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The Cayley plane and string bordism

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This paper shows that, away from 6, the kernel of the Witten genus is precisely the ideal consisting of (bordism classes of) Cayley plane bundles with connected structure group, but only after restricting the Witten genus to string bordism. It does so by showing that the divisibility properties of Cayley plane bundle characteristic numbers arising in Borel–Hirzebruch Lie group-theoretic calculations correspond precisely to the divisibility properties arising in the Hovey–Ravenel–Wilson BP Hopf ring-theoretic calculation of string bordism at primes greater than 3.

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Introduction

This paper shows that an affinity between bordism rings and projective spaces extends further than previously known.

The first manifestation of the affinity is the fact that *every positive-dimensional element of the unoriented bordism ring* $\pi_*\mathrm{MO}$ *is represented by a real projective bundle.* In more detail, Thom [30] showed that $\pi_*\mathrm{MO}$ is a polynomial ring over $\mathbf{Z}/2$ with one generator in each dimension not of the form 2^k-1. Milnor [21] showed that a smooth degree–$(1,1)$ hypersurface $H \hookrightarrow \mathbf{RP}^i \times \mathbf{RP}^j$ can serve as generator if $1 < i < j$ and if $\binom{i+j}{i}$ is not divisible by 2 (equivalently, if there are no "carries" when adding i to j in base 2; see Milnor and Stasheff [22, Problem 16-F]. If $i \le j$, then the projection $H \to \mathbf{RP}^i$ is a fiber bundle with fiber $\mathbf{RP}^{j-1}$. In fact, Stong [29, Proposition 8.1] showed that every positive-dimensional element of $\pi_*\mathrm{MO}$ is represented by an $\mathbf{RP}^2$ bundle.

The second manifestation of the affinity is the fact that *every positive-dimensional element of the oriented bordism ring* $\pi_*\mathrm{MSO}$ *is represented by a complex projective bundle.* In more detail, $\pi_*\mathrm{MSO}/\mathrm{Torsion}$ is a polynomial ring over $\mathbf{Z}$ with one generator in each dimension $4k$. In each such dimension, a $\mathbf{Z}$–linear combination of smooth degree-$(1,1)$ hypersurfaces $H \hookrightarrow \mathbf{CP}^i \times \mathbf{CP}^j$ can serve as generator. If $i \le j$, then the projection $H \to \mathbf{CP}^i$ is a fiber bundle with fiber $\mathbf{CP}^{j-1}$. Wall [31] showed that these generators, together with certain of Dold's [10] generators for $\pi_*\mathrm{MO}$ (all of which

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are complex projective bundles), generate $\pi_*\mathrm{MSO}$. In fact, Führing [11] showed that every positive-dimensional element of $\pi_*\mathrm{MSO}$ is represented by a $\mathbf{CP}^2$ bundle. We shall return to this manifestation in more detail in the next section.

The third manifestation of the affinity is the fact that *almost every element of the spin bordism ring* $\pi_*\mathrm{MSpin}$ *is represented by a quaternionic projective bundle*; specifically, the set of quaternionic projective bundles with connected structure group is an ideal of $\pi_*\mathrm{MSpin}$ (indeed, for any space M, if $F \to E \to B$ is a fiber bundle, then so is $F \to E \times M \to B \times M$), and this ideal is precisely the kernel of the Atiyah invariant:

$$\alpha\colon \pi_*\mathrm{MSpin} \to \pi_*\mathrm{ko} \cong \mathbf{Z}[\eta, \omega, \mu]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\mu)$$

Here η, ω, μ have degree 1, 4, 8 respectively. In more detail, Anderson, Brown and Peterson [1] computed $\pi_*\mathrm{MSpin}_{(2)}$ and the forgetful homomorphism $\pi_*\mathrm{MSpin} \to \pi_*\mathrm{MSO}$ becomes an isomorphism after inverting 2. Stolz [26], together with Kreck [18], used this to show that every element of the kernel of the Atiyah invariant is represented by an $\mathbf{HP}^2$ bundle. ($\mathbf{HP}^2$ is 8–dimensional, so $\mathbf{HP}^2$ bundles cannot possibly represent every element of the spin bordism ring.) The Atiyah invariant is thus a complete obstruction for the representability of a spin bordism class by an $\mathbf{HP}^2$ bundle.

The fourth manifestation of the affinity is the subject of this paper: *almost every element of the string bordism ring* $\pi_*\mathrm{MO}\langle 8\rangle$ *is represented, at least up to powers of* 2 *and* 3, *by a Cayley plane, ie an octonionic projective plane (*$\mathbf{CaP}^2$*)—bundle.* (The Cayley plane is 16–dimensional so Cayley plane bundles cannot possibly represent every element of the string bordism ring.) Specifically, we prove that:

Theorem 1 *Away from* 6, *the ideal of* $\pi_*\mathrm{MO}\langle 8\rangle$ *consisting of (bordism classes of) Cayley plane bundles with connected structure group is precisely the kernel of the Witten genus. In other words, the extension of this ideal in* $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$ *is precisely the kernel of*

$$\phi_{\mathrm{W}} \otimes \mathbf{Z}\left[\tfrac{1}{6}\right]\colon \pi_*\mathrm{MO}\langle 8\rangle\left[\tfrac{1}{6}\right] \to \pi_*\mathrm{tmf}\left[\tfrac{1}{6}\right] \cong \mathbf{Z}\left[\tfrac{1}{6}\right][\mathbf{G}_4, \mathbf{G}_6],$$

where $\mathbf{G}_4, \mathbf{G}_6$ *have degree* 8, 12, *respectively.*

The Witten genus is thus a complete obstruction for the representability of a string bordism class by a $\mathbf{CaP}^2$ bundle, at least up to a powers of 2 and 3.

An interesting complication here is that Theorem 1 only appears to be true after restricting the Witten genus to string bordism. In other words, not every element of the kernel of the quasi-modular-form-valued Witten genus $\pi_*\mathrm{MSO}\left[\frac{1}{6}\right] \to \mathbf{Z}\left[\frac{1}{6}\right][\mathbf{G}_2, \mathbf{G}_4, \mathbf{G}_6]$ appears to be represented by a $\mathbf{CaP}^2$ bundle. Far from it, in fact: the subring of $\pi_*\mathrm{MSO}\left[\frac{1}{6}\right]$ generated by total spaces of oriented $\mathbf{CaP}^2$ bundles (and string manifolds of dimension less than 16) appears to coincide with the image of the forgetful homomorphism $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right] \to \pi_*\mathrm{MSO}\left[\frac{1}{6}\right]$. As we shall see, this homomorphism is the inclusion of an intricate, non-polynomial subring.

That Cayley plane bundles lie in the kernel of the Witten genus is already known:

Theorem *If* $\mathbf{CaP}^2 \to E \to W$ *is a Cayley plane bundle with connected structure group, then the Witten genus of* E *vanishes.*

This result was often proved in the 1990s — by Jung, Kreck–Singhof–Stolz, Dessai, Höhn — but rarely published. Rainer Jung's proof, which has yet to appear in print, used the work of Borel and Hirzebruch summarized below to show that the vanishing of the Witten genus on Cayley plane bundles is equivalent to the Jacobi triple identity for the Weierstrass sigma function. A little later Anand Dessai proved, using results of Kefeng Liu [19], that if S^3 acts nontrivially on a string manifold E, then the Witten genus of E vanishes. (This generalizes the theorem above since S^3 acts nontrivially on the total space of any Cayley plane bundle.) Dessai's work appeared in the preprint [7], in his PhD thesis [8], and in the conference proceedings [9]. Around the same time Gerald Höhn proved, again using results of Liu, that the Witten genus of any string homogeneous manifold vanishes. These results helped inspire Stephan Stolz's conjecture (see [27, Theorem 3.1]) that the Witten genus of a closed $4k$–dimensional string manifold vanishes if and only if it admits a Riemannian metric of positive Ricci curvature. (The author thanks Dessai for informing him of the history of these results.)

In fact, Jung and Dessai both proved the rational version of Theorem 1:

Theorem *Rationally, the ideal of* $\pi_*\mathrm{MO}\langle 8\rangle$ *consisting of (bordism classes of) Cayley plane bundles with connected structure group is precisely the kernel of the Witten genus. In other words, the extension of this ideal in* $\pi_*\mathrm{MO}\langle 8\rangle \otimes \mathbf{Q}$ *is precisely the kernel of*

$$\phi_{\mathrm{W}} \otimes \mathbf{Q}\colon \pi_*\mathrm{MO}\langle 8\rangle \otimes \mathbf{Q} \to \pi_*\mathrm{tmf} \otimes \mathbf{Q} \cong \mathbf{Q}[\mathbf{G}_4, \mathbf{G}_6].$$

Since stable rational homotopy theory is trivial, rational results are unsatisfying to homotopy theorists. This paper does not tackle the primes 2 or 3, the primes at which tmf is most interesting. But the author has no reason to be pessimistic about those primes and hopes that homotopy theorists will be pleased to see geometry in alignment at the primes greater than 3. As far as the author knows, this paper gives the first geometrically explicit list of generators for $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$.

Note that $\mathrm{tmf}\left[\frac{1}{6}\right]$ is not a ring spectrum quotient of $\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$. In fact, for any prime $p>3$ and any sequence X in $\pi_*\mathrm{MO}\langle 8\rangle$, the $\pi_*\mathrm{MO}\langle 8\rangle$–module

$$\pi_*\big(\mathrm{MO}\langle 8\rangle_{(p)}/X\big)$$

is not (even abstractly) isomorphic to $\pi_*\mathrm{tmf}_{(p)}$ (McTague [20]).

Throughout this paper, the italic letter p will denote a prime number. The roman letter p will denote the Pontrjagin class.

1 Pontrjagin numbers and oriented bordism

This section briefly reviews background material on Pontrjagin classes and the oriented bordism ring. This serves both to fix notation as well as to illustrate how the results of this paper extend well-known calculations.

The i^{th} Pontrjagin class of a real vector bundle V is by definition

$$\mathrm{p}_i(V)=(-1)^i\mathrm{c}_{2i}(V\otimes\mathbf{C}).$$

It pulls back from the universal i^{th} Pontrjagin class p_i in $\mathrm{H}^*(\mathrm{BO}(4n),\mathbf{Z})$ for $n\geq i$, which in turn may be identified with the i^{th} elementary symmetric polynomial. This is because the i^{th} Pontrjagin class of a sum of complex line bundles is the i^{th} elementary symmetric polynomial in the first Pontrjagin classes of the individual line bundles, $\mathrm{p}(L_1\oplus\cdots\oplus L_n)=\prod(1+\mathrm{p}_1(L_i))$. (The driving force behind this is the fact that, in ordinary cohomology, the total Chern class is exponential, $\mathrm{c}(V_1\oplus V_2)=\mathrm{c}(V_1)\cdot\mathrm{c}(V_2)$.)

It is a basic fact that the ring of symmetric polynomials is a polynomial ring on the elementary symmetric polynomials. There are other symmetric polynomials of geometric interest, though. Given a partition $I=i_1,\dots,i_r$ let s_I denote the polynomial $\sum\mathrm{p}_1(L_1)^{i_1}\cdots\mathrm{p}_1(L_r)^{i_r}$, where the sum runs over all distinct monomials obtained by permuting $L_1,\dots,L_n$. Each s_I is a symmetric polynomial, so may be written as a polynomial in the elementary symmetric polynomials. Thus we may associate to each s_I a polynomial in the Pontrjagin classes, which we also denote s_I. Note in particular that $\mathrm{s}_1,\mathrm{s}_{1,1},\mathrm{s}_{1,1,1},\dots$ are the Pontrjagin classes $\mathrm{p}_1,\mathrm{p}_2,\mathrm{p}_3,\dots$ themselves. The geometric significance of the classes s_I comes from the following lemma [22, Lemma 16.2].

Lemma 2 (Thom) *If $0\to V_1\to W\to V_2\to 0$ is an exact sequence of vector bundles, then*

$$\mathrm{s}_I(W)=\sum_{JK=I}\mathrm{s}_J(V_1)\,\mathrm{s}_K(V_2),$$

where the sum ranges over all partitions J and K with juxtaposition JK equal to I.

