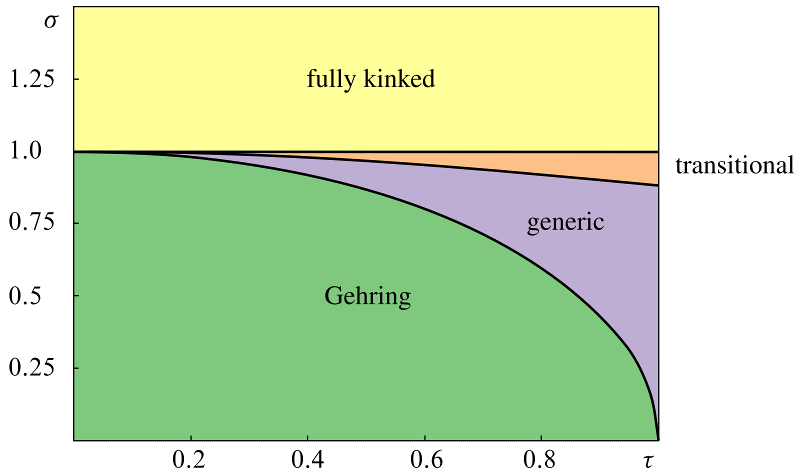

Figure 5: This phase diagram shows the domain of the various types of solutions to the clasp problem as the values of τ (the sine of the angle made by the endpoints of the clasp with the horizontal) and σ (the stiffness parameter) change. In the uppermost "fully kinked" region, the clasp is a pair of circle arcs of radius σ joined with straight segments. There is a single strut connecting these arcs. In the next "transitional" region, the clasp consists of arcs of circles of radius σ at the tips joined by straight segments to arcs of circles of radius 1 at the shoulders of the clasp, finally joined by straight segments to the endpoints. In the third "generic" region, the curve is piecewise analytic, with eleven analytic pieces: a circle arc of radius σ at the tip, joined by straight segments to arcs of the "Gehring clasp" from [4]. These arcs are joined by straight segments to circular arcs of unit radius, which are joined by straight segments to the endpoints of the clasp. In the last "Gehring" region, the solution is the same as that from [4].

The fully kinked regime: $\sigma \geq 1$ Here the curve consists of a kink of total angle $2\arcsin\tau$, with straight segments attached to the endpoints. There is exactly one strut between the two components, joining their tips (the points lying on the z–axis).

The transitional regime: $(\sqrt{4+\tau^2}-2)/(2-\sqrt{4-\tau^2}) \leq \sigma < 1$ In this case the curve consists of a kink of angle $2\arcsin\tau/2$ joined by line segments to two circular arcs of radius 1 and angle $\arcsin\tau - \arcsin\tau/2$, each centered at the tip of the other component. There is a one-parameter family of struts connecting each point of the latter arcs to the tip of the other component.

The generic regime: $\sqrt{1-\tau^2} < \sigma < (\sqrt{4+\tau^2}-2)/(2-\sqrt{4-\tau^2})$ This is the most complicated possibility, of which the others may be regarded as degenerations. The curve is piecewise analytic, with eleven analytic pieces, described by four parameters

a, b, α, β (determined in Section 7.5 below): a kink of angle 2α at the tip; joined to two straight segments of length a; each joined to a section of the Gehring τ–clasp described by the parameter interval $[\sin\alpha, \sin\beta]$; each joined to another straight segment of length b; joined to a circular arc of radius 1, centered at the tip of the other component, and of angle $\arcsin\tau - \beta$; each joined finally to a straight segment connected to a constraining plane. There are two types of one-parameter families of struts connecting the two components: first, those connecting the arcs of radius 1 to the tip of the other component; second, each point of each Gehring arc shares struts with its conjugate points (in the sense of [4]) on the two Gehring arcs of the other component.

The Gehring regime: $0 \le \sigma \le \sqrt{1-\tau^2}$ For these parameter values the critical curves are identical to those described in [4].

The clasp problem was analyzed earlier by Starostin [20]. While Starostin did not have a general criticality theory to work with, and so could not prove that his configurations were fully ropelength-critical, he derived a solution equivalent to our "generic" clasp by considering the problem of length-critical curves with a fixed contact set. Very recently, the clasp has been numerically analyzed with extremely high resolution by Pieranski and Przybyl [17]. Their results (at least for the generic regime), agree very closely with both Starostin's work and the conclusions here.

7.1 General results on clasp-type curves

We start with some useful lemmas about configurations of circular arcs.

Lemma 7.2 *Suppose a σ–critical link L passes through the origin and includes the circular arc $C := \{(\sin\theta, 0, \cos\theta) \mid \theta_0 \le \theta \le \theta_1\}$. If $\sigma < 1$ so that C is not kinked and if C has no struts except those to the origin, then these struts generate an atom of strut force measure at the origin whose vertical component has magnitude $\sin\theta_1 - \sin\theta_0$.*

Proof Since C has no kinks, $\Omega(C)$ is the difference in the tangent vectors at the two ends of C. This force all gets transmitted to the origin. □

Lemma 7.3 *Let C be circle in the xz–plane, centered at a point c on the z–axis, and let B be a C^1 arc in the yz–plane. If $(p,q) \in B \times C$ is critical for distance, and p is an interior point of B, then either $p = c$ or q lies on the z–axis.*

Proof Since (p,q) is critical for distance, the segment $\overline{pq}$ is normal to B and C. Therefore, if q does not lie on the z–axis then the projection of p to the xz–plane must be the center c of C. It follows that all points of C are equidistant from p. However, unless $p = c$ then not all of the segments $\overline{pr}$ joining p to $r \in C$ are normal to B at p, contradicting the criticality of the pair (p,r). □

To fix the symmetry of our clasps in coordinates, let one component lie in the xz–plane while the other lies in yz–plane. Our symmetry group $2 * 2$ (using the Conway–Thurston orbifold notation) is then the dihedral point group of order eight in $O(3)$ generated by mirror reflections across the xz– and yz–planes, together with a four-fold rotary reflection around the z–axis. To describe a symmetric clasp, it suffices to describe half of one component: the arc from the "tip" on the z–axis (where the curve is horizontal) to the endpoint (on a face of the enclosing tetrahedron); this convex arc has total curvature $\arcsin\tau$.

