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Geometry & Topology 18 (2014) 2477–2486

Homogeneous Ricci solitons are algebraic

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In this short note, we show that homogeneous Ricci solitons are algebraic. As an application, we see that the generalized Alekseevskii conjecture is equivalent to the Alekseevskii conjecture.

53C25; 22E25, 53C30

1 Introduction

A Riemannian manifold (M, g) is said to be a Ricci soliton if it satisfies the equation

$$\mathrm{ric}_g = cg + L_X g \tag{1-1}$$

for some $c \in \mathbb{R}$ and some smooth vector field $X \in \mathfrak{X}(M)$. Such metrics are of interest as they correspond to self-similar solutions of the Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{ric}_g\,.$$

That is, g is the initial value of a solution to the Ricci flow of the form $g_t = c(t)\varphi_t^* g$, where $c(t) \in \mathbb{R}$ and $\varphi_t \in \mathrm{Diff}(M)$. In this way, Ricci solitons are geometric fixed points of the flow and so are special metrics.

Homogeneous Ricci solitons arise naturally as limits under the Ricci flow (Lott [15], Lauret [14]) and, independently, hold a distinguished place apart from other homogeneous metrics. For example, nilmanifolds cannot admit Einstein metrics, but do often admit Ricci solitons (Jensen [9], Jablonski [6]), Ricci solitons on nilmanifolds are precisely the minima of a natural geometric functional (Lauret [13]), and Ricci solitons are metrics of maximal symmetry on certain solvmanifolds (Jablonski [5]).

One natural kind of example arises as follows. Consider a homogeneous space G/K, where K is closed and connected. For every derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\colon \mathfrak{k} \to \mathfrak{k}$, we have a well-defined map $D_{\mathfrak{g}/\mathfrak{k}}\colon \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k}$. Denote such derivations of $\mathfrak{g}$ by $\mathrm{Der}(\mathfrak{g}/\mathfrak{k})$. A homogeneous Ricci soliton $(G/K, g)$ is called *G–semialgebraic* if the $(1,1)$ Ricci tensor is of the form

$$\mathrm{Ric} = c\,\mathrm{Id} + \tfrac{1}{2}\big(D_{\mathfrak{g}/\mathfrak{k}} + {D_{\mathfrak{g}/\mathfrak{k}}}^t\big) \tag{1-2}$$

Published: 2 October 2014 DOI: 10.2140/gt.2014.18.2477

on $\mathfrak{g}/\mathfrak{k} \simeq T_e G/K$, for some $c \in \mathbb{R}$ and some $D \in \mathrm{Der}(\mathfrak{g}/\mathfrak{k})$. This definition is motivated by the idea of taking our family of diffeomorphisms $\{\varphi_t\}$ above to come from automorphisms of the group G which leave K invariant; see Jablonski [7] or Lafuente and Lauret [12] for more details.

If our semialgebraic Ricci soliton satisfies the seemingly stronger condition that $D_{\mathfrak{g}/\mathfrak{k}}$ is symmetric, then it is called a *G–algebraic Ricci soliton*. Up to this point, all known examples of semialgebraic Ricci solitons were in fact algebraic and isometric to solvmanifolds. (This follows from [7] together with [11] by Lafuente and Lauret.) Further, it was known that every homogeneous Ricci soliton must be semialgebraic relative to its full isometry group [7]. We now present our main result.

Theorem 1 *Every G–semialgebraic Ricci soliton is necessarily G–algebraic.*

Corollary 2 *Let (M, g) be a homogeneous Ricci soliton. There exists a transitive group G of isometries such that $M = G/K$ is a G–algebraic Ricci soliton.*

The theorem above resolves questions raised by Lafuente and Lauret [12] and He, Petersen and Wylie [4]. In these works, it was shown that one can always extend a simply connected, algebraic soliton to an Einstein metric on a larger homogeneous space. There the goal was to relate the classical Alekseevskii conjecture on Einstein metrics to a more general version for Ricci solitons. More precisely, they showed that (among simply connected manifolds) the Alekseevskii conjecture for Einstein metrics is equivalent to the (a priori) more general conjecture in the case of algebraic Ricci solitons. We state these conjectures for completeness.