This implies that s_n of the tangent bundle of a nontrivial product of closed oriented manifolds vanishes. In fact, a closed oriented manifold M^{4n} is decomposable in $\pi_*\mathrm{MSO}\big[\frac{1}{2}\big]$ if and only if the number $\mathrm{s}_n[M^{4n}] = \int_M \mathrm{s}_n(\mathrm{T}M)$ equals zero. (The integral $\int_M$ here denotes the pushforward to a point $\mathrm{H}^{4n}(M) \to \mathrm{H}^0(\mathrm{pt}) \cong \mathbf{Z}$, or equivalently, the Kronecker pairing $\langle \mathrm{s}_n(\mathrm{T}M), [M]\rangle$ with the fundamental class $[M] \in \mathrm{H}_{4n}(M, \mathbf{Z})$.) Since $\pi_*\mathrm{MSO} \otimes \mathbf{Q}$ is a polynomial ring over $\mathbf{Q}$ with one generator in each dimension $4n \geq 4$, a sequence $\{M^{4n}\}_{n\geq 1}$ therefore generates $\pi_*\mathrm{MSO} \otimes \mathbf{Q}$ if and only if $\mathrm{s}_n[M^{4n}] \neq 0$ for each $n \geq 1$. As mentioned in the introduction, however, inverting just the prime 2 is enough to make $\pi_*\mathrm{MSO}$ a polynomial ring. It follows that the numbers s_n suffice to recognize a sequence of generators for $\pi_*\mathrm{MSO}\big[\frac{1}{2}\big]$ but it turns out that these numbers have unexpected divisibility properties.

For any integer n and any prime p, let $\mathrm{ord}_p(n)$ denote the p–adic order of n, that is, the largest integer ν such that p^ν divides n.

Theorem 3 (cf Stong [28, page 180]) *A sequence $\{M^{4n}\}_{n\geq 1}$ generates $\pi_*\mathrm{MSO}\big[\frac{1}{2}\big]$ if and only if for any integer $n > 0$ and any odd prime p,*

$$\mathrm{ord}_p\big(\mathrm{s}_n[M^{4n}]\big) = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ for some integer } i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, if p is odd then the Hurewicz homomorphism $\pi_*\mathrm{MSO}_{(p)} \to \mathrm{H}_*\mathrm{MSO}_{(p)}$, after passing to indecomposable quotients, is multiplication by $\pm p$ in degrees of the form $2(p^i - 1)$ and is an isomorphism otherwise. (See [24, Theorem 3.1.5] where the special behavior in degrees $2(p^i - 1)$ ultimately comes from the degrees of the generators v_i of $\pi_*\mathrm{BP}$.)

Now we return to the second manifestation of the affinity discussed in the introduction.

Proposition *If $H \hookrightarrow \mathbf{CP}^i \times \mathbf{CP}^{2n-i+1}$ is a smooth complex hypersurface of degree $(1,1)$ and $1 < i < 2n$, then*

$$\mathrm{s}_n[H] = -\binom{2n+1}{i}.$$

Proof Since the tangent bundle of the ambient manifold $\mathbf{CP}^i \times \mathbf{CP}^{2n-i+1}$ splits nontrivially, Lemma 2 implies that $\mathrm{s}_n(\mathrm{T}H) = -\,\mathrm{s}_n(\mathrm{N}H)$, where the normal bundle $\mathrm{N}H$ is isomorphic to the complex line bundle

$$\mathrm{O}(1,1)|_H = \big(\pi_1^*\mathrm{O}(1) \otimes \pi_2^*\mathrm{O}(1)\big)\big|_H$$

and π_1, π_2 are the projections of the ambient manifold. Since, for a complex line bundle $\mathrm{p}_1 = \mathrm{c}_1^2$, and since in ordinary cohomology $\mathrm{c}_1(L_1 \otimes L_2) = \mathrm{c}_1(L_1) + \mathrm{c}_1(L_2)$, it follows that

$$\mathrm{s}_n(\mathrm{O}(1,1)) = \mathrm{p}_1(\mathrm{O}(1,1))^n = \mathrm{c}_1(\mathrm{O}(1,1))^{2n} = (x_1+x_2)^{2n},$$

where $x_j = \pi_j^*\mathrm{c}_1(\mathrm{O}(1))$. Thus

$$\mathrm{s}_n[H] = -\int_H \mathrm{s}_n(\mathrm{O}(1,1)|_H) = -\int_H (x_1+x_2)^{2n}\Big|_H.$$

By Poincaré duality, then (see [22, Problem 16-D])

$$\mathrm{s}_n[H] = -\int_{\mathbf{CP}^i\times\mathbf{CP}^{2n-i+1}} (x_1+x_2)^{2n+1} = -\binom{2n+1}{i}. \qquad \square$$

Kummer's theorem, which states that $\mathrm{ord}_p\big[\binom{n}{i}\big]$ equals the number of "carries" when adding i to $n-i$ in base p (see Granville [13, Section 1]), can be used to show that:

Lemma *For any integer $n>0$ and any odd prime p,*

$$\mathrm{ord}_p\left[\operatorname*{GCD}_{1<i<2n}\binom{2n+1}{i}\right] = \begin{cases} 1 & \text{if } 2n+1 = p^i \text{ for some integer } i>0,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $\mathbf{Z}$–linear combinations of the hypersurfaces appearing in the proposition generate $\pi_*\mathrm{MSO}\big[\frac{1}{2}\big]$, as asserted in the introduction.

In short, then, the divisibility properties of s_n for oriented manifolds, deduced from homotopy theory, align perfectly with the divisibility properties of s_n for $\mathbf{CP}^n$ bundles, deduced from divisibility properties of binomial coefficients.

This paper will follow the same outline. First we will deduce the divisibility properties of s_n (and $\mathrm{s}_{n,n'}$) for string manifolds from known results in homotopy theory. Then we will show that these divisibility properties align perfectly with the divisibility properties of s_n (and $\mathrm{s}_{n,n'}$) for Cayley plane bundles, which we will in turn deduce from divisibility properties of binomial coefficients. The arguments and calculations will at each stage be more complicated than for oriented bordism and complex projective bundles, but the outline and spirit will be the same.

2 How to recognize generators for string bordism

In Section 1 we stated a criterion (Theorem 3), involving the number s_n, which ensures that a sequence $\{M^{4n}\}_{n\geq 1}$ generates $\pi_*\mathrm{MSO}\big[\frac{1}{2}\big]$. The purpose of this section is to establish an analogous criterion (Theorem 4) for the string bordism ring $\pi_*\mathrm{MO}\langle 8\rangle\big[\frac{1}{6}\big]$. It turns out that Pontrjagin numbers still suffice to distinguish elements of $\pi_*\mathrm{MO}\langle 8\rangle\big[\frac{1}{6}\big]$, but, since this ring is not a polynomial ring, the numbers s_n do not suffice to recognize generators; certain numbers of the form $\mathrm{s}_{n,n'}$ are also needed. As we shall see, the criterion is a consequence of Hovey's calculation [15] of $\pi_*\mathrm{MO}\langle 8\rangle_{(p)}$ for $p > 3$.

First recall what string bordism is. Any real vector bundle $V \to X$ of rank k pulls back from the universal rank-k bundle over the classifying space $\mathrm{BO}(k)$ by a map $f\colon X \to \mathrm{BO}(k)$.

- An *orientation* of V is a (homotopy class of) lift f_2 of f to the 1–connected cover $\mathrm{BSO} \to \mathrm{BO}$. Such a lift exists if and only if the generator w_1 of $\mathrm{H}^1(\mathrm{BO}, \mathbf{Z}/2)$ pulls back to 0 in $\mathrm{H}^1(X, \mathbf{Z}/2)$.
- A *spin structure* on V is a (homotopy class of) lift f_4 of f_2 to the 3–connected cover $\mathrm{BSpin} \to \mathrm{BSO}$. Such a lift exists if and only if the generator w_2 of $\mathrm{H}^2(\mathrm{BSO}, \mathbf{Z}/2)$ pulls back to 0 in $\mathrm{H}^2(X, \mathbf{Z}/2)$.
- A *string structure* on V is a (homotopy class of) lift f_8 of f_4 to the 7–connected cover $\mathrm{BO}\langle 8\rangle \to \mathrm{BSpin}$. Such a lift exists if and only if the generator $\frac{1}{2}\mathrm{p}_1$ of $\mathrm{H}^4(\mathrm{BSpin}, \mathbf{Z})$ pulls back to 0 in $\mathrm{H}^4(X, \mathbf{Z})$.

$$\begin{tikzcd}
 & \mathrm{BO}\langle 8\rangle \arrow[d] \\
 & \mathrm{BSpin} \arrow[d] \\
 & \mathrm{BSO} \arrow[d] \\
X \arrow[r, "f"] \arrow[ur, dashed, bend left, "f_2"] \arrow[uur, dashed, bend left, "f_4"] \arrow[uuur, dashed, bend left, "f_8"] & \mathrm{BO}
\end{tikzcd}$$

The bordism spectrum of string manifolds $\mathrm{MO}\langle 8\rangle$ is the Thom spectrum of the map $\mathrm{BO}\langle 8\rangle \to \mathrm{BO}$. Its coefficient ring $\pi_*\mathrm{MO}\langle 8\rangle$ is the bordism ring of manifolds equipped with a string structure on their stable normal bundle.

Theorem 4 *A set S generates $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$ if:*

(1) *For each integer $n > 1$, there is an element M^{4n} of S such that for any prime $p > 3$,*

$$\mathrm{ord}_p\big(\mathrm{s}_n[M^{4n}]\big) = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ or } 2n = p^i + p^j \text{ for some integers } 0 \le i \le j, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For each prime $p > 3$ and each pair of integers $0 < i < j$, there is an element $N^{2(p^i+p^j)}$ of S such that*

$$\mathrm{s}_{(p^i+p^j)/2}[N^{2(p^i+p^j)}] = 0,$$
$$\mathrm{s}_{(p^i+1)/2,(p^j-1)/2}[N^{2(p^i+p^j)}] \not\equiv 0 \mod p^2.$$

We prove this in stages.

Proposition *The forgetful homomorphism* $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]\to\pi_*\mathrm{MSpin}\left[\frac{1}{6}\right]$ *is injective.*

Proof It is injective tensor $\mathbf{Q}$, so its kernel is torsion (since $\mathbf{Q}$ is a flat $\mathbf{Z}$–module). Giambalvo, however, showed that $\pi_*\mathrm{MO}\langle 8\rangle$ has no p–torsion for $p>3$ [12, Theorem 4.3]. □

Since $\pi_*\mathrm{MSpin}\left[\frac{1}{2}\right]\cong\pi_*\mathrm{MSO}\left[\frac{1}{2}\right]$ and since Pontrjagin numbers detect equality in $\pi_*\mathrm{MSO}\left[\frac{1}{2}\right]$ we get the next two corollaries.

Corollary *Any two string structures for an oriented manifold determine the same element of* $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$.

Corollary *Pontrjagin numbers detect equality in* $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$.

To prove Theorem 4 it therefore suffices to determine the image of $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]\to\pi_*\mathrm{MSpin}\left[\frac{1}{6}\right]$ or, equivalently, to determine the image of $\pi_*\mathrm{MO}\langle 8\rangle_{(p)}\to\pi_*\mathrm{MSpin}_{(p)}$ for each prime $p>3$. The Hovey–Ravenel–Wilson approach [16; 25] to $\mathrm{BO}\langle 4k\rangle$ reduces $\pi_*\mathrm{MO}\langle 8\rangle_{(p)}\to\pi_*\mathrm{MSpin}_{(p)}$ to the homomorphism $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2(p+1)}\to\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4$, and Hovey's description [15] of these rings reveals enough information about the image to prove Theorem 4. What follows is a brief summary of the results of [15; 16; 25] needed to prove Theorem 4.

First some standard notation. Let BP denote the Brown–Peterson spectrum [6]; its coefficient ring is $\pi_*\mathrm{BP}\cong\mathbf{Z}_{(p)}[v_1,v_2,\dots]$, where $\deg(v_i)=2(p^i-1)$. Let $\mathrm{BP}\langle 1\rangle$ denote the Johnson–Wilson spectrum obtained from BP by killing the ideal $(v_2,v_3,\dots)$ of $\pi_*\mathrm{BP}$; its coefficient ring is $\pi_*\mathrm{BP}\langle 1\rangle\cong\mathbf{Z}_{(p)}[v_1]$ and its homotopy type is independent of the polynomial generators $v_2,v_3,\dots$ chosen [17]. The infinite loop space obtained by applying the k^{th} space functor to a spectrum X will be denoted $\boldsymbol{X}_k$.

Recall that the ring homomorphism $\pi_*\mathrm{BP}\langle 1\rangle\to\pi_*\mathrm{ku}_{(p)}$ taking v_1 to v^{p-1} lets one identify $\pi_*\mathrm{ku}_{(p)}\cong\mathbf{Z}_{(p)}[v]$ with $\pi_*\mathrm{BP}\langle 1\rangle[v]/(v_1-v^{p-1})$. This identification extends to a multiplicative splitting of spectra

$$\mathrm{ku}_{(p)}\cong\prod_{i=1}^{p-2}\Sigma^{2i}\mathrm{BP}\langle 1\rangle.$$

Multiplication by v on the left corresponds to the (upward) shift of factors on the right, the shift from top to bottom factor being accompanied by multiplication by v_1.