In each of our descriptions of a clasp, we will describe only the portion of the clasp in a fundamental domain for this symmetry. This will be a convex curve in the half-plane of the xz–plane with positive x; its endpoint on the z–axis will be called the *tip* of the clasp. It will sometimes be convenient for us to parametrize this curve by the sine u of the angle that its tangent makes with the x–axis.

We will be interested in proving that the minimum distance between two such arcs is at least 1. To this end we adapt [4, Lemma 9.3].

Lemma 7.4 *Let γ_1 and γ_2 be two convex curves lying in the xz– and yz–planes respectively. Suppose there is a critical pair (p_1, p_2) of length ρ connecting these components. Write x_i for the distance from p_i to the z–axis, and u_i for the sine of the angle between the tangent to γ_i and the horizontal. Then $0 \le x_i/\rho \le u_i \le 1$, and any two of the numbers x_1, x_2, u_1, u_2 determine the other two according to the formulas*

$$x_i^2 = \rho^2 - \frac{x_j^2}{u_j^2} = \rho^2 \frac{u_i^2(1-u_j^2)}{1-u_i^2 u_j^2}, \quad u_i^2 = \frac{\rho^2 - x_j^2/u_j^2}{\rho^2 - x_j^2} = \frac{x_i^2}{\rho^2 - x_j^2},$$

where $j \ne i$. The height difference between p_1 and p_2 is $\Delta z = \dfrac{x_i}{u_i}\sqrt{1-u_i^2}$.

Proof The difference vector is $p_1 - p_2 = (x_1, x_2, \Delta z)$. Since this strut has length ρ and is perpendicular to each γ_i, we get

$$\Delta z^2 + x_1^2 + x_2^2 = \rho^2, \quad \Delta z = \frac{x_i}{u_i}\sqrt{1-u_i^2}.$$

Simple algebraic manipulations, eliminating Δz, yield the other given equations. □

7.2 The fully kinked regime

We first consider a clasp constructed of very stiff rope, consisting of circular arcs and line segments (see Figure 6, left).

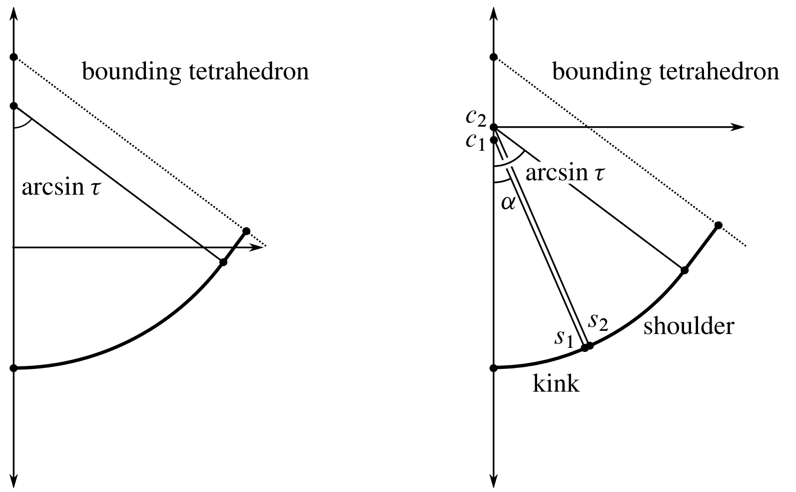

Figure 6: At the left, we see the fully kinked clasp of Proposition 7.5 with $(\tau, \sigma) = (0.8, 1.1)$. At the right, we see the transitional clasp of Proposition 7.6 with $(\tau, \sigma) = (0.8, 0.95)$. In each diagram, the upper (closely dotted) line is the intersection of a face of the bounding tetrahedron with the xz–plane. The entire curved portion of the kinked clasp (left) is a single circular arc of radius σ; the tips of the two components are at unit distance. The transitional clasp (right) consists of a lower "kinked" circular arc of radius σ joined by a short straight segment to an upper "shoulder" circular arc of radius 1. The kink extends to an angle $\alpha = \arcsin \tau/2$, while the shoulder extends to angle $\arcsin \tau$. The tip of the other component is at the center c_2 of the shoulder.

Proposition 7.5 *Let C_K be the curve in the right half-plane of the xz–plane consisting of*

- *a circular arc of radius σ of angle* $\arcsin \tau$ *centered at* $(0, 0, \sigma - \frac{1}{2})$,
- *joined to a line segment in the xz–plane.*

If $\sigma \geq 1$, the corresponding $2 * 2$ symmetric curve $\widetilde{C}_K$, where the tips of the two components lie at unit distance, is critical for the (τ, σ)–clasp problem.

Proof We must check that (i) $\widetilde{C}_K$ obeys the endpoint constraints, (ii) $\widetilde{C}_K$ obeys the thickness constraint, and (iii) $\widetilde{C}_K$ is σ–critical. The first is clear from the construction. For the second, we first note that the radius of curvature is always at least σ by construction, so that if the struts have length at least 1, the thickness constraint is satisfied. In fact, by Lemma 7.3 and symmetry, if $\sigma > 1$ the only strut is the one joining the tip points $(0,0,\frac{1}{2})$ and $(0,0,-\frac{1}{2})$. (If $\sigma = 1$, there is a family of struts joining each point on each circular arc to the tip of the other component of the clasp.)

To check that our configuration is σ–critical, since the hypotheses are clearly satisfied we may apply the final version of the balance criterion. We let the strut measure be an atom of mass 2τ on the unique strut. The arcs are then balanced against each other by the kink tension function ϕ of Lemma 4.18. On the straight segments, $T' = 0$ and $\phi = 0$, so the balance equation is clearly satisfied. At the endpoints, $\phi = 0$ and there is no strut force measure, so we require only that the curve be normal to the constraint plane, which is true by construction. □

We note that Lemma 4.18 tells us that such a configuration of circular arcs of turning angles $2\theta_0$ and $2\theta_1$ and lines is σ–critical as above if and only if $\sin\theta_0 = \sin\theta_1$. This means that in addition to the configuration above, where $\theta_0 = \theta_1 \le \pi/2$, there are balanced solutions with $\theta_0 \le \pi/2 \le \theta_1$ where a short circular arc balances a longer one, as well as balanced solutions with $\theta_0 = \theta_1 > \pi/2$. These are interesting σ–critical curves, but they do not satisfy the boundary conditions of the (τ,σ)–clasp problems.

