msp
Geometry & Topology 18 (2014) 2189–2243

H–spaces, loop spaces and the space of positive scalar curvature metrics on the sphere

MARK WALSH

For dimensions $n \geq 3$, we show that the space $\mathcal{R}\mathrm{iem}^+(S^n)$ of metrics of positive scalar curvature on the sphere S^n is homotopy equivalent to a subspace of itself which takes the form of an H–space with a homotopy commutative, homotopy associative product operation. This product operation is based on the connected sum construction. We then exhibit an action on this subspace of the operad obtained by applying the bar construction to the little n–disks operad. Using results of Boardman, Vogt and May we show that this implies, when $n \geq 3$, that the path component of $\mathcal{R}\mathrm{iem}^+(S^n)$ containing the round metric is weakly homotopy equivalent to an n–fold loop space. Furthermore, we show that when $n = 3$ or $n \geq 5$, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is weakly homotopy equivalent to an n–fold loop space provided a conjecture of Botvinnik concerning positive scalar curvature concordance is resolved in the affirmative.

53C99; 55S99

1 Introduction

This work is motivated by the problem of understanding the topology of the space of metrics of positive scalar curvature (*psc-metrics*) on the sphere S^n. This space is denoted $\mathcal{R}\mathrm{iem}^+(S^n)$ and is an open subspace of the space $\mathcal{R}\mathrm{iem}(S^n)$ of all Riemannian metrics on S^n, equipped with its standard smooth topology. It is known that when $n = 2$, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is contractible; see Rosenberg and Stolz [15]. When $n = 3$, we know from a recent result of Marques that this space is path connected; see [11]. In fact it is thought by experts that the space is contractible in this case also. When $n \geq 4$ however, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is usually not path connected; see for example Carr [3]. Furthermore for $k \geq 1$, the groups $\pi_k(\mathcal{R}\mathrm{iem}^+(S^n))$ are often non-trivial; see Hitchin [9], Crowley and Schick [4], and Hanke, Schick and Steimle [7]. In this paper we make the following contribution.

Main results (i) *When $n \geq 3$, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is homotopy equivalent to a subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$ that admits a homotopy product (ie is an H–space). Furthermore this product is homotopy commutative and homotopy associative.*

Published: 2 October 2014 DOI: 10.2140/gt.2014.18.2189

(ii) *When $n \geq 3$, the path component of the space $\mathcal{R}\mathrm{iem}^+(S^n)$ containing the round metric is weakly homotopy equivalent to an n–fold loop space. As $\mathcal{R}\mathrm{iem}^+(S^n)$ is path connected when $n = 3$, this implies that $\mathcal{R}\mathrm{iem}^+(S^3)$ is weakly homotopy equivalent to a 3–fold loop space.*

(iii) *When $n \geq 5$, the problem of whether or not the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is weakly homotopy equivalent to an n–fold loop space depends only on the problem of whether or not psc-concordant metrics on S^n are psc-isotopic.*

Definitions of the terms H–space and loop space are given in Section 2. We will not discuss the topological implications of such structure on $\mathcal{R}\mathrm{iem}^+(S^n)$ other than to point out that the condition that a topological space is an H–space, or especially an iterated loop space, imposes significant restrictions on its homotopy type. For more on this, see Markl, Shnider and Stasheff [10, Chapter 4].

The main idea is as follows. We specify certain subspaces of $\mathcal{R}\mathrm{iem}^+(S^n)$ consisting of psc-metrics that take a "standard form" near a fixed base point $p_0 \in S^n$. It is known from results in Walsh [20] that these subspaces are all homotopy equivalent to the space $\mathcal{R}\mathrm{iem}^+(S^n)$. We then construct products on these spaces, in the case where $n \geq 3$, based on the Gromov–Lawson connected sum construction in [6]. Roughly speaking, these products involve removing standard "caps" around the point p_0 of the factor metrics and then taking a connected sum via some appropriate connecting metric. There is one important caveat. We need to ensure that the metric obtained by this product has a base point, something which the individual factors lose once we remove the standard caps. Hence we use, as an intermediary metric, a psc-metric on the sphere containing 3 such standard caps, two of which will be removed for the attachments and the third which will be a base point cap. In all cases, we will show that this determines a homotopy product (ie makes the subspace an H–space) that is homotopy commutative and homotopy associative.

We then focus on one such subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$: the space of psc-metrics that take the form of a round hemisphere of radius 1 near a fixed base point p_0, denoted $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$. On this space, we show that the homotopy product generalises to an action of a certain operad. This operad is obtained via a process called the bar construction, from the *operad of little n–dimensional disks*; see below for a description of this object and the bar construction. It follows from results of Boardman, Vogt and May that a path connected space Z that admits such an action is weakly homotopy equivalent to an n–fold loop space. This allows us to conclude that the path component of $\mathcal{R}\mathrm{iem}^+(S^n)$ that contains the round metric is weakly homotopy equivalent to an n–fold loop space. When $n = 3$, $\mathcal{R}\mathrm{iem}^+(S^n)$ is path connected and so the conclusion holds for the entire space. In the case when the space Z is not path connected, we may still conclude that Z is weakly homotopy equivalent to an n–fold loop space provided Z satisfies the condition of being group-like with respect to the operad action. Roughly, this means that multiplication induced on Z by appropriate restriction of the operad action gives rise to a group structure on $\pi_0(Z)$. At this stage we have reduced the question of whether or not the entire space $\mathcal{R}\mathrm{iem}^+(S^n)$ is weakly homotopic to an n–fold loop space to the question of whether or not $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ is group-like. In Section 9 we show that $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ is indeed group-like provided a certain conjecture of Botvinnik is resolved in the affirmative.

1.1 Organisation of the paper

The paper is organised as follows. After recalling the definitions of an H–space and a loop space in Section 2, we proceed in Section 3 to describe two types of psc-metric on the disk that are well-behaved near the boundary. Roughly, these are metrics that are either cylindrical or sphere-like near the boundary and may be appropriately combined to obtain psc-metrics on the sphere. In later sections we make use of this in specifying product structures on certain subspaces of psc-metrics on the sphere that are standard near a fixed base point. In particular, in Section 4, we consider the most elementary of these subspaces: the space of psc-metrics on S^n that have a "torpedo cap" around some fixed base point $p_0 \in S^n$. We denote this space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$. After specifying a multiplication map on this space, we review the Gromov–Lawson connected sum construction as generalised in Walsh [19] for compact families of psc-metrics. In particular, we show in Lemma 4.7 that the subspace of psc-metrics on S^n that have torpedo caps at a fixed base point is actually homotopy equivalent, when $n \geq 3$, to the space of all psc-metrics on S^n. In Section 5 we prove the first of our main results: the H–space theorem. This is Theorem 5.1, where we show that the multiplication map discussed above gives the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$ the structure of an H–space. We also show that the product is both homotopy commutative and homotopy associative. Thus, $\mathcal{R}\mathrm{iem}^+(S^n)$ is homotopy equivalent to an H–space when $n \geq 3$. One consequence of this, Corollary 5.2, is that the fundamental group of $\mathcal{R}\mathrm{iem}^+(S^n)$, with base point the standard round metric, is abelian.

In Section 6, we describe slightly more sophisticated subspaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0)}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0)}(S^n)$ of $\mathcal{R}\mathrm{iem}^+(S^n)$, consisting of psc-metrics that have the form of "bulbs" and "heads" near a base point $p_0 \in S^n$. These spaces are shown also to be homotopy equivalent to $\mathcal{R}\mathrm{iem}^+(S^n)$ in Lemma 6.6. We then define multiplication maps, analogous to the one above, which give these spaces an H–space structure. In the case of $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0)}(S^n)$, there is a deformation retract down to the subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$, of psc-metrics that take the form of a round hemisphere of radius 1 near p_0. On this space, we will show that the homotopy product generalises nicely to a certain operad action. Before doing this, we spend some time in Section 7 reviewing the various operads we will require. In particular, we consider the operad of little n–dimensional disks $\mathcal{D}_n$ (as well as a variant of this operad on the round hemisphere) and the bar construction for operads. We recall relevant results of Boardman, Vogt and May: Theorems 7.3 and 7.4. These results allow us to conclude that the existence of an appropriate action of the operad $W\mathcal{D}_n$, obtained from $\mathcal{D}_n$ via the bar construction, on a group-like space Z implies that Z is weakly homotopy equivalent to an n–fold loop space. In Section 8 we exhibit, for $n \geq 3$, precisely such an action on the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$. This is Lemma 8.2. Finally, in Section 9, we reduce the problem of determining whether or not the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$ is indeed group-like in the case when $n \geq 5$ to a conjecture of Botvinnik on psc-concordance; see Lemma 9.4. This allows us to conclude the main result, Theorem 9.6.

Acknowledgements

I would like to thank Boris Botvinnik at the University of Oregon for suggesting this problem, Kirk Lancaster and Philip Parker at Wichita State University, David Wraith at NUI Maynooth, Ireland, and especially Victor Turchin at Kansas State University, for some helpful conversations.

2 *H*–spaces and loop spaces

A topological space Z is an *H–space* if Z is equipped with a continuous multiplication map $\mu\colon Z \times Z \to Z$ and an identity element $e \in Z$ so that the maps from Z to Z given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are both homotopy equivalent to the identity map $x \mapsto x$. There are stronger versions of this definition where the above homotopies to the identity map are required to be homotopic through pointed maps $(Z, e) \to (Z, e)$ or where multiplication by the identity is the identity map. It is well known that in the case when Z is homotopy equivalent to a CW–complex, Z admits a product that agrees with one of these definitions if and only if it admits products agreeing with the other two; see Hatcher [8, Chapter 3.C]. Moreover, it follows from the work of Palais in [13] that for any smooth compact manifold X, $\mathcal{R}\mathrm{iem}^+(X)$ is homotopy equivalent to a CW–complex. Thus, we feel justified in using the weaker definition. An H–space Z is said to be homotopy commutative if the maps μ and $\mu \circ \omega$, where $\omega\colon Z \times Z \to Z \times Z$ is the "flip" map defined by $\omega(x, y) = (y, x)$, are homotopy equivalent. Finally, Z is a homotopy associative H–space if the maps from $Z \times Z \times Z$ to Z given by $(x, y, z) \mapsto \mu(\mu(x, y), z)$ and $(x, y, z) \mapsto \mu(x, \mu(y, z))$ are homotopy equivalent.

Given a topological space Z with a prescribed base point $z_0 \in Z$, we may consider the space of all loops based at z_0. This is the space of all continuous maps $\gamma\colon [0,1] \to Z$ so that $\gamma(0) = \gamma(1) = z_0$. This space is known as the *loop space of* Z, denoted $\Omega(Z, z_0)$, with base point the constant loop at z_0. Assuming the base point to be understood, we simply write ΩZ. Repeated application of this construction yields the k^{th} iterated loop space $\Omega^k Z = \Omega(\Omega \cdots (\Omega Z))$ where at each stage the new base point is simply the constant loop at the old base point. We close by pointing out that a loop space is also an H–space with the multiplication determined by concatenation of loops. Whether or not a given H–space has the structure of a loop space is a more complicated problem concerning certain "coherence" conditions on the homotopy associativity of the multiplication. A space that satisfies these conditions is known as an A_∞–space. A theorem of Stasheff says that a space is an A_∞–space if and only if it is a loop space; see [10, Theorem 4.18]. The notion of an *operad* was constructed to more efficiently describe these coherence conditions, something we will return to in Section 7.

3 Metrics on the disk and sphere

For a smooth n–dimensional manifold M, possibly with non-empty boundary, we denote by $\mathcal{R}\mathrm{iem}(M)$ the space of Riemannian metrics on M equipped with its standard C^∞–topology; see [19, Section 1.1] for a description. Contained inside $\mathcal{R}\mathrm{iem}(M)$ as an open subspace is the space of psc-metrics on M, denoted $\mathcal{R}\mathrm{iem}^+(M)$. A path in this space is known as a *psc-isotopy* while metrics that lie in the same path component are said to be *psc-isotopic*. In the case when $\partial M \neq \varnothing$, it is common to consider only a subspace of $\mathcal{R}\mathrm{iem}^+(M)$ of metrics that satisfy some constraint near the boundary. We will need such a constraint in this paper also and will return to this issue shortly.

We will mostly focus on the case when M is either D^n or S^n, the standard smooth disk or sphere of dimension n. Usually n is assumed to be at least three. We denote by ds_n^2 the standard round metric of radius 1 on S^n. As smooth topological objects, we model the disk $D^n = D^n(1)$ as the set of points $\{x \in \mathbb{R}^n : |x| \le 1\}$ and the sphere $S^n = S^n(1)$ as the set $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. In constructing metrics on these spaces, we will often work on an underlying disk or sphere $D^n(r)$ or $S^n(r)$, where the radius $r \neq 1$. The re-scaling function $x \mapsto rx$ gives a canonical way of pulling back metrics to the standard disk or sphere. Thus, we will often declare a metric that has been constructed on a general $D^n(r)$ or $S^n(r)$ to be a metric on D^n or S^n, assuming the metric to be pulled back in this way. Finally, we respectively denote by D^n_- and D^n_+ the spaces $\{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \le 0\}$ and $\{x \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \ge 0\}$, ie the southern and northern hemispheres of S^n. These hemispheres will be identified with the disk D^n via the obvious map that sends geodesic rays emanating from the points $(0,0,\ldots,0,\pm 1)$ to the corresponding ray on the flat disk $D^n(\frac{\pi}{2})$ followed by the above rescaling map.

We now return to the question of boundary conditions on certain metrics. Our various constructions will involve attaching Riemannian disks along their boundaries to obtain new metrics on the sphere. On the smooth topological level, this involves gluing a pair disks together by identifying the boundary spheres via some diffeomorphism of S^{n-1}. In our case, the boundary sphere is canonically identified with the standard unit $(n-1)$–sphere in $\mathbb{R}^n$ and we always assume that we are gluing with the identity diffeomorphism. Of course, we need to ensure smooth attachment at the metric level. There are two ways of doing this that we will explore. The first is to work only with metrics that take the form of a round cylinder near, or at least infinitesimally at, the boundary. The second is to consider metrics that, near their boundaries, agree with a geodesic ball from a round sphere (for example a hemisphere) near its boundary, at least infinitesimally; see Figure 1 for a rough depiction. We will now describe these metrics in more detail.

Figure 1: Metrics on the disk which are cylindrical (left) and spherical (right) near the boundary

3.1 Metrics that are cylindrical near the boundary

The usual method is to restrict ourselves to working with metrics that take the structure of a standard round cylinder near the boundary, at least infinitesimally. With this in mind we specify a subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n) \subset \mathcal{R}\mathrm{iem}^+(D^n)$ as follows. Beginning with the disk D^n, let $\epsilon > 0$ and consider $D^n = D^n(1)$ as a submanifold of the disk $D^n(1+\epsilon)$ of radius $1+\epsilon$. Let t denote the radial distance from the origin. Furthermore let $\mathrm{Ann}(1,1+\epsilon)$ denote the closure of the annulus $D^n(1+\epsilon) \setminus D^n(1)$. Let $\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(\epsilon)}(D^n(1+\epsilon))$ denote the space of psc-metrics on $D^n(1+\epsilon)$ defined as

$$\begin{aligned}&\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(\epsilon)}(D^n(1+\epsilon))\\ &\qquad := \{g \in \mathcal{R}\mathrm{iem}^+(D^n(1+\epsilon)) : g|_{\mathrm{Ann}(1,1+\epsilon)} = dt^2 + \delta^2 ds^2_{n-1} \text{ for some } \delta > 0\}.\end{aligned}$$

We next consider the restriction map

$$\begin{aligned}\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(\epsilon)}(D^n(1+\epsilon)) &\to \mathcal{R}\mathrm{iem}^+(D^n(1)),\\ g &\mapsto g|_{D^n(1)}.\end{aligned}$$

We then define the space $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n)$ to be the image of this restriction map.

We now consider a pair of psc-metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n)$. These metrics are very well behaved along the boundary. Indeed the only obstruction to simply gluing them together in the usual way is that the radii of their boundary spheres may not agree. There are two obvious ways we might proceed. The simplest is to rescale one or both of these metrics (multiplying the metric by an appropriate constant) so that the boundaries are compatible. Alternatively, one may wish to leave g_0 and g_1 unscathed and connect them via an appropriate warped round cylinder metric whose ends correspond to the respective boundaries. This second method seems a little cumbersome, but has the advantage that it does not require a global adjustment of g_0 or g_1. For our purposes however, the rescaling method will suffice.

We begin with an elementary fact. For any Riemannian metric g on a smooth n–dimensional manifold M, the scalar curvature R_{cg} of the metric cg obtained by multiplying g by a constant $c > 0$ is given by the formula

$$R_{cg} = \frac{1}{c} R_g.$$

Thus, rescaling a psc-metric by a positive constant results in another psc-metric. We now define the *radius measuring map* ρ as

$$\begin{aligned}(3\text{-}1) \qquad \rho\colon \mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n) &\to (0,\infty),\\ g &\mapsto \rho(g) = \text{Radius of sphere } g|_{\partial D^n}.\end{aligned}$$

Let f be any function $f\colon (0,\infty)\times(0,\infty) \to (0,\infty)$. We now replace the metrics g_0 and g_1 respectively with the metrics

$$\frac{f(\rho_0,\rho_1)^2}{{\rho_0}^2}\, g_0 \quad \text{and} \quad \frac{f(\rho_0,\rho_1)^2}{{\rho_1}^2}\, g_1,$$

where $\rho_i = \rho(g_i)$ for $i = 0, 1$. These replacement metrics are still elements of $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n)$ but with boundary radii both equal to $f(\rho_0,\rho_1)$. We attach these metrics in the obvious way to obtain a new psc-metric on S^n, which we denote $g_0 \cup_f g_1$; see Figure 2.

Thus, for each f, the construction gives rise to a continuous *joining map*, $J^{\mathrm{cyl}(f)}$, defined by

$$\begin{aligned}(3\text{-}2) \qquad J^{\mathrm{cyl}(f)}\colon \mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n) \times \mathcal{R}\mathrm{iem}^{+}_{\mathrm{cyl}(0)}(D^n) &\to \mathcal{R}\mathrm{iem}^{+}(S^n),\\ (g_0, g_1) &\mapsto g_0 \cup_f g_1.\end{aligned}$$

It is probably easiest to apply this construction when $f = \pi_{\mathrm{L}}$ or π_{R}, the projection function onto the left or right factor. Here we write J^{L} or J^{R} to mean $J^{\mathrm{cyl}(\pi_{\mathrm{L}})}$ or

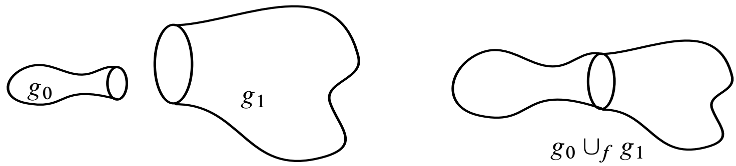

Figure 2: The sphere metric $g_0 \cup_f g_1$ (right) is formed by rescaling and gluing g_0 and g_1 (left)

$J^{\mathrm{cyl}(\pi_{\mathrm{R}})}$, respectively. Thus J^{L} fixes the size of the left input metric g_0 and rescales the right input g_1 while J^{R} fixes g_1 and rescales g_0.