Alekseevskii conjecture *Every homogeneous Einstein metric with negative scalar curvature is isometric to a simply connected solvmanifold.*

Generalized Alekseevskii conjecture *Every expanding homogeneous Ricci soliton is isometric to a simply connected solvmanifold.*

Until now, it was not clear if these conjectures were equivalent. Applying the results of Lafuente and Lauret [12] or He, Petersen and Wylie [4] in the simply connected case, together with those from Jablonski [8] and the results presented here, we now know the following:

Theorem 3 *The generalized Alekseevskii conjecture is equivalent to the Alekseevskii conjecture.*

Remark It is important to note that the Alekseevskii conjecture stated above is a more modern, geometric version than that given by Besse in [2]. The version in [2] has the weaker, topological conclusion that a noncompact, homogeneous, Einstein space is only diffeomorphic to $\mathbb{R}^n$. It is still an open question whether the classical version stated in [2] is equivalent to the stronger version we pose above.

Acknowledgments It is our pleasure to thank Ramiro Lafuente for providing useful comments on a draft of this manuscript. This work was supported in part by NSF grant DMS-1105647

2 Ricci solitons by type

The analysis of (homogeneous) Ricci solitons varies depending on which of the following categories the metric falls into. A Ricci soliton is called *shrinking*, *steady* or *expanding* if the cosmological constant c appearing in (1-1) satisfies $c > 0$, $c = 0$ or $c < 0$, respectively.

Shrinking solitons The simplest example of a non-Einstein, homogeneous, shrinker is obtained by considering a compact homogeneous Einstein space M' (which necessarily has positive scalar curvature) and taking a product with $\mathbb{R}^n$, ie $M = M' \times \mathbb{R}^n$. Here the vector field $X \in \mathfrak{X}(M)$ appearing in (1-1) generates a family of diffeomorphisms which simply dilate the $\mathbb{R}^n$ factor. Examples of this type are called trivial Ricci solitons and a result of Petersen and Wylie [16] says that every homogeneous shrinking Ricci soliton is finitely covered by a trivial one. Observe that such spaces are algebraic Ricci solitons.

Steady solitons A homogeneous steady soliton is necessarily flat. This well-known fact is proved as follows. Along the Ricci flow of any homogeneous manifold, the scalar curvature sc evolves by the ODE

$$\frac{d}{dt}\,\mathrm{sc} = 2|\mathrm{Ric}|^2.$$

As the scalar curvature of a steady soliton does not change along the flow, we see that the homogeneous, steady solitons are Ricci flat and so flat by [1]. Such spaces are trivially algebraic Ricci solitons.

Expanding solitons Every homogeneous, expanding Ricci soliton is necessarily noncompact, nongradient and all known examples of such spaces are isometric to solvable Lie groups with left-invariant metrics. While there is no characterization in this case as nice as in the previous two cases, new structural results have recently appeared in a paper by Lafuente and Lauret [12]. The results obtained there are essential in our proof and we briefly recall those which we need.

We first observe that it suffices to prove the theorem for simply connected manifolds. Now consider a simply connected, expanding, semialgebraic Ricci soliton on G/K. As G/K is endowed with a G–invariant metric, $\mathrm{Ad}(K)$ is contained in a compact subgroup of $\mathrm{Aut}(G)$ and so we have a decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{p}$ is an $\mathrm{Ad}(K)$–complement to $\mathfrak{k}$. We fix the point $p = eK \in M = G/K$ and naturally identify $\mathfrak{p}$ with T_pM by

$$X \in \mathfrak{p} \quad \leftrightarrow \quad \frac{d}{ds}\Big|_{s=0} \exp(sX)\cdot p = \frac{d}{ds}\Big|_{s=0} \exp(sX)K.$$

Although there is more than one possible choice of $\mathfrak{p}$, in the sequel we apply some results of [12] and so we choose, as they do, to have $B(\mathfrak{k},\mathfrak{p}) = 0$, where B is the Killing form of $\mathfrak{g}$.

