Geometry & Topology 18 (2014) 2547–2681

Computing $\widehat{HF}$ by factoring mapping classes

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Bordered Heegaard Floer homology is an invariant for 3–manifolds with boundary. In particular, this invariant associates to a handle decomposition of a surface F a differential graded algebra, and to an arc-slide between two handle decompositions, a bimodule over the two algebras. In this paper, we describe these bimodules for arc-slides explicitly, and then use them to give a combinatorial description of $\widehat{HF}$ of a closed 3–manifold, as well as the bordered Floer homology of any 3–manifold with boundary.

57M27; 53D40

1 Introduction

Heegaard Floer homology is an invariant of 3–manifolds defined using Heegaard diagrams and holomorphic disks; see the second author and Szabó [11]. This invariant is the homotopy type of a chain complex over a polynomial algebra in a formal variable U. The present paper will focus on $\widehat{HF}(Y)$ (with coefficients in $\mathbb{F}_2$), which is the homology of the $U=0$ specialization. This variant is simpler to work with, but it still encodes interesting information about the underlying 3–manifold Y (for instance, the Thurston norm [10]). Although the definition of $\widehat{HF}$ involves holomorphic disks, the work of Sarkar and Wang [13] allows one to calculate $\widehat{HF}$ explicitly from a Heegaard diagram for Y satisfying certain properties. (See also the second author, Stipsicz and Szabó [9].)

Bordered Heegaard Floer homology (see the authors [6]) is an invariant for 3–manifolds with parameterized boundary. A pairing theorem from [6] allows one to reconstruct the invariant $\widehat{HF}(Y)$ of a closed 3–manifold Y which is decomposed along a separating surface F in terms of the bordered invariants of the pieces.

In this paper we use bordered Floer theory to give another algorithm to compute $\widehat{HF}(Y)$ (with $\mathbb{F}_2$–coefficients). This algorithm is logically independent of [13], and quite natural (both aesthetically and mathematically; see the authors [7]). It is also

Published: 1 December 2014 DOI: 10.2140/gt.2014.18.2547

practical for computer implementation; some computer computations are described in Section 9. A Heegaard decomposition of Y is determined by an automorphism ϕ of the Heegaard surface, and the complex we describe here is associated to a suitable factorization of ϕ. To explain this in slightly more detail, we recall some of the basics of the bordered Heegaard Floer homology package.

We represent oriented surfaces by *pointed matched circles*, which are essentially handle decompositions with a little extra structure; see [6] or the review in Section 1.1 below. To a pointed matched circle $\mathcal{Z}$, bordered Heegaard Floer theory associates a differential graded algebra $\mathcal{A}(\mathcal{Z})$. A *$\mathcal{Z}$–bordered 3–manifold* is a 3–manifold Y_0 equipped with an orientation-preserving identification of its boundary ∂Y_0 with the surface associated to the pointed matched circle $\mathcal{Z}$. To a $\mathcal{Z}$–bordered 3–manifold, bordered Heegaard Floer theory associates modules over the algebra $\mathcal{A}(\mathcal{Z})$. Specifically, if Y_1 is a $\mathcal{Z}$–bordered 3–manifold, there is an associated module $\widehat{CFA}(Y_1)$, which is a right $\mathcal{A}_\infty$–module over $\mathcal{A}(\mathcal{Z})$. Similarly, if Y_2 is a $(-\mathcal{Z})$–bordered 3–manifold, we obtain a different module $\widehat{CFD}(Y_2)$ which is a left differential module over $\mathcal{A}(\mathcal{Z})$. A key property of the bordered invariants [6] states that the Heegaard Floer complex $\widehat{CF}(Y)$ of the closed 3–manifold obtained by gluing Y_1 and Y_2 along the above identifications is calculated by the (derived) tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$; we call results of this sort "pairing theorems".

In [5] the authors construct bimodules which can be used to change the boundary parameterizations. These are special cases of bimodules associated to 3–manifolds with two boundary components. In that paper, a duality property is also established, which allows us to formulate all aspects of the theory purely in terms of $\widehat{CFD}$. This has two advantages: $\widehat{CFD}$ involves fewer (and simpler) holomorphic curves than $\widehat{CFA}$ does, and $\widehat{CFD}$ is always an honest differential module, rather than a more general $\mathcal{A}_\infty$–module (like $\widehat{CFA}$). This point of view is further pursued by the authors in [8]. Suppose that a closed 3–manifold Y is separated into two pieces Y_1 and Y_2 along a separating surface F. It is shown in [8] that $\widehat{HF}(Y)$ is obtained as the homology of the chain complex of maps from the bimodule associated to the identity cobordism of F into $\widehat{CFD}(Y_1) \otimes_{\mathbb{F}_2} \widehat{CFD}(Y_2)$; this result is restated as Theorem 2.19 below. (It is also shown in [8] that the Heegaard Floer invariant for $Y = Y_1 \cup_\partial Y_2$ is identified via a "Hom pairing theorem" with $\mathrm{Ext}_{\mathcal{A}(\mathcal{Z})}(\widehat{CFD}(-Y_1), \widehat{CFD}(Y_2))$, the homology of the chain complex of homomorphisms from $\widehat{CFD}(-Y_1)$ to $\widehat{CFD}(Y_2)$. We will usually work with the other formulation in order to avoid orientation reversals.)

A Heegaard decomposition of a closed 3–manifold Y is a decomposition of Y along a surface into two particular simple pieces: handlebodies. The invariants of handlebodies with suitable boundary parametrizations are easy to calculate. Thus, the key step to calculating $\widehat{HF}$ from a Heegaard splitting is calculating the bimodule associated to a surface automorphism ψ to allow us to match up the two boundary parametrizations. We approach this problem as follows. Any diffeomorphism ψ between two surfaces associated to two (possibly different) pointed matched circles can be factored into a sequence of elementary pieces, called *arc-slides*. Then Theorem 2.19 allows us to compute the bimodule associated to ψ as a suitable composition of bimodules associated to arc-slides.

Thus, the primary task of the present paper is to calculate the type-*DD* bimodule associated to an arc-slide. These calculations turn out to follow from simple geometric constraints coming from the Heegaard diagram, combined with algebraic constraints imposed by the relation $\partial^2 = 0$.

For most of this paper, we work entirely within the context of "type-D" invariants. This allows us to avoid much of the algebraic complication of $\mathcal{A}_\infty$–modules which are built into the bordered theory: the bordered invariants we consider are simply differential modules over a differential algebra. (In practice, it may be convenient to take homology to make our complexes smaller; the cost of doing this is to work with $\mathcal{A}_\infty$–modules. We return to this point in Section 9.)

We now turn to an explicit description of all the ingredients in our calculation of $\widehat{HF}(Y)$.

1.1 Algebras for pointed matched circles

As defined in [6], *a pointed matched circle* $\mathcal{Z}$ is an oriented circle Z, equipped with a basepoint $z \in Z$ and $4k$ basepoints $\boldsymbol{a}$, which we label in order $1, \dots, 4k$ and which are matched in pairs. This matching is encoded in a two-to-one function $M = M_{\mathcal{Z}}\colon 1, \dots, 4k \to 1, \dots, 2k$. We sometimes denote $\boldsymbol{a}$ by $[4k]$ and write $[4k]/M$ for the range of the matching M. This matching is further required to satisfy the following combinatorial property: surgering out the $2k$ pairs of matched points (thought of as embedded zero-spheres in Z) results in a one-manifold which is connected.

A pointed matched circle specifies a surface $F^\circ(\mathcal{Z})$ with a single boundary component by filling in Z with a disk and then attaching 1–handles along the pairs of matched points in $\boldsymbol{a}$. Capping off the boundary component with a disk, we obtain a compact, oriented surface $F(\mathcal{Z})$.

Given $\mathcal{Z}$, there is an associated differential graded algebra $\mathcal{A}(\mathcal{Z})$, introduced in [6]. We briefly recall the construction here, and describe some more of its properties in Section 2.2; for more, see [6].

The algebra $\mathcal{A}(\mathcal{Z})$ is generated as an $\mathbb{F}_2$–vector space by strands diagrams υ, defined as follows. First, cut the circle along z so that it is an interval I_Z. Strands diagrams are collections of nondecreasing, linear paths in $[0,1]\times I_Z$, each of which starts and ends in one of the distinguished points in the matched circle. The strands are required to satisfy the following properties:

- Horizontal strands come in matching pairs: ie if $i\neq j$ and $M(i)=M(j)$, then υ contains a horizontal strand starting at i if and only if it contains the corresponding strand at j.
- If there is a nonhorizontal strand in υ starting at some point i, then there is no other strand in υ starting at either j with $M(j)=M(i)$.
- If there is a nonhorizontal strand in υ ending at some point i, then there is no other strand in υ ending at either j with $M(j)=M(i)$.

When we think of a strands diagram as an element of the algebra, we call the corresponding element a *basic generator*.

Multiplication in this algebra is defined in terms of basic generators. Given two strands diagrams, their product is gotten by concatenating the diagrams when possible, throwing out one of any given pair of horizontal strands if necessary, and then homotoping straight the piecewise linear juxtaposed paths (while fixing endpoints) and declaring the result to be 0 if this homotopy decreases the total number of crossings. More precisely, suppose υ and τ are two strands diagrams. We declare the product to be zero if any of the following conditions are satisfied:

(1) υ contains a nonhorizontal strand whose terminal point is not the initial point of any strand in τ.

(2) τ contains a nonhorizontal strand whose initial point is not the terminal point of any strand in υ.

(3) υ contains a horizontal strand which has the property that both it and its matching strand have terminal points which are not initial points of strands in τ.

(4) τ contains a horizontal strand which has the property that both it and its matching strand have initial points which are not terminal points of strands in υ.

(5) The concatenation $\upsilon*\tau$ of υ and τ contains two piecewise linear strands which cross each other twice.

(See Figure 1 for an illustration.) Otherwise, we take the resulting diagram $\upsilon*\tau$, remove any horizontal strands which do not go all the way across, and then pull all piecewise linear strands straight (fixing the endpoints).

The differential of a strands diagram is a sum of terms, one for each crossing. The term corresponding to a crossing c is gotten by forming the upward resolution at c (ie if two strands meet at c, we replace them by a nearby approximation by two nondecreasing paths which do not cross at c, in such a manner that the two initial points and two terminal points are the same). If this resolved diagram has a double-crossing, we set it equal to zero. Otherwise, once again, the corresponding term is gotten by pulling the strands straight and dropping any horizontal strand whose mate is no longer present. We denote the differential on these algebras by d; differentials on modules will usually be denoted ∂.

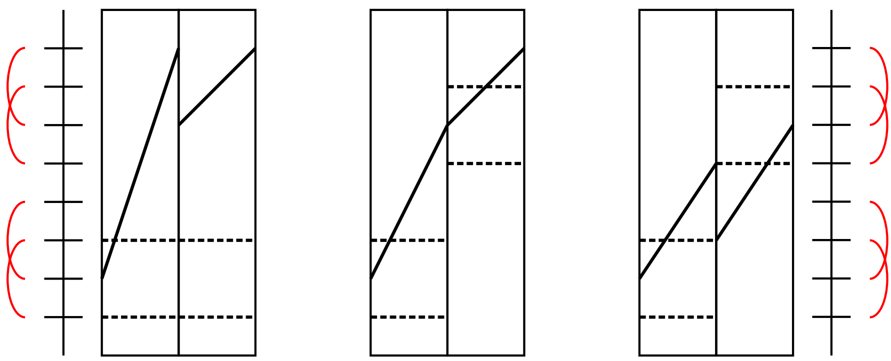


Figure 1: *Vanishing products on $\mathcal{A}$* The five cases in which the multiplication on $\mathcal{A}$ vanishes are illustrated. The product on the left vanishes for both of the first two reasons, the product in the center for both the third and fourth reasons, and the product on the right for the last reason. Horizontal strands are drawn dashed, to illustrate that they have weight $\frac{1}{2}$. All pictures are in $\mathcal{A}(\mathcal{Z}, 0)$, where $\mathcal{Z}$ represents a surface of genus 2.

The product and differential endow $\mathcal{A}(\mathcal{Z})$ with the structure of a differential algebra; ie $d(a \cdot b) = (da) \cdot b + a \cdot (db)$.

A strands diagram υ has a total weight, which is gotten by counting each nonhorizontal strand with weight 1, each horizontal strand with weight $\frac{1}{2}$, and then subtracting the genus k. Let $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(\mathcal{Z})$ be the subalgebra generated by weight i strands diagrams. This, of course, is a differential subalgebra.

Note also that for each subset $\boldsymbol{s}$ of $[4k]/M$, there is a corresponding idempotent $I(\boldsymbol{s})$, consisting of the collection of horizontal strands $[0,1] \times M^{-1}(\boldsymbol{s})$. These are the minimal idempotents of $\mathcal{A}(\mathcal{Z})$, with respect to the partial order $I_1 \leq I_2$ if $I_1 I_2 = I_1$.

A strands diagram υ also has an underlying one-chain in $H_1(Z, \boldsymbol{a})$, which we denote in this paper by $\operatorname{supp}(\upsilon)$ and called the *support of* υ. At any position q between two consecutive marked points p_i and p_{i+1} in Z, the local multiplicity of $\operatorname{supp}(\upsilon)$ is the intersection number of υ with $[0,1]\times q$.

A *chord* is an interval $[i,j]$ connecting two elements in $[4k]$. A chord ξ determines an algebra element $a(\xi)$, which is represented by the sum of all strands diagrams in which the strand from i to j is the only nonhorizontal strand. We denote the set of chords by $\mathcal{C}(\mathcal{Z})$, or simply $\mathcal{C}$.

1.2 The identity type-*DD* bimodule

Before introducing the bimodules for arc-slides, we first describe a simpler bimodule $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ associated, in a suitable sense, to the identity map. Motivation for calculating this invariant comes from its prominent role in one version of the pairing theorem, quoted as Theorem 2.19, below.

Definition 1.1 Let $\boldsymbol{s},\boldsymbol{t}\subset[4k]/M_{\mathcal{Z}}$ be subsets with the property that $\boldsymbol{s}$ and $\boldsymbol{t}$ form a partition of $[4k]/M_{\mathcal{Z}}$. Then we say that the corresponding idempotents $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$ are *complementary idempotents*.

Our bimodules have the following special form:

Definition 1.2 Let $\mathcal{A}$ and $\mathcal{B}$ be two dg–algebras. A *DD bimodule* over $\mathcal{A}$ and $\mathcal{B}$ is a dg–bimodule M which, as a bimodule, splits into summands isomorphic to $\mathcal{A}i\otimes j\mathcal{B}$ for various choices of idempotent i and j (but the differential need not respect this splitting). (See also Section 2.3.2.)

Definition 1.3 The module $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is generated by all pairs of complementary idempotents. This means that its elements are of the form $r\,i\otimes i's$, where $r,s\in\mathcal{A}(\mathcal{Z})$, and i and i' are complementary idempotents. The differential on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is determined by the Leibniz rule and the fact that

$$\partial(i\otimes i')=\sum_{\xi\in\mathcal{C}} i\,a(\xi)\otimes a(\xi)i'.$$

(Here and later, the symbol $\otimes$ denotes tensor product over $\mathbb{F}_2$, unless otherwise specified.)

In particular, the differential on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is determined by an element

$$A=\sum_{\xi\in\mathcal{C}} a(\xi)\otimes a(\xi)\in\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}).$$

If we let $*$ denote the action of $\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z})$ on itself by multiplication on the outside then the fact that $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$, as defined above, is a chain complex is equivalent to the fact that $dA + A * A = 0$. (See Proposition 3.4 for an algebraic verification that $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is, indeed, a chain complex.) The relevance of $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ to bordered Floer homology arises from the following:

Theorem 1 *The bimodule $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is canonically homotopy equivalent to the type-DD bimodule of the identity map defined using pseudoholomorphic curves in [5].*

More precisely, in the notation of [5],

$$\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}}) \cong (\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(-\mathcal{Z})) \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}),$$

where we identify right actions by $\mathcal{A}(\mathcal{Z})$ with left actions by $\mathcal{A}(-\mathcal{Z}) = \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$.

1.3 *DD* bimodules for arc-slides

We turn now to bimodules for arc-slides. Before doing this, we recall briefly the notion of an arc-slide, and introduce some notation.

Let $\mathcal{Z}$ be a pointed matched circle, and fix two matched pairs $C = \{c_1, c_2\}$ and $B = \{b_1, b_2\}$. Suppose moreover that b_1 and c_1 are adjacent, in the sense that there is an arc σ connecting b_1 and c_1 which does not contain the basepoint z or any other point $p_i \in \boldsymbol{a}$. Then we can form a new pointed matched circle $\mathcal{Z}'$ which agrees everywhere with $\mathcal{Z}$, except that b_1 is replaced by a new distinguished point b_1', which now is adjacent to c_2 and b_1' is positioned so that the orientation on the arc from b_1 to c_1 is opposite to the orientation of the arc from b_1' to c_2. In this case, we say that $\mathcal{Z}'$ and $\mathcal{Z}$ differ by an *arc-slide m of b_1 over C at c_1* or, more succinctly, an *arc-slide of b_1 over c_1*, and write $m\colon \mathcal{Z}\to\mathcal{Z}'$. Let σ' denote the arc connecting c_2 and b_1'. See Figure 2 for two examples.

Note that if $\mathcal{Z}$ and $\mathcal{Z}'$ differ by an arc-slide, then there is a canonical diffeomorphism from $F^\circ(\mathcal{Z})$ to $F^\circ(\mathcal{Z}')$; see Figure 3. We will denote this diffeomorphism $F^\circ(m)$.

Let m be an arc-slide taking the pointed matched circle $\mathcal{Z}$ to the pointed matched circle $\mathcal{Z}'$. Our next goal is to describe an $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(\mathcal{Z}')$–bimodule associated to m, which we denote ${}_{\mathcal{A}(\mathcal{Z})}\widehat{\mathcal{DD}}(m\colon \mathcal{Z}\to\mathcal{Z}')_{\mathcal{A}(\mathcal{Z}')}$, or just $\widehat{\mathcal{DD}}(m)$.

To describe the generators of $\widehat{\mathcal{DD}}(m\colon \mathcal{Z}\to\mathcal{Z}')$ we need two extensions of the notion of complementary idempotents to the case of arc-slides.

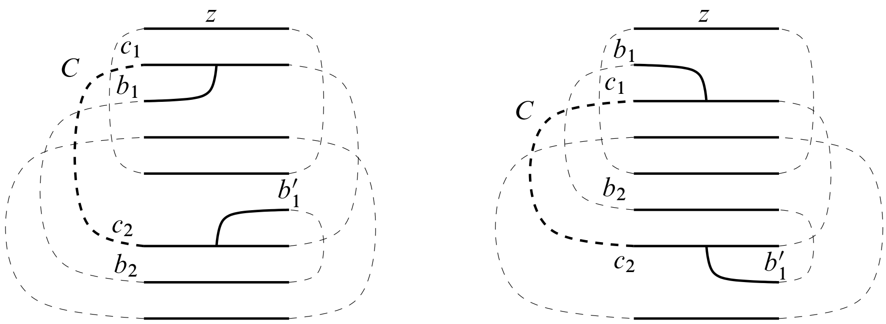


Figure 2: *Arc-slides* Two examples of arc-slides connecting pointed matched circles for genus-2 surfaces are shown: in both cases, the foot b_1 is sliding over the matched pair $C = \{c_1, c_2\}$ (indicated by the darker dotted matching) at c_1.

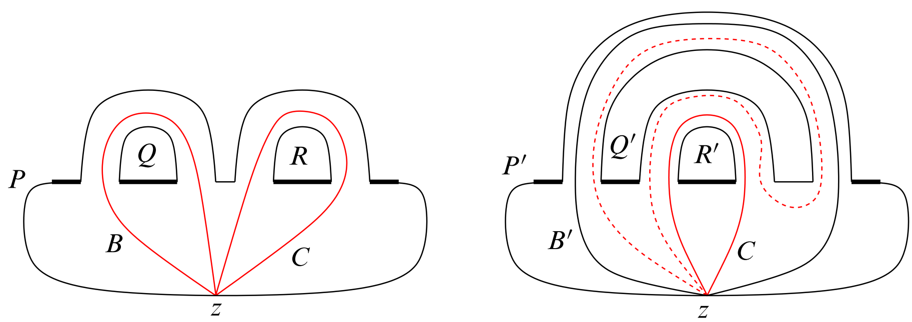


Figure 3: *The local case of an arc-slide diffeomorphism* On the left, we have a pair of pants with boundary components labeled P, Q, and R, and two distinguished curves B and C; on the right, we have another pair of pants with boundary components P', Q', R' and distinguished curves B' and C. The arc-slide diffeomorphism carries B to the dotted curve on the right, the curve labeled C on the left to the curve labeled C on the right, and boundary components P, Q and R to P', Q' and R', respectively. This diffeomorphism can be extended to a diffeomorphism between surfaces associated to pointed matched circles: in such a surface there are further handles attached along the four dark intervals; however, our diffeomorphism carries the four dark intervals on the left to the four dark intervals on the right and hence extends to a diffeomorphism as stated. (This is only one of several possible configurations of B and C: they could also be nested or linked.)

Definition 1.4 Let $\boldsymbol{s} \subset [4k]/M_{\mathcal{Z}}$ and $\boldsymbol{t} \subset [4k]/M'_{\mathcal{Z}'}$ be subsets with the property that $\boldsymbol{s}$ and $\boldsymbol{t}$ form a partition of $[4k]/M_{\mathcal{Z}'}$ (where we have suppressed the identification between the matched pairs $[4k]/M_{\mathcal{Z}}$ and $[4k]/M_{\mathcal{Z}'}$). We say that the corresponding idempotents $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$ in $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$ are *complementary idempotents*. An idempotent in $\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}')$ of the form $i\otimes i'$, where i and i' are complementary idempotents, is also called an *idempotent of type X*.

In a similar vein, we have the following:

Definition 1.5 Two elementary idempotents i of $\mathcal{A}(\mathcal{Z})$ and i' of $\mathcal{A}(\mathcal{Z}')$ are *subcomplementary idempotents* if $i=I(\boldsymbol{s})$ and $i'=I(\boldsymbol{t})$ where $\boldsymbol{s}\cap\boldsymbol{t}$ consists of the matched pair of the feet of C, while $\boldsymbol{s}\cup\boldsymbol{t}$ contains all the matched pairs, except for the pair of feet of B. An idempotent in $\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}')$ of the form $i\otimes i'$, where i and i' are subcomplementary idempotents is also called an *idempotent of type Y*. Two elementary idempotents i of $\mathcal{A}(\mathcal{Z})$ and i' of $\mathcal{A}(\mathcal{Z}')$ are said to be *near-complementary* if they are either complementary or subcomplementary.

Given a chord ξ for $\mathcal{Z}$, let $a(\xi)$ be the algebra element in $\mathcal{A}(\mathcal{Z})$ associated to ξ. Similarly, given a chord ξ for $\mathcal{Z}'$, let $a'(\xi)$ be the algebra element in $\mathcal{A}(\mathcal{Z}')$ associated to the chord ξ.

Definition 1.6 The *restricted support* $\operatorname{supp}_{\boldsymbol{R}}(a)$ of a basic generator $a\in\mathcal{A}(\mathcal{Z})$ is the image of $\operatorname{supp}(a)$ under the map $H_1(Z,\boldsymbol{a})\to H_1(Z,\boldsymbol{a}\backslash b_1)$ gotten by contracting σ to a point. In other words, the restricted support of a is the collection of local multiplicities of the associated one-chain at all the regions except σ. Similarly, if $a\in\mathcal{A}(\mathcal{Z}')$, then the restricted support $\operatorname{supp}_{\boldsymbol{R}}(a)$ of a is the image of $\operatorname{supp}(a)$ under the map $H_1(Z,\boldsymbol{a}')\to H_1(Z,\boldsymbol{a}'\setminus b_1')$ gotten by contracting σ' to a point.

A *short near-chord* is a nonzero algebra element of the form $(i\cdot a\cdot j)\otimes(j'\cdot b'\cdot i')$ with the following four properties:

1. The pairs $(i\otimes i')$ and $(j\otimes j')$ are near-complementary idempotents.
2. $\operatorname{supp}_{\boldsymbol{R}}(a)=\operatorname{supp}_{\boldsymbol{R}}(b')$.
3. The support of at least one of a or b' is nonzero.
4. The lengths of the (unrestricted) support of a and the (unrestricted) support of b' are both no greater than 1.

(In a particular degenerate case, we also allow one more kind of short near-chord; see Definition 4.6.)

The above definition of short near-chords includes elements of the form $(i\cdot a(\sigma)\cdot j)\otimes j'$, where both (i,j') and (j,j') are near-complementary idempotents; and also $i\otimes(j'\cdot a'(\sigma')\cdot i')$, where both (i,i') and (i,j') are near-complementary. (Note that an element $a\otimes b$ with $\operatorname{supp}(a)=\sigma$ and $\operatorname{supp}(b)=\sigma'$ does not satisfy Property (1) of Definition 1.6.)

Definition 1.7 Let $m\colon \mathcal{Z} \to \mathcal{Z}'$ be an arc-slide. Let N be any type-*DD* bimodule over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$. Suppose N satisfies the following properties:

(AS-1) As an $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(\mathcal{Z}')$–bimodule, N has the form

$$N = \bigoplus_{i\otimes i' \text{ near-complementary}} \mathcal{A}(\mathcal{Z})i \otimes i'\mathcal{A}(\mathcal{Z}').$$

(AS-2) For each generator $I = i \otimes i'$ of N the differential of I has the form

$$\partial(I) = \sum_{J=j\otimes j'} \sum_k (i\cdot \nu_k \cdot j) \otimes (j' \cdot \nu'_k \cdot i'), \tag{1-1}$$

where the ν_k and ν'_k are strand diagrams with the same restricted support (and k ranges over some index set) and J runs through the generators of N. (For the ν_k as in (1-1), we will say that the differential on N *contains* $(i\cdot \nu_k \cdot j) \otimes (j' \cdot \nu'_k \cdot i')$.)

(AS-3) N is graded (see Section 2.2) by a λ–free grading set S (see Definition 2.8).

(AS-4) All short near-chords appear in the differential; ie given a generator $I = i \otimes i'$ of N, the differential of I contains all short near-chords of the form $(i\cdot\nu\cdot j)\otimes(j'\cdot\nu'\cdot i')$.

Then we say that N is an *arc-slide bimodule* for m.

Let $\mathcal{Z}$ and $\mathcal{Z}_0$ be pointed matched circles. We can form their connected sum $\mathcal{Z}\#\mathcal{Z}_0$. Given any idempotent $I(\boldsymbol{s}_0)$ of $\mathcal{A}(\mathcal{Z}_0)$, we have a quotient map

$$q\colon \mathcal{A}(\mathcal{Z}\#\mathcal{Z}_0) \to \mathcal{A}(\mathcal{Z}).$$

(For more on this, see Section 2.2.3.)

Definition 1.8 We say that an arc-slide bimodule N for $m\colon \mathcal{Z}\to\mathcal{Z}'$ is *stable* if for any other $\mathcal{Z}_0$ and idempotent in $\mathcal{A}(\mathcal{Z}_0)$, and either choice of connect sum $\mathcal{Z}\#\mathcal{Z}_0$, there is an arc-slide bimodule M for $m_0\colon \mathcal{Z}\#\mathcal{Z}_0 \to \mathcal{Z}'\#\mathcal{Z}_0$ with the property that $N = q_*(M)$, where q_* denotes induction of bimodules, ie $q_*(M) = \mathcal{A}(\mathcal{Z}) \otimes_{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}_0)} M \otimes_{\mathcal{A}(\mathcal{Z}'\#\mathcal{Z}_0)} \mathcal{A}(\mathcal{Z}')$.

Remark 1.9 In fact, stability is much weaker than it might appear from the above definition. From the proof of Proposition 1.10, one can see that N is stable if there exists some pointed matched circle $\mathcal{Z}_0$ of genus greater than one and a single associated idempotent I in $\mathcal{A}(\mathcal{Z}_0)$ with weight zero, so that for both choices of connected sum, there are arc-slide bimodules M and N as in Definition 1.8 with $N = q_*(M)$.

The following is proved in Section 4.

Proposition 1.10 *Let m: $\mathcal{Z} \to \mathcal{Z}'$ be an arc-slide. Then, up to isomorphism, there is a unique stable type-DD arc-slide bimodule for m (as defined in Definitions 1.7 and 1.8).*

The proof is constructive: after making some explicit choices, the coefficients in the differential of the arc-slide bimodule are uniquely determined.

Definition 1.11 Let $\widehat{\mathcal{DD}}(m\colon \mathcal{Z} \to \mathcal{Z}')$ be the arc-slide bimodule for m.

In [5], it is shown that for any mapping class ϕ: $F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z}')$ which fixes the boundary, there is an associated type-*DD* bimodule $\widehat{CFDD}(\phi)$. Given an arc-slide m: $\mathcal{Z} \to \mathcal{Z}'$, let $\widehat{CFDD}(F^\circ(m))$ denote this construction, applied to the canonical diffeomorphism $F^\circ(m)$: $F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z}')$ specified by m.

Theorem 2 *The bimodule $\widehat{\mathcal{DD}}(m\colon \mathcal{Z} \to \mathcal{Z}')$ is canonically homotopy equivalent to the type-DD bimodule $\widehat{CFDD}(F^\circ(m))$ associated in [5] to the arc-slide diffeomorphism from $F^\circ(\mathcal{Z})$ to $F^\circ(\mathcal{Z}')$.*

More precisely, in the notation of [5], if $\widehat{\mathcal{DD}}(m)$ is an arc-slide bimodule given in Proposition 1.10, then (up to homotopy) there is a canonical homotopy equivalence

$$\widehat{\mathcal{DD}}(m) \simeq (\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')) \boxtimes \widehat{CFDD}(F^\circ(m)),$$

where we identify right actions by $\mathcal{A}(\mathcal{Z}')$ with left actions by $\mathcal{A}(-\mathcal{Z}') = \mathcal{A}(\mathcal{Z}')^{\mathrm{op}}$.

1.4 Modules associated to a handlebody

We now describe the modules associated to a handlebody. First we consider the case of a handlebody with a standard framing, and then we show how the arc-slide bimodules can be used to change the framing.

1.4.1 The 0–framed handlebody We start by fixing some notation. Let $\mathcal{Z}^1$ denote the unique genus-1 pointed matched circle. $\mathcal{Z}^1$ consists of an oriented circle Z equipped with a basepoint z and two pairs $\{a, a'\}$ and $\{b, b'\}$ of matched points. As we travel along Z in the positive direction starting at z we encounter the points a, b, a', b' in that order. Note that the pair $\{a, a'\}$ specifies a simple closed curve on $F(\mathcal{Z}^1)$, as does the pair $\{b, b'\}$.

Let $\mathcal{Z}_0^g = \#^g\, \mathcal{Z}^1$ be the *split pointed matched circle* describing a surface of genus g, which is obtained by taking the connect sum of g copies of $\mathcal{Z}^1$. The circle $\mathcal{Z}_0^g$

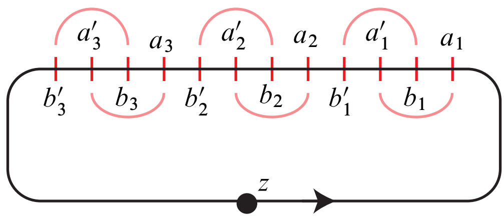


Figure 4: Split pointed matched circle: the genus-3 case is illustrated.

has $4g$ marked points, which we label in order $a_1, b_1, a'_1, b'_1, a_2, \dots, b'_g$, as well as a basepoint z. See Figure 4.

The 0*–framed solid torus* $\mathsf{H}^1 = (H^1, \phi_0^1)$ is the solid torus with boundary $-F(\mathcal{Z}^1)$ in which $\{a, a'\}$ bounds a disk; let ϕ_0^1 denote the preferred diffeomorphism $-F(\mathcal{Z}^1) \to \partial H^1$. The 0*–framed handlebody of genus* g $\mathsf{H}^g = (H^g, \phi_0^g)$ is a boundary connect sum of g copies of H^1. Our conventions are illustrated by the bordered Heegaard diagrams in Figure 5.

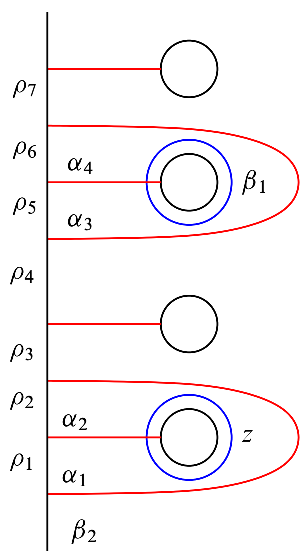


Figure 5: Heegaard diagram for the 0–framed genus-two handlebody

We next give a combinatorial model $\widehat{\mathcal{D}}(\mathsf{H}^g)$ for the type-D module $\widehat{CFD}(\mathsf{H}^g)$ associated to H^g. Let $\boldsymbol{s} = \{a_i, a'_i\}_{i=1}^g$. The module $\widehat{\mathcal{D}}(\mathsf{H}^g)$ is generated over the algebra by a single idempotent $I = I(s)$, and equipped with the differential determined by

$$\partial(I) = \sum_{i=1}^{g} a(\xi_i)I,$$

where ξ_i is the arc in $\mathcal{Z}^g$ connecting a_i and a_i'.

A straightforward calculation (see Section 7) shows:

Proposition 1.12 *We have that the module $\widehat{\mathcal{D}}(\mathsf{H}^g)$ is homotopy equivalent to the module $\widehat{CFD}(\mathsf{H}^g) = \widehat{CFD}(H^g, \phi_0^g)$ as defined (via holomorphic curves) in [6].*

1.4.2 Handlebodies with arbitrary framings Before turning to handlebodies with arbitrary framings, we pause for an algebraic interlude. Let M and N be type-DD bimodules over A and B. Define $\mathrm{Mor}(M, N)$, the *chain complex of bimodule morphisms from M to N*, to be the space of bimodule maps from M to N, equipped with a differential given by

$$\partial(f) = f \circ \partial_M + \partial_N \circ f.$$

(Under technical assumptions on A and B satisfied by the algebras in bordered Floer theory, the homology of $\mathrm{Mor}(M, N)$ is the *Hochschild cohomology $HH^*(M, N)$* of M with N. See [8] for a little further discussion.)

Now, let $(H^g, \phi_g^0 \circ \psi)$ be a handlebody with arbitrary framing. Here, we have $\psi\colon -F(\mathcal{Z}) \to -F(\mathcal{Z}_0^g)$ for some genus-g pointed matched circle $\mathcal{Z}$. Fix a factorization $\psi = \psi_1 \circ \cdots \circ \psi_n$ of ψ into arc-slides. Let $\psi_i\colon -F(\mathcal{Z}_i) \to -F(\mathcal{Z}_{i-1})$. Here, $\mathcal{Z}_0 = \mathcal{Z}_0^g$ and $\mathcal{Z}_n = \mathcal{Z}$.

As discussed in Section 1.3, associated to each ψ_i is a bimodule ${}_{\mathcal{A}(\mathcal{Z}_i)}\widehat{\mathcal{DD}}(\psi_i)_{\mathcal{A}(\mathcal{Z}_{i-1})}$. Define

$$\widehat{\mathcal{D}}(H^g, \phi_g^0 \circ \psi) = \mathrm{Mor}\big(\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_{n-1}}) \otimes \cdots \otimes \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_0}), \widehat{\mathcal{DD}}(\psi_n) \otimes \widehat{\mathcal{DD}}(\psi_{n-1}) \otimes \cdots \otimes \widehat{\mathcal{DD}}(\psi_1) \otimes \widehat{\mathcal{D}}(\mathsf{H})\big), \tag{1-2}$$

the chain complex of morphisms of $\mathcal{A}(\mathcal{Z}_{n-1}) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_0)$–bimodules. This complex retains a left action by $\mathcal{A}(\mathcal{Z})$, from the left action on $\widehat{\mathcal{DD}}(\psi_n)$. (This is illustrated schematically in Figure 6.)

Theorem 3 *The module $\widehat{\mathcal{D}}(H^g, \phi_g^0 \circ \psi)$ is homotopy equivalent to the module $\widehat{CFD}(H^g, \phi_g^0 \circ \psi)$ as defined (via holomorphic curves) in [6].*

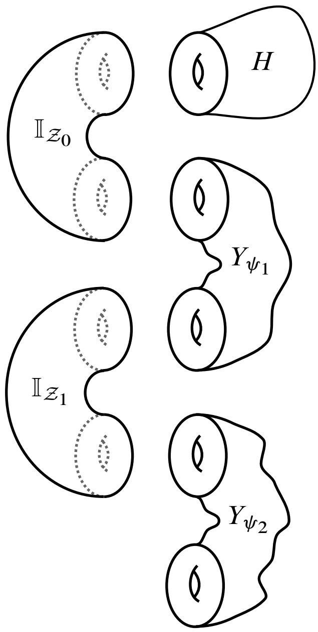


Figure 6: *Changing framing by gluing mapping cylinders* This is an illustration of (1-2). The pieces labeled Y_{ψ_1} and Y_{ψ_2} represent the mapping cylinders of ψ_1 and ψ_2 (see Section 2.1), and the pieces labeled $\mathbb{I}_{\mathcal{Z}_i}$ represent copies of $F(\mathcal{Z}_i) \times [0,1]$. The fact that the $\mathbb{I}$ pieces face right indicates they have been reflected, ie dualized.

A priori, the module $\widehat{\mathcal{D}}(H^g, \phi_g^0 \circ \psi)$ depends not just on ψ but also the factorization into arc-slides. Theorem 3 implies that, up to homotopy equivalence, $\widehat{\mathcal{D}}(H^g, \phi_g^0 \circ \psi)$ is independent of the factorization. In fact, this homotopy equivalence is canonical up to homotopy, a fact that will be used (and proved) in [7].

1.5 Assembling the pieces: Calculating $HF^\wedge$ from a Heegaard splitting

Let Y be a closed, oriented 3–manifold presented by a Heegaard splitting $Y = H_1 \cup_\Sigma H_2$, where H_1 and H_2 are handlebodies, with $\partial H_1 = -\Sigma$ and $\partial H_2 = \Sigma$. Thinking of both H_1 and H_2 as a standard bordered handlebody H_0, we can think of the gluing map identifying the two boundaries as a map $\psi\colon F(\mathcal{Z}_0^g) \to -F(\mathcal{Z}_0^g)$.

Using ψ we get a module ${}_{\mathcal{A}(-\mathcal{Z}_0^g)}\widehat{\mathcal{D}}(H^g, \phi_g^0 \circ \psi)$, which is a left module over $\mathcal{A}(-\mathcal{Z}_0^g)$. Using the identification

$$\mathcal{A}(-\mathcal{Z}_0^g) = \mathcal{A}(\mathcal{Z}_0^g)^{\mathrm{op}},$$

view $\widehat{\mathcal{D}}(H^g,\phi_g^0\circ\psi)_{\mathcal{A}(\mathcal{Z}_0^g)}$ as a right module over $\mathcal{A}(\mathcal{Z}_0^g)$. So, ${}_{\mathcal{A}(\mathcal{Z}_0^g)}\widehat{\mathcal{D}}(H^g,\phi_g^0)\otimes\widehat{\mathcal{D}}(H^g,\phi_g^0\circ\psi)_{\mathcal{A}(\mathcal{Z}_0^g)}$ becomes an $\mathcal{A}(\mathcal{Z}_0^g)$–bimodule.

Theorem 4 *The chain complex $\widehat{CF}(Y)$, as defined in [11] via holomorphic curves, is homotopy equivalent to*

$$\mathrm{Mor}\big({}_{\mathcal{A}(\mathcal{Z}_0^g)}\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_0^g})_{\mathcal{A}(\mathcal{Z}_0^g)},\, {}_{\mathcal{A}(\mathcal{Z}_0^g)}\widehat{\mathcal{D}}(H^g,\phi_g^0)\otimes\widehat{\mathcal{D}}(H^g,\phi_g^0\circ\psi)_{\mathcal{A}(\mathcal{Z}_0^g)}\big),$$

which is the chain complex of bimodule morphisms from $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_0^g})$ to $\widehat{\mathcal{D}}(H^g,\phi_g^0)\otimes\widehat{\mathcal{D}}(H^g,\phi_g^0\circ\psi)$.

(Compare to Theorem 2.19 in Section 2.5.)

To keep the exposition simple, we have suppressed relative gradings and spin^c–structures from the introduction. However, this information can be extracted in a natural way from the tensor products, once the gradings on the constituent modules have been calculated, ie once we have graded analogues of Theorems 2 and 3. We return to a graded analogue of Theorem 4 in Section 7.

1.6 More computations: Open books, bordered invariants

Theorem 2 can be combined with slight variants on Proposition 1.12 for other kinds of computations as well.

As a first example, suppose one is given an open book decomposition of a 3–manifold Y, with connected binding. Let F denote the fiber of the open book and $\psi\colon F\to F$ the monodromy. Let $\overline{\psi}\colon F\cup_\partial -F\to F\cup_\partial -F$ be the result of extending ψ by the identity map of $-F$. Fix a pointed matched circle $\mathcal{Z}$ representing F. There is a particular bordered handlebody, which we call the self-gluing handlebody H_{sg}, so that $\partial\mathsf{H}_{sg}=\mathcal{Z}\cup(-\mathcal{Z})$ and

$$Y=\mathsf{H}_{sg}\cup_{\overline{\psi}}\mathsf{H}_{sg};$$

see Definition 9.2. Factoring ψ into arc-slides gives a formula for $\widehat{CF}(Y)$, analogous to Theorem 4, in terms of the module $\widehat{CFD}(\mathsf{H}_{sg})$ and the bimodules for the arc-slides. The invariant $\widehat{CFD}(\mathsf{H}_{sg})$ is computed in Theorem 9.3, completing this algorithm for computing $\widehat{CF}(Y)$ from an open book decomposition. See Section 9.5 for a (computer-assisted) example.

As a second example, one can also compute the bordered invariants of arbitrary bordered 3–manifolds Y. The new ingredient here is a computation of the invariants of elementary cobordisms, Proposition 8.2. (The answer is an amalgam of $\widehat{CFD}(\mathsf{H}^1)$ and $\widehat{CFDD}(\mathbb{I})$.) From there, one decomposes an arbitrary bordered 3–manifold as a composition of elementary cobordisms with standard framings and mapping cylinders of arc-slides, and again uses an analogue of the formula from Theorem 4 to obtain $\widehat{CFD}(Y)$. See Section 8 for more details.

One can obtain $\widehat{CFA}(Y)$ from $\widehat{CFD}(Y)$ via the duality theorem [5, Proposition 9.2]; see also Section 9.1. This gives a finite-dimensional but fairly large for model for $\widehat{CFA}$. One can obtain a smaller model by using homological perturbation theory; this is reviewed, with an example, in Sections 9.2–9.4.

1.7 Organization

In Section 2, we give some of the background on bordered Floer theory needed for this paper; for further details the reader is referred to [6; 5]. In Section 3, we calculate the *DD* bimodule for the identity map, verifying Theorem 1. The proof follows from inspecting the relevant Heegaard diagram and applying the relations which are forced by $\partial^2 = 0$. In Section 4, we calculate the *DD* bimodules for arc-slides. This uses similar reasoning to the proof of Theorem 1. In Section 5, we explain the specialization of these to the genus-one case, for concreteness. In Section 6 we compute gradings on the arc-slide bimodules (needed for a suitably graded analogue of Theorem 4). Note that we do not at present have a conceptual description of the bimodules for arbitrary surface diffeomorphisms (rather, they have to be factored into arc-slides, and the corresponding bimodules have to be composed); however we do give an intrinsic description of its corresponding grading set. This is done in Proposition 6.13. In Section 7, we compute the invariant for handlebodies with the preferred framing ϕ_0, which is quite easy, and assemble the ingredients to prove the main result, Theorem 4 (as well as a graded version).

The computations of the arc-slide bimodules also lead quickly to a description of the bordered Heegaard Floer invariants for arbitrary bordered 3–manifolds. The main ingredient beyond what we have explained so far is the invariant associated to an elementary cobordism that adds or removes a handle. This is an easy generalization of the calculations for handlebodies, and is discussed in Section 8.

The point of view of $\mathcal{A}_\infty$–modules, which we have otherwise avoided in this paper, allows one to trade generators for complexity of the differential, and is useful in practice. This is discussed in Section 9, along with some examples.

1.8 Further remarks

Theorem 4 gives a purely combinatorial description of $\widehat{HF}(Y)$, with coefficients in $\mathbb{Z}/2\mathbb{Z}$, in terms of a mapping class of a corresponding Heegaard splitting. We point out again that this calculation is independent of the methods of [13]. Indeed, the methods of this paper are based on general properties of bordered invariants, together with some very crude input coming from the Heegaard diagrams (see especially Theorems 1 and 2). The particular form of the bimodules is then forced by algebraic considerations (notably $\partial^2 = 0$).

The *DD* bimodule for the identity (as described in Theorem 1) was also calculated in [8, Theorem 14], by different methods. The proof of Theorem 1 is included here (despite its redundancy with results of [8]), since it is a model for the more complicated Theorem 2.

In the present paper, we have calculated the $\widehat{HF}$ variant of Heegaard Floer homology for closed 3–manifolds. This is also a key component in the combinatorial description of the invariant for cobordisms, which will be given in a future paper [7].

Acknowledgements We thank Bohua Zhan for helpful comments on a draft of this paper. We also thank the referee for a thorough reading of the paper and many suggestions and corrections.

The first author was supported by NSF grant DMS-0905796, the Mathematical Sciences Research Institute and a Sloan Research Fellowship. The second author was supported by NSF grant DMS-0505811, the Mathematical Sciences Research Institute and a Clay Senior Scholar Fellowship. The third author was supported by NSF grant DMS-1008049, the Mathematical Sciences Research Institute and a Sloan Research Fellowship.

2 Preliminaries

In this section we will review most of the background on bordered Floer theory needed later in the paper. In Section 2.1, we recall the mapping class group (or rather, groupoid) relevant to our considerations. In Section 2.2, we amplify the remarks in the introduction regarding the algebras $\mathcal{A}(\mathcal{Z})$ associated to pointed matched circles. In Section 2.3, we review the basics of the type-D modules associated to 3–manifolds with one boundary component. We also introduce the notion of the *coefficient algebra* of a type-D structure, which is used later in the calculation of arc-slide bimodules. In Section 2.4, we review the case of type-DD modules for 3–manifolds with two boundary components, and introduce their coefficient algebras. In Section 2.5, we turn to the versions of the pairing theorem that will be used in this paper. For more details on any of these topics, the reader is referred to [5; 6; 8].

This section does not discuss the type-A module associated to a bordered 3–manifold with one boundary component, nor the type-DA or -AA modules associated to a bordered 3–manifold with two boundary components. By [5, Proposition 9.2] (or any of several results from [8]), these invariants can be recovered from the type-D and type-DD invariants. We use these results to circumvent explicitly using type-A modules in most of the paper, though we return to them in Section 9.

2.1 The (strongly-based) mapping class groupoid

As discussed in the introduction, the main work in this paper consists in computing the bimodules associated to arc-slides. Since arc-slides connect different pointed matched circles, they correspond to maps between different (though homeomorphic) surfaces. To put this phenomenon in a more general context, we recall some basic properties of a certain mapping class groupoid.

Fix an integer k. Let $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$ be a pointed matched circle on $4k$ points. We can associate to $\mathcal{Z}$ a surface $F^\circ(\mathcal{Z})$ as follows. Let D be a disk with boundary Z. Attach a 2–dimensional 1–handle to ∂D along each pair of matched points in $\boldsymbol{a}$. The result is a surface $F^\circ(\mathcal{Z})$ with one boundary component, and a basepoint z on that boundary component. Let $F(\mathcal{Z})$ denote the result of filling the boundary component of $F^\circ(\mathcal{Z})$ with a disk; we call the disk $F(\mathcal{Z}) \setminus F^\circ(\mathcal{Z})$ the *preferred disk* in $F(\mathcal{Z})$; the basepoint z lies on the boundary of the preferred disk.

(The construction of $F(\mathcal{Z})$ given here agrees with [5; 6], and differs superficially from the construction in [8].)

Given pointed matched circles $\mathcal{Z}_1$ and $\mathcal{Z}_2$, the set of *strongly-based mapping classes from $\mathcal{Z}_1$ to $\mathcal{Z}_2$*, denoted $MCG_0(\mathcal{Z}, \mathcal{Z}_2)$, is the set of orientation-preserving, isotopy class of homeomorphisms $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ carrying z_1 to z_2, where $z_i \in \partial F^\circ(\mathcal{Z}_i)$ is the basepoint:

$$MCG_0(\mathcal{Z}_1, \mathcal{Z}_2) = \{\phi\colon F^\circ(\mathcal{Z}_1) \xrightarrow{\cong} F^\circ(\mathcal{Z}_2) \mid \phi(z_1) = z_2\} \text{ / isotopy.}$$

(The subscript 0 on the mapping class group indicates that maps respect the boundary and the basepoint.) In the case where $\mathcal{Z}_1 = \mathcal{Z}_2$, this set naturally forms a group, which we call the *strongly-based mapping class group*.

More generally, the *strongly-based genus-k mapping class groupoid $MCG_0(k)$* is the category whose objects are pointed matched circles with $4k$ points and with morphism set between $\mathcal{Z}_1$ and $\mathcal{Z}_2$ given by $MCG_0(\mathcal{Z}_1, \mathcal{Z}_2)$.