7.3 The transitional regime

In the transitional regime, the clasp is a circle-line-circle-line curve as in Figure 6(right).

Proposition 7.6 *Suppose $\tau \le 2$. Let C_T be the C^1 curve in the right half-plane of the xz–plane consisting of the following pieces joined in succession:*

- *A (kinked) circular arc of angle* $\arcsin\tau/2$ *and radius* σ
- *A line segment of length* $\tau(1-\sigma)/\sqrt{4-\tau^2}$
- *A circular arc of radius* 1 *and angle* $\arcsin\tau - \arcsin\tau/2$ *(which we will refer to as the* shoulder*)*
- *A ray attached to the other end of the shoulder*

If

$$1 > \sigma \ge \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}} \tag{8}$$

then this curve exists, and the corresponding $2*2$ symmetric curve $\widetilde{C}_T$, the tip of whose second component lies at the center of the shoulder of the first, is a critical curve for the (τ,σ)–clasp problem.

Remark Since $(\sqrt{4+\tau^2}-2)/(2-\sqrt{4-\tau^2}) < 1$ for $\tau \in (0,1]$, we see that for each such τ the condition (8) is not vacuous.

Proof We first show that C_T exists. Referring to Figure 6, we choose coordinates so that the center of the shoulder arc lies at the origin of the xz–plane. Then the endpoints of the shoulder arc are

$$(\tau, 0, -\sqrt{1-\tau^2}), \quad s_2 := \Big(\frac{\tau}{2}, 0, -\sqrt{1-\tau^2/4}\Big). \tag{9}$$

One endpoint of the segment is s_2, and the segment has slope

$$m := \frac{\tau}{\sqrt{4-\tau^2}} \iff \tau = \frac{2m}{\sqrt{1+m^2}}. \tag{10}$$

Thus the x and z coordinates of a point on the segment are related by

$$z = \frac{\tau}{\sqrt{4-\tau^2}}\Big(x - \frac{\tau}{2}\Big) - \frac{\sqrt{4-\tau^2}}{2}. \tag{11}$$

From the value for the length of the segment given in the Proposition it is easily computed that its other endpoint is

$$s_1 := \Big(\frac{\sigma\tau}{2}, 0, \frac{\sigma\tau^2-4}{2\sqrt{4-\tau^2}}\Big). \tag{12}$$

This endpoint coincides with one endpoint of the kinked arc of radius σ. Putting c_1 for the center of this arc, the radial vector $s_1 - c_1$ is parallel to the radial vector s_2 of the shoulder, that is, makes the angle $\arcsin\frac{\tau}{2}$ with the vertical. Thus the center of this arc is

$$c_1 := \Big(0, 0, \frac{\sigma\tau^2-4}{2\sqrt{4-\tau^2}} + \sigma\frac{\sqrt{4-\tau^2}}{2}\Big) = \Big(0, 0, \frac{2\sigma-2}{\sqrt{4-\tau^2}}\Big)$$

and the tip of C is $p_0 := (0,0,z_0)$, where

$$z_0 := \frac{2\sigma-2}{\sqrt{4-\tau^2}} - \sigma. \tag{13}$$

Next we show that if (8) holds then $\widetilde{C}_T$ has $\mathrm{Thi}_\sigma \geq 1$. It is easy to see that its curvature satisfies $\kappa \leq 1/\sigma$ (since $\sigma < 1$), so we need only show that all the critical pairs have length at least 1. Let us call the two components of the curve C and C^*, and put $p_0^* = (0,0,0)$ for the tip point of C^*.

If $(p, p^*) \in C \times C^*$ is a critical pair with p on the kink arc of C, then $p = p_0$ by Lemma 7.3, since C^* does not pass through the center of the kink. The shoulders of C^* lie on the boundary of the ball of radius 1 about p_0, and by elementary geometry the rest of C^* lies strictly outside it. Therefore any such pair has length at least 1.

If (p, p^*) is a critical pair with p on the shoulder of C, then $p^* = p_0^*$ by Lemma 7.3 again, so $|p - p^*| = 1$.

By symmetry it remains to consider the case of critical pairs (p, p^*) where the points lie on the respective straight segments of C and C^*. We show that if (8) holds then $\rho := |p - p^*| \geq 1$. In the notation of Lemma 7.4, put

$$p =: (x_1, 0, z_1), \quad p^* =: (0, x_2, z_2).$$

By (10), the sine of the angle made by the respective segments with the x– and y–axes is $u := \tau/2$. Then by Lemma 7.4,

$$x_1^2 = x_2^2 = \frac{\rho^2 u^2}{1+u^2} = \frac{\rho^2 (\tau/2)^2}{1+(\tau/2)^2} = \frac{\rho^2 \tau^2}{4+\tau^2}. \tag{14}$$

In particular p and p^* correspond to one another under the symmetry of the clasp, and the midpoint of the segment pp^* lies on the horizontal plane equidistant from the two tips p_0, p_0^*. Therefore the difference in heights between p and p_0^* is equal to the difference in heights between p_0 and p^*, that is,

$$z_1 + z_2 = z_0 + 0. \tag{15}$$

On the other hand, by Lemma 7.4 the difference in the heights of p, p^* is

$$\Delta z := z_2 - z_1 = \frac{x_1}{u}\sqrt{1-u^2} = \frac{x_1}{\tau}\sqrt{4-\tau^2}. \tag{16}$$

Substituting (13) and solving the system (15), (16) we obtain

$$x_1 = \frac{\tau}{\tau^2+4}[2+\sigma(2-\sqrt{4-\tau^2})] \tag{17}$$

and from (14)

$$\rho = \frac{2+\sigma(2-\sqrt{4-\tau^2})}{\sqrt{\tau^2+4}}. \tag{18}$$

The thickness condition is violated if and only if both $\rho < 1$ and the point p lies on the segment of C (rather than somewhere on the rest of the line it determines). The latter condition is equivalent to the condition that x_1 lie between the x coordinates of s_1 and s_2, that is,

$$\frac{\tau\sigma}{2} < x_1 < \frac{\tau}{2}$$

in view of (9), (12), or by (17), (14),

$$\frac{\sigma}{2} < \frac{\rho}{\sqrt{\tau^2+4}} < \frac{1}{2}. \tag{19}$$

The second inequality of (19) is a clear consequence of $\rho < 1$, which may in turn be expressed as

$$\sigma < \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}. \tag{20}$$

Substituting (18), the first inequality of (19) is equivalent to

$$\sigma < \frac{4}{\tau^2+2\sqrt{4-\tau^2}}. \tag{21}$$

We claim that the right-hand side of (21) dominates that of (20) in the relevant range $0 \le \tau \le 2$. Putting $t := \tau^2/4$ this is equivalent to the inequality

$$t+\sqrt{1-t} \le \frac{1-\sqrt{1-t}}{\sqrt{1+t}-1} = \frac{(\sqrt{1+t}-\sqrt{1-t})+(1-\sqrt{1-t^2})}{t}, \quad 0 \le t \le 1. \tag{22}$$