Since, for k even, $\mathrm{BU}\langle k\rangle$ can be taken as the k^{th} space of ku, this implies that there is a p–local decomposition of H–spaces

$$\mathrm{BU}\langle k\rangle_{(p)} \cong \prod_{i=1}^{p-2} \mathbf{BP}\langle\mathbf{1}\rangle_{k+2i}\,.$$

There is an analogous splitting of $\mathrm{BO}\langle k\rangle_{(p)}$ for $p>2$:

Theorem [16, Corollary 1.5] *If k is divisible by 4 and $p>2$, then there is a p–local decomposition of H–spaces*

$$\mathrm{BO}\langle k\rangle_{(p)} \cong \prod_{i=0}^{(p-3)/2} \mathbf{BP}\langle\mathbf{1}\rangle_{k+4i}.$$

Under this decomposition the map $\mathrm{BO}\langle k+4\rangle \to \mathrm{BO}\langle k\rangle$ corresponds to the identity map on the factors $\mathbf{BP}\langle\mathbf{1}\rangle_{k+4i}$ for $0<i<\frac{1}{2}(p-3)$ and to $[v_1]\colon \mathbf{BP}\langle\mathbf{1}\rangle_{k+2p-2} \to \mathbf{BP}\langle\mathbf{1}\rangle_k$ on the remaining factor.

If $k=4$, then the situation looks like this:

$$\begin{tikzcd}
\mathrm{BO}\langle 8\rangle_{(p)} \arrow[d] & \cong & \mathbf{BP}\langle\mathbf{1}\rangle_8 \arrow[drr] & \times & \mathbf{BP}\langle\mathbf{1}\rangle_{12} \arrow[drr] & \times & \cdots & & \cdots \arrow[drr] & \times & \mathbf{BP}\langle\mathbf{1}\rangle_{2p+2} \arrow[dllllllll, dashed] \\
\mathrm{BO}\langle 4\rangle_{(p)} & \cong & \mathbf{BP}\langle\mathbf{1}\rangle_4 & \times & \mathbf{BP}\langle\mathbf{1}\rangle_8 & \times & \mathbf{BP}\langle\mathbf{1}\rangle_{12} & \times & \cdots & \times & \mathbf{BP}\langle\mathbf{1}\rangle_{2p-2}
\end{tikzcd}$$

Hovey shows that $\pi_*\mathrm{MO}\langle 8\rangle_{(p)}$ is (abstractly) isomorphic as a ring to a quotient of the BP–homology of this decomposition, the ring structure of the latter coming from the infinite loop space structures of the factors. To state his result precisely, we need to introduce some notation. If $p>2$ then there is a natural map of ring spectra $\mathrm{MO}\langle 8\rangle \to \mathrm{MSO} \to \mathrm{BP}$. If $p>3$ then the induced homomorphism $\mathrm{BP}_*\mathrm{MO}\langle 8\rangle \to \mathrm{BP}_*\mathrm{BP}$ is surjective [15, Lemma 2.1]. For each positive integer i, choose a generator u_i in $\mathrm{BP}_{2(p^i-1)}\mathrm{MO}\langle 8\rangle$ mapping to the generator t_i of $\mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1,t_2,\dots]$. For dimensional reasons, each u_i must lie in the tensor factor $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p-2}$ of $\mathrm{BP}_*\mathrm{MO}\langle 8\rangle$.

Theorem [15, Theorem 2.4] *If $p>3$ then there are (abstract) isomorphisms of rings*

$$\begin{aligned}
\pi_*\mathrm{MO}\langle 8\rangle_{(p)} &\cong \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_8 \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{12} \otimes_{\mathrm{BP}_*} \cdots \\
&\qquad \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p-2}/(u_1,u_2,\dots) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2}, \\
\pi_*\mathrm{MSpin}_{(p)} &\cong \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p-2}/(u_1,u_2,\dots).
\end{aligned}$$

So to understand the forgetful homomorphism $\pi_*\mathrm{MO}\langle 8\rangle_{(p)} \to \pi_*\mathrm{MSpin}_{(p)}$ it suffices to understand the ring homomorphism induced by the dotted arrow above, ie

$$[v_1]_*\colon \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2} \to \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4 .$$

As we shall see, it is the inclusion of a non-polynomial subring into a polynomial ring. A toy model worth bearing in mind is the inclusion $\mathbf{Z}[5x, y, xy] \hookrightarrow \mathbf{Z}[x, y]$.

Instead of studying each ring $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_n$ individually, Hovey exploits the fact that they fit together to form a Hopf ring $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_*$. In particular there is a circle product

$$\circ\colon \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_m \otimes \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_n \to \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{m+n}$$

corresponding to the ring spectrum structure of $\mathrm{BP}\langle 1\rangle$. It gives an inductive way to construct elements in the increasingly complicated rings $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{m+n}$. In fact, all the elements we will need can be constructed that way from just two kinds of elements, $b_{(i)}$ and $[v_1^i]$, defined as follows. The complex orientation gives a map $\mathbf{CP}^\infty \to \mathbf{BP}\langle\mathbf{1}\rangle_2$. Let $b_i \in \mathrm{BP}_{2i}\mathbf{BP}\langle\mathbf{1}\rangle_2$ be the image under this map of the BP–homology generator of degree $2i$. Let $b_{(i)}$ denote the generator b_{p^i} (generators not of this form are decomposable). The homotopy class v_1^i is represented by a map $\mathrm{S}^0 \to \mathbf{BP}\langle\mathbf{1}\rangle_{-2i(p-1)}$. Let $[v_1^i] \in \mathrm{BP}_0\mathbf{BP}\langle\mathbf{1}\rangle_{-2i(p-1)}$ denote the image under this map of the BP–homology generator.

Wilson [32, Corollary 5.1] showed that, for $n < 2p+2$, the p–local homology of $\mathbf{BP}\langle\mathbf{1}\rangle_n$ is an evenly graded torsion-free polynomial algebra with one generator in each dimension corresponding to $s^n v_1^k$ for $k \geq 0$. The Atiyah–Hirzebruch spectral sequence therefore collapses and the BP homology of $\mathbf{BP}\langle\mathbf{1}\rangle_n$ has the same properties. In fact:

Theorem [15, Theorem 1.2] *If $n < 2p$, then $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_n$ is a polynomial algebra over BP_* with one generator in each positive even degree congruent to n mod $2p-2$. In a degree $2m$ of that form, one can take*

$$x_{2m} = [v_1^i] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ b_{(k)}^{\circ j_k}$$

as a generator, where $m = \sum j_l p^l$ is the p–adic expansion and $i = \frac{1}{p-1}(\alpha(m) - \frac{1}{2}n)$ with $\alpha(m) = \sum_l j_l$.

If $n = 2p+2$, then $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_n$ is *not* a polynomial ring over BP_*. It has a generator in each degree congruent to 4 mod $2p-2$ (and greater than 4) but it has two generators in some of these dimensions, and these generators satisfy a relation. Specifically:

- In each degree $4p^i$ for $i > 0$ there is one generator

$$w_{4p^i} = b_{(i)} \circ b_{(i-1)}^{\circ p}.$$

- In each degree $2(p^i+p^j)$ for $0\le i<j$ there is a generator
$$y_{2(p^i+p^j)} = b_{(i)} \circ b_{(j-1)}^{\circ p}.$$
- In each degree $2(p^i+p^j)$ for $0<i<j$ there is a second generator
$$z_{2(p^i+p^j)} = b_{(i-1)}^{\circ p} \circ b_{(j)}.$$
To simplify formulas later on, let $z_{2(1+p^j)} = 0$ for $j>0$.
- In each of the other degrees — that is, in each degree $2m$ congruent to 4 mod $2p-2$, but not of the form $2(p^i+p^j)$ for any $0\le i\le j$ — there is a single generator of the form x_{2m}, defined as in the preceding theorem.

Hovey constructs, for each $0<i<j$, a relation r_{ij} involving $y_{2(p^i+p^j)}$, $z_{2(p^i+p^j)}$ and p. To express it, let I be the ideal of BP_* generated by $(p, v_1, v_2, \dots)$, and let $\mathrm{I}(n)$ be the kernel of $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_n \to \mathrm{BP}_*$.

Proposition [15, Corollary 1.6] *For any pair of integers $0<i<j$ there is a relation in $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2}$ of the form*

$$p(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) \equiv v_j\, y_{2(1+p^i)} - v_i \cdot y_{2(1+p^j)} + y^p_{2(p^{i-1}+p^{j-1})} - z^p_{2(p^{i-1}+p^{j-1})} \mod \mathrm{I}^2\cdot \mathrm{I}(2p+2) + \mathrm{I}\cdot\mathrm{I}(2p+2)^{*2} + \mathrm{I}(2p+2)^{*p+1}.$$

Considering each of these relations as an element r_{ij} of the BP_*–polynomial ring R on all the generators w_{4p^i}, $y_{2(1+p^i)}$, $y_{2(p^i+p^j)}$, $z_{2(p^i+p^j)}$, x_{2m} for $0<i<j$ and $2m$ of the form described above, Hovey shows that:

Theorem [15, Theorem 1.7] $\mathrm{R}/(r_{ij} \mid 0<i<j) \to \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2}$ *is an isomorphism of* BP_**–algebras.*

Remember that we want to understand the homomorphism:

$$[v_1]_*\colon \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2} \to \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4$$

If $0<i<j$ then by definition:

$$\begin{aligned}
[v_1]_* w_{4p^i} &= [v_1]\circ b_{(i)} \circ b_{(i-1)}^{\circ p}\\
[v_1]_* y_{2(1+p^i)} &= [v_1]\circ b_{(0)} \circ b_{(i-1)}^{\circ p}\\
[v_1]_* y_{2(p^i+p^j)} &= [v_1]\circ b_{(i)} \circ b_{(j-1)}^{\circ p}
\end{aligned}$$

$$[v_1]_* z_{2(p^i+p^j)} = [v_1] \circ b_{(i-1)}^{\circ p} \circ b_{(j)}$$

$$[v_1]_* x_{2m} = \underbrace{[v_1] \circ [v_1^i]}_{=[v_1^{i+1}]} \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots \circ b_{(k)}^{\circ j_k}$$

Recall that the exponent i of v_1 appearing in the generator x_{2m} depends on both m and n, specifically $i = i(m,n) = \frac{1}{(p-1)}(\alpha(m) - \frac{1}{2}n)$. So $i(m,4) = i(m,2p+2)+1$ and the homomorphism carries each generator of $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p+2}$ of the form x_{2m} to the corresponding generator x_{2m} of $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4$. To relate the images of the other generators to the generators x_{2m} of $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4$, we rely on the following proposition.

Proposition [15, Corollary 1.5] *For each integer $i > 0$, there is a relation in $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_2$ of the form*

$$[v_1] \circ b_{(i-1)}^{\circ p} \equiv v_i \cdot b_{(0)} - p \cdot b_{(i)} - b_{(i-1)}^{*p} \quad \mathrm{mod}\ \mathrm{I}^2 \cdot \mathrm{I}(2) + \mathrm{I} \cdot \mathrm{I}(2)^{*2} + \mathrm{I}(2)^{*p+1}.$$

If we $\circ$–multiply this relation by $b_{(j)}$ then we obtain a relation in $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4$:

$$[v_1] \circ b_{(i-1)}^{\circ p} \circ b_{(j)} \equiv v_i \cdot b_{(0)} \circ b_{(j)} - p \cdot b_{(i)} \circ b_{(j)} - \underbrace{b_{(i-1)}^{*p} \circ b_{(j)}}_{=(b_{(i-1)} \circ b_{(j-1)})^{*p}}$$

$$\mathrm{mod}\ \mathrm{I}^2 \cdot \mathrm{I}(4) + \mathrm{I} \cdot I(4)^{*2} + \mathrm{I}(4)^{*p+1}$$

The bracketed equality is a consequence of the Hopf ring distributive law (see the discussion just before Lemma 1.7 of [16]). If $j = 0$, then (as that discussion points out) the bracketed quantity equals 0. The fact that $\mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_m \circ \mathrm{I}(n)^{*k} \subseteq \mathrm{I}(n+m)^{*k}$ is also a consequence of the Hopf ring distributive law.

Substituting $(i,j) \mapsto (i,i), (1,i), (j,i), (i,j)$ (and subtracting) produces, for $0 < i < j$, the following congruences mod $\mathrm{I}^2 \cdot \mathrm{I}(4) + \mathrm{I} \cdot \mathrm{I}(4)^{*2} + \mathrm{I}(4)^{*p+1}$:

$$[v_1]_* w_{4p^i} \equiv v_i \cdot x_{2(1+p^i)} - p \cdot x_{4p^i} - x_{4p^{i-1}}^p$$

$$[v_1]_* y_{2(1+p^i)} \equiv v_i \cdot x_4 - p \cdot x_{2(1+p^i)}$$

$$[v_1]_* y_{2(p^i+p^j)} \equiv v_j \cdot x_{2(1+p^i)} - p \cdot x_{2(p^i+p^j)} - x_{2(p^{i-1}+p^{j-1})}^p$$

$$[v_1]_* (z_{2(p^i+p^j)} - y_{2(p^i+p^j)}) \equiv v_i \cdot x_{2(1+p^j)} - v_j \cdot x_{2(1+p^i)}$$

These congruences suffice for computing characteristic numbers of the form s_n and $\mathrm{s}_{n,n'}$ since, by Lemma 2, such numbers vanish on the ideal $\mathrm{I}^2 \cdot \mathrm{I}(4) + \mathrm{I} \cdot \mathrm{I}(4)^{*2} + \mathrm{I}(4)^{*p+1}$.