3.2 Metrics that are sphere-like near the boundary

There is another approach to this problem, one which will be of great use to us later on. We begin with a round n–dimensional sphere of radius λ. We denote by $d_\lambda(a,b)$ the usual distance between points a and b on this sphere and by $B_\lambda(p,r)$ the closed geodesic ball of radius $r \in (0, \lambda\pi)$ about $p \in S^n$. We identify $B_\lambda(p,r)$ with the northern hemisphere D^n_+ of the standard unit sphere in the following way. We move p by the obvious rigid rotation of S^n along the great circle containing p and the north pole into the north pole position. We then rescale the sphere to make its radius $\lambda = 1$. The ball $B_\lambda(p,r)$ is therefore replaced by the ball $B_1(p, \frac{r}{\lambda})$. Next, we identify $B_1(p, \frac{r}{\lambda})$ with the northern hemisphere D^n_+ by moving each point $x \in B_1(p, \frac{r}{\lambda})$, along the great circle through p and x, to the point whose distance from p is $(\lambda\pi/2r)d_1(p,x)$. All of this is depicted in Figure 3. Finally, we identify the northern hemisphere D^n_+ with the disk D^n in the obvious way described at the beginning of this section. By pulling back the restriction of the round metric of radius λ on the ball $B_\lambda(p,r)$ via this composition of identifications, we obtain a metric on the disk D^n. This metric is known as the (λ,r)–*lens metric* on D^n and denoted $g^n_{\mathrm{lens}}(\lambda,r)$. Note that, context permitting, we will sometimes refer to the ball $B_\lambda(p,r)$ as the (λ,r)–*lens* at p also. Each lens $g^n_{\mathrm{lens}}(\lambda,r)$ has a *lens metric complement*, namely the metric $g^n_{\mathrm{lens}}(\lambda, \lambda\pi - r)$, which may be attached to $g^n_{\mathrm{lens}}(\lambda,r)$ in the obvious way to reconstitute the round sphere metric of radius λ.

In spherical coordinates on the disk $D^n(r)$, the metric $g^n_{\mathrm{lens}}(\lambda,r)$ takes the form

$$g^n_{\mathrm{lens}}(\lambda,r) = ds^2 + \lambda^2 \sin^2 \frac{s}{\lambda}\, ds^2_{n-1},$$

where $s \in (0,r]$ denotes the radial distance coordinate. After pulling back to D^n this metric then takes the form

$$g^n_{\mathrm{lens}}(\lambda,r) = r^2 dt^2 + \lambda^2 \sin^2 \frac{rt}{\lambda}\, ds^2_{n-1},$$

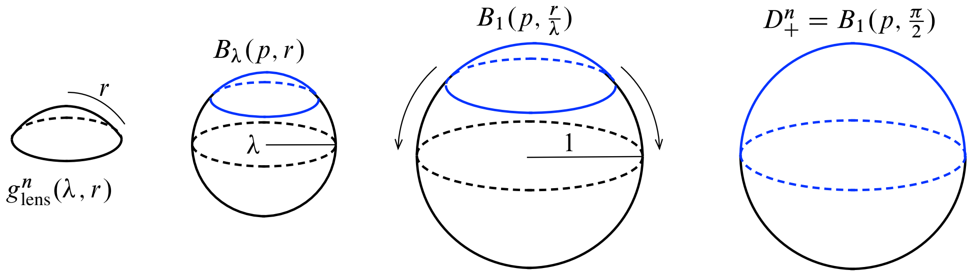

Figure 3: The metric $g^n_{\text{lens}}(\lambda, r)$ (left) and the rescaling to identify the ball $B_\lambda(p, r)$ with D^n_+ (right)

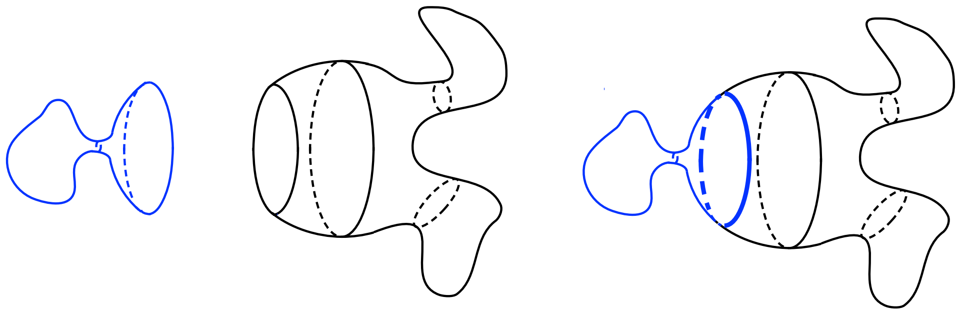

Figure 4: Combining metrics in $\mathcal{R}\mathrm{iem}^+_{\text{lens}(0)}(D^n)$ that have complementary boundaries (left) gives rise to a psc-metric on S^n (right)

where $t \in (0, 1]$ is the new radial distance coordinate. We consider the space of psc-metrics on $D^n(1+\epsilon)$ defined as follows. For each pair $\lambda > 0$, $r \in (0, \lambda\pi)$ and each $\epsilon \in (0, \lambda\pi - r)$, let $\mathcal{R}\mathrm{iem}^+_{(\lambda,r)-\text{lens}(\epsilon)}(D^n(1+\epsilon))$ denote the space

$$\mathcal{R}\mathrm{iem}^+_{(\lambda,r)-\text{lens}(\epsilon)}(D^n(1+\epsilon)) \\ := \left\{ g \in \mathcal{R}\mathrm{iem}^+(D^n(1+\epsilon)) : g|_{\mathrm{Ann}(1,1+\epsilon)} = r^2\, dt^2 + \lambda^2 \sin^2 \frac{rt}{\lambda}\, ds^2_{n-1} \right\},$$

where $t \in (1, 1+\epsilon]$ here. The outer annulus metric here is isometric to the annulus of width $r\epsilon$ which extends the ball of radius r in $S^n(\lambda)$. As before, we consider the restriction map

$$\begin{aligned} \mathcal{R}\mathrm{iem}^+_{(\lambda,r)-\text{lens}(\epsilon)}(D^n(1+\epsilon)) &\to \mathcal{R}\mathrm{iem}^+(D^n(1)), \\ g &\mapsto g|_{D^n(1)}, \end{aligned}$$

and define $\mathcal{R}\mathrm{iem}^+_{(\lambda,r)-\text{lens}(0)}(D^n)$ to be the image of this map. Finally, we define $\mathcal{R}\mathrm{iem}^+_{\text{lens}(0)}(D^n)$ to be the union of the spaces $\mathcal{R}\mathrm{iem}^+_{(\lambda,r)-\text{lens}(0)}(D^n)$ over all pairs $\lambda > 0, r \in (0, \lambda\pi)$. Recall that our original motivation was in gluing disk metrics together to obtain metrics on the sphere. In this case, it is clear that elements of

$\mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n)$ may be smoothly attached to other elements of $\mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n)$ provided their boundaries correspond to complementary lenses; see Figure 4.

It is possible to specify a map from $\mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n)$ to $\mathcal{R}\mathrm{iem}^+(S^n)$ that is analogous to the map defined in (3-2). Indeed, such a map will be very important for us. The construction in this case is more complicated and so we will postpone it until Section 6.

4 Torpedo metrics and the Gromov–Lawson construction

We now turn our attention to the problem of combining pairs of psc-metrics on the sphere S^n in order to obtain new psc-metrics, also on S^n. The best known example of this is the connected sum construction of Gromov and Lawson; see [6]. This is at the heart of our work. Essentially, for any metrics g_0 and g_1 on S^n and provided $n \geq 3$, one may use this construction to obtain a new psc-metric on S^n, which is obtained by taking a geometric connected sum $g_0 \# g_1$. We will shortly revisit this construction and so we will postpone the details until then. However, we should point out that the Gromov–Lawson construction requires we make a number of choices. Thus, it does not give rise to a well-defined binary operation on the space $\mathcal{R}\mathrm{iem}^+(S^n)$. It does in fact give rise to a multiplication on certain quotient spaces of $\mathcal{R}\mathrm{iem}^+(S^n)$, such as the spaces of psc-concordance classes or psc-isotopy classes of metrics; this is something we will return to later on. There are ways, however, of achieving such an operation without taking a quotient, provide we restrict to certain subspaces of $\mathcal{R}\mathrm{iem}^+(S^n)$. Over the next two sections, we will spend time defining some of these subspaces as well as recalling the Gromov–Lawson construction.

4.1 Warped product metrics on the disk

We begin by constructing a particular family of rotationally symmetric warped product metrics on the disk D^n. For our purposes, this is a metric on the disk which takes the form

$$g^\eta = dt^2 + \eta^2(t)\, ds_{n-1}^2, \tag{4-1}$$

where t denotes the radial distance coordinate and where for some $b > 0$, $\eta\colon [0,b) \to [0,\infty)$ is a smooth function satisfying

(i) $\eta(0) = \delta\sin(\frac{t}{\delta})$ for some $\delta > 0$, when t is near 0,

(ii) $\eta(t) > 0$ when $t > 0$.

Although technically $dt^2 + \eta^2(t)\, ds_{n-1}^2$ degenerates at $t = 0$, the radius of the sphere factor closes in at the end $\{0\} \times S^{n-1}$ in such a way as to uniquely determine a smooth Riemannian metric on the disk D^n that is rotationally symmetric with respect to the obvious action of orthogonal group $\mathrm{O}(n)$. This follows from the results of Petersen [14, Chapter 1, Section 3.4].

Remark 4.1 We will intermittently regard g^η as a metric on both $(0, \delta\frac{\pi}{2}] \times S^{n-1}$ and on D^n depending on our circumstances.

Finally, the scalar curvature R of the metric g^η at the point $(t, \theta) \in (0, \frac{\pi}{2}] \times S^{n-1}$ is given by the formula

$$R(t,\theta) = -2(n-1)\frac{\eta''(t)}{\eta(t)} + (n-1)(n-2)\frac{1-(\eta'(t))^2}{\eta(t)^2}. \tag{4-2}$$

4.2 Infinitesimal torpedo metrics on the disk

Recall that a torpedo metric on the disk D^n is an $\mathrm{O}(n)$–symmetric metric that is round near the centre of the disk but transitions to a standard round Riemannian cylinder (neck) $[0, \epsilon] \times S^{n-1}$ near the boundary. For a detailed discussion of these metrics and their variants, see [19, Chapter 1]. Roughly speaking, an infinitesimal torpedo metric takes this product structure only infinitesimally at the boundary of D^n. Importantly however, it smoothly attaches along the boundary to an end of a round cylinder. With this in mind, we fix a smooth function $\eta_1\colon [0, \frac{\pi}{2}] \to [0, \infty)$, which satisfies the following requirements:

(i) $\eta_1(t) = \sin t$ when t is near 0.

(ii) $\eta_1(\frac{\pi}{2}) = 1$.

(iii) $\eta_1{}''(t) < 0$ when $0 < t < \frac{\pi}{2}$.

(iv) $\eta_1{}_-^{(k)}(\frac{\pi}{2}) = 0$ for all $k \geq 1$.

Here $\eta_1{}_-^{(k)}$ represents the left-sided k^{th} derivative of η_1. The graph of η_1 is depicted in Figure 5.

Essentially, we want functions that behave like sine for the most part but end with all zero derivatives, as illustrated in Figure 5. More generally, we obtain a family of functions $\{\eta_\delta\}_{\delta>0}$ defined as

$$\begin{aligned} \eta_\delta\colon \left[0, \delta\frac{\pi}{2}\right] &\to [0,\infty), \\ t &\mapsto \delta\eta_1\Big(\frac{t}{\delta}\Big). \end{aligned}$$

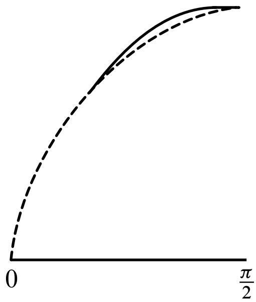

Figure 5: Comparing the graph of η_1 with the graph of the standard sin function represented by the dashed curve

Given any such function η_δ we obtain a metric $g^{\eta_\delta} = dt^2 + {\eta_\delta}^2(t)\, ds_{n-1}^2$ as described above. It is clear from formula (4-2) and the conditions on the second derivative of η_δ that the scalar curvature of g^{η_δ} is always positive. (Recall we assume $n \geq 3$. When $n = 2$ the best we can say is that $R \geq 0$.) We then obtain a space of metrics $\mathcal{T}^+_{\mathcal{R}\mathrm{iem}}$, the subspace of $\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n)$ defined by

$$\mathcal{T}^+_{\mathcal{R}\mathrm{iem}} := \{g^{\eta_\delta} \in \mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n) : \delta > 0\}, \tag{4-3}$$

where g^{η_δ} is given by formula (4-1) above on $(0, \delta\frac{\pi}{2}] \times S^{n-1}$ but of course extends uniquely onto D^n. We make one final elementary observation concerning the fact that the restriction of the radius measuring map $\rho\colon \mathcal{T}^+_{\mathcal{R}\mathrm{iem}} \to (0, \infty)$ is a bijection.

Proposition 4.1 *For any constant $c > 0$ and any $g^{\eta_\delta} \in \mathcal{T}^+_{\mathcal{R}\mathrm{iem}}$, the metric $c^2 g^{\eta_\delta}$ is exactly the element $g^{\eta_{c\delta}} \in \mathcal{T}^+_{\mathcal{R}\mathrm{iem}}$.*

Proof The element g takes the form $g = dt^2 + \eta_\delta(t)^2$ on $(0, \delta\frac{\pi}{2}] \times S^{n-1}$. Replacing t with $\frac{s}{c}$, we see that $c^2 g = ds^2 + \eta_{c\delta}(s)^2\, ds_{n-1}^2$ on $(0, c\delta\frac{\pi}{2}] \times S^{n-1}$. □

4.3 A space of psc-metrics with torpedos

We now describe a subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$ on which the construction described in Section 3.1 yields a product. When studying spaces of metrics on a manifold, one often fixes a particular metric, called a reference metric, to be used to unambiguously specify coordinate balls or an exponential map. Although in principle the choice of reference metric does not matter, it is convenient in our case to use the standard round metric of radius 1, ds_n^2, as a reference metric on S^n. Let p be a point in S^n. A choice of orthonormal (with respect to ds_n^2) basis for the tangent space T_pS^n to S^n at p gives rise to an isomorphism from $\mathbb{R}^n$ to T_pS^n. Composing this with the exponential map gives rise to a smooth pointed map $(\mathbb{R}^n, 0) \to (S^n, p)$, which restricts to an embedding on small disks around $0 \in \mathbb{R}^n$. By precomposing with an appropriate rescaling, we obtain an embedding of the standard unit disk D^n into S^n. We will call this embedding ϕ_p and let D_p denote its image in S^n. Finally, let p' denote the antipodal point to p on S^n, let $D'_p = \mathrm{closure}(S^n \setminus D_p)$ and let $\phi'_p\colon D^n \to D'_p$ denote the corresponding complementary embedding obtained from an appropriate restriction of the exponential map at p'. We now define a subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$, which we denote $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$, as

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n) = \{g \in \mathcal{R}\mathrm{iem}^+(S^n) : \phi_p^*(g|_{D_p}) \in \mathcal{T}^+_{\mathcal{R}\mathrm{iem}}\}, \tag{4-4}$$

where, recall, $\mathcal{T}^+_{\mathcal{R}\mathrm{iem}}$ is the space of infinitesimal torpedo metrics on D^n defined in (4-3). This is known as the space of *psc-metrics with a torpedo at* p. Thus, each element of $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ is a metric which has an infinitesimal torpedo-like "cap" at the point p. Furthermore, there is an "uncapping" map Unc_p, which removes this torpedo cap around p by restricting such metrics to the complementary disk D'_p (and then pulling back to the standard D^n). This map is defined as

$$\begin{aligned} \mathrm{Unc}_p\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n) &\to \mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n), \\ g &\mapsto (\phi'_p)^*(g|_{D'_p}). \end{aligned} \tag{4-5}$$

Thus, Unc_p sends certain psc-metrics on S^n to psc-metrics on D^n with (infinitesimal) cylindrical boundaries.

Naively, in constructing a product on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$, one might consider taking two psc-metrics with caps, removing the caps and then gluing them together after some appropriate rescaling. Using the joining map $J^{\mathrm{cyl}(f)}$ from (3-2) for some rescaling function $f\colon (0,\infty)\times(0,\infty) \to (0,\infty)$, and the uncapping map Unc_p defined above in (4-5), the composition map $J^{\mathrm{cyl}(f)} \circ (\mathrm{Unc}_p \oplus \mathrm{Unc}_p)$ does precisely this. Unfortunately this produces a metric on a sphere with no base point (and thus no torpedo cap) and so a slightly more intricate multiplication is required. Before describing the more intricate construction, it is worth describing a version of this naive construction as it gets us most of the way there.

We will begin with a slight generalisation of the idea of a psc-metric with a torpedo cap. Suppose $\boldsymbol{p} = \{p_0, p_1, \dots, p_k\} \subset S^n$ is a finite collection of points on S^n. We may specify, around each of these points, closed disjoint normal coordinate neighbourhoods $D_{p_0}, \dots, D_{p_k}$ of the type described above, with corresponding diffeomorphisms $\phi_{p_i}\colon D^n \to D_{p_i}$ and complementary diffeomorphisms $\phi'_{p_i}\colon D^n \to D'_{p_i}$ for each $i \in \{0, 1, \ldots, k\}$. We now define the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$ as

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n) = \bigcap_{i=0}^{k} \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_i)}(S^n).$$

In Figure 6, we represent an element of $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$, where $\boldsymbol{p} = \{p_0, p_1, p_2, p_3\}$ is a set of four distinct points on S^n.

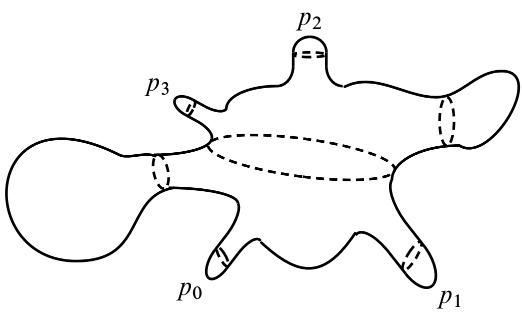

Figure 6: A sphere with four torpedo caps

We will now make a couple of technical observations about the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$. Firstly, the choice of orthonormal basis for each tangent space $T_{p_i}S^n$ is unimportant.

Lemma 4.2 *For $n \geq 3$, the subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n) \subset \mathcal{R}\mathrm{iem}^+(S^n)$ remains fixed if we vary the choice of orthonormal basis for $T_{p_i}S^n$ for each $p_i \in \boldsymbol{p}$.*

Proof This essentially follows from the rotational symmetry of the caps. For a detailed proof, see Walsh [21]. □

For a single point p, the topology of $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ is also unaffected by the choice of p. In particular we have the following lemma.

Lemma 4.3 *For $n \geq 3$ and for any $p, q \in S^n$, the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(q)}(S^n)$ are homeomorphic.*

Proof Let $\mathrm{ro}_{(p,q)}$ denote the rotation from p to q along the great circle (with respect to the standard round metric) containing p and q. Then the map which sends a metric g on S^n to the pull-back metric $\mathrm{ro}^*_{(p,q)}g$ defines a homeomorphism from $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(q)}(S^n)$ to $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$. □

Remaining a little longer with the case when $\boldsymbol{p}$ is a single point p in S^n, it is worth pointing out that this space may be simplified somewhat by considering only torpedo caps of a fixed radius. For any $\delta > 0$, let $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n)$ be the subspace of $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ that is defined as

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n) := \{g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)\} : \rho(g|_{D_p}) = \delta\}.$$

Equivalently, this is the space of psc-metrics on S^n so that $\phi_p^*(g|_{D_p}) = g^{\eta_\delta}$, or more simply, the space of psc-metrics on S^n with a torpedo cap of radius δ about the point p. As the following lemma shows, up to homotopy, this space is no different from $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$.

Lemma 4.4 *For $n \geq 3$ and any $\delta > 0$, there is a deformation retract from the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ onto its subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n)$.*

Proof Let $i\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n) \hookrightarrow \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ denote the inclusion map. For each $s \in [0,1]$, let r_s be the map defined by

$$\begin{aligned} r_s\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n) &\to \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n), \\ g &\mapsto \frac{\delta^2 g}{((1-s)\rho(g|_{D_p}) + s\delta)^2}. \end{aligned}$$

It follows from Proposition 4.1 that for each $s \in [0,1]$, this map is well-defined. Furthermore, it is immediate that r_1 is the identity map on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$, r_0 maps into $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n)$ and that the composition $r_0 \circ i$ is the identity map on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p,\delta)}(S^n)$. □

Remark 4.2 With somewhat more sophisticated tools, such as those used in the Gromov–Lawson construction as described in [19], one could prove a more general version of this homotopy equivalence for psc-metrics with multiple torpedo caps.