As G/K admits an expanding Ricci soliton, we know from [12] that the group G decomposes as $N \rtimes U$, where N is the nilradical and U is a reductive subgroup which contains the stabilizer K. Thus the underlying manifold of M may be considered as $N \times U/K$ and we naturally identify the point $p = eK \in G/K$ with $(e, eK) \in N \times U/K$. The subalgebra $\mathfrak{u}$ contains a subspace $\mathfrak{h}$ which is complementary to $\mathfrak{k}$, and so we have $T_pM \simeq \mathfrak{p} = \mathfrak{n} \oplus \mathfrak{h}$. Furthermore, $\mathfrak{n}$ and $\mathfrak{h}$ are orthogonal subspaces of T_pM. For more details, see [12].

Denote the restriction of our metric g to $\mathfrak{p} \simeq T_eG/K$ by $\langle\cdot,\cdot\rangle$. Denote by $H \in \mathfrak{p}$ the "mean curvature vector" of G/K defined by

$$\langle H, X\rangle = \mathrm{tr}(\mathrm{ad}\, X) \quad \text{for all } X \in \mathfrak{p}.$$

Observe that $H \in \mathfrak{h}$. It is a useful fact that the subspace $\mathfrak{h}$ of $\mathfrak{u}$ is $(\mathrm{ad}\, H)$–stable [12, Proposition 4.1]. If D is the soliton derivation appearing in (1-2), then we have

$$D = -\,\mathrm{ad}\, H + D_1,$$

where D_1 is the derivation which vanishes on $\mathfrak{u}$ and restricts to the nilsoliton derivation on $\mathfrak{n}$.

In [12, Proposition 4.14], several equivalent conditions are given for when a semialgebraic Ricci soliton is actually algebraic. One of those conditions is

$$S(\mathrm{ad}\, H|_{\mathfrak{h}}) = 0, \tag{2-1}$$

where $S(A) = \frac{1}{2}(A + A^t)$. This is the technical result that we will prove, from which the theorem follows.

3 The proof of Theorem 1

The soliton inner product $\langle\cdot,\cdot\rangle$ on T_pM above gives rise to a natural inner product on the endomorphisms of T_pM given by $\langle A, B\rangle = \mathrm{tr}(AB^t)$, where B^t denotes the metric adjoint of B relative to $\langle\cdot,\cdot\rangle$.

Lemma 4 *Using the above inner product on endomorphisms we have*

$$\big\langle(0, \mathrm{ad}\, H|_{\mathfrak{h}}), \mathrm{Ric}\big\rangle = 0,$$

where $(0, \mathrm{ad}\, H|_{\mathfrak{h}})$ *is the map on* T_pM *defined as* 0 *on* $\mathfrak{n}$ *and* $\mathrm{ad}\, H|_{\mathfrak{h}}$ *on* $\mathfrak{h}$.

Remark As has been observed by Lafuente [10], our proof of the lemma holds more generally. In fact, one simply needs the group to satisfy $G = U \ltimes N$ with N nilpotent, U reductive and $K < U$, and the metric to satisfy $N \perp U/K$ at eK, whereas the element H may be replaced by any $Y \in \mathfrak{u}$ satisfying $[Y, \mathfrak{k}] \subset \mathfrak{k}$.

Before proving the lemma, we use it to verify that (2-1) holds.

Verification of (2-1) Consider the mean curvature vector $H \in \mathfrak{u}$. As $\mathfrak{u}$ is reductive, $\mathrm{ad}\, H|_{\mathfrak{u}}$ is traceless. Furthermore, since $\mathrm{ad}\, H$ vanishes on the stabilizer $\mathfrak{k}$ (see [12, Equation 26]) and $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{h}$, we see that $\mathrm{tr}(\mathrm{ad}\, H|_{\mathfrak{h}}) = 0$. Together with the above lemma we have

$$\begin{aligned} 0 = \langle(0, \mathrm{ad}\, H|_{\mathfrak{h}}), \mathrm{Ric}\rangle &= \langle(0, \mathrm{ad}\, H|_{\mathfrak{h}}), c\,\mathrm{Id} - S(\mathrm{ad}\, H) + D_1\rangle \\ &= \langle \mathrm{ad}\, H|_{\mathfrak{h}}, c\,\mathrm{Id}|_{\mathfrak{h}} - S(\mathrm{ad}\, H|_{\mathfrak{h}})\rangle \\ &= c\,\mathrm{tr}(\mathrm{ad}\, H|_{\mathfrak{h}}) - \mathrm{tr}\, S(\mathrm{ad}\, H|_{\mathfrak{h}})^2 = 0 - \mathrm{tr}\, S(\mathrm{ad}\, H|_{\mathfrak{h}})^2. \end{aligned}$$