Recall that when $\mathcal{Z}$ and $\mathcal{Z}'$ differ by an arc-slide, there is a canonical strongly-based diffeomorphism $F^\circ(m)\colon F(\mathcal{Z}) \to F(\mathcal{Z}')$, as pictured in Figure 3.

Any morphism in the mapping class groupoid can be factored as a product of arc-slides; see, for example, Bene [2, Theorem 5.3]. One proof: Consider Morse functions f on $F^\circ(\mathcal{Z})$ and f' on $F^\circ(\mathcal{Z}')$ inducing the pointed matched circles. Let $\phi\colon F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z}')$ be an orientation-preserving diffeomorphism. The Morse functions f and $\phi^*(f') = f' \circ \phi$ can be connected by a generic one-parameter family of Morse functions f_t. The finitely many times t for which f_t has a flow-line from between two index-one critical points give the sequence of arc-slides connecting $\mathcal{Z}$ and $\mathcal{Z}'$.

For instance, any Dehn twist can be factored as a product of arc-slides. The key point to doing this in practice is the following:

Lemma 2.1 *Let $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$ be a pointed matched circle and $\{b, b'\} \subset \boldsymbol{a}$ a matched pair in $\mathcal{Z}$. Consider the sequence of arc-slides where one slides each of the points in $\boldsymbol{a}$ between b and b' over $\{b, b'\}$ once, in turn. This product of arc-slides is a factorization of the Dehn twist around the curve in $F^\circ(\mathcal{Z})$ specified by $\{b, b'\}$.*

(See Figure 7 and compare Andersen, Bene and Penner [1, Lemma 8.3].)

Proof The proof is left to the reader. □

In particular, for a genus-1 pointed matched circle, arc-slides are Dehn twists. For an illustration of the factorization of a more interesting Dehn twist in the genus-2 case, see Figure 7.

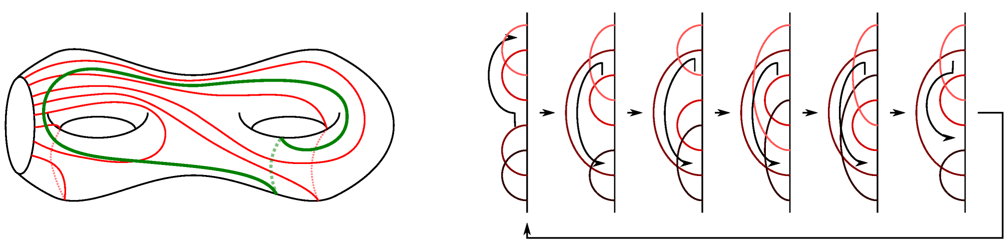


Figure 7: *Factoring a Dehn twist into arc-slides* On the left, we have a genus-2 surface specified by a pointed matched circle, and a curve γ (drawn in thick green) in it; on the right, we have a sequence of arc-slides whose composition is a Dehn twist around γ.

2.1.1 Strongly bordered 3–manifolds and mapping cylinders It will be convenient to think of strongly-based diffeomorphisms in terms of their mapping cylinders. Given a strongly-based diffeomorphism $\psi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we can extend ψ

by the identity map on $\mathbb{D}^2$ to a diffeomorphism $\psi\colon F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$. Consider the 3–manifold $[0,1] \times F(\mathcal{Z}_2)$. This manifold is equipped with orientation-preserving identifications

$$\psi\colon -F(\mathcal{Z}_1) \to \{0\} \times F(\mathcal{Z}_2) \subset [0,1] \times F(\mathcal{Z}_2),$$
$$\mathbb{I}\colon F(\mathcal{Z}_2) \to \{1\} \times F(\mathcal{Z}_2) \subset [0,1] \times F(\mathcal{Z}_2).$$

It is also equipped with a cylinder $[0,1] \times (F(\mathcal{Z}_2) \setminus F^\circ(\mathcal{Z}_2))$, which is essentially the same data as a framed arc γ connecting the two boundary components of $[0,1]\times F(\mathcal{Z}_2)$. We call the data $([0,1] \times F(\mathcal{Z}_2), \psi, \mathbb{I}, \gamma)$ the *strongly bordered* 3*–manifold associated to* ψ or the *mapping cylinder of* ψ; compare [5, Construction 5.27]. Let Y_ψ denote the mapping cylinder of ψ.

Observe that for $\psi_{12}\colon F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$ and $\psi_{23}\colon F(\mathcal{Z}_2) \to F(\mathcal{Z}_3)$, $Y_{\psi_{23}\circ\psi_{12}}$ is orientation-preserving homeomorphic to $Y_{\psi_{12}} \cup_{F(\mathcal{Z}_2)} Y_{\psi_{23}}$.

More generally, a *strongly bordered* 3*–manifold with two boundary components* consists of a 3–manifold Y with two boundary components $\partial_L Y$ and $\partial_R Y$, diffeomorphisms $\phi_L\colon F(\mathcal{Z}_L) \to \partial_L Y$ and $\phi_R\colon F(\mathcal{Z}_R) \to \partial_R Y$ for some pointed matched circles $\mathcal{Z}_L$ and $\mathcal{Z}_R$, and a framed arc in Y connecting the basepoints z in $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$, and so that the framing points into the preferred disk of $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$ at the two boundary components.

2.2 More on the algebra associated to a pointed matched circle

Fix a pointed matched circle $\mathcal{Z}$, as in Section 1.1, with basepoint z. Let $\boldsymbol{a} \subset \mathcal{Z}$ denote the set of points which are matched.

Each strands diagram υ has an associated one-chain $\mathrm{supp}(\upsilon)$, which is an element of $H_1(\mathcal{Z} \setminus \{z\}, \boldsymbol{a})$. This is gotten by projecting the strands diagram, thought of as a one-chain in $[0,1] \times (\mathcal{Z} \setminus \{z\})$, onto $\mathcal{Z}$. (The one-chain $\mathrm{supp}(\upsilon)$ is denoted $[\upsilon]$ in [6]; we have chosen to change notation here in order to avoid a conflict with the standard notation for a closed interval.)

Recall that if υ is a strands diagram then we call its associated algebra element a *basic generator* for the algebra $\mathcal{A}(\mathcal{Z})$; we usually do not distinguish between the strands diagram and its associated algebra element, writing, for example $\mathrm{supp}(a)$ when a is a basic generator.

In general, for a set $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_k\}$ of chords on $\mathcal{Z}$ with endpoints on $\boldsymbol{a}$, there is an algebra element $a(\boldsymbol{\xi})$, in which the moving strands correspond to the ξ_i and we sum over all valid ways of adding horizontal strands. (If some ξ_i, ξ_j, $i \neq j$, have their initial (respectively terminal) endpoints the same or matched, we define $a(\boldsymbol{\xi}) = 0$: in this case there are no valid ways of adding horizontal strands.) We will also abuse notation slightly, and write $a(X)$ for X a subset of $\mathcal{Z}$ with boundary only at points in $\boldsymbol{a}$: this means $a(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the set of connected components of X. (Each connected component is an interval, of course.)

An element of the algebra is called *homogeneous* if it can be written as a sum of basic generators so that each basic generator in the sum:

- Has the same associated one-chain.
- Has the same initial (and hence, in view of the previous condition, terminal) idempotent, and in particular has the same weight.
- Has the same number of crossings.

2.2.1 The opposite algebra Suppose that $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$ is a pointed matched circle. Let $-\mathcal{Z}$ denote its reverse, ie the pointed matched circle obtained by reversing the orientation on Z. There is an obvious orientation-reversing map $r\colon \mathcal{Z} \to -\mathcal{Z}$, and hence an identification between chords for $\mathcal{Z}$ and chords for $-\mathcal{Z}$. It is easy to see that this map r induces an isomorphism $\mathcal{A}(\mathcal{Z})^{\mathrm{op}} \cong \mathcal{A}(-\mathcal{Z})$, where $\mathcal{A}(\mathcal{Z})^{\mathrm{op}}$ denotes the opposite algebra to $\mathcal{A}(\mathcal{Z})$.

In particular, left $\mathcal{A}(-\mathcal{Z})$–modules correspond to left $\mathcal{A}(\mathcal{Z})^{\mathrm{op}}$–modules, and hence to right $\mathcal{A}(\mathcal{Z})$–modules.

2.2.2 Gradings The algebra $\mathcal{A}(\mathcal{Z})$ is graded in the following sense. There is a group $G(\mathcal{Z})$, equipped with a distinguished central element λ, and a function gr from basic generators of $\mathcal{A}(\mathcal{Z})$ to $G(\mathcal{Z})$, with the following properties:

- If υ and τ are basic generators of $\mathcal{A}(\mathcal{Z})$, and $\upsilon \cdot \tau \neq 0$, then $\mathrm{gr}(\upsilon \cdot \tau) = \mathrm{gr}(\upsilon) \cdot \mathrm{gr}(\tau)$.
- If τ appears with nonzero multiplicity in $d\upsilon$ then $\mathrm{gr}(\upsilon) = \lambda \cdot \mathrm{gr}(\tau)$.

In fact, there are two choices of grading group for $\mathcal{A}(\mathcal{Z})$. The smaller one, which is more natural from the point of view the pairing theorem, is a Heisenberg group on the first homology of the underlying surface. Gradings in this smaller set depend on a further universal choice of grading refinement data, as in [5, Section 3.2] (see also Section 6.1.3), although different choices of refinement data lead to canonically equivalent module categories. However, we will generally work with the big grading group $G'(\mathcal{Z})$ in this paper.

More precisely, the big grading group $G'(\mathcal{Z})$ is a $\mathbb{Z}$ central extension of $H_1(Z\setminus\{z\},\boldsymbol{a})$, realized explicitly as pairs $(j,\alpha) \in \frac{1}{2}\mathbb{Z} \times H_1(Z \setminus \{z\},\boldsymbol{a})$ subject to a congruence condition $j \equiv \epsilon(\alpha) \pmod 1$, for the function $\epsilon\colon H_1(Z \setminus \{z\},\boldsymbol{a}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ given by $\frac{1}{4}$ the number of parity changes in the support of a; see [5, Section 3.2]. The multiplication is given by

$$(j_1,\alpha_1)\cdot(j_2,\alpha_2) = (j_1 + j_2 + m(\alpha_2,\partial\alpha_1),\alpha_1 + \alpha_2),$$

where $m(\alpha,x)$ is the local multiplicity of α at x; $m(\alpha_2,\partial\alpha_1)$ is a $\frac{1}{2}\mathbb{Z}$–valued extension of the intersection form on $H_1(F(\mathcal{Z})) \subset H_1(Z \setminus \{z\},\boldsymbol{a})$. The distinguished central element is $\lambda = (1,0)$. The $G'(\mathcal{Z})$–grading of a strands diagram is given by

$$\mathrm{gr}'(a) := (\iota(a),\mathrm{supp}(a)),$$

where $\mathrm{supp}(a) \in H_1(\mathcal{Z} \setminus \{z\},\boldsymbol{a})$ is as defined above, and $\iota(a)$ records the number of crossings plus a correction term:

$$\iota(a) := \mathrm{inv}(a) - m(\mathrm{supp}(a),\boldsymbol{s}),$$

where $I(\boldsymbol{s})$ is the initial idempotent of a. See [6, Section 3.3] for further details.

Homogeneous algebra elements (as defined earlier) live in a single grading. For a sum of basic generators with the same left and right idempotents, the converse is true: homogeneity with respect to the grading (for either grading group) implies homogeneity as defined above.

Lemma 2.2 *If a basic generator a is not an idempotent, then $\iota(a) \leq -k/2$, where k is the number of intervals in a minimal expression of* $\mathrm{supp}(a)$ *as a sum of intervals.*

Proof This is essentially [5, Lemma 3.6]. The argument there shows that $\iota(a) \leq -k'/2$, where k' is the number of moving strands in a; but if k is as given in the statement, then $k' \geq k$. □

2.2.3 The quotient map Recall that if $\mathcal{Z}$ and $\mathcal{Z}_0$ are pointed matched circles then we can form their connected sum $\mathcal{Z} \# \mathcal{Z}_0$. Note that there are two natural choices of where to put the basepoint in $\mathcal{Z} \# \mathcal{Z}_0$.

Given any idempotent $I_0 = I(\boldsymbol{s}_0)$ for $\mathcal{Z}_0$, we have a quotient map

$$q\colon \mathcal{A}(\mathcal{Z} \# \mathcal{Z}_0) \to \mathcal{A}(\mathcal{Z})$$

defined as follows. The idempotents for $\mathcal{Z} \# \mathcal{Z}_0$ have the form $I(\boldsymbol{s} \amalg \boldsymbol{t})$, where $\boldsymbol{s}$ (respectively $\boldsymbol{t}$) is a subset of the matched pairs in $\mathcal{Z}$ (respectively $\mathcal{Z}_0$). The quotient map q is determined by its action on the idempotents:

$$q(I(\mathbf{s} \amalg \mathbf{t})) = \begin{cases} I(\mathbf{s}) & \text{if } \mathbf{t} = \mathbf{s}_0, \\ 0 & \text{otherwise,} \end{cases}$$

and also the property that $q(a) = 0$ unless $\mathrm{supp}(a) \subset \mathcal{Z} \subset \mathcal{Z} \# \mathcal{Z}_0$.

The map q can be promoted to a map

$$Q\colon \mathcal{A}(\mathcal{Z} \# \mathcal{Z}_0) \otimes \mathcal{A}(-\mathcal{Z}' \# \mathcal{Z}_0) \to \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}').$$

The map q is used in the definition of stability for arc-slide bimodules. The notation is somewhat lacking, since we have not specified how we have taken the connected sum of $\mathcal{Z}$ and $\mathcal{Z}_0$ (ie in which of the two possible regions in the connect sum we have placed the basepoint). This information is not important, however, since stability uses both possible choices.

2.3 Bordered invariants of 3–manifolds with connected boundary

We recall the basics of the bordered Heegaard Floer invariant $\widehat{CFD}(Y)$ for a bordered 3–manifold.

2.3.1 Bordered Heegaard diagrams and *CFD*^

Definition 2.3 A *bordered Heegaard diagram*, is a quadruple $\mathcal{H} = (\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ consisting of:

- A compact, oriented surface $\bar{\Sigma}$ with one boundary component, of some genus g.
- A g–tuple of pairwise-disjoint circles $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ in the interior of Σ.
- A $(g+k)$–tuple of pairwise-disjoint curves $\bar{\boldsymbol{\alpha}}$ in $\bar{\Sigma}$, consisting of $g-k$ circles $\boldsymbol{\alpha}^c = (\alpha_1^c, \dots, \alpha_{g-k}^c)$ in the interior of $\bar{\Sigma}$ and $2k$ arcs $\bar{\boldsymbol{\alpha}}^a = (\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a)$ in $\bar{\Sigma}$ with boundary on $\partial\bar{\Sigma}$ (and transverse to $\partial\bar{\Sigma}$).
- A point z in $(\partial\bar{\Sigma}) \setminus (\boldsymbol{\alpha} \cap \partial\bar{\Sigma})$.

Also, $\boldsymbol{\beta} \pitchfork \bar{\boldsymbol{\alpha}}$ and $\bar{\Sigma} \setminus \bar{\boldsymbol{\alpha}}$ and $\bar{\Sigma} \setminus \boldsymbol{\beta}$ must be connected.

(As in [6], we let Σ denote the interior of $\bar{\Sigma}$ and $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}} \cap \Sigma$; and will often blur the distinction between $(\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.)

The boundary of a Heegaard diagram $\mathcal{H} = (\Sigma, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ is naturally a pointed, matched circle as follows. The boundary $(\partial\Sigma)$ inherits its basepoint from $z \in \partial\Sigma$, and the points $(\bar{\boldsymbol{\alpha}} \cap \partial\bar{\Sigma})$, can be paired off according to which arc they belong.

A bordered Heegaard diagram specifies an oriented 3–manifold with boundary Y, along with an orientation preserving diffeomorphism $\phi\colon F(\mathcal{Z}) \to \partial Y$; ie a $\mathcal{Z}$–bordered 3–manifold.

We briefly recall the construction of the type-D module associated to a bordered Heegaard diagram $\mathcal{H}$. Let $\mathcal{Z}$ be the matched circle appearing on the boundary of $\mathcal{H}$. The type-D module associated to $\mathcal{H}$ is a left module over $\mathcal{A}(-\mathcal{Z})$, where $-\mathcal{Z}$ is the reverse of $\mathcal{Z}$.

Let $\mathfrak{S}(\mathcal{H})$ be the set of subsets $\boldsymbol{x} \subset \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ with the following properties:

- $\boldsymbol{x}$ contains exactly one element on each β circle.
- $\boldsymbol{x}$ contains exactly one element on each α circle.
- $\boldsymbol{x}$ contains at most one element on each α arc.

Let $X(\mathcal{H})$ be the $\mathbb{F}_2$–vector space spanned by $\mathfrak{S}(\mathcal{H})$. For $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$, let $o(\boldsymbol{x}) \subset [2k]$ be the set of α–arcs occupied by $\boldsymbol{x}$. Define $I_D(\boldsymbol{x})$ to be $I([2k] \setminus o(\boldsymbol{x}))$; that is, the idempotent corresponding to the complement of $o(\boldsymbol{x})$. We can now define an action of the subalgebra of idempotents $\mathcal{I}$ inside $\mathcal{A}(-\mathcal{Z})$ on $X(\mathcal{H})$ via

$$I(\boldsymbol{s}) \cdot \boldsymbol{x} = \begin{cases} \boldsymbol{x} & I(\boldsymbol{s}) = I_D(\boldsymbol{x}), \\ 0 & \text{otherwise}, \end{cases} \tag{2-1}$$

where $\boldsymbol{s}$ is a k–element subset of $[2k]$. As a module, let $\widehat{CFD}(\mathcal{H}) = \mathcal{A}(-\mathcal{Z}) \otimes_{\mathcal{I}} X(\mathcal{H})$.

Fix generators $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}(\mathcal{H})$. Two-chains in Σ which connect $\boldsymbol{x}$ and $\boldsymbol{y}$ in a suitable sense can be organized into *homology classes*, denoted $\pi_2(\boldsymbol{x}, \boldsymbol{y})$; we say elements of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ *connect* $\boldsymbol{x}$ *to* $\boldsymbol{y}$. (To justify the terminology "homology class", note that the difference between any two elements of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ can be thought of as a two-dimensional homology class in Y. The notation is justified by its interpretation in terms of the symmetric product; see [11].) Given a homology class $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ and asymptotics specified by a vector $\vec{\rho}$, there is an associated moduli space of holomorphic curves $\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}, \vec{\rho})$. Counting points in this moduli space gives rise to an algebra element

$$n^B_{\boldsymbol{x},\boldsymbol{y}} = \sum_{\{\vec{\rho} \mid \mathrm{ind}(B,\vec{\rho})=1\}} \#(\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \vec{\rho}))a(-\vec{\rho}) \in \mathcal{A}(-\mathcal{Z}_L). \tag{2-2}$$

The algebra elements $n^B_{\boldsymbol{x},\boldsymbol{y}}$ can be assembled to define an operator

$$\delta^1\colon X(\mathcal{H}) \to \mathcal{A}(\mathcal{Z}) \otimes X(\mathcal{H})$$

by

$$\delta^1(\boldsymbol{x}) := \sum_{\boldsymbol{y}\in\mathfrak{S}(\mathcal{H})} \sum_{B\in\pi_2(\boldsymbol{x},\boldsymbol{y})} n^{\boldsymbol{B}}_{\boldsymbol{x},\boldsymbol{y}} \otimes \boldsymbol{y}. \tag{2-3}$$

Let $\widehat{CFD}(\mathcal{H})$ denote the space $\mathcal{A}(-\mathcal{Z})\otimes X(\mathcal{H})$. We endow $\widehat{CFD}(\mathcal{H})$ with a differential ∂ induced from the above map δ^1, and the differential on the algebra $\mathcal{A}(-\mathcal{Z})$, via the Leibniz rule

$$\partial(a\otimes \boldsymbol{x}) = (da)\otimes \boldsymbol{x} + a\cdot\delta^1(\boldsymbol{x}).$$

The differential, together with the obvious left action on $\mathcal{A}(-\mathcal{Z})$, gives $\widehat{CFD}(\mathcal{H})$ the structure of a left differential module over $\mathcal{A}(-\mathcal{Z})$. (The proof involves studying one-parameter families of holomorphic curves; see [6, Section 6.2].)

The differential on $\widehat{CFD}(\mathcal{H})$ has the following key property. Suppose that $B\in\pi_2(\boldsymbol{x},\boldsymbol{y})$ gives a nonzero contribution of $a\otimes\boldsymbol{y}$ to $\partial\boldsymbol{x}$. Then, supp(a) is calculated by the local multiplicities of B at $\partial\overline{\Sigma}$.

Up to homotopy equivalence, the module $\widehat{CFD}(\mathcal{H})$ depends only on the bordered 3–manifold specified by $\mathcal{H}$. Thus, given a bordered 3–manifold Y we will write $\widehat{CFD}(Y)$ to denote the homotopy type of $\widehat{CFD}(\mathcal{H})$ for any bordered Heegaard diagram $\mathcal{H}$ representing Y.

2.3.2 Type-*D* structures The special structure of $\widehat{CFD}(\mathcal{H})$ can be formalized in the following:

Definition 2.4 Let $\mathcal{A}$ be a dg–algebra over a ground ring $\boldsymbol{k}$. A left *type-D structure* over $\mathcal{A}$ is a left $\boldsymbol{k}$–module ${}^{\mathcal{A}}N$ together with a degree-0 map $\delta^1\colon N\to A[1]\otimes N$ satisfying the structural equation

$$(\mu_2\otimes\mathbb{I}_N)\circ(\mathbb{I}_{\mathcal{A}}\otimes\delta^1)\circ\delta^1 + (\mu_1\otimes\mathbb{I}_N)\circ\delta^1 = 0. \tag{2-4}$$

(Here, μ_1 and μ_2 denote the differential and multiplication on $\mathcal{A}$, respectively; the notation is drawn from the theory of $\mathcal{A}_\infty$–algebras.)

Given a type-D structure as above, we can form the associated module denoted ${}_{\mathcal{A}}N$ or $\mathcal{A}\boxtimes N$, whose generators are $a\otimes\boldsymbol{x}$ with $a\in\mathcal{A}$ and $\boldsymbol{x}\in N$, algebra action by

$$a\cdot(b\otimes\boldsymbol{x}) = (a\cdot b)\otimes\boldsymbol{x},$$

and differential given by

$$\partial(a\otimes\boldsymbol{x}) = (da)\otimes\boldsymbol{x} + a\cdot(\delta^1\boldsymbol{x}).$$

(Here, $\otimes$ denotes tensor product over $\boldsymbol{k}$, not $\mathbb{F}_2$. The structural equation for δ^1, (2-4) is equivalent to the condition that $\partial^2 = 0$.)

The bordered invariant $\widehat{CFD}(\mathcal{H})$ is naturally a type-D structure over $\mathcal{A}(-\mathcal{Z})$.

The notion of a type-D structure has an obvious analogue for bimodules: a *type-DD structure* over dg–algebras $\mathcal{A}$ and $\mathcal{B}$ is just a type-D structure over $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$.

2.3.3 Some particular holomorphic curves We have not explained here precisely which curves contribute to $\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}, \vec{\rho})$. Rather than reviewing the general case, we restrict our discussion to the main examples we will need in this paper. (See [6] for further details.)

Definition 2.5 Suppose that P is a connected component of $\bar{\Sigma} \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, and P does not contain z. Suppose moreover that P is a $2n$–gon. Each side of P is one of three kinds:

(P-1) An arc contained in some β_i.

(P-2) An arc contained in some α_i (which might be of the form α_i^c, α_i^a).

(P-3) An arc contained in $\partial\Sigma$.

Traversing the boundary of P with its induced orientation, one alternates between meeting sides of type (P-1) and sequences of sides of types (P-2) and (P-3). Suppose that P has only one side of type (P-3). We call such a component a *fundamental polygon*.

The intersection points of arcs of type (P-1) and those of type (P-2), which we call *corners*, can be partitioned into two types: those which lie at the initial point of the arc of type (P-2) (with its induced orientation from ∂P), and those which lie at the terminal point of the arc of type (P-2). Let $\boldsymbol{x}_0$ denote the set of corners of the first type, and let $\boldsymbol{y}_0$ denote the set of corners of the second type. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two generators with the property that $\boldsymbol{x} \setminus (\boldsymbol{x} \cap \boldsymbol{y}) = \boldsymbol{x}_0$ and $\boldsymbol{y} \setminus (\boldsymbol{x} \cap \boldsymbol{y}) = \boldsymbol{y}_0$. Then, P determines a homology class $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$.

Definition 2.6 In the above situation, we say that the fundamental polygon P *connects $\boldsymbol{x}$ to $\boldsymbol{y}$ in the homology class B*, or simply that P *connects $\boldsymbol{x}$ to $\boldsymbol{y}$*.

Lemma 2.7 *Suppose P is a fundamental polygon that connects $\boldsymbol{x}$ and $\boldsymbol{y}$ in the homology class B. Let ξ be the chord in $\partial\bar{\Sigma}$ which lies on ∂P. Then $n^B_{\boldsymbol{x},\boldsymbol{y}} = I(\boldsymbol{x}) \cdot a(-\xi) \cdot I(\boldsymbol{y})$.*

Proof This is an easy consequence of the definitions (see [6, Section 6])) and a little complex analysis (cf Rasmussen [12, Section 9.5]). □

2.3.4 Gradings and the coefficient algebra In addition to the group-valued grading on the algebra as described in Section 2.2.2, the modules $\widehat{CFD}(\mathcal{H})$ are G–set graded modules. This means that there is a set $S'(\mathcal{H})$ with a left $G'(-\mathcal{Z})$–action and a grading function $\mathrm{gr}'\colon \mathfrak{S}(\mathcal{H}) \to S'(\mathcal{H})$ satisfying the following compatibility conditions: if $m \in \widehat{CFD}(\mathcal{H})$ is an $S'(\mathcal{H})$–homogeneous element, and a is a $G'(-\mathcal{Z})$–homogeneous algebra element with $a \cdot m \neq 0$, then $a \cdot m$ is $S'(\mathcal{H})$–homogeneous and $\mathrm{gr}'(a \cdot m) = \mathrm{gr}'(a) \cdot \mathrm{gr}'(m)$ (where $\cdot$ means the left translation of $G'(-\mathcal{Z})$ on $S'(\mathcal{H})$); and if m is an $S'(\mathcal{H})$–homogeneous element then ∂m is also $S'(\mathcal{H})$–homogeneous and $\mathrm{gr}'(\partial m) = \lambda^{-1} \cdot \mathrm{gr}'(m)$.

To be more explicit about these gradings in the case of $\widehat{CFD}(\mathcal{H})$ for a Heegaard diagram $\mathcal{H}$ with boundary $\mathcal{Z}$, for any generators $\boldsymbol{x}$, $\boldsymbol{y}$ and any $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, define

$$g'(B) := (-e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B), \partial^{\partial}(B)) \in G'(\mathcal{Z});$$

cf [6, Equation 10.2]. Here $\partial^{\partial}(B)$ is $\partial(B) \cap \partial\Sigma$, the portion of ∂B on the boundary of the Heegaard diagram. To relate this to the grading on the algebra $\mathcal{A}(-\mathcal{Z})$, let $R\colon G'(\mathcal{Z}) \to G'(-\mathcal{Z})$ be the map induced by the orientation-reversing map $r\colon \mathcal{Z} \to -\mathcal{Z}$ via the formula $R(k, \alpha) = (k, r_*(\alpha))$. The map R is a group antihomomorphism. (Note that if α has positive multiplicities, then $r_*(\alpha)$ has negative multiplicities. In particular, $\mathrm{gr}'(a(-\rho)) = (-\frac{1}{2}, -r_*(\mathrm{supp}(\rho)))$ while $R(\mathrm{gr}'(a(\rho))) = (-\frac{1}{2}, r_*(\mathrm{supp}(\rho)))$.)

The function $g'(B)$ satisfies the crucial property that if $\vec{\rho}$ is any set of asymptotics compatible with B and $a(-\vec{\rho}) \neq 0$ then

$$R(g'(B))\,\mathrm{gr}'(a(-\vec{\rho})) = \lambda^{-\operatorname{ind}(B,\vec{\rho})};$$

see [6, Lemma 10.20]. The grading set $S'(\mathcal{H})$ is therefore chosen in a suitable way so that $\mathrm{gr}'(\boldsymbol{x})$ and $\mathrm{gr}'(\boldsymbol{y})$ are in the same $G'(-\mathcal{Z})$–orbit if and only if there is a domain $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, and if there is such a B, then

$$R(g'(B))\,\mathrm{gr}'(\boldsymbol{x}) = \mathrm{gr}'(\boldsymbol{y}) \tag{2-5}$$

(see [6, Equation 10.27]); this guarantees that the grading on $\widehat{CFD}(\mathcal{H})$ is compatible with the grading on $\mathcal{A}(-\mathcal{Z})$. Explicitly, after fixing a base generator $\boldsymbol{x}_0$ for each spin^c–structure $\mathfrak{s}$ we can set

$$S'(\mathcal{H}, \mathfrak{s}) = G'(-\mathcal{Z})/\langle R(g'(P)) \mid P \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)\rangle.$$

We then let $S'(\mathcal{H}) = \coprod_{\mathfrak{s} \in \mathrm{spin}^c(\mathcal{H})} S'(\mathcal{H}, \mathfrak{s})$. See [6, Chapter 10] for more details.

The $G'(\mathcal{Z})$–sets for the mapping cylinders will have the following convenient property:

Definition 2.8 A G–set S is said to be λ*–free* if for any $s \in S$ and $n \in \mathbb{Z}$, $\lambda^n \cdot s \neq s$.

In this paper, we will also use an alternate way of thinking about gradings, which we define in slightly greater generality than we use in this paper.

Definition 2.9 A *based algebra* is an algebra over $\mathbb{F}_2$ with a distinguished finite set of *basic idempotents*, which are primitive, pairwise-orthogonal idempotents whose sum is the identity.

A based algebra can also be thought of as a dg–category with a finite number of elements. The algebra $\mathcal{A}(\mathcal{Z})$ is a based algebra with basic idempotents the idempotents $I(\boldsymbol{s})$.

Definition 2.10 For a type-D structure ${}^{A}M$ over a based algebra A, where A is graded by G and M is graded by a G–set S, the *coefficient algebra* $\operatorname{Coeff}(M)$ of M is the differential algebra spanned by triples $(\boldsymbol{x}, a, \boldsymbol{y})$ with $a \in A$ and $\boldsymbol{x}, \boldsymbol{y} \in M$ so that

- $a = I \cdot a \cdot J$, $I\boldsymbol{x} = \boldsymbol{x}$ and $J\boldsymbol{y} = \boldsymbol{y}$ for some basic idempotents I and J, and
- there is a $k \in \mathbb{Z}$ so that $\lambda^k \operatorname{gr}(\boldsymbol{x}) = \operatorname{gr}(a)\operatorname{gr}(\boldsymbol{y})$,

modulo the relations that the triples are linear in each factor:

$$\begin{aligned} (\boldsymbol{x}+\boldsymbol{x}', a, \boldsymbol{y}) &= (\boldsymbol{x}, a, \boldsymbol{y}) + (\boldsymbol{x}', a, \boldsymbol{y}), \\ (\boldsymbol{x}, a+a', \boldsymbol{y}) &= (\boldsymbol{x}, a, \boldsymbol{y}) + (\boldsymbol{x}, a', \boldsymbol{y}), \\ (\boldsymbol{x}, a, \boldsymbol{y}+\boldsymbol{y}') &= (\boldsymbol{x}, a, \boldsymbol{y}) + (\boldsymbol{x}, a, \boldsymbol{y}'). \end{aligned} \tag{2-6}$$

The differential is $\partial(\boldsymbol{x}, a, \boldsymbol{y}) = (\boldsymbol{x}, \partial a, \boldsymbol{y})$, and the product is given by

$$(\boldsymbol{x}_1, a_1, \boldsymbol{y}_1)\cdot(\boldsymbol{x}_2, a_2, \boldsymbol{y}_2) = \begin{cases} (\boldsymbol{x}_1, a_1\cdot a_2, \boldsymbol{y}_2) & \boldsymbol{y}_1 = \boldsymbol{x}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus each generator of ${}^{A}M$ gives an idempotent of the coefficient algebra. For $A = \mathcal{A}(\mathcal{Z})$ and $M = {}^{A}\widehat{CFD}(\mathcal{H})$, this means that there is an idempotent of the coefficient algebra corresponding to each element of $\mathfrak{S}(\mathcal{H})$. The elements of $\operatorname{Coeff}({}^{A}M)$ record algebra coefficients whose gradings do not prevent them from appearing in the differential, as we see in the next lemma.

Lemma 2.11 *If ${}^{A}M$ is a type-D structure over A, where A is graded by G and M is graded by S, let $n = \gcd\{m \in \mathbb{N} \mid \lambda^m s = s$ for some $s \in S\}$ (or 0 if S is λ–free). Then the coefficient algebra $\operatorname{Coeff}(M)$ has a canonical grading $\operatorname{gr}(\boldsymbol{x}, a, \boldsymbol{y}) \in \mathbb{Z}/n\mathbb{Z}$, characterized as follows. By definition, there is some $k \in \mathbb{Z}$ such that $\lambda^k \operatorname{gr}(\boldsymbol{x}) = \operatorname{gr}(a)\operatorname{gr}(\boldsymbol{y})$. Then $\operatorname{gr}(\boldsymbol{x}, a, \boldsymbol{y}) \equiv k \pmod{n}$.*

With this grading, if $a\cdot\boldsymbol{y}$ appears in $\partial\boldsymbol{x}$ then $(\boldsymbol{x}, a, \boldsymbol{y})$ has grading -1.

(Note that the divisibility of λ in its action on S is constant on each G–orbit, since λ is central in G.)

Proof Since, by definition of $\mathrm{Coeff}(M)$, there is a k so that $\lambda^k\,\mathrm{gr}(\boldsymbol{x}) = \mathrm{gr}(a)\,\mathrm{gr}(\boldsymbol{y})$, this defines $\mathrm{gr}(\boldsymbol{x}, a, \boldsymbol{y})$ as an element of the cyclic subgroup of G generated by λ, up to indeterminacy given by the divisibility of λ in its action on $\mathrm{gr}(\boldsymbol{x})$. By assumption, n divides this divisibility, so we get a well-defined element of $\mathbb{Z}/n\mathbb{Z}$, as claimed. It is elementary to check that this is a grading. The last statement follows from the assumption that M is a graded differential module: $\mathrm{gr}(\partial\boldsymbol{x}) = \lambda^{-1}\,\mathrm{gr}(\boldsymbol{x})$. □

Thus, for a module M graded by a λ–free G–set, $\mathrm{Coeff}(M)$ is $\mathbb{Z}$–graded.

If ${}^{A}M$ has at most one generator per idempotent, then we can view $\mathrm{Coeff}(M)$ as a subalgebra of A.

Recall that there are two different gradings on our algebras $\mathcal{A}(\mathcal{Z})$, one by $G(\mathcal{Z})$ and one by $G'(\mathcal{Z})$. These induce the same grading on the coefficient algebra of any type-D structure over $\mathcal{A}(\mathcal{Z})$:

Lemma 2.12 *Let ${}^{\mathcal{A}(\mathcal{Z})}M$ be a $G'(\mathcal{Z})$–set graded type-D structure over $\mathcal{A}(\mathcal{Z})$. Fix some collection of grading refinement data Ξ for $\mathcal{Z}$. Let $\mathrm{Coeff}(M)$ denote the coefficient algebra of M as a G–set graded module and $\mathrm{Coeff}'(M)$ the coefficient algebra of M as a G'–set graded module. Then $\mathrm{Coeff}(M) = \mathrm{Coeff}'(M)$, and this identification respects the gradings on the two sides.*

Proof Recall that for a generator $\boldsymbol{x}$ of M with $\boldsymbol{x} = i\cdot\boldsymbol{x}$ for some minimal idempotent i, $\mathrm{gr}(\boldsymbol{x}) = \Xi(i)\cdot\mathrm{gr}'(\boldsymbol{x})$; and if $a \in \mathcal{A}(\mathcal{Z})$ is such that $j\cdot a\cdot i = a$ for minimal idempotents i and j then $\mathrm{gr}(a) = \Xi(j)\,\mathrm{gr}'(a)\,\Xi(i)^{-1}$. The result follows. □

We now compute the grading on the coefficient algebra for a Heegaard diagram more explicitly. Loosely speaking, it is the Maslov component of the grading on the algebra plus a correction term.

Lemma 2.13 *Suppose $(\boldsymbol{x}, a, \boldsymbol{y}) \in \mathrm{Coeff}(\widehat{CFD}(\mathcal{H}))$. Then there is a $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ so that $-r_*(\partial^{\partial}(B)) = \mathrm{supp}(a)$. Moreover, for any such B,*

$$\mathrm{gr}(\boldsymbol{x}, a, \boldsymbol{y}) = \iota(a) - e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B).$$

(The map r_* appears in this lemma because $a \in \mathcal{A}(\mathcal{Z})$ where $\mathcal{Z} = -\partial\mathcal{H}$.)

Proof By definition of $\mathrm{Coeff}(\widehat{CFD}(\mathcal{H}))$, $\lambda^k \,\mathrm{gr}'(\boldsymbol{x}) = \mathrm{gr}'(a)\,\mathrm{gr}'(\boldsymbol{y})$. In particular, $\mathrm{gr}'(\boldsymbol{x})$ and $\mathrm{gr}'(\boldsymbol{y})$ are in the same $G'(-\mathcal{Z})$ orbit, so there is a domain B connecting them. By definition of the grading on the coefficient algebra, we have

$$\lambda^{\mathrm{gr}(\boldsymbol{x},a,\boldsymbol{y})}\,\mathrm{gr}'(\boldsymbol{x}) = \mathrm{gr}'(a)\,\mathrm{gr}'(\boldsymbol{y}) = \mathrm{gr}'(a)R(g'(B))\,\mathrm{gr}'(\boldsymbol{x}).$$

By assumption, the homological components of $\mathrm{gr}'(a)$ and $R(g'(B))$ cancel each other, and give no correction to the Maslov component of the grading. The Maslov components sum to the stated total. □

Remark 2.14 The coefficient algebra is not invariant under homotopy equivalences of modules, as can be seen by comparing the coefficient algebra of an acyclic but nonzero module with that of the zero module (with no generators).

Remark 2.15 One can give a more abstract definition of the coefficient algebra as follows. Let I denotes the subring of A generated by the (distinguished) orthogonal idempotents and let M^* denote the dual of M over I. Then (2-6) is equivalent to saying that the coefficient algebra is a subring of the tensor product $M^* \otimes_I A \otimes_I M$. The multiplication is induced by the obvious pairing $M \otimes M^* \to I$ and the multiplication on A.

We can identify $M^* \otimes_I A \otimes_I M$ with the space $\mathrm{Mor}^A({}^AM, {}^AM)$ of type-D structure morphisms (as in [5, Section 2.2.3]); multiplication corresponds to composition. The space $\mathrm{Mor}^A({}^AM, {}^AM)$ is graded by a $\mathbb{Z}$–set. The coefficient algebra $\mathrm{Coeff}(M)$ is the subring of $\mathrm{Mor}^A({}^AM, {}^AM)$ generated by elements whose gradings lie in the same $\mathbb{Z}$–orbit as the identity map. Note, however, that the differential we have specified on the coefficient algebra is not induced by the differential on $\mathrm{Mor}^A({}^AM, {}^AM)$.

2.4 Bordered invariants of manifolds with two boundary components

The ideas from Section 2.3 were extended to 3–manifolds with two boundary components in [5]. This extension takes the form of bimodules of various types; we will focus on the type-*DD* bimodules. The most important case for us is the case of mapping cylinders of diffeomorphisms, though in Section 8 we will also use elementary cobordisms.

2.4.1 Arced bordered Heegaard diagrams and $CFDD^\wedge$ As explained in [5], bordered Heegaard Floer homology admits a fairly straightforward generalization to the case of several boundary components:

Definition 2.16 An *arced bordered Heegaard diagram with two boundary components* (or just *arced bordered Heegaard diagram*) is a quadruple $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{z})$, where:

- $\overline{\Sigma}$ is a compact surface of some genus g with two boundary components, $\partial_L \overline{\Sigma}$ and $\partial_R \overline{\Sigma}$.
- $\boldsymbol{\beta}$ is a g–tuple of pairwise-disjoint curves in the interior Σ of $\overline{\Sigma}$.
- $\overline{\boldsymbol{\alpha}}$ is a collection

$$\overline{\boldsymbol{\alpha}} = \{\overbrace{\overline{\alpha}_1^{a,L}, \dots, \overline{\alpha}_{2g_L}^{a,L}}^{\overline{\boldsymbol{\alpha}}^{a,L}}, \overbrace{\overline{\alpha}_1^{a,R}, \dots, \overline{\alpha}_{2g_R}^{a,R}}^{\overline{\boldsymbol{\alpha}}^{a,R}}, \overbrace{\alpha_1^c, \dots, \alpha_{g-g_L-g_R}^c}^{\boldsymbol{\alpha}^c}\}$$

  of pairwise disjoint embedded arcs with boundary on $\partial_L \overline{\Sigma}$ (the $\overline{\alpha}_i^{a,L}$), arcs with boundary on $\partial_R \overline{\Sigma}$ (the $\overline{\alpha}_i^{a,R}$), and circles (the α_i^c) in the interior Σ of $\overline{\Sigma}$.
- $\boldsymbol{z}$ is a path in $\overline{\Sigma} \setminus (\overline{\boldsymbol{\alpha}} \cup \boldsymbol{\beta})$ between $\partial_L \overline{\Sigma}$ and $\partial_R \overline{\Sigma}$.

These are required to satisfy:

- $\overline{\Sigma} \setminus \overline{\boldsymbol{\alpha}}$ and $\overline{\Sigma} \setminus \boldsymbol{\beta}$ are connected.
- $\overline{\boldsymbol{\alpha}}$ intersects $\boldsymbol{\beta}$ transversely.

In this case, there are two pointed matched circles,

$$\mathcal{Z}_L = (\partial_L \overline{\Sigma}, \overline{\alpha} s^{a,L} \cap \partial_L \overline{\Sigma}, \boldsymbol{z} \cap \partial_L \overline{\Sigma}),$$
$$\mathcal{Z}_R = (\partial_R \overline{\Sigma}, \overline{\alpha} s^{a,R} \cap \partial_R \overline{\Sigma}, \boldsymbol{z} \cap \partial_R \overline{\Sigma}).$$

An arced bordered Heegaard diagram specifies a compact, oriented 3–manifold Y with two boundary components, $\partial Y = \partial_L Y \amalg \partial_R Y$, along with identifications

$$\phi_L \colon F(\mathcal{Z}_L) \to \partial Y, \quad \phi_R \colon F(\mathcal{Z}_R) \to \partial Y.$$

The data also specifies a framed arc connecting the two boundary components of Y, as explained in [5, Section 5], and hence specifies Y as a strongly bordered 3–manifold.

To an arced bordered Heegaard diagram with two boundary components $\mathcal{H}$ we associate a left $\mathcal{A}(-\mathcal{Z}_L) \otimes \mathcal{A}(-\mathcal{Z}_R)$ module $\widehat{CFDD}(\mathcal{H})$, where $\mathcal{Z}_L$ and $\mathcal{Z}_R$ are the pointed matched circles appearing on the boundary of the Heegaard diagram at $\partial_L \overline{\Sigma}$ and $\partial_R \overline{\Sigma}$ respectively.

The module $\widehat{CFDD}(\mathcal{H})$ has generating set $\mathfrak{S}(\mathcal{H})$ defined exactly as in the one boundary component case. If $\boldsymbol{x}$ is a generator, let $o_L(\boldsymbol{x})$ (respectively $o_R(\boldsymbol{x})$) denote the set of α^L–arcs (respectively α^R–arcs) occupied by $\boldsymbol{x}$. Let $I_{D,L}(\boldsymbol{x})$ (respectively

$I_{D,R}(\boldsymbol{x})$) denote the idempotent in $\mathcal{A}(-\mathcal{Z}_L)$ (respectively $\mathcal{A}(-\mathcal{Z}_R)$) corresponding to the complement of $o_L(\boldsymbol{x})$ (respectively $o_R(\boldsymbol{x})$).

Given a sequence $\vec{\rho}$ of chords in $\mathcal{Z}_L \amalg \mathcal{Z}_R$ let $\vec{\rho}_L$ (respectively $\vec{\rho}_R$) denote the subsequence of $\vec{\rho}$ consisting of the chords lying in $\mathcal{Z}_L$ (respectively $\mathcal{Z}_R$). Modify (2-2) as

$$n^B_{\boldsymbol{x},\boldsymbol{y}} = \sum_{\{\vec{\rho} \mid \mathrm{ind}(B,\vec{\rho})=1\}} \#(\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}))a(-\vec{\rho}_L)\otimes a(-\vec{\rho}_R) \in \mathcal{A}(-\mathcal{Z}_L)\otimes\mathcal{A}(-\mathcal{Z}_R). \tag{2-7}$$

Exactly as in (2-2), this determines a map

$$\delta^1\colon X(\mathcal{H}) \to \mathcal{A}(-\mathcal{Z}_L)\otimes\mathcal{A}(-\mathcal{Z}_R)\otimes X(\mathcal{H}),$$

which can be used to build a differential on the space

$$\widehat{CFDD}(\mathcal{H}) = \mathcal{A}(-\mathcal{Z}_L)\otimes_{\mathcal{I}(-\mathcal{Z}_L)}\mathcal{A}(-\mathcal{Z}_R)\otimes_{\mathcal{I}(-\mathcal{Z}_R)} X(\mathcal{H}).$$

It is proved in [5, Theorem 10] that the homotopy type of $\widehat{CFDD}(\mathcal{H})$ depends only on the strongly bordered 3–manifold represented by $\mathcal{H}$. So, if Y is a strongly bordered 3–manifold then we will often write $\widehat{CFDD}(Y)$ to denote the module $\widehat{CFDD}(\mathcal{H})$ for some arced bordered Heegaard diagram $\mathcal{H}$ representing Y.

Let Y be a strongly bordered 3–manifold with boundary parameterized by $F(-\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$; ie a $(-\mathcal{Z}_L)$–$\mathcal{Z}_R$–bordered 3–manifold. Using the identification $\mathcal{A}(-\mathcal{Z}_R) = \mathcal{A}(\mathcal{Z}_R)^{\mathrm{op}}$, we can view the $\mathcal{A}(\mathcal{Z}_L)\otimes\mathcal{A}(-\mathcal{Z}_R)$–module $\widehat{CFDD}(Y)$ as an $\mathcal{A}(\mathcal{Z}_L)$–$\mathcal{A}(\mathcal{Z}_R)$–bimodule. When it is important to indicate in which way we are viewing $\widehat{CFDD}(Y)$, we will write either ${}_{\mathcal{A}(\mathcal{Z}_L),\mathcal{A}(-\mathcal{Z}_R)}\widehat{CFDD}(Y)$ (for the bimodule with two left actions) or ${}_{\mathcal{A}(\mathcal{Z}_L)}\widehat{CFDD}(Y)_{\mathcal{A}(\mathcal{Z}_R)}$ (for the bimodule with one left and one right action).

As a special case, we obtain bimodules associated to strongly-based diffeomorphisms:

Definition 2.17 Suppose $\psi\colon F(-\mathcal{Z}_1)\to F(-\mathcal{Z}_2)$ is a strongly-based diffeomorphism. Let Y_ψ denote the mapping cylinder of ψ. Then define

$${}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\psi)_{\mathcal{A}(\mathcal{Z}_1)} = {}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(Y_\psi)_{\mathcal{A}(\mathcal{Z}_1)}.$$

2.4.2 Polygons in diagrams with two boundary components Again, fundamental polygons (in the sense of Definition 2.5) contribute to the differential. In the present case, chords on the boundary can be of two types: chords contained in $\partial_L\overline{\Sigma}$, and chords contained in $\partial_R\overline{\Sigma}$. We assume that, for our polygon, there is at most one edge of each type, ξ_L and ξ_R. The associated algebra element is $a_L(-\xi_L)\otimes a_R(-\xi_R)$ if both ξ_L and ξ_R are present, $a(-\xi_L)\otimes 1$ or $1\otimes a_R(-\xi_R)$ if only ξ_L or ξ_R is present, or $1\otimes 1$ if neither is.

For a concrete example, the reader is invited to look ahead to the bordered diagram displayed in Figure 14. There two generators are indicated: $\boldsymbol{x}$ (which is indicated by black circles) and $\boldsymbol{y}$ (which is indicated by white ones). There is a shaded octagon from $\boldsymbol{x}$ to $\boldsymbol{y}$, which goes out to $\partial\overline{\Sigma}$ in two chords, denoted ρ_5 and σ_3. This shows that $\partial\boldsymbol{x}$ contains a term $(\rho_5 \otimes \sigma_3) \otimes \boldsymbol{y}$.

This notion of polygons is still a little too restrictive for our purpose. In some cases, we will need to consider polygonal regions which are obtained as unions of closures of components in $\overline{\Sigma} \setminus \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. In order for such more general polygons to contribute, we must have that P is a union of R_i which meet along edges which do not contain any component of $\boldsymbol{x}$ or $\boldsymbol{y}$.

