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Asymptotic behaviour and the Nahm transform of doubly periodic instantons with square integrable curvature

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We study the asymptotic behaviour of doubly periodic instantons with square-integrable curvature. Then we establish an equivalence given by the Nahm transform between the doubly periodic instantons with square integrable curvature and wild harmonic bundles on the dual torus. We also introduce algebraic Nahm transforms, which describe the transformations of the underlying filtered objects.

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Dedicated to Professor Shigeru Mukai on the occasion of his 60^{th} *birthday*

1 Introduction

Set $T := \mathbb{C}/L$, where L is a lattice of $\mathbb{C}$. The product $T \times \mathbb{C}$ is equipped with the standard metric $dz\,d\bar{z} + dw\,d\overline{w}$, where (z, w) is the standard local coordinate of $T \times \mathbb{C}$. In this paper, we shall study L^2 instantons (E, ∇, h) on $T \times \mathbb{C}$, ie triples where the curvature $F(\nabla)$ satisfies the equation $\Lambda F(\nabla) = 0$ and is L^2.

There is a natural decay condition around ∞, the quadratic curvature decay, ie $|F(\nabla)| = O(|w|^{-2})$ with respect to h and the Euclidean metric $dz\,d\bar{z} + dw\,d\overline{w}$. M Jardim [24] studied the Nahm transform of some kinds of harmonic bundles with tame singularities on the dual torus $T^\vee$ to produce instantons on $T \times \mathbb{C}$ satisfying quadratic curvature decay. O Biquard and Jardim [8] studied the asymptotic behaviour of such instantons with rank 2. Upon on those results, Jardim [26] constructed an inverse transform, ie the Nahm transform of such instantons on $T \times \mathbb{C}$, to produce some types of harmonic bundles with tame singularities on $T^\vee$. See also Jardim [25; 27] and Ford and Pawlowski [16].

In this paper we aim to to generalise these results. Namely, we will study the asymptotic behaviour of L^2 instantons and establish an equivalence between the L^2 instantons on $T \times \mathbb{C}$ and harmonic bundles with wild singularities on $T^\vee$. We shall also introduce algebraic counterparts of the transforms. They are useful for describing induced transformations of the singular data.

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1.1 Asymptotic behaviour of L^2 instantons

1.1.1 The dimensional reduction of N Hitchin Briefly said, our goal in the study of asymptotic behaviour of L^2 instantons is to show that they behave like wild harmonic bundles around ∞ (see Section 1.1.3). As a preliminary, we recall the dimensional reduction of N Hitchin [22; 23].

Let U be any open subset of $\mathbb{C}$. Let $(V, \overline{\partial}_V)$ be a holomorphic vector bundle on U with a Higgs field θ. Let h be a Hermitian metric on V. We have the Chern connection $\nabla_{V,h} = \overline{\partial}_V + \partial_{V,h}$. We have the adjoint $\theta^\dagger$ of θ with respect to h. The tuple $(V, \overline{\partial}_V, h, \theta)$ is called a harmonic bundle if the Hitchin equation $F(\nabla_{V,h}) + [\theta, \theta^\dagger] = 0$ is satisfied.

Let $p\colon T \times U \to U$ be the projection. We have the expressions $\theta = f\,dw$ and $\theta^\dagger = f^\dagger\,d\overline{w}$, where f is a holomorphic endomorphism of V and $f^\dagger$ is the adjoint of f. We set $(E, h_E) := p^*(V, h)$. Let ∇_E be the unitary connection given by $\nabla_E = p^*(\nabla_{V,h}) + f\,d\bar{z} - f^\dagger dz$. Then (E, h_E, ∇_E) is an instanton if and only if $(V, \overline{\partial}_V, h, \theta)$ is a harmonic bundle. Indeed, Hitchin discovered that the above procedure gives an equivalence between harmonic bundles on U and T–equivariant instantons on $T \times U$.

1.1.2 Examples and remarks We set $U := \{w \in \mathbb{C} \mid |w| > R\}$. The dimensional reduction allows us to construct easy examples of L^2 instantons on $T \times U$. Let $\mathfrak{a}$ be any holomorphic function on U. We obtain a harmonic bundle $\mathcal{L}(\mathfrak{a})$ as the tuple of the trivial line bundle $\mathcal{O}_U e$, the trivial metric $h(e,e) = 1$ and the Higgs field $d\mathfrak{a}$. By using the dimensional reduction above, we get an associated instanton on $T \times U$. Its curvature is $\partial_w^2 \mathfrak{a}\,dw\,d\bar{z} + \overline{\partial_w^2 \mathfrak{a}\,dz\,d\overline{w}}$. In this case, the curvature is L^2 if and only if it has quadratic decay.