To prove (22), we note

$$\frac{t}{2} \le 1-\sqrt{1-t}, \quad 0 \le t \le 1, \tag{23}$$

so the left-hand side of (22) is dominated by $1+t/2$. On the other hand (23) also yields immediately

$$\frac{t^2}{2} \le 1-\sqrt{1-t^2}, \quad t \le \sqrt{1+t}-\sqrt{1-t},$$

for $0 \le t \le 1$, so $1+t/2$ is dominated by the right-hand side of (22) in turn.

Thus (20) is the effective condition. But this is precisely the negation of (8) (assuming we are not in the fully kinked case). So we have now shown that if (τ,σ) obey our conditions then $\mathrm{Thi}_\sigma(\widetilde{C}_T) \ge 1$.

Finally we show that the curve is (strongly) σ–critical with the given endpoint constraints by showing it is regularly balanced.

There is a one-parameter family of struts joining each point on the shoulder arcs to the opposite tip. By Lemma 7.2, the strut measure ds on these struts balances the shoulders. Further, this measure generates a strut force measure of magnitude τ at the tip. By Lemma 4.18, this is balanced by a ϕ function on the kink if and only if the angle of the kink is $\arcsin(\tau/2)$. But this is true by construction. As before, $\widetilde{C}_T$ is normal to the constraint planes at the endpoints of the arc, so the endpoint conditions of Theorem 4.13 are satisfied as well.

This completes the proof of Proposition 7.6. □

7.4 The Gehring regime

We have now described the clasp structures in very stiff rope with

$$\sigma > \frac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}}.$$

These are characterized by kinked circular arcs in balance with shoulder arcs. We now jump to the opposite end of the spectrum and describe clasps in very flexible rope with

$\sigma < \sqrt{1-\tau^2}$. The generic clasp described in Section 7.5 will combine features from both of these situations.

In [4], we described critical τ–clasps for the Gehring problem. We check below that the maximum curvature of those Gehring τ–clasps is $\sqrt{1-\tau^2}$ (at their tips). This is all that is needed to strengthen [4, Theorem 9.5] to yield the following result.

Theorem 7.7 *Suppose $\sigma \le \sqrt{1-\tau^2}$. Consider the curve C_1 in the xz–plane given parametrically for $u \in [-\tau, \tau]$ by*

$$x = x_\tau(u) := \frac{u\sqrt{1-(\tau-|u|)^2}}{\sqrt{1-u^2(\tau-|u|)^2}}, \tag{24}$$

$$z = z_\tau(u) := \int \frac{dz}{dx}\,dx = \int \frac{u}{\sqrt{1-u^2}}\frac{du}{\kappa_\tau(u)},$$

where

$$\kappa_\tau(u) := \frac{\sqrt{(1-u^2(\tau-|u|)^2)^3(1-(\tau-|u|)^2)}}{1-(\tau-|u|)^2+(\tau-|u|)|u|(1-u^2)} \tag{25}$$

and the constant of integration for z is chosen so that

$$z(0)+z(\tau) = -\sqrt{1-\tau^2}.$$

There is a curve C_2 in the yz–plane, congruent to C_1 and lying at distance exactly 1 from C_1, such that $\widetilde{C}_{Ge} := C_1 \cup C_2$ is $2*2$ symmetric, with $\mathrm{Thi}_\sigma(\widetilde{C}_{Ge}) = 1$, and is critical for the (τ,σ)–clasp problem.

Remark As described in [4], the parameter u equals the sine of the angle between the tangent to C_1 and the x–axis. The function κ_τ is the curvature. Each point $(x(u), 0, z(u)) \in C_1$ is connected by two struts of length 1 to symmetrically located points $(0, \pm x(u^*), -z(u^*)) \in C_2$, where $u+u^* = \tau$. These struts bear a strut measure which balances the curvature measure on each arc of the curve.

Following [4], the parameters u, u^* as above are said to be *conjugate*. Likewise, a subarc $A \subset C_1$ corresponding to $c \le u \le d$ is said to be conjugate to the subarcs of C_2 corresponding to $\tau-d \le u^* \le \tau-c$. In other words the conjugate arcs to A are precisely the subarcs of C_2 that are joined to A by struts.

Proof The only thing to check is that the curvature function $\kappa_\tau(u) \le 1/\sigma$ when $u \in [0,\tau]$. To prove it, it will be convenient to define $\alpha, \beta, \gamma \in [0, \pi/2]$ by

$$\sin\alpha = u, \quad \sin\beta = u^* = \tau - \sin\alpha, \quad \sin\gamma = \sin\alpha\sin\beta.$$

Then by (25)

$$\kappa_\tau(u) = \kappa_\tau(\sin\alpha) = \frac{\cos\beta\cos^3\gamma}{\cos^2\beta + \sin\gamma\cos^2\alpha} \le \frac{\cos^3\gamma}{\cos\beta} \le \frac{\cos\gamma}{\cos\beta}. \tag{26}$$

Furthermore

$$\frac{1}{\sigma} \ge \frac{1}{\sqrt{1-\tau^2}} \ge \frac{1}{\sqrt{1-\sin^2\beta}} = \frac{1}{\cos\beta}$$

since $\tau \ge \sin\beta$. Therefore

$$\frac{1}{\sigma} \ge \frac{1}{\cos\beta} \ge \frac{\cos\gamma}{\cos\beta} \ge \kappa_\tau(u),$$

as desired. □

7.5 The generic regime

We now describe the most complicated clasps. As the stiffness of the curve decreases from the transitional regime, the transitional clasp develops a self-contact in the middle of the straight segment. This contact causes the straight segment to split into two straight segments, with an arc of the Gehring clasp of Theorem 7.7 between them. The kink and shoulder arcs remain, though they become smaller (they will eventually vanish) as the stiffness continues to decrease. These clasps are pictured in Figure 7.