2.4.3 Gradings and the coefficient algebra for bimodules As was the case for modules (Section 2.3.4), the bimodules $\widehat{CFDD}(\mathcal{H})$ are set-graded. Suppose that $\mathcal{H}$ has boundary $\mathcal{Z}_L \cup \mathcal{Z}_R$. Then there is a set $S'(\mathcal{H})$ with commuting left actions of $G'(-\mathcal{Z}_L)$ and $G'(-\mathcal{Z}_R)$ (in which the two actions of λ agree), and the generators of $\widehat{CFDD}(\mathcal{H})$ have gradings in $S'(\mathcal{H})$ which are compatible with the differential and algebra actions in the natural sense. If we extend R to a map $R\colon G'(\mathcal{Z}_L) \times_{\mathbb{Z}} G'(\mathcal{Z}_R) \to G'(-\mathcal{Z}_L) \times_{\mathbb{Z}} G'(-\mathcal{Z}_R)$ (applying the map R from Section 2.3.4 to both factors), (2-5) remains true. See [5, Section 6.5] for further details.

(In Sections 4 and 6, we will also use the analogous extension of r_* from Section 2.3.4 to the disconnected case, gotten by applying r_* to each component. In other words, $r_*\colon H_1(Z_L, \boldsymbol{a}_L) \times H_1(Z_R, \boldsymbol{a}_R) \to H_1(-Z_L, \boldsymbol{a}_L) \times H_1(-Z_R, \boldsymbol{a}_R)$ is induced by the orientation-reversing map $r\colon (Z_L \amalg Z_r) \to -(Z_L \amalg Z_r)$.)

The coefficient algebra of a type-*DD* structure ${}^{\mathcal{A}(\mathcal{Z})}M^{\mathcal{A}(\mathcal{Z}')}$ is defined as it is defined in Definition 2.10: $\mathrm{Coeff}(M)$ is generated by triples $(\boldsymbol{x}, a_1 \otimes a_2, \boldsymbol{y})$, where

- if $a_1 = I(\boldsymbol{s}_1) \cdot a_1 \cdot I(\boldsymbol{t}_1)$ and $a_2 = I(\boldsymbol{s}_2) \cdot a_2 \cdot I(\boldsymbol{t}_2)$, then $\boldsymbol{x} = I(\boldsymbol{s}_1)I(\boldsymbol{s}_2)\boldsymbol{x}$ and $\boldsymbol{y} = I(\boldsymbol{t}_1)I(\boldsymbol{t}_2)\boldsymbol{y}$, and
- there is a $k \in \mathbb{Z}$ so that $\lambda^k \,\mathrm{gr}'(\boldsymbol{x}) = \mathrm{gr}'(a_1)\,\mathrm{gr}'(a_2)\,\mathrm{gr}'(\boldsymbol{y})$,

modulo an analogue of (2-6). The differential is $\partial(\boldsymbol{x}, a_1 \otimes a_2, \boldsymbol{y}) = (\boldsymbol{x}, \partial(a_1 \otimes a_2), \boldsymbol{y})$, and the product is

$$(\boldsymbol{x}_1, a_1 \otimes b_1, \boldsymbol{y}_1) \cdot (\boldsymbol{x}_2, a_2 \otimes b_2, \boldsymbol{y}_2) = \begin{cases} (\boldsymbol{x}_1, (a_1 \cdot a_2) \otimes (b_1 \cdot b_2), \boldsymbol{y}_2) & \boldsymbol{y}_1 = \boldsymbol{x}_2, \\ 0 & \text{otherwise,} \end{cases}$$

just as before.

The rest of the theory carries through, as follows:

- The idempotents of the coefficient algebra correspond to the generating set of the type-DD structure M. In particular, if M has at most one generator per idempotent then we can view $\mathrm{Coeff}(M)$ as a subalgebra of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$.
- As in Lemma 2.11, the coefficient algebra is graded by $\mathbb{Z}/n$, where n is the divisibility of the kernel of the action of $\mathbb{Z} = \langle \lambda \rangle$ on the grading set S of M. This action is characterized by $\mathrm{gr}(\boldsymbol{x}, a_1 \otimes a_2, \boldsymbol{y}) \equiv k$, where

$$\lambda^k \,\mathrm{gr}(\boldsymbol{x}) = (\mathrm{gr}(a_1) \times \mathrm{gr}(a_2))\,\mathrm{gr}(\boldsymbol{y}). \tag{2-8}$$

  In particular, if S is λ–free then $\mathrm{Coeff}(M)$ is $\mathbb{Z}$–graded.
- The following analogue of Lemma 2.13 holds.

Lemma 2.18 *Suppose $(\boldsymbol{x}, a_1 \otimes a_2, \boldsymbol{y}) \in \mathrm{Coeff}(\widehat{CFDD}(\mathcal{H}))$. Then there is a $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ so that $-r_*(\partial^\partial(B)) = \mathrm{supp}(a_1) \amalg \mathrm{supp}(a_2)$. Moreover, for any such B,*

$$\mathrm{gr}(\boldsymbol{x}, a_1 \otimes a_2, \boldsymbol{y}) = \iota(a_1) + \iota(a_2) - e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B).$$

2.5 A pairing theorem

In this paper, we will employ a particular version of the pairing theorem for reconstructing Heegaard Floer homology from the bordered invariants. Before stating it, suppose that (Y_1, ϕ_1) and (Y_2, ϕ_2) are bordered 3–manifolds with boundaries parameterized by $F(\mathcal{Z})$ and $F(-\mathcal{Z})$; ie we have homeomorphisms $\phi_1\colon F(-\mathcal{Z}) \to \partial Y_1$ and $\phi_2\colon F(\mathcal{Z}) \to \partial Y_2$. In this case, the bordered invariant $\widehat{CFD}(Y_1)$ is a left module over $\mathcal{A}(\mathcal{Z})$, while the bordered invariant for $\widehat{CFD}(Y_2)$ is a left module over $\mathcal{A}(-\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$ and hence can be viewed as a right module over $\mathcal{A}(\mathcal{Z})$. In particular, $\widehat{CFD}(Y_1) \otimes \widehat{CFD}(Y_2)$ can be viewed as an $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(\mathcal{Z})$ bimodule.

Theorem 2.19 *Let (Y_1, ϕ_1) and (Y_2, ϕ_2) be bordered 3–manifolds with parameterizations $\phi_1\colon -F(\mathcal{Z}) \to \partial Y_1$ and $\phi_2\colon F(\mathcal{Z}) \to \partial Y_2$, and let*

$$Y = Y_1 \,{}_{\partial Y_1}\!\cup_{\partial Y_2} Y_2.$$

Then the chain complex $\widehat{CF}(Y)$ calculating $\widehat{HF}(Y)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is homotopy equivalent to

$$\mathrm{Mor}\big({}_{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}\,(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(\mathcal{Z})}, {}_{\mathcal{A}(\mathcal{Z})}\widehat{CFD}\,(Y_1) \otimes \widehat{CFD}(Y_2)_{\mathcal{A}(\mathcal{Z})}\big),$$

that is, the chain complex of $\mathcal{A}(\mathcal{Z})$–bimodule maps from ${}_{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}\,(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(\mathcal{Z})}$ to ${}_{\mathcal{A}(\mathcal{Z})}\widehat{CFD}\,(Y_1) \otimes \widehat{CFD}(Y_2)_{\mathcal{A}(\mathcal{Z})}$.

Proof This is a special case of [8, Corollary 7], where one boundary component of Y_1 and Y_2 is empty. (Note that the boundary Dehn twist appearing in [8, Corollary 7] acts trivially on the invariant of a 3–manifold with a single boundary component.) □

There is an analogue when Y_2 is a 3–manifold with two boundary components:

Theorem 2.20 *Let $(Y_1, \phi_1\colon -F(\mathcal{Z}_1) \to Y_1)$ be a bordered 3–manifold with one boundary component and let $(Y_2, \phi_2\colon F(\mathcal{Z}_1) \to \partial_L Y_2, \phi_3\colon -F(\mathcal{Z}_2) \to \partial_R Y_2)$ be a strongly bordered 3–manifold with two boundary components. Let*

$$Y = Y_1 \,{}_{\partial Y_1}\!\cup_{\partial_L Y_2} Y_2,$$

a bordered 3–manifold with boundary parameterized by $\phi_3\colon F(\mathcal{Z}_2) \to \partial Y$. Then $\widehat{CFD}(Y, \phi_3)$ is homotopy equivalent, as a differential $\mathcal{A}(\mathcal{Z}_2)$–module, to

$$\operatorname{Mor}\big({}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(\mathcal{Z}_1)}, {}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFD}(Y_1) \otimes {}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(Y_2)_{\mathcal{A}(\mathcal{Z}_1)}\big),$$

which is the chain complex of $\mathcal{A}(\mathcal{Z})$–bimodule maps from ${}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(\mathcal{Z}_1)}$ to ${}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFD}(Y_1) \otimes {}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(Y_2)_{\mathcal{A}(\mathcal{Z}_1)}$.

Proof Again, this is a special case of [8, Corollary 7]. □

In particular, for mapping classes we have:

Corollary 2.21 *Let $(Y_1, \phi_1\colon -F(\mathcal{Z}_1) \to Y_1)$ be a bordered 3–manifold with one boundary component and $\psi\colon -F^\circ(\mathcal{Z}_2) \to -F^\circ(\mathcal{Z}_1)$ be a strongly-based diffeomorphism. Then $\widehat{CFD}(Y, \phi_1 \circ \psi)$ is homotopy equivalent, as a differential $\mathcal{A}(\mathcal{Z}_2)$–module, to*

$$\operatorname{Mor}\big({}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(\mathcal{Z}_1)}, {}_{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFD}(Y_1) \otimes {}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\psi)_{\mathcal{A}(\mathcal{Z}_1)}\big).$$

Remark 2.22 The obvious analogue of Theorem 2.20 when both Y_1 and Y_2 are strongly bordered 3–manifolds with two boundary components is false. Rather, the chain complex of bimodule morphisms picks up an extra boundary Dehn twist; see [8, Corollary 7] for more details.

The isomorphisms in Theorems 2.19 and 2.20 are graded isomorphisms in the following sense. In Theorem 2.19, $\widehat{CFD}(Y_1)$ (respectively $\widehat{CFD}(Y_2)$) is graded by a set S_1 (respectively S_2) with a left (respectively right) action of $G(\mathcal{Z})$, the (small) grading group associated to $\mathcal{Z}$. The space of bimodule homomorphisms is then graded by the set

$$S_2 \times_G G \times_G S_1 = S_2 \times_G S_1 = S_2 \times S_1 / [(xg, y) \sim (x, gy)].$$

The center $\mathbb{Z}$ of G acts on $S_2 \times_G S_1$. As a $\mathbb{Z}$–set, $S_2 \times_G S_1$ decomposes into orbits

$$S_2 \times_G S_1 = \coprod_i \mathbb{Z}/n_i.$$

Each orbit corresponds to a spin^c–structure on $Y = Y_1 \, {}_{\partial Y_1}\!\cup_{\partial Y_2} Y_2$. If $\mathbb{Z}/n_i$ corresponds to the spin^c–structure $\mathfrak{s}$ then $n_i = \operatorname{div}(c_1(\mathfrak{s}))$, and the relative $\mathbb{Z}/n_i$–grading from $S_2 \times_G S_1$ corresponds to the relative $\mathbb{Z}/\operatorname{div}(c_1(\mathfrak{s}))$–grading in Heegaard Floer homology.

The story for Theorem 2.20 is the same, except that $S_1 \times_G S_2$ retains a left action by $G(\mathcal{Z}_2)$; and the isomorphism of Theorem 2.20 covers an isomorphism of $G(\mathcal{Z}_2)$–sets (in all orbits where the modules are nontrivial).

See [5, Section 7.1.1] for further discussion in a closely related context.

3 The type-*DD* bimodule for the identity map

Recall that Theorem 1 provides a model for the type-*DD* bimodule for the identity map. The aim of the present section is to prove that theorem. First we set up some notation.

In the standard Heegaard diagram for the identity map (Figure 13), the generators are in one-to-one correspondence with pairs of complementary idempotents, and each domain has the same multiplicities on the left and right of the diagram. It follows that the type-*DD* bimodule for the identity diagram has a special form. To make this precise, think of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ as a left-left bimodule (compare Section 2.2.1). Then $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is induced from a module over a preferred subalgebra of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$:

Definition 3.1 The *diagonal subalgebra* of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$ is the algebra generated by elements of the form $(j \cdot a \cdot i) \otimes (j_o \cdot b \cdot i_o)$, where:

- The support of a is identified with the corresponding one for b; ie in the notation of Section 2.2, $r_*(\operatorname{supp}(a)) = -\operatorname{supp}(b)$.
- The elements $i \in \mathcal{A}(\mathcal{Z})$ and $i_o \in \mathcal{A}(-\mathcal{Z})$ are complementary idempotents.
- The elements $j \in \mathcal{A}(\mathcal{Z})$ and $j_o \in \mathcal{A}(-\mathcal{Z})$ are complementary idempotents.

Note that in view of the first condition above, the second two conditions are redundant with one another.

Some definitions for $\mathcal{A}(\mathcal{Z})$ extend in obvious ways to the diagonal subalgebra; for instance, a *basic generator* of the diagonal subalgebra is an element $a \otimes a'$ of the diagonal subalgebra so that a and a' are basic generators of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$.

We next rephrase the module $\widehat{\mathcal{DD}}(\mathbb{I})$ from the introduction as a left-left module. Letting $\mathcal{C}$ denote the set of connected chords for $\mathcal{Z}$, we have a map

$$\widetilde{a}\colon \mathcal{C} \to \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$$

(where here $\otimes$ is taken over $\mathbb{F}_2$) defined by

$$\widetilde{a}(\xi) = a(\xi) \otimes a_o(\xi),$$

where here $a(\xi)$ denotes the algebra element in $\mathcal{A}(\mathcal{Z})$ associated to ξ, and $a_o(\xi)$ denotes the algebra element in $\mathcal{A}(-\mathcal{Z})$ specified by the chord $r(\xi)$. Nonzero elements of the form $I \cdot \widetilde{a}(\xi) \cdot J$, where ξ is a chord and each of I and J is a pair of complementary idempotents (Definition 1.1) in $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$, are called *chord-like*. (The orientation-reversing map r is discussed in Section 2.2.1.)

Fix a chord diagram $\mathcal{Z}$, and consider the left-left $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(-\mathcal{Z})$–bimodule $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ defined in Definition 1.3. In our present notation, the element $A \in \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$ which determines the differential can be written as

$$A = \sum_{\xi \in \mathcal{C}} \widetilde{a}(\xi).$$

More explicitly, write a typical element of $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ as $c \cdot I$, where c is an element of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$, and $I = i \otimes i_o$ is a pair of complementary idempotents. The differential on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is given by

$$\partial(c \cdot I) = (dc) \cdot I + \sum_{\xi \in \mathcal{C}} c \cdot \widetilde{a}(\xi) \cdot I.$$

Lemma 3.2 *Let ξ and η be chords, and $I = (i, i_o)$ and $J = (j, j_o)$ be pairs of complementary idempotents. If ξ and η share an endpoint and $J \cdot \widetilde{a}(\xi) \cdot \widetilde{a}(\eta) \cdot I$ is nonzero, then there is a unique nontrivial factorization of $J \cdot \widetilde{a}(\xi) \cdot \widetilde{a}(\eta) \cdot I$ into homogeneous elements with nontrivial support in the diagonal subalgebra. Moreover, $J \cdot \widetilde{a}(\xi) \cdot \widetilde{a}(\eta) \cdot I$ appears in the differential $\partial(J \cdot \widetilde{a}(\xi \cup \eta) \cdot I)$.*

Proof Let a be a basic generator of $\mathcal{A}(\mathcal{Z})$ (ie one represented by a strands diagram). We say that a has a *break at* p if p is the initial point of some strand in a and also the terminal point of some strand in a. Similarly, let $x \otimes x'$ be an element of the diagonal subalgebra, where x and x' are basic generators. We say that $x \otimes x'$ has a break at p if either x has a break at p or x' has a break at $r(p)$.

Let $\widetilde{x} = x \otimes x'$ and $\widetilde{y} = y \otimes y'$ be a pair of basic generators in the diagonal subalgebra, with nonzero product. Suppose that there is some position p in the boundary of the support of both x and y. We claim then that $\widetilde{x} \cdot \widetilde{y}$ has a break at p. There are two cases: either p is an initial endpoint of the support of x, or it is a terminal endpoint of the support of x. If p is an initial endpoint in the support of x then it must also be a terminal endpoint of the support of y; thus, $x \cdot y$ has a break at p. Symmetrically, if p is a terminal endpoint of the support of x, then it must be an initial endpoint in the support of x', and hence $x' \cdot y'$ has a break at $r(p)$.

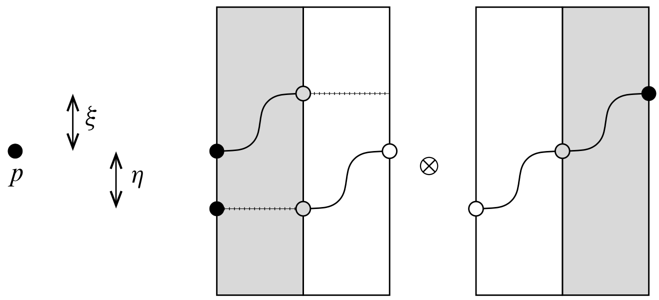


Figure 8: *Break in product* The algebra element $\widetilde{a}(\xi)$ is contained in the two shaded boxes; the algebra element $\widetilde{a}(\eta)$ is contained in the two unshaded boxes. The chords ξ and η share a boundary point p. The product $\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)$ is gotten from the illustrated juxtaposition, and has a break at p. Note that the juxtaposition appears in $d\widetilde{a}(\xi \cup \eta)$.

Suppose now that ξ and η are chords which share an endpoint. Then, $t = J\cdot\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)\cdot I$ has a unique break. Now, consider a factorization of t into $\widetilde{x} \cdot \widetilde{y}$ in the diagonal subalgebra. As in the previous paragraph, t must have a break at any point q where the support of $\widetilde{x}$ and $\widetilde{y}$ meet; since the product has a unique break, there must be a single such point q, and so q agrees with p. From this, it is straightforward to see that the factorization coincides with the initial one, ie $\widetilde{x} = J \cdot \widetilde{a}(\xi)$ and $\widetilde{y} = \widetilde{a}(\eta) \cdot I$.

To see that $J\cdot\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)\cdot I$ appears in the differential of $J\cdot\widetilde{a}(\xi\cup\eta)\cdot I$, suppose without loss of generality that the terminal point p of ξ coincides with the initial point of η, so that $j \cdot a(\xi)\cdot a(\eta)\cdot i$ has a break. In this case, $a(\xi)\cdot a(\eta)$ appears in the differential $da(\xi \cup \eta)$, and $\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)$ appears in $d\widetilde{a}(\xi \cup \eta)$. □

Lemma 3.3 *Let $I = (i, i_o)$ and $J = (j, j_o)$ be pairs of complementary idempotents and ξ, η be chords. If ξ and η do not share an endpoint then*

$$J \cdot \widetilde{a}(\xi)\cdot\widetilde{a}(\eta)\cdot I = J \cdot \widetilde{a}(\eta)\cdot\widetilde{a}(\xi)\cdot I.$$

Proof There are three cases on the endpoints of ξ which are handled differently: the boundaries can be linked, an endpoint of ξ can be matched with an endpoint of η, and the case where neither of the above two phenomena occur.

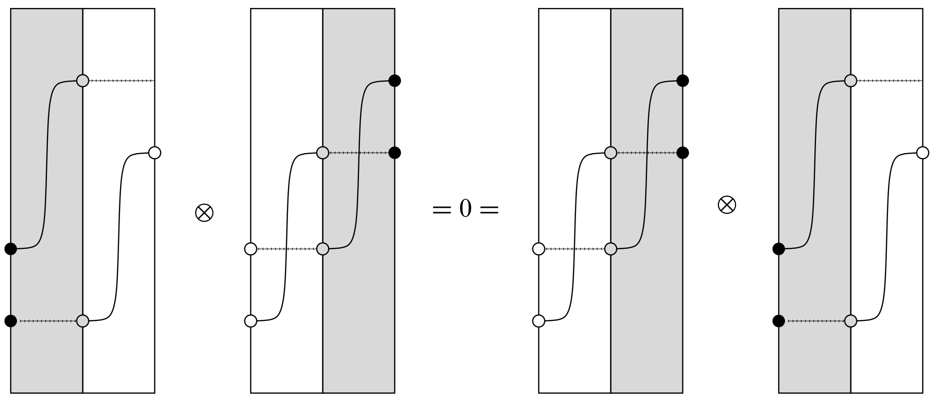


Figure 9: *Boundaries of ξ and η are linked* We have illustrated here chords ξ and η whose boundaries are linked. We have $\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)=0=\widetilde{a}(\eta)\cdot\widetilde{a}(\xi)$, since one of the two sides always has a double-crossing in it.

In the first case, where the boundaries of ξ and η are linked (ie exactly one of the endpoints of ξ is contained in the interior of η), we claim that $\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)=0$. To see this, write $x=i\cdot a(\xi)$, $x'=i_o\cdot a_o(\xi)$, $y=j\cdot a(\eta)$ $y_o=j_o\cdot a(\eta)$. Observe that all of these are basic generators. Next, note that exactly one of the juxtapositions $x*y$ or x_o*y_o contains a double-crossing. Thus, $\widetilde{a}(\xi)\cdot\widetilde{a}(\eta)=0$. By the same reasoning, $\widetilde{a}(\eta)\cdot\widetilde{a}(\xi)=0$.

Consider next the second case, where some endpoint of ξ is matched with some endpoint of η. If the initial boundary of ξ is matched with the initial point of η then $a(\xi)\cdot a(\eta)=a(\eta)\cdot a(\xi)=0$, so once again both terms vanish. The same reasoning applies if the terminal points are matched. Finally, suppose that the terminal point p of ξ is matched with the initial point p' of η. Thus, $a(\xi)\cdot a(\eta)=0$. However, it is not guaranteed that $a(\eta)\cdot a(\xi)=0$. But for $j\cdot a(\eta)\cdot a(\xi)\cdot i$ to be nonzero, both i and j must contain the matched pair $\{p,p'\}$, and hence neither i_o nor j_o contains the matched pair $\{p,p'\}$. It follows that $j_o\cdot a_o(\eta)\cdot a_o(\xi)\cdot i_o=0$, and hence $\widetilde{a}(\eta)\cdot\widetilde{a}(\xi)=0$.

In the third case, where none of the endpoints of ξ are matched to endpoints of η, and their boundaries are unlinked, we have that $a(\xi)\cdot a(\eta)=a(\eta)\cdot a(\xi)$ and $a_o(\xi)\cdot a_o(\eta)=a_o(\eta)\cdot a_o(\xi)$, so the stated equality holds. □

It follows from the identification of $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ with $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ that $\partial^2=0$ on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$. Although it is not strictly necessary for this paper, it is not hard to give a combinatorial proof of this fact:

Proposition 3.4 *The endomorphism ∂ on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is a differential, ie $\partial^2=0$.*

Proof Let $I = (i, i_o)$. The $J = (j, j_o)$ coefficient of $\partial^2 I$ is given by

$$\sum_{\xi_1,\xi_2 \in \mathcal{C}} (j \cdot a(\xi_2) \cdot a(\xi_1) \cdot i) \otimes (j_o \cdot a_o(\xi_2) \cdot a_o(\xi_1) \cdot i_o) + d \sum_{\xi \in \mathcal{C}} (j \cdot a(\xi) \cdot i) \otimes (j_o \cdot a_o(\xi) \cdot i_o).$$

By Lemma 3.3, many of the terms in the first sum cancel in pairs (and some are individually zero), leaving only those terms where the ξ_1 and ξ_2 are two chords which share an endpoint. Indeed, according to Lemma 3.2 all such terms are left over in the sum. The last statement in Lemma 3.2 implies that all of these terms occur in the second sum; moreover, that these terms exactly cancel with the second sum. □

We investigate some of the algebraic properties of the diagonal subalgebras. These will be useful when considering gradings. It is not true that any homogeneous element in $\mathcal{A}(\mathcal{Z})$ can be factored as a product of chord-like elements. The corresponding fact is, however, true for the diagonal subalgebra:

Lemma 3.5 *Any homogeneous element of the diagonal subalgebra can be factored as a product of chord-like elements.*

Proof Fix a basic element of the diagonal subalgebra $\xi \otimes \xi'$ with nontrivial support. It suffices to factor off a chord-like element; ie exhibit a factorization $\xi \otimes \xi' = (\eta \otimes \eta') \cdot (\zeta \otimes \zeta')$, where $\eta \otimes \eta'$ is also in the diagonal subalgebra and $\zeta \otimes \zeta'$ is chord-like. (Since this expresses $\xi \otimes \xi'$ as a product of an element which has strictly smaller support with a chord-like element, induction on the total support then gives the factorization claimed in the lemma.)

To factor off the chord-like element, we proceed as follows. Let s be a moving strand in ξ and t be a moving strand in ξ'. For the strand s, the initial point s^- and the terminal point s^+ are points in $\mathcal{Z}$ with $s^- < s^+$; thinking of the initial and terminal points of t (t^- and t^+ respectively) as points in $\mathcal{Z}$, as well, we have that $t^+ < t^-$. Now, consider pairs of such strands (s,t) with the property that $t^+ < s^+$; and choose among these a pair of strands for which the distance $s^+ - t^+$ is minimized. We call such a strand pair (s,t) *minimal*.

For a minimal strand pair (s,t), we claim that $s^- \leq t^+ < s^+ \leq t^-$. This follows from the condition on the support of elements of the diagonal subalgebra. Specifically, if the stated inequalities do not hold, then either $t^- < s^+$ or $t^+ < s^-$. The case where $t^- < s^+$ can be divided into two subcases, depending on whether or not the support of ξ' jumps at t^-, ie $\partial \operatorname{supp}(\xi')$ has a nonzero coefficient at t^-. (Recall that $\operatorname{supp}(\xi) = \operatorname{supp}(\xi')$.) If the support of ξ' jumps at t^-, there is a different strand u in ξ ending at t^-. But in this case (u,t) satisfies $t^+ < u^+ < s^+$, contradicting minimality of (s,t). If the support of ξ' does not jump at t^-, there must be a different strand v in ξ' which ends in t^-. But in this case (s, v) satisfies $t^+ < v^+ < s^+$, again contradicting minimality of (s, t). The case where $t^+ < s^-$ is excluded similarly.

For a minimal strand pair (s, t), let ζ be the algebra element in $\mathcal{A}(\mathcal{Z})$ associated to a single moving strand from t^+ to s^+, and whose terminal idempotent coincides with that for ξ. Since t^+ appears in the terminal idempotent of ξ', it does not appear in the terminal idempotent of ξ, and hence we can consider the algebra element η which consists of all the moving strands in ξ, except that the strand s is terminated at t^+ instead of at s^-. Now, $\eta \cdot \zeta = \xi$, unless $\eta \cdot \zeta = 0$ because of the introduction of a double-crossing. But $\eta \cdot \zeta$ cannot introduce a double-crossing, for that would mean that there is some strand u in ξ with $t^+ < u^+ < s^+$, contradicting the minimality of the strand pair (s, t). Similarly, we can factor $\xi' = \eta' \cdot \zeta'$, where ζ' is the algebra element consisting of a single moving strand from s^+ to t^+ and whose terminal idempotent coincides with that for ξ'. So, $\zeta \otimes \zeta'$ is the desired chord-like element. $\square$

The above proof in fact gives an algorithm for factoring any given element of the diagonal subalgebra as a product of chord-like elements. For an illustration of this algorithm, see Figure 10.

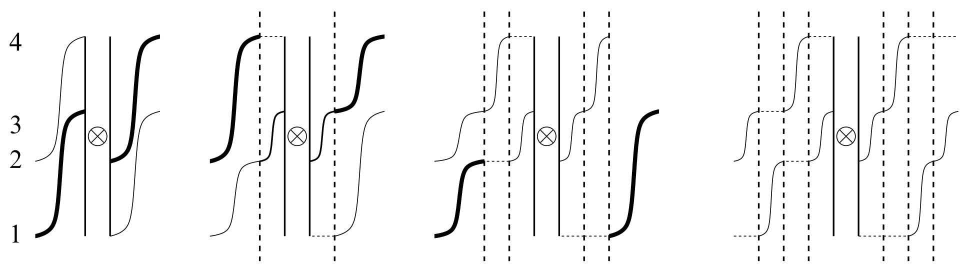


Figure 10: *Factoring into chord-like elements* We start with the element consisting of two strands, one from 1 to 3 and another from 2 to 4 on the $\mathcal{Z}$ side, and a similar element on the $\mathcal{Z}'$ side, as illustrated on the left. We then apply successively the algorithm from the proof of Lemma 3.5, to factor off chord-like elements. At each stage, a minimal strand pair is illustrated with darker strands.

Definition 3.6 A chord ξ is said to be a *special length-three chord* if:

- ξ has length three.
- The terminal point p of ξ is matched with some other point p' in the interior of ξ.
- The initial point q of ξ is matched with another point q' in the interior of ξ.

See Figure 11 for an illustration.

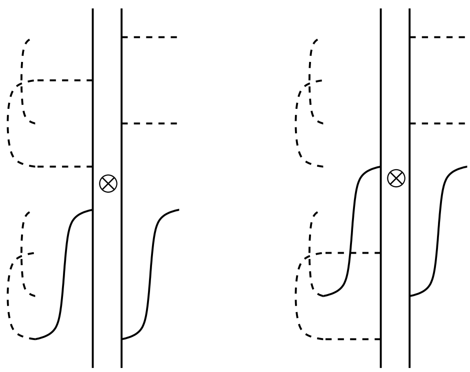


Figure 11: *Length-three chords* On the left, we have a special length-three chord: this chord is a cycle. On the right, we have a length-three chord which is not special: this chord is not a cycle.

Lemma 3.7 *If ξ is a chord which is not length-one or special length-three, and $I\widetilde{a}(\xi)J \neq 0$, then $d(I\widetilde{a}(\xi)J) \neq 0$.*

Proof By hypothesis, there must be some position p in the interior of support of ξ which is not matched with either an initial or terminal point of ξ. The initial idempotent $I = i \otimes i_o$ must contain this position either in i or in i_o (since i and i_o are complementary idempotents). Thus the differential of either $a(\xi)$ or $a_o(\xi)$ must contain a term corresponding to the resolution at p. □

Not every pointed matched circle has special length-three chords. For instance, there are none in the antipodally matched circle for a surface of genus $k > 1$. (The *antipodally matched circle* is the one where the matched pairs of points are antipodal on the circle.) On the other hand, for the surface of genus $k = 1$, there is a unique pointed matched circle, and it does have a special length-three chord, so the split pointed matched circle with any genus has special length-three chords.

Proposition 3.8 *Let M be any type-DD bimodule over $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$, where $\mathcal{Z}$ is any pointed matched circle with genus greater than one. Suppose M satisfies the following properties:*

(1) *Generators of M are in one-to-one correspondence with pairs $I = (i, i_o)$ of complementary idempotents, in such a manner that if $\boldsymbol{x}(I)$ is the generator corresponding to $I = (i, i_o)$, then $I \cdot \boldsymbol{x}(I) = \boldsymbol{x}(I)$.*

(2) *The coefficients in the differential of M all lie in the diagonal subalgebra of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$; ie writing*

$$\partial \boldsymbol{x}(I) = \sum_{J} a(I, J) \cdot \boldsymbol{x}(J),$$

where $a(I, J) \in \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$, then all of the $a(I, J)$ lie in the diagonal subalgebra of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$.

(3) *M is graded by a λ–free G–set S.*

(4) *The differential of $\boldsymbol{x}(I)$ contain all nonzero elements of the form $I \cdot \widetilde{a}(\xi) \cdot J \cdot \boldsymbol{x}(J)$ where ξ is any length-one chord.*

Then, M is isomorphic to the bimodule $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$.

Proof First, we prove by induction on the length of the support of $\xi \in \mathcal{C}$ that all terms of the form $a(\xi) \otimes a_o(\xi)$ appear in the coefficient of the differential.

Assume for simplicity that our diagram does not have any special length-three chords; we return to the general case later. Our goal is to argue that, if ξ is a chord of length $n > 1$, then $\widetilde{a}(\xi)$ appears in the differential. We proceed by induction on the length of the support of ξ. Consider the part of the coefficient of $\partial^2 \boldsymbol{x}(I)$ which is spanned by algebra elements whose supports coincides with the support of ξ. By the inductive hypothesis, the $\boldsymbol{x}(J) = \boldsymbol{x}(j, j_o)$ coefficient of $\partial^2 \boldsymbol{x}(I)$ with total support ξ has the form

$$\Bigg(\sum_{\{\xi_1, \xi_2 \in \mathcal{C} \mid \xi_1 \cup \xi_2 = \xi\}} I \cdot \widetilde{a}(\xi_2) \cdot \widetilde{a}(\xi_1) \cdot J \Bigg) + \Bigg(\sum I \cdot \widetilde{x} \cdot \widetilde{y} \cdot J \Bigg) + d(I \cdot \widetilde{z} \cdot J),$$

where $\widetilde{z}$ is the $\boldsymbol{x}(J)$ component of $\partial \boldsymbol{x}(I)$ with support equal to ξ, $\widetilde{x}$ and $\widetilde{y}$ are other basic elements in the diagonal algebra where at least one of $\widetilde{x}$ or $\widetilde{y}$ has at least one break (and $\operatorname{supp}(\widetilde{x}) + \operatorname{supp}(\widetilde{y}) = \operatorname{supp}(\xi)$).

According to Lemma 3.2, terms appearing in the first sum cannot cancel with other terms appearing in the first sum or with terms appearing in the second sum; thus, they must cancel with terms in the third. Moreover, as in the proof of Proposition 3.4, every term in $d(I\widetilde{a}(\xi)J)$ occurs in the first sum; in particular, since $d(I\widetilde{a}(\xi)J)$ is nontrivial by Lemma 3.7, the first sum is nontrivial. This forces $d(I \cdot \widetilde{z} \cdot J)$ to contain the terms in $d(I \cdot \widetilde{a}(\xi) \cdot J)$ with nonzero multiplicity. It follows that $I \cdot \widetilde{z} \cdot J$ must contain $I \cdot a(\xi) \cdot J$ with nonzero multiplicity: as in the proof of Lemma 3.2, the nonzero terms in the first sum have three nonhorizontal strands in them (two on one side and one on the other), so if they appear in the differential of a homogeneous element, then that element must have exactly two moving strands in it, ie it must be of the form $I \cdot \widetilde{a}(\xi) \cdot J$.

Having verified that the differential contains $\sum_{\xi\in\mathcal{C}}\widetilde{a}(\xi)$, we must verify that it contains no other terms. This follows from grading reasons. Having established that if $I\cdot\widetilde{a}(\xi)\cdot J\neq 0$ then $I\cdot\widetilde{a}(\xi)\cdot\boldsymbol{x}(J)$ appears in $\partial\boldsymbol{x}(I)$, we can conclude that

$$\lambda^{-1}\,\mathrm{gr}'(\boldsymbol{x}(I))=\mathrm{gr}'(I\cdot\widetilde{a}(\xi)\cdot J)\cdot\mathrm{gr}'(\boldsymbol{x}(J)) \tag{3-1}$$

for any chord ξ. Let a be any basic generator of the diagonal subalgebra, and suppose that $a\boldsymbol{x}(J)$ occurs in $\partial\boldsymbol{x}(I)$. Then, in particular, $\mathrm{gr}'(a)\,\mathrm{gr}'(\boldsymbol{x}(J))=\lambda^{-1}\,\mathrm{gr}'(\boldsymbol{x}(I))$. According to Lemma 3.5, there is a sequence of chords $\{\xi_i\}_{i=1}^n$ with the property that $a=I\cdot\prod_{i=1}^n\widetilde{a}(\xi_i)\cdot J$. By (3-1),

$$\lambda^{-n}\cdot\mathrm{gr}'(\boldsymbol{x}(I))=\mathrm{gr}'(a)\cdot\mathrm{gr}'(\boldsymbol{x}(J));$$

it follows that $n=1$ (thanks in part to Proposition 3.8), and hence a had to be a chord-like element.

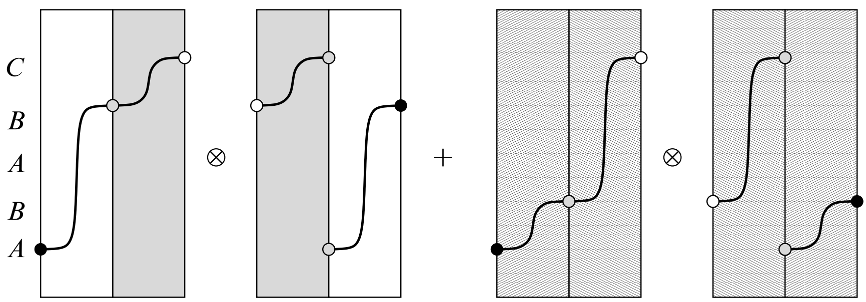


Figure 12: *Special length-three chords* The matching is indicated by the letters on the rows. The algebra element in the white box represents the algebra element associated to a special length-three chord; we multiply it by the gray-shaded length-one chord, to get a term which evidently appears in the differential of the algebra element associated to the length-four chord. The other term is illustrated to the right (in the two hatched boxes). It decomposes as a product of length-three and length-one chords, but now the length-three chord is not special.

This completes the proof of the proposition for pointed matched circles without special length-three chords. When there are special length-three chords ξ, we must show $\widetilde{a}(\xi)$ also appears in the differential. This is seen by considering a length-four chord η which contains the given special length-three chord ξ. Such a chord can be found since the genus k is bigger than 1. Now we have that

$$d\widetilde{a}(\eta)=\widetilde{a}(\xi_1)\cdot\widetilde{a}(\xi_3)+\widetilde{a}(\xi_3')\cdot\widetilde{a}(\xi_1'),$$

where ξ_i and ξ_i' have length i, and exactly one of ξ_3 or ξ_3' ξ, while the other of ξ_3 or ξ_3' is not special. Suppose for concreteness that $\xi=\xi_3$. Our inductive hypothesis ensures that $\widetilde{a}(\xi_3')$ appears in the differential, and hence the term $\widetilde{a}(\xi_3')\cdot\widetilde{a}(\xi_1')$, appears in ∂^2, and must cancel against something. According to Lemma 3.2, the only term it can cancel is $d\widetilde{a}(\eta)$. Hence, we have established that $\widetilde{a}(\eta)$ appears with nonzero multiplicity in ∂. Thus, since $d\widetilde{a}(\eta)$ appears in ∂^2, we see that $\widetilde{a}(\xi)\cdot\widetilde{a}(\xi_3)$ appears in ∂^2. According to Lemma 3.2, the only way this can cancel is if $\widetilde{a}(\xi_3)$ — the algebra element associated to our special length-three chord — also appears with nonzero multiplicity in the differential. Repeating this argument for each special length-three chord, we conclude that all length-three chords appear in the differential. Now, we can proceed with the same induction as before. □

Proof of Theorem 1 Consider the standard genus-$2k$ Heegaard diagram for the identity map of S, as pictured in Figure 13. We verify that $\widehat{CFDD}$ of this Heegaard diagram satisfies the hypotheses of Proposition 3.8.

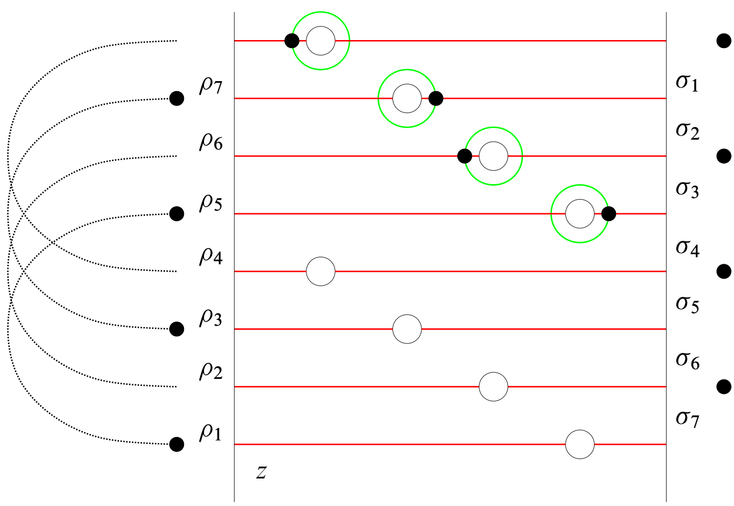


Figure 13: *Heegaard diagram for the identity map* This is a Heegaard diagram for the identity cobordism of the genus-two diagram with antipodal matching, as indicated by the arcs to the left of the diagram. To the left and the right of the diagram, we have also indicated a pair of complementary idempotents, along with its unique extension into the diagram as a generator for the complex.

The generators $\mathfrak{S}(\mathcal{H})$ of the $\mathcal{A}(-\mathcal{Z})\oplus\mathcal{A}(\mathcal{Z})$–module $\widehat{CFDD}(\mathcal{H})$ are in one-to-one correspondence with pairs of complementary idempotents, as follows. There are no α–circles, and each β_i meets exactly two α–arcs, α_i^L and α_i^R, where we label the α–curves so that if α_i^L is the α–arc which meets $\partial_L\mathcal{H}$ in $\{p_i,q_i\}$ for $\mathcal{Z}_L$ then α_i^R

is the α–arc which meets $\partial_R\mathcal{H}$ in $\{r(p_i), r(q_i)\}$. The map $\boldsymbol{x} \mapsto I_{D,L}(\boldsymbol{x}) \times I_{D,R}(\boldsymbol{x})$ sets up the one-to-one correspondence.

Next, we claim that the coefficients of the boundary operator

$$\delta^1\colon X \to (\mathcal{A}(\mathcal{Z}) \oplus \mathcal{A}(-\mathcal{Z})) \otimes X$$

take values in the diagonal subalgebra. We have already checked this on the level of idempotents. To see that coefficients lie in the diagonal subalgebra, notice that if q_i^L is a position between two consecutive places p_i and p_{i+1} on $\mathcal{Z}_L$, and q_i^R is a position between the corresponding places $r(p_i)$ and $r(p_{i+1})$ on $\mathcal{Z}_R = -\mathcal{Z}_L$, then q_i^L and q_i^R can be connected by an arc in $\overline{\Sigma}$ which does not cross any of the α– or β–circles. Thus, if $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ is any homotopy class then the local multiplicity of B at q_i^L coincides with the local multiplicity of B at q_i^R. It follows that the coefficients for $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ lie in the diagonal subalgebra.

A periodic domain in $\mathcal{H}$ is uniquely determined by its local multiplicities at the boundary of the Heegaard diagram. From this, and the definition of the grading set for bimodules, it follows that the grading set for the identity bimodule is λ–free.

If ξ is a length-one chord, the domains contributing to the $J \cdot \widetilde{a}(\xi) \cdot I$ component of ∂I are all octagons; see Figure 14. These have holomorphic representatives for any conformal structure, cf Lemma 2.7.

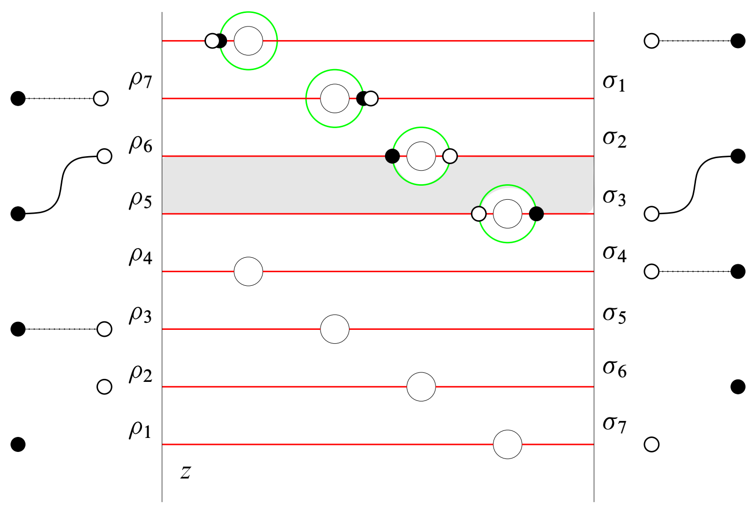


Figure 14: *A differential* Here is another picture of the Heegaard diagram for the identity cobordism on the antipodally matched circle with genus two. The shaded rectangle region represents a term in the differential of the black idempotent, which has the form $\rho_5 \otimes \sigma_3$ times the white idempotent.

If the genus k is bigger than 1 then we have verified that $M = \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ satisfies all the hypotheses of Proposition 3.8. So, that proposition implies $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is isomorphic to $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$, as desired.

The case that the genus $k = 1$ is established as follows. Embed the given genus-1 diagram for $\mathcal{Z}$ inside a genus-two diagram for $\mathcal{Z}\#\mathcal{Z}_0$ (where here $\mathcal{Z}_0$ is another genus-one diagram). This is shown in Figure 15. The diagram defines a type-D structure over $\mathcal{A}(\mathcal{Z}\#\mathcal{Z}_0)\otimes\mathcal{A}(-\mathcal{Z}\#\mathcal{Z}_0)$. Setting all the algebra elements with nontrivial support in $\mathcal{Z}_0$ to zero, and restricting to elements with some fixed idempotent $I(\boldsymbol{s}_0)$ in $\mathcal{Z}_0$ and its complementary idempotent in $-\mathcal{Z}_0$, we obtain an induced module over $\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(-\mathcal{Z})$. (In the figure, this corresponds to setting $\rho_4 = \rho_5 = \rho_6 = \rho_7 = \sigma_4 = \sigma_5 = \sigma_6 = \sigma_7 = 0$ and restricting to generators whose coordinates in the portion corresponding to $\mathcal{Z}_0$ and $-\mathcal{Z}_0$ are at the displayed black dots.) The holomorphic curve counts in this portion of the diagram correspond exactly to holomorphic curve counts for the smaller genus-one diagram gotten by excising the portion corresponding to $\mathcal{A}(\mathcal{Z}_0)\otimes\mathcal{A}(-\mathcal{Z}_0)$. (Algebraically, this corresponds to the statement that the type-DD identity bimodule for the genus-one diagram coincides with the induced module $Q_*\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}\#\mathcal{Z}_0})$; compare Definition 1.8.) Thus, the genus-1 case follows from the genus-2 case.

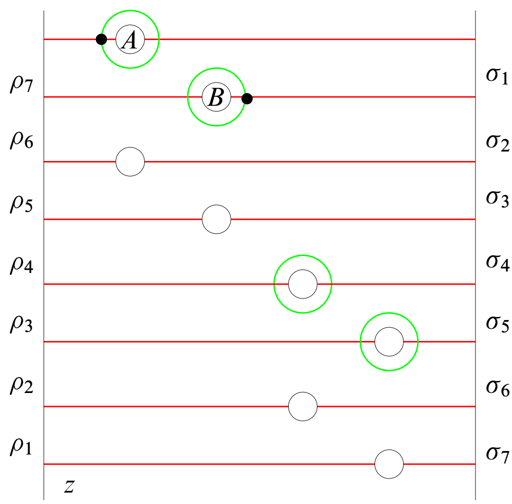


Figure 15: *Split genus-two diagram* A genus-one diagram involving ρ_i for $i = 1, \dots, 3$ and σ_j $j = 5, \dots, 7$ is stabilized by adding another genus-one diagram and, for example, the fixed generator.

The fact that the homotopy equivalences $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}}) \simeq \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ are canonical, ie unique up to homotopy, follows from [5, Corollary 8.1 and Lemma 8.15]. □

Remark 3.9 The grading set for the identity *DD* bimodule can be explicitly determined from the Heegaard diagram, and is given as follows. Recall from Section 2.3.4 that the map

$$R\colon G'(-\mathcal{Z}) \to G'(\mathcal{Z})^{\mathrm{op}}$$

defined by $R(k,\alpha) = (k, r_*(\alpha))$ is a group isomorphism. Using this, the grading set for $\widehat{CFDD}(\mathbb{I})$ is $G'(\mathcal{Z})$ with the structure of a left $G'(\mathcal{Z}) \times_{\mathbb{Z}} G'(-\mathcal{Z})$–set given by the rule

$$(g_1 \times_{\mathbb{Z}} g_2) * h := g_1 \cdot h \cdot R(g_2)$$

(where the operation $\cdot$ on the right-hand side refers to multiplication in $G'(\mathcal{Z})$); for the proof, see [5, Lemma 8.13].

It follows that, in the language of Sections 2.3.4 and 2.4.3, the diagonal subalgebra is the coefficient algebra of $\widehat{CFDD}(\mathbb{I})$.

4 Bimodules for arc-slides

Recall that Theorem 2 provides a model for the type-*DD* bimodule for an arc-slide. The aim of the present section is to prove that theorem. As discussed in Section 4.6, the proof proceeds in two steps:

(1) Proving that the type-*DD* module associated to the standard Heegaard diagram for an arc-slide (Definition 4.4) is a stable arc-slide bimodule (Definitions 1.7 and 1.8). This is proved in Section 4.2. The proof relies only on coarse properties of the Heegaard diagram (identification of generators, combinatorics of domains, and the existence of a few of the in principle many holomorphic curves which need to be computed).