To compute s_n and $\mathrm{s}_{n,n'}$ of the right-hand sides of these congruences, note that, by the construction of u_i, the image of v_i in

$$\pi_*\mathrm{MSpin}_{(p)} \cong \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_4 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathbf{BP}\langle\mathbf{1}\rangle_{2p-2}/(u_1, u_2, \ldots)$$

can serve as the $\mathbf{Z}_{(p)}$–polynomial algebra generator of degree $2(p^i-1)$. So by Theorem 3, p divides $s_{(p^i-1)/2}[v_i]$ to order 1. Similarly, if $2m$ is not of the form $2(p^i-1)$, then the image of x_{2m} may serve as the $\mathbf{Z}_{(p)}$–polynomial algebra generator of degree $2m$. So by Theorem 3, p does not divide $s_{m/2}[x_{2m}]$.

Thus, by Lemma 2,

$$\begin{aligned} s_{p^i}([v_1]_* w_{4p^i}) &= s_{p^i}(v_i\cdot x_{2(1+p^i)} - p\cdot x_{4p^i} - x^p_{4p^i-1}) \\ &= \underbrace{s_{p^i}(v_i\cdot x_{2(1+p^i)})}_{=0} - p\cdot s_{p^i}(x_{4p^i}) - \underbrace{s_{p^i}(x^p_{4p^i-1})}_{=0} \end{aligned}$$

and since $\mathrm{ord}_p(a\cdot b) = \mathrm{ord}_p(a)+\mathrm{ord}_p(b)$, it follows that

$$\mathrm{ord}_p\bigl[s_{p^i}([v_1]_* w_{4p^i})\bigr] = \underbrace{\mathrm{ord}_p[p]}_{=1} + \underbrace{\mathrm{ord}_p\bigl[s_{p^i}(x_{4p^i})\bigr]}_{=0} = 1.$$

Similarly, by Lemma 2,

$$\begin{aligned} &s_{(p^i+1)/2,(p^j-1)/2}\bigl(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}\bigr) \\ &\qquad = s_{(p^i+1)/2,(p^j-1)/2}\bigl(v_i\cdot x_{2(1+p^j)} - v_j\cdot x_{2(1+p^i)}\bigr) \\ &\qquad = -\, s_{(p^j-1)/2}\bigl(v_j\bigr)\cdot s_{(p^i+1)/2}\bigl(x_{2(1+p^i)}\bigr) \end{aligned}$$

and it follows that

$$\begin{aligned} &\mathrm{ord}_p\bigl[s_{(p^i+1)/2,(p^j-1)/2}\bigl(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}\bigr)\bigr] \\ &\qquad = \underbrace{\mathrm{ord}_p\bigl[s_{(p^j-1)/2}\bigl(v_j\bigr)\bigr]}_{=1} + \underbrace{\mathrm{ord}_p\bigl[s_{(p^i+1)/2}\bigl(x_{2(1+p^i)}\bigr)\bigr]}_{=0} = 1. \end{aligned}$$

These, and similar calculations, show that p divides

$$\begin{aligned} s_{p^i}\bigl([v_1]_* w_{4p^i}\bigr) &\quad \text{to order } 1, \\ s_{(1+p^i)/2}\bigl([v_1]_* y_{2(1+p^i)}\bigr) &\quad \text{to order } 1, \\ s_{(p^i+p^j)/2}\bigl([v_1]_* y_{2(p^i+p^j)}\bigr) &\quad \text{to order } 1, \\ s_{(p^i+p^j)/2}\bigl([v_1]_* (z_{2(p^i+p^j)} - y_{2(p^i+p^j)})\bigr) &\quad \text{to order } \infty, \\ s_{(p^i+1)/2,(p^j-1)/2}\bigl(z_{2(p^i+p^j)} - y_{2(p^i+p^j)}\bigr) &\quad \text{to order } 1, \\ s_{(p^i+1)/2,(p^j-1)/2}\bigl([v_1]_* (v_j\cdot y_{2(1+p^i)})\bigr) &\quad \text{to order } 2. \end{aligned}$$

(Recall that by definition $\mathrm{ord}_p(0)=\infty$.)

Theorem 4 follows from these six facts: (1) from the first three and (2) from the last three. In more detail, the last three facts imply that the image of $z_{2(p^i+p^j)} - y_{2(p^i+p^j)}$ can be distinguished from the image of $y_{2(p^i+p^j)}$ and from the images of degree–$2(p^i+p^j)$

products of lower degree generators by the vanishing of the number $s_{(p^i+p^j)/2}$ together with the nonvanishing mod p^2 of the number $s_{(p^i+1)/2,(p^j-1)/2}$.

3 Cayley plane bundles

In this section we summarize work of Borel and Hirzebruch [3; 4] on characteristic classes of homogeneous spaces, which we will use in the next section to prove Theorem 1.

The Cayley plane is the homogeneous space $\mathbf{CaP}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$. Much of what follows applies to any bundle with fiber a homogeneous space G/H, so we begin in that generality and later specialize to the case $G/H = \mathrm{F}_4/\mathrm{Spin}(9)$.

Throughout this section, let G be a compact connected Lie group, let $\mathrm{i}_{H,G}\colon H \hookrightarrow G$ be a maximal rank subgroup, and let $\mathrm{i}_{T,H}\colon T \to H$ and $\mathrm{i}_{T,G}\colon T \to G$ be a common maximal torus:

$$\begin{tikzcd} & H \arrow[d, "\mathrm{i}_{H,G}"] \\ T \arrow[ur, "\mathrm{i}_{T,H}"] \arrow[r, "\mathrm{i}_{T,G}"'] & G \end{tikzcd}$$

Every G/H bundle (with structure group G) pulls back from the universal G/H bundle $\mathrm{B}H \to \mathrm{B}G$. That is, every G/H bundle (with structure group G) fits into a pullback square

$$\begin{tikzcd} E \arrow[r, "\tilde{g}"] \arrow[d, "\pi"'] & \mathrm{B}H \arrow[d, "\mathrm{Bi}_{H,G}"] \\ Z \arrow[r, "g"] & \mathrm{B}G \end{tikzcd}$$

where g is unique up to homotopy.

Let η denote the bundle of tangents along the fibers of $\mathrm{B}H \to \mathrm{B}G$. Then the bundle of tangents along the fibers of $E \to Z$ is the pullback $\tilde{g}^*(\eta)$ and there is an exact sequence

$$0 \to \tilde{g}^*(\eta) \to \mathrm{T}E \to \pi^*\mathrm{T}Z \to 0.$$

This enables us to compute the characteristic classes of $\mathrm{T}E$ from those of η and $\mathrm{T}Z$, eg

$$\mathrm{p}_1(\mathrm{T}E) = \tilde{g}^*\mathrm{p}_1(\eta) + \pi^*\mathrm{p}_1(\mathrm{T}Z).$$

The characteristic classes of η, or rather their pullbacks to $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$, may in turn be computed using the beautiful methods of Borel and Hirzebruch. To state their results precisely, we need to introduce some notation (see [3, Chapter 1] for more detail).

Let V be the universal cover of the maximal torus T. Let Γ be the unit lattice of V, ie, the inverse image of the identity element of T. A real-valued linear form on V is called *integral* if it takes integral values on Γ; the group of all such forms $\mathrm{Hom}(\Gamma,\mathbf{Z})$ is naturally isomorphic to $\mathrm{H}^1(T,\mathbf{Z}) \cong \mathrm{Hom}(\pi_1(T),\mathbf{Z})$. The adjoint representation of T on the Lie algebra $\mathfrak{g}$ of G is fully reducible, and there is a direct sum decomposition of $\mathfrak{g}$ into invariant subspaces,

$$\mathfrak{g} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_m + \mathfrak{t},$$

where $\dim(\mathfrak{a}_i) = 2$. The action on $\mathfrak{a}_i$ of an element t of T may be written as

$$\begin{pmatrix} \cos 2\pi a_i(t) & -\sin 2\pi a_i(t) \\ \sin 2\pi a_i(t) & \cos 2\pi a_i(t) \end{pmatrix}.$$

The function $a_i\colon T \to \mathbf{R}$ lifts to a nonzero integral linear form on V, also denoted a_i. The linear forms $\pm a_1,\dots,\pm a_m$ on V are called the *roots* of G. The decompositions of $\mathfrak{g}$ and $\mathfrak{h}$ may be chosen compatibly so that we may speak of the roots $\pm\overline{a}_1,\dots,\pm\overline{a}_k$ of G *complementary* to those of H.

Transgression in a principal T–bundle $P \to P/T$ associates to each element of $\mathrm{H}^1(T,\mathbf{Z})$ an element of $\mathrm{H}^2(P/T,\mathbf{Z})$. Since $\mathrm{H}^1(T,\mathbf{Z}) \cong \mathrm{Hom}(\Gamma,\mathbf{Z})$ (as discussed above), this associates to each root of G, and more generally to each integral form, an element of $\mathrm{H}^2(P/T,\mathbf{Z})$. For the universal T–bundle $\mathrm{E}T \to \mathrm{B}T$, we obtain an isomorphism $\mathrm{H}^2(\mathrm{B}T,\mathbf{Z}) \cong \mathrm{Hom}(\Gamma,\mathbf{Z})$.

Theorem [3, Theorem 10.7] *Let $P \to P/G$ be a principal G–bundle, ρ the projection $P/T \to P/H$, and η the bundle of tangents along the fibers of the G/H bundle $P/H \to P/G$. Then*

$$\rho^*(\mathrm{p}(\eta)) = \prod (1+\overline{a}_j^2),$$

where $\{\pm\overline{a}_j\}_{1\le j\le k}$ are the roots of G complementary to those of H, regarded as elements of $\mathrm{H}^2(P/T,\mathbf{Z})$.

Applied to the principal G–bundle $\mathrm{e}\,G \to \mathrm{B}G$, for which $\rho = \mathrm{Bi}_{T,G}\colon \mathrm{B}T \to \mathrm{B}G$, this gives a formula for (the pullback to $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$ of) the characteristic class $\mathrm{s}_I(\eta)$ of the bundle η of tangents along the fibers of the universal G/H bundle $\mathrm{B}H \to \mathrm{B}G$, namely

$$\mathrm{Bi}_{T,G}^*\big(\mathrm{s}_I(\eta)\big) = \mathrm{s}_I\big(\overline{a}_1^2,\dots,\overline{a}_k^2\big) \in \mathrm{H}^*(\mathrm{B}T,\mathbf{Z}).$$

This formula together with the following Lie-theoretic description of the pushforward

$$\mathrm{Bi}_{H,G*}\colon \mathrm{H}^*(\mathrm{B}H,\mathbf{Z}) \to \mathrm{H}^*(\mathrm{B}G,\mathbf{Z})$$

will enable us to prove Theorem 1. To describe the latter, we need to introduce further notation (again, see [3, Chapter 1] for more detail).

Fix a positive-definite metric on $\mathfrak{g}$, invariant under the adjoint representation of G. It determines a metric on V and hence a canonical isomorphism between V and its dual space V^* as well as a metric on V^*. A symmetry S_a of V with respect to a hyperplane $a = 0$ induces a symmetry of V^*, also denoted S_a, defined by

$$\mathrm{S}_a(b) = b - 2(a,b)(a,a)^{-1} \cdot a.$$

The Weyl group $\mathrm{W}(G)$ of G is the group of automorphisms of T induced by inner automorphisms of G that leave T invariant. It may also be viewed as the group of isometries of V that leave Γ and the root diagram invariant. It is generated by the symmetries S_{a_i} to the hyperplanes $a_i = 0$ $(i = 1, \dots, m)$. The *sign* of an element w of $\mathrm{W}(G)$, denoted $\mathrm{sgn}(w)$, is the determinant of w viewed as a linear transformation of V; it always equals ± 1. Choose a basis $e_1, \dots, e_l$ for V^*. Call a root $a = a_1 e_1 + \dots + a_l e_l$ *positive* if the first nonvanishing coefficient a_i is positive. Call a positive root *simple* if it is not the sum of two positive nonzero roots. The simple roots form a basis for V^*, and every root is a linear combination, with integral coefficients of the same sign, of simple roots.