Theorem 7.8 *Suppose* $\dfrac{\sqrt{4+\tau^2}-2}{2-\sqrt{4-\tau^2}} > \sigma > \sqrt{1-\tau^2}$.

(1) *There exists a unique solution* $(\alpha,\beta,\gamma,a,b)$ *to the system of equations*

$$\sin\alpha + \sin\beta = \tau, \tag{27a}$$

$$\sin\gamma = \sin\alpha\sin\beta, \tag{27b}$$

$$\frac{b}{\sin\beta} = a\sin\alpha + \sigma(1-\cos\alpha), \tag{27c}$$

$$b\cos\beta = \sin\beta - \frac{\cos\alpha\sin\beta}{\cos\gamma}, \tag{27d}$$

$$a\cos\alpha = \frac{\sin\alpha\cos\beta}{\cos\gamma} - \sigma\sin\alpha, \tag{27e}$$

with $\alpha,\beta,\gamma \in [0,\pi/2]$, $\sin\alpha \le \tau/2$, *and* $a,b > 0$.

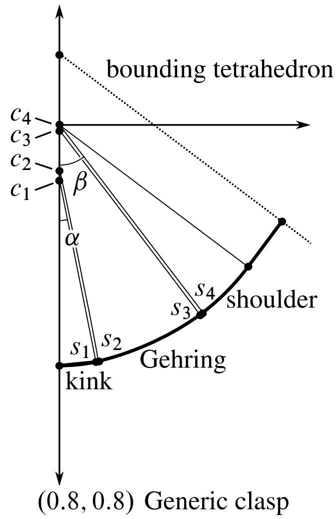

Figure 7: This diagram shows the construction of the generic clasp of Theorem 7.8 with $(\tau, \sigma) = (0.8, 0.8)$. The top (closely dotted) line is the intersection of a face of the bounding tetrahedron with the xz–plane. The generic clasp consists of a kinked circular arc of radius σ, a straight segment, an arc of the Gehring clasp, another straight segment, and a "shoulder" circular arc of radius 1. The length of the straight segments is exaggerated on this picture; their true length is close to the width of the lines used to draw the radii. The tip of the other component is located at the center c_4 of the shoulder; the remaining c_i are used in the proof below.

(2) *Given this solution, there is a C^1 curve C_Γ in the right half-plane of the xz–plane as shown in Figure 7, consisting of the following pieces joined in succession:*

- *A kinked circular arc of angle α, meeting the z–axis orthogonally*
- *A straight segment of length a,*
- *The arc $\sin\alpha \le u \le \sin\beta$ arc of the Gehring clasp of Theorem 7.7*
- *A straight segment of length b*
- *A "shoulder" circular arc of radius 1 from angle β to angle $\arcsin\tau$*

Furthermore, if we denote by $\widetilde{C}_\Gamma$ the corresponding $(2 * 2)$–symmetric curve, the tip of whose second component lies at the center of the shoulder arc of the first, then the Gehring arcs of the two components of $\widetilde{C}_\Gamma$ are conjugate.

(3) *We have $\mathrm{Thi}_\sigma(\widetilde{C}_\Gamma) = 1$.*

(4) *The curve $\widetilde{C}_\Gamma$ is critical for the (τ, σ)–clasp problem.*

Proof (1) Let us change our point of view by taking τ as given, and viewing (27) as a 1–parameter family of systems in the unknowns $\sigma, \beta, \gamma, a, b$ as the parameter α varies from 0 to $\arcsin \tau/2$. It is clear that (27a), (27b), (27d) determine β, γ, b uniquely, with $b > 0$ since

$$\cos\gamma = \sqrt{1-\sin^2\gamma} = \sqrt{1-\sin^2\alpha\sin^2\beta} > \sqrt{1-\sin^2\alpha} = \cos\alpha. \tag{28}$$

Solving (27c), (27e) for a, σ and substituting the value for b arising from (27d), we obtain

$$\begin{aligned} \sigma &= \frac{\sin^2\alpha\cos^2\beta+\cos^2\alpha-\cos\alpha\cos\gamma}{(1-\cos\alpha)\cos\beta\cos\gamma} \qquad (29)\\ &= \frac{\cos\gamma-\cos\alpha}{(1-\cos\alpha)\cos\beta} = \frac{(1+\cos\alpha)\cos\beta}{\cos\gamma+\cos\alpha}, \\ a &= \tan\alpha\cos\beta\Big(\frac{1}{\cos\gamma} - \frac{1+\cos\alpha}{\cos\gamma+\cos\alpha}\Big) \qquad (30)\\ &= \tan\alpha\cos\beta\frac{\cos\alpha(1-\cos\gamma)}{\cos\gamma(\cos\gamma+\cos\alpha)} > 0. \end{aligned}$$

Thus we may show that (27) is uniquely solvable in the original sense, with σ given and α unknown, by establishing that (29) expresses σ as a continuous strictly increasing function of α, with $\sigma(\arcsin(\tau/2)) = (\sqrt{4+\tau^2}-2)/(2-\sqrt{4-\tau^2})$ and $\sigma(0) = \sqrt{1-\tau^2}$. The latter relations may be verified directly, and continuity of σ is trivial. To prove that σ is strictly increasing, since $\sin\alpha$ and $\sin\gamma = \sin\alpha(\tau-\sin\alpha)$ are both increasing in the range $0 \le \sin\alpha \le \tau/2$, it is clear that both $\cos\alpha$ and $\cos\gamma$ are decreasing functions of α. Thus it remains only to show that the numerator $(1+\cos\alpha)\cos\beta$ of (29) is increasing as a function of $u := \sin\alpha \in [0, \tau/2]$. Since

$$\frac{d}{du}\cos\alpha = -\tan\alpha, \quad \frac{d}{du}\sin\beta = -1, \quad \frac{d}{du}\cos\beta = \tan\beta,$$

we compute

$$\frac{d}{du}(1+\cos\alpha)\cos\beta = -\tan\alpha\cos\beta + (1+\cos\alpha)\tan\beta > \tan\beta - \tan\alpha.$$