(2) Proving the uniqueness theorem for arc-slide bimodules (Proposition 1.10). There are two combinatorially different cases of arc-slides: *under-slides* and *over-slides* (see Definition 4.2). The argument is easier in the first case, where uniqueness is true in a slightly stronger form. In both cases, however, the proof is broken down as follows:

   (a) The gradings on the modules restrict what terms can occur in the differential. The terms that are in correct gradings to occur are called *near-chords* (for under-slides or over-slides, see Definitions 4.15 and 4.25, respectively). The key properties of gradings required for this argument are collected in Section 4.3, and the restrictions are obtained in Lemmas 4.17 and 4.32 (Sections 4.4 and 4.5). We use a comparison with gradings in the standard Heegaard diagram from Section 4.2.

(b) In the case of under-slides the equation $\partial^2 = 0$ and the hypothesis that short near-chords appear in the differential implies that all near-chords appear; this is discussed in Section 4.4.

(c) In the case of over-slides there is some indeterminacy in which near-chords appear. Near-chords which may or may not appear are called *indeterminate* (Definition 4.26); which indeterminate near-chords appear is determined by a so-called *basic choice* (Definition 4.27). This is discussed in Section 4.5.

Before turning to these proofs, we introduce a little more notation, in Section 4.1.

4.1 More arc-slide notation and terminology

Let m be an arc-slide taking a pointed matched circle $\mathcal{Z}$ to another pointed matched circle $\mathcal{Z}'$. Here, as in the introduction, $\mathcal{Z}'$ is obtained from $\mathcal{Z}$ by sliding one of the feet b_1 of an arc B over another arc C; see Figure 2. The foot b_1 is connected to one of the feet c_1 of C by an arc σ in $\mathcal{Z}$; in $\mathcal{Z}'$, b_1 is replaced by the new foot b_1' of B', which is connected by an arc σ' to the foot c_2 of C in $\mathcal{Z}'$.

Convention 4.1 We focus on the case that c_1 is above c_2 in the $\mathcal{Z}$ matching, with respect to the orientation of Z; the case that c_1 is below c_2 is symmetric. With respect to a Heegaard diagram $\mathcal{H}$ for m, $\partial\mathcal{H} = -\mathcal{Z} \amalg \mathcal{Z}'$, so if we draw $\mathcal{H}$ in the plane with handles attached, with $\mathcal{Z}$ on the left and $\mathcal{Z}'$ on the right, then c_1 is above c_2 in the plane as well; see Figure 16.

When the matching does not satisfy the assumption from Convention 4.1, we will switch the roles of $\mathcal{Z}$ and $\mathcal{Z}'$; see for example the remarks at the end of Definition 4.15; see also Remark 4.19.

Definition 4.2 An arc-slide is called an *over-slide* if b_1 is contained in the same component of $\mathcal{Z} \setminus \{c_1, c_2\}$ as z. (Note that this condition is symmetric in the roles of $\mathcal{Z}$ and $\mathcal{Z}'$.) Otherwise it is called an *under-slide*.

The arc-slide on the left in Figure 2 is an under-slide, while the one on the right is an over-slide.

We find it convenient to think of arc-slide bimodules as left-left $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(-\mathcal{Z}')$–bimodules (analogously to what was done in Section 3) rather than left-right $\mathcal{A}(\mathcal{Z})$–$\mathcal{A}(\mathcal{Z}')$–bimodules (the point of view taken in the introduction). Correspondingly, we can reformulate Property (AS-2) for arc-slide bimodules in terms of a subalgebra of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')$:

Definition 4.3 The *near-diagonal subalgebra* of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')$ is the algebra generated by pairs of algebra elements of the form $(j \cdot a \cdot i) \otimes (j' \cdot a' \cdot i')$, where each of (j, j') and (i, i') is a pair of near-complementary idempotents, and also $\operatorname{supp}_R(a) = \operatorname{supp}_R(a')$.

With this definition, property (AS-2) of Definition 1.7 can be reformulated as stating that the $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')$–module N is induced from a module over the near-diagonal subalgebra.

Some definitions for $\mathcal{A}(\mathcal{Z})$ extend in obvious ways to the near-diagonal subalgebra; for instance, a *basic generator* of the near-diagonal subalgebra is an element $a \otimes a'$ of the near-diagonal subalgebra so that a and a' are basic generators of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$.

The restrictions placed by $\partial^2 = 0$ are strongest when we restrict attention to the part of $\mathcal{A}(\mathcal{Z})$ with the property that both the idempotents and the complementary idempotents have at least two occupied positions; this is the portion with weight $-g+1 < i < g-1$. Since we are working with stable bimodules (in Propositions 4.16 and 4.30), we can always stabilize (in the sense of Definition 1.8) so that we are working in this portion of the algebra. This will be the main way that we use stability (see for example the proofs of Lemmas 4.22 and 4.35; we possibly stabilize more in the proof of Lemma 4.28).

4.2 Heegaard diagrams for arc-slides

It is convenient to represent arc-slides by graphs embedded in an annulus, as follows. Thinking of $\mathcal{Z}$ and $\mathcal{Z}'$ as two different markings on the same circle Z, consider the annulus $[0,1] \times Z$. This annulus has marked points on its boundary corresponding to the positions in $\mathcal{Z}$ (in $0 \times Z$) and $\mathcal{Z}'$ (in $1 \times Z$), and a special horizontal arc corresponding to the basepoint $z \in \mathcal{Z}$. Each position $p \in \mathcal{Z}$, other than b_1, determines a horizontal segment $[0,1] \times \{p\}$, connecting p to its corresponding point $p' \in \mathcal{Z}'$. These horizontal segments, except for the two horizontal segments corresponding to c_1 and c_2, are edges for the graph. The horizontal segments for c_1 and c_2 are both subdivided, by points p_1 and p_2 respectively, and we draw two additional edges, one connecting p_1 to $\{0\} \times b_1$, and another connecting p_2 to $\{1\} \times b_1'$. The pictures in Figure 2 can be thought of as illustrations of these graphs (where the annuli have been cut along the horizontal arcs corresponding to the basepoint z).

The graph for an arc-slide can be turned into a Heegaard diagram $\mathcal{H}(m)$, as follows.

Definition 4.4 Let $m\colon \mathcal{Z} \to \mathcal{Z}'$ be an arc-slide. The *standard Heegaard diagram for the arc-slide* $\mathcal{H}(m)$ is the Heegaard diagram obtained as follows. Start from the graph associated to m, as defined above. Attach a one-handle with feet at the two trivalent points of the graph, in effect surgering out the two disks containing trivalent points, and replacing them with an annulus equipped with three arcs running along it. These three arcs naturally extend to an arc connecting c_1 and c_2, an arc connecting c_1' and c_2', and an arc connecting b_1 and b_1'. Add a one-handle for each matched pair $\{p,q\}$ in $\mathcal{Z}$ other than $\{c_1,c_2\}$ and $\{b_1,b_2\}$ with feet at the midpoints of the edges corresponding to p and q. The edges corresponding to p and q are surgered to get a pair of arcs, one connecting p to q and the other connecting p' to q'. Performing one more such handle addition, one of whose feet is on the edge corresponding to b_2 and the other at the edge connecting b_1 to b_1', we obtain the desired Heegaard surface. The surgered arcs are the α–arcs, and β–circles are chosen to be meridians of the attached one-handles.

Figure 16 illustrates the result of this procedure.

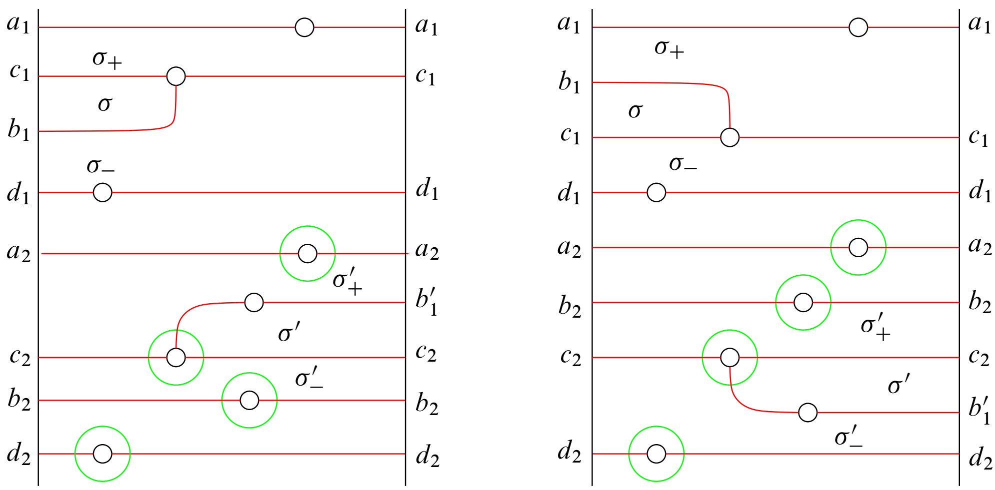


Figure 16: *Heegaard diagram for an arc-slide* Heegaard diagrams for the arc-slides from Figure 2. The one on the left represents an under-slide, and the one on the right is an over-slide. Both satisfy Convention 4.1. In both cases, the basepoint z in the pointed matched circle separates d_2 and a_1, and the picture is obtained by cutting the Heegaard diagram along the corresponding arc z.

Recall that if $m\colon \mathcal{Z} \longrightarrow \mathcal{Z}'$, then there is an associated strongly-based mapping class $F^\circ(m)\colon F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z}')$. In [5, Definition 5.35] we constructed a bordered Heegaard diagram from each strongly-based diffeomorphism. The diagram $\mathcal{H}(m)$ is the Heegaard diagram associated to the diffeomorphism $F^\circ(m)$.

For an arc-slide $m\colon \mathcal{Z} \to \mathcal{Z}'$, the chord σ in $\mathcal{Z}$ lies on the boundary of a unique region (component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$) in $\mathcal{H}(m)$; abusing notation, we will denote this region σ, as well. Similarly, the chord σ' in $\mathcal{Z}'$ lies on the boundary of a region σ' in $\mathcal{H}(m)$. Name the regions just above and below σ by σ_+ and σ_-, and the regions just above and below σ' by σ'_+ and σ'_-. All other regions look the same, and are strips across the diagram. (For this notation, we are implicitly using the hypothesis from Convention 4.1; for the other case, compare Figure 17.)

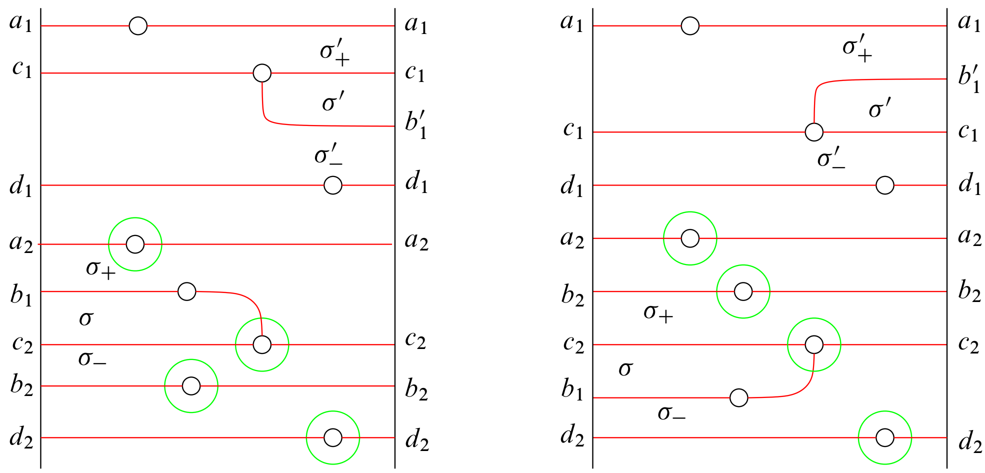


Figure 17: *Heegaard diagram for an arc-slide which does not satisfy Convention 4.1* This is a reflection of Figure 16.

Definition 4.5 We call an arc-slide *degenerate* if there is only one position between c_1 and c_2. (In the under-slide case which satisfies Convention 4.1, this is equivalent to $\sigma_- = \sigma'_+$.) See Figure 18.

For degenerate under-slides we need to allow one more kind of short near-chord. We recall the definition of short near-chords (Definition 1.6), extended to include this case:

Definition 4.6 A *short near-chord* is a nonzero algebra element of the form $(i \cdot a \cdot j) \otimes (j' \cdot b' \cdot i')$ with the following four properties:

(1) The pairs $(i \otimes i')$ and $(j \otimes j')$ are near-complementary idempotents.

(2) $\operatorname{supp}_R(a) = \operatorname{supp}_R(b)$.

(3) The support of at least one of a or b is nonzero.

(4) The lengths of the (unrestricted) support of a and the (unrestricted) support of b are both no greater than 1.

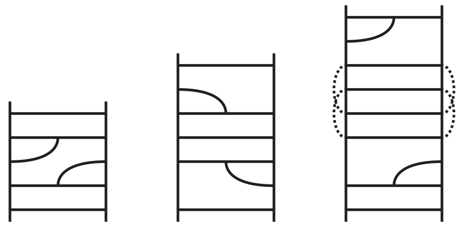


Figure 18: *Degenerate arc-slides and very special length-*3 *chords* On the left, we have a degenerate under-slide (Definition 4.5); in the center, a degenerate over-slide (Definition 4.5); on the right, an under-slide containing a very special length-three chord (proof of Lemma 4.20).

In the degenerate case of under-slides with $\sigma_- = \sigma'_+$, we also call elements of the form $(i \cdot a(\sigma_- \cup \sigma) \cdot j) \otimes (j' \cdot a'_o(\sigma_-) \cdot i')$ and $(i \cdot a(\sigma_-) \cdot j) \otimes (j' \cdot a'_o(\sigma' \cup \sigma_-) \cdot i')$ short near-chords. Similarly, if $\sigma_+ = \sigma'_-$, then we call $(i \cdot a(\sigma \cup \sigma_+) \cdot j) \otimes (j' \cdot a'_o(\sigma_+) \cdot i')$ and $(i \cdot a(\sigma_+) \cdot j) \otimes (j' \cdot a'_o(\sigma_+ \cup \sigma') \cdot i')$ a short near-chord.

Proposition 4.7 *If $m\colon \mathcal{Z} \to \mathcal{Z}'$ is an arc-slide and $\mathcal{H} = \mathcal{H}(m)$ is its associated standard Heegaard diagram (in the sense of Definition 4.4), then the type-DD bimodule $\widehat{CFDD}(\mathcal{H})$ is a stable arc-slide bimodule (in the sense of Definitions 1.7 and 1.8).*

Proof The generators correspond to near-complementary idempotents, as can be seen by adapting the corresponding fact for the identity type-*DD* bimodule (see the proof of Theorem 1). Generators of type Y occur because now there is a β–circle which intersects three (rather than two) α–arcs: two of those intersection points were of the type already encountered in the type-*DD* identity bimodule; the third, however, represents an intersection point of the α–arc for the B–matched pair with the β–circle for the C and C' matched pair. See Figure 19.

Since every region touches the boundary, the multiplicities at the boundary together with a choice of initial generator $\boldsymbol{x}$ determine a domain $Q \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. Furthermore, local multiplicities at the boundary are possible if and only if the restricted supports agree on the two sides. (Compare the proof of Theorem 1.) Thus the coefficients of the differential on $\widehat{CFDD}(\mathcal{H}(m))$ lie in the near-diagonal subalgebra. Moreover, for each basic generator $I \cdot a \cdot J$ of the near-diagonal subalgebra there is a unique domain $B \in \pi_2(\boldsymbol{x}(I), \boldsymbol{x}(J))$ which could contribute $a\boldsymbol{x}(J)$ to $\partial \boldsymbol{x}(I)$.

In particular, as in the proof of Theorem 1, the grading set of $\widehat{CFDD}(\mathcal{H})$ is λ–free, as periodic domains are determined by their local multiplicities near the boundary.

In the nondegenerate case, the differential on $\widehat{CFDD}(\mathcal{H})$ contains all short near-chords (Definition (AS-4)), as they are represented by polygons (compare Lemma 2.7).

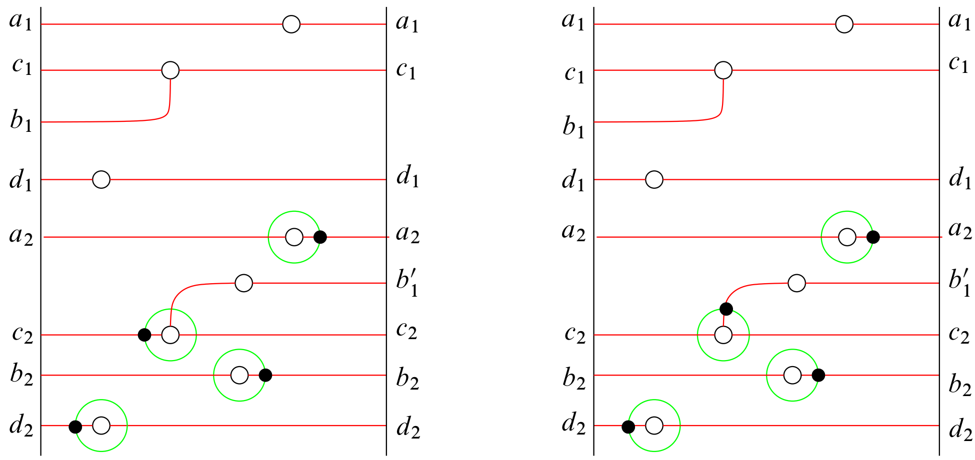


Figure 19: *Generators for arc-slide bimodule* At the left is a generator of type X (indeed, X_C, for the notation used in the proof of Proposition 4.14); at the right is a generator of type Y.

In the degenerate case, the domains corresponding to the additional near-chords $(i\cdot a(\sigma_-\cup\sigma)\cdot j)\otimes(j'\cdot a_o'(\sigma_-)\cdot i')$ and $(i\cdot a(\sigma_-)\cdot j)\otimes(j'\cdot a_o'(\sigma'\cup\sigma_-)\cdot i')$ are annuli. The combinatorics of these annuli (one $270°$ corner, at which the α– and β–curves go out to the other boundary component) are such that they always have algebraically one holomorphic representative; see, for instance, [12, Lemma 9.10] or the ρ_3 case of the proof of [5, Proposition 10.6].

Thus, we have verified that $\widehat{CFDD}(\mathcal{H})$ is an arc-slide bimodule in the sense of Definition 1.7. To see it is stable in the sense of Definition 1.8, embed the Heegaard diagram $\mathcal{H}$ for $m\colon \mathcal{Z}\to\mathcal{Z}'$ in the Heegaard diagram $\mathcal{H}_\#$ for the $\mathcal{Z}_0$–stabilized arc-slide $m_\#$. Setting to zero algebra elements whose support intersects $\mathcal{Z}_0$, and restricting the generator in the new region, we obtain a module representing $Q_*(\widehat{CFDD}(m_\#))$. If we choose almost-complex structures for $\mathcal{H}$ and $\mathcal{H}_\#$ compatibly then the holomorphic curve counts involved in $Q_*(\widehat{CFDD}(m_\#))$ coincide with the curve counts for $\widehat{CFDD}(\mathcal{H})$. □

Remark 4.8 The grading sets for $\widehat{CFDD}(\mathcal{H})$ are determined explicitly in Section 6.

4.3 Gradings on the near-diagonal subalgebra

The near-diagonal subalgebra has a $\mathbb{Z}$–grading, which can be used to exclude the appearance of many of its elements from the differential in an arc-slide bimodule. This comes from thinking of it as a coefficient algebra: we will see (Lemma 4.12 below) that for any arc-slide bimodule N (including a standard one, $\widehat{CFDD}(\mathcal{H}(m))$), Coeff($N$) contains the near-diagonal subalgebra, which is therefore $\mathbb{Z}$–graded. Furthermore, we will see (Proposition 4.13) that this $\mathbb{Z}$–grading is independent of the choice of N. Thus we can compute the $\mathbb{Z}$–grading on the near-diagonal subalgebra by looking at the grading set S_{std} for $\widehat{CFDD}(\mathcal{H}(m))$; we do this explicitly in Proposition 4.14.

By Lemma 2.12, we are free here to choose the grading group to be $G(\mathcal{Z})$ or $G'(\mathcal{Z})$. For definiteness, we work with the latter, ie $\mathcal{A}(\mathcal{Z})$ is thought of as graded by $G'(\mathcal{Z})$, and the grading set of our module N is a $G'(\mathcal{Z})$–set. We will be working with both the grading set S'_{std} for $\widehat{CFDD}(\mathcal{H}(m))$, and the grading set S'_N for an arbitrary arc-slide bimodule N. Denote the gradings with values in S'_{std} and S'_N by $\mathrm{gr}'_{\mathrm{std}}$ and gr'_N, respectively. Also note that we can naturally identify the generators of N and the generators for $\widehat{CFDD}(\mathcal{H}(m))$, since both are identified with idempotents for the near-diagonal subalgebra.

Definition 4.9 For a pair of basic idempotents I, J in a based algebra A (as in Definition 2.9), a *chain of algebra elements* K connecting I and J is a sequence $(a_1,\epsilon_1),(a_2,\epsilon_2),\dots,(a_n,\epsilon_n)$ of pairs of algebra elements a_i and signs $\epsilon_i\in\{-1,+1\}$ so that there is a sequence of basic idempotents

$$I=I_0,I_1,\dots,I_{n-1},I_n=J$$

with

$$a_i=\begin{cases} I_{i-1}\cdot a_i\cdot I_i & \text{if } \epsilon_i=+1,\\ I_i\cdot a_i\cdot I_{i-1} & \text{if } \epsilon_i=-1,\end{cases}$$

for each i. (Note that the I_i can be recovered from the a_i.)

The *inverse* of K is

$$K^{-1}=\big((a_n,-\epsilon_n),\dots,(a_1,-\epsilon_1)\big),$$

which is a chain connecting J to I.

If A is graded, the *grading* of K is

$$\mathrm{gr}'(K)=\prod_{i=1}^{n}(\lambda\,\mathrm{gr}'(a_i))^{\epsilon_i}.$$

Similarly, if A is $\mathcal{A}(\mathcal{Z})$ or the near-diagonal subalgebra, the *support* of K is

$$\mathrm{supp}(K)=\sum_{i=1}^{n}\epsilon_i\,\mathrm{supp}(a_i).$$

We say that an idempotent $I = (i \otimes i')$ for the near-diagonal subalgebra has *extremal weight* if the weight of i is $\pm k$ (and hence the weight of i' is $\mp k$). Note that there are exactly two idempotents of extremal weight. For definiteness, we will say that the weight of $I = (i \otimes i')$ is the weight of i.

Lemma 4.10 *For any two idempotents I and J for the near-diagonal subalgebra, with the same nonextremal weight, there is a chain K of short near-chords connecting I and J. Furthermore, if $\mathbf{x}$ and $\mathbf{y}$ are the corresponding generators for $\mathcal{H}(m)$ and $Q \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we can choose the chain K so that*

$$\operatorname{supp}(K) = -r_*(\partial^\partial(Q)).$$

(Recall that $\partial\mathcal{H} = -\mathcal{Z} \amalg \mathcal{Z}'$, while the near-diagonal subalgebra lives inside $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')$. This is the reason for the presence of the map r_* from Section 2.4.3.)

Proof For definiteness, we will discuss the case of under-slides; the case of over-slides is similar. Let d_1 be the point in $\boldsymbol{a}$ just below b_1 and d_2 the point matched with d_1. (For a degenerate arc-slide, $d_1 = c_2$.)

For the first part, connect any pair of idempotents by a chain of short near-chords by swapping adjacent pairs. More precisely, we may assume that I and J are both idempotents of type X, by choosing a_1 and/or a_n to be the short near-chord $\sigma \otimes 1$ if necessary (and $\epsilon_1 = 1$ and/or $\epsilon_n = -1$). Define a multigraph[1] Γ as follows:

- Γ has one vertex A_i for each matched pair in $\mathcal{Z}$ (or equivalently $\mathcal{Z}'$).
- Γ has an edge connecting A_i and A_j each time a foot of A_i is adjacent to a foot of A_j in $\boldsymbol{a} \setminus \{b_1\}$. In particular, the point d_1 is adjacent to c_1, and b_1 is not viewed as adjacent to anything.
  Let (x_1, y_1) denote the edge connecting the matched pair $\{x_1, x_2\}$ to the matched pair $\{y_1, y_2\}$ coming from the fact that x_1 is adjacent to y_1.
- Add one more edge from the matched pair $\{d_1, d_2\}$ to the matched pair $\{b_1, b_2\}$. Call this edge (d_1, b_1).

See Figure 20.

The type-X idempotents of weight n correspond to subsets S of the vertices of Γ with $|S| = n + k$. Most of the edges in Γ correspond to short near-chords in an obvious way: except for the edges (d_1, c_1) and (d_1, b_1) the edges in Γ come from length-1 intervals in $(Z, \boldsymbol{a})$. We will see that the edge (d_1, b_1) corresponds to the pair of short

[1] A *multigraph* is a graph which may have multiple edges connecting the same pair of vertices.

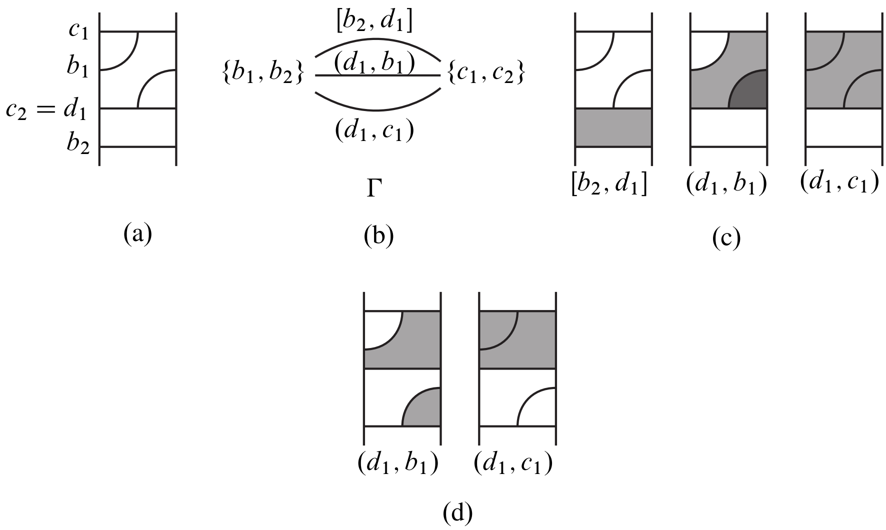


Figure 20: *The adjacency graph for an arc-slide* (a) A genus-one arc-slide; (b) the corresponding graph; (c) the domains corresponding to the edges in the graph; the darkly-shaded region is covered with multiplicity 2; (d) the domains corresponding to (d_1, b_1) and (d_1, c_1) in the nondegenerate case.

near-chords $(\sigma_- \otimes \sigma_-, 1 \otimes \sigma')$ and the edge (d_1, c_1) corresponds to the pair of short chords $(\sigma_- \otimes \sigma_-, \sigma \otimes 1)$ (in the nondegenerate case).

Consider the symmetric group G on the vertices of Γ. Since type-X idempotents correspond to subsets of the vertices of Γ, the group G acts transitively on the set of type-X idempotents. Since Γ is connected, G is generated by those transpositions that interchange vertices connected by an edge. Let γ be an edge of Γ, connecting some A_i to some A_j, and $\tau_\gamma \in G$ the corresponding transposition. Let I be a indecomposable idempotent.

- If $\gamma \notin \{(d_1, b_1), (d_1, c_1)\}$ and exactly one of A_i or A_j is occupied in I then the action of τ_γ on I can be achieved by multiplying by the corresponding short near-chord $a(\xi) \otimes a'_o(\xi)$ (with $\epsilon_i = \pm 1$).
- If $\gamma = (d_1, b_1)$ (so $\{A_i, A_j\} = \{\{d_1, d_2\}, \{b_1, b_2\}\}$) and exactly one of A_i or A_j is occupied in I then the action of τ_γ on I can be achieved by multiplying by either $(\sigma_- \otimes \sigma_-, \pm 1), (1 \otimes \sigma', \pm 1)$ or $(1 \otimes \sigma', \pm 1), (\sigma_- \otimes \sigma_-, \pm 1)$ (depending on the occupancy of $\{c_1, c_2\}$ in I). (For a degenerate arc-slide, $\sigma_- \otimes \sigma_-$ is replaced by $(\sigma_- \cup \sigma) \otimes \sigma_-$.)

- If $\gamma = (d_1, c_1)$ (so $\{A_i, A_j\} = \{\{d_1, d_2\}, \{c_1, c_2\}\}$) and exactly one of A_i or A_j is occupied in I then the action of τ_γ on I can be achieved by multiplying by either $(\sigma_- \otimes \sigma_-, \pm 1), (\sigma \otimes 1, \pm 1)$ or $(\sigma \otimes 1, \pm 1), (\sigma_- \otimes \sigma_-, \pm 1)$, depending on the occupancy of $\{b_1, b_2\}$ in I. (For a degenerate arc-slide, $\sigma_- \otimes \sigma_-$ is replaced by $\sigma_- \otimes (\sigma' \cup \sigma_-)$.)
- If neither A_1 nor A_2 is occupied in I then the action of γ on I is trivial.
- If both A_1 and A_2 are occupied in I then the action of γ on I is trivial.

Any two $(n+k)$–element sets of vertices of Γ are related by an element of the symmetric group, so the first part of the claim follows.

For the second part, it is enough to see that we can obtain the boundary of any periodic domain as the support of a chain. Then, since we can connect I and J by some chain K_0, to get a chain representing any other domain connecting I and J we merely concatenate a chain representing the appropriate periodic domain. Moreover:

- We may work with any convenient starting and ending idempotent: given a periodic domain Q, a chain K_1 connecting I_0 to I_0 with $\operatorname{supp}(K) = -r_*(\partial^\partial Q)$, and another idempotent J, choose a chain L connecting I_0 to J. Then $L^{-1}K_1L$ is a chain connecting J to J with $-r_*(\partial^\partial Q) = \operatorname{supp}(L^{-1}K_1L) = \operatorname{supp}(K_1)$.
- Given a basis $\{Q_i\}$ for the space of periodic domains, it suffices to find a chain representing each basis element.

We will now show how cycles in the graph Γ give periodic domains. Pick a basis of $H_1(\Gamma)$ consisting of simple cycles (edge loops with no repeated edges). For each such basis element, we can find a chain of short near-chords as follows. Start in a type-X idempotent where there is at least one occupied and at least one unoccupied vertex in the cycle. (This is possible since we are not in the extremal weight and the cycles in Γ have at least 2 vertices.) Then swap any consecutive (occupied, unoccupied) vertex pair where the unoccupied vertex is clockwise from the occupied one. Repeat until each occupied vertex has moved clockwise to the next occupied slot, or equivalently until we have swapped once on each edge. This sequence of swappings gives a periodic domain. The $2k$ independent cycles give a basis for the space of periodic domains. $\square$

Lemma 4.11 *For any arc-slide bimodule N for m, near-complementary idempotents I and J, and chain K of short near-chords connecting I and J,*

$$\operatorname{gr}'_N(I) = \operatorname{gr}'(K)\operatorname{gr}'_N(J).$$

Proof If a_i is a short near-chord with $I_i \cdot a_i \cdot I_{i+1} = a_i$ then $a_i I_{i+1}$ appears in ∂I_i in N (by hypothesis of an arc-slide bimodule) and so

$$\mathrm{gr}'_N(I_i) = \lambda\, \mathrm{gr}'(a_i)\, \mathrm{gr}'_N(I_{i+1}).$$

Similarly, if $I_{i+1} \cdot a_i \cdot I_i = a_i$ then $a_i \boldsymbol{x}_i$ appears in ∂I_{i+1} and

$$\mathrm{gr}'_N(I_i) = (\lambda\, \mathrm{gr}'(a_i))^{-1}\, \mathrm{gr}'_N(I_{i+1}).$$

The result follows by induction. □

Note that Lemma 4.11 applies to any arc-slide bimodule. In particular, $\widehat{CFDD}(\mathcal{H}(m))$ is an arc-slide bimodule, and so it applies there.

Lemma 4.12 *For any arc-slide bimodule N, the near-diagonal subalgebra is contained in* $\mathrm{Coeff}(N)$.

In principle, $\mathrm{Coeff}(N)$ could be larger than the near-diagonal subalgebra, but by Definition 1.7, we are guaranteed that any elements not in the near-diagonal subalgebra do not appear in the differential.

Proof Let a be an element of the near-diagonal subalgebra, with initial idempotent I and final idempotent J. Then, as each region in $\mathcal{H}(m)$ touches the boundary, there is a unique domain Q connecting the generators $\boldsymbol{x}(I)$, $\boldsymbol{x}(J)$ of $\mathcal{H}(m)$ corresponding to I, J so that $\mathrm{supp}(a) = -r_*(\partial^\partial Q)$. Let K be a chain of algebra elements connecting $\boldsymbol{x}(I)$ to $\boldsymbol{x}(J)$ with support $\mathrm{supp}(K) = -r_*(\partial^\partial Q)$, whose existence is guaranteed by Lemma 4.10. Then

$$\mathrm{supp}(a) = -r_*(\partial^\partial Q) = \mathrm{supp}(K),$$
$$\mathrm{gr}'(a) = \lambda^m\, \mathrm{gr}'(K),$$

for some integer m. Thus by Lemma 4.11,

$$\mathrm{gr}'_N(\boldsymbol{x}(I)) = \mathrm{gr}'(K)\, \mathrm{gr}'_N(\boldsymbol{x}(J)) = \lambda^{-m}\, \mathrm{gr}'(a)\, \mathrm{gr}'_N(\boldsymbol{x}(J)),$$

which says that $(\boldsymbol{x}(I), a, \boldsymbol{x}(J)) \in \mathrm{Coeff}(N)$, as desired. □

Proposition 4.13 *Let N be an arc-slide bimodule for m in the sense of Definition 1.7. Then there is a G'–set map $f\colon S'_{\mathrm{std}}(m) \to S'_N$ so that for each generator $\boldsymbol{x}(I)$ for N,*

$$f(\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}_{\mathrm{std}}(I))) = \mathrm{gr}'_N(\boldsymbol{x}(I)),$$

where $\boldsymbol{x}_{\mathrm{std}}(I)$ is the corresponding generator for $\widehat{CFDD}(\mathcal{H}(m))$ with idempotent I. Furthermore, the $\mathbb{Z}$–gradings on the near-diagonal subalgebra from the two bimodules agree.

Proof Pick a base generator $\boldsymbol{x}_0$. We first compare the stabilizer $\mathrm{Stab}_{\mathrm{std}}(\boldsymbol{x}_0)$ of $\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}_0)$ in $S'_{\mathrm{std}}(m)$ and the stabilizer $\mathrm{Stab}_N(\boldsymbol{x}_0)$ of $\mathrm{gr}'_N(\boldsymbol{x}_0)$ in S'_N. For any $Q \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$, there is a chain K of short near-chords with $-r_*(\partial^\partial Q) = \mathrm{supp}(K)$ by Lemma 4.10, and so by Lemma 4.11,

$$\mathrm{gr}'(\boldsymbol{x}_0) = \mathrm{gr}'(K)\,\mathrm{gr}'(\boldsymbol{x}_0),$$

where $\mathrm{gr}'(\boldsymbol{x}_0)$ denotes either $\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}_0)$ or $\mathrm{gr}'_N(\boldsymbol{x}_0)$. Thus $\mathrm{gr}'(K)$ is in both $\mathrm{Stab}_{\mathrm{std}}(\boldsymbol{x}_0)$ and $\mathrm{Stab}_N(\boldsymbol{x}_0)$. From $\mathcal{H}(m)$ we see that $\mathrm{gr}'(K)\,\mathrm{gr}'(\boldsymbol{x}_0)$ must be $R(g'(Q))$: the homological components of $R(g'(Q))$ and of $\mathrm{gr}'(K)$ agree, and since the grading set on $\mathcal{H}(m)$ is λ–free, there can be at most one such element in $\mathrm{Stab}_{\mathrm{std}}(\boldsymbol{x}_0)$.

By hypothesis, $\mathrm{Stab}_{\mathrm{std}}(\boldsymbol{x}_0)$ is generated by elements of the form $R(g'(Q))$, and so we have $\mathrm{Stab}_{\mathrm{std}}(\boldsymbol{x}_0) \subset \mathrm{Stab}_N(\boldsymbol{x}_0)$. We can therefore define the map $f\colon S'_{\mathrm{std}}(m) \to S'_N$ in a canonical way.

Now, for any other generator $\boldsymbol{x}$, connect $\boldsymbol{x}$ to $\boldsymbol{x}_0$ by a chain K of short near-chords. Then

$$\begin{aligned} f(\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x})) = f(\mathrm{gr}'(K)\,\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}_0)) &= \mathrm{gr}'(K) f(\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}_0)) \\ &= \mathrm{gr}'(K)\,\mathrm{gr}'_N(\boldsymbol{x}_0) = \mathrm{gr}'_N(\boldsymbol{x}), \end{aligned} \tag{4-1}$$

as desired. We used Lemma 4.11 twice (once in each grading set), as well as the fact that f is a $G'(\mathcal{H})$–set map.

Now consider any element of the near-diagonal subalgebra, which we can think of as a triple $(\boldsymbol{x}, a, \boldsymbol{y})$ in the coefficient algebra of either $\widehat{CFDD}(\mathcal{H}(m))$ or N. We have

$$\begin{aligned} \lambda^{\mathrm{gr}_{\mathrm{std}}(\boldsymbol{x},a,\boldsymbol{y})}\,\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x}) &= \mathrm{gr}'(a)\,\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{y}), \\ \lambda^{\mathrm{gr}_{\mathrm{std}}(\boldsymbol{x},a,\boldsymbol{y})} f(\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{x})) &= \mathrm{gr}'(a) f(\mathrm{gr}'_{\mathrm{std}}(\boldsymbol{y})), \\ \lambda^{\mathrm{gr}_{\mathrm{std}}(\boldsymbol{x},a,\boldsymbol{y})}\,\mathrm{gr}'_N(\boldsymbol{x}) &= \mathrm{gr}'(a)\,\mathrm{gr}'_N(\boldsymbol{y}), \end{aligned}$$

by (in order) the definition of $\mathrm{gr}'_{\mathrm{std}}$ on $\mathrm{Coeff}(\widehat{CFDD}(\mathcal{H}(m)))$ (Lemma 2.11), the fact that f is a $G'(\mathcal{H})$–set map, and (4-1). Since by assumption S'_N is λ–free, the last equation implies that $\mathrm{gr}_N(\boldsymbol{x}, a, \boldsymbol{y}) = \mathrm{gr}_{\mathrm{std}}(\boldsymbol{x}, a, \boldsymbol{y})$. □

Recall from Definition 1.5 that the near-complementary idempotents are either the analogues of complementary idempotents, called type X, or are subcomplementary idempotents, called type Y. Idempotents of type X are further divided into idempotents where there are horizontal strands at C in $\mathcal{Z}$, which we call *type* ${}_CX$, and idempotents where there are horizontal strands at C in $\mathcal{Z}'$, which we call *type* X_C.

Recall that σ, σ', σ_+, σ_-, σ'_+ and σ'_- denote regions in the standard Heegaard diagram $\mathcal{H}(m)$ for m; see Section 4.2. For a region $\mathcal{R}$ in $\mathcal{H}(m)$, $n_{\mathcal{R}}$ is the multiplicity with which $\mathcal{R}$ appears in a domain compatible with a. Since every region of $\mathcal{H}(m)$ touches the boundary, this is actually a function of $\operatorname{supp}(a)$. These multiplicities of a are well defined only if a is in the near-diagonal subalgebra, not for general a in $\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(-\mathcal{Z}')$: if a is not in the near-diagonal subalgebra then there is no domain Q compatible with a.

Proposition 4.14 *Consider an arc-slide satisfying Convention 4.1. In the $\mathbb{Z}$–grading on the near-diagonal subalgebra for an arc-slide of b_1 over c_1, the grading of a basic generator a with initial idempotent I and final idempotent J is*

$$\iota(a)+c(I,\operatorname{supp}(a))+c(J,\operatorname{supp}(a)), \tag{4-2}$$

where $c(I,\operatorname{supp}(a))$ is a correction term given by

$$c(I,\operatorname{supp}(a))=\begin{cases}\frac{1}{4}(n_{\sigma'_+}-n_{\sigma'}) & I \text{ of type } {}_CX,\\ \frac{1}{4}(-n_{\sigma}+n_{\sigma_-}) & I \text{ of type } X_C,\\ \frac{1}{4}(n_{\sigma_+}-n_{\sigma}-n_{\sigma'}+n_{\sigma'_-}) & I \text{ of type } Y,\end{cases} \tag{4-3}$$

for an under-slide and by

$$c(I,\operatorname{supp}(a))=\begin{cases}\frac{1}{4}(-n_{\sigma'}+n_{\sigma'_-}) & I \text{ of type } {}_CX,\\ \frac{1}{4}(n_{\sigma_+}-n_{\sigma}) & I \text{ of type } X_C,\\ \frac{1}{4}(-n_{\sigma}+n_{\sigma_-}+n_{\sigma'_+}-n_{\sigma'}) & I \text{ of type } Y,\end{cases} \tag{4-4}$$

for an over-slide.

(In the degenerate case for under-slides in which we have $\sigma_-=\sigma'_+$, the formulas in Proposition 4.14 hold without change.)

Proof By Proposition 4.13, we can compute the grading either in the coefficient algebra of the bimodule N or from the Heegaard diagram $\mathcal{H}(m)$. Inside $\mathcal{H}(m)$, we can apply Lemma 2.18, as follows. The α– and β–curves divide the Heegaard surface Σ into regions. Most of these regions $\mathcal{R}$, other than the six regions neighboring σ or σ', are strips across the diagram; these are all octagons, so have $e(\mathcal{R})=-1$, and have $n_{\boldsymbol{x}}(\mathcal{R})=n_{\boldsymbol{y}}(\mathcal{R})=\frac{1}{2}$, for a total contribution of 0. To analyze the six special regions more conveniently, divide the correction into two pieces, associated to the initial and final generators: for $Q\in\pi_2(\boldsymbol{x},\boldsymbol{y})$ with $\partial^\partial Q=\operatorname{supp}(a)$,

$$\operatorname{gr}(\boldsymbol{x},a,\boldsymbol{y})=\iota(a)+\bigl(-\tfrac{1}{2}e(Q)-n_{\boldsymbol{x}}(Q)\bigr)+\bigl(-\tfrac{1}{2}e(Q)-n_{\boldsymbol{y}}(Q)\bigr).$$

Then, for instance, if b_1 is below c_1, σ_+ is an octagon, and we find

$$-\tfrac{1}{2}e(\sigma_+)-n_I(\sigma_+)=\tfrac{1}{2}+\begin{cases}-\frac{1}{2} & I \text{ of type } X,\\ -\frac{1}{4} & I \text{ of type } Y.\end{cases}$$

This gives the contribution of n_{σ_+} to $c(I, \mathrm{supp}(a))$ in this case. The other contributions are similar. $\square$

4.4 Under-slides

We explicitly describe the differential for the arc-slide bimodule for under-slides in Proposition 4.16. We first define near-chords in this case.

Definition 4.15 A (nonzero) basic algebra element x in the near-diagonal subalgebra for an under-slide satisfying Convention 4.1 is called a *near-chord* if it satisfies any of the following conditions:

(U-1) It has the form $x = I\cdot(a(\xi)\otimes a'_o(\xi))\cdot J$, where I and J are near-complementary idempotents and ξ is some chord in $\mathcal{Z}$ neither of whose endpoints is b_1 (so that it can be interpreted, as it is in the above expression, as a chord also in $\mathcal{Z}'$); furthermore, ξ is required to be different from the chord $[c_2, c_1]$.

(U-2) It has the form $x = I\cdot(a(\sigma)\otimes 1)\cdot J$ or $I\cdot(1\otimes a'_o(\sigma'))\cdot J$, where I and J are near-complementary idempotents.

(U-3) There is a chord ξ for $\mathcal{Z}$ so that the interior of ξ is disjoint from σ and the support of $\xi\cup\sigma$ is connected, and x has the form $x = I\cdot(a(\xi\cup\sigma)\otimes a'_o(\xi))\cdot J$; or there is a chord ξ for $\mathcal{Z}'$ so that the interior of ξ is disjoint from σ', and the support of $\xi\cup\sigma'$ is connected, and x has the form $x = I\cdot(a(\xi)\otimes a'_o(\xi\cup\sigma'))\cdot J$.

(U-4) It has the form $x = I\cdot(a(\xi\setminus\sigma)\otimes a'_o(\xi))\cdot J$ where $\sigma\subset\xi$; or $x = I\cdot(a(\xi)\otimes a'_o(\xi\setminus\sigma'))\cdot J$ where $\sigma'\subset\xi$. (The element x can have two or three moving strands.)

(U-5) It has the form $x = I\cdot(a(\xi\cup\eta)\otimes a'_o(\xi\cup\eta))\cdot J$, where:

- ξ and η are chords.
- Neither b_1 nor b_2 appears in the boundary of ξ.
- c_1 appears in the boundary of ξ and c_2 appears in the boundary of η with the opposite orientation.
- The points b_1 and b'_1 do not appear in the support of ξ (or, consequently, η).

(U-6) It has the form $x = I\cdot(a(\xi\cup\sigma)\otimes a'_o(\xi\setminus\sigma'))\cdot J$, where ξ is a chord other than $[c_1, c_2]$ such that $\sigma'\subset\xi$ but σ' is not contained in the interior of ξ (so ξ has c_2 as one endpoint); or $x = I\cdot(a(\xi\setminus\sigma)\otimes a'_o(\xi\cup\sigma'))\cdot J$, where ξ is a chord such that $\sigma\subset\xi$ but σ is not contained in the interior of ξ (so ξ has c_1 as an endpoint). (There are two subcases: either all local multiplicities are 0 or 1, or there is some local multiplicity of two, which occurs when σ or σ' is contained in ξ.)

Near-chords for under-slides are illustrated in Figure 21. In this subsection we will almost always call these elements simply "near-chords." (The reader is forewarned that there is a different set of near-chords for over-slides, Definition 4.25, used in Section 4.5.)

When Convention 4.1 does not hold, in the definition of "near-chords", we switch the roles of the two tensor factors.

Note that near-chords of type (U-2) are short, so they appear in the differential of any arc-slide bimodule, by hypothesis.

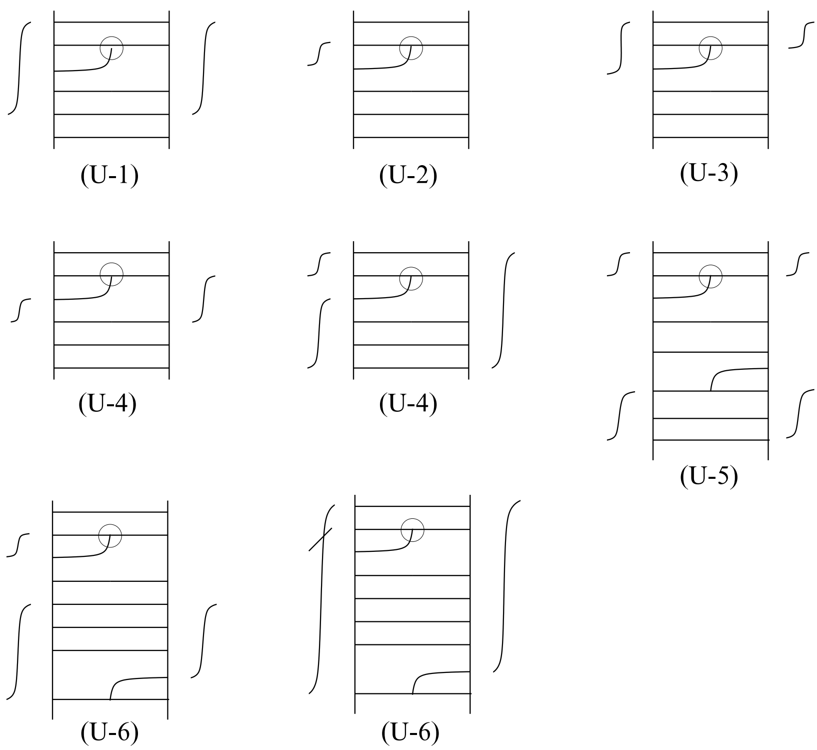


Figure 21: *Near-chords for under-slides* We have illustrated here examples of all the types of near-chords for under-slides, Definition 4.15. (Note that there are two near-chords of type (U-4) and of type (U-6) since there are two distinct subtypes of these.)

The aim of the present subsection is to verify the following, which is a special case of Proposition 1.10:

Proposition 4.16 *If N is a stable arc-slide bimodule for an under-slide then the differential on N contains precisely the near-chords.*

We return to the proof at the end of the subsection, after some preliminary results. We first establish that the near-chords listed above are all the elements of the near-diagonal subalgebra of grading -1.

Lemma 4.17 *In the near-diagonal subalgebra of an under-slide $m\colon \mathcal{Z} \to \mathcal{Z}'$, there are no elements of positive grading; the elements of grading 0 are the idempotents; and the basic elements of grading -1 are the near-chords for under-slides.*

Proof Assume that the arc-slide satisfies Convention 4.1. Let $a = a_L \otimes a_R$ be a basic generator of the near-diagonal subalgebra of grading greater than or equal to -1. Then a uniquely determines a pair of generators $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}(\mathcal{H}(m))$ and a nonnegative domain $Q \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ in $\mathcal{H}(m)$, with $\mathrm{supp}(a) = \partial^{\partial} Q$. By Proposition 4.14, the grading of a is determined by Q, via formula (4-2). We will proceed by cases on the types of $\boldsymbol{x}$ and $\boldsymbol{y}$ (whether they are of type ${}_CX$, X_C, or Y) and the multiplicities of Q in the 6 regions σ_+, σ, σ_-, σ'_+, σ', and σ'_-.