A little later we will revisit the fact, proved in [20], that when $n \geq 3$, the subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$ is homotopy equivalent to the space $\mathcal{R}\mathrm{iem}^+(S^n)$. Now however, we will return to the problem of defining a product on the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ for the case when $\boldsymbol{p}$ is a single point p. To do this we will need to spend a little more time on the more general case where $\boldsymbol{p}$ contains several points.

Let $\boldsymbol{p} = \{p_0, p_1, \dots, p_k\}$ and $\boldsymbol{q} = \{q_0, q_1, \dots, q_l\}$ be two finite sets of points on S^n. We will assume that $p_i \neq p_j$ when $i \neq j$ and that $q_i \neq q_j$ when $i \neq j$, but make no assumptions about whether or not $p_i = q_j$. We now consider the corresponding spaces

$\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{q})}(S^n)$. For each pair of integers (i,j) with $0\le i\le k$ and $0\le j\le l$, we define the *ij–uncapping map* Unc_{ij} as

$$\text{(4-6)}\quad \begin{aligned}\mathrm{Unc}_{ij}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{q})}(S^n)&\\ \to \mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n)&,\\ (g,h)\mapsto (\mathrm{Unc}_{p_i}(g),\mathrm{Unc}_{q_j}(h))&.\end{aligned}$$

In simple terms, the map Unc_{ij} removes the torpedo caps at p_i and q_j on the metrics g and h, respectively. After an appropriate rescaling, the resulting disks can be glued together along their boundaries. Thus, for each map $f\colon (0,\infty)\times(0,\infty)\to(0,\infty)$, we obtain the *ij–joining map*, $J_{ij}^{\mathrm{cyl}(f)}$, defined as

$$\text{(4-7)}\quad \begin{aligned}J_{ij}^{\mathrm{cyl}(f)}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{q})}(S^n)&\\ \to \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\{\boldsymbol{p}\setminus\{p_i\}\}\cup\{\boldsymbol{q}\setminus\{q_j\}\})}(S^n)&,\\ (g,h)\mapsto J^{\mathrm{cyl}(f)}(\mathrm{Unc}_{ij}(g,h))&,\end{aligned}$$

where $J^{\mathrm{cyl}(f)}$ is the map defined in (3-2). Henceforth, we will usually suppress the function f and simply write J for $J^{\mathrm{cyl}(f)}$ and J_{ij} for $J_{ij}^{\mathrm{cyl}(f)}$, knowing that some f is fixed in the background. This construction is illustrated schematically (for some unspecified f) in Figure 7 where $\boldsymbol{p}=\{p_0,p_1,p_2,p_3\}$, $\boldsymbol{q}=\{q_0,q_1,q_2\}$, $i=1$ and $j=2$.

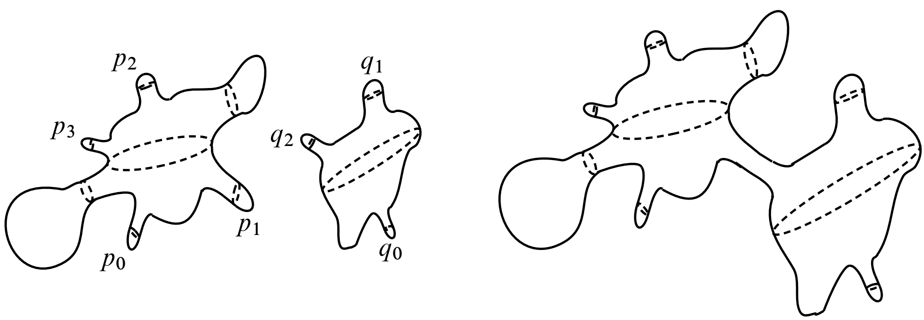

Figure 7: The metric $J_{12}(g,h)$ (right) formed by combining g and h (left)

In the case when $\boldsymbol{p}=\boldsymbol{q}=\{p_0\}$, the image of J_{00} does not lie in $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$. Thus, to define any sort of product we need to do some additional work. Henceforth we fix a *base point* $p_0\in S^n$. Let p_1 and p_2 be two distinct points on $S^n\setminus\{p_0\}$ and let $\boldsymbol{p}=\{p_0,p_1,p_2\}$. We now define a product on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$ as follows. Consider for each $j=1,2$ the map

$$J_{0j}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)\to\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p}\setminus\{p_j\})}(S^n)$$

defined as in formula (4-7). Suppose we fix the second input metric as some $g_3 \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$. Then for each of $j = 1, 2$ we obtain maps

$$J^3_{0j}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p}\setminus\{p_j\})}(S^n), \qquad g \mapsto J_{0j}(g, g_3). \tag{4-8}$$

Finally, we define a product on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$ by means of the continuous map

$$\mu^{\mathrm{torp}}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n) \times \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n), \qquad (g,h) \mapsto J_{02}(h, J_{01}(g, g_3)). \tag{4-9}$$

Example 4.5 As an illustration, consider the metric $\bar{g}_\lambda$ obtained by gluing two copies the infinitesimal torpedo metric on the disk D^n, g^{η_λ}, together along the boundary in the obvious way. This metric is represented in Figure 8.

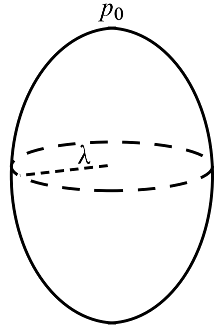

Figure 8: The metric $\bar{g}_\lambda$

Strictly speaking, this is a metric with two torpedo caps, one at the north pole, which we denote by p_0, and one at the south pole, p_0'. Now choose two points p_1 and p_2 in the interior of the southern hemisphere. For simplicity, we may as well choose p_1 and p_2 so that p_0, p_1 and p_2 are equidistant along a great circle. As the dimension of the underlying sphere n is at least three, it is possible to "push out" two torpedo caps of radius $\delta > 0$ (for some sufficiently small $\delta > 0$) at each of the points p_1 and p_2 to construct the psc-metric g_3^m illustrated in Figure 9. This follows from the work of Gromov and Lawson in [6] and is something we will discuss in more detail in the next section. Finally, in Figure 10, we depict the result of multiplying a pair of metrics $g, h \in \mathcal{R}\mathrm{iem}_{\mathrm{torp}(p_0)}$ via the multiplication μ^{torp} in (4-9), where $g_3 = g_3^m$, the metric constructed above.

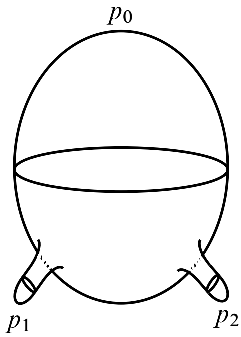

Figure 9: The metric g_3^m with three torpedo caps

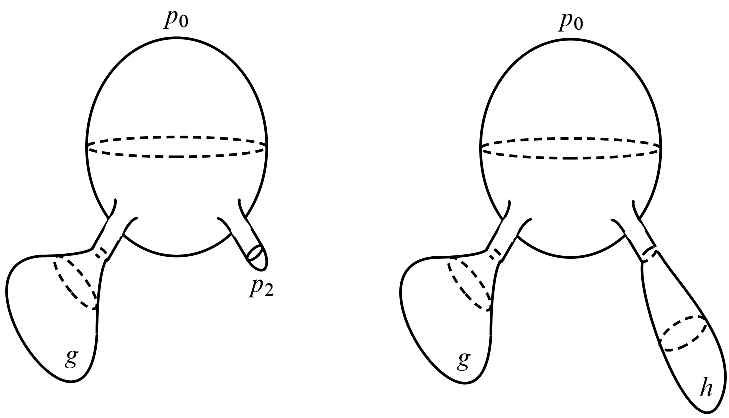

Figure 10: The metrics $J_{01}(g, g_3)$ (left) and $\mu^{\mathrm{torp}}(g, h) = J_{02}(h, J_{01}(g, g_3))$ (right)

Of course there are several choices to be made in specifying such a map. For a start, there is the choice of rescaling function f, whose notation we have suppressed. Furthermore, there is the choice of distinct points $p_1, p_2 \in S^n \setminus \{p_0\}$, and the choice of metric $g_3 \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$, where $\boldsymbol{p} = \{p_0, p_1, p_2\}$. We are assuming that p_0 is fixed throughout and so the map μ^{torp} can be thought of as dependent on the choices of f, p_1, p_2 and g_3. As one may suspect, up to homotopy, many of these choices have no impact. We will shortly state a lemma, Lemma 4.8 below, which helps clarify the situation. Before doing this, it is time to revisit a construction that is at the heart of this paper: the Gromov–Lawson connected sum construction.

4.4 The Gromov–Lawson connected sum construction

We now consider a special case of the well-known surgery theorem due to Gromov and Lawson [6] and proved independently by Schoen and Yau [17]. In this paper we will concern ourselves only with the simplest type of surgery on smooth manifolds: the connected sum. Let M^n be a smooth manifold with $n \geq 3$, g a psc-metric on M, $p \in M$ a point and $B_g(p, \epsilon)$ a closed geodesic ball about p with respect to the metric g. By specifying a curve $\gamma = \gamma_{g,p,\epsilon}$ of the type shown on the left of Figure 11, it is possible to adjust the metric g inside $B_g(p, \epsilon)$ by pushing out geodesic spheres in the space $B_g(p, \epsilon) \times [0, \infty)$ in a way that is determined by γ. More precisely, a geodesic sphere of radius r is pushed out to lie in the slice $B_g(p, \epsilon) \times \{t\}$, where $(t, r) \in \gamma$. The induced metric on the resulting hypersurface M_γ, shown in the right image of Figure 11, extends smoothly onto the rest of M as a new metric g'. Essentially, the shape of γ means that the resulting metric, g', is very close to an infinitesimal torpedo metric with radius $\delta > 0$ (which may be very small) on a neighbourhood of p. This is indicated by the shaded region of Figure 11.

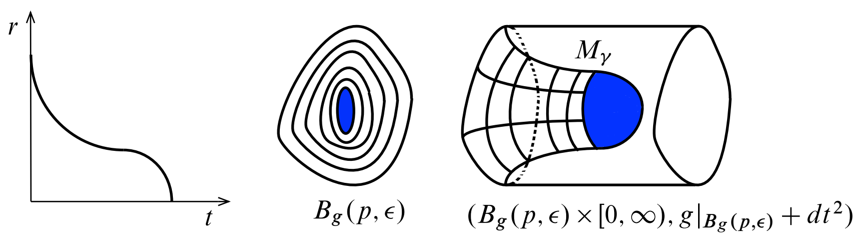

Figure 11: The curve γ (left), geodesic ball $B_g(p, \epsilon)$ (middle) and the hypersurface obtained by pushing out geodesic spheres with respect to γ

One of the main challenges of this construction was in demonstrating the fact that γ can always be chosen so that positivity of the scalar curvature is maintained. As the space of psc-metrics is open and as the psc-metric induced on M_γ is close to being "standard" near p, a psc-isotopy (obtained from a linear homotopy through metrics) is then used to adjust g' so that, near p, it is precisely an infinitesimal torpedo metric of radius δ. Of course, if the geodesic spheres around p are already standard spheres (as in the case when the original metric g is a round metric) then no such final adjustment is necessary. This will very often be the case for us in this paper. Note that if $\delta > 0$ is sufficiently small for the construction to work, then it works also for all δ' so that $0 < \delta' < \delta$. Thus, given a pair of Riemannian manifolds (M^n, g) and (N^n, h), where g and h are psc-metrics and $n \geq 3$, and a pair of points $p \in M$ and $q \in N$, a sufficiently small δ may be found for the construction of both g' and h'. Removing the infinitesimal torpedo caps from these metrics gives rise to psc-metrics, which may be identified along their respective boundaries to obtain a psc-metric on the connected sum $M \# N$.

In [19], we further show that a psc-isotopy may be obtained between the metrics g and g'. The main work is in contracting the curve γ back to the vertical axis. Note that we show in [19, Theorem 2.13] that the whole construction (including this psc-isotopy) actually goes through for a compact family of psc-metrics. We restate the relevant aspects of this theorem as Lemma 4.6 below.

Lemma 4.6 *Let M^n be a compact smooth manifold of dimension $n \geq 3$. Suppose g_t, $t \in K$, is a compact family of psc-metrics on M, indexed by a compact space K. Suppose also that p_s, $s \in I$, is a path in M. Then there is a continuously varying family of neighbourhoods U_s of p_s in M as s varies along I, a constant $\delta > 0$ and a compact family of psc-metrics g'_{st}, $(s,t) \in I \times K$, which satisfies the following:*

(1) *For each $(s,t) \in I \times K$, $g'_{st} = g_t$ outside of U_s.*

(2) *For $s \in I$, there is a continuously varying family of neighbourhoods D_s of p_s, with $D_s \subset U_s$, so that for each $(s,t) \in I \times K$, g'_{st} takes the form of an infinitesimal torpedo metric of radius δ on D_s.*

(3) *For each $s_0 \in I$ there is a continuous homotopy through families of psc-metrics on M^n, which deforms the family $\{g_t : t \in K\}$ into $\{g'_{s_0 t} : t \in K\}$.*

Proof This is a special case of Theorem 2.13 from [19]. □

We have already seen an application of Lemma 4.6 in the construction of the metric g_3^m in Example 4.5. Another important application for us is the following lemma. This is actually a special case of a more general result proved in [20, Lemma 3.3].

Lemma 4.7 *When $n \geq 3$ and $\boldsymbol{p}$ is a finite collection of points on the sphere S^n, the spaces $\mathcal{R}\mathrm{iem}_{\mathrm{torp}^+(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\mathrm{iem}^+(S^n)$ are homotopy equivalent.*

Proof The idea is to use the construction in Lemma 4.6 to show that elements of the relative homotopy groups $\pi_k(\mathcal{R}\mathrm{iem}^+(S^n), \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n))$ are trivial. This, coupled with a result of Palais that indicates that these spaces of psc-metrics are dominated by CW–complexes, allows us to conclude the result via a well-known theorem of Whitehead. Details can be found in [20, Lemma 3.3]. □

We close this section by clarifying an earlier comment concerning the multiplication map μ^{torp} in (4-9). Recall that this map depended on several choices: a rescaling function $f\colon (0,\infty) \times (0,\infty) \to (0,\infty)$, a pair $p_1, p_2 \in S^n \setminus \{p_0\}$ determining a triple of distinct points $\boldsymbol{p} = \{p_0, p_1, p_2\}$ (assuming p_0 is already fixed) and a psc-metric $g_3 \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$. We now state a theorem, which makes use of Lemma 4.6 above.

Lemma 4.8 *Let f_1, f_2 be continuous maps $(0,\infty)\times(0,\infty)\to(0,\infty)$ and let $p_1, p_2, p_1', p_2' \in S^n \setminus \{p_0\}$. Suppose also that*

$$g_3 \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n) \quad \text{and} \quad g_3' \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p}')}(S^n),$$

where $\boldsymbol{p} = \{p_0, p_1, p_2\}$ and $\boldsymbol{p}' = \{p_0, p_1', p_2'\}$. Then the corresponding maps

$$\mu^{\mathrm{torp}} = \mu^{\mathrm{torp}}_{(f_1,p_1,p_2,g_3)} \quad \text{and} \quad \mu^{\mathrm{torp}'} = \mu^{\mathrm{torp}}_{(f_2,p_1',p_2',g_3')}$$

are homotopic if and only if the metrics g_3 and g_3' are psc-isotopic.

Proof Convexity of the space of maps $(0,\infty)\times(0,\infty)\to(0,\infty)$ means that the choice of rescaling map has no effect up to homotopy. Furthermore, it is an immediate consequence of Lemma 4.6 that the process of pushing out a cap while maintaining positive scalar curvature can be done "on the move", ie while the point is moving along a continuous path. Thus the choice of points, whether $\{p_1, p_2\}$ or $\{p_1', p_2'\}$, has no effect up to homotopy either.

Now suppose g_3 and g_3' are psc-isotopic. It is a consequence of Lemma 4.6 that g_3 and g_3' can be connected by a continuous path in $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$. In this case the interval of definition of this path plays the role of the compact set K in the statement of Lemma 4.6. Three applications of Lemma 4.6 where p_s takes the form of the constant path p_0 and paths from p_1 to p_1' and p_2 to p_2' then give the desired psc-isotopy in $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$. Homotopy equivalence of the maps μ^{torp} and $\mu^{\mathrm{torp}'}$ follows easily.

Now suppose μ^{torp} and $\mu^{\mathrm{torp}'}$ are homotopic. Let $\mu^{\mathrm{torp}}_t, t \in I$, be a continuous family of maps

$$\mu^{\mathrm{torp}}_t\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)\to\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n),$$

where $\mu^{\mathrm{torp}}_0 = \mu^{\mathrm{torp}}$ and $\mu^{\mathrm{torp}}_1 = \mu^{\mathrm{torp}'}$. Consider the path in $\mathcal{R}\mathrm{iem}^+(S^n)$ given by

$$t \mapsto \mu^{\mathrm{torp}}_t(\bar{g}_1, \bar{g}_1),$$

where $\bar{g}_1$ is the metric we constructed before stating this lemma. This path is now easily deformed, using Lemma 4.6, into a path which connects the metrics g_3 and g_3'. □

As the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is often not path connected, it is evident that there are many non-homotopic possibilities for the map μ^{torp}. In the next section, we will see that in order to make $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$ into an H–space, we will have to restrict our choice of g_3 to metrics which lie in the path component of the standard round metric.

5 The H–space theorem

We are now able to state and prove the first of our main results. Let μ^{torp} be the map defined in (4-9) above with respect to some 4–tuple (f, p_1, p_2, g_3), where as before $f\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ is a continuous map, $p_1, p_2 \in S^n \setminus \{p_0\}$ are distinct points and $g_3 \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(\boldsymbol{p})}(S^n)$. We obtain the following theorem.

Theorem 5.1 *Let $n \geq 3$ and let $\mu^{\mathrm{torp}} = \mu^{\mathrm{torp}}_{(f,p_1,p_2,g_3)}$ be the multiplication map given by formula (4-9). In the case when the metric g_3 is psc-isotopic to the round metric ds_n^2, μ^{torp} defines a homotopy product on $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$ with homotopy identity $\bar{g}_1$, giving it the structure of an H–space. Furthermore, this product is both homotopy commutative and homotopy associative.*

Proof For convenience, we take g_3 to be the 3–cap metric g_3^m constructed in the previous section. This is reasonable given Lemma 4.8 and our hypothesis that all choices of g_3 are psc-isotopic to the standard round metric ds_n^2. We begin by showing that the metric $\bar{g}_1$ constructed earlier plays the role of homotopy identity. We will show that the map $g \mapsto \mu^{\mathrm{torp}}(g, \bar{g}_1)$ is homotopic to the identity map $g \mapsto g$. The case of $g \mapsto \mu^{\mathrm{torp}}(\bar{g}_1, g)$ is completely analogous. For the most part this will involve a psc-isotopy of the metric g_3. To help see this we represent, for an arbitrary $g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p_0)}(S^n)$, the element $\mu^{\mathrm{torp}}(g, \bar{g}_1)$ in Figure 12.