Thus $S(\mathrm{ad}\, H|_{\mathfrak{h}}) = 0$, as claimed. □

Proof of Lemma 4 We now prove the lemma by considering a certain deformation of the metric g on M. As $\mathrm{ad}\, H$ vanishes on $\mathfrak{k}$ and K is connected, the family of automorphisms $\Phi_t = C_{\exp(tH)} \in \mathrm{Aut}(U)$ is the identity on K and hence gives rise to well-defined diffeomorphisms ϕ_t on U/K given by

$$\phi_t(uK) = \Phi_t(u)K \quad \text{for } u \in U.$$

Note that $(\Phi_t)_* = \mathrm{Ad}(\exp(tH)) = e^{t\,\mathrm{ad}\, H} \in \mathrm{Aut}(\mathfrak{u})$. On the manifold $M = N \times U/K$, we consider the family of diffeomorphisms given by

$$\varphi_t = (\mathrm{id}, \phi_t) \quad \text{on } N \times U/K.$$

The deformations of g of interest are $g_t = {\varphi_t}^* g$.

As φ_t fixes the point $p := eK = (e, eK) \in M = N \times U/K$, and scalar curvature is an invariant, we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{sc}({\varphi_t}^* g)_p = 0.$$

We use this in the equation (which holds for any family of metrics $\{g_t\}$ with variation $h = \frac{\partial}{\partial t} g_t$; see [3, Lemma 3.7])

$$\frac{\partial}{\partial t}\,\mathrm{sc} = -\Delta \overline{H} + \mathrm{div}(\mathrm{div}\, h) - \langle h, \mathrm{ric}\rangle, \tag{3-1}$$

where in local coordinates we have

$$\Delta \overline{H} = g^{ij} g^{kl} \nabla_i \nabla_j h_{kl}\,, \tag{3-2}$$

$$\mathrm{div}(\mathrm{div}\, h) = g^{ij} g^{kl} \nabla_i \nabla_k h_{jl}\,. \tag{3-3}$$

Note that at the point $p := eK = (e, eK)$ of M we have $\frac{\partial}{\partial t}\big|_{t=0} (\varphi_t)_* = (0, \mathrm{ad}\, H|_{\mathfrak{h}})$ and hence the lemma follows from (3-1) (evaluated at p) upon showing the terms $\Delta \overline{H}$ and $\mathrm{div}(\mathrm{div}\, h)$ vanish.

Remark Recall that, in local coordinates, we define the metric inverse g^{ij} as the function satisfying $\delta_i^l = g^{ij} g_{jl}$. By choosing a frame which is g–orthonormal at every point, one would have that both g_{ij} and g^{ij} are the identity. We make such a choice below.

To ease computational burden, we build a frame which is g–orthonormal at every point and exploits the property that our metric g is G–invariant. We start with an orthonormal basis of $T_p M$. As $T_p M = \mathfrak{n} \oplus \mathfrak{h}$, we may choose a basis $\{e_i\}$ which is the union of an orthonormal basis of $\mathfrak{n}$ together with an orthonormal basis of $\mathfrak{h}$.

Next, we extend the basis $\{e_i\}$ to a local frame near $p \in M$. To do this, we first consider a slice $\mathfrak{S}$ of the right K–action on G through $e \in G$. That is, we have a submanifold $\mathfrak{S}$ of G containing e such that $\dim \mathfrak{S} = \dim G/K$ and the map

$$s \mapsto sK, \quad s \in \mathfrak{S}$$

is a diffeomorphism of a neighborhood of $e \in \mathfrak{S}$ to a neighborhood of $eK \in G/K$. Now, for $q \in M$ near p, there exists $s \in \mathfrak{S}$ such that $q = s \cdot p$ and we define

$$e_i(q) = s_* e_i,$$

where s_* denotes the differential of the translation $s\colon p \mapsto q$. We note that the frame is well-defined as our choice of $s \in \mathfrak{S}$ is unique, since $\mathfrak{S}$ is a slice. Furthermore, our frame is g–orthonormal as g is G–invariant.