Recall from Lemma 2.2 that for any $a_0 \in \mathcal{A}(\mathcal{Z})$, we have

$$\iota(a_0) \leq -m/2,$$

where m is the minimal number of intervals needed to get the multiplicities in $\mathrm{supp}(a_0)$. Thus for an element a of the near-diagonal subalgebra,

$$\iota(a) \leq -(m_L + m_R)/2,$$

where m_L and m_R are the number of intervals needed to cover $\mathrm{supp}(a_L)$ and $\mathrm{supp}(a_R)$, respectively. Note that $m_L = m_R$ unless $n_\sigma(a)$ or $n_{\sigma'}(a)$ is nonzero.

If all six multiplicities $n_{\sigma_+}, \dots, n_{\sigma'_-}$ are 0 then Q consists of a union of horizontal strips. Since the correction term is zero in this case, Q can have at most one connected set of horizontal strips, as in type (U-1), or no strips at all, in which case a is an idempotent. In the case analysis that follows we will assume that not all multiplicities are zero.

In general, there are other constraints on these multiplicities:

- The multiplicity difference across any α–arc is at most 1.
- The multiplicity differences are constrained by the idempotents

$$n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - n_{\sigma'_-} = -1, 0 \text{ or } +1, \tag{4-5}$$

$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'} - n_{\sigma'_-} = -1, 0 \text{ or } +1, \tag{4-6}$$

  where the right-hand side is determined by what happens to the occupancy of the C idempotent on the left in (4-5) and on the right in (4-6). This will be spelled out case by case below.

There are also some more constraints that depend more closely on the idempotents. For instance, if C is not occupied on the left in either the starting or ending idempotents, then $n_{\sigma_+} = n_\sigma$ and $n_{\sigma'_+} = n_{\sigma'_-}$, as no strand can start or end at the endpoints of C.

${}_C X \to {}_C X$ In this case we have

$$n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} = n_{\sigma'} - n_{\sigma'_-} = 0.$$

(The second set of equations come from the fact that the pair C is not occupied on the right in either the initial or final idempotent, so there can be no strand starting or ending there.) According to Proposition 4.14, the correction to the grading is given by $c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \frac{1}{2}(n_{\sigma'_+} - n_{\sigma'})$.

The linear equations tell us that the multiplicities are of the following forms:

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$\epsilon/2$	m	$m+\epsilon$	m	$l+\epsilon$	l	l

Here $\epsilon \in \{-1, 0, 1\}$ (as the difference in multiplicities is at most one), and the correction to the grading is $\epsilon/2$, as indicated.

If $\epsilon = -1$, we would need to have $\iota(a) = -\frac{1}{2}$, which is not possible with the given multiplicities.

If $\epsilon = 0$, we have complete horizontal strips, giving a near-chord of type (U-1) or an idempotent (if there are no strips at all).

If $\epsilon = +1$, the left side of Q will need at least two intervals to cover it, leaving only one interval for the right. This implies that $l = 0$ and $m \in \{0, 1\}$. In both cases, this gives a near-chord of type (U-6).

In summary, the possibilities are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(U-1)	0	-1	1	1	1	0	0	0
(U-1)	0	-1	0	0	0	1	1	1
(U-1)	0	-1	1	1	1	1	1	1
(U-6)	$+\frac{1}{2}$	-1	0	1	0	1	0	0
(U-6)	$+\frac{1}{2}$	-1	1	2	1	1	0	0

${}_C X \to X_C$ In this case we have

$$n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - n_{\sigma'_-} = 1,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'} - n_{\sigma'_-} = 1.$$

The correction to the grading is given by

$$c(I,\mathrm{supp}(a))+c(J,\mathrm{supp}(a))=\tfrac{1}{4}(-n_\sigma+n_{\sigma_-}+n_{\sigma'_+}-n_{\sigma'})=0.$$

The linear equations tell us that the multiplicities are given by

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
0	m	$m+\epsilon-1$	$m+\delta-1$	$l+\epsilon$	$l+\delta$	l

where $\epsilon,\delta\in\{0,1\}$.

The only solutions to these equations which yield a connected domain on both sides are the following:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(U-1)	0	-1	1	0	0	0	0	0
(U-1)	0	-1	1	1	1	1	1	0
(U-1)	0	-1	0	0	0	1	1	0

$\boldsymbol{{}_C X\to Y}$ In this case we have

$$n_{\sigma_+}-n_\sigma+n_{\sigma'_+}-n_{\sigma'_-}=0,$$
$$n_{\sigma_+}-n_{\sigma_-}+n_{\sigma'}-n_{\sigma'_-}=1.$$

The correction to the grading is given by

$$c(I,\mathrm{supp}(a))+c(J,\mathrm{supp}(a))=\tfrac{1}{4}(n_{\sigma_+}-n_\sigma+n_{\sigma'_+}-2n_{\sigma'}+n_{\sigma'_-})=\tfrac{1}{2}(-n_{\sigma'}+n_{\sigma'_-}).$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$-\delta/2$	m	$m+\epsilon$	$m+\delta-1$	$l+\epsilon$	$l+\delta$	l

with $\delta\in\{0,1\}$ and $\epsilon\in\{-1,0,1\}$.

The solutions to these equations which can have grading greater than or equal to -1 are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(U-2)	$-\frac{1}{2}$	-1	0	0	0	0	1	0
(U-3)	0	-1	1	1	0	0	0	0

$\boldsymbol{X_C\to {}_C X}$ This is related to the case ${}_C X\to X_C$ by rotating the diagram 180°. Again, the solutions are all of type (U-1).

$\boldsymbol{X_C\to X_C}$ This is related to the case ${}_C X\to {}_C X$ by rotating the diagram 180°. The solutions are idempotents or near-chords of type (U-1) or (U-6).

$\boldsymbol{X_C \to Y}$ This is related to the case ${}_CX \to Y$ by rotating the diagram $180°$. The solutions are of type (U-2) or (U-3).

$\boldsymbol{Y \to {}_CX}$ In this case we have

$$n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'} - n_{\sigma'_-} = -1.$$

The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{4}(n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - 2n_{\sigma'} + n_{\sigma'_-}) = \tfrac{1}{2}(-n_{\sigma'} + n_{\sigma'_-}).$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$-\delta/2$	m	$m+\epsilon$	$m+\delta+1$	$l+\epsilon$	$l+\delta$	l

with $\delta \in \{-1, 0\}$ and $\epsilon \in \{-1, 0, 1\}$.

The solutions which can have grading greater than or equal to -1 are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(U-4)	0	-1	0	1	1	1	0	0
(U-4)	0	-1	0	0	1	0	0	0
(U-4)	0	-1	0	0	1	1	1	1
(U-4)	$\frac{1}{2}$	-1	0	0	0	1	0	1
(U-4)	$\frac{1}{2}$	-1	1	1	1	1	0	1

$\boldsymbol{Y \to X_C}$ This is related to the case $Y \to {}_CX$ by rotating the diagram $180°$. The solutions are of type (U-1) and (U-4).

$\boldsymbol{Y \to Y}$ In this case we have

$$n_{\sigma_+} - n_\sigma + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'} - n_{\sigma'_-} = 0,$$
$$n_\sigma - n_{\sigma_-} = n_{\sigma'_+} - n_{\sigma'} = 0.$$

(The last equations come from the fact that the B strand is not occupied in either the initial or final idempotent on either the left or right.) The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{2}(n_{\sigma_+} - n_\sigma - n_{\sigma'} + n_{\sigma'_-}) = n_{\sigma_+} - n_\sigma.$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$-\epsilon$	m	$m+\epsilon$	$m+\epsilon$	$l+\epsilon$	$l+\epsilon$	l

with $\epsilon \in \{-1, 0, 1\}$.

The solutions which can have grading greater than or equal to -1 are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(U-1)	0	-1	1	1	1	0	0	0
(U-1)	0	-1	0	0	0	1	1	1
(U-1)	0	-1	1	1	1	1	1	1
(U-5)	1	-1	1	0	0	0	0	1

We have shown that only the idempotents have degree 0 and that the elements of degree -1 are all near-chords. Checking through the various cases of near-chords verifies that they all appear in one of the cases above. This completes the proof of Lemma 4.17, under the hypotheses of Convention 4.1.

When Convention 4.1 does not hold, the above discussion nonetheless applies, by reflecting the Heegaard diagram vertically. For example, the analogues of the formulas from Proposition 4.14 (Equations (4-3) and (4-4)) hold, where we relabel regions after reflection, as indicated in Figure 17. For the new formulas, we swap the roles of σ_+, σ_-, and σ with σ'_+, σ'_-, and σ', respectively; so, for instance, the new correction formula for underslides (replacing (4-3)) reads

$$c(I, \operatorname{supp}(a)) = \begin{cases} \frac{1}{4}(n_{\sigma_+} - n_\sigma) & I \text{ of type } {}_C X, \\ \frac{1}{4}(-n_{\sigma'} + n_{\sigma'_-}) & I \text{ of type } X_C, \\ \frac{1}{4}(n_{\sigma'_+} - n_{\sigma'} - n_\sigma + n_{\sigma_-}) & I \text{ of type } Y. \end{cases} \tag{4-7}$$

With this understood, the above case analysis holds as stated, in the case where Convention 4.1 is not met. □

Remark 4.18 Instead of considering the grading on the near-diagonal subalgebra, we could instead prove an analogue of Lemma 3.5: every element of the near-diagonal subalgebra can be factored into near-chords.

Remark 4.19 In the proofs of the following sequence of lemmas, we will implicitly assume Convention 4.1. This is not essential for the statements, and the proofs adapt easily to the other case (sometimes at the cost of switching the orders of products, and replacing "initial idempotents" by "terminal idempotents").

We break Proposition 4.16 into a series of lemmas. We focus first on the nondegenerate case (in which there are at least two positions between c_1 and c_2; see Definition 4.5), and return to the degenerate case in Lemma 4.24.

Lemma 4.20 *Let N be a stable arc-slide bimodule for a nondegenerate under-slide. Then the differential in N contains all near-chords of type (U-1).*

Proof Consider the intervals connecting b_2 (which is matched with b_1) to the basepoint z. There are two such intervals; we call the one which does not contain b_1 the *outside region*. We say that a pointed matched circle $\mathcal{Z}$ is *big enough* for the arc-slide if the number of matched pairs in the outside region exceeds the number of positions in the complement of the outside region. By stabilizing, it suffices to consider the case of pointed matched circles which are big enough for the arc-slide.

A *very special length-three chord* is a special length-three chord (Definition 3.6) contained between c_1 and c_2, which is adjacent to b_1 in $\mathcal{Z}$ and to b_1' in $\mathcal{Z}'$. (See Figure 18.) Very special length-three chords, when they exist, need special attention.

Claim 1 *Suppose x is a near-chord of type (U-1) which is not a very special length-three chord, and suppose moreover that neither b_1 nor b_1' is contained in the interior of (the support of) x. Then x appears in the differential.*

Claim 1 follows from the same argument used to prove Proposition 3.8, by induction on the length of the support. Note that each special length-three chord which is not very special is adjacent to a length-one chord which appears in the differential by hypothesis. Thus, these chords all appear in the differential, by the same principle which established the existence of special length-three chords in the *DD* identity bimodule (see the end of the proof of Proposition 3.8).

Consider now a near-chord $x = I \cdot (a(\xi) \otimes a_o'(\xi')) \cdot J$ of type (U-1) whose restricted support has length one. This near-chord appears in the differential by hypothesis except in the special cases when ξ contains either b_1 or b_1' in its interior.

In the case that ξ contains b_1 or b_1' in its interior there are two subcases, according to whether or not $dx = 0$. If $dx \neq 0$ it is straightforward to see that dx factors as a product $y \cdot z$ of two short near-chords, and hence dx appears in the expression for ∂^2. (See the first row of Figure 22.) This product has no alternative factorization, and hence x must appear in the differential.

By induction on the length of the support (again, see the proof of Proposition 3.8), we can conclude the following.

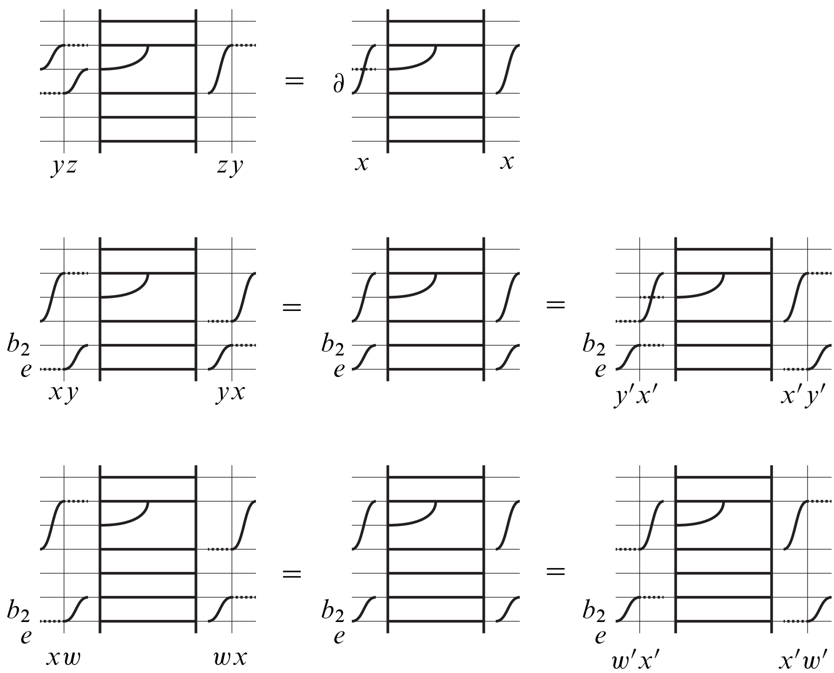


Figure 22: Existence of near-chords of type (U-1) with length-1 restricted support

Claim 2 *Suppose x is any near-chord of type (U-1) with the following properties:*

- *The support of x contains exactly one of b_1 or b_1'.*
- *The position at b_1 or b_1' (whichever is contained in the support of x) is occupied in both the initial and terminal idempotents for x (on the $\mathcal{Z}$ or the $\mathcal{Z}'$ side, respectively).*

Then x appears in the differential.

(Note that any near-chord satisfying the criteria of Claim 2 has $dx \neq 0$.)

Next we consider near-chords x of type (U-1) whose restricted support has length 1 and with $dx = 0$. We consider two subcases: either b_2 lies below b_1 or above it. For the time being, we make the following:

Assumption 1 Suppose that b_2 lies below b_1.

Let e denote the position immediately below b_2. Clearly, $e \neq c_2$ (otherwise $\mathcal{Z}'$ would not be a valid pointed matched circle); by Assumption 1, $e \neq c_1$. In addition, e cannot be matched with the position directly below b_1 (since otherwise $\mathcal{Z}$ would not be a valid pointed matched circle).

Claim 3 *Under Assumption 1, if x is a type (U-1) near-chord whose restricted support has length 1, b_1 is in the support of x, and e is contained in the left (hence also the right) idempotent of x on the $\mathcal{Z}$ side, then x appears in the differential.*

There is a short near-chord y with support $[e, b_2]$ and $x \cdot y \neq 0$. Now, there are type (U-1) near-chords x' and y' with $x \cdot y = y' \cdot x'$ with $\mathrm{supp}(x) = \mathrm{supp}(x')$, $\mathrm{supp}(y) = \mathrm{supp}(y')$ and $dx' \neq 0$. (See the second row in Figure 22.) In fact, this is the only other way to factor $x \cdot y$ in the near-diagonal subalgebra. We already showed that x' occurs in the differential, and y' occurs in the differential by hypothesis; it follows that x occurs in the differential as well. Thus, we have established Claim 3.

We now generalize Claim 3.

Claim 4 *Under Assumption 1, if x is a type (U-1) near-chord whose restricted support has length 1 and b_1 is contained in the support of x, then x appears in the differential.*

If e does not appear in the left idempotent of x on the $\mathcal{Z}$ side, since our pointed matched circle is big enough, we can find some near-chord w of type (U-1) supported entirely in the outside region, terminating at e, and so that $x \cdot w \neq 0$. Again, the product $x \cdot w$ does not appear in the differential of any algebra element, and it has a unique alternative factorization as $w' \cdot x'$, with $\mathrm{supp}(w) = \mathrm{supp}(w')$ and $\mathrm{supp}(x) = \mathrm{supp}(x')$. In Claim 3, we established that x' appears in the differential (since e is in the left idempotent of x' on the $\mathcal{Z}$ side). So, to show that x appears in the differential, it suffices to show that w' appears in the differential. Note that b_1 is not contained in the support of w. If b_1' is also not in the support of w, then w' appears in the differential by Claim 1; if b_1' is contained in the support of w, then, since dx was assumed to vanish, b_1 is not contained in the left idempotent of x (and hence of w') on the $\mathcal{Z}$ side, so b_1' is contained in the left idempotent of w' on the $\mathcal{Z}'$ side; and hence w' appears in the differential by Claim 2. We conclude that x appears in the differential, proving Claim 4. (See the third line of Figure 22.)

Next we consider the case when Assumption 1 does not hold, ie:

Assumption 1′ *Suppose that b_2 lies above b_1.*

Let f denote the position immediately above b_2. The same argument used to establish Claim 3 (with some products reversed) shows the following:

Claim 3′ *Under Assumption 1′, if x is a type (U-1) near-chord whose restricted support has length 1, b_1 is in the support of x, and f is contained in the left idempotent of x on the $\mathcal{Z}$ side, then x appears in the differential.*

Using this, the argument used to deduce Claim 4 can be modified to give the following:

Claim 4′ *Under Assumption 1′, if x is a type (U-1) near-chord whose restricted support has length 1 and b_1 is contained in the support of x, then x appears in the differential.*

Claim 5 *If x is any type (U-1) near-chord whose restricted support has length 1, then x appears in the differential of N.*

The only cases of Claim 5 not covered by hypothesis are those where b_1 or b_1' are contained in the interior of x; but in this case Claim 4 or Claim 4′ applies (perhaps after reversing the roles of $\mathcal{Z}$ and $\mathcal{Z}'$).

Claim 6 *If x is a near-chord of type (U-1) which is not a special length 3 chord, then x appears in the differential.*

If the restricted support of x has length 1, this is covered by Claim 5. Otherwise, we claim that dx contains terms of the form $y \cdot t$ with the following properties:

- Each of y and t is of type (U-1).
- The product $y \cdot t$ has no alternative factorization.
- The product $y \cdot t$ does not appear in the differential of any other basic generator of the near-diagonal subalgebra.

To ensure that the factorization is into two near-chords of type (U-1), we break at a position in x other than b_1, b_2, c_1 and c_2, which in turn can be done since we assumed the arc-slide is nondegenerate. Therefore an induction on the length of the restricted support establishes Claim 6.

Finally, since any special length-three chord x is adjacent to a near-chord of type (U-1) with length-one restricted support, which in turn appears in the differential according to Claim 5 established above (even in the case of a very special length-three chord), the argument from Proposition 3.8 ensures the existence of x in the differential, as well. □

By hypothesis, the differential on N contains all near-chords of type (U-2).

Lemma 4.21 *Let N be a stable arc-slide bimodule for a nondegenerate under-slide. Then the differential on N contains all near-chords of types (U-3) and (U-6).*

Proof Let x be a near-chord of type (U-3). As illustrated in the first line in Figure 23, we can find near-chords y and z so that

- y is of type (U-1),

- z is of type (U-2),
- $y \cdot z$ has no alternative factorizations as a product of two homogeneous elements in the near-diagonal subalgebra, with nontrivial support, and
- $y \cdot z$ appears in dx (and not in dw for any basic generator $w \neq x$).

It follows that x must appear in the differential, so that $\partial^2 = 0$.

Next, let x be a near-chord of type (U-6). There are near-chords y_1, y_2 of type (U-2) and a near-chord z of type (U-1) with the property that $z \cdot y_1 = x \cdot y_2$ has exactly these two factorizations (with nontrivial support in the near-diagonal subalgebra), and $z \cdot y_1$ does not appear in the differential of any other algebra element. Since z and y_1 appear in the differential (according to Lemma 4.20; and the fact that all near-chords of type (U-2) are short), it follows that x must, as well. This is illustrated in the second line of Figure 23. □

Lemma 4.22 *Let N be a stable arc-slide bimodule for a nondegenerate under-slide. Then the differential on N contains all near-chords of type (U-4).*

Proof We will show that all near-chords of type (U-4) appear in the differential of any idempotent which has at least two occupied positions on both the $\mathcal{Z}$ and the $\mathcal{Z}'$ sides. This assumption on the idempotent can be made freely, in view of the stability hypothesis for N.

There are two types of near-chords of type (U-4): those for which the support has three moving strands, and those for which it has only two. In other words, after rotating 180° if necessary, near-chords of type (U-4) have the form $I \cdot (a(\xi \setminus \sigma) \otimes a_o(\xi)) \cdot J$. For the first type, σ is a subset of the interior of ξ; for the second type, σ is a subset of ξ, but one of its boundary points is on the boundary of ξ.

We handle first the case where the near-chord x contains three strands. In this case, we can find a near-chord z of type (U-1) and a near-chord y of type (U-2) with the properties that dz contains $y \cdot x$, as in the third line in Figure 23. We claim that $y \cdot x$ has no alternative factorizations as a product of two near-chords with nontrivial support. More specifically, $y \cdot x$ has three moving strands, like x. The only other products of basic elements of the near-diagonal subalgebra with the same support as $y \cdot x$ and three moving strands are products $y' \cdot x'$ where y' has type (U-3) and x' has type (U-4). However, a closer look at the idempotents shows that $y \cdot x$ does not equal $y' \cdot x'$. Specifically, suppose with out loss of generality that two of the moving strands of x are on the $\mathcal{Z}$–side. Then, the final idempotent of $y \cdot x$ contains C on the $\mathcal{Z}'$ side (x is of

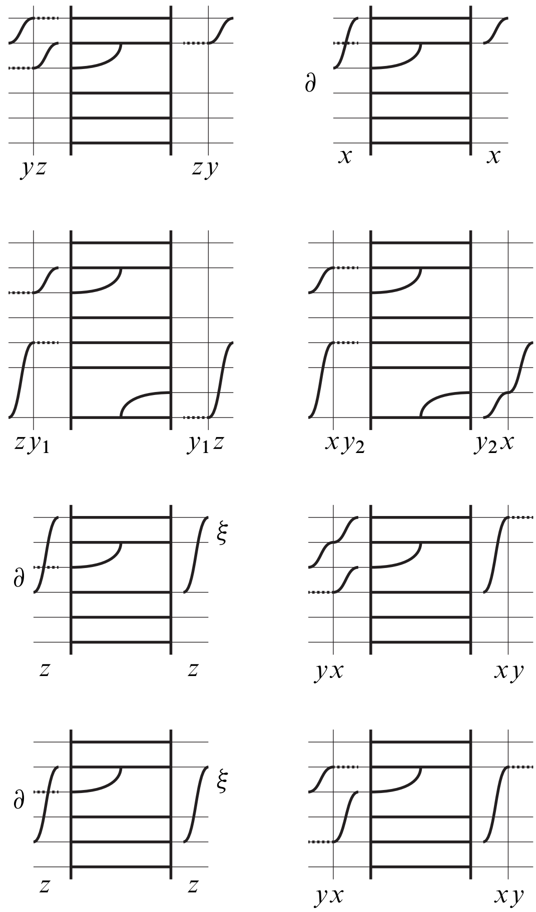


Figure 23: *Existence of near-chords of type (U-3) and (U-6), and certain near-chords of type (U-4)* In the left column, we illustrate terms we already know contribute to ∂^2. On the right, we have the only possible alternative terms which could cancel (some) terms on the left. Recall that we draw $a \cdot b$ with a on the outside and b on the inside; compare Figure 8. Here and later, some of the horizontal lines in the algebra elements have been suppressed.

type $Y \to X_C$), whereas the final idempotent of $y' \cdot x'$ contains C on the $\mathcal{Z}$ side (x' is of type $Y \to {}_C X$).

It follows that $\partial^2 = 0$ forces x to appear in the differential.

With one exception, the same argument also applies to near-chords x of type (U-4) with only two moving strands, as illustrated in the fourth line in Figure 23. The exception is

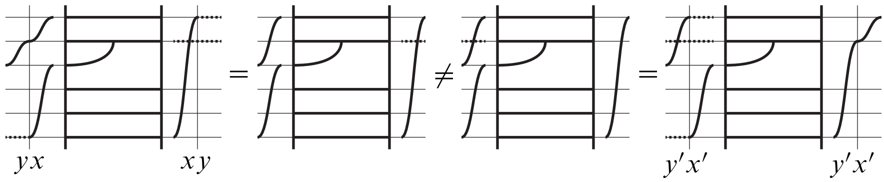


Figure 24: *Not an alternative factorization* One of the cases in Figure 23 (third line, right hand column) might appear to have an alternate factorization; however, a more careful look at idempotents (as indicated) shows that this alternative factorization does not exist.

when the restricted support of ξ is $[c_2, c_1]\times[c_2, c_1]$. In this case, there is no near-chord z of type (U-1) as required for the argument above, and we use a different argument.

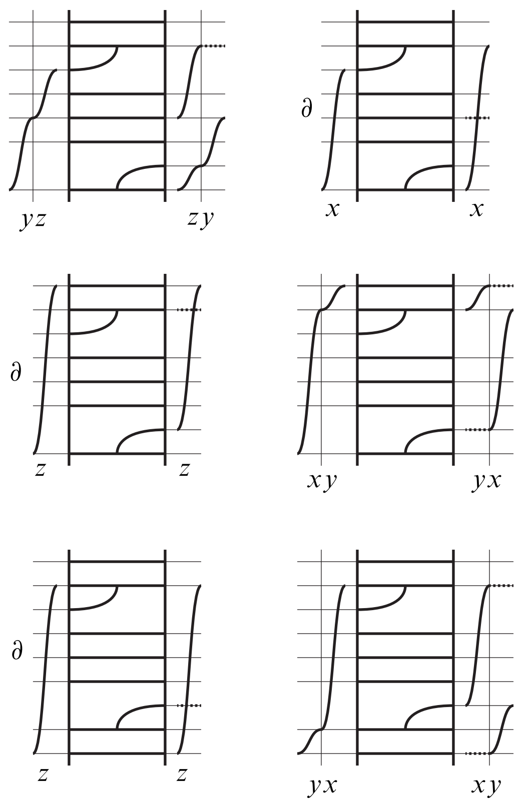


Figure 25: *Existence of remaining near-chords of type (U-4)* We continue with the conventions from Figure 23. This establishes the remaining cases of near-chords of type (U-4) which have not already been covered in Figure 23.

There are three subcases of type (U-4) with restricted support $[c_2, c_1] \times [c_2, c_1]$, according to the placement of the initial idempotent. Specifically, if we rotate so that the support of the near-chord is $[c_2, c_1] \times [b_1', c_1]$, then the three (not necessarily distinct) subcases are:

(1) The initial idempotent on the $\mathcal{Z}'$ side contains a position in the open interval (c_2, c_1).

(2) The initial idempotent on the $\mathcal{Z}'$ side contains a position strictly above c_1.

(3) The initial idempotent on the $\mathcal{Z}'$ side contains a position strictly below c_2.

The three cases are illustrated in Figure 25. In the above verification, we are using the hypothesis on our idempotent that in both $\mathcal{Z}$ and $\mathcal{Z}'$ there are at least two unoccupied positions.

In the first of these cases, we can find near-chords y and z with the following properties:

- y is of type (U-4), but of the type which we have already verified appear in the differential.
- z is of type (U-6) (and hence appears in the differential, by Lemma 4.21).
- The product $y \cdot z$ has no alternative factorizations into homogeneous elements in the near-diagonal subalgebra.
- The term $y \cdot z$ appears in the differential dx of our near-chord x (and $y \cdot z$ does not appear in dx' for any other homogeneous element of the near-diagonal subalgebra).

It follows from the above properties that x appears in the differential.

In the second case, we find a near-chord y of type (U-1) for which $x \cdot y$ has no alternative factorization, but $x \cdot y$ appears in the differential of another near-chord z of type (U-4), which we have already verified occurs in the differential, and $x \cdot y$ does not appear in the differential of any other basic algebra element.

In the third case, we find a near-chord y of type (U-3) for which $y \cdot x$ has no alternative factorization, and $y \cdot x$ appears in dz for a near-chord z of type (U-1) which we have already verified occurs in the differential, and $y \cdot x$ does not appear in the differential of any other basic algebra element. $\square$

Lemma 4.23 *Let N be a stable arc-slide bimodule for a nondegenerate under-slide. Then, the differential in N contains all near-chords of type (U-5).*

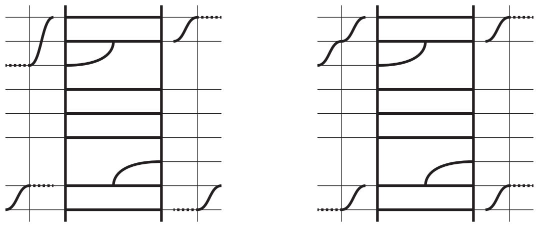


Figure 26: *Proof of Lemma 4.23* Verifying the fact that near-chords of type (U-5) occur in the differential

Proof This follows from the observation that a near-chord of type (U-2) times one of type (U-5) has a unique alternative factorization as a near-chord of type (U-1) times a near-chord of type (U-3) (and it does not appear in the differential of any algebra element); together with Lemmas 4.20 and 4.21. (See Figure 26.) □

We return now to the degenerate case.

Lemma 4.24 *Let N be a stable arc-slide bimodule for a degenerate under-slide. Then the differential on N contains all near-chords.*

Proof We prove this in a sequence of claims:

Claim 1 *The differential contains all near-chords of type (U-1) which are supported outside $[c_1, c_2]$.*

This follows from a straightforward induction on the length of the support, as in Proposition 3.8.

Claim 2 *The differential contains all near-chords of type (U-3).*

Apply the argument from Lemma 4.21 and Claim 1.

Claim 3 *The differential contains all near-chords x of type (U-1) with support $[e, c_2]$ such that b_1 is contained in the idempotent of x on the $\mathcal{Z}$ side, for any e above c_1.*

In this case, (a term in) dx factors as $dx = y \cdot z$, where y is of type (U-3), while z is the short near-chord added in the degenerate case (see Definition 4.6); see Figure 27. Thus, we know that y and z appear in the differential. Again, since there is no alternative factorization of this element, we conclude that x, too, must appear in the differential.

Claim 4 *The differential contains all near-chords x of type (U-1) with support $[e, c_2]$ for any e above c_1.*

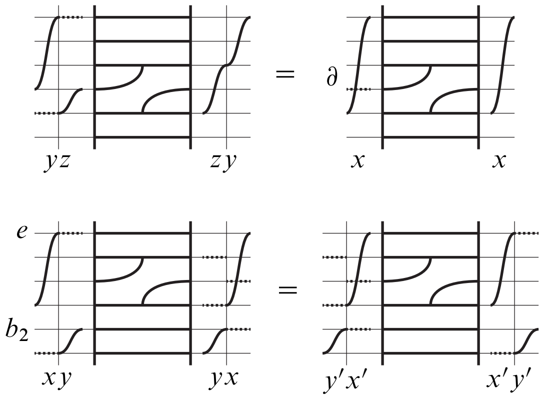


Figure 27: *Existence of type (U-1) chords in the degenerate case* On the top line, we have Claim 3 of the proof of Lemma 4.24; on the bottom line, we have Claim 4 of the proof of Lemma 4.24.

It remains to consider the case where b_1 is not in the initial idempotent x. After stabilizing, it suffices to consider the case that the diagram is big enough, in the sense described in the proof of Lemma 4.20. Assume that b_2 is below c_2; the case that b_2 is above c_1 is similar. In this case, there is some chord y of type (U-1) in the outside region (in the sense of the proof of Lemma 4.20) which terminates at b_2, so that $x \cdot y$ does not vanish. Now, there is a unique alternative factorization $x \cdot y = y' \cdot x'$, where $\mathrm{supp}(x) = \mathrm{supp}(x')$ and $\mathrm{supp}(y) = \mathrm{supp}(y')$. The existence of x' and y' in the differential are ensured by Claims 3 and 1, respectively. The product $x \cdot y$ does not appear in the differential of any algebra element, and hence x must appear in the differential. (Again, see Figure 27.)

Claim 5 *The differential contains all near-chords x of type (U-1) with support $[c_1, e]$ for any e below c_2.*

The proof of Claim 4 applies *mutatis mutandis*.

Claim 6 *The differential contains all near-chords x of type (U-1).*

It remains to consider cases where $[c_1, c_2]$ is contained in the interior of the support. For such a near-chord x, the differential has a term with a unique factorization as $y \cdot z$, where y z exist because of Claims 4 and 5.

Having established the existence of all near-chords of type (U-1), we can apply the proofs of Lemmas 4.21, 4.22 and 4.23 to establish the existence of the remaining near-chords. □

Proof of Proposition 4.16 Lemmas 2.11 and 4.17 imply that the near-chords are the only elements in the near-diagonal subalgebra whose gradings are compatible with appearing in the differential. On the other hand, by Lemmas 4.20, 4.21, 4.22, 4.23 (all in the nondegenerate case) and 4.24 (in the degenerate case), we know that each near-chord appears in the differential. □

4.5 Over-slides

We now turn to bimodules for over-slides. The appropriate notion of near-chords in this case is given in Definition 4.25. The first 6 types correspond to the types of near-chords for under-slides (except that one of the kinds of (U-6) near-chords has no analogue for over-slides). There are two kinds of near-chords for over-slides which have no under-slide analogues, types (O-7) and (O-8).

Definition 4.25 A (nonzero) basic algebra element x in the near-diagonal subalgebra for an over-slide satisfying Convention 4.1 is called a *near-chord* (for the over-slide) if it satisfies any of the following eight conditions:

(O-1) It has the form $x = I \cdot (a(\xi) \otimes a'_o(\xi)) \cdot J$, where ξ is some chord in $\mathcal{Z}$ neither of whose endpoints is b_1 (so that it can be interpreted, as it is in the above expression, as a chord in $\mathcal{Z}'$); furthermore, ξ is required to be different from the chord $[c_2, c_1]$.

(O-2) It has the form $x = I \cdot (a(\sigma) \otimes 1) \cdot J$ or $I \cdot (1 \otimes a'_o(\sigma')) \cdot J$, where I and J are near-complementary idempotents.

(O-3) There is a chord ξ with the property that the interior of ξ is disjoint from σ and the support of $\xi \cup \sigma$ is connected, and $x = I \cdot (a(\xi \cup \sigma) \otimes a'_o(\xi)) \cdot J$; or the interior of $r(\xi)$ is disjoint from σ' and $r(\xi) \cup \sigma'$ is connected, and $x = I \cdot (a(\xi) \otimes a'_o(r(\xi) \cup \sigma')) \cdot J$.

(O-4) It has the form $x = I \cdot (a(\xi \setminus \sigma) \otimes a'_o(\xi)) \cdot J$ where $\sigma \subset \xi$ or $x = I \cdot (a(\xi) \otimes a'_o(\xi \setminus \sigma')) \cdot J$ where $\sigma' \subset \xi$. (Note that $\xi \setminus \sigma$ or $\xi \setminus \sigma'$ can be disconnected in this case).

(O-5) $x = I \cdot (a(\xi \cup \eta) \otimes a'_o(r(\xi) \cup r(\eta))) \cdot J$, where here:

- ξ and η are disjoint chords, or one is contained in the other.
- neither b_1 nor b_2 appear in the boundary of ξ.
- c_1 appears in the boundary of ξ.
- c_2 appears in the boundary of η with the opposite orientation.

(O-6) The nonzero element x has the form $x = I \cdot (a(\xi \cup \sigma) \otimes a'_o(\xi \setminus \sigma')) \cdot J$ where $\sigma' \subset \xi$ but σ' is not contained in the interior of ξ, and $\xi \cap \sigma = \varnothing$; or $x = I \cdot (a(\xi \setminus \sigma) \otimes a'_o(\xi \cup \sigma')) \cdot J$, where the $\sigma \subset \xi$ but σ is not contained in the interior of ξ, and $\xi \cap \sigma' = \varnothing$.

(O-7) x has support $[c_2, c_1] \times [c_2, c_1]$ and exactly three moving strands.

(O-8) x factors as a product of $I \cdot (a([c_2, c_1]) \otimes a'_o([c_2, c_1])) \cdot I$ and $I \cdot (a(\xi) \otimes a_o(\xi)) \cdot J$, where ξ is disjoint from $[c_2, c_1]$ or ξ is properly contained inside $[c_2, c_1]$.

Near-chords for over-slides (satisfying Convention 4.1) are illustrated in Figure 28.

When Convention 4.1 does not hold for the over-slide, as before we switch the roles of the two tensor factors in the definition of near-chords.

The calculation of the arc-slide bimodule for an over-slide is not quite as straightforward as for under-slides: we cannot say that all near-chords appear in the differential, and indeed the bimodule is determined uniquely only up to isomorphism. The near-chords which might or might not appear in the differential for a given arc-slide bimodule are the following:

Definition 4.26 A near-chord for an over-slide is called *indeterminate* if it is of one of the following types:

- It is a near-chord of type (O-3), and its restricted support is $[c_2, c_1]$.
- It is a near-chord of type (O-4), and the boundary of its restricted support contains c_1 or c_2, and the restricted support contains the interval $[c_2, c_1]$.
- It is a near-chord of type (O-7).
- It is a near-chord of type (O-8).

These cases are illustrated in Figure 29.

A given indeterminate near-chord might or might not appear in the differential of an arc-slide bimodule N. Exactly which ones appear are governed by the following:

Definition 4.27 For arc-slides satisfying Convention 4.1, a *basic choice* is a collection $\mathcal{B}$ of indeterminate near-chords of type (O-3), satisfying the following condition: if x and x' are two distinct indeterminate near-chords of type (O-3) with the same initial idempotent (ie there is some I with $I \cdot x = x$ and $I \cdot x' = x'$), then exactly one of x or x' is in $\mathcal{B}$.

For arc-slides not satisfying Convention 4.1, a basic choice is analogous, but with the terminal idempotent now playing the crucial role; ie if x and x' are two distinct indeterminate near-chords of type (O-3) with the same terminal idempotent (ie there is some I with $x = x \cdot I$ and $x' = x' \cdot I$), then exactly one of x or x' is in $\mathcal{B}$.

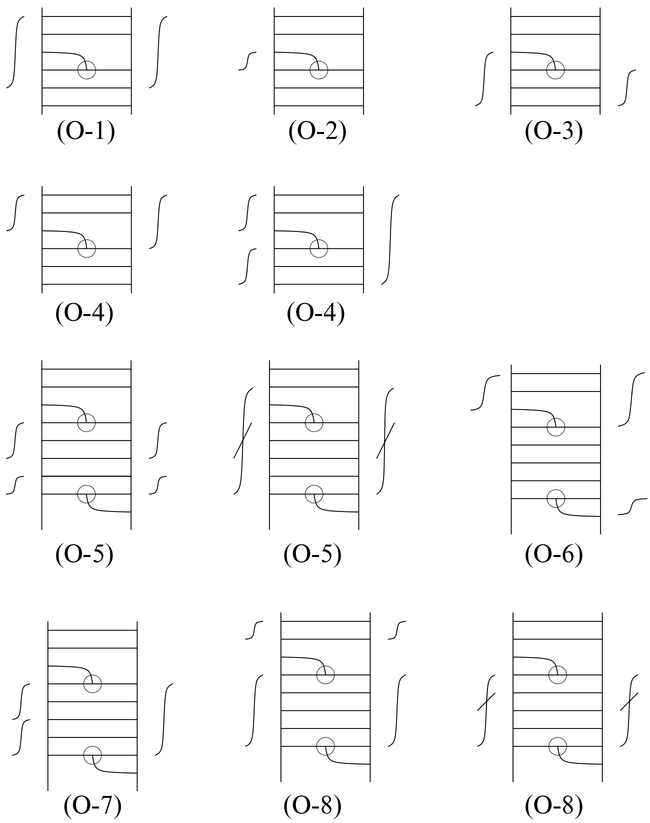


Figure 28: *Near-chords for over-slides* We have illustrated here examples of all the types of near-chords for over-slides appearing in Definition 4.25. (Note that there are two illustrations for type (O-4) and type (O-8).)

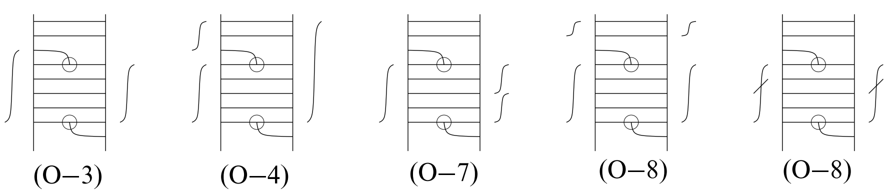


Figure 29: *Indeterminate near-chords* Examples of the different kinds of indeterminate near-chords

Lemma 4.28 *If N is a stable arc-slide bimodule, then the set of indeterminate near-chords of type (O-3) which are contained in the differential on N forms a basic choice, in the sense of Definition 4.27.*

We defer the proof of Lemma 4.28 until page 2639.

Definition 4.29 Let N be a stable arc-slide bimodule. If $\mathcal{B}$ denotes the basic choice of indeterminate near-chords which appear as coefficients in the boundary operator for N then we say that N is *compatible with the basic choice* $\mathcal{B}$; we also say that $\mathcal{B}$ *is the basic choice of* N.

Proposition 4.30 *If N is a stable arc-slide bimodule which is compatible with a basic choice $\mathcal{B}$ then only near-chords can appear in the differential on N, and precisely which ones do appear are uniquely determined by the basic choice $\mathcal{B}$. If N_1 and N_2 are two arc-slide bimodules which are compatible with basic choices $\mathcal{B}_1$ and $\mathcal{B}_2$ then there is an isomorphism between N_1 and N_2.*

Before proving Proposition 4.30 we establish some preliminary results.

As in the case of under-slides, we study the elements of the near-diagonal subalgebra of grading greater than or equal to -1. This time, there are some elements of grading 0, which are responsible for the indeterminacy. We give them a name:

Definition 4.31 A basic generator of the near-diagonal subalgebra with support $[c_2, c_1] \times [c_2, c_1]$ and exactly two moving strands (one in $\mathcal{Z}$ and one in $\mathcal{Z}'$) is called a *dischord.*

Lemma 4.32 *In the near-diagonal subalgebra of an over-slide $m\colon \mathcal{Z} \to \mathcal{Z}'$, there are no elements of positive grading; the basic elements of grading 0 are the idempotents and the dischords (Definition 4.31); and the basic elements of grading -1 are the near-chords for over-slides.*

Proof The proof is similar to the under-slide case (Lemma 4.17). Again, we assume Convention 4.1, reducing to this case via reflection if it is needed. As there, let a be a basic generator in the near-diagonal subalgebra of grading greater than or equal to -1, and let Q be the corresponding domain in the standard Heegaard diagram $\mathcal{H}(m)$ for m. If all six multiplicities $n_{\sigma_+}, \dots, n_{\sigma'_-}$ are 0 then Q consists of a union of horizontal strips, and a is of type (O-1) or an idempotent.

In general, the constraints are:

- The multiplicity difference along any line is at most 1.
- The multiplicity differences from the idempotents are

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = -1, 0 \text{ or } +1, \tag{4-8}$$

$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'} = -1, 0 \text{ or } +1, \tag{4-9}$$

  where the right-hand side is determined by what happens to the occupancy of the C idempotent on the left in (4-8) and on the right in (4-9).

${}_C X \to {}_C X$ In this case we have

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$

$$n_{\sigma_+} - n_{\sigma_-} = n_{\sigma'_+} - n_{\sigma'} = 0.$$

(The second set of equations come from the fact that the strand C is not occupied on the right in either the initial or final idempotent, so there can be no strand starting or ending there.) According to Proposition 4.14, the correction to the grading is given by $c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \frac{1}{2}(-n_{\sigma'} + n_{\sigma'_-})$.

The linear equations tell us that the multiplicities are of the following forms:

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$\epsilon/2$	m	$m+\epsilon$	m	l	l	$l+\epsilon$

Here, $\epsilon \in \{-1, 0, 1\}$ (as the difference in multiplicities is at most one).

If $\epsilon = -1$, we would need to have $M(\mathrm{gr}'(a)) = -\frac{1}{2}$, which is not possible with the given multiplicities.

If $\epsilon = 0$, we have complete horizontal strips, giving near-chords of type (O-1) or, in the case of no strips at all, an idempotent.

If $\epsilon = +1$, the left side of Q will need at least two intervals to cover it, leaving only one interval for the right. This implies that $m = l = 0$, which gives a domain of type (O-6).

In summary, the possibilities are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(O-1)	0	-1	1	1	1	0	0	0
(O-1)	0	-1	0	0	0	1	1	1
(O-1)	0	-1	1	1	1	1	1	1
(O-6)	$+\frac{1}{2}$	-1	0	1	0	0	0	1

$_{C}X \to X_{C}$ In this case we have

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = 1,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'} = 1.$$

The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{4}(n_{\sigma_+} - n_\sigma - n_{\sigma'} + n_{\sigma'_-}) = 0.$$

The linear equations tell us that the multiplicities are given by

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
0	$m+\delta$	$m+\epsilon$	m	l	$l+\delta-1$	$l+\epsilon-1$

where $\epsilon, \delta \in \{0, 1\}$.

The only solutions to these equations which yield a connected domain on both sides (as required by the gradings) are the following:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(O-1)	0	-1	1	1	0	0	0	0
(O-1)	0	-1	1	1	1	1	0	0
(O-1)	0	-1	0	0	0	1	0	0

$_{C}X \to Y$ In this case we have

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'} = 1.$$

The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{4}(-n_\sigma + n_{\sigma_-} + n_{\sigma'_+} - 2n_{\sigma'} + n_{\sigma'_-}) = \tfrac{1}{2}(n_{\sigma'_+} - n_{\sigma'}).$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$(1-\delta)/2$	$m+\delta$	$m+\epsilon$	m	l	$l+\delta-1$	$l+\epsilon$

with $\delta \in \{0, 1\}$ and $\epsilon \in \{-1, 0, 1\}$.

The solutions to these equations that can have grading greater than or equal to -1 are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(O-4)	0	-1	1	0	0	0	0	0
(O-4)	$\frac{1}{2}$	-1	0	0	0	1	0	1
(O-4)	$\frac{1}{2}$	-1	1	1	1	1	0	1
(O-4)	$\frac{1}{2}$	-1	1	0	1	1	0	0

(The last case is an indeterminate near-chord.)

$\boldsymbol{X_C \to {}_C X}$ This is related to the case ${}_C X \to X_C$ by rotating the diagram $180°$. Again, the solutions are all of type (O-1).

$\boldsymbol{X_C \to X_C}$ This is related to the case ${}_C X \to {}_C X$ by rotating the diagram $180°$. The solutions are idempotents and near-chords of types (O-1) and (O-6).

$\boldsymbol{X_C \to Y}$ This is related to the case ${}_C X \to Y$ by rotating the diagram $180°$. The solutions are of types (O-1) and (O-4).

$\boldsymbol{Y \to {}_C X}$ In this case we have

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'} = -1.$$

The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{4}(-n_\sigma + n_{\sigma_-} + n_{\sigma'_+} - 2n_{\sigma'} + n_{\sigma'_-}) = \tfrac{1}{2}(n_{\sigma'_+} - n_{\sigma'}).$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$-\delta/2$	$m+\delta-1$	$m+\epsilon$	m	l	$l+\delta$	$l+\epsilon$

with $\delta \in \{0,1\}$ and $\epsilon \in \{-1,0,1\}$.

The solutions which can have grading greater than or equal to -1 are

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(O-2)	$-\frac{1}{2}$	-1	0	0	0	0	1	0
(O-3)	0	-1	0	1	1	0	0	0
(O-3)	0	-1	0	1	1	1	1	1
(O-3)	0	-1	0	0	1	1	1	0

(The last case is an indeterminate near-chord.)

$Y \to X_C$ This is related to the case $Y \to {}_C X$ by rotating the diagram $180°$. The solutions are of types (O-2) and (O-3).

$Y \to Y$ In this case we have

$$n_\sigma - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'_-} = 0,$$
$$n_{\sigma_+} - n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'} = 0,$$
$$n_{\sigma_+} - n_\sigma = n_{\sigma'} - n_{\sigma'_-} = 0.$$

(As in the under-slide case, the last equations come from the fact that the B strand is not occupied in either idempotent on either side.) The correction to the grading is given by

$$c(I, \mathrm{supp}(a)) + c(J, \mathrm{supp}(a)) = \tfrac{1}{2}(-n_\sigma + n_{\sigma_-} + n_{\sigma'_+} - n_{\sigma'}) = -n_\sigma + n_{\sigma_-}.$$

The linear equations tell us that the multiplicities are

Corr	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
$-\epsilon$	$m+\epsilon$	$m+\epsilon$	m	l	$l+\epsilon$	$l+\epsilon$

with $\epsilon \in \{-1, 0, 1\}$.