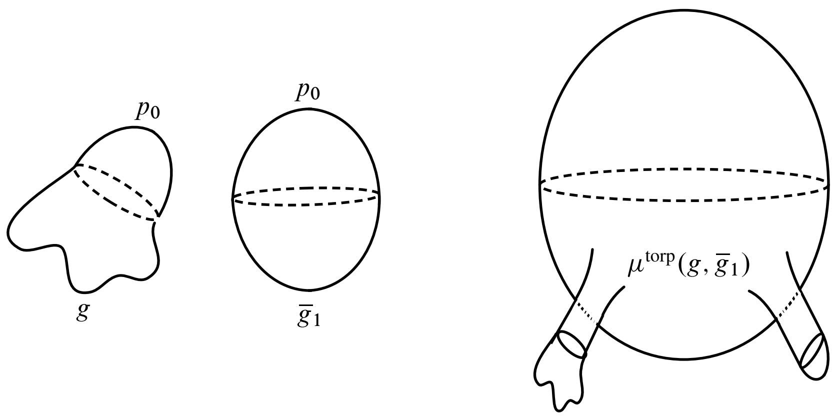

Figure 12: The metrics $g, \bar{g}_1$ and $\mu^{\mathrm{torp}}(g, \bar{g}_1)$

We will now construct a deformation of the map $g \mapsto \mu^{\mathrm{torp}}(g, \bar{g}_1) = J_{01}(g, J_{02}(\bar{g}_1, g_3))$ to the identity map. Recall that the map J_{0j} is really $J_{0j}^{\mathrm{cyl}(f)}$, where $f\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ is the rescaling map defined earlier. The first step is to replace $J_{01}^{\mathrm{cyl}(f)}$ with $J_{0j}^{\mathrm{cyl}((1-t)f+t\pi_{\mathrm{L}})}$, where $t\in[0,1]$. Let $\pi_{\mathrm{L}}\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ denote projection onto the left factor. This induces a homotopy of the map $g \mapsto \mu^{\mathrm{torp}}(g, \overline{g}_1)$ to one that fixes the size of the left input metric g during attachment to the right metric $J_{02}(\overline{g}_1, g_3)$. We once again suppress the scaling function in our notation. The main step now involves constructing a psc-isotopy of the metric $J_{02}(\overline{g}_1, g_3)$, which will induce a psc-isotopy on $J_{01}(g, J_{02}(\overline{g}_1, g_3))$, turning it into g. Importantly, this construction uses no data arising from the metric g and so easily goes through for all choices of g. As a result, this psc-isotopy induces a homotopy of the map $g \mapsto \mu^{\mathrm{torp}}(g, \overline{g}_1)$ to the identity map $g \mapsto g$.

We will now describe the psc-isotopy from $J_{01}(g, J_{02}(\overline{g}_1, g_3))$ to g. To aid the reader, a step by step description of this psc-isotopy is depicted in Figure 13. The first image in Figure 13 is the metric $J_{01}(g, J_{02}(\overline{g}_1, g_3))$. As a first step, we use two applications of Lemma 4.6 to contract the cap at p_2 and obtain a psc-isotopy of the remaining metric to one where the cap at p_0 is antipodal to the connecting metric with g. This is done with the compact family K as the singleton set $\{J_{01}(g, J_{02}(\overline{g}_1, g_3))\}$ and with the path p_s first playing the role of the constant path at p_1 and then a path that moves p_2 to the point antipodal to p_0. The process is depicted in the second and third images in Figure 13. The resulting metric then easily contracts, via the results of [19, Chapter 1], to the one shown in the fourth image in Figure 13, which takes the form of a standard torpedo metric on $S^n \setminus D_{p_1}$ and is connected to $g|_{D'_{p_1}}$ along the boundary. Finally, the neck of this torpedo is contracted down to yield precisely the metric g.

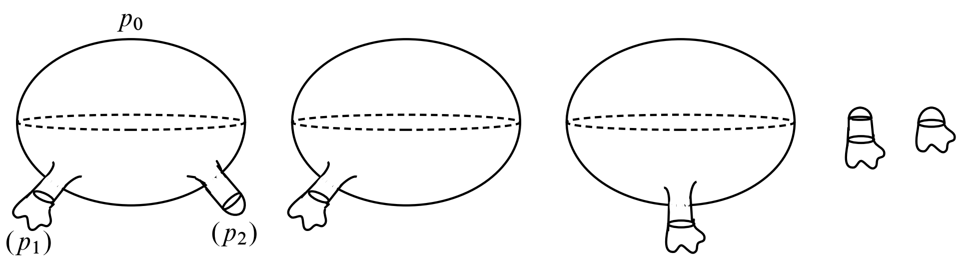

Figure 13: An isotopy of the metric $J_{02}(\overline{g}_1, g_3)$ back to g. The brackets indicate the places where p_1 and p_2 lay before attachments.

Homotopy commutativity follows immediately from Lemma 4.8. For a general g_3, just choose $p'_1 = p_2$ and $p'_2 = p_1$ in the statement of that lemma. In the case when $g_3 = g_3^m$, this is most easily induced by continuous rotation of the sphere, which swaps p_1 with p_2 as shown in Figure 14 below.

It remains to show homotopy associativity. This is similar to the proof of homotopy commutativity in that it involves moving the metrics which are attached to g_3, by their attaching tubes, along a closed bounded arc while adjusting the radius if necessary.

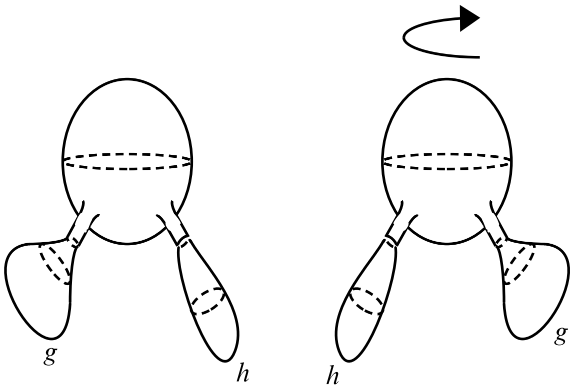

Figure 14: The metrics $\mu^{\mathrm{torp}}(g,h)$ and $\mu^{\mathrm{torp}}(h,g)$

Again we use Lemma 4.6 to move the metrics continuously and maintain positivity of the scalar curvature. In particular, this means that for any $g,h,h' \in \mathcal{R}\mathrm{iem}^{+}_{\mathrm{torp}(p_0)}(S^n)$ the metric $\mu^{\mathrm{torp}}(\mu^{\mathrm{torp}}(g,h),h')$ is psc-isotopic to $\mu^{\mathrm{torp}}(g,\mu^{\mathrm{torp}}(h,h'))$, as shown in Figure 15.

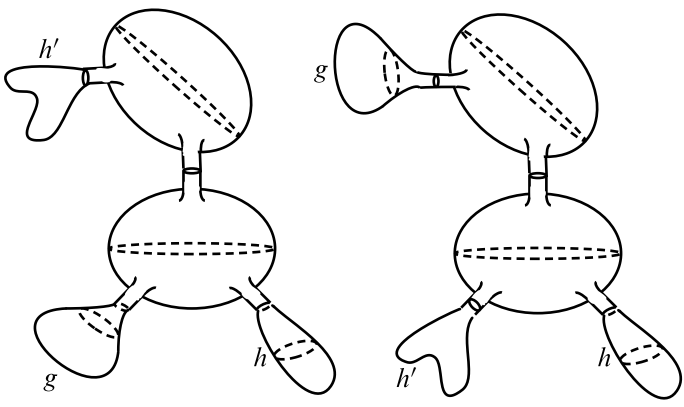

Figure 15: The metrics $\mu^{\mathrm{torp}}(\mu^{\mathrm{torp}}(g,h),h')$ and $\mu^{\mathrm{torp}}(g,\mu^{\mathrm{torp}}(h.h'))$

With this in mind, let g_3 and g_3' be two copies of the same intermediary metric with cap points $\{p_0,p_1,p_2\}$ and $\{p_0',p_1',p_2'\}$, respectively. Obviously p_0' is identified with the base point $p_0 \in S^n$. We now fix a psc-isotopy $g(t)$, which moves $g(0) = J_{01}(g_3,g_3')$ to the metric $g(1)$ obtained by swapping the p_1 cap of g_3 with the p_2' cap of g_3'. Now consider the map $(g,h,h') \mapsto \mu_0^{\mathrm{torp}}(g,h,h') = \mu^{\mathrm{torp}}(\mu^{\mathrm{torp}}(g,h),h')$. Viewing the metric $J_{01}(g_3,g_3)$ as the metric $J_{01}(g_3,g_3')$ and then moving it by $g(t)$ induces a homotopy of maps μ_t^{torp} between μ_0^{torp} which is defined above and μ_1^{torp} defined by $\mu_1^{\mathrm{torp}}(g,h,h') = \mu^{\mathrm{torp}}(g,\mu^{\mathrm{torp}}(h,h'))$. □

Corollary 5.2 *For $n \geq 3$, the group $\pi_1(\mathcal{R}\mathrm{iem}^+(S^n), ds_n^2)$ is abelian.*

Proof This is a well-known fact about H–spaces and so we will be terse. Suppose Z is an H–space with multiplication μ and homotopy identity $e \in Z$. Let $\alpha, \beta\colon [0,1] \to Z$ be loops based at e representing classes of $\pi_1(Z, e)$. Now consider the map $F\colon [0,1] \times [0,1] \to Z$, which is defined $F(s,t) = \mu(\alpha(s), \beta(t))$. Depending on one's choice of parametrisation, the restriction of F, firstly to the left and bottom sides of the domain square and secondly to the top and right sides of the square, gives rise to maps that are respectively homotopy equivalent to the loop concatenations $\alpha \circ \beta$ and $\beta \circ \alpha$. Constructing the appropriate homotopy is then a straightforward exercise. □

6 Bulb metrics

In Section 3, we discussed two types of psc-metric on the disk D^n that we could use in appropriate gluing constructions to obtain psc-metrics on the sphere S^n. The first of these, metrics that are cylindrical (at least infinitesimally) near the boundary, motivated the construction of the space of psc-metrics on S^n with torpedo caps, and ultimately a product on this space. We will now carry out an analogous project for the second of our families of disk metrics with well-behaved boundaries: metrics that are sphere-like near the boundary. The construction here is more complicated. However, it will allow us to expose extra structure, beyond the H–space structure, on the space of psc-metrics on S^n.

6.1 The space of psc-metrics with bulbs

Suppose we have a psc-metric g on a smooth manifold M^n, a point $x \in M$ and a geodesic ball $B_g(x, \epsilon)$ of radius $\epsilon > 0$ about x. Recall that when $n \geq 3$, we may construct a psc-isotopy starting at the metric g, which fixes g outside of $B_g(x, \epsilon)$, and which pushes out an infinitesimal torpedo metric of radius $\delta > 0$ (dependent on g, x and ϵ), on a disk around x and inside $B_g(x, \epsilon)$. This follows immediately from Lemma 4.6. We have called this process *pushing out a torpedo cap* around x. When the starting metric g is the round metric of radius $\lambda > 0$ on the sphere S^n, we can be very explicit about the construction. With this in mind, we will describe a family of psc-metrics that we call "bulb" metrics. These are obtained by pushing out a torpedo cap of radius δ inside a geodesic ball of radius r on a round sphere of radius λ. We make this precise in the following lemma.

Lemma 6.1 *Let $n \geq 3$, $\lambda > 0$ and $r < (0, \lambda\frac{\pi}{2}]$. Let p and p' be antipodal points in S^n. Then there is a psc-metric $g_{\text{bulb}}(p, \lambda, r)$ on the sphere S^n which satisfies the following conditions:*

(1) *The metric $g_{\text{bulb}}(p, \lambda, r)$ is rotationally symmetric about the line in $\mathbb{R}^{n+1}$ that connects p to its antipodal point p'.*

(2) *There are continuous parameters $r' = r'(\lambda, r) \in (0, r]$ and $\delta = \delta(\lambda, r) \in (0, r]$ so that on the ball $B_\lambda(p', r')$, the metric $g_{\text{bulb}}(p, \lambda, r)$ restricts as an infinitesimal torpedo metric of radius δ.*

(3) *On the annular region $\text{Ann}_\lambda(p'; r', r)$ (taken with respect to the round metric of radius λ), the metric $g_{\text{bulb}}(p, \lambda, r)$ takes the form of the connecting tube from the Gromov–Lawson construction.*

(4) *Outside of the ball $B_\lambda(p', r)$, the metric $g_{\text{bulb}}(p, \lambda, r)$ is precisely the lens metric $g_{\text{lens}}(\lambda, \lambda\pi - r)$, the lens complement obtained by removing the geodesic ball $B_\lambda(p', r)$ from the original round sphere of radius λ.*

(5) *The metric $g_{\text{bulb}}(p, \lambda, \lambda\frac{\pi}{2})$ is the original round metric of radius λ.*

Proof This is an easy consequence of Lemma 4.6. □

The metric $g_{\text{bulb}}(p, \lambda, r)$ obtained by this construction will be known as a (λ, r)–*bulb metric*, or simply a *bulb*. Needless to say, all such metrics are psc-isotopic to the original round metric via Lemma 4.6. We point out that this metric decomposes the sphere naturally into regions

$$S^n = B_\lambda(p, \lambda\pi - r) \cup \text{Ann}_\lambda(p'; r', r) \cup B_\lambda(p', r), \tag{6-1}$$

on which the metric takes the forms indicated in Figure 16. We will refer to these pieces respectively as the *head*, *neck* and *cap* of the bulb metric $g_{\text{bulb}}(p, \lambda, r)$. Furthermore the quantities λ, r and $\delta = \delta(\lambda, r)$ will be known as the *head radius*, *head angle* and *cap radius* respectively. The head metric, which is of course the metric $\lambda^s ds_n^2|_{B_\lambda(p, \lambda\pi - r)}$, will be denoted $g_{\text{head}}(p, \lambda, r)$. Note that in the case where $r = \lambda\frac{\pi}{2}$, the head and cap are the respective hemispheres of radius λ about p and p' while the neck is simply the $(n-1)$–dimensional equator.

It will be useful to think of the construction in Lemma 6.1 as the following continuous map. Consider the subspace $\mathcal{B}$ of $(0, \infty) \times (0, \infty)$ consisting of pairs (λ, r), which satisfy $0 < r < \lambda\frac{\pi}{2}$. We define the map Bulb_p as

$$\begin{aligned} \text{Bulb}_p \colon S^n \times \mathcal{B} &\to \mathcal{R}\text{iem}^+_{\text{torp}(p')}(S^n), \\ (p, (\lambda, r)) &\mapsto g_{\text{bulb}}(p, \lambda, r). \end{aligned} \tag{6-2}$$

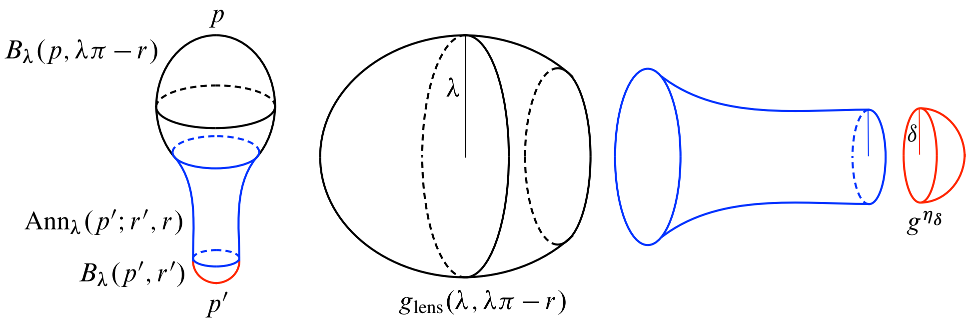

Figure 16: The metric $g_{\mathrm{bulb}}(p, \lambda, r)$ (left) and its decomposition into head, neck and cap (right)

The reader should note that the image consists of metrics that are torpedo at p', not p. In most cases the choice of p is unimportant (unless of course we start pushing out extra torpedo caps). Thus, we will declare the "standard case" to be the one where p is the north pole, p' the south pole, and write bulb_p simply as bulb and $g_{\mathrm{bulb}}(p, \lambda, r)$ simply as $g_{\mathrm{bulb}}(\lambda, r)$ in this case.

We denote by $\mathcal{B}\mathrm{ulb}_p(S^n)$ the subspace in $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p')}(S^n)$ that is the image of the map Bulb_p. Recall, we defined an "uncapping" map $\mathrm{Unc}_{p'}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p')}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{cyl}(0)}(D^n)$ in (4-5), which removes the torpedo cap about p' and pulls the remaining metric back to the standard disk D^n in the usual way (sending p to the origin 0). We thus denote by $\mathcal{NH}^+_{\mathcal{R}\mathrm{iem}}$ the image of the composition $\mathrm{Unc}_{p'} \circ \mathrm{Bulb}_p$, which uncaps the bulb around p', leaving only the neck and head, and pulls the metric back to the standard disk D^n. The corresponding space of all head metrics, obtained by restricting each $g_{\mathrm{bulb}}(p, \lambda, r)$ to the ball $B_\lambda(p, \lambda\pi - r)$ and pulling back to D^n in the usual way, is denoted $\mathcal{H}^+_{\mathcal{R}\mathrm{iem}}$. Thus, an element of $\mathcal{NH}^+_{\mathcal{R}\mathrm{iem}}$ is the union of the head and neck of the bulb metric on D^n and an element of $\mathcal{H}^+_{\mathcal{R}\mathrm{iem}}$ is simply a head metric on D^n.

The reader should consider these space to be the analogues, with respect to bulbs, of the space of infinitesimal torpedo metrics $\mathcal{T}^+_{\mathcal{R}\mathrm{iem}}$. This allows us define two important subspaces of the space of psc-metrics on S^n, in a similar vein to the definition of the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{torp}(p)}(S^n)$ in equation (4-4). Letting D_p denote a normal coordinate ball around p and $\phi_p\colon D^n \to D_p$ the corresponding diffeomorphism, we define the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p)}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p)}(S^n)$ of *psc-metrics on S^n with bulbs at p* and *psc-metrics with heads at p* as

$$\begin{aligned} \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p)}(S^n) &= \{g \in \mathcal{R}\mathrm{iem}^+(S^n) : \phi_p^*(g|_{D_p}) \in \mathcal{NH}^+_{\mathcal{R}\mathrm{iem}}\}, \\ \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p)}(S^n) &= \{g \in \mathcal{R}\mathrm{iem}^+(S^n) : \phi_p^*(g|_{D_p}) \in \mathcal{H}^+_{\mathcal{R}\mathrm{iem}}\}. \end{aligned} \tag{6-3}$$

Moreover, for a finite collection of points on S^n, $\boldsymbol{p} = \{p_0, p_1, \dots, p_k\}$, with corresponding coordinate diffeomorphisms $\phi_i\colon D^n \to D_{p_i}$, we obtain the more general spaces of psc-metrics with bulbs or heads around $\boldsymbol{p}$:

$$\begin{aligned} \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(\boldsymbol{p})}(S^n) &= \bigcap_{i=0}^{k} \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_i)}(S^n), \\ \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n) &= \bigcap_{i=0}^{k} \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_i)}(S^n). \end{aligned} \tag{6-4}$$

Ultimately, we will be more interested in the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n)$. However, both spaces will have their uses for us. As much of what we are going to say applies equally to both spaces, we let $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)$ denote either of these spaces. Notice that each element $g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)$ restricts near p_i to a metric that is precisely the head of a (λ_i, r_i)–bulb metric for some $\lambda_i > 0$ and $r_i \in (0, \lambda_i \frac{\pi}{2}]$. In particular, this means that if we cut off the head on such a metric, the resulting metric is, infinitesimally at least, sphere-like at the boundary. Recall that the head of the bulb at such a point p_i is the part of the metric defined on the region $B_g(p_i, r_i)$ $(= B_\lambda(p_i, r_i)$, as g and $\lambda^2\, ds_n^2$ agree on this region.) We now define, for any $\rho > 0$, the map $\mathrm{Cut}_{p_i,\rho}$ as

$$\mathrm{Cut}_{p_i,\rho}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n) \tag{6-5}$$
$$g \mapsto \begin{cases} g|_{S^n \setminus B_{\lambda_i(g)}(p_i,\rho)} & \text{if } 0 < \rho \le \lambda_i(g)\pi - r_i(g), \\ g|_{S^n \setminus B_{\lambda_i(g)}(p_i,\lambda_i(g)\pi - r_i(g))} & \text{if } \rho > \lambda_i(g)\pi - r_i(g), \end{cases}$$

where $\lambda_i(g)$ and $r_i(g)$ are the head radius and head angle of g around p_i. In the case when $\rho = \lambda_i(g)\pi - r_i(g)$, we simply remove the actual i^{th} head from g. To aid the reader we depict the resulting metric $\mathrm{Cut}_{p_i,\lambda_i(g)\pi - r_i(g)}(g)$, where $g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(\boldsymbol{p})}(S^n)$ in Figure 17. By allowing ρ to vary, we give ourselves the option of cutting through the head along various geodesic spheres about p_i. For example, choosing $\rho = \lambda_i(g)\frac{\pi}{2}$ would cut the round hemisphere of radius $\lambda_i(g)$ from the head at p_i. This will be important later on.