But $\sin\alpha + \sin\beta = \tau$ and $\sin\alpha < \tau/2$, so

$$\sin\beta > \sin\alpha \Longrightarrow \beta > \alpha \Longrightarrow \tan\beta > \tan\alpha.$$

(2) Letting $x(u) = x_\tau(u)$ denote the parametrization of the Gehring arc given in (24), the x–coordinates of the two endpoints of this arc are

$$x(\sin\alpha) = \frac{\sin\alpha\cos\beta}{\cos\gamma}, \quad x(\sin\beta) = \frac{\cos\alpha\sin\beta}{\cos\gamma},$$

by (27a) and (24). On the other hand the x–coordinates of the inner endpoints of the kink and the shoulder arcs are given by $\sigma \sin\alpha, \sin\beta$ respectively. Since by part (1)

$$a\cos\alpha = x(\sin\alpha) - \sigma\sin\alpha = \frac{\sin\alpha\cos\beta}{\cos\gamma} - \sigma\sin\alpha > 0,$$
$$b\cos\beta = \sin\beta - x(\sin\beta) = \sin\beta - \frac{\cos\alpha\sin\beta}{\cos\gamma} > 0,$$

we may interpolate straight segments of lengths a, b between the kink and the Gehring arc, and between the Gehring arc and the shoulder, respectively, to obtain a C^1 curve C_Γ as described.

Next we show that the Gehring arcs of the two components of $\widetilde{C}_\Gamma$ are conjugate to each other provided the components are situated with the tip of one at the center of the shoulder of the other. Referring to Figure 7, this is to say that the point c_3 is the projection to the xz–plane of the point s_2^* of the other component that corresponds to s_2. If the center of the shoulder arc (which is the tip of the other component) is the origin then the z–coordinate of c_3 is clearly $b/\sin\beta$. On the other hand, since the two components are congruent the z–coordinate of s_2^* equals the difference in the z–coordinates of s_2 and the tip of C_Γ. Equating these two,

$$\frac{b}{\sin\beta} = a\sin\alpha + \sigma(1-\cos\alpha),$$

which is (27c).

(3) We show first that the curvature of C_Γ is no more than $1/\sigma$. The kink, shoulder, and straight segments clearly obey this bound, so we need only check the Gehring clasp arc. We parametrize this arc by $u \in [\sin\alpha, \sin\beta]$ as in Theorem 7.7. Viewing $\sigma = \sigma(\alpha)$ as in (29) above, we must check that

$$\kappa_\tau(u) \le 1/\sigma(\alpha) \tag{31}$$

on this interval. We carry this out for the two subintervals $[\sin\alpha, \tau/2]$ and $[\tau/2, \sin\beta]$ separately.

Since $\sigma(\alpha)$ is strictly increasing in α for $\sin\alpha \in [0, \tau/2]$, for u in this range we have $1/\sigma(u) \le 1/\sigma(\alpha)$ and it suffices to show $\kappa_\tau(u) \le 1/\sigma(u)$. Define α' by $\sin\alpha' = u$, and β', γ' analogously to (27a) and (27b). Then

$$\kappa_\tau(u) = \kappa_\tau(\sin\alpha') = \frac{\cos\beta'\cos^3\gamma'}{\cos^2\beta' + \sin\gamma'\cos^2\alpha'} \le \frac{\cos\beta'\cos^3\gamma'}{\cos^2\beta'} \le \frac{\cos\gamma'}{\cos\beta'}.$$

On the other hand, by (29)

$$\frac{1}{\sigma(u)} = \frac{\cos\gamma' + \cos\alpha'}{(1+\cos\alpha')\cos\beta'}$$

and (31) follows easily.

To cover the range $u \in [\tau/2, \sin\beta]$ it suffices to prove that $\kappa_\tau(u^*) \le 1/\sigma(u)$ for $u \in [\sin\alpha, \tau/2]$, where $u + u^* = \tau$ (that is, u, u^* are conjugate). Since replacing u by u^* exchanges the variables α' and β' and leaves γ' unchanged,

$$\kappa_\tau(u^*) = \frac{\cos\alpha'\cos^3\gamma'}{\cos^2\alpha' + \sin\gamma'\cos^2\beta'} \le \frac{\cos^3\gamma'}{\cos\alpha'} \le \frac{\cos\gamma'}{\cos\alpha'}.$$

On the other hand,

$$\frac{1}{\sigma(u)} = \frac{\cos\gamma' + \cos\alpha'}{(1+\cos\alpha')\cos\beta'} \ge \frac{\cos\gamma' + \cos\gamma'\cos\alpha'}{(1+\cos\alpha')\cos\beta'} = \frac{\cos\gamma'}{\cos\beta'}.$$

Now (31) follows from the fact that $\sin\alpha' \le \tau/2 \le \sin\beta'$.

Next we claim that all critical pairs (p, p^*) of the distance between the components of $\widetilde{C}_\Gamma$ satisfy $|p - p^*| \ge 1$. To simplify the discussion we will put C_Γ^* for the part of the second component lying in the $y \ge 0$ part of the yz–plane, and consider only those pairs with $p \in C_\Gamma$, $p^* \in C_\Gamma^*$.

The claim is clearly true if p lies on the Gehring arc, since in this case p^* is the conjugate point of the Gehring arc of C_Γ^*.

Note that if (p, p^*) is a critical pair then the projection of the segment pp^* to the xz–plane is a line segment perpendicular to C_Γ at p and with the other endpoint on the z–axis. Now if we denote by $z^*(p)$ the z–intercept of the normal line through C_Γ at p, then z^* is an increasing function of the x–coordinate of p. (This is obvious for the circular arcs and line segments, and true for the Gehring arc by construction.)

By Lemma 7.3, if p lies on the shoulder arc or the kink then p^* is the tip of C_Γ^*. In the shoulder case $|p - p^*| = 1$ by construction. To handle the kink case we note that every point of C_Γ lies at distance ≥ 1 from the tip of C_Γ^*: otherwise C_Γ crosses the circle of radius 1 about the origin in the xz–plane at some point p. Since the slope of C_Γ must be less than the slope of the circle at this point, it follows that $z^*(p) > z^*(s_4) = 0$. But $z^*(p) \le 0$ by monotonicity.