The solutions which can have grading greater than or equal to -1 are:

Type	Corr	Grading	n_{σ_+}	n_σ	n_{σ_-}	$n_{\sigma'_+}$	$n_{\sigma'}$	$n_{\sigma'_-}$
(O-1)	0	-1	1	1	1	0	0	0
(O-1)	0	-1	0	0	0	1	1	1
(O-1)	0	-1	1	1	1	1	1	1
Dischord	-1	0	0	0	1	1	0	0
(O-5), (O-7), (O-8)	-1	-1	0	0	1	1	0	0
(O-5)	-1	-1	1	1	2	1	0	0
(O-5)	-1	-1	0	0	1	2	1	1

Here there is a grading 0 solution, the dischord (with support $[c_2, c_1] \times [c_2, c_1]$), which can be modified without changing the multiplicities near σ or σ' in a variety of ways: introducing a break in the support (type (O-5)), introducing a break in the chord on one side without changing the support (type (O-7)), or adding a new chord somewhere else, either overlapping with the existing support or not (type (O-8)).

This is the end of the case analysis. It is straightforward (if somewhat tedious) to verify that every near-chord for over-slides appears in the list of grading -1 elements. The idempotents and dischords were exactly the grading 0 elements which occurred in the case analysis, and no positive grading elements occurred in the case analysis. This concludes the proof. □

Lemma 4.33 *If N is an arc-slide bimodule then the only algebra elements which appear in the differential are near-chords.*

Proof This is an immediate consequence of the definition of the grading on the coefficient algebra and Lemma 4.32. □

Lemma 4.34 *Let N be a stable arc-slide bimodule for an over-slide. Then the differential on N contains all near-chords of type (O-1).*

Proof The proof follows along the lines of Lemma 4.20.

We define the *outside region* and *big enough* as in the proof of that lemma; and by stability, we restrict attention to the case where the pointed matched circle is big enough. By stabilizing, we can further assume that any special length-three chord which is adjacent to one of b_1 or b_1' is not adjacent to the basepoint.

Now the arguments proving Lemma 4.20 give the following sequence of claims:

Claim 1 *If x is a near-chord of type (O-1) and the support of x is disjoint from b_1 and b_1' then x appears in the differential.*

The case of special length-three chords are handled as they were in the proof of Proposition 3.8. (There is no analogue of very special length-three chords in the over-slide case.)

Claim 2 *Suppose x is any near-chord of type (O-1) with the following properties:*

- *The support of x contains exactly one of b_1 or b_1'.*
- *The position at b_1 or b_1' (whichever is contained in the support of x) is occupied in both the initial and terminal idempotents for x (on the $\mathcal{Z}$ or the $\mathcal{Z}'$ side, respectively).*

Then x appears in the differential.

Claim 3 *If x is a type (O-1) chord with restricted support of length 1, then x appears in the differential.*

This is the analogue of Claim 5 from the proof of Lemma 4.20, and its proof is similar (eg establishing first the case where $dx \neq 0$).

We turn next to the inductive proof that longer near-chords of type (O-1) appear. We call a near-chord x of type (O-1) *simplifiable* if dx contains terms of the form $y \cdot t$ with the following properties:

- Each of y and t is of type (O-1).
- The product $y \cdot t$ has no alternative factorization.
- The product $y \cdot t$ does not appear in the differential of any other basic generator of the near-diagonal subalgebra.

The proof has one extra complication for degenerate over-slides (Definition 4.5), so we separate out that case.

Nondegenerate case In this case, all near-chords of type (O-1) either have restricted support of length-one (so they appear in the differential by Claim 3); they are special length-three chords (which can be shown to appear as they were in the proof of Proposition 3.8); or they are simplifiable, in which case we can apply induction on the length of the restricted support of the near-chord, to conclude that all near-chords of type (O-1) appear in the differential.

Degenerate case There are two subcases, depending on whether the unique position between c_1 and c_2 is b_2.

Subcase: b_2 lies between c_1 and c_2 In this case, there is one more distinguished type of near chords: chords x of type (O-1) such that:

- x has length 5 on both $\mathcal{Z}$ and $\mathcal{Z}'$.
- x contains all of c_1, c_2, and b_2 in the interior of its support (and hence has restricted length three).
- The initial idempotent of x has type Y (and hence so does the terminal idempotent of x).

We call such a near-chord *extra special*. Under the present assumptions, there are three kinds of nonsimplifiable chords of type (O-1): those which have length-one restricted support (which appear in the differential by Claim 3), those which are special length-three chords (which appear in the differential by the usual arguments), and those which are extra special. (Chords like extra special ones but with idempotent of type X do not need a special argument, but see the first line of Figure 30.)

We must argue that every extra-special near-chord x appears in the differential. Note that there is a chord y of type (O-2) with $x \cdot y \neq 0$. Now, $x \cdot y$ has a unique alternate factorization, which has the form $x \cdot y = y' \cdot x'$ (with $\mathrm{supp}(x) = \mathrm{supp}(x')$ and $\mathrm{supp}(y) = \mathrm{supp}(y')$, so that x' is of type (O-1) and y' of type (O-2)). Moreover, the term in dx' corresponding to the crossing at b_2 factors as a product of two (not extra special) near-chords of type (O-1), so x' appears in the differential. The near-chord y'

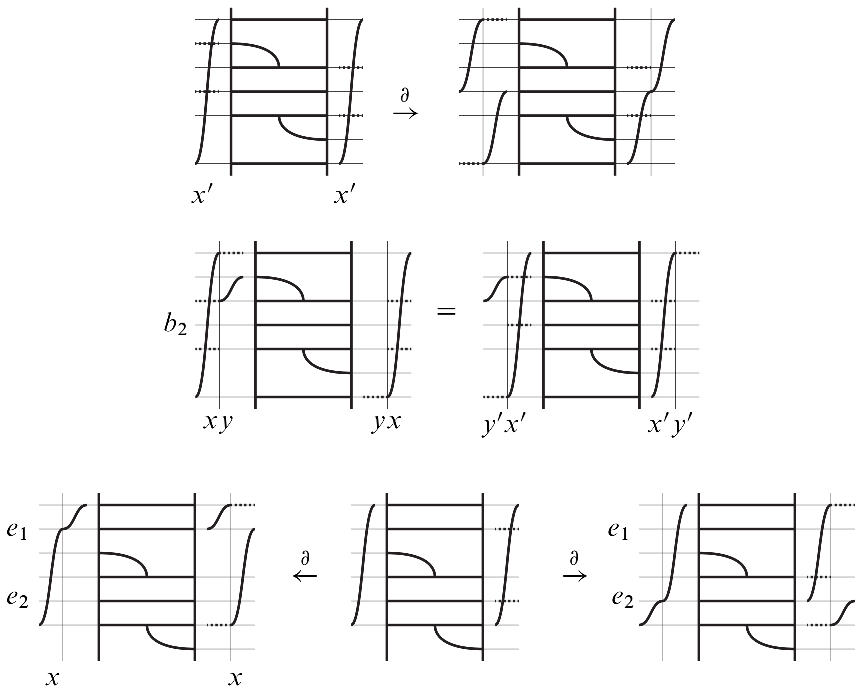


Figure 30: *Existence of extra-special chords* x The chord in the top line is not extra-special: the idempotent has type X. The second and third lines demonstrate the existence of extra-special chords in the cases that the position between c_1 and c_2 is or is not b_2, respectively.

appears in the differential by hypothesis. Moreover, $x \cdot y$ does not appear in the differential of any other algebra element, so it follows that x appears in the differential. See the second line of Figure 30.

Now induction on the length of the restricted support establishes the proposition in this case.

Subcase: Some $e_2 \neq b_2$ lies between c_1 and c_2 If the position e_1 matched with e_2 is adjacent to neither b_1 nor b_1' then the same argument as in the nondegenerate case applies. Special attention is needed when the position e_1 matched with e_2 is adjacent to either b_1 or b_1'. Assume for definiteness that e_1 is above and adjacent to b_1; the other cases are similar. In this case, a near chord x of type (O-1) is called *extra special* if its support is $[e_1, c_2]$. For an extra special near chord x, $dx = 0$.

Under the current assumptions, any near-chord x of type (O-1) either has restricted support of length one (in which case x appears in the differential by Claim 3), is a special length-three chord (in which case x appears in the differential by the argument from Proposition 3.8), is an extra-special chord, or is simplifiable. Extra special chords appear in the differential by the same argument used for special length-three chords. (See the last line in Figure 30.) So, again, induction on the length of the restricted support establishes the proposition. □

By hypothesis, the differential on N contains all near-chords of type (O-2).

Lemma 4.35 *Let N be a stable arc-slide bimodule for an over-slide. Then the differential on N contains all nonindeterminate near-chords of types (O-3) and (O-4).*

Proof We describe the proof; but indeed, most of it is encapsulated in Figure 31, and it runs parallel to the proof of the corresponding fact in the under-slide case (Lemma 4.22).

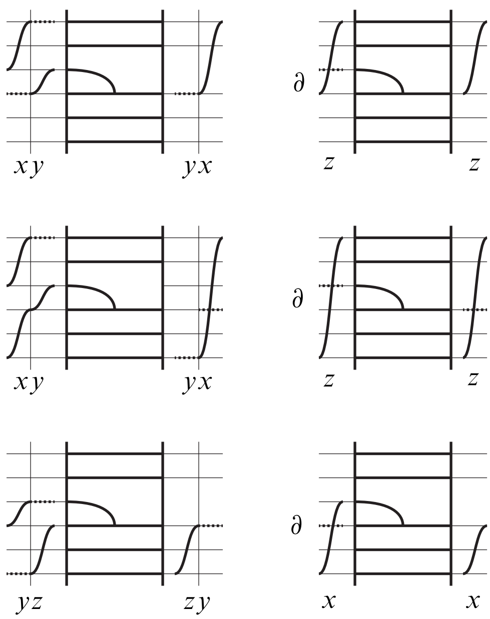


Figure 31: The three cases of Lemma 4.35

For any near-chord x of type (O-4), we can find a near-chord y of type (O-2) with the property that $x \cdot y$ appears in the differential of a near-chord z of type (O-1), as in the top two rows of Figure 31. This product $x \cdot y$ has no alternative factorization as a product of two near-chords with nontrivial support. (This is where we use that x is not indeterminate.) More specifically, the case where x has only two moving chords is obvious. In the case where x has three moving chords, $x \cdot y$ has three moving chords as well. There is exactly one other factorization of $x \cdot y$, into a near-chord x' of type (O-4) and another y' of type (O-3) whose product $x' \cdot y'$ has the same three moving chords as $x \cdot y$. (See Figure 32.) However, a closer look at the idempotents shows that $x \cdot y$ does not equal $x' \cdot y'$. Specifically, suppose, without loss of generality, that two of the moving strands of x are on the $\mathcal{Z}$–side. Then, since x is not indeterminate, the initial idempotent of x contains C on the $\mathcal{Z}'$–side (x has type $X_C \to Y$). On the other hand, the initial idempotent of x' contains C on the $\mathcal{Z}$ side (x' has type ${}_CX \to Y$). (Again, see Figure 32.)

Since, by Lemma 4.34, z appears in the differential, we conclude that x must, as well.

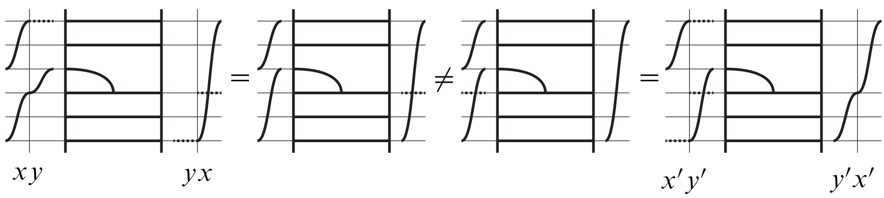


Figure 32: *Not an alternative factorization* One of the cases in Figure 31 might appear to have an alternate factorization; however, a more careful look at idempotents (as indicated) shows that this alternative factorization does not exist.

To prove the other half of the lemma, consider a near-chord x of type (O-3) which is not indeterminate. Then, the differential of x contains an element which factors as $y \cdot z$, where y is of type (O-2) and z is of type (O-1), as in the third line of Figure 31. (This is where we use the fact that x is not indeterminate.) That factorization is unique in the near-diagonal subalgebra, and $y \cdot z$ does not appear in the differential of any other algebra element. According to Lemma 4.34, $y \cdot z$ appears in ∂^2, so x must appear in ∂. $\square$

Lemma 4.36 *Let N be a stable arc-slide bimodule for an over-slide. Then the differential on N contains all near-chords of types (O-5) and (O-6).*

Proof The proof is illustrated in Figure 33.

Let x be a near-chord of type (O-5). Postmultiply x by a near-chord y of type (O-2). The resulting algebra element $x \cdot y$ does not appear in the differential of any other algebra element. Moreover, $x \cdot y$ has a unique alternative factorization as a product of two elements of the near-diagonal subalgebra, $x' \cdot y'$, where x' has type (O-3), y' has type (O-1), and the boundary of y' meets C. (In the case where the chords ξ and η are neither disjoint nor nested, this alternative factorization does not make sense. In fact, if x is analogous to a type (O-5) near-chord, except that the chords ξ and η are

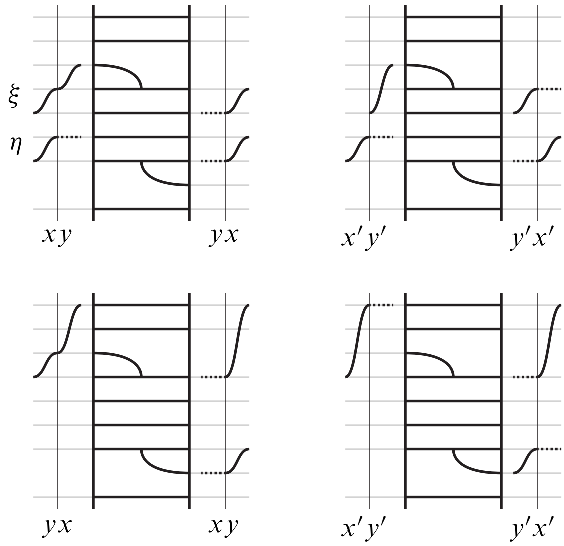


Figure 33: *Proof of Lemma 4.36* The top line is for near-chords of type (O-5). The bottom line is for near-chords of type (O-6).

neither disjoint nor nested, then x already admits a factorization into near-chords.) This is illustrated in the first row of Figure 33.

If x is of type (O-6), we premultiply x by a near-chord y of type (O-2). The resulting algebra element $y \cdot x$ has an alternative factorization as $x' \cdot y'$ where y' is of type (O-2) and x' is of type (O-1). See the second row of Figure 33. □

Next, we turn our attention to the indeterminate near-chords.

Lemma 4.37 *Let N be a stable arc-slide bimodule for an over-slide. Which indeterminate near-chords of type (O-4) appear in the differential is uniquely determined by which indeterminate near-chords of type (O-3) appear.*

Proof Let x be an indeterminate near-chord of type (O-4). We can find a near-chord y of type (O-2) with the property that $x \cdot y$ appears in the differential of a near-chord z of type (O-1). The term $x \cdot y$ has a unique alternative factorization as $x' \cdot y'$, where x' is of type (O-4) (but x' is not indeterminate), and y' is an indeterminate near-chord of type (O-3). Since dz appears in ∂^2, it follows that the term y' appears in the differential if and only if x does not. See Figure 34 for an illustration. □

Lemma 4.37, or rather its proof, can be used to establish Lemma 4.28. To this end, note that an indeterminate near-chord of type (O-3) has the form $\flat \cdot (\sigma \otimes 1)$ or $\flat \cdot (1 \otimes \sigma')$, where $\flat$ is a dischord (in the sense of Definition 4.31).

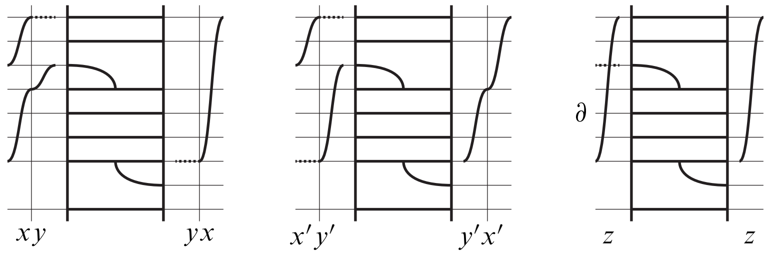


Figure 34: The three types of terms in Lemma 4.37

Proof of Lemma 4.28 Our aim is to show the following: for each dischord $\flat$, the indeterminate near-chord $y = \flat \cdot (\sigma \otimes 1)$ appears in the differential of a stable arc-slide bimodule if and only if $y' = \flat \cdot (1 \otimes \sigma')$ does not. (Again, we are using Convention 4.1, bearing in mind Remark 4.19.)

After stabilizing, we work in a portion of the algebra where the number of occupied strands exceeds the number of positions between c_1 and c_2. We also stabilize so that there is some position above b_1.

Consider first the case where the initial idempotent I of y (ie the basic idempotent I so that $I \cdot y = y$) contains some occupied position in $\mathcal{Z}$ which is above b_1. In this case, the proof of Lemma 4.37 gives an indeterminate near-chord x of type (O-4) which appears in the differential precisely if y does not. (The initial idempotent of y differs from the initial idempotent of x in only one position.) Now, $x \cdot (1 \otimes \sigma')$ does not appear in the differential of any element, and it has a unique alternative factorization as $x' \cdot y'$, where x' is a (not indeterminate) near-chord of type (O-4) which appears in the differential, by Lemma 4.35. It follows that y' appears in the differential if and only if x does, which in turn appears in the differential if and only if y does not. See Figure 35 for an illustration.

Suppose instead that in I there is no occupied position above b_1. By our hypothesis on the total number of occupied positions, this forces there to be some occupied position d_1 below c_2. Thus, we can find a near-chord x of type (O-1) (connecting d_1 to some position above b_1) so that:

- $y \cdot x$ is nonzero, and has a unique alternate factorization as $x_2 \cdot y_2$, where x and x_2 have the same support, and y and y_2 have the same support.
- $y' \cdot x$ is nonzero, and has a unique alternate factorization as $x_2 \cdot y_2'$.
- $y \cdot x$ and $y' \cdot x$ do not appear in the differential of any other basic algebra element.

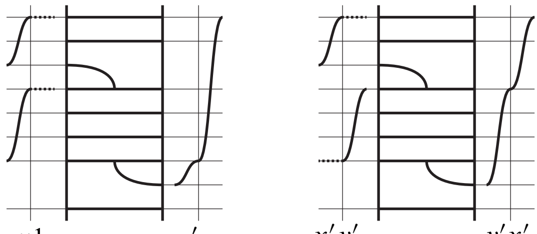


Figure 35: *Proof of Lemma 4.28* The existence of the term on the left in ∂^2 is determined by the appearance in the differential of a corresponding near-chord y of type (O-3) according to Lemma 4.37; the second factorization $(x' \cdot y')$ is determined by the existence of a different near-chord y' which is of type (O-3).

- The initial idempotent of y_2 (which is the same as the initial idempotent of y_2') contains some occupied position in $\mathcal{Z}$ which is above b_1.

In particular, the pair y_2, y_2' fit into the case of the lemma that we already established, so y_2 is contained in the differential if and only if y_2' is not. By $\partial^2 = 0$ we see that y is contained in the differential if and only if y_2 is, and y' is contained in the differential if and only if y_2' is. It follows that y is contained in the differential if and only if y' is not, as claimed. □

Using the terminology of Definition 4.29, Lemma 4.37 says that for a stable arc-slide bimodule, its basic choice $\mathcal{B}$ uniquely specifies which indeterminate near-chords of type (O-4) appear in the differential. In the same vein, we have:

Lemma 4.38 *If N is a stable arc-slide bimodule then its basic choice $\mathcal{B}$ uniquely specifies which indeterminate near-chords of type (O-7) appear in the differential on N.*

Proof We consider terms in ∂^2 which have support, say, $[c_2, b_1] \times [c_2, c_1]$. Let α denote the sum of all terms of type (O-7) which appear in the differential. The terms in ∂^2 coming from near-chords are of the following types:

- Terms of type (O-3) times terms of type (O-1).
- Terms of type (O-7) times terms of type (O-2).
- Differentials of indeterminate near-chords of type (O-3).

See Figure 36 for an illustration. Since any near-chord of type (O-7) has nontrivial

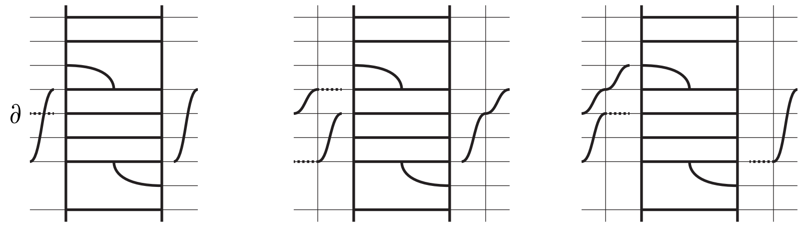


Figure 36: *The three types of terms in Lemma 4.38* On the left, we have a type (O-3) near-chord; in the center, we have the product of a type (O-3) near-chord and a type (O-1) near-chord; on the right, we have the product of a type (O-7) near-chord and a type (O-2) near-chord.

product with any near chord of type (O-2), it follows at once that the set of elements of type (O-7) are determined by the terms of type (O-3) which appear in the differential; and this latter is the basic choice. □

Lemma 4.39 *If N is a stable arc-slide bimodule then its basic choice $\mathcal{B}$ uniquely specifies which near-chords of type (O-8) appear in the differential on N.*

Proof Suppose x is a near-chord of type (O-8). We can find a near-chord y of type (O-2) so that $x \cdot y$ has exactly two alternate factorizations: $x \cdot y = w \cdot z = z' \cdot w'$, where w and w' are indeterminate of type (O-3) and z and z' are of type (O-1). Moreover, $x \cdot y$ does not appear in the differential of any other algebra element. From this (and Lemma 4.34), we see that x appears in the differential if exactly one of w or w' appears in the differential. See Figure 37 for an illustration. □

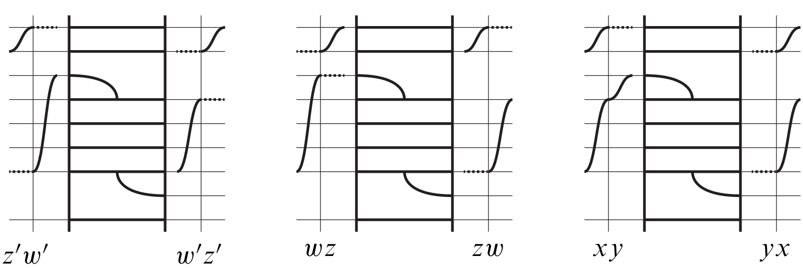


Figure 37: *The three types of terms in Lemma 4.39* On the left, we have $z' \cdot w'$; in the center, we have $w \cdot z$; on the right, we have $x \cdot y$.

Lemma 4.40 *Let x be a near-chord and y be a dischord. If $x \cdot y$ is nonzero then $x \cdot y$ is a near-chord as well, and similarly for $y \cdot x$.*

Proof This is a simple case analysis on the type of x.

If x has type (O-1) and $x \cdot y \neq 0$ then $x \cdot y$ has type (O-8). If x has type (O-2) and $x \cdot y \neq 0$ then $x \cdot y$ has type (O-3). If x has type (O-4) and $y \cdot x \neq 0$ then $y \cdot x$ has type (O-3). All other cases vanish.

(Alternately, the result follows from noting that the grading of $x \cdot y$ in the near-diagonal subalgebra must be -1, and so by Lemma 4.32 $x \cdot y$ must be a near-chord.) $\square$

Proof of Proposition 4.30 Let N be a stable arc-slide bimodule and let $\mathcal{B}$ be its associated basic choice (whose existence is guaranteed by Lemma 4.28). By hypothesis, near-chords of type (O-2) exist in the differential; combining this with Lemmas 4.34, 4.35 and 4.36, we conclude that all the near-chords which are not indeterminate appear in the differential. According to Lemmas 4.37, 4.38 and 4.39, the basic choice $\mathcal{B}$ uniquely determines all the indeterminate near-chords which contribute to the differential. According to Lemma 4.33 no other terms can contribute to the differential. In sum, the basic choice $\mathcal{B}$ uniquely determines N.

Next, let N be compatible with a basic choice $\mathcal{B}$, and let $\mathcal{B}'$ be a different basic choice. Then we can find a sum of dischords $\mathcal{Q}$ with the property that

$$\mathcal{B} + \mathcal{B}' = \mathcal{Q} \cdot (a(\sigma) \otimes 1 + 1 \otimes a(\sigma')), \tag{4-10}$$

where in the above equation we do not distinguish between a basic choice $\mathcal{B}$ and its associated algebra element

$$\sum_{b \in \mathcal{B}} b.$$

We use $\mathcal{Q}$ to construct a new bimodule N', with the same generators as N and differential given by

$$\partial'_N = (\mathbb{I} + \cdot\mathcal{Q}) \circ \partial_N \circ (\mathbb{I} + \cdot\mathcal{Q}). \tag{4-11}$$

Here, $\mathbb{I}$ denotes the identity map $N \to N$ while $\cdot\mathcal{Q}$ denotes the map induced by $a \cdot \boldsymbol{x}(I) \mapsto a \cdot \mathcal{Q} \cdot \boldsymbol{x}(I)$ (where $\boldsymbol{x}(I)$ is the generator of N corresponding to the near-complementary idempotent I). Since $(\mathbb{I} + \cdot\mathcal{Q})^2 = \mathbb{I}$ (as $\mathcal{Q}^2 = 0$), it follows that ∂'_N is a differential. Note that N' is also an arc-slide bimodule: Properties (AS-1) and (AS-2) are clear; Property (AS-3) follows from the fact that the elements appearing in ∂'_N are elements of the near-diagonal subalgebra with grading -1 (see Lemma 4.40); Property (AS-4) continues to hold since the operation of replacing ∂_N by ∂'_N does not affect the short chords which appear in the differential. Similarly, N' is also stable. Now, if N is compatible with $\mathcal{B}$ then, according to (4-10), N' is compatible with $\mathcal{B}'$. Moreover, the map $f\colon N \to N'$ induced by

$$(\mathbb{I} + \cdot\mathcal{Q})\colon N \to N'$$

is an isomorphism of chain complexes. □

4.6 Arc-slide bimodules

We put together the results (Propositions 4.16 and 4.30) from the previous sections to deduce Theorem 2. First, we have the following:

Proof of Proposition 1.10 For under-slides, this is Proposition 4.16.

For over-slides, this is Proposition 4.30. (Recall that the exact form of the differential is then determined by the basic choice, but by Proposition 4.30, all of these choices give isomorphic bimodules.) The result follows. □

And next:

Proof of Theorem 2 This is immediate from the fact that $\widehat{CFDD}(\mathcal{H}(m))$ is a stable arc-slide bimodule (Proposition 4.7) and the fact that all such bimodules are isomorphic (Proposition 1.10). Moreover, it follows from [5, Corollary 8.1 and Lemma 8.15] that the homotopy equivalence $\widehat{\mathcal{DD}}(m) \cong (\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z}')) \boxtimes \widehat{CFDD}(F^\circ(m))$ is unique up to homotopy. □

Remark 4.41 It is worth noting that the above proof in fact performs all the holomorphic curve counts for $\widehat{CFDD}(\mathcal{H}(m))$ when m is an under-slide. This is not the case for over-slides. For instance, disconnected domains never contribute to a differential in $\widehat{CFDD}(\mathcal{H}(m))$. This can be used to give a criterion which ensures that certain near-chords of type (O-8) do not contribute to the differential on $\widehat{CFDD}(\mathcal{H}(m))$ (although one can arrange that they do for other stable arc-slide bimodules). However, this point is irrelevant for our purposes: we are interested in $\widehat{CFDD}(\mathcal{H}(m))$ only up to homotopy equivalence.

5 The genus-one case

In this section, we illustrate the computations of the arc-slide bimodules by spelling out the answers explicitly in the genus-one case. We will focus on the part of the algebra with weight zero (ie one moving strand), as $\mathcal{A}(T^2,-1) = \mathbb{F}_2$ and $\mathcal{A}(T^2,1) \simeq \mathbb{F}_2$, and so the bimodules over these algebras are not very interesting (free of rank 1 with trivial differential, when viewed as bimodules modules over $\mathbb{F}_2$).

The (unique) pointed matched circle for the torus has 4 matched points, $1,2,3,4$, with 1 matched to 3 and 2 matched to 4. Let ι_0 denote the idempotent in $\mathcal{A}(T^2,0)$

corresponding to $\{1,3\}$ and ι_1 the idempotent corresponding to $\{2,4\}$. Let ρ_i denote the (short) chord from i to $i+1$, so

$$\iota_0\rho_1\iota_1 = \rho_1, \quad \iota_1\rho_2\iota_0 = \rho_2, \quad \iota_0\rho_3\iota_1 = \rho_3.$$

Let

$$\rho_{12} = \rho_1\rho_2, \quad \rho_{23} = \rho_2\rho_3, \quad \rho_{123} = \rho_1\rho_2\rho_3.$$

Of course, we are considering two copies of $\mathcal{A}(T^2)$. In the second copy, we will denote the idempotents corresponding to ι_0 and ι_1 by j_0 and j_1, and the chord corresponding to ρ_i by σ_i.

With this notation in hand, consider the *DD* identity module, which was already computed in this case in [5, Proposition 10.1]. The module $\widehat{CFDD}(\mathbb{I})$ has two generators, $p = (\iota_0 \otimes j_0)$ and $q = (\iota_1 \otimes j_1)$, and differential

$$\partial \boldsymbol{p} = (\rho_1\sigma_3 + \rho_3\sigma_1 + \rho_{123}\sigma_{123}) \otimes \boldsymbol{q}, \quad \partial \boldsymbol{q} = (\rho_2\sigma_2) \otimes \boldsymbol{p}.$$

This is in obvious agreement with Theorem 1. Note that the term $\rho_{123}\sigma_{123}$ comes from a special length-three chord, and is not forced by $\partial^2 = 0$.

Next we turn to the arc-slide bimodules. The genus-1 mapping class group is generated by Dehn twists τ_m, τ_l around two curves m, l in T^2. We can view τ_m and τ_l as underslides; see Figure 38. We give the answers, and then explain the terms.

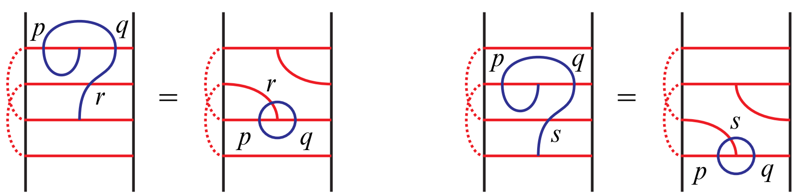


Figure 38: *Generators of the genus-one mapping class group* This figure may be compared with [5, Figure 25].

The module $\widehat{CFDD}(\tau_m)$ has three generators, $p = \iota_0 \otimes j_0$, $q = \iota_1 \otimes j_1$ and $r = \iota_1 \otimes j_0$. The differential is given by

$$\begin{aligned}
\partial(p) &= \big(\overset{\text{(U-1)}}{\rho_1\sigma_3} + \overset{\text{(U-1)}}{\rho_{123}\sigma_{123}}\big) \otimes q + \big(\overset{\text{(U-4)}}{\rho_3\sigma_{12}}\big) \otimes r, \\
\partial(q) &= \big(\overset{\text{(U-4)}}{\rho_{23}\sigma_2}\big) \otimes r, \\
\partial(r) &= \big(\overset{\text{(U-2)}}{\rho_2}\big) \otimes p + \big(\overset{\text{(U-2)}}{\sigma_1}\big) \otimes q.
\end{aligned}$$

The type of the near-chord corresponding to each term is indicated above it. There are no contributions from chords of types (U-3), (U-5) or (U-6).

The terms $\rho_1\sigma_3$, ρ_2 and σ_1 are short near-chords, and so appear by hypothesis / directly counting holomorphic curves. The remaining terms are not forced by $\partial^2 = 0$ in this diagram, but instead follow from the fact that our bimodule is stable, together with $\partial^2 = 0$ on a bigger diagram.

Similarly, the module $\widehat{CFDD}(\tau_l)$ has three generators, $p = \iota_0 \otimes j_0$, $q = \iota_1 \otimes j_1$ and $s = \iota_0 \otimes j_1$. The differential is given by

$$\begin{aligned}
\partial(p) &= \big(\overset{\text{(U-1)}}{\rho_3\sigma_1} + \overset{\text{(U-1)}}{\rho_{123}\sigma_{123}}\big) \otimes q + \big(\overset{\text{(U-4)}}{\rho_{12}\sigma_3}\big) \otimes s,\\
\partial(q) &= \big(\overset{\text{(U-4)}}{\rho_2\sigma_{23}}\big) \otimes s,\\
\partial(s) &= \big(\overset{\text{(U-2)}}{\sigma_2}\big) p + \big(\overset{\text{(U-2)}}{\rho_1}\big) \otimes q.
\end{aligned}$$

(The modules $\widehat{CFDA}(\tau_m)$ and $\widehat{CFDA}(\tau_l)$ were computed directly in [5, Section 10.2]. Tensoring these modules with $\widehat{CFDD}(\mathbb{I})$ gives another computation of $\widehat{CFDD}(\tau_m)$ and $\widehat{CFDD}(\tau_l)$.)

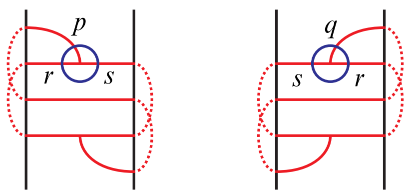


Figure 39: Genus-one overslides

There are also two overslides in genus 1; they are shown in Figure 39. The bimodule $\widehat{CFDD}$ associated to the overslide on the left of Figure 39 has three generators, $p = \iota_0 \otimes j_0$, $r = \iota_1 \otimes j_0$ and $s = \iota_0 \otimes j_1$, and differential given by

$$\begin{aligned}
\partial(p) &= \big(\overset{\text{(O-2)}}{\rho_3} + \overset{\text{(O-3)}}{\rho_{123}\sigma_{12}}\big) \otimes r + \big(\overset{\text{(O-2)}}{\sigma_3} + \overset{\text{(O-3)}}{\rho_{12}\sigma_{123}}\big) \otimes s,\\
\partial(r) &= \big(\overset{\text{(O-1)}}{\rho_2\sigma_1}\big) \otimes s,\\
\partial(s) &= \big(\overset{\text{(O-1)}}{\rho_1\sigma_2}\big) \otimes r,
\end{aligned}$$

with the understanding that exactly one of the terms labeled (O-3) occurs.

The (O-1) and (O-2) terms are short near-chords, and so occur by hypothesis / direct computation. Both (O-3) terms are indeterminate; the fact that exactly one of them occurs is part of the definition of a basic choice (Definition 4.27). (In terms of holomorphic curves, the (O-3) near-chords do not follow from $\partial^2 = 0$ in this diagram, but do follow from $\partial^2 = 0$ after stabilizing; see the proof of Lemma 4.28.)

We can switch between basic choices by making the change of variables $p' = (1 + \rho_{12}\sigma_{12}) \otimes p$ (compare (4-11)).

For completeness, the bimodule corresponding to the overslide on the right has generators $q = \iota_1 \otimes j_1$, $r = \iota_1 \otimes j_0$ and $s = \iota_0 \otimes j_1$, and differential given by

$$\begin{aligned}
\partial(q) &= 0,\\
\partial(r) &= \big(\overset{\text{(O-1)}}{\rho_2\sigma_3}\big) \otimes s + \big(\overset{\text{(O-2)}}{\sigma_1} + \overset{\text{(O-3)}}{\rho_{23}\sigma_{123}}\big) \otimes q,\\
\partial(s) &= \big(\overset{\text{(O-1)}}{\rho_3\sigma_2}\big) \otimes r + \big(\overset{\text{(O-2)}}{\rho_1} + \overset{\text{(O-3)}}{\rho_{123}\sigma_{23}}\big) \otimes q,
\end{aligned}$$

again with the understanding that exactly one of the terms labeled (O-3) contributes. (Which one contributes is determined by the basic choice; bear in mind that this diagram is mirror to Convention 4.1.)

6 Gradings on bimodules for arc-slides and mapping classes

In this section, we discuss further the gradings on the type-*DD* modules associated to arc-slides. The main goal is to compute explicitly the gradings on $\widehat{CFDD}(F^\circ(m))$, the bimodule associated to an arc-slide m; this is done in Section 6.1. As we explain in Section 7, this allows one to compute both the (relative) Maslov gradings on $\widehat{HF}(Y)$ and the decomposition of $\widehat{HF}(Y)$ into spin^c–structures. Section 6.2 is a brief digression to compute the grading sets for general surface homeomorphisms; Section 6.2 is not needed for the rest of the paper, but answers a question which arises naturally.

6.1 Gradings on arc-slide bimodules

In Section 6.1.1, we finish computing the gradings of periodic domains for the standard Heegaard diagrams for arc-slides; this was begun in Section 4.3. In Section 6.1.2, we give the grading set for $\widehat{CFDD}(F^\circ(m))$ (ie the range of the grading function), with respect to both the big and small grading groups. In Section 6.1.3 we compute the gradings of generators of $\widehat{CFDD}(F^\circ(m))$, ie the grading function itself.

In this section, we will work with the bimodules $\widehat{CFDD}(F^\circ(m))$ associated to a Heegaard diagram, rather than a general arc-slide bimodule $\widehat{\mathcal{DD}}(m)$. No generality is lost, according to the following:

Proposition 6.1 *Suppose that $\widehat{\mathcal{DD}}(m\colon \mathcal{Z} \to \mathcal{Z}')$ is a stable arc-slide bimodule, and that the actions of $G'(\mathcal{Z})$ and $G'(-\mathcal{Z}')$ on the grading set for $\widehat{\mathcal{DD}}(m)$ are free and transitive. Then the homotopy equivalence $\widehat{\mathcal{DD}}(m) \simeq \widehat{CFDD}(m)$ is a G'–set graded homotopy equivalence.*

Proof This follows from Proposition 4.13. □

6.1.1 Gradings of periodic domains Consider the standard Heegaard diagram $\mathcal{H}(m)$ associated to an arc-slide $m\colon \mathcal{Z} \to \mathcal{Z}'$. Label the matched pairs in $\mathcal{Z}$ by $B_1, \dots, B_{2k}$, with $B_{2k} = B$ (in the notation of Section 4).

There are various combinatorial cases for the arc-slide m. For the remainder of Section 6.1, we will assume that c_1 is above c_2 in $\mathcal{Z}$, as in Section 4; the other case is obtained by reflecting the diagrams horizontally, or equivalently by replacing m by m^{-1}. Then we have the following cases:

- Under-slides: b_1 is between c_1 and c_2. This case is further divided as follows:
  - (U.I) c_1 is between b_1 and b_2.
  - (U.II) b_2 is between c_1 and c_2.
  - (U.III) c_2 is between b_1 and b_2.
- Over-slides: b_1 is above c_1. This case is further divided as follows:
  - (O.I) b_2 is above c_1.
  - (O.II) b_2 is between c_1 and c_2 (so c_1 is between b_1 and b_2).
  - (O.III) c_1 and c_2 are both between b_1 and b_2.

These cases are illustrated in Figure 40.

We can find a basis for the space of periodic domains on $\mathcal{H}(m)$, $\pi_2(\boldsymbol{x}, \boldsymbol{x})$ (for any generator $\boldsymbol{x}$), given by elements $P_1, \dots, P_{2k}$ with the following properties. For $i < 2k$, $\partial P_i \cap \mathcal{Z}$ is the interval between the two points in B_i, and $\partial P_i \cap \mathcal{Z}'$ is the interval between the two points in B_i'. The domain P_{2k} is such that:

- (U.I) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_1, b_2]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([c_1, b_2]) - r([c_2, b_1'])$. (The region σ' is covered with multiplicity -1.)
- (U.II) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_2, b_1]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([b_2, c_1]) + r([c_2, b_1'])$.
- (U.III) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_2, b_1]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([b_2, c_1]) + r([c_2, b_1'])$. (The region σ' is covered with multiplicity two.)
- (O.I) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_1, b_2]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([b_1', c_2]) + r([c_1, b_2])$.
- (O.II) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_2, b_1]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([b_2, c_1]) - r([b_1', c_2])$. (The region σ' is covered with multiplicity -1.)
- (O.III) $\partial P_{2k} \cap \mathcal{Z}$ consists of $[b_2, b_1]$, while $\partial P_{2k} \cap \mathcal{Z}'$ consists of $r([b_2, b_1']) + r([c_2, c_1])$.

See Figure 40 for an illustration.

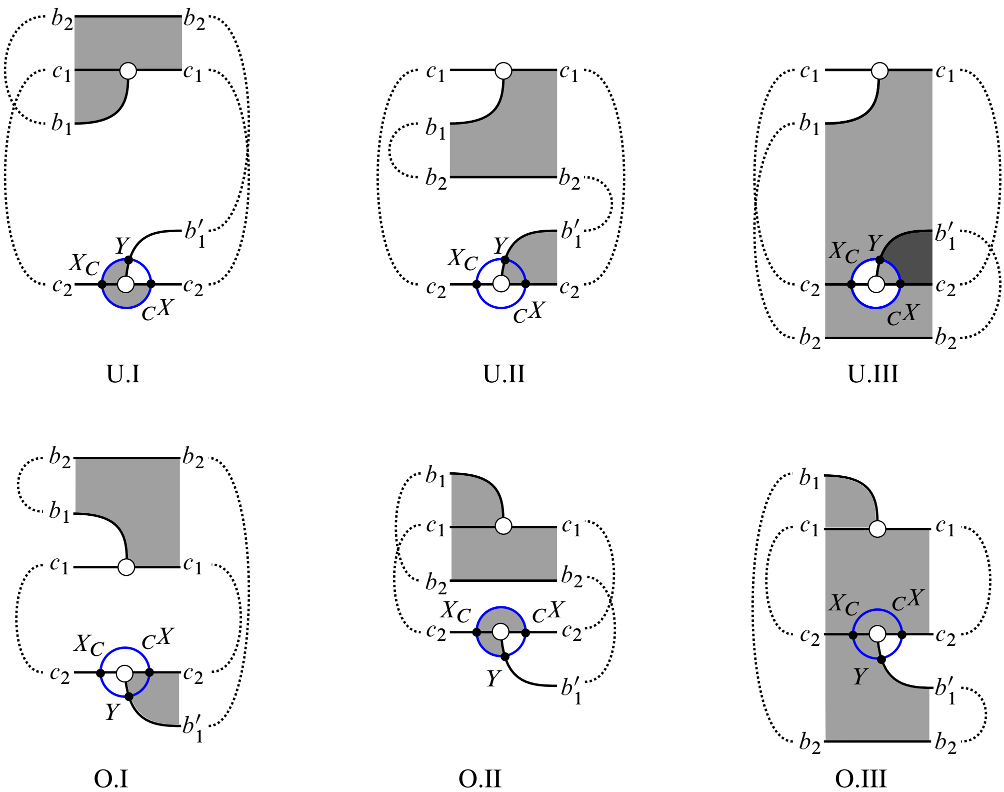


Figure 40: *Combinatorial cases of arc-slides* The domain P_{2k} is shaded in each diagram. Light gray shading indicates multiplicity 1, dark gray shading indicates multiplicity 2, and red, checkered shading indicates multiplicity -1.

As discussed in Section 2.3.4, for each generator $\boldsymbol{x}$ and each $P_i \in \pi_2(\boldsymbol{x}, \boldsymbol{x})$ there is an element

$$g'(P_i) = (-e(P_i) - 2n_{\boldsymbol{x}}(P_i), \partial^{\partial} P_i) \in G'(\mathcal{Z}) \times_{\mathbb{Z}} G'(-\mathcal{Z}').$$

We compute these elements $g'(P_i)$:

Lemma 6.2 *For the domain $P_i \in \pi_2(\boldsymbol{x}(I), \boldsymbol{x}(I))$ with corresponding matched pair B_i, the Maslov component of $g'(P_i)$ is:*

- 0 *if* $B_i \neq B, C$.
- 0 *if* $B_i = C$ *and* I *has type* ${}_CX$ *or* X_C.
- 1 *if* $B_i = C$, m *is an over-slide, and* I *has type* Y.
- -1 *if* $B_i = C$, m *is an under-slide, and* I *has type* Y.
- *Given by the tables below for* P_{2k}:

I has type X_C		I has type ${}_C X$		I has type Y	
Handleslide type	Maslov component	Handleslide type	Maslov component	Handleslide type	Maslov component
U.I	$-\frac{1}{2}$	U.I	$\frac{1}{2}$	U.I	$\frac{1}{2}$
U.II	$\frac{1}{2}$	U.II	$-\frac{1}{2}$	U.II	$-\frac{1}{2}$
U.III	$\frac{1}{2}$	U.III	$-\frac{1}{2}$	U.III	$-\frac{1}{2}$
O.I	$\frac{1}{2}$	O.I	$-\frac{1}{2}$	O.I	$-\frac{1}{2}$
O.II	$-\frac{1}{2}$	O.II	$\frac{1}{2}$	O.II	$\frac{1}{2}$
O.III	$-\frac{1}{2}$	O.III	$\frac{1}{2}$	O.III	$\frac{1}{2}$

Proof This follows by inspecting the periodic domains in Figure 40 and applying the calculations from Proposition 4.14: in the notation from that proposition, the Maslov component of $g'(P_i)$ is $2c(I(\boldsymbol{x}), \mathrm{supp}(a))$. □

6.1.2 The grading set and refined grading set Following [5, Section 6.5], as a G'–set graded module, the module $\widehat{CFDD}(\mathcal{H}(m))$ is graded by

$$G'(\mathcal{Z}) \times_{\mathbb{Z}} G'(-\mathcal{Z}')/\langle R(g'(P_i))\rangle,$$

where $\langle g'(P_i)\rangle$ denotes the subgroup of $G'(-\mathcal{Z}) \times_{\mathbb{Z}} G'(\mathcal{Z}')$ generated the gradings of the periodic domains, and R is the orientation-reversal map from Sections 2.3.4 and 2.4.3. The Maslov components of these gradings are computed in Lemma 6.2; the homology class component of $R(g'(P_i))$ is simply $r_*(\partial^\partial P_i)$.

In the notation of Section 6.1.1, since P_i is a periodic domain, $R(g'(P_i))$ lies in the smaller grading group $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}')$ [5, Lemma 6.29]. As a G–set graded module, the module $\widehat{CFDD}(\mathcal{H}(m))$ is graded by

$$S = G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}')/\langle R(g'(P_i))\rangle.$$

Let $H = \langle R(g'(P_i))\rangle$. The subgroup H is a complement to $G(\mathcal{Z})$ (respectively $G(-\mathcal{Z}')$), in the sense that $G(\mathcal{Z})$ (respectively $G(-\mathcal{Z}')$) intersects each coset of H exactly once. So, each of $G(\mathcal{Z})$ and $G(-\mathcal{Z}')$ act freely and transitively on S, ie S is a *principal left-right $G(\mathcal{Z})$–$G(\mathcal{Z}')$–set*. (Here we interpret the left action of $G(-\mathcal{Z}')$ as a right action of $G(\mathcal{Z}')$.) We spend the next few paragraphs reinterpreting S as a map.

Definition 6.3 Let G_1 and G_2 be isomorphic groups. Let $\mathrm{Isom}(G_1, G_2)$ denote the set of isomorphisms from G_1 to G_2. Then G_1 acts on $\mathrm{Isom}(G_1, G_2)$ as follows: For $\phi \in \mathrm{Isom}(G_1, G_2)$ and $g \in G_1$ define

$$(g \cdot \phi)(h) = \phi(g^{-1}hg).$$

Let $\mathrm{Out}(G_1, G_2) = G_1 \backslash \mathrm{Isom}(G_1, G_2)$. An element of $\mathrm{Out}(G_1, G_2)$ is an *outer isomorphism from G_1 to G_2*.

In Definition 6.3, one could alternatively take the quotient $G_2 \backslash \mathrm{Isom}(G_1, G_2)$, with the action given by $(g' \cdot \phi)(h) = g'\phi(h)(g')^{-1}$. This gives the same equivalence relation on $\mathrm{Isom}(G_1, G_2)$.

Composition of maps induces a map $\mathrm{Out}(G_1, G_2) \times \mathrm{Out}(G_2, G_3) \to \mathrm{Out}(G_1, G_3)$. The following lemma is straightforward:

Lemma 6.4 *For any (isomorphic) groups G_1 and G_2, there is a canonical bijection between isomorphism classes of principal left-right G_1–G_2–sets S and outer isomorphisms $\mathrm{Out}(G_1, G_2)$. Given a left-right G_1–G_2–set S, let $\phi_S \in \mathrm{Out}(G_1, G_2)$ denote its corresponding outer isomorphism under this canonical bijection. If S is a left-right principal G_1–G_2–set, and T is a left-right principal G_2–G_3–set, we can form the orbit space $S \times_{G_2} T$. This is a left-right principal G_1–G_3–set. Moreover, $\phi_{S \times_{G_2} T} = \phi_T \circ \phi_S$.*

Proof For a given principal left-right G_1–G_2–set S, pick any $x \in S$ and define ϕ_S by

$$g \cdot x = x \cdot \phi_S(g).$$

(This defines ϕ_S uniquely up to conjugation because the actions of both G_1 and G_2 on S are simply transitive.) The map ϕ_S is a homomorphism:

$$x \cdot \phi_S(gh) = (gh) \cdot x = (g \cdot x) \cdot \phi_S(h) = (x \cdot \phi_S(g)) \cdot \phi_S(h) = x \cdot \phi_S(g)\phi_S(h).$$

The map ϕ_S is also clearly bijective.