Before discussing the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)$ any further, there is another construction we must attend to. We return once more to the general case of a smooth Riemannian manifold (M, g), a point $x \in M$ and closed geodesic ball $B_g(x, \epsilon)$. Let g' denote the metric obtained by pushing out an infinitesimal torpedo metric of radius δ, for some $\delta > 0$, on a disk around x inside $B_g(x, \epsilon)$. Using Lemma 4.6, we will perform an adjustment to the metric g' near x. Essentially, we construct a psc-isotopy, starting at g', which is trivial outside of $B_g(x, \epsilon)$ and which results in a metric that is the result of a Gromov–Lawson connected sum between (M, g) and the standard sphere.

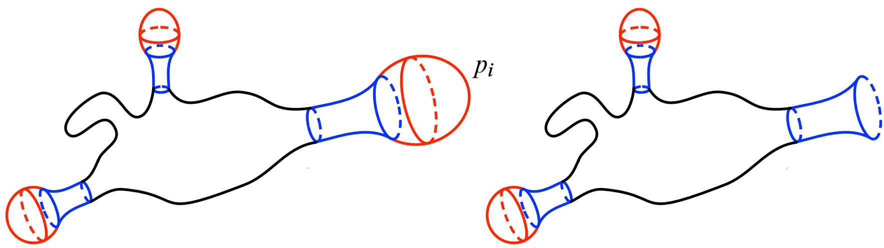

Figure 17: A metric $g \in \mathcal{R}\mathrm{iem}^{+}_{\mathrm{bulb}(\boldsymbol{p})}(S^n)$ (left) and $\mathrm{Cut}_{p_i,\lambda_i(g)\pi - r_i(g)}(g)$, the metric on D^n obtained by cutting the i^{th} head from g (right)

Importantly, it is the metric obtained by removing the torpedo cap of radius δ about x, removing a corresponding torpedo cap of radius δ from a bulb metric and gluing them together in the obvious way. This is made more precise in Lemma 6.2 below. In Figure 18 we depict the restrictions of these metrics to $B_g(x,\epsilon)$.

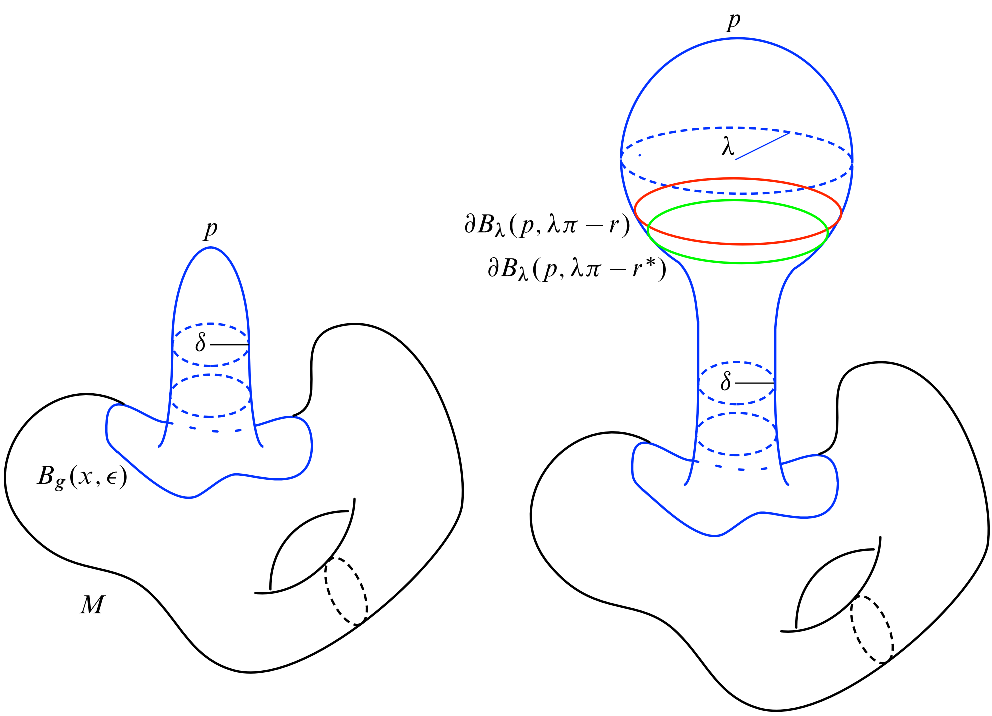

Figure 18: Pushing out a bulb (right) around x in $B_g(x,\epsilon)$

Lemma 6.2 *Let M^n be a smooth compact manifold with dimension $n \geq 3$ and g a psc-metric on M. Suppose $x \in M$ and $B_g(x,\epsilon)$ is a geodesic ball around p for some*

$\epsilon > 0$. For any $\lambda > 0$ and any $r \in (0, \lambda\frac{\pi}{2}]$, there is a psc-isotopy $g_{\text{bulb}}(t)$, $t \in I$ so that the following conditions hold:

1. *The psc-isotopy $g_{\text{bulb}}(t)$, $t \in I$ varies continuously with respect to g, p, ϵ, r and λ.*
2. *Outside of $B_g(p, \epsilon)$, $g_{\text{bulb}}(t) = g$ for all $t \in I$.*
3. *There are continuous parameters ϵ^* satisfying $0 < \epsilon^* < \epsilon$ and r^* satisfying $0 < r^* \leq r$, which depend continuously on g, p, ϵ, r and λ, so that the restriction metric $g_{\text{bulb}}(1)|_{B_g(p,\epsilon^*)}$ is the head of the bulb metric $g_{\text{bulb}}(\lambda, r^*)$.*

Proof Most of the work here lies in continuously pushing out torpedo caps. This is done in original Gromov–Lawson construction as described in [19, Theorem 2.13]. As we already discussed, these torpedo caps have a nice standard structure that is easily manipulated to obtain the desired bulb structure; see [19]. □

Remark 6.1 The reader may wonder why we require the parameter r^* with the property that $0 < r^* \leq r$ and why we only end up by pushing out a (λ, r^*)–bulb metric, in the lemma above. This is a consequence of the Gromov–Lawson construction. The radius of the connecting tube, δ, may have to be very small. Thus, we cannot guarantee that it connects up correctly with the head and neck of the (λ, r)–bulb. However, for any λ there is always a small enough head angle r^* that works.

We are now ready to make some observations about the spaces $\mathcal{R}\text{iem}^+_{\text{bulb}(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\text{iem}^+_{\text{head}(\boldsymbol{p})}(S^n)$ defined above. Firstly, the obvious analogues of Lemmas 4.2 and 4.3 hold here also.

Proposition 6.3 *Lemmas 4.2 and 4.3 hold if we replace $\mathcal{R}\text{iem}^+_{\text{torp}(\boldsymbol{p})}(S^n)$ with the spaces $\mathcal{R}\text{iem}^+_{\text{bulb}(\boldsymbol{p})}(S^n)$ or $\mathcal{R}\text{iem}^+_{\text{head}(\boldsymbol{p})}(S^n)$ for relevant collections of points $\boldsymbol{p}$.*

More importantly, we have the following lemma, which concerns the homotopy type of $\mathcal{R}\text{iem}_{\text{bulb}(\boldsymbol{p})}(S^n)$.

Lemma 6.4 *If n is at least 3, the spaces $\mathcal{R}\text{iem}^+_{\text{torp}(\boldsymbol{p})}(S^n)$, $\mathcal{R}\text{iem}^+_{\text{bulb}(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\text{iem}^+_{\text{head}(\boldsymbol{p})}(S^n)$ are homotopy equivalent.*

Proof The construction in Lemma 6.2 gives rise to a map from $\mathcal{R}\text{iem}^+_{\text{torp}(\boldsymbol{p})}(S^n)$ to $\mathcal{R}\text{iem}^+_{\text{bulb}(\boldsymbol{p})}(S^n)$. Conversely one can easily construct a map in the opposite direction, which involves pushing out a cap on the bulb head of each element in $\mathcal{R}\text{iem}^+_{\text{bulb}(\boldsymbol{p})}(S^n)$. Showing that compositions of these maps are homotopy equivalent to the appropriate identity map is then straightforward given the psc-isotopies constructed in Lemma 6.2.

Moreover, a similar argument using the construction done in Lemma 6.2 may be used to add necks to the heads of psc-metrics in $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n)$ in order to demonstrate a homotopy equivalence between $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n)$. □

Corollary 6.5 *For $n \geq 3$, the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(\boldsymbol{p})}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n)$ are homotopy equivalent to $\mathcal{R}\mathrm{iem}^+(S^n)$.*

Proof This is immediate by Lemma 4.7. □

In fact, we can say a little more. To simplify notation, we will work in the case when $\boldsymbol{p}$ is just a single point p_0, although there is an obvious generalisation to the case when $\boldsymbol{p}$ has many points. As in the case of psc-metrics with torpedo caps, we may also define, for each pair $\lambda_0, r_0 > 0$ with $r_0 \in (0, \lambda_0 \frac{\pi}{2}]$, the following subspaces of psc-metrics with a fixed bulb or fixed head:

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0,\lambda_0,r_0)}(S^n) := \{g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0)}(S^n) : \phi^*_{p_0}(g|_{D_{p_0}}) = \mathrm{Unc}_{p_0'} \circ \mathrm{Bulb}_{p_0}(\lambda_0, r_0)\},$$

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0,\lambda_0,r_0)}(S^n) := \{g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0)}(S^n) : \phi^*_{p_0}(g|_{D_{p_0}}) = g_{\mathrm{lens}}(\lambda_0, r_0)\}.$$

There are also intermediary spaces where the head radius is fixed at λ_0 but the head angle r is allowed to vary in the interval $(0, \lambda_0 \frac{\pi}{2}]$. We denote these spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0)}(S^n)$, simply dropping the head radius coordinate. To clarify, we have the sequence of inclusions

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0,r_0)}(S^n) \subset \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0)}(S^n) \subset \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n). \tag{6-6}$$

Given a metric a metric $g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$, there is a canonical way of moving it in the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$ to a metric that lies in $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0,r_0)}(S^n)$. This method takes the form of a map, which we denote

$$\mathrm{Mov}^{\mathrm{b/h}} = \mathrm{Mov}^{\mathrm{b/h}}_{p_0,\lambda_0,r_0}$$

and define as

$$\mathrm{Mov}^{\mathrm{b/h}}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0,\lambda_0,r_0)}(S^n), \tag{6-7}$$

where for each input metric g whose bulb about p_0 takes the form $g_{\mathrm{bulb}}(p_0, \lambda_1, r_1)$, the output $\mathrm{Mov}^{\mathrm{b/h}}(g)$ is obtained by the construction described below.

(1) Replace the entire metric g with the metric $(\lambda_0/\lambda_1)^2 g$ to obtain a new metric, whose bulb head has radius λ_0. We denote by δ_1 the neck radius of this metric.

(2) The next part is more delicate as the cases of $\mathrm{Mov}^{\mathrm{bulb}}$ and $\mathrm{Mov}^{\mathrm{head}}$ are slightly different. We begin with the map $\mathrm{Mov}^{\mathrm{bulb}}$. By Lemma 6.1, we may replace the bulb component of the newly scaled metric $(\lambda_0/\lambda_1)^2 g$ with the unique one with head radius λ_0 and head angle r_0. This metric has a neck whose radius we denote $\delta_0 > 0$. The problem is that this may not agree with the neck radius, δ_1, of the connecting tube which connects the bulb with the rest of the metric g. Thus, to compensate, we replace the restriction metric $g|_{S^n \setminus D_p}$ with $(\delta_0/\delta_1)^2 g|_{S^n \setminus D_p}$. This latter restriction metric attaches smoothly to the head and neck of the new bulb component, resulting in an element of $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0,\lambda_0,r_0)}$. In the case of $\mathrm{Mov}^{\mathrm{head}}$ we first use the techniques of Lemma 6.2 to push out a neck on the metric $(\lambda_0/\lambda_1)^2 g$ before proceeding as above. We emphasise that the pushing out of such a neck can be done in a canonical way inside the standard head, relying only on input data arising from the head angle and radius, and so does not require any non-standard metric data from g.

This gives rise to the following lemma, an analogue of Lemma 4.4.

Lemma 6.6 *For $n \geq 3$, $\lambda_0 > 0$ and $r_0 \in (0, \lambda_0 \frac{\pi}{2}]$, there is a deformation retract from the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$ onto its corresponding subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0)}(S^n)$, and then a further deformation retract to the corresponding subspace $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0,r_0)}(S^n)$.*

Proof This is similar to the proof of Lemma 4.4. The map $\mathrm{Mov}^{\mathrm{b/h}}$ fixes metrics that lie in $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0,r_0)}(S^n)$, and its composition with the inclusion of $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0,\lambda_0,r_0)}(S^n)$ into $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$ is easily shown to be homotopic to the identity. □

6.2 A product on the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$

We now return to a construction alluded to at the end of Section 3. Once again, we let $\boldsymbol{p}$ and $\boldsymbol{q}$ denote finite collections of points $\{p_0, p_1, \dots, p_k\}$ and $\{q_0, q_1, \dots, q_l\}$ on S^n. For some $i \in \{0, 1, \dots, k\}$, we begin by composing the maps $\mathrm{Cut}_{p_i,\rho}$ and $\mathrm{Mov}^{\mathrm{b/h}}_{p_i,\lambda,r}$, defined in (6-5) and (6-7), for some $\rho \geq 0$ and $\lambda, r > 0$ with $r \in (0, \lambda \frac{\pi}{2}]$. Note that the values λ and r are not a priori connected with p. Any bulb around p will be made to "fit" those values. With this in mind we define a new map $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\rho} = \mathrm{Cut}_{p_i,\rho} \circ \mathrm{Mov}^{\mathrm{b/h}}_{p_i,\lambda,r}$. Just to clarify, the map $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\rho}$ takes the form

(6-8)
$$\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\rho}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n) \to \mathcal{R}\mathrm{iem}^+_{\mathrm{lens}(0)}(D^n),$$
$$g \mapsto \begin{cases} \mathrm{Mov}^{\mathrm{b/h}}_{p_i,\lambda,r}(g)|_{S^n \setminus B_\lambda(p_i,\rho)}, & \text{if } 0 < \rho \leq \lambda\pi - r, \\ \mathrm{Mov}^{\mathrm{b/h}}_{p_i,\lambda,r}(g)|_{S^n \setminus B_\lambda(p_i,\lambda\pi - r)}, & \text{if } \rho > \lambda\pi - r. \end{cases}$$

For example, the metric $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\lambda\pi/2}(g)$ is obtained by removing the round northern hemisphere of radius λ about p_i from $\mathrm{Mov}^{\mathrm{b/h}}_{p_i,\lambda,r}(g)$. On the other hand, the output metric $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\lambda\pi-r}(g)$ is obtained by removing the head of the metric $\mathrm{Mov}_{p_i,\lambda,r}(g)^{\mathrm{b/h}}$ about p_i, but not the neck. In particular, we see that the boundary of the output metric $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\lambda\pi-r}(g)$ smoothly attaches to the boundary metric $g_{\mathrm{lens}}(\lambda,\lambda\pi-r)$. We therefore obtain, for each pair (i,j) with $i\in\{p_0,p_1,\dots,p_k\}$ and $j\in\{q_0,q_1,\dots,q_l\}$, the following "joining" map for psc-metrics on S^n with bulbs or heads:

$$
\begin{aligned}
(6\text{-}9)\quad J^{\mathrm{b/h}(\lambda,r)}_{ij}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{q})}(S^n)&\\
\to \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\{\boldsymbol{p}\setminus\{p_i\}\}\cup\{\boldsymbol{q}\setminus\{q_j\}\})}(S^n),&\\
(g,h)\mapsto \mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\lambda\pi-r}(g)\cup\mathrm{Fit}^{\mathrm{b/h}}_{q_j,\lambda,\lambda(r_j(h)/\lambda_j(h)),r}(h),&
\end{aligned}
$$

where $r_j(h)$ is the head angle and $\lambda_j(h)$ is the head radius at q_j of the metric h. Note that the output metric is obtained by gluing the metrics $\mathrm{Fit}^{\mathrm{b/h}}_{p_i,\lambda,r,\lambda\pi-r}(g)$ and $\mathrm{Fit}^{\mathrm{b/h}}_{q_j,\lambda,\lambda(r_j(h)/\lambda_j(h)),r}(h)$ together in the obvious way. Here, the figure $\lambda(r_j(h)/\lambda_j(h))$ is the new head angle obtained after rescaling h. This choice of head angle is somewhat arbitrary. However, it is smaller than $\lambda\frac{\pi}{2}$ (something we require) and seems the most natural choice. To aid the read we depict an example in the case of psc-metrics with bulbs in Figure 19.

We now return to the case when $\boldsymbol{p}=\boldsymbol{q}=\{p_0\}$. As in the case of psc-metrics with torpedo caps, the maps $J^{\mathrm{b/h}(\lambda,r)}_{00}$ for $\lambda>0$, $r\in(0,\lambda\frac{\pi}{2}]$, do not quite give us the product we need. We solve this problem as we did in the torpedo case, when defining the multiplication μ^{torp} in (4-9). Let p_1 and p_2 be two distinct points on $S^n\setminus\{p_0\}$ and redefine $\boldsymbol{p}=\{p_0,p_1,p_2\}$. We now define products on the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)$ as follows. Consider for each $j=1,2$ the map

$$J^{\mathrm{b/h}(\lambda,r)}_{0j}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)\times\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)\to\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p}\setminus\{p_j\})}(S^n)$$

defined as in formula (6-9). We now fix a psc-metric $g_3\in\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p})}(S^n)$ as the second input. Then for each of $j=1,2$, we obtain maps

$$
\begin{aligned}
(6\text{-}10)\qquad J^{\mathrm{b/h}(g_3)}_{0j}\colon \mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(p_0)}(S^n)&\to\mathcal{R}\mathrm{iem}^+_{\mathrm{b/h}(\boldsymbol{p}\setminus\{p_j\})}(S^n),\\
g&\mapsto J^{\mathrm{b/h}(\lambda_j(g_3),r_j(g_3))}_{0j}(g,g_3).
\end{aligned}
$$

Finally, we define products μ^{bulb} and μ^{head} (which we notationally combine as $\mu^{\mathrm{b/h}}$) on the spaces $\mathcal{R}\mathrm{iem}^+_{\mathrm{bulb}(p_0)}(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0)}(S^n)$ by means of the continuous maps

$$\mu^{\mathrm{b/h}}\colon \mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)\times\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)\to\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n),\qquad (g,h)\mapsto J^{g_3}_{02}(h, J^{g_3}_{01}(g,g_3)). \tag{6-11}$$

There are obvious analogues of Lemma 4.8 that clarify the role played by the various choices in determining these maps up to homotopy type, but we will not state them here. We close by pointing out that for certain choices of g_3, the maps $\mu^{\mathrm{b/h}}$ determine an H–space structure on the respective spaces $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)$.