By monotonicity of z^* again, and symmetry, it remains only to consider the case where $p \in s_1 s_2$ and $p^* \in s_3^* s_4^*$. However, since the lines generated by these segments are skew, there is at most one such critical pair. This pair is $p = s_2$, $p^* = s_3^*$, that is, the common endpoints of the segments and the Gehring arcs.

(4) We will show $\widetilde{C}_\Gamma$ is regularly balanced.

There is a one-parameter family of struts joining each point on the shoulder arcs to the opposite tip. By Lemma 7.2, the strut measure ds on these struts balances the shoulders. Further, this measure generates a strut force measure of magnitude τ at the tip. By Lemma 4.18, this is balanced by a ϕ function on the kink if and only if the angle of the kink is $\arcsin(\tau/2)$. But this is true by (27a). The straight segments bear no strut force and have $T' = 0$, so they obey the balance equation as well. Further, the Gehring arcs obey the balance equation by construction.

As before, $\widetilde{C}_\Gamma$ is normal to the constraint planes at the endpoints of the arc, so the endpoint conditions of Theorem 4.13 are satisfied as well.

This completes the proof of Theorem 7.8. A picture of the clasp appears in Figure 8. □

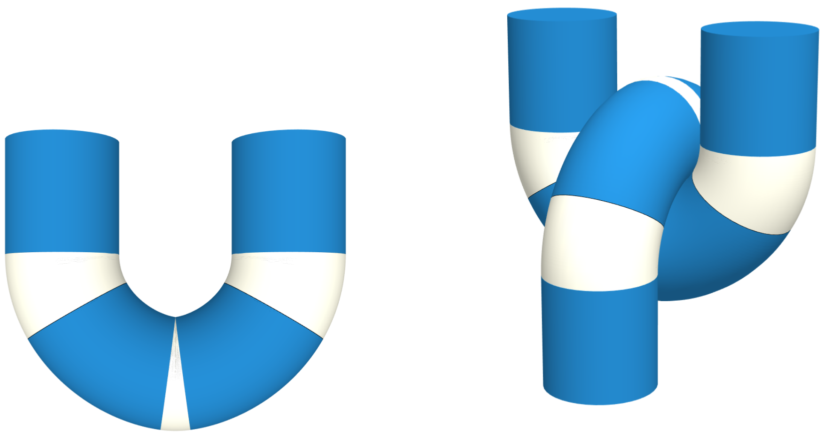

Figure 8: These figures show the $(1, \frac{1}{2})$ clasp. From left to right, the straight "tail," shoulder, Gehring, and kinked arcs of the clasp are shown in alternating blue and white colors. The two straight segments are included in black. The longer segment of length $b \sim 0.003878$ between the Gehring and shoulder sections is barely visible as a thin black border about one pixel wide. The much shorter segment of length $a \sim 0.000224$ between the kink and Gehring regions is too narrow to show up.

7.6 Geometry of the tight clasps

To compare the length of various clasps with the same τ but different σ, in a way independent of a particular bounding tetrahedron, we define the *excess length* $\ell(\tau,\sigma)$ of our (τ,σ) clasp to be the difference between the length of the clasp and four times the inradius of the bounding tetrahedron, which would be the infimal length in the absence of any thickness constraint. As σ increases, we are strengthening the curvature constraint, so the excess length must be monotonically increasing.

While the excess length of the kinked and transitional clasps can be computed exactly, the length of the Gehring clasp (and the generic clasp, which includes a Gehring arc) is only known as the solution of a certain hyperelliptic integral [4]. We constructed all of our clasps numerically, checking the thickness and curvature of each with `octrope` (see Ashton and Cantarella [1]), and computing the excess length by numerical integration. The results are shown in Figure 9, which shows how the excess length increases with σ for $\tau = 0.8$. For a kinked clasp we find $\ell(0.8, 1) \approx 2.109180872$, while for the Gehring clasp we get $\ell(0.8, \frac{1}{2}) \approx 2.103080861$; these differ by about 0.3%.

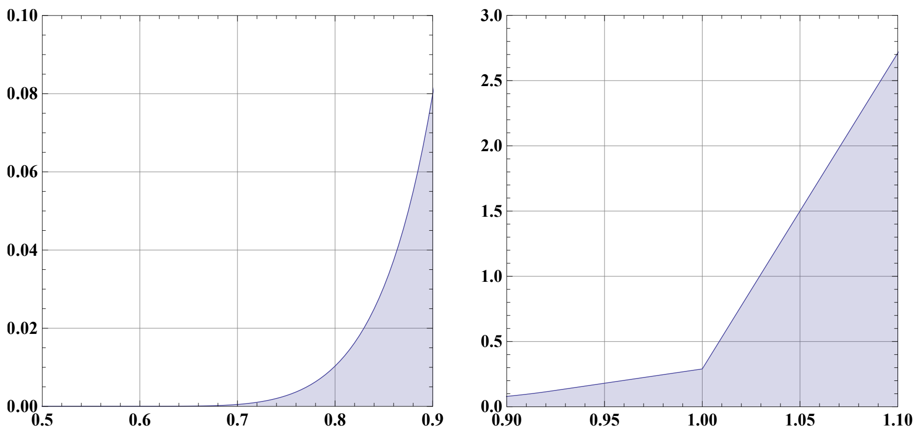

Figure 9: This pair of graphs shows how the excess length $\ell(0.8, \sigma)$ increases for $\sigma \in (0.5, 1.1)$. In the Gehring regime $0 \leq \sigma \leq 0.6$, the (τ, σ) clasp is of course just the Gehring τ–clasp, independent of σ, so $\ell(0.8, \sigma)$ stays constant at about 2.10308. The graphs plot $100(\ell(0.8, \sigma)/\ell(0.8, 0) - 1)$, that is the percentage increase of $\ell(0.8, \sigma)$ over the Gehring excess length. For example, at $\sigma = 1.05$, our (fully kinked) solution is a clasp with 1.5% more excess length than the Gehring clasp. We have changed the scale of the plot at $\sigma = 0.9$ in order to make the behavior for smaller σ easier to see. From the graphs, it seems the excess length function may be C^1 across the Gehring/generic boundary at $\sigma = 0.6$ and the generic/transitional boundary at $\sigma \approx 0.927$, but clearly has a corner at the transitional/kinked boundary at $\sigma = 1$.