If we choose a different element $x' \in S$ to define ϕ_S, the isomorphism changes by an inner automorphism, as follows. Let ϕ'_S be the automorphism defined with respect to x'. Choose g'_0 so that $x' = g_0 x$. Then

$$x' \cdot \phi'_S(g) = g \cdot x' = g \cdot g_0 x = x \cdot \phi_S(g g_0) = (g_0)^{-1} x' \phi_S(g g_0) = x' \phi_S((g_0)^{-1} g g_0).$$

Thus the element $\phi_S \in \mathrm{Out}(G_1, G_2)$ is well defined.

We claim the assignment $S \mapsto \phi_S$ gives a bijection between the set of isomorphism classes of left-right transitive G_1–G_2–sets and the set $\mathrm{Out}(G_1, G_2)$. To invert this bijection, suppose that $\phi\colon G_1 \to G_2$ is a group isomorphism and consider the associated principal G_1–G_2–set S_ϕ defined to be the quotient of $G_1 \times G_2$ by the equivalence relation $(g_1 \cdot g, g_2) \sim (g_1, \phi(g) \cdot g_2)$ where $g \in G_1$. This quotient clearly retains its left-right G_1–G_2–set structure. We leave it as an exercise for the reader to verify the following facts:

- The quotient space S_ϕ defined above is indeed a transitive G_1–G_2–set.
- Given two isomorphisms ϕ and ϕ' from G_1 to G_2 representing the same outer isomorphism, there is a canonical isomorphism of left-right G_1–G_2–sets $S_\phi \cong S_{\phi'}$.
- Given any transitive G_1–G_2–set S, there is a canonical isomorphism $S_{\phi_S} \cong S$.
- Given an isomorphism $\phi\colon G_1 \to G_2$, $\phi_{S_\phi} = \phi$, as elements of $\mathrm{Out}(G_1, G_2)$.

Finally, let S_{12} be a principal G_1–G_2–set and S_{23} a principal G_2–G_3–set, with corresponding automorphisms ϕ_{12} and ϕ_{23}, defined with respect to $x_{12} \in S_{12}$ and $x_{23} \in S_{23}$. Then

$$\begin{aligned}(x_{12} \times x_{23}) \cdot \phi_{S_{12} \times_{G_2} S_{23}}(g) = g \cdot (x_{12} \times x_{23}) &= (x_{12} \cdot \phi_{12}(g)) \times x_{23} \\ &= x_{12} \times (\phi_{12}(g) \cdot x_{23}) \\ &= (x_{12} \times x_{23}) \cdot \phi_{23}(\phi_{12}(g)).\end{aligned}$$

This verifies the last part of the claim. □

Our next goal is to describe explicitly the map $\phi = \phi_S = \phi_m\colon G(\mathcal{Z}) \to G(\mathcal{Z}')$ associated to the grading set for the arc-slide m. For each matched pair B_i in $\mathcal{Z}$ there is an associated homology class $h(B_i) \in H_1(F(\mathcal{Z}))$, namely the sum of the core of the handle attached to the pair $B_i \subset Z$ and the segment in Z between the points in B_i. We orient $h(B_i)$ so that the induced orientation of $h(B_i) \cap Z$ agrees with the orientation of Z. The group $G(\mathcal{Z})$ is generated by the element λ and the $2k$ elements

$$\gamma(B_i) = (-\tfrac{1}{2}; h(B_i)),$$

where we write elements of $G(\mathcal{Z})$ as pairs $(m; h)$ where $m \in \frac{1}{2}\mathbb{Z}$ and $h \in H_1(F(\mathcal{Z}))$ subject to a congruency condition (compare Section 2.2.2 and [5, Section 3.1]); in this notation, $\lambda = (1; 0)$. Recall that for $B_i \neq B$ (ie $i \neq 2k$), there is a corresponding matched pair B_i in $\mathcal{Z}'$, while B corresponds to a matched pair B'. We use $h'(B_i)$ (respectively $\gamma'(B_i)$) to denote the homology class in $F(\mathcal{Z}')$ (respectively group element in $G(\mathcal{Z}')$) associated to the matched pair B_i.

Lemma 6.5 *The map ϕ is given (up to inner isomorphism) by $\phi(\lambda) = \lambda$, $\phi(\gamma(B_i)) = \gamma'(B_i)$ if $B_i \neq B$, and*

$$\phi(\gamma(B)) = \begin{cases} \lambda^{-1}\gamma'(B')\gamma'(C)^{-1} = (0; h'(B') - h'(C)) & \text{in case (U.I)}, \\ \lambda^{-1}\gamma'(B')^{-1}\gamma'(C) = (-1; -h'(B') + h'(C)) & \text{in case (U.II)}, \\ \lambda^{-1}\gamma'(B')\gamma'(C) = (-1; h'(B') + h'(C)) & \text{in case (U.III)}, \\ \lambda^{-1}\gamma'(B')\gamma'(C)^{-1} = (-1; h'(B') - h'(C)) & \text{in case (O.I)}, \\ \lambda^{+1}\gamma'(B')^{-1}\gamma'(C) = (0; -h'(B') + h'(C)) & \text{in case (O.II)}, \\ \lambda^{+1}\gamma'(B')\gamma'(C) = (0; h'(B') + h'(C)) & \text{in case (O.III)}. \end{cases}$$

Proof Choose a base idempotent of type ${}_C X$. By Lemma 6.2, the grading set for $\mathcal{H}(m)$ is given by $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}')/H$ where H is generated by $(0; h(B_i), -h(B_i'))$ for $i \neq 2k$, and one additional element:

Handleslide type	$g(P_{2k})$
U.I	$\bigl(-\frac{1}{2}; h(B), -h'(B') + h'(C')\bigr)$
U.II	$\bigl(\frac{1}{2}; h(B), h(B') - h'(C')\bigr)$
U.III	$\bigl(\frac{1}{2}; h(B), -h(B') - h(C')\bigr)$
O.I	$\bigl(\frac{1}{2}; h(B), -h'(B') + h'(C')\bigr)$
O.II	$\bigl(-\frac{1}{2}; h(B), h'(B') - h'(C')\bigr)$
O.III	$\bigl(-\frac{1}{2}; h(B), -h'(B') - h'(C')\bigr)$

(Note that we have reversed all of the signs from Lemma 6.2; the domains shown there have boundary $-h(B)$ in $H_1(F(\mathcal{Z}))$.) By Lemma 6.4, this data is reinterpreted as a map. Using $(0;0,0) \in G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}')/H$ as the element x in the proof of Lemma 6.4, for the case (U.II), say, we have

$$\begin{aligned}(\tfrac{1}{2}; h(B)) \cdot (0;0,0) \cdot (0; h'(B') - h'(C)) &= (0;0,0),\\ (-\tfrac{1}{2}; h(B)) \cdot (0;0,0) &= (0;0,0) \cdot (-1; -h'(B') + h'(C)),\end{aligned}$$

which corresponds to the statement that $\phi((-\frac{1}{2}; h(B))) = (-1; -h'(B') + h'(C))$. □

6.1.3 Gradings of generators and algebra elements Now that we understand the grading sets for an arc-slide $m\colon \mathcal{Z} \to \mathcal{Z}'$, the next task is to understand the gradings of generators of $\widehat{CFDD}(\mathcal{H}(m))$ and of algebra elements of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$.

Fix a base idempotent I_0 in the near-diagonal subalgebra of $\mathcal{A}(\mathcal{Z}, i) \otimes \mathcal{A}(-\mathcal{Z}', -i)$. We choose I_0 to have type X. Recall that any such idempotent corresponds to a unique generator $\boldsymbol{x}_0 = \boldsymbol{x}(I_0) \in \mathfrak{S}(\mathcal{H}(m))$. Choose for each other idempotent J an element $Q_J \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}(J))$. The unrefined ($G'$–set) grading of generators is given by

$$\begin{aligned}\mathrm{gr}'(\boldsymbol{x}(J)) &= R(g'(Q_J)) \cdot \langle R(g'(P_i))\rangle\\ &= (-e(Q_J) - n_{\boldsymbol{x}(I)}(Q_J) - n_{\boldsymbol{x}(J)}(Q_J); r_*\partial^{\partial_L} Q_J, r_*\partial^{\partial_R} Q_J) \cdot \langle R(g'(P_i))\rangle,\end{aligned}$$

where $\partial^{\partial_L} Q_J$ and $\partial^{\partial_R} Q_J$ denote the intersections of Q_J with the left and right boundaries of $\mathcal{H}(m)$, and we view $G'(\mathcal{Z}) \times_{\mathbb{Z}} G'(\mathcal{Z})$ as a subset of $\frac{1}{2}\mathbb{Z} \times H_1(Z, \boldsymbol{a}) \times H_1(Z', \boldsymbol{a}')$. These cosets are independent of the choices of the Q_J.

For definiteness, for idempotents J of type X, take Q_J to be a union of horizontal strips in the graph for the arc-slide (Figure 3), so that

$$R(g'(Q_J)) = (0; r_*\partial^{\partial_L} Q_J, r_*\partial^{\partial_R} Q_J),$$
$$\mathrm{gr}'(\boldsymbol{x}(J)) = (0; r_*\partial^{\partial_L} Q_J, r_*\partial^{\partial_R} Q_J) \cdot \langle R(g'(P_i))\rangle.$$

For an idempotent J of type Y, there is an associated idempotent J_X of type X_C obtained by replacing the C in the left of J by a B. Then $Q_{J_X} * \sigma^{\epsilon}$ is a domain connecting I_0 to J, where ϵ is ± 1, depending on the geometry of the arc-slide: ϵ is $+1$ if b_1 is below c_1, and -1 if b_1 is above c_1. We will choose Q_J to be $Q_{J_X} * \sigma^{\epsilon}$. We then have

$$R(g'(Q_J)) = R(g'(\sigma))^{\epsilon} R(g'(Q_{J_X})) = (-\epsilon/2; \epsilon r_*[\sigma] + r_*\partial^{\partial_L} Q_J, r_*\partial^{\partial_R} Q_J),$$
$$\mathrm{gr}'(\boldsymbol{x}(J)) = R(g'(Q_J)) \cdot \langle R(g'(P_i))\rangle.$$

Next we turn to the refined gradings. To specify grading refinement data for $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z}')$ (as in [5, Section 3.2.1]), recall that for each idempotent J of type X there are corresponding idempotents j_L and j_R of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z}')$, respectively, and that every idempotent of $\mathcal{A}(\mathcal{Z})$ (respectively $\mathcal{A}(-\mathcal{Z}')$) arises as j_L (respectively j_R) for a unique idempotent J of type X.

We will use the $Q_J \in \pi_2(\boldsymbol{x}(I_0), \boldsymbol{x}(J))$ for idempotents J of type X to define grading refinement data. Specifically, define grading refinement data Ξ for $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z}')$ by setting, for idempotents J of type X,

$$\Xi(j_L) = (\epsilon(\partial^{\partial_L} Q_J); -r_*\partial^{\partial_L} Q_J),$$
$$\Xi(j_R) = (\epsilon(\partial^{\partial_R} Q_J); -r_*\partial^{\partial_R} Q_J),$$

where $\epsilon\colon H_1(Z, \boldsymbol{a}) \to \frac{1}{2}\mathbb{Z}$ is as in Section 2.2.2. (The minus signs arise because we are giving grading refinement data on $\mathcal{A}(\mathcal{Z})$, not on $\mathcal{A}(\partial_L \mathcal{H}(m)) = \mathcal{A}(-\mathcal{Z}) = \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$, and similarly for $\mathcal{A}(-\mathcal{Z}')$.)

With respect to this choice, if J is an idempotent of type X then

$$\mathrm{gr}(\boldsymbol{x}(J)) = \Xi(j_L) \cdot \Xi(j_R) \cdot \mathrm{gr}'(\boldsymbol{x}(J)) = (0;0) \cdot \langle R(g'(P_i))\rangle;$$

compare [6, Section 10.5; 5, Section 6.5].

For idempotents J of type Y, we again use $Q_{J_X} * \sigma^{\epsilon} \in \pi_2(I_0, J)$ to define the grading refinement data. Then

$$\mathrm{gr}(\boldsymbol{x}(J)) = \Xi(J) R(g'(\sigma))^{\epsilon} \Xi(J_X)^{-1} \cdot \langle R(g'(P_i))\rangle.$$

We can furthermore choose the domains Q_J for all J of type X to have equal multiplicities at σ, σ_+ and σ_-. (This can be achieved by adding or subtracting the periodic domain corresponding to C.) Then $\Xi(j_R)$, $\Xi((J_X)_R)$, $\Xi(j_L)$ and $\Xi((J_X)_L)$ all commute with $R(g'(\sigma))^\epsilon$, and we can write

$$\mathrm{gr}(\boldsymbol{x}(J)) = R(g'(\sigma))^\epsilon \,\Xi(j_L)\Xi((J_X)_L)^{-1}\cdot\langle R(g'(P_i))\rangle.$$

With respect to the refined grading, the grading of an algebra element $a \in \mathcal{A}(\mathcal{Z})$ (say) with $j\cdot a\cdot k = a$ (for primitive idempotents j and k) is given by (see [5, Definition 3.9])

$$\mathrm{gr}_\Xi(a) = \Xi(j)\,\mathrm{gr}'(a)\,\Xi(k)^{-1}.$$

6.2 The refined grading set for general surface homeomorphisms

Much of the discussion of gradings for arc-slides extends to arbitrary surface homeomorphisms. In particular, given a strongly-based mapping class $\psi\colon F(\mathcal{Z}_1)\to F(\mathcal{Z}_2)$ there is an associated outer isomorphism of grading groups $\phi_\psi\colon G(\mathcal{Z}_1)\to G(\mathcal{Z}_2)$, gotten from the grading set of any Heegaard diagram for ψ. These assemble to give an action of the mapping class group by outer automorphisms on $G(\mathcal{Z})$, as follows:

Proposition 6.6 *For any pointed matched circle $\mathcal{Z}$, there is a homomorphism*

$$\phi\colon MCG_0(F(\mathcal{Z}))\to \mathrm{Out}(G(\mathcal{Z})),$$

extending the standard action on the homology

$$MCG_0(F(\mathcal{Z}))\to \mathrm{Sp}(2k,\mathbb{Z})\subset \mathrm{Aut}(H_1(F(\mathcal{Z}))).$$

More generally, let $\{\mathcal{Z}_i^k\}$ be the set of all pointed matched circles for surfaces of genus k, and $\mathrm{Out}(\{G(\mathcal{Z}_i^k)\})$ be the groupoid with object set $\{\mathcal{Z}_i^k\}$ and morphisms $\mathrm{Hom}(\mathcal{Z}_i,\mathcal{Z}_j)=\mathrm{Out}(G(\mathcal{Z}_i),G(\mathcal{Z}_j))$ (see Definition 6.3). Then there is a homomorphism

$$\phi\colon MCG_0(k)\to \mathrm{Out}(\{G(\mathcal{Z}_i^k)\})$$

extending the standard action on H_1.

Proof Let $\psi\in MCG_0(F(\mathcal{Z}),F(\mathcal{Z}'))$. Then the mapping cylinder Y_ψ of ψ has grading set S_ψ which is isomorphic to $G(\mathcal{Z})$ (respectively $G(\mathcal{Z}')$) as a left $G(\mathcal{Z})$–set (respectively right $G(\mathcal{Z}')$–set), which by Lemma 6.4 gives a map $\phi_\psi\in\mathrm{Out}(G(\mathcal{Z}),G(\mathcal{Z}'))$. If ψ' is an element of $MCG_0(F(\mathcal{Z}'),F(\mathcal{Z}''))$ then $S_{\psi'\circ\psi}=S_\psi\times_{G(\mathcal{Z})}S_{\psi'}$. Thus, Lemma 6.4 applied to the S_ψ gives a map $MCG_0(k)\to\mathrm{Out}(\{G(\mathcal{Z}_i^k)\})$.

It is immediate from the form of S_ψ that the action on $\mathrm{Out}(\{G(\mathcal{Z}_i^k)\})$ projects to the standard action of the mapping class group on H_1. □

The main goal of this subsection is to compute ϕ_ψ for general mapping classes ψ.

We start with some lemmas regarding the structure of $G(\mathcal{Z})$.

Recall from Section 6.1.2 that the pointed matched circle $\mathcal{Z}$ gives a canonical basis $\{h(B_i)\}$ for $H_1(F(\mathcal{Z}))$, gotten by joining the cores of the handles attached to $\mathcal{Z}$ with corresponding segments in $\mathcal{Z}$. These can be upgraded to a generating set of $G(\mathcal{Z})$ as follows:

Lemma 6.7 *Let $\{B_i\}_{i=1}^{2k}$ denote the set of matched pairs for $\mathcal{Z}$, and $\gamma(B_i) = (-\frac{1}{2}, h(B_i))$ denote the corresponding group element of $G(\mathcal{Z})$. Then $G(\mathcal{Z})$ is generated by the elements $\{\gamma(B_i)\}_{i=1}^{2k}$ and λ, subject to the relations:*

- *λ is central.*
- *$\gamma(B_i)\cdot\gamma(B_j) = \lambda^{2h(B_i)\cdot h(B_j)}\gamma(B_j)\cdot\gamma(B_i)$, where $\cdot$ denotes the intersection product in $H_1(F(\mathcal{Z}))$.*

Proof It is immediate from the definition of ϵ that $\{\gamma(B_i)\}_{i=1}^{2k}$ lies in $G(\mathcal{Z})$. Together with λ, the elements generate an index-two subgroup of $\frac{1}{2}\mathbb{Z}\times H_1(F(\mathcal{Z}))$ (with its twisted multiplication), which must be $G(\mathcal{Z})$. The relations follow from the description of $G(\mathcal{Z})$ as a central extension of $H_1(F(\mathcal{Z}))$ [6, Section 3.3.2]. □

Lemma 6.8 *For any pointed matched circle $\mathcal{Z}$ and any matched pair B_i in $\mathcal{Z}$, the map $\psi\colon G(\mathcal{Z})\to G(\mathcal{Z})$ given by*

$$\begin{aligned}\psi(\lambda) &= \lambda,\\ \psi(\gamma(B_i)) &= \lambda^2\gamma(B_i),\\ \psi(\gamma(B_j)) &= \gamma(B_j) \quad \text{if } j\neq i,\end{aligned}$$

is an inner automorphism.

Proof Consider first the case when $\mathcal{Z}$ is the split pointed matched circle, for which the argument is more concrete. For each B_i there is a corresponding $B_{\tau(i)}$ so that $h(B_i)\cdot h(B_{\tau(i)}) = \pm 1$ and $h(B_i)\cdot h(B_j) = 0$ for $j\neq\tau(i)$, where $\cdot$ denotes the intersection product on $H_1(F(\mathcal{Z}))$. The inner automorphism of $G(\mathcal{Z})$ given by conjugating by $\gamma(B_{\tau(i)})$ is given by

$$\begin{aligned}\gamma(B_i) &\mapsto \lambda^{\mp 2}\gamma(B_i),\\ \gamma(B_j) &\mapsto \gamma(B_j) \quad \text{if } j\neq i,\end{aligned}$$

as desired.

For general pointed matched circles, instead of conjugating by $\gamma(B_{\sigma(i)})$, let $V \subset H_1(F(\mathcal{Z}))$ denote the subspace generated by all $h(B_j)$ for $j \neq i$. Choose a primitive vector $v \in H_1(F(\mathcal{Z}))$ symplectically orthogonal to V (so $v \cdot h(B_j) = 0$ for $j \neq i$ and $v \cdot h(B_i) = \pm 1$), and conjugate by an element of G of the form $(m, \pm v)$ (for some $m \in \frac{1}{2}\mathbb{Z}$). □

Let $G_{\mathbb{Z}/2}(\mathcal{Z}) = (\mathbb{Z}/2) \times H_1(F(\mathcal{Z}))$, which we view as the trivial $\mathbb{Z}/2$–central extension of $H_1(F(\mathcal{Z}))$. Since the cocycle defining the $\mathbb{Z}$–central extension $G(\mathcal{Z})$ of $H_1(F(\mathcal{Z}))$ takes values in $2\mathbb{Z} \subset \mathbb{Z}$, there is a homomorphism $\Upsilon\colon G(\mathcal{Z}) \to G_{\mathbb{Z}/2}(\mathcal{Z})$ extending the projection map $\mathbb{Z} \to \mathbb{Z}/2$. Explicitly, we may take Υ to be defined by $\Upsilon(\gamma(B_i)) = (0, h(B_i))$ and $\Upsilon(\lambda) = (1, 0)$.

Lemma 6.9 *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be pointed matched circles. Let $\phi\colon G(\mathcal{Z}_1) \to G(\mathcal{Z}_2)$ be an isomorphism so that $\phi(\lambda) = \lambda$. Then:*

(1) The isomorphism ϕ descends to an isomorphism $\overline{\phi}\colon G_{\mathbb{Z}/2}(\mathcal{Z}_1) \to G_{\mathbb{Z}/2}(\mathcal{Z}_2)$.

(2) As an outer isomorphism with $\phi(\lambda) = \lambda$, ϕ is determined by the induced map $\overline{\phi}$.

Proof For part (1), since the kernel of Υ is the subgroup generated by λ^2, ϕ descends to a homomorphism $\overline{\phi}\colon G_{\mathbb{Z}/2}(\mathcal{Z}_1) \to G_{\mathbb{Z}/2}(\mathcal{Z}_2)$. Since ϕ is an isomorphism and $\phi(\lambda) = \lambda$, the elements $[\phi(0, h(B_i)]$ form a basis for $H_1(F(\mathcal{Z}_2))$. It follows that the map $\overline{\phi}$ is an isomorphism.

Part (2) follows from Lemma 6.8. □

Remark 6.10 In particular, we have that Lemma 6.9 gives an injective homomorphism $\mathrm{Out}_\lambda(G(\mathcal{Z})) \to \mathrm{Aut}(G_{\mathbb{Z}/2}(\mathcal{Z}))$, where $\mathrm{Out}_\lambda(G(\mathcal{Z})) = \{\phi \in \mathrm{Out}(G(\mathcal{Z})) \mid \phi(\lambda) = \lambda\}$. The image of this homomorphism is all automorphisms of $G_{\mathbb{Z}/2}$ which cover symplectomorphisms of $H_1(F(\mathcal{Z}))$ (with respect to the intersection form).

Definition 6.11 For any class $a \in H_1(F(\mathcal{Z}))$, write $a = \sum_i a_i h(B_i)$, and define the ℓ^1–*norm* of a to be

$$\|a\|_1 = \sum_{i=1}^{2k} |a_i|.$$

Given a map $\psi\colon F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$, define a map $\xi_\psi\colon G_{\mathbb{Z}/2}(\mathcal{Z}_1) \to G_{\mathbb{Z}/2}(\mathcal{Z}_2)$ by

$$\xi_\psi(m, a) = (m + \|a\|_1 + \|\psi_*(a)\|_1, \psi_*(a)). \tag{6-1}$$

Lemma 6.12 *The map ξ_ψ is a group homomorphism.*

Proof This is a direct computation:

$$\begin{aligned}
&\xi_\psi(m,a)\xi_\psi(m',a')\\
&\qquad= \big(m+\|a\|_1+\|\psi_*(a)\|_1, \psi_*(a)\big)\big(m'+\|a'\|_1+\|\psi_*(a')\|_1, \psi_*(a')\big)\\
&\qquad= \big(m+m'+\|a\|_1+\|a'\|_1+\|\psi_*(a)\|_1+\|\psi_*(a')\|_1, \psi_*(a)+\psi_*(a')\big)\\
&\qquad= \big(m+m'+\|a+a'\|_1+\|\psi_*(a+a')\|_1, \psi_*(a+a')\big)\\
&\qquad= \xi_\psi(m+m',a+a') = \xi_\psi((m,a)(m',a'))
\end{aligned}$$

(The third equality uses the fact that ψ_* is linear, and the modulo 2 reduction of $\|\cdot\|_1$ is also linear.) □

Proposition 6.13 *Let $\psi\colon F(\mathcal{Z}_1)\to F(\mathcal{Z}_2)$ be a strongly-based mapping class. Then the outer isomorphism $\phi_\psi\colon G(\mathcal{Z}_1)\to G(\mathcal{Z}_2)$ is characterized by $\overline{\phi}_\psi=\xi_\psi$ (as defined in* (6-1)*).*

Proof The proof is in two steps:

(1) Verify that if $\psi = F^\circ(m)$ is the diffeomorphism induced by an arc-slide, then $\xi_\psi=\overline{\phi}_\psi$.

(2) Verify that if $\psi_1\colon F(\mathcal{Z}_1)\to F(\mathcal{Z}_2)$ and $\psi_2\colon F(\mathcal{Z}_2)\to F(\mathcal{Z}_3)$, then $\xi_{\psi_1}=\overline{\phi}_{\psi_1}$ and $\xi_{\psi_2}=\overline{\phi}_{\psi_2}$ imply $\xi_{\psi_2\circ\psi_1}=\overline{\phi}_{\psi_2\circ\psi_1}$.

Since the arc-slides generate the strongly-based mapping class group, it follows from steps (1) and (2) that $\overline{\phi}_\psi=\xi_\psi$ for any mapping class ψ.

We start with part (1). Since both $\overline{\phi}_\psi$ and ξ_ψ are group homomorphisms, it suffices to verify that $\overline{\phi}_\psi(\lambda)=\xi_\psi(\lambda)$ and $\overline{\phi}_\psi(\gamma(B_i))=\xi_\psi(\gamma(B_i))$. The statement about λ holds trivially. For $i\neq 2k$,

$$\begin{aligned}
\phi_\psi(\gamma(B_i)) &= \gamma'(B_i),\\
\overline{\phi}_\psi(\gamma'(B_i)) &= (0,h'(B_i)),\\
\xi_\psi(\gamma(B_i)) &= (0,h'(B_i)),
\end{aligned}$$

as desired. For $i=2k$, it follows from the formulas in Lemma 6.5 that

$$\overline{\phi}_\psi(\gamma(B_i)) = (1,\psi_*(h(B_i))),$$

where $\psi_*(h(B_i))$ has ℓ^1–norm 2. Similarly,

$$\xi_\psi(\gamma(B_i)) = (0+1+2,\psi_*(h(B_i))) = (1,\psi_*(h(B_i))) = \overline{\phi}_\psi(\gamma(B_i)),$$

as desired.

Recall that the cases (U.I)–(O.III) are only half of the combinatorial configurations for arc-slides; the other half are their inverses. But $\phi_{\psi^{-1}} = \phi_{\psi}^{-1}$ and $\xi_{\psi^{-1}} = \xi_{\psi}^{-1}$, so it follows that $\overline{\phi}_{\psi}$ and ξ_{ψ} agree on the remaining six types of arc-slides, as well.

For part (2), since we already know (by Proposition 6.6) that $\phi_{\bullet}$ is a groupoid homomorphism $MCG_0(k) \to \mathrm{Out}\{G(\mathcal{Z})\}$, and so $\overline{\phi}_{\bullet}$ is a groupoid homomorphism as well, it suffices to verify that $\xi_{\bullet}$ is a groupoid homomorphism. That is, we must check that

$$\begin{aligned} \xi_{\psi_2 \circ \psi_1} &= \xi_{\psi_2} \circ \xi_{\psi_1}, \\ \xi_{\psi^{-1}} &= \xi_{\psi}^{-1}. \end{aligned}$$

The second property is obvious (and we already used it, above). For the first property,

$$\begin{aligned} &\xi_{\psi_2} \circ \xi_{\psi_1}(m, a) \\ &\qquad = \xi_{\psi_2}\big(m + \|a\|_1 + \|(\psi_1)_*(a)\|_1, (\psi_1)_*(a)\big) \\ &\qquad = \big(m + \|a\|_1 + \|(\psi_1)_*(a)\|_1 + \|(\psi_1)_*(a)\|_1 + \|(\psi_2)_* \circ (\psi_1)_*(a)\|_1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\psi_2)_* \circ (\psi_1)_*(a)\big) \\ &\qquad = \big(m + \|a\|_1 + \|(\psi_2 \circ \psi_1)_*(a)\|_1, (\psi_2 \circ \psi_1)_*(a)\big) \\ &\qquad = \xi_{\psi_2 \circ \psi_1}(m, a), \end{aligned}$$

as desired. □

Remark 6.14 Since the G'–set grading $\widehat{CFDD}(\psi)$ is not transitive for the left and right actions, it does not correspond to a map $G'(\mathcal{Z}) \to G'(\mathcal{Z}')$. Note also that the G–set grading induces the G'–set grading (cf [5, Lemma 3.15]), so Proposition 6.13 determines the G'–set which grades $\widehat{CFDD}(\psi)$.

7 Assembling the pieces to compute $HF^{\wedge}$

Using the computations from Section 4 and the pairing theorems (Theorems 2.19 and 2.20), we finish the proofs of Theorems 3 and 4. Then, in Section 7.1, we use the computations of gradings from Section 6 to compute the decomposition of $\widehat{HF}(Y)$ according to spin^c–structures, and the relative Maslov grading inside each spin^c–structure.

First, we turn to the calculation of $\widehat{CFD}$ for a handlebody. We start with the standard "0–framed" handlebody:

Proof of Proposition 1.12 Recall from Section 1.4 that H^g denotes the 0–framed handlebody of genus g, the boundary sum of g copies of the 0–framed solid torus. A bordered Heegaard diagram $\mathcal{H}_0^g$ for H^g can be constructed as the boundary sum of g bordered Heegaard diagrams for the 0–framed solid torus. We can draw this diagram $\mathcal{H}_0^g$ on a genus-g surface with $2g$ alpha-arcs $\{\alpha_1^c, \alpha_2^c, \dots, \alpha_{2g-1}^c, \alpha_{2g}^c\}$ and g beta-circles, so that β_i meets α_{2i}^c transversely in a single point (and is disjoint from all other α–arcs). The $g=2$ case $\mathcal{H}_0^2$ is illustrated in Figure 5.

The type-D module $\widehat{CFD}(\mathcal{H}_0^g)$ has a single generator $\boldsymbol{x}$. Its associated idempotent $I_D(\boldsymbol{x})$ is the idempotent where all the odd-numbered strands are occupied. The differential is specified by

$$\partial \boldsymbol{x} = \left(\sum_{i=1}^{g} a(\rho_{4i-3}) \cdot a(\rho_{4i-2})\right) \cdot \boldsymbol{x},$$

where ρ_j denotes the j^{th} short chord in $-\partial\mathcal{H}_0^g$. To see this, note that there are g connected components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, each of which contributes one term in the sum. Although these disks do not quite fit the conditions of Definition 2.6, it is easy to see that they each contribute as indicated. Moreover, there are no other contributions to the differential.

The module $\widehat{CFD}(\mathcal{H}_0^g)$ we have just described is exactly the module $\widehat{\mathcal{D}}(\mathsf{H}^g)$ of Section 1.4. $\square$

Proof of Theorem 3 This follows immediately from Proposition 1.12 and Theorems 1 and 2, by inductively applying the relation

$$\begin{aligned}\operatorname{Mor}_{D,C}\big({}_{D,C}X, {}_{D}M \otimes_{\mathbb{F}_2} \operatorname{Mor}_{B,A}({}_{B,A}Y, {}_{C,B}N \otimes_{\mathbb{F}_2} {}_{A}P)\big) \\ \simeq \operatorname{Mor}_{D,C,B,A}\big({}_{D,C}X \otimes_{\mathbb{F}_2} {}_{B,A}Y, {}_{D}M \otimes_{\mathbb{F}_2} {}_{C,B}N \otimes_{\mathbb{F}_2} {}_{A}P\big).\end{aligned}$$

and the pairing theorem, Corollary 2.21. $\square$

Proof of Theorem 4 This follows immediately from Theorems 3 and 2.19. $\square$

7.1 Gradings

The one missing ingredient to compute the spin^c and Maslov gradings on $\widehat{HF}(Y)$ is the computation of the gradings on $\widehat{CFD}(\mathcal{H}_0^g)$. Number the points in $\mathcal{Z}_0^g = -\partial\mathcal{H}_0^g$ by $1, \dots, 4g$, from bottom to top. Write elements of $G'(\mathcal{Z})$ as pairs $(m; a)$ where $m \in \frac{1}{2}\mathbb{Z}$ and a is a linear combination of intervals $[i, j]$ in $\mathcal{Z}$, subject to the congruence condition $m \equiv \epsilon(a) \pmod 1$.

Proposition 7.1 *As a $G'(\mathcal{Z}_0^g)$–set graded module, $\widehat{CFD}(\mathcal{H}_0^g)$ is graded by $S'_0 = G'(\mathcal{Z}_0^g)/H$, where*

$$H = \big\langle \big\{(-\tfrac{1}{2}; -[4i+1, 4i+3])\big\}_{i=0}^{g-1} \big\rangle.$$

The generator $\boldsymbol{x}$ has grading $(0;0)$ in $G'(\mathcal{Z})$.

Proof Each generator of H corresponds to a periodic domain P consisting of a single component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, with multiplicity 1. Each of these domains P has Euler measure $-\frac{1}{2}$ and point measure 1. □

Now, to compute the gradings on $\widehat{CF}(Y)$, one follows a simple, seven-step process:

(1) Take a Heegaard decomposition $Y = \mathsf{H}_0^g \cup_\psi \mathsf{H}_0^g$ and factoring the gluing map into arc-slides, $\psi = \psi_1 \circ \cdots \circ \psi_n$, where $\psi_i \colon -F(\mathcal{Z}_i) \to -F(\mathcal{Z}_{i-1})$. By Theorem 4, $\widehat{CF}(Y)$ is homotopy equivalent to the complex

$$\operatorname{Mor}\big(\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_n}) \otimes \cdots \otimes \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_1}), \widehat{\mathcal{D}}(\mathsf{H}_0^g) \otimes \widehat{\mathcal{DD}}(\psi_n) \otimes \cdots \otimes \widehat{\mathcal{DD}}(\psi_1) \otimes \widehat{\mathcal{D}}(\mathsf{H}_0^g)\big), \tag{7-1}$$

where the Mor–complex is over $\mathcal{A}(\mathcal{Z}_n) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_1)$.

(2) Choose grading refinement data Ξ_i for $\mathcal{A}(\mathcal{Z}_i)$, as in [5, Section 3.2.1]. Each homogeneous element $a = j \cdot a \cdot j' \in \mathcal{A}(\mathcal{Z}_i)$ (where j and j' are primitive idempotents) gets a grading $\mathrm{gr}_\Xi(a) = \Xi(j)^{-1}\,\mathrm{gr}'(a)\,\Xi(j')$. (Sometimes we will denote $\mathrm{gr}_\Xi(a)$ simply by $\mathrm{gr}(a)$.)
(In Section 6.1.2, we explain how to use ψ_i to choose convenient grading refinement data for $\mathcal{Z}_i$ and $\mathcal{Z}_{i-1}$. But the data chosen this way by ψ_i and ψ_{i+1} for $\mathcal{Z}_i$ may not agree.)

(3) As discussed in Section 6.1.3, each module $\widehat{\mathcal{DD}}(\psi_i) \simeq \widehat{CFDD}(F^\circ(\psi_i))$ is graded by a set S'_i with a left action by $G'(\mathcal{Z}_i) \times G'(-\mathcal{Z}_{i-1})$. (These sets are computed in Section 6.1.2, and the gradings of generators are computed in Section 6.1.3.)
Consequently,

$$R = \widehat{\mathcal{D}}(\mathsf{H}_0^g) \otimes \widehat{\mathcal{DD}}(\psi_n) \otimes \cdots \otimes \widehat{\mathcal{DD}}(\psi_1) \otimes \widehat{\mathcal{D}}(\mathsf{H}_0^g)$$

is graded by the $G'(-\mathcal{Z}_n) \times G'(\mathcal{Z}_n) \times G'(-\mathcal{Z}_{n-1}) \times \cdots \times G'(-\mathcal{Z}_1) \times G'(\mathcal{Z}_1)$–set

$$S'_R = S'_0 \times S'_n \times \cdots \times S'_1 \times S'_0.$$

(Here, $\times$ means $\times_{\mathbb{Z}}$; we suppress the subscript in this section to keep the notation manageable.)

(4) Using the grading refinement data Ξ_i, we can turn S'_R into a $G(-\mathcal{Z}_n)\times G(\mathcal{Z}_n)\times G(-\mathcal{Z}_{n-1})\times\cdots\times G(-\mathcal{Z}_1)\times G(\mathcal{Z}_1)$–set S_R, as in [5, Lemma 3.15]. In this set, the grading of a generator $y_0\otimes y_1\otimes\cdots\otimes y_n\otimes y_{n+1}$, where $j_i\cdot y_i\cdot j_{i+1}=y_i$ for primitive idempotents $j_1,\dots,j_n$, is given by

$$\begin{aligned}&\mathrm{gr}(y_0\otimes y_1\otimes\cdots\otimes y_n\otimes y_{n+1})\\ &\qquad=\mathrm{gr}'(y_0)\Xi_1(j_1)\otimes\Xi_1(j_1)^{-1}\,\mathrm{gr}'(y_1)\Xi_2(j_2)\\ &\qquad\qquad\otimes\cdots\otimes\Xi_{n-1}(j_{n-1})^{-1}\,\mathrm{gr}'(y_n)\Xi_n(j_n)\otimes\Xi_n(j_n)^{-1}\,\mathrm{gr}'(y_{n+1}).\end{aligned}$$

(5) The module $L=\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_n})\otimes\cdots\otimes\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_1})$ is graded by the $G(-\mathcal{Z}_n)\times G(\mathcal{Z}_n)\times G(-\mathcal{Z}_{n-1})\times\cdots\times G(-\mathcal{Z}_1)\times G(\mathcal{Z}_1)$–set

$$S_L=G(\mathcal{Z}_n)\times G(\mathcal{Z}_{n-1})\times\cdots\times G(\mathcal{Z}_1).$$

Each complementary idempotent I generating $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_i})$ has grading $(0;0)$. (This argument is a simpler version of the discussion in Section 6.1, and is also given in [5, Lemma 8.13].)

(6) Finally, as discussed in Section 2.5, the Mor–complex (7-1) is graded by

$$S=S_L^{\mathrm{op}}\times_{G(\mathcal{Z}_n)\times G(\mathcal{Z}_n)^{\mathrm{op}}\times G(\mathcal{Z}_{n-1})\times\cdots\times G(\mathcal{Z}_1)^{\mathrm{op}}}S_R.$$

The Mor–complex is generated, as an $\mathbb{F}_2$–vector space, by maps of the form

$$f=x_1\otimes\cdots\otimes x_n\mapsto y_0\otimes a_1\otimes y_1\otimes\cdots\otimes y_n\otimes a_n\otimes y_{n+1},$$

where $x_1,\dots,x_n$ are generators of $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_n}),\dots,\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}_1})$, y_i is a generator for $\widehat{\mathcal{DD}}(\psi_{n+1-i})$ if $0<i<n+1$, y_0,y_n are generators of $\widehat{\mathcal{D}}(\mathsf{H}_0^g)$, and $a_i\in\mathcal{A}(\mathcal{Z}_{n+1-i})$ is a basic generator (strands diagram). The grading of such a map f is

$$\begin{aligned}\mathrm{gr}(f)&=\big(\mathrm{gr}(x_1)^{\mathrm{op}}\times\cdots\times\mathrm{gr}(x_n)^{\mathrm{op}}\big)\\ &\qquad\times\big(\mathrm{gr}(y_0)\,\mathrm{gr}(a_1)\times\mathrm{gr}(a_1')\,\mathrm{gr}(y_1)\,\mathrm{gr}(a_2)\times\cdots\times\mathrm{gr}(a_{n+1}')\,\mathrm{gr}(y_{n+1})\big)\\ &=\mathrm{gr}(y_0)\,\mathrm{gr}(a_1)\times\mathrm{gr}(x_1)^{\mathrm{op}}\times\mathrm{gr}(a_1')\,\mathrm{gr}(y_1)\,\mathrm{gr}(a_2)\\ &\qquad\qquad\times\cdots\times\mathrm{gr}(x_n)^{\mathrm{op}}\times\mathrm{gr}(a_{n+1}')\,\mathrm{gr}(y_{n+1})\\ &=\mathrm{gr}(y_0)\,\mathrm{gr}(a_1)\times\mathrm{gr}(a_1')\,\mathrm{gr}(y_1)\,\mathrm{gr}(a_2)\times\cdots\times\mathrm{gr}(a_{n+1}')\,\mathrm{gr}(y_{n+1}),\end{aligned}$$

where the last equality uses the fact that each x_i has grading $(0;0)\in G(\mathcal{Z}_i)$. All the $\times$ are $\times_{G(\mathcal{Z}_i)}$ for appropriate i.

(7) The set S retains an action by the central element λ. By the graded version of the pairing theorem, each spin^c–structure on $\widehat{CF}(Y)$ corresponds to a λ–orbit. If f

and g lie in the same spin^c–structure then $\mathrm{gr}(f) = \lambda^i \mathrm{gr}(g)$, for some i, and the grading difference between f and g is i. (If λ acts nonfreely on that orbit then this grading difference is only well defined modulo $\min\{n \mid \lambda^n \mathrm{gr}(f) = \mathrm{gr}(f)\}$; this n is the divisibility of the first Chern class of the corresponding spin^c–structure.)

For practical computations, it is generally not necessary to refine the gradings. Working with the larger grading groups G', let $S'_L = G'(\mathcal{Z}_n) \times G'(\mathcal{Z}_{n-1}) \times \cdots \times G'(\mathcal{Z}_1)$, a set with a left action of $G'(\mathcal{Z}_n) \times G'(\mathcal{Z}_n)^{\mathrm{op}} \times \cdots \times G'(\mathcal{Z}_1) \times G'(\mathcal{Z}_1)^{\mathrm{op}}$. Then, without worrying about grading refinements, elements of the Mor complex (7-1) are graded by

$$S' = (S'_L)^{\mathrm{op}} \times_{G'(\mathcal{Z}_n) \times G'(\mathcal{Z}_n)^{\mathrm{op}} \times G'(\mathcal{Z}_{n-1}) \times \cdots \times G'(\mathcal{Z}_1)^{\mathrm{op}}} S'_R.$$

The grading of a generator f is given by

$$\mathrm{gr}'(f) = \big(\mathrm{gr}'(x_1)^{\mathrm{op}} \times \cdots \times \mathrm{gr}'(x_n)^{\mathrm{op}}\big) \times \big(\mathrm{gr}'(y_0) \times \mathrm{gr}'(a_1) \times \cdots \times \mathrm{gr}'(y_{n+1})\big).$$

Two generators f and g lie in the same λ–orbit of S' if and only if they represent the same spin^c–structure. In this case, $\mathrm{gr}'(f) = \lambda^i \mathrm{gr}'(g)$, where i is the grading difference between f and g. The only part that breaks is that with respect to the unrefined grading there will be λ–orbits with no generators, which do not correspond to spin^c–structures on Y.

8 Elementary cobordisms and bordered invariants

Although we have focused so far on computing $\widehat{HF}$ for a closed 3–manifold, the techniques of this paper can also be used to calculate $\widehat{CFD}$ and $\widehat{CFDD}$ for bordered 3–manifolds, as well. (One can then compute the type-A, -DA and -AA invariants, too; see Section 9.)

Let Y be a bordered 3–manifold with boundary ∂Y parameterized by $F(\mathcal{Z})$. The goal is to break up Y into basic pieces, calculate the bimodules associated to those pieces, and then calculate $\widehat{CFD}(Y)$ by composing the individual bimodules. In addition to arc-slides, we need one new kind of basic piece: an elementary cobordism. Since we already know how to change the parametrizations of the boundary by using arc-slides, we will only need elementary cobordisms with a particular bordering. In Section 8.1 we compute the invariants of these elementary cobordisms. In Section 8.2 we discuss how to compute the invariant $\widehat{CFD}$ of a bordered 3–manifold with connected boundary. Because of the particular form of the pairing theorem we have been using, there are mild technical complications for the invariant $\widehat{CFDD}$ of a 3–manifold with two boundary components; we discuss (and overcome) these in Section 8.3.

8.1 The invariant of a split elementary cobordism

Definition 8.1 Let F_1 be a connected surface of genus g and F_2 be a surface of genus $g+1$. Let $M_1 = M_1(g)$ be the cobordism from F_1 to F_2 gotten by adding a three-dimensional one-handle to $[0,1] \times F_1$ at a pair of points in $\{1\} \times F_2$. Dually, let $M_2 = M_2(g)$ be the cobordism from F_2 to F_1 gotten by adding a three-dimensional two-handle to $[0,1] \times F_2$ along a nonseparating curve in $\{1\} \times F_2$. The 3–manifolds $M_1(g)$ and $M_2(g)$ are called *elementary cobordisms.*

Up to diffeomorphism, an elementary cobordism is uniquely determined by the genera of its boundary. Also, $M_1(g)$ is simply the orientation-reverse of $M_2(g+1)$.

Next, we compute the invariant of an elementary cobordism with a particular bordering. Recall from Section 1.4 that $\mathcal{Z}_1$ denotes the (unique) pointed matched circle for a genus-1 surface. Given a pointed matched circle $\mathcal{Z}$ for a surface of genus g, the *split bordering by* $\mathcal{Z}$ of $M_1(g)$ is the bordering by $F(\mathcal{Z})$ and $F(\mathcal{Z} \# \mathcal{Z}_1)$ so that the circle in $F(\mathcal{Z}_1)$ specified by $\{a, a'\}$ bounds a disk in $M_1(g)$. The *split bordering by* $\mathcal{Z}$ of $M_2(g)$ is the orientation-reverse of the split bordering of $M_1(g-1)$ by $-\mathcal{Z}$.

Let $\mathcal{Z}$ be a pointed matched circle, and let $\mathcal{Z}^1$ denote the pointed matched circle for a genus-one surface. Let H^1 denote the 0–framed solid torus, as in Section 1.4.1. There is an inclusion map

$$i\colon \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}^1) \to \mathcal{A}(\mathcal{Z} \# \mathcal{Z}^1).$$

Now, $\widehat{\mathcal{D}}(\mathbb{I}_{\mathcal{Z}}) \otimes \widehat{\mathcal{D}}(\mathsf{H}^1)$ can be viewed as a type-D structure over $\mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}^1)$. Promoting the homomorphism i by the identity map on $\mathcal{A}(-\mathcal{Z})$, we obtain a homomorphism

$$j\colon \mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}^1) \to \mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z} \# \mathcal{Z}^1).$$

Proposition 8.2 *The induced module $j_*(\widehat{\mathcal{D}}(\mathbb{I}_{\mathcal{Z}}) \otimes \widehat{\mathcal{D}}(\mathsf{H}^1))$, which is a (left-right) type-DD bimodule over $\mathcal{A}(\mathcal{Z} \# \mathcal{Z}^1)$ and $\mathcal{A}(-\mathcal{Z})$, is isomorphic to the bimodule $\widehat{CFDD}$ of the elementary cobordism M_1, endowed with the split bordering by $\mathcal{Z}$. Similarly, the induced bimodule $j_*(\widehat{\mathcal{D}}(\mathbb{I}_{-\mathcal{Z}}) \otimes \widehat{\mathcal{D}}(-\mathsf{H}^1))$, which is a type-DD bimodule over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z} \# \mathcal{Z}^1)$, is isomorphic to the bimodule $\widehat{CFDD}$ of the elementary cobordism M_2, endowed with the split bordering by $\mathcal{Z}$.*

Proof Draw a Heegaard diagram for $(F(\mathcal{Z}) \times [0,1]) \#_\partial \mathsf{H}^1$ by forming the boundary sum of a Heegaard diagram $\mathcal{H}(\mathbb{I}_{\mathcal{Z}})$ for $\mathbb{I}_{\mathcal{Z}}$ with the Heegaard diagram $\mathcal{H}_0^1$ for the 0–framed genus-one handlebody, where the boundary sum is taken near $\boldsymbol{z} \cap (\partial_L \mathcal{H}(\mathbb{I}_{\mathcal{Z}})) \in \partial \mathcal{H}(\mathbb{I}_{\mathcal{Z}})$ and $z \in \partial \mathcal{H}_0^1$. See Figure 41. Holomorphic curves which contribute to the differential must have connected domains. For this reason, the holomorphic curves in the differential are either supported in the identity *DD* bimodule region, or in the handlebody region. This implies the result for M_1. The result for M_2 is obtained by replacing $\mathcal{Z}$ with $-\mathcal{Z}$ and reflecting the picture horizontally. $\square$

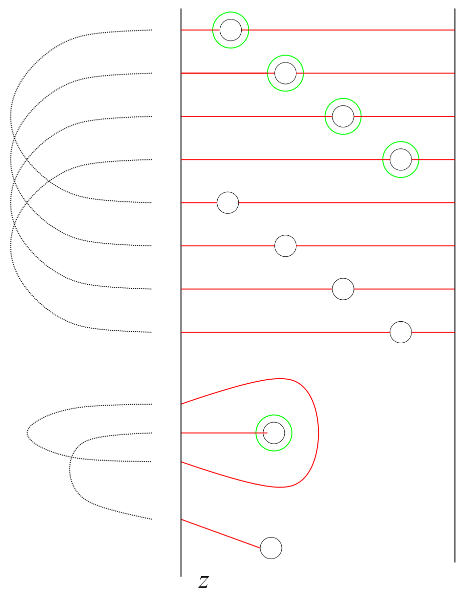


Figure 41: *An elementary cobordism* A diagram for an elementary cobordism from a genus-three surface to a genus-two surface: the genus-two surface here has the antipodal matching.