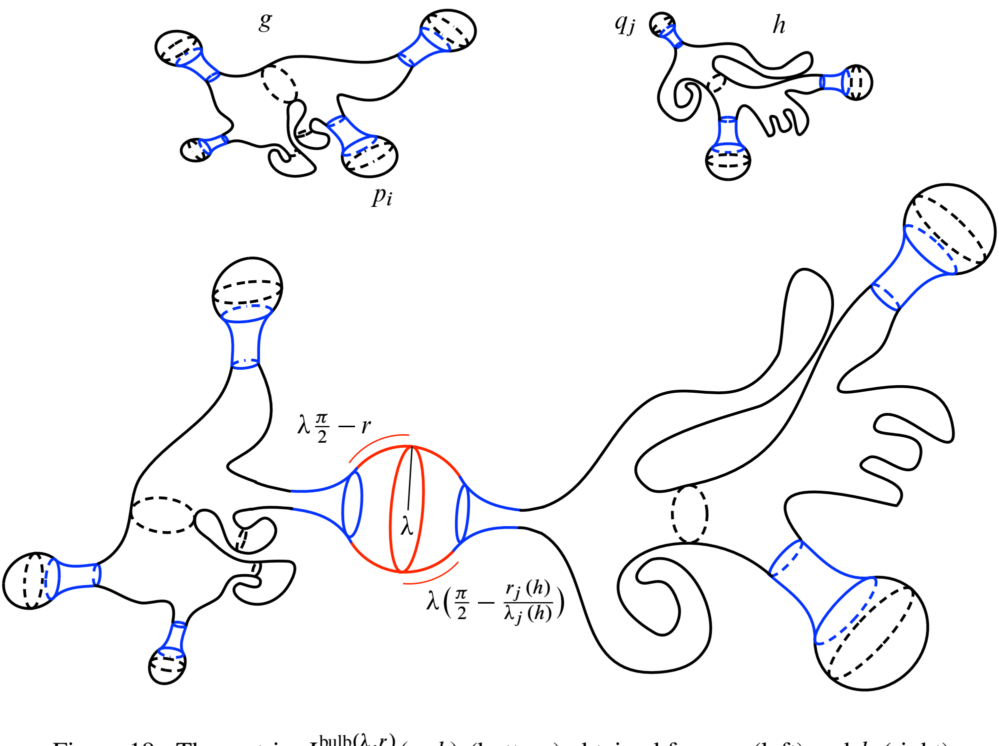

Figure 19: The metric $J^{\mathrm{bulb}(\lambda,r)}_{ij}(g,h)$ (bottom) obtained from g (left) and h (right)

Theorem 6.7 *Let $n\geq 3$ and let $\mu^{\mathrm{b/h}}$ be the multiplication map given by (6-11) with respect to a psc-metric $g_3^{\mathrm{b/h}}\in\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)$. When the metric $g_3^{\mathrm{b/h}}$ is psc-isotopic to the round metric ds_n^2, $\mu^{\mathrm{b/h}}$ defines a homotopy product on $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)$ with homotopy identity $g_{\mathrm{id}}^{\mathrm{b/h}}=ds_n^2$, the round metric of radius 1. This gives $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)$ the structure of an H–space. Furthermore, this product is both homotopy commutative and homotopy associative.*

Remark 6.2 The reader should note that the standard round metric, of any radius, is an element of both spaces $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{b/h}(p_0)}(S^n)$, where p_0 is the north pole and D_{p_0} is the northern hemisphere.

Proof The proof is completely analogous to that of Theorem 5.1. □

In Section 8, we will make considerable use of the multiplication μ^{head} on a subspace of the space $\mathcal{R}\mathrm{iem}^{+}_{\mathrm{head}(p_0)}(S^n)$ to prove our main result. In the mean time we will switch our focus somewhat and discuss a collection of objects known as operads.

7 Little disks, trees and the *W*–construction for operads

We now turn our attention to the second of our main results, Theorem 9.6, which states that $\mathcal{R}\mathrm{iem}^{+}(S^n)$ is weakly homotopy equivalent to an n–fold loop space when $n = 3$ and $n \geq 5$. The problem of recognising when an H–space is an iterated loop space is an old problem in algebraic topology; see [10] for an account of this story. A key step in tackling this problem was the discovery by Boardman and Vogt that an n–fold loop space is a $\mathcal{D}_n$–space. That is, it admits an action of the *operad of little n–dimensional disks*. Before explaining what this means, we should point out that we are interested in a converse to this proposition. That is, given a space which admits such an action, is it an n–fold loop space? Under reasonable conditions such a converse holds. This is the subject of a theorem of Boardman, Vogt and May, which we will state shortly as Theorem 7.4 and which helps us to prove our main result, Theorem 9.6. Before that however, we need to discuss the aforementioned operad of little disks. Much of this section is based on the work of Boardman and Vogt [1; 18], and May [18].

The following definition of an operad is due to P May [12]. The definition itself is rather involved; however, it is followed by two very illustrative examples that the reader may wish to study first. These examples, the operad of little disks and the operad of grown trees, will play a central role in our main construction. An *operad* $\mathcal{P}$ consists of a collection of compactly generated Hausdorff topological spaces $\mathcal{P}(j)$, $j \in \{0, 1, 2, \dots\}$, together with the following data:

(1) The space $\mathcal{P}(0)$ is a single point $*$.

(2) There are continuous maps (known as *composition maps*)

$$\gamma\colon \mathcal{P}(k) \times \mathcal{P}(j_1) \times \mathcal{P}(j_2) \times \cdots \times \mathcal{P}(j_k) \to \mathcal{P}(j),$$

where $\Sigma j_s = j$, and which satisfy the following associativity condition for all $c \in \mathcal{P}(k)$, $d_s \in \mathcal{P}(j_s)$ and $e_t \in \mathcal{P}(i_t)$:

$$\gamma(\gamma(c; d_1, \dots, d_k); e_1, \dots, e_j) = \gamma(c; f_1, \dots, f_k),$$

where $f_s = \gamma(d_s; e_{j_1+j_2+\dots+e_{s-1}+1}, \dots, e_{j_1+\dots+j_s})$, and $f_s = *$ if $j_s = 0$.

(3) There is an identity element $1 \in \mathcal{P}(1)$ so that $\gamma(1; d) = d$ for all $d \in \mathcal{P}(j)$ and $\gamma(c; 1^k)$ for $c \in \mathcal{P}(k)$, $1^k = (1, \dots, 1) \in \mathcal{P}(k)$.

(4) There is a right operation of the symmetric group Σ_j on $\mathcal{P}(j)$ so that the following equivariance formulae are satisfied for all $c \in \mathcal{P}(k)$, $d_s \in \mathcal{P}(j_s)$, $\sigma \in \Sigma_k$ and $\tau_s \in \Sigma_{j_s}$:

$$\gamma(c\sigma; d_1, \dots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(k)})\sigma(j_1, \dots, j_k),$$
$$\gamma(c; d_1\tau_1, \dots, d_k\tau_k) = \gamma(c; d_1, \dots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

where $\sigma(j_1, \dots, j_k)$ denotes the permutation of k letters that permutes the k blocks of letters determined by the given partition of j, and $\tau_1 \oplus \cdots \oplus \tau_k$ denotes the image of $(\tau_1, \dots, \tau_k)$ under the obvious inclusion of $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ into Σ_j.

7.1 The operad of little n–dimensional disks

We now consider a very important example. For $n \geq 1$ we recall that D^n denotes the standard closed unit disk in $\mathbb{R}^n$. For each point p in the interior of D^n and each quantity ϵ where $0 < \epsilon \leq 1 - |p|$, let $D(p, \epsilon)$ denote the round disk of radius ϵ that is centred at p. Let $j \geq 0$ be an integer. We denote by $\mathcal{D}(j)_n$ the set of ordered j–tuples of closed round disks $D(p_i, \epsilon_i)$ where $i = 1, \dots, j$, and which satisfy the condition

$$\mathring{D}(p_i, \epsilon_i) \cap \mathring{D}(p_k, \epsilon_k) = \varnothing \quad \text{for all } i, k \in \{1, \dots, j\}.$$

In the case when $j = 0$, $\mathcal{D}(j)_n$ is just the single point. To ease notation we will fix an n and simply write $\mathcal{D}(j)$ instead of $\mathcal{D}(j)_n$. Each element of $\mathcal{D}(j)$ is therefore an ordered j–tuple of little disks. By viewing each such element as a collection of pairs (p_i, ϵ_i), we may topologise $\mathcal{D}(j)$ by identifying it with an appropriate subspace of the space $(D^n \times I)^j$. There is an obvious action of the permutation group Σ_j on $\mathcal{D}(j)$, where for any pair $c \in \mathcal{D}(j)$, $\sigma \in \Sigma_j$, the element $c\sigma$ has little disks labelled $\sigma(1), \sigma(2), \dots, \sigma(j)$. We illustrate this for an element of $\mathcal{D}(3)$, where $\sigma = (1\,2\,3)$, in Figure 20.

Notice that for each little disk in an element of $\mathcal{D}(j)$, there is a canonical homeomorphism that identifies it with the larger unit disk D^n, ie shrink D^n and translate. This allows us to construct the following "fitting" map. Consider the product space $\mathcal{D}(k) \times \mathcal{D}(j_1) \times \cdots \times \mathcal{D}(j_k)$. Suppose $\{c, d_{j_1}, \dots, d_{j_k}\}$ is an element of this space. The first component c consists of k ordered little disks in D^n. By appropriately shrinking and translating the standard unit disk, we may "fit" each of the elements d_{j_r} into the corresponding r^{th} little disk of c. The resulting object now consists of $j = \Sigma j_s$

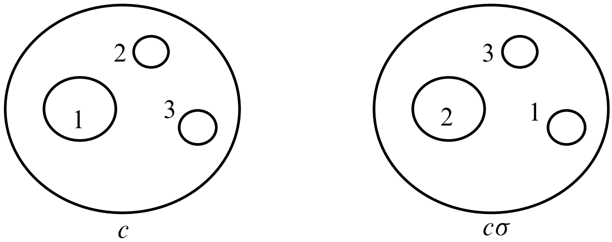

Figure 20: The action of Σ_3 on $\mathcal{D}(3)$

little disks. Regarding the labelling, we apply the following rule. For each element $d_{j_k} \in \mathcal{D}(j_k)$, the corresponding k^{th} little disk in $\mathcal{D}(j)$ obtains its labels from the map

$$(1, 2, \dots, j_k) \mapsto (j_1 + \dots + j_{k-1} + 1, j_1 + \dots + j_{k-1} + 2, \dots, j_1 + \dots + j_{k-1} + j_k).$$

This is shown for a particular example when $k = 2$, $j_1 = 3$ and $j_2 = 2$ in Figure 21. The result is an element of $\mathcal{D}(j)$, which we denote $c(d_{j_1}, \dots, d_{j_k})$. We summarise the *fitting map*, which we denote γ, as

$$\begin{aligned} \gamma\colon \mathcal{D}(k) \times \mathcal{D}(j_1) \times \dots \times \mathcal{D}(j_k) &\to \mathcal{D}(j_1 + \dots + j_k), \\ (c; (d_{j_1}, \dots, d_{j_k})) &\mapsto c(d_{j_1}, \dots, d_{j_k}). \end{aligned}$$

It is a straightforward (albeit tedious) exercise to show that the fitting maps γ satisfy the associativity and permutation equivariance conditions described in properties (1) and (3) of the definition on an operad above. Furthermore, property (2) is satisfied by taking the identity element $1 \in \mathcal{D}(1)$ as the disk D^n with itself as the lone little disk. Finally, we define the space $\mathcal{D}$ as

$$\mathcal{D} = \bigcup_{j \geq 0} \mathcal{D}(j),$$

where $\mathcal{D}(0)$ is the single point $*$. Recall that we have suppressed the dimension, n, of the underlying disk. For each n, the space $\mathcal{D} = \mathcal{D}_n$, along with the appropriate collection of fitting maps γ, is known as the *operad of little n–dimensional disks*. Before discussing our second example, the operad of grown trees, it is worth considering a variation on the little-disks operad, which will be useful for us later on.

7.2 Little disks, little lenses and little bulbs

Recall from Section 3.2, that on a round n–dimensional sphere of radius λ, we described a canonical way of identifying the (λ, r)–lens at the point $p \in S^n$, $B_\lambda(p, r)$, with the disk D^n. We will assume here that $r \in (0, \lambda \frac{\pi}{2}]$. Using this identification we may, for

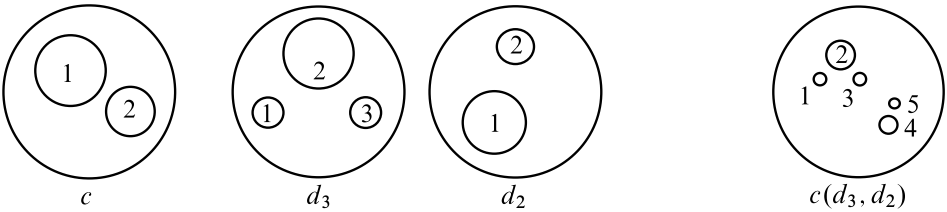

Figure 21: The fitting map γ in action

any point $p \in S^n$ and such a pair λ, r, reinvent the operad of little n–dimensional disks on D^n as an operad of little n–dimensional lenses on $B_\lambda(p, r)$. The centre points of little disks are determined by this identification while the radii are determined by the map $\epsilon \mapsto r\epsilon$, where $\epsilon \in (0, 1]$ denotes the radius of a little disk in D^n. In Figure 22, we depict an example of this where p is the north pole.

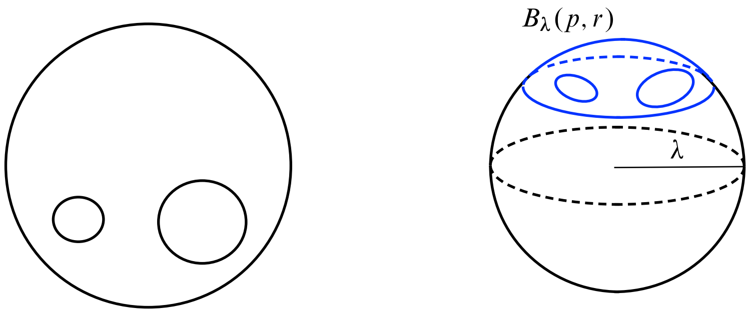

Figure 22: Reinventing the little disks operad as a little lens operad

Thus, instead of thinking of j–tuples of little disks in D^n we may substitute j–tuples of little lenses in $B_\lambda(p, r)$. Furthermore, we can obtain a psc-metric representative of each such element using the work done in Lemmas 6.1 and 6.2. We will state this in the form of a lemma below.

Lemma 7.1 *Let $n \geq 3, p \in S^n, \lambda > 0$ and $r \in (0, \lambda\frac{\pi}{2}]$. Let $c \in \mathcal{D}(j)$ and let $\{B_\lambda(p_i, r_i)\}$ denote the collection of little lenses in $B_\lambda(p, r)$ arising from the identification above. Then there is a psc-metric $g_c \in \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(\boldsymbol{p})}(S^n)$ and a continuous family of psc-metrics $g_c(t_1, \dots, t_j)$ where each $t_i \in I$, so that the following conditions hold:*

1. $g_c(0, \dots, 0) = \lambda^2\, ds_n^2$ *and* $g_c(1, \dots, 1) = g_c$.
2. *Outside of* $\bigcup_{i=1}^{j}\{B_{\lambda(p_i, r_i)}\}$, *the metric* $g_c(t_1, \dots, t_j)$ *is the standard round metric of radius* λ *for all* $(t_1, \dots, t_j) \in I^j$.

(3) *On each ball $B_\lambda(p_i, r_i)$, the metric $g_c(t_1, \dots, t_i = 1, \dots, t_j)$ is precisely the metric obtained by pushing out a bulb with head radius 1 using the method of Lemma 6.2.*

(4) *For each i, there are continuous parameters $\lambda_i(t) > 0$, $r_i(t) > 0$ and $\epsilon_i(t) \in (0, r_i]$, where $t \in I$, so that the restriction $g_c(t_1, \dots, t_j)$ to the ball $B_\lambda(p_i, \epsilon_i)$ is the lens metric $g_{\mathrm{lens}}(\lambda_i(t_i), r_i(t_i))$.*

(5) *The parameters r_i, λ_i and ϵ_i above satisfy $\lambda_i(0) = \lambda$, $\lambda_i(1) = 1$, $r_i(0) = r_i$, $r_i(1) = \frac{\pi}{2}$ and $\epsilon_i(0) = r_i$.*

Proof This is an easy consequence of Lemmas 6.1 and 6.2. □

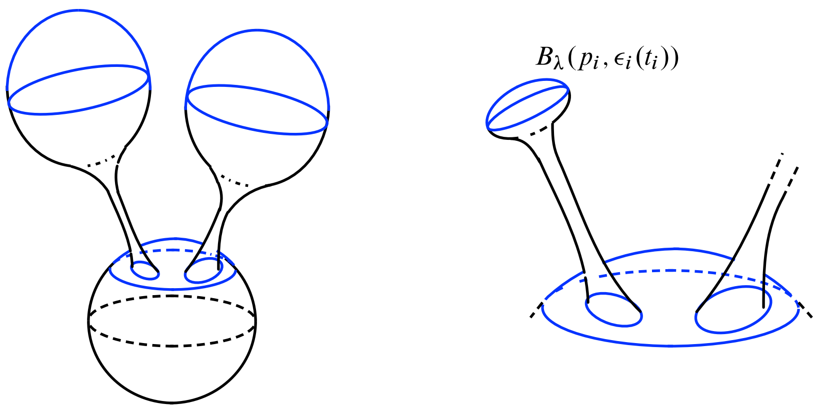

Figure 23: The metric $g_c = g_c(1, \dots, 1)$ is depicted (left) for an element $c \in \mathcal{D}(2)$ along with a special focus on the i^{th} bulb as it undergoes a continuous deformation via the psc-isotopy $g_c(1, \dots t_i, \dots, 1), t_i \in I$ (right). The neighbourhood $B_\lambda(p_i, \epsilon_i(t_i))$ on which the metric takes the form $g_{\mathrm{lens}}(\lambda_i(t_i), r_i(t_i))$ is highlighted.

The point of this rather technical lemma concerns operad composition. Roughly speaking, if we used the round hemisphere of radius 1 at p_i to push out bulbs corresponding to another element $c' \in \mathcal{D}(j)$ for some j, we could continuously deform the resulting psc-metric back to the one which we would have obtained by simply composing the elements c and c' at p_i. Furthermore, conditions (4) and (5) of Lemma 7.1 mean that at each stage in the deformation the metric restricts on a lens cap around p_i, as precisely the psc-metric corresponding to the operad c' on that particular lens. This will be of immense benefit later on. We now consider a second example of an operad.

7.3 The operad of grown trees

A *tree* T is a finite contractible planar graph with the exception that edges may have less than two adjacent vertices. Every tree must have at least one edge; the tree consisting of just one edge and no vertices is called the *trivial tree* and is depicted in Figure 24. Edges that have two adjacent vertices are called *internal edges* while edges with only one adjacent vertex are called *external edges* or *legs*. The edges are oriented in the following way. Each vertex v in T has a set of incoming edges, denoted $\mathrm{In}(v)$, and exactly one outgoing edge. We allow for the case when $\mathrm{In}(v) = \varnothing$. In particular, this means that an internal edge is both an outgoing edge for one of its vertices (the starting vertex) and an incoming edge for the other (ending) vertex. Moreover, the set of external edges of T consists of two mutually disjoint subsets: the set of *inputs* $\mathrm{In}(T)$ of all incoming edges of T that have no starting vertices and the set consisting of the single outgoing edge or *output*, which has no end vertex. We typically depict trees with the edges directed from bottom to top and with inputs ordered from left to right. The orientation of the trivial tree is ambiguous so we simply choose one.

To aid the reader we provide an example. Consider the tree T shown on the right of Figure 24. This tree has 3 internal edges and 7 external edges. The output is the edge at the top adjacent to the vertex v_1. The remaining 6 external edges are inputs. Notice also that, in the case of this tree, $|\mathrm{In}(v_1)| = 2$, $|\mathrm{In}(v_2)| = 4$, $|\mathrm{In}(v_3)| = 3$, $|\mathrm{In}(v_4)| = 0$ and $|\mathrm{In}(T)| = 6$, where $|S|$ is the cardinality of a set S.