Let $\tilde{\mathrm{e}}(G/T) \in \mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$ be the Euler class of the bundle of tangents along the fibers of $\mathrm{B}T \to \mathrm{B}G$. Up to sign, it is the product of a set of positive roots of G, regarded as elements of $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$. More precisely, it is the product of the roots of an invariant almost complex structure on G/T. Note that G/T always admits a complex structure and that although the individual roots associated to an almost complex structure depend on the almost complex structure, their product does not. (See [3, Sections 12.3, 13.4].)

The key to describing $\mathrm{Bi}_{H,G*}$ is the following:

Theorem 5 (Borel and Hirzebruch [4, Theorem 20.3]) *If $t \in \mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$, then*

$$\sum_{w \in \mathrm{W}(G)} \mathrm{sgn}(w) \cdot w(t) = \mathrm{Bi}^*_{T,G}\big(\mathrm{Bi}_{T,G*}(t)\big) \cdot \tilde{\mathrm{e}}(G/T).$$

Corollary 6 *If $h \in \mathrm{H}^*(\mathrm{B}H,\mathbf{Z})$, then*

$$\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h) = \sum_{[w] \in \mathrm{W}(G)/\mathrm{W}(H)} w\left(\frac{\tilde{\mathrm{e}}(H/T)}{\tilde{\mathrm{e}}(G/T)}\,\mathrm{Bi}^*_{T,H}(h)\right)$$

where the sum runs over the cosets of $\mathrm{W}(H)$ in $\mathrm{W}(G)$.

(Note that this is a formula in the polynomial ring $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$.)

Proof Since $\mathrm{Bi}_{T,H*}\big(\tilde{\mathrm{e}}(H/T)\big)=\chi(H/T)=|\mathrm{W}(H)|\in\mathrm{H}^0(\mathrm{B}H,\mathbf{Z})$, write

$$\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h)=\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}\left(\frac{\mathrm{Bi}_{T,H*}\big(\tilde{\mathrm{e}}(H/T)\big)}{|\mathrm{W}(H)|}\cdot h\right).$$

Apply the projection formula [3, Proposition 8.2] to obtain

$$\begin{aligned}\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h)&=\frac{1}{|\mathrm{W}(H)|}\,\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}\,\mathrm{Bi}_{T,H*}\big(\tilde{\mathrm{e}}(H/T)\cdot\mathrm{Bi}^*_{T,H}(h)\big)\\&=\frac{1}{|\mathrm{W}(H)|}\,\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{T,G*}\big(\tilde{\mathrm{e}}(H/T)\cdot\mathrm{Bi}^*_{T,H}(h)\big).\end{aligned}$$

Apply Theorem 5 to obtain

$$\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h)=\frac{1}{|\mathrm{W}(H)|}\cdot\frac{1}{\tilde{\mathrm{e}}(G/T)}\sum_{w\in\mathrm{W}(G)}\mathrm{sgn}(w)\cdot w\big(\tilde{\mathrm{e}}(H/T)\cdot\mathrm{Bi}^*_{T,H}(h)\big).$$

Since $w\big(\tilde{\mathrm{e}}(G/T)\big)=\mathrm{sgn}(w)\,\tilde{\mathrm{e}}(G/T)$,

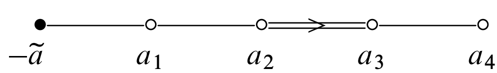
$$\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h)=\frac{1}{|\mathrm{W}(H)|}\sum_{w\in\mathrm{W}(G)}w\left(\frac{\tilde{\mathrm{e}}(H/T)}{\tilde{\mathrm{e}}(G/T)}\,\mathrm{Bi}^*_{T,H}(h)\right).$$

Since $\mathrm{W}(G)$ acts on $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$ by ring homomorphisms, since if $w\in\mathrm{W}(H)$ then $w(\tilde{\mathrm{e}}(H/T))=\mathrm{sgn}(w)\,\tilde{\mathrm{e}}(H/T)$ and $w(\tilde{\mathrm{e}}(G/T))=\mathrm{sgn}(w)\,\tilde{\mathrm{e}}(G/T)$, and since $\mathrm{Bi}^*_{T,H}$ maps to the $\mathrm{W}(H)$–invariant subring of $\mathrm{H}^*(\mathrm{B}T,\mathbf{Z})$, this sum may be written over the cosets of $\mathrm{W}(H)$ in $\mathrm{W}(G)$

$$\mathrm{Bi}^*_{T,G}\,\mathrm{Bi}_{H,G*}(h)=\sum_{[w]\in\mathrm{W}(G)/\mathrm{W}(H)}w\left(\frac{\tilde{\mathrm{e}}(H/T)}{\tilde{\mathrm{e}}(G/T)}\,\mathrm{Bi}^*_{T,H}(h)\right).\qquad\square$$

Now we specialize to the Cayley plane $G/H=\mathrm{F}_4/\mathrm{Spin}(9)$ (see [3, Section 19] and [5, Plate VIII] for more detail).

The extended Dynkin diagram of the root system F_4 is:



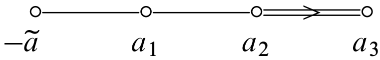
A choice of simple roots is

$$a_1=e_2-e_3,\quad a_2=e_3-e_4,\quad a_3=e_4,\quad a_4=\tfrac{1}{2}(e_1-e_2-e_3-e_4).$$

Since the coefficient of a_4 in the maximal root

$$\widetilde{a} = 2a_1 + 3a_2 + 4a_3 + 2a_4 = e_1 + e_2$$

is prime, a theorem of Borel and de Siebenthal [2] implies that erasing a_4 from the extended Dynkin diagram gives the Dynkin diagram



of a subgroup of the compact Lie group F_4. This type–B_4 subgroup is globally isomorphic to $\mathrm{Spin}(9)$, the 1–connected double cover of $\mathrm{SO}(9)$.

The roots of this type–B_4 root subsystem are

$$\begin{cases} \pm e_i & 1 \leq i \leq 4, \\ \pm e_i \pm e_j & 1 \leq i < j \leq 4. \end{cases}$$

The roots of F_4 are these roots together with the complementary roots

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

Let T be the standard maximal torus of $\mathrm{SO}(9)$ (see [3, Section 19.2]). Its preimage T' under the double covering $\mathrm{Spin}(9) \to \mathrm{SO}(9)$ is a maximal torus of $\mathrm{Spin}(9)$ and hence also of F_4. The double covering $\mathrm{T}' \to \mathrm{T}$ determines an index-2 sublattice

$$\mathrm{H}^1(\mathrm{T}, \mathbf{Z}) \hookrightarrow \mathrm{H}^1(\mathrm{T}', \mathbf{Z})$$

corresponding, under the identification $\mathrm{H}^1(T, \mathbf{Z}) \cong \mathrm{Hom}(\Gamma, \mathbf{Z})$, to the index-2 sublattice

$$\mathbf{Z}\langle e_1, e_2, e_3, e_4 \rangle \hookrightarrow \mathbf{Z}\left\langle \begin{matrix} \frac{1}{2}(e_1+e_2+e_3+e_4), \frac{1}{2}(e_1+e_2+e_3-e_4) \\ \frac{1}{2}(e_1+e_2-e_3+e_4), \frac{1}{2}(e_1-e_2+e_3+e_4) \end{matrix} \right\rangle.$$

The following positive roots determine an almost complex structure on $\mathrm{Spin}(9)/\mathrm{T}'$:

$$\begin{cases} e_i & 1 \leq i \leq 4, \\ e_i \pm e_j & 1 \leq i < j \leq 4. \end{cases}$$

These, together with the following complementary positive roots, determine an almost complex structure on F_4/T':

$$\{\overline{a}_1, \ldots, \overline{a}_8\} = \{\tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$$

(The ordering of these roots will not matter.)

The 3 cosets of $\mathrm{W}(\mathrm{Spin}(9))$ in $\mathrm{W}(\mathrm{F}_4)$ are represented by the reflections

$$\{1, \mathrm{S}_{a_4}, \mathrm{S}_{a_4}\mathrm{S}_{a_3}\mathrm{S}_{a_4}\}$$

which act, with respect to the basis $(e_1,\dots,e_4)$, by the matrices:

$$\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&1\\-1&1&1&1\end{pmatrix}\right\}$$

In particular, they act on the set of positive complementary roots $\{\overline{a}_1,\dots,\overline{a}_8\}$ by:

$$\begin{aligned}
\{\overline{a}_1,\dots,\overline{a}_8\} &= \left\{\tfrac{1}{2}(e_1\pm e_2\pm e_3\pm e_4)\right\}\\
\mathrm{S}_{a_4}(\{\overline{a}_1\dots,\overline{a}_8\}) &= \Big\{e_1,e_2,e_3,e_4,\\
&\qquad \tfrac{1}{2}(e_1+e_2+e_3-e_4), \tfrac{1}{2}(\ \ e_1+e_2-e_3+e_4)\\
&\qquad \tfrac{1}{2}(e_1-e_2+e_3+e_4), \tfrac{1}{2}(-e_1+e_2+e_3+e_4)\Big\}\\
\mathrm{S}_{a_4}\mathrm{S}_{a_3}\mathrm{S}_{a_4}(\{\overline{a}_1,\dots,\overline{a}_8\}) &= \Big\{e_1,e_2,e_3,e_4,\\
&\qquad \tfrac{1}{2}(e_1+e_2+e_3+e_4), \tfrac{1}{2}(\ \ e_1+e_2-e_3-e_4)\\
&\qquad \tfrac{1}{2}(e_1-e_2+e_3-e_4),\ \tfrac{1}{2}(-e_1+e_2+e_3-e_4)\Big\}
\end{aligned}$$

Thus we conclude:

Proposition 7

$$\begin{aligned}
&\mathrm{Bi}^*_{\mathrm{T}',\mathrm{F}_4}\,\mathrm{Bi}_{\mathrm{Spin}(9),\mathrm{F}_4*}\,\mathrm{s}_I(\eta)\\
&\qquad = \frac{\mathrm{s}_I\left(\overline{a}_1^2,\dots,\overline{a}_8^2\right)}{\prod_i \overline{a}_i} + \mathrm{S}_{a_4}\left(\frac{\mathrm{s}_I\left(\overline{a}_1^2,\dots,\overline{a}_8^2\right)}{\prod_i \overline{a}_i}\right) + \mathrm{S}_{a_4}\mathrm{S}_{a_3}\mathrm{S}_{a_4}\left(\frac{\mathrm{s}_I\left(\overline{a}_1^2,\dots,\overline{a}_8^2\right)}{\prod_i \overline{a}_i}\right)
\end{aligned}$$

where the complementary roots $\{\pm\overline{a}_1,\dots,\pm\overline{a}_8\} = \left\{\frac{1}{2}(\pm e_1\pm e_2\pm e_3\pm e_4)\right\}$ are regarded as elements of $\mathrm{H}^2(\mathrm{BT}',\mathbf{Z})$ and $\mathrm{S}_{a_4}, \mathrm{S}_{a_4}\mathrm{S}_{a_3}\mathrm{S}_{a_4}$ act on them as described above.

4 Proof of Theorem 1

Theorem 1 *Away from 6, the ideal of $\pi_*\mathrm{MO}\langle 8\rangle$ consisting of (bordism classes of) Cayley plane bundles with connected structure group is precisely the kernel of the Witten genus. In other words, the extension of this ideal in $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$ is precisely the kernel of*

$$\phi_{\mathrm{W}}\otimes\mathbf{Z}\left[\tfrac{1}{6}\right]\colon \pi_*\mathrm{MO}\langle 8\rangle\left[\tfrac{1}{6}\right]\to\pi_*\mathrm{tmf}\left[\tfrac{1}{6}\right]\cong\mathbf{Z}\left[\tfrac{1}{6}\right][\mathbf{G}_4,\mathbf{G}_6],$$

where $\mathbf{G}_4,\mathbf{G}_6$ have degrees $8, 12$ respectively.

Since the Witten genus carries the subring of $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$ generated by elements of degree at most 12 isomorphically to the polynomial ring $\mathbf{Z}\left[\frac{1}{6}\right][\mathbf{G}_4, \mathbf{G}_6]$, and since (as discussed in the introduction) the Witten genus of any $\mathbf{CaP}^2$ bundle with connected structure group vanishes, Theorem 1 can be proved by showing that $\mathbf{CaP}^2$ bundles with connected structure group can serve as generators for $\pi_*\mathrm{MO}\langle 8\rangle\left[\frac{1}{6}\right]$ in dimensions greater than 12. And this can be done by constructing a set S of such $\mathbf{CaP}^2$ bundles that satisfy the conditions of Theorem 4 in all dimensions except 8 and 12.

Construction of M^{4n}

The first step is to construct, for each $n \geq 4$, a $\mathbf{CaP}^2$ bundle M^{4n} that satisfies condition (1) of Theorem 4. It will be a $\mathbf{Z}$–linear combination (topologically, a disjoint union with some string structures possibly reversed) of total spaces of $\mathbf{CaP}^2$ bundles whose base spaces are products of two carefully chosen complete intersections.