Using the above choice of frame near $p \in M$, we now study (3-2) and (3-3). We begin by computing the variation h of $g_t = {\varphi_t}^* g$ in terms of $\{e_i\}$. For a point $q \in M$ near p we have

$$h_{ij}(q) = \frac{\partial}{\partial t}\Big|_{t=0} (g_t)_{ij}(q) = \frac{\partial}{\partial t}\Big|_{t=0} (g_t)\big(e_i(q), e_j(q)\big) \tag{3-4}$$
$$= \frac{\partial}{\partial t}\Big|_{t=0} g\big((\varphi_t)_* e_i(q), (\varphi_t)_* e_j(q)\big).$$

Next we compute $(\varphi_t)_* v_q$ for a vector $v_q \in T_q M$.

As $G = NU$, there exist $n \in N$ and $u \in U$ such that $s \in \mathfrak{S}$ may be written as $s = nu$ and $q = (nu) \cdot p$. Furthermore, there exists $X \in \mathfrak{p} = \mathfrak{n} \oplus \mathfrak{h}$ such that

$$v_q = (nu)_* \frac{d}{ds}\Big|_{s=0} \exp(sX) \cdot p.$$

To understand (3-4), we analyze separately the cases when X is an element of $\mathfrak{n}$ or of $\mathfrak{h}$.

For $X \in \mathfrak{n}$, we have

$$\begin{aligned} (\varphi_t)_* v_q &= (\varphi_t)_*(nu)_* X \\ &= \frac{d}{ds}\Big|_{s=0} \varphi_t\big(nu \exp(sX) \cdot p\big) \\ &= \frac{d}{ds}\Big|_{s=0} \varphi_t\big(nu \exp(sX) u^{-1} u \cdot p\big) \\ &= \frac{d}{ds}\Big|_{s=0} \varphi_t\big(nu \exp(sX) u^{-1}, uK\big) \\ &= \frac{d}{ds}\Big|_{s=0} \big(n \exp(s \operatorname{Ad}_u X), \Phi_t(u) K\big) \\ &= \frac{d}{ds}\Big|_{s=0} \big(n \Phi_t(u) \Phi_t(u)^{-1} \exp(s \operatorname{Ad}_u X) \Phi_t(u) K\big) \\ &= \frac{d}{ds}\Big|_{s=0} \big(n \Phi_t(u) \exp(s \operatorname{Ad}_{\Phi_t(u)^{-1}} \operatorname{Ad}_u X) K\big) \\ &= \big(n\Phi_t(u)\big)_* \operatorname{Ad}_{\Phi_t(u)^{-1} u} X. \end{aligned} \tag{3-5}$$

Here we have used that N is normal in G. Note also that $\operatorname{Ad}_{\Phi_t(u)^{-1} u} X \in \mathfrak{n}$.

In the case when $X \in \mathfrak{h} \subset \mathfrak{u}$, we have

$$\begin{aligned} (\varphi_t)_* v_q = (\varphi_t)_*(nu)_* X &= \frac{d}{ds}\Big|_{s=0} \varphi_t\big(nu \exp(sX) \cdot p\big) \\ = \frac{d}{ds}\Big|_{s=0} \varphi_t\big(nu \exp(sX) K\big) &= \frac{d}{ds}\Big|_{s=0} \big(n \Phi_t\big(u \exp(sX)\big) K\big) \\ = \frac{d}{ds}\Big|_{s=0} \big(n \Phi_t(u) \exp\big(s (\Phi_t)_* X\big) K\big) &= \big(n\Phi_t(u)\big)_* (\Phi_t)_* X. \end{aligned} \tag{3-6}$$

Observe that since $\operatorname{ad} H$ preserves $\mathfrak{h}$ [12, Equation 32], $(\Phi_t)_* X \in \mathfrak{h}$ and so the last line is consistent with our identification of $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{h}$ with $T_p M$.