We can obtain more examples by considering ramifications along ∞. We set $U_\eta := \{\eta \in \mathbb{C} \mid |\eta| > R^{1/2}\}$. We consider a harmonic bundle $\mathcal{L}(\mathfrak{a})$, where $\mathfrak{a}$ is a holomorphic function on U_η. Let $\varphi\colon U_\eta \to U$ be given by $\varphi(\eta) = \eta^2$. We obtain a harmonic bundle $\varphi_* \mathcal{L}(\mathfrak{a})$ of rank 2 on U by pushforward. It is easy to check that the associated instanton is L^2 if and only if $\eta^{-2}\mathfrak{a}(\eta)$ is holomorphic at ∞. In that case, the curvature F satisfies the decay condition $O(|w|^{-3/2})$. If $\mathfrak{a} = \alpha\eta$ for $\alpha \neq 0$, we have $0 < C_1 < |F||w|^{3/2} < C_2$ for some constants C_i.

More generally, for any positive integer p, we set $U^{\langle p \rangle} := \{w_p \in \mathbb{C} \mid |w_p| > R^{1/p}\}$. For a covering $\varphi_p\colon U^{\langle p \rangle} \to U$ given by $\varphi_p(w_p) = w_p^p$ and for a holomorphic function $\mathfrak{a}$ on $U^{\langle p \rangle}$, we obtain a harmonic bundle $\varphi_{p*}\mathcal{L}(\mathfrak{a})$ of rank p on U. The associated instanton is L^2 if and only if $\varphi_p^*(w)^{-1}\mathfrak{a}$ is holomorphic at ∞. If $\mathfrak{a}$ is polynomial in w_p, then holomorphicity is described as a condition $\deg_{w_p}(\mathfrak{a})/p \le 1$. In that case, the curvature F satisfies $O(|w|^{-1-1/p})$. It is easy to construct an example satisfying $0 < C_1 < |F||w|^{1+1/p} < C_2$ for some $C_i > 0$.

Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on U, ie the (possibly multivalued) eigenvalues of θ are meromorphic at ∞. As the above examples suggest, the L^2 and quadratic decay conditions for the associated instanton can be described in terms of the eigenvalues of θ. For simplicity, by shrinking U, we assume that the ramification of the eigenvalues of θ may happen at most along ∞. If we take an appropriate covering $\varphi_p\colon U^{\langle p\rangle} \to U$, we have a holomorphic decomposition

$$\varphi_p^*(E,\theta) = \bigoplus_{\mathfrak{a}\in w_p\mathbb{C}[w_p]} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}) \tag{1}$$

such that $\theta_{\mathfrak{a}} - d\mathfrak{a}$ are tame, ie for the expression $\theta_{\mathfrak{a}} - d\mathfrak{a} = f_{\mathfrak{a}}\, dw_p/w_p$, the eigenvalues of $f_{\mathfrak{a}}$ are bounded. We set $\mathrm{Irr}(\theta) := \{\mathfrak{a} \mid E_{\mathfrak{a}} \neq 0\}$. Then by using results of the author on the asymptotic behaviour of wild harmonic bundles [36], it is not difficult to prove that the instanton associated to $(E, \overline{\partial}_E, \theta, h)$ is L^2 if and only if $\deg_{w_p}(\mathfrak{a})/p \le 1$ for any $\mathfrak{a} \in \mathrm{Irr}(\theta)$. It moreover satisfies the quadratic decay condition if and only if the harmonic bundle is unramified, ie it has a decomposition as in (1) on U.

The condition can also be described in terms of the spectral variety of θ. We have the expression $\theta = f\, dw$. Let $\mathcal{S}p(f) \subset \mathbb{C}_\zeta \times U$ denote the support of the cokernel of $\zeta - f\colon \mathcal{O}_{\mathbb{C}_\zeta\times U} \to \mathcal{O}_{\mathbb{C}_\zeta\times U}$. It induces a subvariety $\Phi(\mathcal{S}p(f))$, where $\Phi\colon \mathbb{C}_\zeta \times U \to T^\vee \times U$ denotes the projection. Then the instanton associated to $(E, \overline{\partial}_E, \theta, h)$ is L^2 if and only if the closure of $\Phi(\mathcal{S}p(f))$ in $T \times \overline{U}$ is a complex subvariety, where $\overline{U} = U \cup \{\infty\}$.

1.1.3 Brief description of the asymptotic behaviour of the L^2 instantons Let (E, ∇, h) be an L^2 instanton on $T \times U$. Let $(E, \overline{\partial}_E)$ denote the underlying holomorphic vector bundle on $T\times U$. By using a theorem of Uhlenbeck, we obtain $F(\nabla) = o(1)$. This implies that the restrictions $(E, \overline{\partial}_E)_{|T\times\{w\}}$ are semistable of degree 0 if $|w|$ is sufficiently large. Hence, the relative Fourier–Mukai transform of $(E, \overline{\partial}_E)$ gives an $\mathcal{O}_{T^\vee\times U}$–module whose support $\mathcal{S}p(E)$ is finite and flat over U. Our first important result is the following.