Let $i\colon V^m(d_1, \dots, d_r) \hookrightarrow \mathbf{CP}^{m+r}$ denote a smooth complete intersection of degree $(d_1, \dots, d_r)$ and complex dimension m. Consider the $\mathbf{CaP}^2$ bundle pulling back from the universal bundle $\mathbf{CaP}^2 \to \mathrm{BSpin}(9) \to \mathrm{BF}_4$ by a classifying map g of the form

$$\begin{tikzcd}
E \arrow[rrrr, "\tilde{g}"] \arrow[d, "\pi"] & & & & \mathrm{BSpin}(9) \arrow[d, "\mathrm{Bi}_{\mathrm{Spin}(9),\mathrm{F}_4}"] \\
W = V^m(d_1, \dots, d_r) \times V^{m'}(d'_1, \dots, d'_{r'}) \arrow[r, "i \times i'"] \arrow[rrrr, bend right, "g"'] & \mathbf{CP}^{m+r} \times \mathbf{CP}^{m'+r'} \arrow[r] & \mathbf{CP}^\infty \times \mathbf{CP}^\infty \arrow[r] \arrow[rr, bend left, "f"] & \mathrm{BT}' \arrow[r] & \mathrm{BF}_4
\end{tikzcd}$$

where $m + m' = 2n - 8$.

Let $\mathrm{H}^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong \mathbf{Z}[x_1, x_2]$. Choose the map $f\colon \mathbf{CP}^\infty \times \mathbf{CP}^\infty \to \mathrm{BF}_4$ so that (e_1, e_2, e_3, e_4) pull back to $n_f \cdot (x_1, x_1, x_2, -x_2)$, respectively, for some integer $n_f \geq 1$. The generators $(e_2, e_3, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4))$ of the lattice $\mathrm{H}^2(\mathrm{BT}', \mathbf{Z})$ then pull back to $n_f \cdot (x_1, x_2, -x_2, 0)$, respectively.

The degrees $(d_1, \dots, d_r)$ and $(d'_1, \dots, d'_{r'})$ need to be chosen so that $\mathrm{p}_1(\mathrm{T}E) = 0$, since this implies that E admits a string structure. The exact sequences of vector bundles

$$\begin{gathered}
0 \to \tilde{g}^*(\eta) \to \mathrm{T}E \to \pi^*\mathrm{T}\big(V^m(d_1, \dots, d_r) \times V^{m'}(d'_1, \dots, d'_{r'})\big) \to 0 \\
0 \to i^* \bigoplus_j \mathrm{O}(d_j) \to i^*\mathrm{T}\mathbf{CP}^{m+r} \to \mathrm{T}V^m(d_1, \dots, d_r) \to 0
\end{gathered} \tag{4-1}$$

imply that:

$$\begin{aligned}
&\mathrm{p}_1(\mathrm{T}E)\\
&= \tilde{g}^*\mathrm{p}_1(\eta) + \pi^*\mathrm{p}_1\mathrm{T}\big(V^m(d_1,\dots,d_r)\times V^{m'}(d'_1,\dots,d'_{r'})\big)\\
&= \tilde{g}^*\mathrm{p}_1(\eta) + \pi^*i^*\Big[\mathrm{p}_1\mathrm{T}\mathbf{CP}^{m+r} - \sum_j \mathrm{p}_1\mathrm{O}(d_j)\Big] + \pi^*i'^*\Big[\mathrm{p}_1\mathrm{T}\mathbf{CP}^{m'+r'} - \sum_{j'} \mathrm{p}_1\mathrm{O}(d'_{j'})\Big]\\
&= \tilde{g}^*\mathrm{p}_1(\eta) + \pi^*(i\times i')^*\Big[\Big(m+r+1-\sum_j d_j^2\Big)x_1^2 + \Big(m'+r'+1-\sum_{j'}(d'_{j'})^2\Big)x_2^2\Big]
\end{aligned}$$

The image of $\mathrm{p}_1(\eta)$ in $\mathrm{H}^4(\mathrm{BT}')$ is $\sum \frac{1}{4}(e_1 \pm e_2 \pm e_3 \pm e_4)^2 = 2(e_1^2+e_2^2+e_3^2+e_4^2)$, which pulls back to $4n_f(x_1^2+x_2^2)$. So:

$$\begin{aligned}
&\mathrm{p}_1(\mathrm{T}E)\\
&= \pi^*(i\times i')^*\Big[\Big(4n_f+m+1+r-\sum_j d_j^2\Big)x_1^2 + \Big(4n_f+m'+1+r'-\sum_{j'}(d'_{j'})^2\Big)x_2^2\Big]
\end{aligned}$$

The following lemma shows that, for any given m and m', it is simple to choose degrees $(d_1,\dots,d_r)$ and $(d'_1,\dots,d'_{r'})$ so that this quantity vanishes, *provided n_f is sufficiently large.* (The fact that the degrees can all be taken to be 2 and 3 is relevant since these are the primes inverted in this paper.)

Lemma 8 *For any integer $n \geq 14$, there exist integers $a,b \geq 0$ so that*

$$n+(a+b) = a\cdot 2^2 + b\cdot 3^2.$$

Proof This follows by induction since

$$14+3 = 2^2+2^2+3^2, \quad 15+5 = 2^2+2^2+2^2+2^2+2^2, \quad 16+2 = 3^2+3^2$$

and since

$$n+(a+b) = a\cdot 2^2+b\cdot 3^2 \implies (n+3)+(a+1+b) = (a+1)\cdot 2^2 + b\cdot 3^2. \quad \square$$

As an aside, the values for a and b constructed in the proof are

$$a(n) = 3n - 8\lceil n/3\rceil, \quad b(n) = 3\lceil n/3\rceil - n.$$

Although the preceding lemma suffices to prove the results of this paper, the reader may find the reliance on complete intersections of arbitrarily high codimension unsatisfying. It is therefore worth noting that the following replacement for Lemma 8 would make it possible to prove the results of this paper using complete intersections of codimension at most 4.

Conjecture 9 *If $n \geq 25$, then the GCD*

$$\mathrm{GCD}\left\{\prod_{i=1}^{4} d_i \;\middle|\; 4n+4+1=\sum_{i=1}^{4} d_i^2,\, d_i>0\right\}$$

has the form $2^a 3^b$ with $a+b>0$. In fact, as n increases from 25, this GCD *takes the values $2^4\cdot 3$, 2^3, $2^4\cdot 3^2$, $2^3\cdot 3$, 2^4, $2^3\cdot 3^2$, and then repeats from the beginning.*

We have to carefully choose the degrees $(d_1,\dots,d_r)$ and $(d'_1,\dots,d'_{r'})$ to ensure that the total space E admits a string structure. However, these degrees have little effect on the Pontrjagin number $\mathrm{s}_n[E]$, which we compute next. Indeed, for dimension reasons,

$$\mathrm{s}_n[E]=(i\times i')^* f^* \,\mathrm{Bi}_{\mathrm{Spin}(9),\mathrm{F}_4 *}\,\mathrm{s}_n(\eta).$$

Since the base space W is a product of complete intersections, the pullback

$$(i\times i')^* x_1^m x_2^{m'}$$

equals $\left(\prod_j d_j\right)\left(\prod_{j'} d'_{j'}\right)$ times the fundamental class $[W]$. So the key is to compute the coefficients of the polynomial $f^*\,\mathrm{Bi}_{\mathrm{Spin}(9),\mathrm{F}_4 *}\,\mathrm{s}_n(\eta)$ or, rather, their GCD as a function of n. This calculation lies at the heart of this paper. (It was the smoking gun that led to Theorem 1.)

Proposition 10

$$f^*\,\mathrm{Bi}_{\mathrm{Spin}(9),\mathrm{F}_4 *}\,\mathrm{s}_n(\eta)=2n_f^{2n-8}\sum_{k=2}^{n-2}\left[\binom{2n}{2}-\binom{2n}{2k}\right]x_1^{2k-4}x_2^{2n-2k-4}$$

Proof Since the polynomial in question is homogeneous in $n_f x_1$ and $n_f x_2$, we can, without loss of generality, simplify notation by setting $n_f=1$ and $(x_1,x_2)=(x,1)$.

Proposition 7 gives the polynomial in the form of a power series:

$$\begin{aligned}-\frac{1}{x^4}&(1+x^2+x^4+\cdots)\\ &\cdot\Bigg[\underbrace{-2+(x+1)^{2n}+(x-1)^{2n}}-x^2\Big[\overbrace{-2+(x+1)^{2n}+(x-1)^{2n}+2\binom{2n}{2}}\Big]\\ &\qquad\qquad\qquad\qquad\qquad\qquad+x^{2n}\Big[2\binom{2n}{2}-2\Big]+2x^{2n+2}\Bigg]\end{aligned}$$

The bracketed quantities differ by $2\binom{2n}{2}$, so the power series simplifies to the polynomial

$$-\frac{1}{x^4}\cdot\left[-2+(x+1)^{2n}+(x-1)^{2n}-2\binom{2n}{2}(x^2+x^4+\cdots+x^{2n-2})-2x^{2n}\right]$$

which simplifies further to

$$2\sum_{k=2}^{n-1}\left[\binom{2n}{2}-\binom{2n}{2k}\right]x^{2k-4}. \qquad \square$$

Proposition 11 *For any integer $n \geq 4$ and any odd prime p,*

$$\operatorname{ord}_p\left[\underset{1<k<n-1}{\mathrm{GCD}}\left\{\binom{2n}{2}-\binom{2n}{2k}\right\}\right] = \begin{cases} 1 & \text{if } 2n = p^i - 1 \text{ or } 2n = p^i + p^j \text{ for some } 0 \leq i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

The key behind this is the following lemma.

Lemma 12 *For any integer $n > 1$ and any odd prime p,*

$$\operatorname{ord}_p\left[\underset{0<k<n}{\mathrm{GCD}}\binom{2n}{2k}\right] = \begin{cases} 1 & \text{if } 2n = p^i + p^j \text{ for some } 0 \leq i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

It is worth comparing this result to the better-known result that for any integer $n > 1$ and any prime p,

$$\operatorname{ord}_p\left[\underset{0<k<n}{\mathrm{GCD}}\binom{n}{k}\right] = \begin{cases} 1 & \text{if } n = p^i \text{ for some integer } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, for any given integer $n > 1$, at most one prime divides the latter GCD whereas several primes may divide the former. For example, if $n = 7$ then $2n = 7^1 + 7^1 = 13^0 + 13^1$, and indeed $\mathrm{GCD}_{0<k<7}\binom{14}{2k} = 7 \cdot 13$.

Proof of Lemma 12 By Kummer's theorem, $\binom{2n}{2k}$ is divisible by p if and only if there is at least 1 carry when adding $2k$ to $2n-2k$. Consider the base-p expansion $\sum n_i p^i$ of an even integer $2n$. If there is a digit $n_i \geq 2$, then there is no carry when adding $2p^i$ to $2n - 2p^i$. If there are 2 distinct nonzero digits n_i, n_j, then there is no carry when adding $p^i + p^j$ to $2n - p^i - p^j$. If $2n = p^i + p^j$ and $0 < 2k < 2n$ then there is always a carry when adding $2k$ to $2n - 2k$, even if $i = j$. These 3 facts together imply the first part of the lemma. The second part of the lemma follows from the fact that if $j > 0$, then there is precisely 1 carry when adding $(p-1)p^{j-1}$ to $p^i + p^j - (p-1)p^{j-1}$. (If $j = 0$, then the second part of the lemma is vacuous.) $\square$

Proof of Proposition 11 If an odd prime p divides the GCD, then all the binomial coefficients $\binom{2n}{2k}$ for $0 < 2k < 2n$ must be congruent mod p. If they are all congruent to 0 mod p, then Lemma 12 applies and $2n = p^i + p^j$ for some $0 \le i \le j$. So suppose that the binomial coefficients are all nonzero mod p. By Kummer's theorem, this happens precisely when for each $0 < 2k < 2n$ there are no carries when adding $2k$ to $2n-2k$. This in turn happens precisely when $2n = l \cdot p^i - 1$ for some $i > 0$ and some (odd) $0 < l < p$. According to Lucas's theorem (see [13, Section 1]), if $l > 1$ then

$$\binom{l \cdot p^i - 1}{p^i + 1} \equiv \binom{p-1}{1}\binom{p-1}{0}\cdots\binom{p-1}{0}\binom{l-1}{1} \equiv 1 - l \mod p.$$

However,

$$\binom{l \cdot p^i - 1}{2} \equiv 1 \mod p.$$

So all the binomial coefficients can be congruent mod p only if $l = 1$, and indeed the congruence $(1+x)^{p^i} \equiv 1 + x^{p^i} \mod p$ implies that

$$(1+x)^{p^i - 1} \equiv (1 + x^{p^i})(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + x^{p^i - 1} \mod p$$

and hence that

$$\binom{p^i - 1}{2k} \equiv 1 \mod p$$

for all $0 < 2k < p^i - 1$.