From (3-4), (3-5), and (3-6) we see that:

(i) If $e_i \in \mathfrak{n}$ and $e_j \in \mathfrak{h}$, then we have $g_{ij}(q) = 0$.

(ii) If $e_i \in \mathfrak{n}$ and $e_j \in \mathfrak{h}$, then we have $h_{ij}(q) = 0$.

(iii) If $e_i, e_j \in \mathfrak{h}$, then $h_{ij}(q)$ does not depend on n and u, and so is constant in q.

(iv) If $e_i, e_j \in \mathfrak{n}$, then $h_{ij}(q)$ does not depend on n, but does depend on u.

Using these observations, we see that the only possibly nonzero terms of

$$\operatorname{div}(\operatorname{div} h) = g^{ij} g^{kl} \nabla_i \nabla_k h_{jl}$$

occur when $e_j, e_l \in \mathfrak{n}$ and $e_i, e_k \in \mathfrak{h}$. However, $(g_{\alpha\beta}) = \mathrm{Id}$ implies $(g^{\alpha\beta}) = \mathrm{Id}$ and so $g^{kl} = 0$. This yields

$$\operatorname{div}(\operatorname{div} h) = 0.$$

Next we study $\Delta \overline{H} = g^{ij} g^{kl} \nabla_i \nabla_j h_{kl}$. As above, the only possibly nonzero terms occur when $e_k, e_l \in \mathfrak{n}$ and $e_i, e_j \in \mathfrak{h}$. Further, as our frame is orthonormal, we have

$$\Delta \overline{H}(q) = g^{ii}(q) g^{kk}(q) (\nabla_i \nabla_i h_{kk})(q) = \sum_i \Big(\nabla_i \nabla_i \sum_k h_{kk} \Big)(q),$$

where the first sum is over the frame from $\mathfrak{h}$ and the second is over the frame from $\mathfrak{n}$. From (3-4) and (3-5) we have

$$\begin{aligned} h_{kk}(q) &= \frac{\partial}{\partial t}\Big|_{t=0} g\big((\varphi_t)_* e_k(q), (\varphi_t)_* e_k(q)\big) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \big\langle \mathrm{Ad}_{\Phi_t(u)^{-1}u}(e_k), \mathrm{Ad}_{\Phi_t(u)^{-1}u}(e_k) \big\rangle \\ &= 2\Big\langle e_k, \Big(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\Phi_t(u)^{-1}u} \Big)(e_k) \Big\rangle = 2\big\langle e_k, \operatorname{ad} M(e_k) \big\rangle, \end{aligned}$$

where $M = \frac{d}{dt}\big|_{t=0} \Phi_t(u)^{-1} u$. To see that this last line makes sense, observe that $\Phi_t(u)^{-1}u$ is a curve in U with $\Phi_0(u)^{-1}u = e$ and thus $\frac{d}{dt}\big|_{t=0} \Phi_t(u)^{-1} u \in \mathfrak{u}$.

Remark Although M is a function of u, we suppress this detail as it does not impact the rest of our proof.

We claim that $\operatorname{ad} M|_{\mathfrak{n}}$ is traceless. To see this, we use the fact that U being reductive and connected implies $U = [U, U] Z(U)$, where $Z(U)$ is the center of U. Thus, we may write $u = u_1 u_2$ where $u_1 \in [U, U]$ and $u_2 \in Z(U)$. As u_2 is central and Φ_t is an inner automorphism, $\Phi_t(u_2) = u_2$ and

$$\Phi_t(u)^{-1} u = \Phi_t(u_1)^{-1} u_1 \in [U, U].$$

This gives $\operatorname{ad} M \in \operatorname{ad}[\mathfrak{u}, \mathfrak{u}]$ from which our claim immediately follows.

Putting the above computations together we get

$$\Delta \overline{H}(q) = \sum_i \Big(\nabla_i \nabla_i \sum_k h_{kk} \Big)(q) = 2 \sum_i \nabla_i \nabla_i \operatorname{tr}(\operatorname{ad} M|_{\mathfrak{n}}) = 0,$$

which completes the proof of the lemma. □

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Proposed: John Lott Received: 16 October 2013
Seconded: Gang Tian, Simon Donaldson Revised: 7 April 2014