For $\tau = 1$, the excess length of the kinked $\sigma = 1$ clasp is $\ell(1, 1) = 2\pi - 2 \approx 4.28318531$, while in the generic regime we have for instance $\ell(1, \frac{1}{2}) \approx 4.2630946$; these differ by about 0.46%. For the Gehring clasp we have $\ell(1, 0) \approx 4.262897$, which is about 0.5% less. We can see, from this example and from the graphs in Figure 9, that very little length is saved over the generic regime.

One of the most striking features of the Gehring clasp is a small gap between the two tubes, forming a small chamber between the two tubes as they are pulled together.

We have already seen that the same gap exists in the generic solutions, as we showed above that the tip-to-tip distance was greater than 1. In fact, the tip-to-tip distance is monotonic in σ for each value of τ, as we see in Figure 10. For smaller values of τ, the maximum tip-to-tip distance decreases as well, reaching 1 only for the trivial $\tau = 0$ clasp. The maximum tip-to-tip distance, about 1.05653, occurs at the Gehring $(1,0)$–clasp. The generic $(1, \frac{1}{2})$ clasp still has tip-to-tip distance about 1.05468.

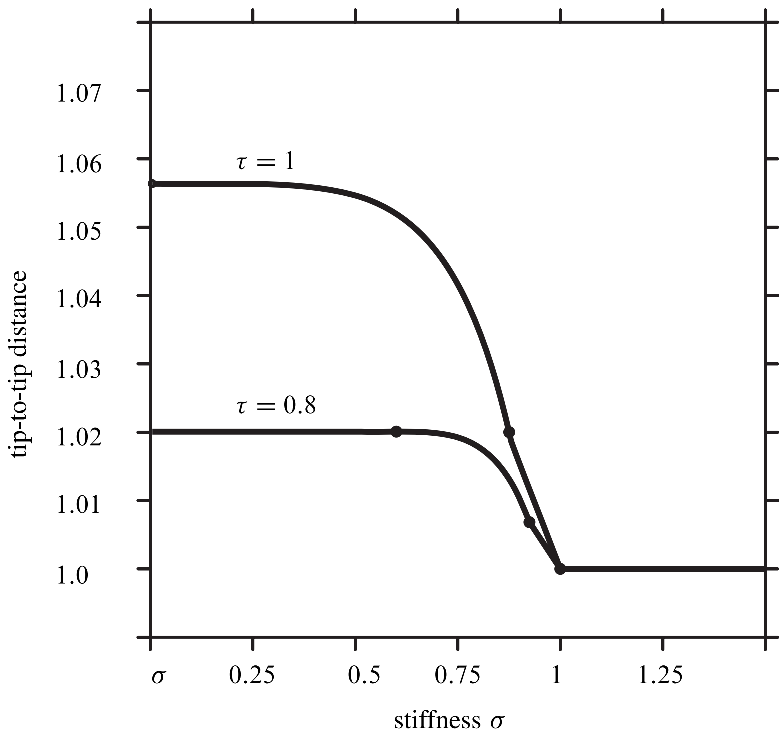

Figure 10: This graph shows the tip-to-tip distance for the clasps with $\tau = 1$ (upper curve) and $\tau = 0.8$ (lower curve). We can see that in all the kinked clasps ($\sigma \geq 1$) the tips are in contact, so the tip-to-tip distance is 1. As the stiffness decreases, the force exerted by the shoulder arcs pushes the tips apart, creating a gap between the tubes. We mark the transition between the kinked, transitional, generic, and Gehring regimes with small dots. For $\tau = 1$, recall that the Gehring regime degenerates to a point, so the corresponding dot appears at $\sigma = 0$. Also, we note that the kinked/transitional boundary occurs at $\sigma = 1$ for all τ so the curves merge at $\sigma = 1$. We can see that the gap size is constant over the Gehring regime (as the curves are not changing with σ) and then decreases monotonically as σ increases until the transition to the kinked regime, which has no gap for any σ or τ.

8 Future directions

A number of interesting questions regarding ropelength remain unanswered by our investigation. First, we note that although every link type has a ropelength minimizer, there are still very few explicit examples of closed links critical for ropelength: only the Borromean rings and the known minimizers from [5]. These have no kinks (so they are critical also for the Gehring problem) and all their components are planar. It would be very interesting to apply our balance criterion to describe further examples.

One way to generate further examples of critical links is to minimize ropelength with some symmetry imposed. The general principle of symmetric criticality suggests that the resulting configurations are still critical when the symmetries are relaxed. For ropelength, the superlinearity of the first variation of thickness (Corollary 3.6) is exactly the technical tool needed to show that symmetric criticality works as expected for ropelength problem, despite the lack of smoothness: the (symmetrized) average of thickening fields is again a thickening field, and thus a link that is critical under the imposition of symmetry remains critical without the symmetry constraint. This means that we now know many knots (including torus knots) with more than one critical configuration. Results of this kind appear in Cantarella, Ellis, Fu and Mastin [3]. It then becomes interesting to ask about second-order behavior, which in particular could determine which are local minima. Although there is a theory of second-order behavior for nonlinear constrained optimization problems in finite dimensions (see for instance Forsgren, Gill and Wright [13, Section 2]) it seems nontrivial to extend this to our infinite-dimensional setting.

It has long been conjectured that any knot — even the unknot — will have multiple local minima for the ropelength problem. Some such unknots have been computed numerically, but proving their existence remains an interesting open question. Promisingly, a solution to a closely related problem — finding distinct configurations of a given link which cannot be isotoped to one another without increasing the ropelength of one component — has recently been given by Coward and Hass [8].

The question of the regularity of ropelength minimizers or critical curves remains a central one in the field. Our regularity results depend on the assumption that kinks are regulated; it would be nice to show this is always the case. Our bootstrapping argument (Corollary 4.15) gives $W^{3,\mathrm{BV}}_{\mathrm{loc}}$ regularity on the kinks. Regularity results for nonkinked regions (and further regularity for kinks) would seem to depend on understanding the possible geometry of how struts can impinge on an arc.

Finally, we note that the supercoiled helices of Section 5 form an interesting family for further investigation. In particular, a comparison of our approach with Sussmann's would be fruitful; there may be borderline cases where solutions to his minimization problem fail to be Thi_σ–regular and thus might not be strongly critical. It would be nice to understand (2) well enough to prove our conjecture that the curves are embedded.

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