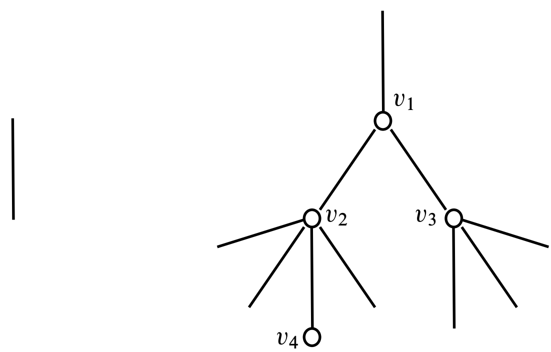

Figure 24: The trivial tree (left) and the tree T (right)

Next, we let $X = \{X_n : n \in \mathbb{N}\}$ be a collection of topological spaces. We will now define a collection of spaces $\mathcal{G}X(j)$ as follows. Let $\mathcal{G}X(0)$ denote the single point $*$. For each $j \in \mathbb{N}$, we let $\mathcal{G}X(j)$ denote the set of all triples (T, α, β), where T is a tree, α is a function that sends each vertex v of T to an element $x \in X_{|\mathrm{In}(v)|}$ and β is a bijection from the set of inputs $\mathrm{In}(T)$ to $\{1, 2, \dots, j\}$. Each space $\mathcal{G}X(j)$ is given the obvious function space topology induced by the vertex labels. This labelling is best thought of as a permutation of the input edges (originally ordered from left to right). Elements of $\mathcal{G}X(j)$ are best thought of as trees with j inputs labelled $1, 2, \dots, j$ and with vertices labelled by elements of X according to the rule that the label associated to a vertex v is an element of $X_{|\mathrm{In}(v)|}$. Finally, we specify composition maps

$$\gamma\colon \mathcal{G}X(k) \times \mathcal{G}X(j_1) \times \mathcal{G}X(j_2) \times \cdots \times \mathcal{G}X(j_k) \to \mathcal{G}X(j_1 + j_2 + \cdots + j_k),$$
$$(T, \psi_1, \psi_2, \dots, \psi_k) \mapsto \phi,$$

where the element ϕ is obtained in the following way. Each ψ_i has inputs labelled $1, 2, \dots, j_i$. For each $i = 1, 2, \dots, k$, relabel these inputs by the rule

$$(1, 2, \dots, j_i)$$
$$\mapsto (j_1 + j_2 + \cdots j_{i-1} + 1, j_1 + j_2 + \cdots + j_{i-1} + 2, \dots, j_1 + j_2 + \cdots + j_{i-1} + j_i).$$

Then identify the lone outgoing edge of each newly labelled ψ_i with the i^{th} input of T. This results in a grown tree ϕ with $j_1 + j_2 + \cdots + j_k$ labelled inputs. Note that the identity element in this case is of course the trivial tree. In Figure 25 we provide an example of this composition with the vertex labels suppressed. In this case $k = 3$, $j_1 = 2$, $j_2 = 2$ and $j_3 = 1$.

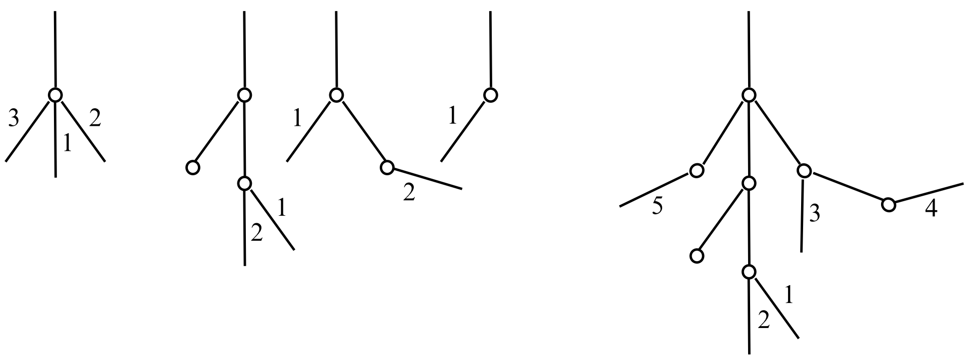

Figure 25: From left to right, the elements T, ψ_1, ψ_2, ψ_3 and ϕ

Finally, we obtain the *operad of grown trees* $\mathcal{G}X$ as the union of the spaces $\mathcal{G}X(j)$ for $j \geq 0$, along with the above composition maps. As with the previous example, it is straightforward to verify that the various operad axioms are satisfied.

7.4 The operad of trees

We now make an important modification to the previous example. Recall that for a collection of topological spaces $X = \{X_n, n \in \mathbb{N}\}$, the space $\mathcal{G}X(j)$ consists of triples

(T, α, β), where T is a tree with j inputs, α is a map that labels each vertex v of T with an element of $X_{|\mathrm{In}(v)|}$ and β is a map that labels the inputs of T with the numbers $1, 2, \ldots, j$. We define the set $\mathcal{T}X(j)$ to be the set of all quadruples $(T, \alpha, \beta, \kappa)$, where T, α and β are as before and κ is a function that assigns to each edge e of T a number $\kappa(e)$ satisfying:

(1) $0 \leq \kappa(e) \leq 1$.

(2) $\kappa(e) = 1$ when e is an external edge (ie input or output) of T.

The number $\kappa(e)$ is called the *length* of the edge e. Each set $\mathcal{T}X(j)$ is then given the obvious function space topology as before. The composition maps defined in the case of spaces $\mathcal{G}X(j)$ are easily generalised to work here. In the case of new internal edges obtained by composing trees, the edges are assigned a length of 1. The union of the resulting collection of spaces, along with the composition maps, gives rise to the *operad of trees* $\mathcal{T}X$. Note that the operad of grown trees $\mathcal{G}X$ can be identified with the suboperad of $\mathcal{T}X$ consisting of trees with only edges of length 1.

7.5 The bar construction

We now consider the case that the collection of spaces $X = \{X_m : m \in \mathbb{N}\}$ introduced above is an operad in its own right, complete with composition maps γ. Of course we need to add in X_0 as the single point space $*$. The example to keep in mind is where X is the collection of spaces $\mathcal{D}(m)$ of m–tuples of little disks with base point the disk of radius 1 in $\mathcal{D}(1)$. We first consider three relations one may impose on the operad $\mathcal{T}X$ above. Note that we will frequently abuse notation and refer to the element $(T, \alpha, \beta, \kappa)$ in $\mathcal{T}X(j)$ as the tree T.

(a) Suppose T is a tree with an internal vertex labelled with the identity element 1 from the space X_1. Furthermore, suppose this vertex's unique incoming edge has length t_1 and its outgoing edge has length t_2. Let T' be the tree that is obtained from T by replacing this vertex and its adjacent edges by a single edge of length $t_1 * t_2 = t_1 + t_2 - t_1 t_2$. This leads to a relation $T \sim T'$ as described in Figure 26.

(b) Let v be a vertex of a tree T in $\mathcal{T}X(j)$ and let T_v be the subtree consisting of v, its unique outgoing edge and all directed paths that end in v. Suppose $\mathrm{In}(v) = k$ and that v is labelled by the element $x.\sigma$, where $x \in X_k$ and $\sigma \in \Sigma_k$ (recall that X is an operad and so there is an action of the symmetric group). Then the relation described in Figure 27 on subtrees induces a relation on trees.

Figure 26: The values t_1, t_2 and $t_1 * t_2$ are the edge lengths

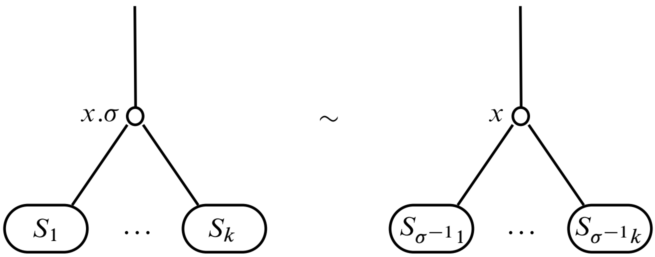

Figure 27: The S_i terms denote subtrees

(c) If T is a tree with an edge of length 0 (this must be an internal edge), then we may shrink this edge away and compose the vertices by means of the operad composition. More precisely, suppose T has edge an e with starting vertex v_2 and ending vertex v_1. Thus e is one of potentially many incoming edges for v_1 but the unique outgoing edge of v_2. Let us say that $|\mathrm{In}(v_1)| = k$. Recall also that v_1 is labelled by an element $x \in X_k$ and v_2 is labelled by some element $y \in X_l$. The value of l is unimportant. Recall that the inputs of v_1 are ordered from left to right and so we assume that $e = e_i$ is the i^{th} edge according to this ordering where $i \in \{1, 2, \dots, k\}$. We replace the edge e_i and its adjacent vertices v_1 and v_2 with a single vertex labelled by the operad element $x \circ_i y$, which is defined as

$$x \circ_i y = \gamma(x; 1, \dots, 1, y, 1, \dots, 1),$$

where the element y appears in the i^{th} position. Note that the resulting element $x \circ_i y$ lies in the space X_{k+l-1}. The edge ordering is repaired in the obvious way. The input edges $e_1, \dots, e_{i-1}$ of the removed vertex v_1 are unaffected. The incoming edges $e'_1, \dots, e'_l$ of the removed vertex v_2 become the incoming edges $e_i, \dots, e_{i+l-1}$ for the new vertex. Finally the edges $e_{i+1}, \dots, e_k$ become the incoming edges $e_{i+l}, \dots, e_{k+l-1}$ for the new vertex. To aid the reader we provide an example where $i = 2$ in Figure 28. Note that in this example

0 represents the edge length, while the numbers $1, 2, 3$ and 4 show the edge ordering.

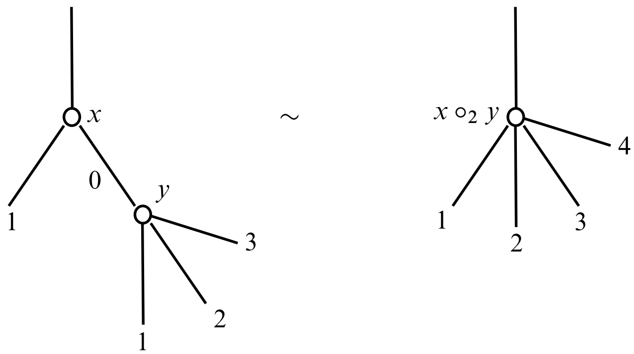

Figure 28: Shrinking an edge of length zero away by composing its vertices

We now state a theorem which follows directly from [10, Theorem 2.20].

Theorem 7.2 *For each operad $\mathcal{P}$, there is an operad $W\mathcal{P}$ defined as*

$$W\mathcal{P} = \mathcal{T}\mathcal{P}/\textit{relations}\ \{(a), (b), (c)\}.$$

The process of constructing $W\mathcal{P}$ from $\mathcal{P}$ is known as the *bar construction* or *W–construction* for operads and is due to Boardman and Vogt. We will now state some important results concerning the relationship between $\mathcal{P}$ and $W\mathcal{P}$.

7.6 Operad actions

Let Z is a topological space and let $\mathcal{P}$ be an operad. We describe Z as a *$\mathcal{P}$–space* if for each integer $k \geq 0$ there are actions

$$\theta\colon \mathcal{P}(k) \times Z^k \to Z$$

so that the following conditions hold:

(1) $\theta(c.\sigma, (z_1, z_2, \dots, z_k)) = \theta(c, (z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(k)}))$ for all $\sigma \in \Sigma_k$, $c \in \mathcal{P}(k)$ and $(z_1, \dots, z_k) \in Z^k$.

(2) The following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(k) \times \mathcal{P}(j_1) \times \mathcal{P}(j_k) \times Z^j \arrow[r, "\gamma \times \mathrm{Id}"] \arrow[d, "\text{shuffle}"] & \mathcal{P}(j) \times Z^j \arrow[r, "\theta"] & Z \arrow[d, "\mathrm{Id}"] \\
\mathcal{P}(k) \times \mathcal{P}(j_1) \times Z^{j_1} \times \cdots \times \mathcal{P}(j_k) \times Z^{j_k} \arrow[r, "\mathrm{Id} \times \theta^k"] & \mathcal{P}(k) \times Z^k \arrow[r, "\theta"] & Z
\end{tikzcd}$$

In the case when $k = 2$, the restriction of this action to an element $\mathcal{P}(2)$ induces a continuous multiplication on Z. This in turn induces a multiplication on $\pi_0(Z)$. We say that Z is *group-like with respect to the action of* $\mathcal{P}$ if this induced multiplication gives $\pi_0(Z)$ the structure of a group for all elements of $\mathcal{P}(2)$. When it is clear which action we are talking about, we will simply say that the space Z is *group-like*. We now state a theorem due to Boardman and Vogt that concerns the bar construction from the previous section.

Theorem 7.3 [1, Theorem 4.37] *A topological space Z is a $\mathcal{P}$–space, for some operad $\mathcal{P}$, if and only it is a $W\mathcal{P}$–space.*

A little later we will show that, for $n \geq 3$, the space of metrics of positive scalar curvature on the sphere S^n is homotopy equivalent to a $W\mathcal{D}_n$–space where $\mathcal{D}_n$ is the operad of little n–dimensional disks. The above theorem allows us to conclude that this space is also homotopy equivalent to a $\mathcal{D}_n$–space. In Section 8 below, we will demonstrate that, for $n \geq 3$, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is homotopy equivalent to a $W\mathcal{D}_n$ and consequently a $\mathcal{D}_n$ space. The importance of this stems from the following theorem due to Boardman, Vogt and May. This is a case of Theorem 13.1 from [12].

Theorem 7.4 (Boardman, Vogt and May [12]) *If a $\mathcal{D}_n$–space Z is group-like, then it is weakly homotopy equivalent to an n–fold loop space.*

Thus, to show that the path component of $\mathcal{R}\mathrm{iem}^+(S^n)$ containing the round metric is weakly homotopy equivalent to an n–fold loop space when $n \geq 3$, it is enough to show that $\mathcal{R}\mathrm{iem}^+(S^n)$ is a $W\mathcal{D}_n$–space. If we wish to use the action of $W\mathcal{D}_n$ to show that the entire space $\mathcal{R}\mathrm{iem}^+(S^n)$ is weakly homotopy equivalent to an n–fold loop space, we must show that $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ is a group under appropriate multiplication. We will return to this problem in the final section, Section 9.

8 Showing that $\mathcal{R}\mathrm{iem}^+(S^n)$ is homotopy equivalent to a $W\mathcal{D}_n$–space

We now return to the sphere S^n, which, as we discussed earlier, is modelled on the standard unit sphere in $\mathbb{R}^{n+1}$. Once again, we assume that $n \geq 3$. We denote by p_0 the north pole $(0, 0, \ldots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$. Recall that immediately preceding (6-6), we defined the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0,1)}(S^n)$ consisting of psc-metrics that take the form of a bulb-head with head radius 1 (but arbitrary head angle $r \in (0, \frac{\pi}{2}]$) on some neighbourhood D_{p_0}. For our purposes, we choose D_{p_0} to a be a geodesic ball

$B_1(p_0, \frac{\pi}{2}+\epsilon)$ for some small $\epsilon \in (0, \frac{\pi}{2})$. The value of ϵ is not important. For each metric $g \in \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0,1)}(S^n)$, the restriction of g to the closed northern hemisphere D_+ is now precisely the round hemisphere of radius 1. To simplify the notation, henceforth we write

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n) = \mathcal{R}\mathrm{iem}^+_{\mathrm{head}(p_0,1)}(S^n).$$

In Lemma 6.6, we showed that, when $n=3$, this space is homotopy equivalent to the space of all psc-metrics on S^n, $\mathcal{R}\mathrm{iem}^+(S^n)$. Thus, in order to demonstrate that $\mathcal{R}\mathrm{iem}^+(S^n)$ has the homotopy type of a $W\mathcal{D}_n$–space (and consequently a $\mathcal{D}_n$–space), it will be sufficient to show this for the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$.

8.1 The action of $W\mathcal{D}_n$ on $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$

We begin by defining a map from $W\mathcal{D}_n$ to $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$. Essentially, psc-metrics in the image of this map will be analogues of the elements of $W\mathcal{D}_n$, which we will use to define the action. In order to define this map, we begin by specifying some rules for associating elements of $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ to certain building blocks of $W\mathcal{D}_n$. To ease notation, we will once more suppress the n, writing $\mathcal{D}_n(j)$ as simply $\mathcal{D}(j)$.

(1) **The trivial tree** We assign the trivial tree to the standard round metric ds_n^2 in $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$.

(2) **A tree with a single vertex** Suppose we have a tree consisting of a single vertex, labelled by the element $c \in \mathcal{D}(j)$. All edges must have length 1. We associate to this tree precisely the element $g_c = g_c(1)$ obtained by Lemma 7.1 with respect to the southern hemisphere D_-.

(3) **A tree with all edges of length 1** We start by associating the root vertex to a psc-metric exactly as in the previous case. This results in a psc-metric with bulb-heads of radius 1 for each of this vertex's input edges. On each of these heads we repeat the previous step. We continue on in this way for all other vertices; see Figure 29 for an example.

(4) **General trees** We must now consider what happens to the psc-metric above if one alters the internal edge lengths. As it stands each edge has length 1 and corresponds to a bulb which has been pushed out by Lemma 7.1. Consider for a moment the i^{th} input edge (currently of length 1) of a vertex v with label $c \in \mathcal{D}(j)$. Recall that the corresponding i^{th} bulb was attained at the $t_i=1$ stage of a psc-isotopy $g_c(1,\dots,t_i,\dots,1)$, $t_i \in I$. If we now replace the edge length of 1 with some other length t_i, we need to perform a corresponding replacement of the metric $g_c(1,\dots,t_i=1,\dots,1)$. One might assume that the metric $g_c(1,\dots,t_i,\dots,1)$ is

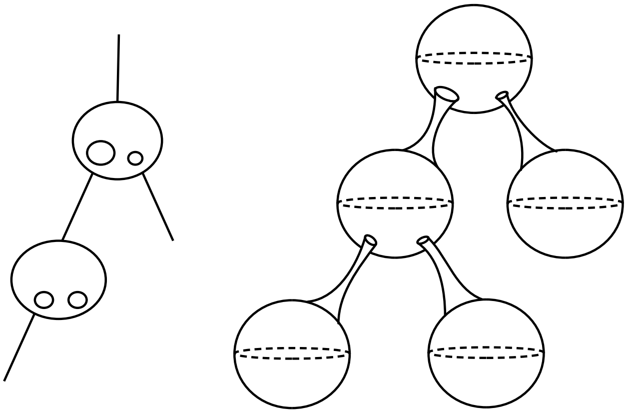

Figure 29: An element of $W\mathcal{D}$ with two vertices and all edges of length 1 (left) and the corresponding psc-metric (right)

the obvious replacement and, usually, this is precisely what we do. Unfortunately, to properly satisfy relation (a) of the bar construction in Section 7.5, there is a case where we must make a tiny adjustment to this association. To deal with this problem, we specify a weighting function $\omega\colon I \to I$; see below. Then, instead of replacing $g_c(1,\dots,t_i = 1,\dots,1)$ with $g_c(1,\dots,t_i,\dots,1)$, we replace it with $g_c(1,\dots,\omega(t_i),\dots,1)$. Of course, this replacement may have the effect of reducing the hemisphere of radius 1 on which operad composition takes place to some general (λ, r)–lens. This is not a problem, given that we have a canonical way of reproducing operad elements on this lens and modifying the construction accordingly, via Lemma 7.1. The weighting function ω satisfies the following properties.

The weighting function ω in the regular case For edges of length $t \in I$, whose non-empty adjacent vertices are labelled by elements of the little disk operad whose little disks all have radius at most $\frac{1}{2}$, we set $\omega(t) = t$. Note that an edge with only one adjacent vertex is an external edge.