Remark 8.3 The above proof can be seen as a special case of a boundary connected sum formula for $\widehat{CFD}$; see Zarev [15] for further generalizations.

8.2 Computing *CFD*^ of a 3–manifold with connected boundary

By standard Morse theory, any connected 3–manifold Y with connected boundary $F(\mathcal{Z})$ can be obtained from the 3–ball by a sequence of elementary cobordisms. That is,

$$Y = \mathbb{D}^3 \cup_{\phi_1} M_{i_1}(g_1) \cup_{\phi_2} M_{i_2}(g_2) \cup_{\phi_3} \cdots \cup_{\phi_n} M_{i_n}(g_n),$$

where each $i_j \in \{1,2\}$ and the genera g_i are determined by the sequence of the i_j in the obvious way (ie $g_i = \sum_{j<i} 2(\frac{3}{2} - i_j)$).

Let Y_k be the part of Y obtained after attaching k of the elementary cobordisms, ie

$$Y_k = \mathbb{D}^3 \cup_{\phi_1} M_{i_1}(g_1) \cup_{\phi_2} M_{i_2}(g_2) \cup_{\phi_3} \cdots \cup_{\phi_n} M_{i_k}(g_k).$$

We compute $\widehat{CFD}(Y_k)$ inductively as follows. The gluing map ϕ_k is a map from $\partial_L M_{i_k}(g_k)$ to $-\partial Y_{k-1}$. Suppose we are given $\widehat{CFD}(Y_{k-1})$ for some bordering $-F(\mathcal{Z}^{g_{k-1}}) \to \partial Y_{k-1}$, and want to compute $\widehat{CFD}(Y_k)$ with respect to a bordering $-F(\mathcal{Z}^{g_k}) \to \partial Y_k$. (Here, $\mathcal{Z}^{g_{k-1}}$ and $\mathcal{Z}^{g_k}$ are arbitrary pointed matched circle for a surface of the right genus.)

Choose the split bordering of $M_{i_k}(g_k)$ by $\mathcal{Z}^{g_{k-1}}$ (say). Then ϕ (respectively ϕ') corresponds to some map $\psi\colon F(\mathcal{Z}^{g_{k-1}}) \to F(\mathcal{Z}^{g_{k-1}})$ (respectively $\psi'\colon F(\mathcal{Z}^{g_k}) \to F(\mathcal{Z}^{g_k,\prime})$). Factor ψ and ψ' into arc-slides,

$$\psi = m_1 \circ \cdots \circ m_l, \quad \psi' = m'_1 \circ \cdots \circ m'_{l'}.$$

Then, by Theorem 2.20,

$$\begin{aligned}(8\text{-}1)\quad \widehat{CFD}(Y_k) \simeq \operatorname{Mor}\big(&\widehat{CFDD}(\mathbb{I}) \otimes \cdots \otimes \widehat{CFDD}(\mathbb{I}), \widehat{CFD}(Y_{k-1})\\ &\otimes \widehat{CFDD}(m_l) \otimes \cdots \otimes \widehat{CFDD}(m_1) \otimes \widehat{CFDD}(M_{i_k}(g_k))\\ &\otimes \widehat{CFDD}(m'_{l'}) \otimes \cdots \otimes \widehat{CFDD}(m'_1)\big).\end{aligned}$$

(As in Section 1.4, Mor denotes the chain complex of bimodule homomorphisms. Also, $\widehat{CFDD}(M_{i_k}(g_k))$ denotes the bimodule computed with respect to the split bordering by $\mathcal{Z}^{g_{k-1}}$, the $\widehat{CFDD}(\mathbb{I})$ in the formula are with respect to the appropriate pointed matched circles.)

We know how to compute all the pieces of (8-1): the bimodule $\widehat{CFDD}(M_{i_k}(g_k))$ is computed in Proposition 8.2; the bimodules $\widehat{CFDD}(m_i)$ are computed in Theorem 2; and the bimodules $\widehat{CFDD}(\mathbb{I})$ are computed in Theorem 1.

Remark 8.4 In the inductive computation, the first step formally uses bordered Floer homology of a manifold with boundary S^2. In this degenerate case, the definitions from bordered Floer homology give $\mathcal{A}(\mathcal{Z}) = \mathbb{F}_2$, and the module $\widehat{CFD}(Y)$ is just $\widehat{CF}(Y \cup_\partial \mathbb{D}^3)$. In particular, $\widehat{CFD}(\mathbb{D}^3) = \mathbb{F}_2$.

Remark 8.5 Another way to compute $\widehat{CFD}(Y)$ is to decompose Y as the union of a handlebody and a compression body. Computing the invariant of a compression body with a standard bordering is a simple extension of Proposition 8.2 (or, in fact, follows from that proposition). Section 1.4 explains how to compute the invariant of a handlebody with an arbitrary framing. Theorem 2.20 then says how to compute $\widehat{CFD}(Y)$.

8.3 Computing *CFDD*^ of a 3–manifold with two boundary components

Recall that associated to a strongly bordered 3–manifold Y with two boundary components is a bordered 3–manifold Y_{dr} with connected boundary, obtained by deleting a neighborhood of the framed arc from Y. Suppose Y is bordered by $-F(\mathcal{Z}_1)$ and $-F(\mathcal{Z}_2)$. Then Y_{dr} is bordered by $-F(\mathcal{Z}_1 \# \mathcal{Z}_2)$. There is a projection map $p\colon \mathcal{A}(\mathcal{Z}_1 \# \mathcal{Z}_2) \to \mathcal{A}(\mathcal{Z}_1) \otimes \mathcal{A}(\mathcal{Z}_2)$ which sets to zero any algebra element crossing between $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Then $\widehat{CFDD}(Y)$ is defined to be the induced module $p_*(\widehat{CFD}(Y_{dr}))$.

Section 8.2 explains how to compute $\widehat{CFD}(Y_{dr})$. Thus, we now know how to compute $\widehat{CFDD}(Y)$.

Remark 8.6 It is possible to give a more direct computation of $\widehat{CFDD}(Y)$, without using Y_{dr}. The version of the pairing theorem we have used so far is inconvenient for this. Suppose Y_1 and Y_2 are strongly bordered 3–manifolds with two boundary components, and $\partial_R(Y_1) = F(\mathcal{Z}) = -\partial_L(Y_2)$. Then

$$\mathrm{Mor}\big({}_{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}\,(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(\mathcal{Z})}, \widehat{CFDD}(Y_1) \otimes \widehat{CFDD}(Y_2)\big)$$

is not $\widehat{CFDD}(Y_1 \cup_\partial Y_2)$, but rather $\widehat{CFDD}(\tau_\partial^{-1}(Y_1 \cup_\partial Y_2))$, the result of gluing Y_1 to Y_2 and then decreasing the framing on the arc by one.

To remedy this, one could use any of several other variants of the pairing theorem. One way forward will be apparent in Section 9.

9 Computing (with) type-A invariants

In [6], we consider two types of modules, $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$. There are analogues $\widehat{CFDD}(Y)$, $\widehat{CFDA}(Y)$ and $\widehat{CFAA}(Y)$ in the two boundary component case [5]. Until now, we have focused exclusively on $\widehat{CFD}(Y)$ and $\widehat{CFDD}(Y)$. There are several reasons for doing this. Type-D modules are easier to compute, as they count fewer holomorphic curves. They are algebraically simpler to describe, because they are ordinary differential modules, rather than $\mathcal{A}_\infty$–modules. Moreover, thanks to duality results from [5], it is possible to formulate the theory purely in terms of $\widehat{CFD}$ and $\widehat{CFDD}$; this formulation serves to shorten the exposition.

However, in some contexts it is useful to think of type-A modules. To this end, we recall how to extract the type-A modules, and type-DA and -AA bimodules, from the type-D modules and -DD bimodules via the duality result, [5, Proposition 9.2]. (Alternatively, one could use any of several results from [8].) This is discussed in Section 9.1.

One advantage of working with type-A modules is that one can work with chain complexes with fewer generators; the cost is more complicated algebra actions. This is a special case of a class of results called homological perturbation theory. We review the relevant instance in Section 9.2. Using this, we discuss reconstructing the closed invariant using type-A modules in Section 9.3. As an example of these techniques, we compute a small (in fact, minimal) model for the type-AA module for the identity map of the torus in Section 9.4 (where here size is measured by the rank of the underlying vector space). In Sections 9.5 and 9.6 we discuss some computer computations.

9.1 Computing type-A invariants

The key step in computing the type-A modules and bimodules from the type-D modules is understanding the type-AA module for the identity map. Fix a pointed matched circle $\mathcal{Z}$. We proved in [5, Proposition 9.2] that

$$\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) = \mathrm{Mor}_{\mathcal{A}(-\mathcal{Z})}\big({}_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}), \mathcal{A}(-\mathcal{Z})\big). \tag{9-1}$$

Here, $\mathrm{Mor}_{\mathcal{A}(-\mathcal{Z})}$ denotes the chain complex of $\mathcal{A}(-\mathcal{Z})$–module maps between the bimodules. This complex retains commuting right actions by $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$. Theorem 1 calculates $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$; this description can be combined with (9-1) to give an explicit description of $\widehat{CFAA}(\mathbb{I})$.

For the purposes of this paper, the reader unfamiliar with the definition of $\widehat{CFAA}$ in terms of holomorphic curves can safely take (9-1) (as well as Equations (9-2), (9-3) and (9-4)) as a definition.

Tensoring with the type-AA bimodule of the identity map turns type-D invariants into type-A invariants. More precisely, suppose Y is a $\mathcal{Z}$–bordered 3–manifold. Then

$$\widehat{CFA}(Y) \simeq \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{CFD}(Y). \tag{9-2}$$

Here, $\boxtimes$, an operation taking in one $\mathcal{A}_\infty$–module and one type-D structure, denotes a particular model for the derived tensor product [5, Section 2.3.2].

Similarly, if Y' is a strongly bordered 3–manifold with boundary components $F(\mathcal{Z}_1)$ and $F(\mathcal{Z}_2)$ then

$${}_{F(\mathcal{Z}_1)}\widehat{CFDA}(Y')_{F(\mathcal{Z}_2)} \simeq \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_1}) \boxtimes \widehat{CFDD}(Y'), \tag{9-3}$$

$${}_{F(\mathcal{Z}_1),F(\mathcal{Z}_2)}\widehat{CFAA}(Y') \simeq \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_2}) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_1}) \boxtimes \widehat{CFDD}(Y'). \tag{9-4}$$

In Section 8, we explained how to compute $\widehat{CFD}(Y)$ and $\widehat{CFDD}(Y')$. So, Equations (9-2), (9-3) and (9-4) give algorithms for computing the type-A, -DA and -AA invariants, as well.

We can describe this procedure for calculating the bimodules in more detail, as follows. Theorem 2 computes the type-DD modules for arc-slides, and Proposition 8.2 computes the type-DD module for a split elementary cobordism. Using (9-3), we can turn each of these bimodules into a type-DA module. Any bordered 3–manifold Y can be factored as

$$Y = Y_1 \,{}_{\partial_R}\!\cup_{\partial_L}\, Y_2 \,{}_{\partial_R}\!\cup_{\partial_L} \cdots {}_{\partial_R}\!\cup_{\partial_L}\, Y_k,$$

where each Y_i is either an arc-slide or a split elementary cobordism. The pairing theorem [5, Theorem 12] then gives

$$\widehat{CFDA}(Y) = \widehat{CFDA}(Y_1) \boxtimes \widehat{CFDA}(Y_2) \boxtimes \cdots \boxtimes \widehat{CFDA}(Y_k). \tag{9-5}$$

We can compute $\widehat{CFDD}(Y)$ or $\widehat{CFAA}(Y)$ by tensoring $\widehat{CFDA}(Y)$ with $\widehat{CFDD}(\mathbb{I})$ or $\widehat{CFAA}(\mathbb{I})$, respectively.

As a special case, (9-5) gives another algorithm for reconstructing $\widehat{CF}(Y)$ for a closed 3–manifold Y via a Heegaard splitting of Y. (Actually, unpacking the definitions, this is the same as the formula in Theorem 4.)

9.2 Homological perturbation theory

One key advantage of $\mathcal{A}_\infty$–modules is that for any $\mathcal{A}_\infty$–module (and in particular, any dg–module) M, the homology $H_*(M)$ of M carries an $\mathcal{A}_\infty$–module structure which is (under mild assumptions) homotopy equivalent to M. We state a version of this result presently; for more on this, see for example Keller [3] (which discusses the case of algebras, rather than modules).

Fix a ground ring (with trivial differential) $\boldsymbol{k}$ of characteristic 2. We consider $\mathcal{A}_\infty$ algebras over $\boldsymbol{k}$, and strictly unital $\mathcal{A}_\infty$–modules. In particular, such $\mathcal{A}_\infty$–modules over $\mathcal{A}$ are honest differential modules over $\boldsymbol{k}$.

Lemma 9.1 *Let $\mathcal{M}$ be an $\mathcal{A}_\infty$–module over an $\mathcal{A}_\infty$–algebra $\mathcal{A}$ over $\boldsymbol{k}$, let M denote its underlying chain complex over $\boldsymbol{k}$, and let $f\colon N \to M$ be a homotopy equivalence of $\boldsymbol{k}$–modules. Then we can find:*

- *An $\mathcal{A}_\infty$–module structure $\mathcal{N}$ on N.*
- *An $\mathcal{A}_\infty$ quasi-isomorphism*

$$F\colon \mathcal{N} \to \mathcal{M}$$

with the property that $F_1 = f$.

(Of course, if $\boldsymbol{k}$ is $\mathbb{F}_2$, or more generally a direct sum of copies of $\mathbb{F}_2$ then we can replace the condition that f be a homotopy equivalence by the condition that f be a quasi-isomorphism: for these rings, any quasi-isomorphism is a homotopy equivalence.)

Proof We recall now some notation. Let $\mathcal{A}[1]$ denote the algebra $\mathcal{A}$ with grading shifted by 1 (or, in the group-graded context, shifted by λ; ie $\mathcal{A}[1]_g = \mathcal{A}_{\lambda^{-1}g}$). If V is a $\boldsymbol{k}$–bimodule, let $\mathcal{T}^*(V)$ denote the tensor algebra on V (this includes the 0^{th} tensor product), and let $\mathcal{T}^+(V) \subset \mathcal{T}^*(V)$ denote the ideal generated by V. Here (and below), tensor products will be taken over $\boldsymbol{k}$. In practice, we will apply this construction in the case where $V = \mathcal{A}[1]$, thought of as a $\boldsymbol{k}$–bimodule.

Let $g\colon M \to N$ be a homotopy inverse to f and $T\colon M \to M$ be a homotopy between $f \circ g$ and $\mathbb{I}_M$.

The $\mathcal{A}_\infty$–module structure on $\mathcal{N}$ is given as follows. We write the $\mathcal{A}_\infty$–module structure on $\mathcal{M}$ as a map $m\colon M \otimes \mathcal{T}^*(A[1]) \to M$. Comultiplication induces a map

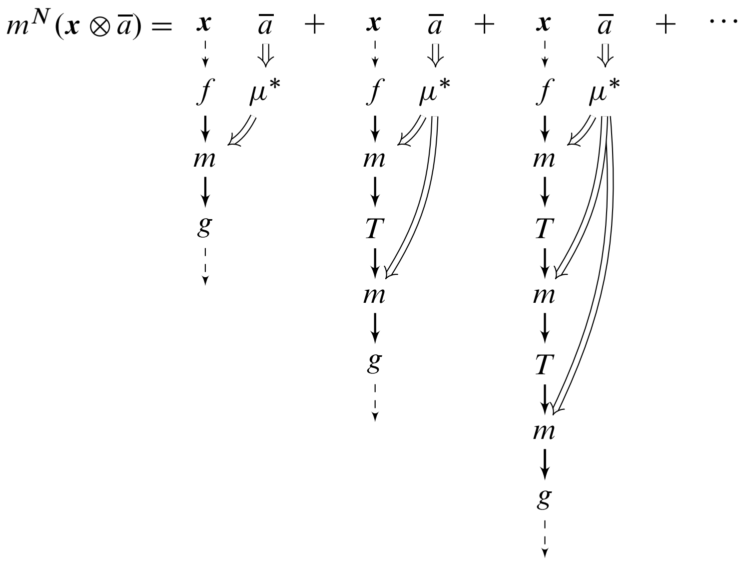
$$\mu^*\colon \mathcal{T}^+(\mathcal{A}) \to \mathcal{T}^*(\mathcal{T}^+(\mathcal{A}[1])),$$

defined as follows: For $\overline{a} = a_1 \otimes \cdots \otimes a_n \in \mathcal{T}^+(\mathcal{A})$, let

$$\mu^*(a_1 \otimes \cdots \otimes a_n) = \sum_{\{i_1,\dots,i_k \mid 1 \le i_1 < \cdots < i_k < n\}} (a_1 \otimes \cdots \otimes a_{i_1}) \otimes (a_{i_1+1} \otimes \cdots \otimes a_{i_2}) \otimes \cdots \otimes (a_{i_k+1} \otimes \cdots \otimes a_n).$$

With these maps in hand, define the operations m_i for $i > 1$ on $\mathcal{N}$ by:

(9-6)



Here, doubled arrows indicate elements of $\mathcal{T}^+\mathcal{A}$; in particular, there is always at least one algebra element present. Dashed lines indicate elements of N, while solid lines indicate elements of M. It is a property of the comultiplication μ^* that for any given $\overline{a}$, there are only finitely many nonzero elements in this sum.

We verify this is indeed an $\mathcal{A}_\infty$–module. As usual, m^N induces an endomorphism $\overline{m}$ of $N \otimes \mathcal{T}^*(\mathcal{A})$, and μ a differential d on $\mathcal{T}^*(\mathcal{A})$. We must check that

$$\overline{m}^2(\boldsymbol{x}, \overline{a}) + \overline{m}(x, d\overline{a}) = 0, \tag{9-7}$$

for any $\overline{a} = a_1 \otimes \cdots \otimes a_n$. We begin this verification by considering terms in the second sum $\overline{m}(x, d\overline{a})$. Applying the $\mathcal{A}_\infty$ relation on M, we see that $\overline{m}(x, d\overline{a})$ can be interpreted as counting the same kinds of trees as in the definition of $m^N(\boldsymbol{x} \otimes \overline{a})$, except for one difference: whereas the m–labeled vertices in the definition of $m^N(\boldsymbol{x} \otimes \overline{a})$ all have incoming algebra elements (and no two of these m–labeled vertices are consecutive), the trees in $\overline{m}(x, d\overline{a})$ have a pair of consecutive m–labeled vertices (one of which may have no incoming algebra elements). Equivalently, we can think of these as counting trees obtained from trees counted in $\overline{m}(x, \overline{a})$, by applying one of the following operations:

- (T-1) Insert an m–labeled vertex with no incoming algebra elements immediately before some T–labeled vertex.
- (T-2) Insert an m–labeled vertex with no incoming algebra elements immediately after some T–labeled vertex.
- (T-3) Insert an m–labeled vertex with no incoming algebra elements immediately after the initial f–labeled vertex.
- (T-4) Insert an m–labeled vertex with no incoming algebra elements before the terminal g–labeled vertex.
- (T-5) Split some m–labeled vertex with at least 2 incoming algebra elements to a pair of consecutive m–labeled vertices, each of which has at least 1 incoming algebra element.

Terms of type (T-1) pair off with terms of type (T-2) (in view of the formula $d \circ T + T \circ d = \mathbb{I} + f \circ g$) to produce a sum of two types of trees: one of these types match those of type (T-5); the other type is gotten by applying the following operation:

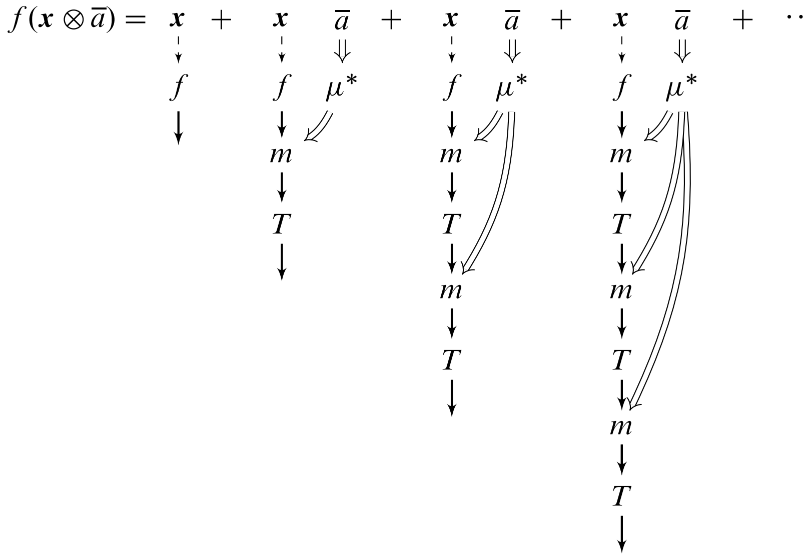
- (T-6) Replace a T–labeled vertex by a vertex labeled by $f \circ g$.

Thus, $\overline{m}(x, d\overline{a})$ counts trees of types (T-3), (T-4) and (T-6). The fact that f and g are chain maps allow us to move the differentials past the f and g–labeled vertices in the trees of types (T-3) and (T-4).

But these terms are precisely the trees counted in $\overline{m}^2(\boldsymbol{x}, \overline{a})$; and hence (9-7) holds.

We claim that f extends to an $\mathcal{A}_\infty$ homomorphism, which has the following graphical representation:

(9-8)



We leave the verification that this extension is, indeed, an $\mathcal{A}_\infty$ homomorphism as an exercise for the reader. The fact that $F_1 = f$ is immediate, and hence F is a quasi-isomorphism. $\square$

9.3 Reconstruction via type-A modules, and its advantages

As discussed in Section 9.1, Theorem 4 can be reformulated as follows. As before, write Y as a union of two 0–framed handlebodies H^g, identified via some gluing map ϕ. Once again, decompose ϕ as a composition of arc-slides, $\phi = m_1 \circ \cdots \circ m_k$. Next, calculate $\widehat{CFDA}(m_i)$ for each arc-slide, using Theorem 2 and (9-3); and $\widehat{CFA}$ for the handlebody H^g using Proposition 1.12 and (9-2) (or by simply writing down $\widehat{CFA}(\mathsf{H}^g)$ directly by counting holomorphic curves). Then, there is a homotopy equivalence

$$\begin{aligned}\widehat{CF}(Y) &\simeq \widehat{CFA}(\mathsf{H}^g) \boxtimes \widehat{CFDA}(m_1) \boxtimes \cdots \boxtimes \widehat{CFDA}(m_k) \widehat{CFD}(\mathsf{H}^g) \\ &= \widehat{CFD}(\mathsf{H}^g) \boxtimes \widehat{CFAA}(\mathbb{I}) \boxtimes \widehat{CFDD}(m_1) \boxtimes \widehat{CFAA}(\mathbb{I}) \\ &\qquad \boxtimes \cdots \boxtimes \widehat{CFDD}(m_k) \boxtimes \widehat{CFAA}(\mathbb{I}) \boxtimes \widehat{CFD}(\mathsf{H}^g).\end{aligned} \tag{9-9}$$

(For clarity, we are viewing the bimodules as each having one left and one right action.)

So far, we have not gained anything computationally. The point is that, since the operation $\boxtimes$ respects quasi-isomorphisms, we are free to replace the bimodule $\widehat{CFAA}(\mathbb{I})$ with a smaller model than the one given by (9-1). Thinking of the type-AA module associated to the standard diagram for the identity map (Figure 13), it is clear that there is a model for $\widehat{CFAA}(\mathbb{I})$ whose rank over $\mathbb{F}_2$ is the number of idempotents in $\mathcal{A}(\mathcal{Z})$. One can arrive at a model of this size by taking the homology of the chain complex $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ (thought of as a vector space over $\mathbb{F}_2$). Although this homology $H_*(\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}))$ is no longer an honest $\mathcal{A}(\mathcal{Z})$ bimodule, by Lemma 9.1 it does retain the structure of an $\mathcal{A}_\infty$–bimodule over $\mathcal{A}(\mathcal{Z})$ (which is quasi-isomorphic to $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$).

Replacing $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ by its homology drastically reduces the number of generators. For instance, we will see in Section 9.4 that in the torus boundary case, we have that $\dim_{\mathbb{F}_2}(\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}},0)) = 30$, while $\dim_{\mathbb{F}_2}(H_*(\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}},0))) = 2$. (The 0 in the notation here refers to the restriction to the portion of the algebra with strands weight zero, ie with exactly one moving strand or two matched horizontal strands.)

In practice, to work with the fewest generators possible, one would want to compute (9-9) by starting at one end, and taking homology (using Lemma 9.1) after each tensor product.

Doing the replacement of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ with $H_*(\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}))$, one has to take care with the boundedness hypotheses needed for $\boxtimes$ to be well defined. If we take a bounded model for $\widehat{CFD}(\mathsf{H}^g)$ on the left of (9-9), and compute the tensor products starting at the far right, the relevant boundedness hypotheses are satisfied throughout, even if one takes homology after each tensor product.

As a last computational point, we can use a smaller model for the algebra $\mathcal{A}(\mathcal{Z})$: we can divide out by the differential ideal of algebra elements with local multiplicity at least 2 somewhere. (See [5, Proposition 4.15].)

9.4 Example: Torus boundary

Let $\mathbb{I} = \mathbb{I}_{\mathcal{Z}^1}$ denote the identity map of the torus. In this section we calculate the AA identity bimodule for the torus with strands weight zero, $\widehat{CFAA}(\mathbb{I}) = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}^1},0)$, using results of the previous subsections. This module was calculated before, by explicitly finding all the relevant holomorphic curves, in [5, Section 10.1]. The method here is more algebraic, and generalizes in a straightforward way to arbitrary genus; direct holomorphic curve counts in higher genus would be complicated at best, and probably intractable.

Let $\mathcal{A} = \mathcal{A}(\mathcal{Z}^1,0)$ be the algebra associated to the torus with strands weight equal to zero, and $\mathcal{A}' = \mathcal{A}(-\mathcal{Z}^1,0)$. (The algebras $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic, but it will be clearer to treat them as distinct.) Note that $\widehat{CFDD}(\mathbb{I})$ has eight generators as a left $\mathcal{A}$ module; it also admits a left action by $\mathcal{A}'$, a ring whose generators we denote λ_i rather than ρ_i.

Similarly, $\mathcal{A}$ has eight generators as an $\mathbb{F}_2$–vector space.

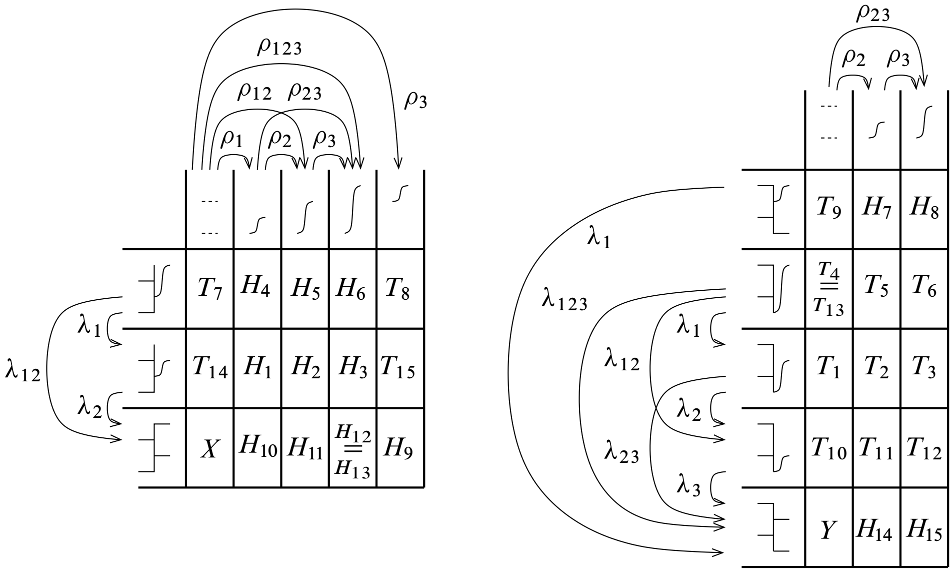


Figure 42: *Genus-one identity* Dualizing the type-DD bimodule, in the case of genus one

By (9-1), $\widehat{CFAA}(\mathbb{I})$ is equivalent to the bimodule of left $\mathcal{A}$–module maps from $\widehat{CFDD}(\mathbb{I})$ to $\mathcal{A}$. Such an $\mathcal{A}$–linear map from $\widehat{CFDD}(\mathbb{I})$ to $\mathcal{A}$ is determined by the image of a basis for $\widehat{CFDD}(\mathbb{I})$; and it is zero unless elements are mapped to elements with compatible idempotents.

Using the description of $\widehat{CFDD}(\mathbb{I})$ from Theorem 1, there are 30 generators of the bimodule $\mathrm{Hom}_{\mathcal{A}}(\widehat{CFDD}(\mathbb{I}), \mathcal{A})$. It is straightforward to find the 15 differentials which connect various generators. The results of this are illustrated in Figure 42, with the following convention. Generators are named H_i, T_i, X or Y, with the understanding that H_i appears in the differential of T_i, while the generators X and Y have no differentials either entering or leaving them. The labels H_{12} and H_{13} refer to the same generator, as do T_4 and T_{13}. In the table, rows are indexed by generators of $\mathcal{A}' \otimes \widehat{CFDD}(\mathbb{I})$, and the columns are indexed by generators of $\mathcal{A}$. Each square corresponds to the morphism which takes the generator in that row to the generator in that column (and all other generators to zero). The right action by $\mathcal{A}$ is indicated by the arrows connecting the columns, while the right action by $\mathcal{A}'$ is indicated by the arrows connecting the rows. (Note that left translation in $\mathcal{A}'$ dualizes to the stated right action.)

The homology of the complex is two-dimensional, generated by the generators X and Y. Thus, $\widehat{CFAA}(\mathbb{I})$ is quasi-isomorphic to an $\mathcal{A}_\infty$–bimodule with just two generators, X_0

and Y_0 (as we already knew from the standard Heegaard diagram for the identity map). Indeed, the quasi-isomorphism

$$f\colon H(\widehat{CFAA}(\mathbb{I})) \to \widehat{CFAA}(\mathbb{I})$$

defined by $f(X_0) = X$, $f(Y_0) = Y$ has a homotopy inverse

$$g\colon \widehat{CFAA}(\mathbb{I}) \to H(\widehat{CFAA}(\mathbb{I}))$$

defined by $g(X) = X_0$, $g(Y) = Y_0$, $g(H_i) = g(T_i) = 0$. Note that $g \circ f = \mathbb{I}_{H_*(CFAA^\wedge(\mathbb{I}))}$, while $f \circ g \simeq \mathbb{I}_{CFAA^\wedge(\mathbb{I})}$ via the map

$$T\colon \widehat{CFAA}(\mathbb{I}) \to \widehat{CFAA}(\mathbb{I})$$

specified by

$$\begin{aligned} T(X) &= 0, \\ T(Y) &= 0, \\ T(T_j) &= 0, \qquad 1 \le j \le 15, \\ T(H_4) &= T_4 + T_{12}, \\ T(H_i) &= T_i, \qquad 1 \le i \le 15,\ i \notin \{4, 13\}, \end{aligned} \tag{9-10}$$

where here $i, j \in 1, \dots, 15$, and $i \neq 4, 13$. (At first glance, it might appear that T is not defined on H_{13}; but in fact $H_{13} = H_{12}$, so $T(H_{13}) = T_{12}$.)

An explicit form of the $\mathcal{A}_\infty$ structure on $H_*(\widehat{CFAA}(\mathbb{I}))$ is constructed in the proof of Lemma 9.1. We can think of this graphically, as follows. Consider the directed graph whose nodes correspond to the 30 generators of $\mathrm{Hom}_{\mathcal{A}}(\widehat{CFDD}(\mathbb{I}), \mathcal{A})$. Draw an edge from v_1 to v_2 labeled by a basis vector a for the algebra if v_2 appears in $m_2(v_1, a)$; include another kind of edge — a *T–labeled edge* — from v_1 to v_2 if v_2 appears in $T(v_1)$. All $\mathcal{A}_\infty$ operations on $H_*(\widehat{CFAA}(\mathbb{I}))$ correspond to paths with the following properties:

- The path starts and ends at vertices labeled X or Y.
- The initial and final edges in the graph are labeled by algebra elements.
- The path alternates between edges labeled by algebra elements and T–labeled edges.

Each such path corresponds to a term in an $\mathcal{A}_\infty$ operation, starting at the initial vertex v, with coefficient 1 in the terminal vertex w; and the sequence of algebra elements gives the sequence of operations. More precisely, if $r_1, \dots, r_m$ and $\ell_1, \dots, \ell_n$ are the sequences of algebra elements in the order they are encountered, where here the r_i correspond to those labeled by algebra elements gotten by products of ρ_j, while the ℓ_i correspond to those gotten as products of the λ_j, then this path gives a contribution of w in the operation $m_{1,n,m}(v,(\ell_1\otimes\cdots\otimes\ell_n),(r_1\otimes\cdots\otimes r_m))$. Note that paths of the above type in this graph coincide with the smaller graph, where we include algebra-labeled edges only in cases where the terminal point of the edge is either the initial vertex of a T–labeled edge, or it is one of X or Y. This smaller graph is illustrated in Figure 43 (except that we have drawn several vertices corresponding to X and Y, for clarity).

For instance, we see that $m_{1,1,1}(X_0,\rho_3,\lambda_2)=Y_0$, via the path through X, H_9, T_9, and Y. More generally, by traveling around the loop through H_9, T_9, H_8, and T_8 we see that

$$m\big(X_0,\rho_3,\overbrace{\rho_{23},\dots,\rho_{23}}^{n},\overbrace{\lambda_{12},\dots,\lambda_{12}}^{n},\lambda_2\big)=Y_0$$

for any $n\geq 0$. (Note that to draw this conclusion, we need to check that there are no other paths in the graph which give rise to a cancelling term.)

9.5 A computer computation

We conclude by describing a computer computation of $\widehat{HF}$ of the Poincaré homology sphere, via an open book decomposition. For this, we will use a particular class of handlebodies, useful for studying open books:

Definition 9.2 Let $\mathcal{Z}$ be a pointed matched circle. The *self-gluing handlebody of* $\mathcal{Z}$, denoted $\mathsf{H}_{sg}(\mathcal{Z})$, is the handlebody corresponding to a bordered Heegaard diagram $\mathcal{H}_{sg}(\mathcal{Z})$ obtained from the standard Heegaard diagram for the identity map of $\mathcal{Z}$ by deleting the arc $\boldsymbol{z}$ and placing a basepoint on one of the two sides.

See the lower right of Figure 44 for the case when $\mathcal{Z}$ is the genus-1 pointed matched circle. Also, observe that $\partial\mathsf{H}_{sg}(\mathcal{Z})=(-\mathcal{Z})\,\#\,\mathcal{Z}$.

Of course, $\mathsf{H}_{sg}(\mathcal{Z})$ can be obtained from the split handlebody of genus g by doing a sequence of arc-slides. Suppose $\mathcal{Z}$ is the pointed matched circle of genus 1. Then we can get $\mathsf{H}_{sg}(\mathcal{Z})$ from the split genus-2 handlebody by performing 8 handleslides. Numbering the points along the boundary from 1 to 8, we perform the handleslides: 5 over 4; 2 over 1; 3 over 2; 4 over 3; 2 over 1; 6 over 5; 7 over 6; 2 over 3. See Figure 44. In Section 9.6, we explain how to find this sequence of handleslides.

We can use a computer to calculate $\widehat{CFD}(\mathcal{Z})$, for instance by computing inductively

$$\widehat{CFD}(\mathsf{H}_i)=\mathrm{Mor}\big(\widehat{CFDD}(-\psi_i),\widehat{CFD}(\mathsf{H}_{i-1})\big),$$

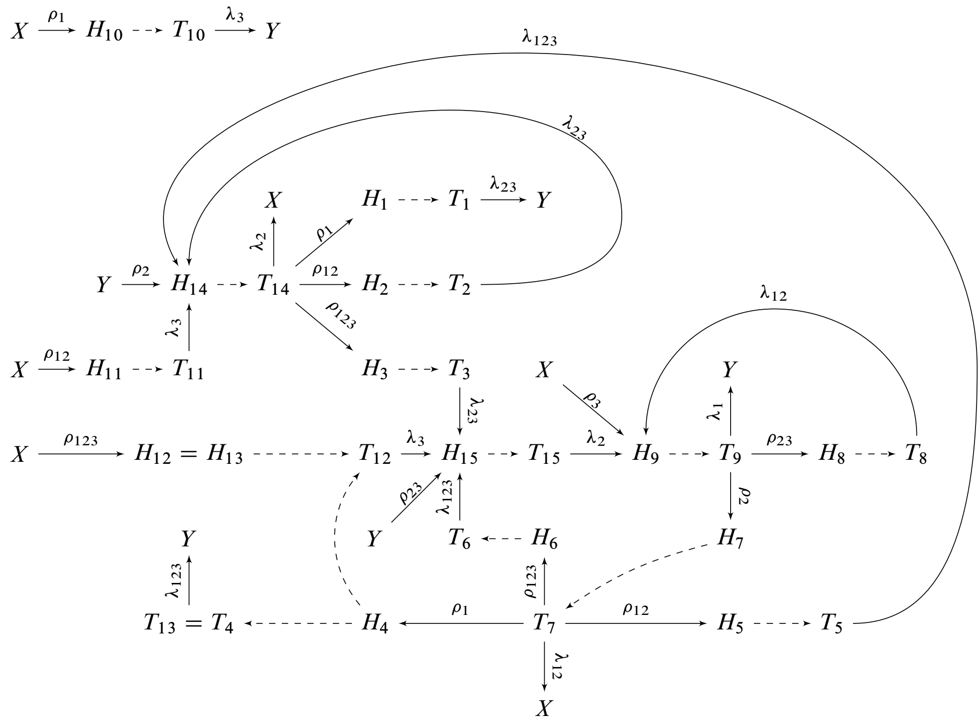


Figure 43: *Genus-one identity type-AA bimodule* A graphical representation of the $\mathcal{A}_\infty$ operations on $\mathit{CFAA}^\wedge(\mathbb{I})$. T–labeled edges are shown dashed. All vertices labeled X are identified, as are all vertices labeled Y. The algebra-labeled edges can be immediately read off from the algebra operations coming from Figure 42; the T–labeled edges are determined by (9-10).

where ψ_i is the i^{th} arc-slide, $-\psi_i$ denotes the same map but between orientation-reversed surfaces, H_0 is the 0–framed, split handlebody, and $\mathsf{H}_i = \psi_i(\mathsf{H}_{i-1})$. This computation leads to type-D structures with the following number of generators before and after simplification (cancelling arrows labeled by idempotents):

Diagram	Gens. before simplifying	Gens. after simplifying	Diagram	Gens. before simplifying	Gens. after simplifying
$\widehat{CFD}(\mathsf{H}_1)$	34	2	$\widehat{CFD}(\mathsf{H}_5)$	33	1
$\widehat{CFD}(\mathsf{H}_2)$	56	2	$\widehat{CFD}(\mathsf{H}_6)$	50	2
$\widehat{CFD}(\mathsf{H}_3)$	57	1	$\widehat{CFD}(\mathsf{H}_7)$	130	2
$\widehat{CFD}(\mathsf{H}_4)$	31	3	$\widehat{CFD}(\mathsf{H}_8)$	134	4

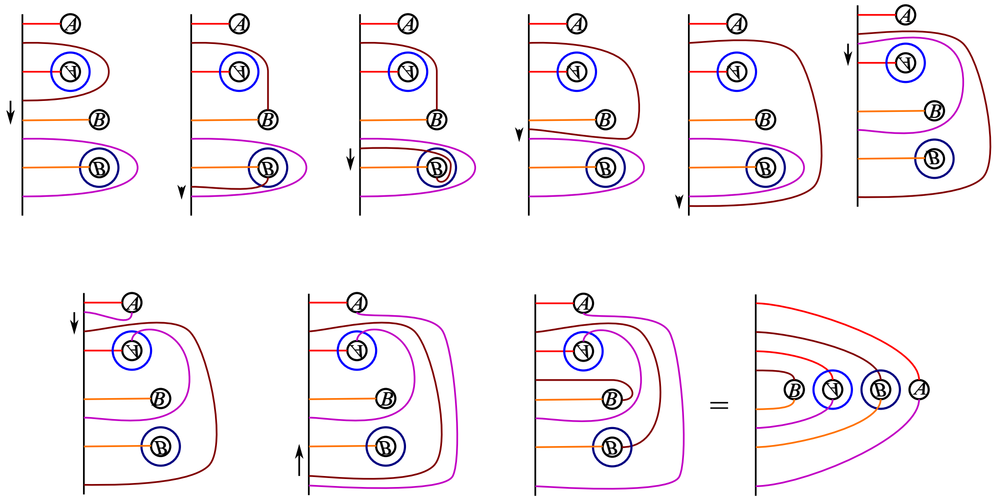


Figure 44: Obtaining the genus-2 self-gluing handlebody by arc-slides

(These computations were done using Sage [14]. Code for doing such computations, together with documentation including this example, is available from [4].)

The result has a more conceptual description, via a mild generalization of Theorem 1. First, let γ be the arc in $(-\mathcal{Z})\#\mathcal{Z}$ which runs from $-\mathcal{Z}$ to $\mathcal{Z}$. There is a map $\mathcal{A}((-\mathcal{Z})\#\mathcal{Z}) \to \mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z})$ gotten by setting to zero any basis element crossing a. Call an element $a \in \mathcal{A}((-\mathcal{Z})\#\mathcal{Z})$:

- *Symmetric of type I* if a does not cross γ, and the image of a in $\mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z})$ is a chord-like element of the diagonal subalgebra.
- *Symmetric of type II* if a consists of a single chord ξ from $4k-i$ to $4k+i$, where k is the genus of $\mathcal{Z}$.

Also, there is an inclusion map $\mathcal{A}(-\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}) \to \mathcal{A}((-\mathcal{Z})\#\mathcal{Z})$.

Theorem 9.3 *Let $\mathsf{H}_{sg}(\mathcal{Z})$ be the self-gluing handlebody of $\mathcal{Z}$. Then $\widehat{CFD}(\mathsf{H}_{sg}(\mathcal{Z}))$ is generated by the images in $\mathcal{A}((-\mathcal{Z})\#\mathcal{Z})$ of all pairs of complementary idempotents. The differential is given by right multiplication by*

$$\sum_{a \text{ symmetric of type I or II}} a.$$

Proof sketch This follows from a factorization lemma just like Lemma 3.5, except that there is an additional type of chord: a chord covering the middle region of the diagram, going between $-\mathcal{Z}$ and $\mathcal{Z}$. $\square$

The Poincaré homology sphere has a genus-1, one boundary component open book decomposition with monodromy $(\tau_a\tau_b)^5$, where τ_a and τ_b are Dehn twists around a pair of dual curves in the punctured torus. Note that these Dehn twists can be viewed as arc-slides in the genus-1 pointed matched circle, of point 2 over point 1 and point 3 over point 2, respectively.

Extend these arc-slides by the identity map to arc-slides A and B of the genus-2 split pointed matched circle. Then the Poincaré homology sphere is given by

$$-\mathsf{H}_{sg}(\mathcal{Z}) \cup Y_{(AB)^5} \cup \mathsf{H}_{sg}(\mathcal{Z})$$

(see Figure 45).

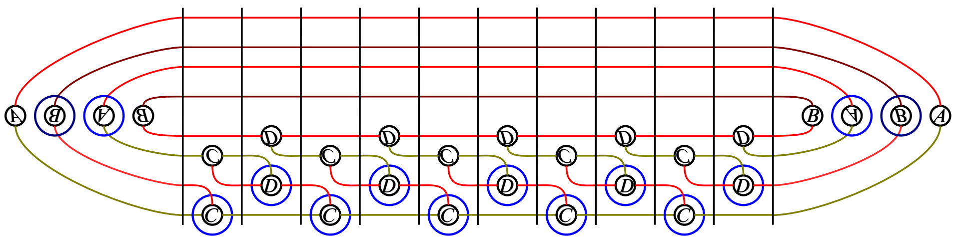


Figure 45: *Heegaard diagram for the Poincaré homology sphere* This diagram is obtained by gluing together a diagram for $-\mathsf{H}_{sg}$ and H_{sg}, the self-gluing handlebody and its reflection, with bordered Heegaard diagrams for ten arc-slides (Dehn twists) in between, and then destabilizing 30 times.

We can again compute this by computer. Let $\mathsf{H}'_i = Y_{B(AB)^{i-1}} \cup \mathsf{H}_{sg}$, $\mathsf{H}''_i = Y_{(AB)^i} \cup \mathsf{H}_{sg}$. Calculating $\widehat{CFD}(\mathsf{H}'_i)$ and $\widehat{CFD}(\mathsf{H}''_i)$ inductively as

$$\widehat{CFD}(\mathsf{H}'_i) = \mathrm{Mor}\big(\widehat{CFDD}(-Y_B), \widehat{CFD}(\mathsf{H}''_{i-1})\big),$$
$$\widehat{CFD}(\mathsf{H}''_i) = \mathrm{Mor}\big(\widehat{CFDD}(-Y_A), \widehat{CFD}(\mathsf{H}'_i)\big),$$

computer computation gives type-D structures with the following ranks before and after simplification:

Diagram	Gens. before simplifying	Gens. after simplifying	Diagram	Gens. before simplifying	Gens. after simplifying
$\widehat{CFD}(\mathsf{H}'_1)$	229	7	$\widehat{CFD}(\mathsf{H}''_1)$	317	5
$\widehat{CFD}(\mathsf{H}'_2)$	250	6	$\widehat{CFD}(\mathsf{H}''_2)$	263	7
$\widehat{CFD}(\mathsf{H}'_3)$	337	9	$\widehat{CFD}(\mathsf{H}''_3)$	374	10
$\widehat{CFD}(\mathsf{H}'_4)$	445	11	$\widehat{CFD}(\mathsf{H}''_4)$	447	13
$\widehat{CFD}(\mathsf{H}'_5)$	586	14	$\widehat{CFD}(\mathsf{H}''_5)$	567	15

Finally, we have

$$\widehat{CF}(Y) = \mathrm{Mor}\big(\widehat{CFD}(\mathsf{H}_{sg}), \widehat{CFD}(\mathsf{H}_5'')\big)$$

(where Y is the Poincaré homology sphere). This is a complex with 405 generators, and 1–dimensional homology.

9.6 Finding handleslide sequences

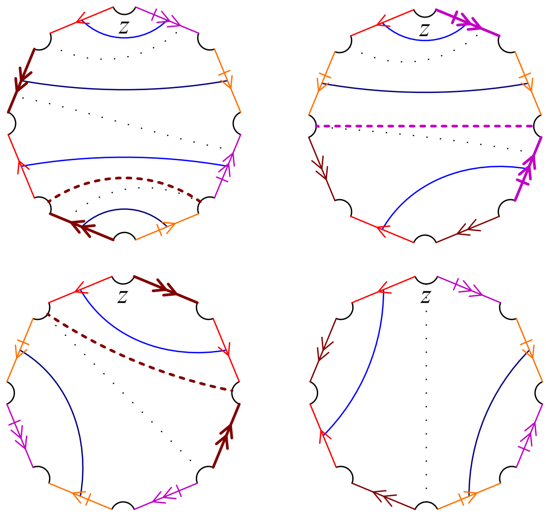


Figure 46: *The source of the handleslides in Figure 44* We start with a Heegaard diagram for the self-gluing handlebody and end up with the split handlebody by a sequence of cuttings and regluings. At each stage, we glue the edges drawn thick, and cut along the dashed line. The dotted line is the desired separating curve.

We now explain how we found the sequence of handleslides used to construct the self-gluing handlebody from the split handlebody, as in Figure 44. It is easiest to compute the sequence of handleslides going in the other direction, starting from a self-gluing handlebody and sliding until we obtain a split handlebody. The split handlebody has two distinguishing characteristics:

- The two β–circles each intersect a single α–arc once.
- There is a separating loop γ from the boundary to itself that does not intersect any α–arcs.

We try to achieve these two features by successively performing arc-slides among the α–arcs, keeping the two β–circles unchanged. To this end we choose also a separating loop γ. Each arc-slide is chosen to decrease the number of intersection points between the α–arcs, the β–circles, and γ. This is illustrated in Figure 46, where the Heegaard surface is shown split open along the α–arcs, rather than along the β–circles as in, for example, Figure 44. The curve γ is indicated by a dotted arc. In this representation, a sequence of handleslides involving a single α–arc α_i sliding over others consists of

- gluing the sides of the diagram corresponding to α_i and
- cutting open the resulting annulus along a new arc connecting the different sides of the annulus.

At each stage, we were able to choose the arc at the second step to reduce the number of intersections of the α_i with the β–circles.

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