It remains to show that the GCD is never divisible by p^2 for p odd. By the preceding argument it remains only to show this when $2n = p^i + p^j$ or $2n = p^i - 1$ for $0 \le i \le j$. Remember that, by assumption, $2n \ge 16$.

Suppose first that $2n = p^i + p^j$. If $i > 1$, then there are at least 2 carries when adding 2 to $p^i + p^j - 2$; so by Kummer's theorem $\binom{2n}{2}$ is congruent to 0 mod p^2, while by Lemma 12 $\binom{2n}{2k}$ is nonzero mod p^2 for some $0 < 2k < 2n$. If $i \le 1$, then since $1 + 1 < 16$ we may assume that $j \ge 1$ and split into 2 cases: $p^j + 1$ and $p^j + p$. When $j \ge 2$ the 1st case can be handled as when $i > 1$. The remaining cases are handled by the following straightforward congruences mod p^2:

$$\binom{p+1}{2} - \binom{p+1}{4} \equiv \frac{5}{12}p, \quad \binom{p^j + p}{2} - \binom{p^j + p}{4} \equiv -\frac{1}{4}(p^j + p)$$

The coefficient $\frac{5}{12}$ is not a problem since $2n = p + 1 \ge 16$ only if $p \ge 17$.

Suppose now that $2n = p^i - 1$. Consider the following congruences mod p^2:

$$\binom{p^i-1}{2} \equiv 1 - \frac{3}{2}p^i, \quad \binom{p-1}{4} \equiv 1 - \frac{25}{12}p, \quad \binom{p^i-1}{p^{i-1}+p^{i-2}} \equiv 1 - p$$

The 1st and 2nd are immediate, and subtracting them gives the desired result for $i = 1$. (The resulting coefficient $-\frac{3}{2} + \frac{25}{12} = \frac{7}{12}$ of p is not a problem since $2n = p - 1 \geq 16$ only if $p \geq 17$.) Subtracting the 3rd congruence from the 1st gives the desired result when $i \geq 2$ but proving the 3rd congruence is more subtle. Here, and quite often in what follows, we rely on the following powerful theorem.

Granville's theorem [13, Theorem 1] *Suppose that a prime power p^q and positive integers $n = m + r$ are given. Write $n = n_0 + n_1 p + \cdots + n_d p^d$ in base p, and let N_j be the least positive residue of $[n/p^j]$ mod p^q for each $j \geq 0$ (so that $N_j = n_j + n_{j+1} p + \cdots + n_{j+q-1} p^{q-1}$); also make the corresponding definitions for m_j, M_j, r_j, R_j. Let e_j be the number of indices $i \geq j$ for which $n_i < m_i$ (that is, the number of "carries", when adding m and r in base p, on or beyond the j^{th} digit). Then*

$$\frac{1}{p^{e_0}} \equiv (\pm 1)^{e_{q-1}} \left(\frac{(N_0!)_p}{(M_0!)_p (R_0!)_p} \right) \left(\frac{(N_1!)_p}{(M_1!)_p (R_1!)_p} \right) \cdots \left(\frac{(N_d!)_p}{(M_d!)_p (R_d!)_p} \right) \mod p^q,$$

where (± 1) is (-1) except if $p = 2$ and $q \geq 3$. Here $(n!)_p$ denotes the product of those integers less than or equal to n that are not divisible by p.

We need to show that the 3rd congruence holds for $i \geq 2$, but assume first that $i \geq 3$. Then according to Granville's theorem, the binomial coefficient

$$\binom{p^i-1}{p^{i-1}+p^{i-2}}$$

is congruent to

$$\frac{((p^2-1)!)_p}{(p!)_p \cdot ((p^2-p-1)!)_p} \cdot \frac{((p^2-1)!)_p}{((p+1)!)_p \cdot ((p^2-p-2)!)_p} \cdot \frac{((p-1)!)_p}{(1!)_p \cdot ((p-2)!)_p} \quad \mod p^2.$$

Gathering common factors gives:

$$\begin{aligned}\binom{p^i-1}{p^{i-1}+p^{i-2}} &\equiv \left(\frac{(1-p)(2-p)\cdots((p-1)-p))}{(p!)_p} \right)^2 \cdot \frac{p^2-p-1}{p+1} \cdot (p-1) \quad \mod p^2 \\ &\equiv \big(1 - \underbrace{p(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{p-1})}_{\equiv 0}\big)^2 \cdot (1-p) \quad \mod p^2\end{aligned}$$

The bracketed quantity is congruent to 0 mod p^2 since by Wolstenholme's theorem [14, Theorem 116],[1] the $(p-1)^{\mathrm{st}}$ harmonic number is congruent to 0 mod p^2 for $p>3$ and to $2p$ for $p=3$. Thus we obtain

$$\binom{p^i-1}{p^{i-1}+p^{i-2}} \equiv 1^2\cdot(1-p) = 1-p \mod p^2.$$

If $i=2$ then the first factor in the congruence provided by Granville's theorem disappears, and the square in the following congruences therefore does too but, since $1^2=1$, this does not affect the final result. □

Construction of $N^{2(p^i+p^j)}$

The second step is to construct, for each prime $p>3$ and $0<i<j$, a $\mathbf{CaP}^2$ bundle $N^{2(p^i+p^j)}$ that satisfies condition (2) of Theorem 4. Throughout this section, let $p>3$ and $0<i<j$ be arbitrary but fixed and, to simplify notation, let

$$n=\tfrac{1}{2}(p^j-1), \quad n'=\tfrac{1}{2}(p^i+1).$$

The goal then is to construct a $\mathbf{CaP}^2$ bundle $N^{4(n+n')}$ with

$$\mathrm{s}_{n+n'}[N^{4(n+n')}]=0, \quad \mathrm{s}_{n,n'}[N^{4(n+n')}]\not\equiv 0 \mod p^2.$$

To do this, we will construct two $\mathbf{CaP}^2$ bundles E_1 and E_2 and define

$$N^{4(n+n')} = \mathrm{LCM}\big(\mathrm{s}_{n+n'}[E_1],\mathrm{s}_{n+n'}[E_2]\big)\cdot\left(\frac{E_1}{\mathrm{s}_{n+n'}[E_1]}-\frac{E_2}{\mathrm{s}_{n+n'}[E_2]}\right).$$

Then $\mathrm{s}_{n+n'}[N^{4(n+n')}]=0$, so all that will remain will be to show that $\mathrm{s}_{n,n'}[N^{4(n+n')}]\not\equiv 0 \mod p^2$. To do so, it will suffice to show that

$$\mathrm{s}_{n,n'}[E_1]\equiv 0 \mod p^2, \quad \mathrm{s}_{n,n'}[E_2]\not\equiv 0 \mod p^2, \quad \mathrm{ord}_p\,\mathrm{s}_{n+n'}[E_1]\le \mathrm{ord}_p\,\mathrm{s}_{n+n'}[E_2].$$

Above, we saw that the characteristic number $\mathrm{s}_n[E]$ depends only on the image of $\mathrm{s}_n(\eta)$ in $\mathrm{H}^*(E)$ and *not* on the Pontrjagin classes of the base W. The characteristic number $\mathrm{s}_{n,n'}[E]$ is more subtle, however. Indeed, for a bundle $\mathbf{CaP}^2\to E\xrightarrow{\pi} W$ classified as before by a map $g\colon W\to \mathrm{BF}_4$, we have

$$\begin{aligned}&\mathrm{s}_{n,n'}(\mathrm{T}E)\\ &\quad= \tilde{g}^*\,\mathrm{s}_{n,n'}(\eta)+\pi^*\,\mathrm{s}_n(\mathrm{T}W)\cdot\tilde{g}^*\,\mathrm{s}_{n'}(\eta)+\pi^*\,\mathrm{s}_{n'}(\mathrm{T}W)\cdot\tilde{g}^*\,\mathrm{s}_n(\eta)+\pi^*\,\mathrm{s}_{n,n'}(\mathrm{T}W).\end{aligned}$$

[1] "Wolstenholme… he was despondent and dissatisfied and consoled himself with mathematics and opium" — Sir Leslie Stephen, Virginia Woolf's father.

Applying the $\mathrm{H}^*(W)$–module homomorphism $\mathrm{B}\pi_*$ (which decreases degrees by 16) gives

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E) = g^*\,\mathrm{Bi}_*\,\mathrm{s}_{n,n'}(\eta) + \mathrm{s}_n(\mathrm{T}W)\cdot g^*\,\mathrm{Bi}_*\,\mathrm{s}_{n'}(\eta) + \mathrm{s}_{n'}(\mathrm{T}W)\cdot g^*\,\mathrm{Bi}_*\,\mathrm{s}_n(\eta).$$

To compute the last two terms, note that the 2^{nd} exact sequence of vector bundles in (4-1) implies that:

$$\begin{aligned}
\mathrm{s}_n(\mathrm{T}W) &= \mathrm{s}_n\big(\mathrm{T}V^m(d_1,\dots,d_r)\times \mathrm{T}V^{m'}(d'_1,\dots,d'_{r'})\big)\\
&= i^*\Big(\mathrm{s}_n(\mathbf{CP}^{m+r}) - \sum_j \mathrm{s}_n\,\mathrm{O}(d_j)\Big) + i'^*\Big(\mathrm{s}_n(\mathbf{CP}^{m'+r'}) - \sum_{j'} \mathrm{s}_n\,\mathrm{O}(d'_{j'})\Big)\\
&= (i\times i')^*\Big[\Big(m+r+1-\sum_j d_j^{2n}\Big)x_1^{2n} + \Big(m'+r'+1-\sum_{j'}(d'_{j'})^{2n}\Big)x_2^{2n}\Big]
\end{aligned}$$

Let E_1 be the $\mathbf{CaP}^2$ bundle obtained by taking $(m,m') = (2n-2, 2n'-6) = (p^j - 3, p^i-5)$ in the construction of E above. Then for dimension reasons, $\mathrm{s}_n(\mathrm{T}V^m) = \mathrm{s}_n(\mathrm{T}V^{m'}) = \mathrm{s}_{n'}(\mathrm{T}V^{m'}) = 0$, and by Proposition 10:

$$\begin{aligned}
&\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_1)\\
&= g^*\,\mathrm{Bi}_*\,\mathrm{s}_{n,n'}(\eta) + \Big(m+r+1-\sum_j d_j^{2n'}\Big)\cdot\Big[\binom{p^j-1}{2} - \binom{p^j-1}{p^j-p^i}\Big]\cdot(i\times i')^* x_1^m x_2^{m'}
\end{aligned}$$

Corollary 15(1) below shows that $g^*\,\mathrm{Bi}_*\,\mathrm{s}_{n,n'}(\eta)\equiv 0 \bmod p^2$ and Granville's theorem can be used to show that both binomial coefficients are congruent to $1 \bmod p^2$, so

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}[E_1] \equiv 0 \mod p^2.$$

Let E_2 be the $\mathbf{CaP}^2$ bundle obtained by taking $(m,m') = (p^{j-1}-3, p^j - p^{j-1} + p^i - 5)$ in the construction of E above. Then for dimension reasons, $\mathrm{s}_n(\mathrm{T}V^m) = \mathrm{s}_n(\mathrm{T}V^{m'}) = 0$. If $i = j-1$ then $\mathrm{s}_{n'}(\mathrm{T}V^m) = 0$ as well. So by Proposition 10,

$$\begin{aligned}
&\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_2)\\
&\quad = g^*\,\mathrm{Bi}_*\,\mathrm{s}_{n,n'}(\eta)\\
&\qquad + \Big(m+r+1-\sum_j d_j^{p^i+1}\Big)\cdot\Big[\binom{p^j-1}{2} - \binom{p^j-1}{p^{j-1}-p^i}\Big]\cdot(1-\delta_{i=j-1})\\
&\qquad + \Big(m'+r'+1-\sum_{j'}(d'_{j'})^{p^i+1}\Big)\cdot\Big[\binom{p^j-1}{2} - \binom{p^j-1}{p^{j-1}+1}\Big]\cdot(i\times i')^* x_1^m x_2^{m'}.
\end{aligned}$$

(Here δ_P equals 1 if P is true and equals 0 otherwise.) Granville's theorem can be used to show that the first three binomial coefficients are congruent to $1 \bmod p^2$ while the last is congruent to $1-p \mod p^2$, so

$$\begin{aligned}&\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_2)\\&\quad\equiv g^*\,\mathrm{B}i_*\,\mathrm{s}_{n,n'}(\eta)+\Big(m'+r'+1-\sum_{j'}(d'_{j'})^{p^i+1}\Big)\cdot p\cdot(i\times i')^*x_1^m x_2^{m'}\mod p^2.\end{aligned}$$