The weighting function ω in the special case We consider paths of the following type on a tree $T \in W\mathcal{D}$. All vertices in the path are labelled by an element of the little disks operad with a little disk of radius at least $\frac{3}{4}$, with the exception of the end vertices. Moreover the end vertices may be empty. By including the possibility of an empty vertex, we allow for paths which include external edges. Suppose the edges of this path have lengths $s_1, s_2, \dots, s_k$, in order of the path direction. Such a situation is illustrated in Figure 30, where we draw the path from left to right.

Finally, we define ω on the edge lengths along this path by the recursive formula

$$\omega(s_1) = s_1, \qquad \omega(s_i + 1) = s_{i+1} + \omega(s_i) - s_{i+1}\omega(s_i). \tag{8-1}$$

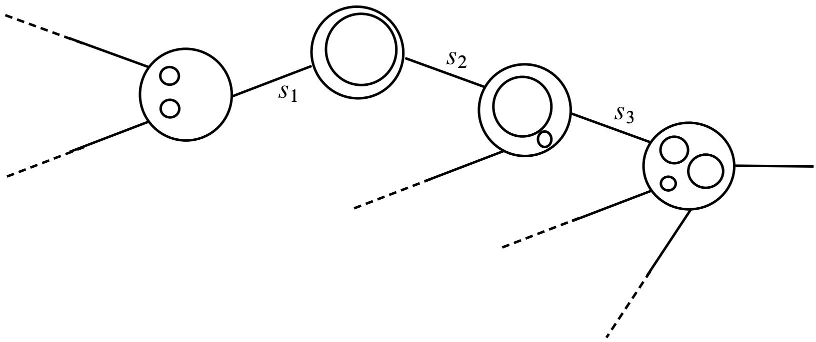

Figure 30: A path in T whose internal vertices are of the type described in the special case

We denote by P the map

$$P\colon W\mathcal{D}_n \to \mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n), \tag{8-2}$$

which sends an element $T \in W\mathcal{D}_n$ to the psc-metric g_T defined by the construction above.

Lemma 8.1 *For $n \geq 3$, the map $P\colon W\mathcal{D}_n \to \mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ is well-defined.*

Proof This involves checking that the relations (a), (b) and (c) of the bar construction in Section 7.5 are satisfied. Relation (c) is satisfied as a result of the construction in Lemma 7.1, which guarantees that shrinking an edge length to zero corresponds to rewinding the psc-isotopy that pushed out a bulb of head radius 1 back to the lens from which it grew. This is precisely the composition we require. It should be obvious that nothing in this construction interferes combinatorially with the tree T and so relation (b) is easily satisfied. Finally, relation (a) is satisfied as a result of the weight function ω on the edge lengths of T. □

We are now in a position to define the action of $W\mathcal{D}_n$ on $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$. Recall that an element $T \in W\mathcal{D}_n(k)$ gives rise to a psc-metric $P(T)$ on S^n with k bulb-heads of radius 1 ordered $1, \dots, k$. Recall that k corresponds to the number of input edges of the tree T. We now define an action θ_{psc} by

$$\begin{aligned} \theta_{\mathrm{psc}}\colon W\mathcal{D}_n(k) \times \mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)^k &\to \mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n), \\ (T; (g_1, g_2, \dots, g_k)) & \\ &\mapsto J_{k0}(J_{(k-1)(0)} \cdots J_{20}(J_{10}(P(T), g_1), g_2) \cdots g_{k-1}), g_k), \end{aligned} \tag{8-3}$$

where $J_{ij} = J_{ij}^{\mathrm{head}(1,\frac{\pi}{2})}$ is the map defined in (6-9). Simply put, we cut off the round radius 1 hemispheres from the k bulb heads on $P(T)$ and on the north pole for each g_i, where $i \in \{1, \dots, k\}$. We then glue in the obvious way according to label. To aid the reader, we depict an example in Figure 31.

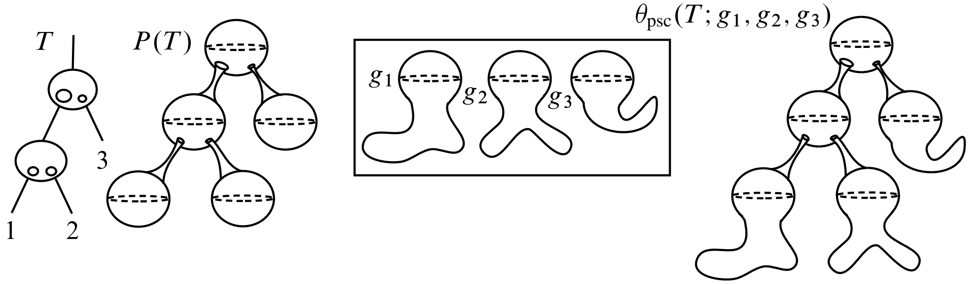

Figure 31: An example of the operad action when $k = 3$

We now state a lemma concerning this action.

Lemma 8.2 *When $n \geq 3$, the action θ_{psc} defined in (8-3) gives $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$ the structure of a $W\mathcal{D}_n$–space.*

Proof We need to verify that the map θ_{psc} satisfies conditions (1) and (2) in the definition of an operad action at the beginning of Section 7.6. Showing that condition (1) is satisfied is an easy combinatorial exercise. The second condition, which concerns composition, is a little more subtle. The main pitfall is as follows. Suppose we compose trees T_1 and T_2 to obtain T_3. We need to be sure that metric $P(T_3)$ is precisely the metric obtained by the corresponding attachment of the metrics $P(T_1)$ and $P(T_2)$. In the case when all edges are of length 1, this is obvious by the construction. However, in the case of more general trees we have to consider the effects of the weighting function ω on the lengths of edges. Recall that for certain edges, ones that are part of a special case described above, we have to ensure that the function ω respects tree composition. The saving grace here is the way in which we compose trees. Recall from Section 7.4 that this composition involves the identification of the outgoing external edge of one tree with an incoming external edge of another. The new edge length is always 1. Now suppose that the edge length directly above this newly formed edge has edge length t and that the edge below has length s. We therefore have a sequence of 3 edges with lengths, listed in order from bottom to top: $s, 1, t$. When we apply ω we obtain the following new values:

$$t \mapsto \omega(t), \qquad 1 \mapsto \omega(1) = 1 + \omega(s) - 1.\omega(s) = 1, \qquad s \mapsto \omega(s).$$

Thus, the edges of length 1 act as "resets" in our recursive formula for ω. In particular, this means that the weight information above the point of composition is unaffected by the newly added subtree. □

Corollary 8.3 *When $n \geq 3$, the space $\mathcal{R}\mathrm{iem}^+(S^n)$ has the homotopy type of a $\mathcal{D}_n$–space.*

Proof This is an immediate consequence of Lemma 8.2 above, Theorem 7.3 and Lemma 6.6. □

We have overcome one significant obstacle to proving our main result. In the next section, we will deal with the other.

9 The group-like condition

In this section we consider the problem of whether or not the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$ is group-like under the action of $W\mathcal{D}_n$. Recall that by restricting the action to an element $T \in W\mathcal{D}_n(2)$, we obtain the multiplication map

$$\begin{aligned} \mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n) \times \mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n) &\to \mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n), \\ (g,h) &\mapsto \theta_{\mathrm{psc}}(T;(g,h)), \end{aligned} \tag{9-1}$$

where θ_{psc} is the map defined in Equation (8-3). In turn this map induces a multiplication on $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n))$. To show that the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$ is group-like under the action of $W\mathcal{D}_n$ means showing that $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n))$ is a group under this multiplication for each element $T \in W\mathcal{D}_n(2)$. To simplify matters we make the following observation.

Proposition 9.1 *Let $n \geq 3$ and let T and T' be any two elements of $W\mathcal{D}_n(2)$. Then the maps $(g,h) \mapsto \theta_{\mathrm{psc}}(T;(g,h))$ and $(g,h) \mapsto \theta_{\mathrm{psc}}(T';(g,h))$, defined in Equation (9-1), induce the same multiplicative structure on $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n))$.*

Proof This is essentially just Lemma 4.8, and is a consequence of the family version of the Gromov–Lawson construction described in Lemma 4.6. □

We now reconsider the more general space $\mathcal{R}\mathrm{iem}^+(S^n)$. The multiplication described above is just a modified version of the Gromov–Lawson connected sum construction described in Section 4.4. In the case of two psc-metrics g_0 and g_1 on S^n, we may form a new psc-metric $g_0 \# g_1$ by removing disks from (S^n, g_0) and (S^n, g_1) and connecting the resulting disks with a cylinder $S^{n-1} \times I$ equipped with an appropriate connecting psc-metric, à la Gromov and Lawson in [6]. Strictly speaking, there are parameter choices to be made in this construction and so it does not initially specify a well-defined multiplication on $\mathcal{R}\mathrm{iem}^+(S^n)$. However, it is an easy corollary of Lemma 6.1 that it does induce a well-defined binary operation on $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$. In other words, after passing to $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$, any choice of parameters gives the same multiplication. At this stage we have described a multiplicative structure on $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n))$ arising from the action of $W\mathcal{D}_n$ and a multiplicative structure on $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ arising from the Gromov–Lawson construction. We now simplify matters still further with the following proposition.

Proposition 9.2 *The inclusion $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n) \subset \mathcal{R}\mathrm{iem}^+(S^n)$ induces a bijection between $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n))$ and $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$, which respects the multiplicative structures.*

Proof We know from Lemmas 4.7 and 6.6 that when $n \geq 3$, the spaces $\mathcal{R}\mathrm{iem}^+(S^n)$ and $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ are homotopy equivalent. The inclusion

$$\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n) \subset \mathcal{R}\mathrm{iem}^+(S^n)$$

therefore induces a bijection between $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n))$ and $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$. The rest follows from the fact that the intermediary metric used in determining the product on $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ is isotopic to the standard round metric. This means metrics resulting from this product are easily deformed by psc-isotopy to a regular Gromov–Lawson style connected sum. Hence the two operations behave in the same way with regard to path components. □

Corollary 9.3 *The space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}_+(1)}(S^n)$ is group-like under the action of $W\mathcal{D}_n$ if and only if $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ is a group under the multiplication induced by the Gromov–Lawson connected sum construction.*

Given Corollary 9.3, it is enough to focus on the problem of whether or not the set $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ is a group under the operation induced by the Gromov–Lawson connected sum construction. The set $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ is of course the set of path components of $\mathcal{R}\mathrm{iem}^+(S^n)$. Earlier in the paper we noted that two metrics in $\mathcal{R}\mathrm{iem}^+(S^n)$ are said to be *psc-isotopic* if they lie in the same path component. The notion of *psc-isotopy* is therefore an equivalence relation on the space $\mathcal{R}\mathrm{iem}^+(S^n)$. A related notion, which we will make use of shortly is the notion of *psc-concordance*. In the case of metrics $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(S^n)$, we say that g_0 and g_1 are *psc-concordant* if there is a psc-metric $\bar{g}$ on $S^n \times I$ that near $S^n \times \{0\}$ takes the form of a product $g_0 + dt^2$

and near $S^n \times \{1\}$ takes the form $g_1 + dt^2$. Again, psc-concordance is an equivalence relation on the set of psc-metrics $\mathcal{R}\mathrm{iem}^+(S^n)$. It is a well known fact that metrics that are psc-isotopic are psc-concordant; see [19, Lemma 1.3] for example. Recent work by Botvinnik in [2] has shown that, under reasonable hypotheses, the converse is true. In particular, we have the following conjecture of Botvinnik.

Conjecture (Botvinni [2]) Let $g_0, g_1 \in \mathcal{R}\mathrm{iem}^+(S^n)$. When $n \geq 5$, g_0 is psc-isotopic to g_1 if and only if g_1 is psc-concordant to g_1.

It is worth noting that the hypothesis that n be at least five cannot be removed as the above result fails to be true when $n = 4$; see Ruberman [16]. In the case when $n = 3$, it is demonstrated by Marques in [11] that the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is path connected, and so Botvinnik's conjecture holds here for trivial reasons. We now return to the problem of equipping $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ with a group structure. We will write $BC(n)$ to denote Botvinnik's conjecture in dimension n.

Lemma 9.4 *For $n = 3$ or $n \geq 5$ and provided $BC(n)$ holds, the set $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$ is a group under the operation induced by Gromov–Lawson connected sum of metrics.*

Proof Corollary 1.1 of [11] states that the space $\mathcal{R}\mathrm{iem}^+(S^n)$ is path connected when $n = 3$. We therefore concentrate on the case when $n \geq 5$. Verifying that the various group axioms hold is mostly straightforward. In particular, it is clear that the class containing the standard round metric, $[ds_n^2]$, is the identity. The only difficulty lies in verifying that each element has an inverse. This is where we make use of Botvinnik's conjecture.

Gajer in [5] shows that the set of psc-concordance classes of $\mathcal{R}\mathrm{iem}^+(S^n)$, which we denote $\pi_0^c(\mathcal{R}\mathrm{iem}^+(S^n))$, forms a group also under the operation induced by connected sum. We will not reprove it here, but it is worth briefly recounting Gajer's method for showing that each psc-concordance class has an inverse, as we will make good use of it. Given a psc-metric g on S^n that represents a particular psc-concordance class, equip $S^n \times I$ with the standard product $g + dt^2$. Let $p \in S^n$ be any point. Consider the arc $\{p\} \times I$ in $S^n \times I$. Using Lemma 6.1 in a slicewise fashion, one can easily adjust the metric $g + dt^2$ in a neighbourhood of this arc to obtain a psc-metric $g' + dt^2$ so that near $\{p\} \times I$, $g' + dt^2 = g^n_{\mathrm{tor}}(\delta) + dt^2$ for some $\delta > 0$. Recall that $g^n_{\mathrm{tor}}(\delta)$ is the standard torpedo metric of radius δ on the disk. Next, we use the Gromov–Lawson method to push out a torpedo cap away from this neighbourhood and preserve positive scalar curvature. By first removing the cap part, then removing the previously constructed "cylinder of caps" and finally smoothing out the inevitable corners, we are left with a manifold that is topologically a cylinder $S^n \times I$ but with a very different metric; see Figure 32. At one end we have a standard round metric of radius δ. At the other end, we have the psc-metric $g \# \tilde{g}$ obtained by taking a connected sum of g and $\tilde{g}$. Here $\tilde{g}$ is isometric to g, but via an orientation-reversing isometry. In Theorem 2.2 of [19], we show in great detail how to adjust a psc-metric on a manifold with boundary in precisely this situation in order to obtain a psc-metric which has a product structure near the boundary. On performing such an adjustment we obtain a psc-concordance between $\delta^2\, ds_n^2$ and $g \# \tilde{g}$ and thus between ds_n^2 and $g \# \tilde{g}$. Thus the classes containing g and $\tilde{g}$ are inverses in the group $\pi_0^c(\mathcal{R}\mathrm{iem}^+(S^n))$. More succinctly, $[g]^{-1} = [\tilde{g}]$.

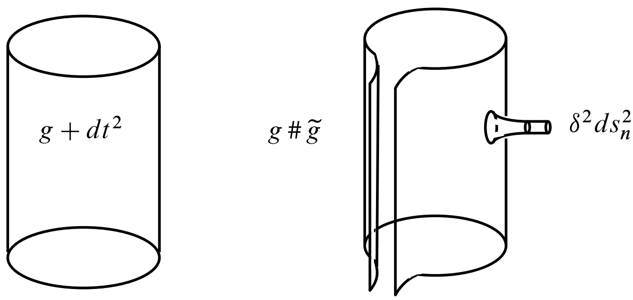

Figure 32: The cylinder $g + dt^2$ (left) and the metric that gives rise, after adjustment, to the concordance between $g \# \tilde{g}$ and $\delta^2 ds_n^2$ (right)

To show that g and $\tilde{g}$ represent inverse elements in $\pi_0(\mathcal{R}\mathrm{iem}^+(S^n))$, we need only show that as well as being psc-concordant, the round metric and the connected sum of g and $\tilde{g}$ are also psc-isotopic. This is immediate if Botvinnik's conjecture holds. □

Corollary 9.5 *If $n = 3$ or $n \geq 5$, and $BC(n)$ holds, the set $\pi_0(\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n))$ is a group under the operation induced by the action of $W\mathcal{D}_n$ defined in Equation (8-3). In other words, the space $\mathcal{R}\mathrm{iem}^+_{\mathrm{D}+(1)}(S^n)$ is group-like with respect to this action.*

By combining Corollary 9.5 with Theorem 7.4, we obtain our final result.

Theorem 9.6 *For $n = 3$ or $n \geq 5$ and provided $BC(n)$ holds, the space of positive scalar curvature metrics on the n–dimensional sphere, $\mathcal{R}\mathrm{iem}^+(S^n)$, is weakly homotopy equivalent to an n–fold loop space.*

Corollary 9.7 *For $n = 3$ or $n \geq 5$ and provided $BC(n)$ holds, all path components of the space $\mathcal{R}\mathrm{iem}^+(S^n)$ are weakly homotopy equivalent.*

References

[1] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics 347, Springer, New York (1973) MR0420609

[2] **B Botvinnik**, *Concordance and isotopy of metrics with positive scalar curvature*, Geom. Funct. Anal. 23 (2013) 1099–1144 MR3077909

[3] **R Carr**, *Construction of manifolds of positive scalar curvature*, Trans. Amer. Math. Soc. 307 (1988) 63–74 MR936805

[4] **D Crowley**, **T Schick**, *The Gromoll filtration, KO–characteristic classes and metrics of positive scalar curvature*, Geom. Topol. 17 (2013) 1773–1789 MR3073935

[5] **P Gajer**, *Riemannian metrics of positive scalar curvature on compact manifolds with boundary*, Ann. Global Anal. Geom. 5 (1987) 179–191 MR962295

[6] **M Gromov**, **H B Lawson, Jr**, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. 111 (1980) 423–434 MR577131

[7] **B Hanke**, **T Schick**, **W Steimle**, *The space of metrics of positive scalar curvature* (2014) arXiv:1212.0068v3

[8] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR1867354

[9] **N Hitchin**, *Harmonic spinors*, Advances in Math. 14 (1974) 1–55 MR0358873

[10] **M Markl**, **S Shnider**, **J Stasheff**, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs 96, Amer. Math. Soc. (2002) MR1898414

[11] **F C Marques**, *Deforming three-manifolds with positive scalar curvature*, Ann. of Math. 176 (2012) 815–863 MR2950765

[12] **J P May**, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271, Springer, New York (1972) MR0420610

[13] **R S Palais**, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966) 1–16 MR0189028

[14] **P Petersen**, *Riemannian geometry*, 2nd edition, Graduate Texts in Mathematics 171, Springer, New York (2006) MR2243772

[15] **J Rosenberg**, **S Stolz**, *Metrics of positive scalar curvature and connections with surgery*, from: "Surveys on surgery theory, Vol. 2", (S Cappell, A Ranicki, J Rosenberg, editors), Ann. of Math. Stud. 149, Princeton Univ. Press (2001) 353–386 MR1818778

[16] **D Ruberman**, *Positive scalar curvature, diffeomorphisms and the Seiberg–Witten invariants*, Geom. Topol. 5 (2001) 895–924 MR1874146

[17] **R Schoen**, **S-T Yau**, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979) 159–183 MR535700

[18] **R M Vogt**, *Cofibrant operads and universal E_∞ operads*, Topology Appl. 133 (2003) 69–87 MR1996461

[19] **M Walsh**, *Metrics of positive scalar curvature and generalised Morse functions, I*, Mem. Amer. Math. Soc. 983, American Mathematical Society (2011) MR2789750

[20] **M Walsh**, *Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics*, Proc. Amer. Math. Soc. 141 (2013) 2475–2484 MR3043028

[21] **M Walsh**, *Metrics of positive scalar curvature and generalised Morse functions, II*, Trans. Amer. Math. Soc. 366 (2014) 1–50 MR3118389

Department of Mathematics, Statistics and Physics, Wichita State University
1845 Fairmount, Wichita, KS 67260, USA

walsh@math.wichita.edu

Proposed: Bill Dwyer Received: 6 February 2013
Seconded: John Lott, Peter Teichner Revised: 3 January 2014