By Fermat's little theorem,

$$\Big(m'+r'+1-\sum_{j'}(d'_{j'})^{p^i+1}\Big)\equiv\Big(m'+r'+1-\sum_{j'}(d'_{j'})^2\Big)\mod p.$$

Recall that the degrees $(d'_1,\dots,d'_{r'})$ are chosen (say using Lemma 8) to make the latter quantity equal $-4n_f$ (since this makes $\mathrm{p}_1(\mathrm{T}E_2)=0$). So the particular degrees chosen are irrelevant here, and

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_2)\equiv g^*\,\mathrm{B}i_*\,\mathrm{s}_{n,n'}(\eta)-4n_f\cdot p\cdot(i\times i')^*x_1^m x_2^{m'}\mod p^2.$$

By Corollary 15(2) below, $g^*\,\mathrm{B}i_*\,\mathrm{s}_{n,n'}(\eta)\equiv 8p\cdot n_f^{m+m'}\cdot(i\times i')^*x_1^m x_2^{m'}\mod p^2$, so

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_2)\equiv(8n_f^{p^i+p^j-8}-4n_f)\cdot p\cdot(i\times i')^*x_1^m x_2^{m'}\mod p^2.$$

By Fermat's little theorem,

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}(\mathrm{T}E_2)\equiv 4n_f(2n_f^{-7}-1)\cdot p\cdot(i\times i')^*x_1^m x_2^{m'}\mod p^2.$$

Since W is a product of complete intersections, $(i\times i')^*x_1^m x_2^{m'}$ equals $\big(\prod_j d_j\big)\big(\prod_{j'}d'_{j'}\big)$ times the fundamental class $[W]$, and the degrees are all chosen to be nonzero mod p. Determining the roots of the polynomial $n_f^7-2 \mod p$ is a delicate task, but certainly if $n_f\equiv 1\mod p$, then

$$\mathrm{B}\pi_*\,\mathrm{s}_{n,n'}[E_2]\not\equiv 0\mod p^2.$$

Lemma 13 $$\mathrm{ord}_p\,\mathrm{s}_{n+n'}[E_1]\le\mathrm{ord}_p\,\mathrm{s}_{n+n'}[E_2]$$

Proof Assuming, as we did above, that $n_f\equiv 1\mod p$, it suffices by Proposition 10 to show that

$$\mathrm{ord}_p\left[\binom{p^i+p^j}{2}-\binom{p^i+p^j}{p^{j-1}+1}\right]\le\mathrm{ord}_p\left[\binom{p^i+p^j}{2}-\binom{p^i+p^j}{p^j+3}\right].$$

By Kummer's theorem,

$$\mathrm{ord}_p\binom{p^i+p^j}{2}=i,\quad\mathrm{ord}_p\binom{p^i+p^j}{p^{j-1}+1}=i+1,\quad\mathrm{ord}_p\binom{p^i+p^j}{p^j+3}=i.$$

So the difference of the 1st and 2nd binomial coefficients has order i while the difference of the 1st and 3rd binomial coefficients has order at least i (in fact it has order $i+2$, as can be shown using Granville's theorem). $\square$

The method used to prove Proposition 10 can be used to establish the following formula (which holds for any integers $n > n'$, not just the integers we are concerned with here).

Proposition 14

$$
\begin{aligned}
&f^* \operatorname{Bi}_* \mathrm{s}_{n,n'}(\eta) \\
&\quad = -4n_f^{n+n'-8} \sum_{k=2}^{n+n'-1} \Biggl[\binom{2n}{2k} + \binom{2n'}{2k} + \binom{2n'}{2k-2n} + \binom{2n}{2k-2n'} \\
&\qquad\qquad + \tfrac{1}{2} \sum_{l=0}^{k} (-1)^l \binom{2n'}{l} \binom{2n-2n'}{2k-2l} \\
&\qquad\qquad - \binom{2n}{2} \sum_{l=1}^{n-1} \binom{2n'}{2k-2l} - \binom{2n'}{2} \sum_{l=1}^{n'-1} \binom{2n}{2k-2l} \\
&\qquad\qquad - \binom{2n'}{2} (1 - \delta_{n' \le k \le n}) - \binom{2n}{2} (1 + \delta_{n'+1 \le k \le n-1}) \\
&\qquad\qquad + \frac{1}{2} \binom{2n+2n'}{2} - 3\delta_{k \in \{n,n'\}} \Biggr] x_1^{2k-4} x_2^{2n+2n'-2k-4}
\end{aligned}
$$

where δ_P equals 1 if P is true and equals 0 otherwise.

Corollary 15 (1) *If $(m,m') = (2n-2, 2n'-6) = (p^j-3, p^i-5)$ then the coefficient of $x_1^m x_2^{m'}$ in $f^* \operatorname{Bi}_* \mathrm{s}_{n,n'}(\eta)$ is congruent to 0 mod p^2.*

(2) *If $(m,m') = (p^{j-1}-3, p^j - p^{j-1} + p^i - 5)$ then the coefficient of $x_1^m x_2^{m'}$ in $f^* \operatorname{Bi}_* \mathrm{s}_{n,n'}(\eta)$ is congruent to $8p \cdot n_f^{m+m'}$ mod p^2.*

Proof Corollary 15(1) If $(m,m') = (2n-2, 2n'-6)$, then the coefficient of $x_1^m x_2^{m'}$ is the $k = n+1$ summand in Proposition 14. It is not difficult to show that this summand is congruent mod p^2 to

$$4n_f^{(p^j+p^i)/2-8}\bigl[0 + 0 + \tfrac{1}{2}p^i + 1 + A - (2^{p^i} - 1 - \tfrac{1}{2}p^i) + \tfrac{1}{4}p^i - \tfrac{1}{2}p^i - 1 - \tfrac{1}{4}p^i - 0\bigr],$$

where

$$A = \frac{1}{2} \sum_{l=0}^{(p^j+1)/2} (-1)^l \binom{p^i+1}{l} \binom{p^j - p^i - 2}{p^j - 2l + 1}.$$

Due to tidy pairwise cancellations, *all that remains is to show that* $A \equiv 2^p - 1 - \frac{1}{2}p^i$ *mod* p^2. (Note that $n^{p^2-p} \equiv 1 \mod p^2$ for any integer $n \not\equiv 0 \mod p$ since the multiplicative group $(\mathbf{Z}/p^2)^\times$ has order $p^2 - p$; it follows by induction that $n^{p^i} \equiv n^p$ mod p^2 for any $i > 0$.)

(a) If $i > 1$ then Granville's theorem can be used to show that

$$A \equiv \sum_{r=0}^{(p-1)/2} (-1)^r \binom{p}{r} \mod p^2.$$

(The key is that

$$\binom{p^i+1}{l} \equiv \begin{cases} \binom{p+1}{l} & \text{if } i = 1, \\ \binom{p}{r} & \begin{array}{l}\text{if } i > 1 \text{ and } l = rp^{i-1} \text{ or} \\ l = rp^{i-1}+1 \text{ with } 0 \le r \le p,\end{array} \\ 0 & \text{otherwise,} \end{cases}$$

mod p^2.) By the identity $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ (proved inductively using Pascal's rule),

$$A \equiv (-1)^{(p-1)/2}\binom{p-1}{(p-1)/2} \mod p^2.$$

By the eponymous congruence of Morley's ingenious 1895 paper [23], $A \equiv 2^{2(p-1)}$ mod p^2. The final step is to show that $2^{2(p-1)} \equiv 2^p - 1 \mod p^2$. Write

$$2^{2(p-1)} = (2^{p-1}+1)(2^{p-1}-1)+1.$$

By Fermat's little theorem, the two factors are congruent to 2 and 0 mod p respectively, so

$$\begin{aligned} A &\equiv 2(2^{p-1}-1)+1 \mod p^2 \\ &= 2^p - 1. \end{aligned}$$

(b) If $i = 1$, then Granville's theorem can be used to show that

$$A \equiv p + \frac{1}{2}\sum_{l=0}^{(p-1)/2}(-1)^l\binom{p+1}{l}\binom{p-2}{2l-1} \mod p^2.$$

Since the 1st binomial coefficient is congruent to 0 mod p for $1 < l < p$, we can simplify the 2nd binomial coefficient mod p via the congruence

$$(1+x)^{p-2} \equiv (1+x^p)(1+x)^{-2} = (1+x^p)\sum_{k=0}^{\infty}(-1)^k(k+1)x^k \mod p$$

and, subtracting a correction factor, obtain

$$A \equiv \frac{1}{2}p - \sum_{l=0}^{(p-1)/2} (-1)^l \binom{p+1}{l} \cdot l \mod p^2.$$

By the identity $\sum_{j=0}^{k}(-1)^j\binom{n}{j}j = (-1)^k\binom{n-2}{k-1}n$ (proved by writing $\binom{n}{j} = \binom{n-1}{j-1}\frac{n}{j}$ and then applying the earlier-cited identity $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$), and by the identity $\binom{n-2}{k-1} = \binom{n-2}{k}\frac{k}{n-k-1}$,

$$A \equiv \tfrac{1}{2}p - (-1)^{(p-1)/2}\binom{p-1}{(p-1)/2} \cdot \frac{p^2-1}{p+1} \mod p^2.$$

By Morley's congruence, we get $A \equiv \frac{1}{2}p + 2^{2(p-1)}(1-p) \mod p^2$. And again since $2^{2(p-1)} \equiv 2^p - 1 \mod p^2$, $A \equiv 2^p - 1 - \frac{1}{2}p \mod p^2$. □

Proof Corollary 15(2) If $(m,m') = (p^{j-1}-3, p^j - p^{j-1} + p^i - 5)$, then the coefficient of $x_1^m x_2^{m'}$ is the $k = \frac{1}{2}(p^{j-1}+1)$ summand in Proposition 14. It is not difficult to show that this summand is congruent mod p^2 to

$$4n_f^{(p^j+p^i)/2-8}\big[(1-p) + \delta_{i=j-1} + 0 + 1 + B - (2^p - \delta_{i=j-1}) - (-\tfrac{1}{4}p^i) \\ - 0 - (2 - \delta_{i=j-1}) - \tfrac{1}{4}p^i - 3\delta_{i=j-1}\big],$$

where

$$B = \frac{1}{2}\sum_{l=0}^{(p^{j-1}+1)/2} (-1)^l \binom{p^i+1}{l}\binom{p^j-p^i-2}{p^{j-1}-2l+1}.$$

Due to tidy cancellations, *all that remains is to show that* $B \equiv 2^p - p$ *mod* p^2.

(a) If $i > 1$, then the above stated fact about $\binom{p^i+1}{l}$ can be used to show that

$$B \equiv \frac{1}{2}\sum_{r=0}^{p}(-1)^r\binom{p}{r}\binom{p^j-p^i-2}{p^{j-1}-2rp^{i-1}+1} \mod p^2.$$

The 1st binomial coefficient is congruent to 0 mod p for $0 < r < p$. The 2nd binomial coefficient is congruent to 0 mod p if $0 < r < \frac{1}{2}(p+1)$ and congruent to -2 mod p if $\frac{1}{2}(p+1) \le r < p$. So

$$B \equiv \frac{1}{2}\binom{p^j-p^i-2}{p^{j-1}+1} - \sum_{r=(p+1)/2}^{p-1}(-1)^r\binom{p}{r} - \frac{1}{2}\binom{p^j-p^i-2}{p^{j-1}-2p^i+1} \mod p^2.$$

Granville's and Wolstenholme's theorems can be used to simplify the first and last terms mod p^2 while the identity $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ can be used to simplify the summation, yielding

$$B \equiv (1+\delta_{i=j-1} - p\delta_{i\neq j-2}) - 1 \\ + (-1)^{(p-1)/2}\binom{p-1}{(p-1)/2} - (\delta_{i=j-1} + p\delta_{i=j-2} - 1) \mod p^2.$$

By Morley's congruence, $B \equiv 2^{2(p-1)} + 1 - p \mod p^2$. And again since $2^{2(p-1)} \equiv 2^p - 1 \mod p^2$, $B \equiv 2^p - p \mod p^2$.

(b) If $i = 1$, then

$$B = \frac{1}{2}\sum_{l=0}^{p+1}(-1)^l\binom{p+1}{l}\binom{p^j - p - 2}{p^{j-1} - 2l + 1}.$$

Granville's theorem can be used to show that

$$B \equiv \frac{1}{2}(p+2) + \frac{1}{2}\sum_{l=0}^{(p-1)/2}(-1)^l\binom{p+1}{l}\binom{p-2}{2l-1} \mod p^2.$$

This summation appeared above in the proof of Corollary 15(1), part (b). In fact, $B \equiv A + 1 - \frac{1}{2}p \mod p^2$. Since we concluded that $A \equiv 2^p - 1 - \frac{1}{2}p^i \mod p^2$, it follows that

$$B \equiv 2^p - p \mod p^2. \qquad \square$$

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