Theorem 5.10 *$\mathcal{S}p(E)$ extends to a complex analytic subvariety $\overline{\mathcal{S}p(E)}$ in $T \times \overline{U}$.*

We shall use an effective control of the spectrum of semistable bundles of degree 0 in terms of the eigenvalues of the monodromy transformations of unitary connections with the small curvature (Corollary 4.10). If we fix an embedding $\mathrm{Sym}^{\mathrm{rank}\, E}(T^\vee) \subset \mathbb{P}^N$, the spectrum induces a holomorphic map from U to $\mathbb{P}^N$, which we can regard as a harmonic map. We will observe that the energy of the harmonic map is dominated by the L^2 norm of the curvature of the instanton. Then we obtain the desired extendability of the spectral curve from the regularity theorem of J Sacks and K Uhlenbeck [42] for harmonic maps with finite energy.

Let $\pi\colon T\times U\to U$ denote the projection. We fix a lift of $\overline{\mathcal{S}p}(E)$ to $\widetilde{\mathcal{S}p}(E)\subset\mathbb{C}\times\bar{U}$. Then we obtain a holomorphic vector bundle V on U with an endomorphism g, with a C^∞ isomorphism $\pi^*V\simeq E$ such that $\pi^*\overline{\partial}_V+g\,d\bar{z}=\overline{\partial}_E$ and $\mathcal{S}p(g)=\widetilde{\mathcal{S}p}(E)$. By the identification $E=\pi^*V$, we obtain a T–action on E.

We set $\widetilde{\mathcal{S}p}_\infty(E):=(\mathbb{C}\times\{\infty\})\cap\widetilde{\mathcal{S}p}(E)$. We have a decomposition $(V,g)=\bigoplus_{\alpha\in\widetilde{\mathcal{S}p}_\infty(E)}(V_\alpha,g_\alpha)$ such that the eigenvalues of $g_\alpha(w)$ go to α when $w\to\infty$. We have a corresponding decomposition $E=\bigoplus_{\alpha\in\widetilde{\mathcal{S}p}_\infty(E)}E_\alpha$.

The Hermitian metric h of E is decomposed into the sum $h=\sum h_{\alpha,\beta}$, where $h_{\alpha,\beta}$ are the sesquilinear pairings of E_α and E_β. By using a Fourier expansion, we decompose $h_{\alpha,\beta}$ into a T–invariant part and its complement. Let h° denote the T–invariant part of $\sum h_{\alpha,\alpha}$. We shall prove that the complement $h^\perp:=h-h^\circ$ and its derivatives have exponential decay.

Theorem 5.11 *For any polynomial $P(a,b,c,d)$ of noncommutative variables, there exists $C>0$ such that*

$$P(\nabla_z,\nabla_{\bar{z}},\nabla_w,\nabla_{\overline{w}})h^\perp=O(\exp(-C|w|)). \tag{2}$$

We have a Hermitian metric h_V on V induced by h°. As a result, $(V,\overline{\partial}_V,h_V,g\,dw)$ satisfies the Hitchin equation up to an exponentially small term (Proposition 5.13). Such a tuple $(V,\overline{\partial}_V,h_V,g\,dw)$ can be studied as in the case of wild harmonic bundles [36] with minor modifications. (See Section 5.5.) Thus, we will arrive at a satisfactory stage of understanding of the asymptotic behaviour of L^2 instantons. We state some significant consequences.

Theorem 5.14 *There exists $\rho>0$ such that*

$$F(\nabla)=O\Big(\frac{dz\,d\bar{z}}{|w|^2(-\log|w|)^2}\Big)+O\Big(\frac{dw\,d\overline{w}}{|w|^2(-\log|w|)^2}\Big)+O\Big(\frac{dw\,d\bar{z}}{|w|^{1+\rho}}\Big)+O\Big(\frac{dz\,d\overline{w}}{|w|^{1+\rho}}\Big). \tag{3}$$

In particular, $F(\nabla)=O(|w|^{-1-\rho})$ for some $\rho>0$ with respect to h and the Euclidean metric $dw\,d\overline{w}+dz\,d\bar{z}$.

The estimate (3) implies that $(E, \overline{\partial}_E, h)$ is acceptable, ie $F(\nabla)$ is bounded with respect to h and the Poincaré-like metric $|w|^{-2}(\log|w|^2)^{-2} dw\, d\overline{w} + dz\, d\bar{z}$ on $T \times U$. By applying a general result in [36], we obtain the following prolongation result.

Corollary 5.16 *The holomorphic bundle $(E, \overline{\partial}_E)$ extends naturally to a filtered bundle $\mathcal{P}_* E$ on $(T \times \overline{U}, T \times \{\infty\})$.*

Here the filtered bundle $\mathcal{P}_* E$ on $(T \times \overline{U}, T \times \{\infty\})$ is an increasing sequence $(\mathcal{P}_a E \mid a \in \mathbb{R})$ of locally free $\mathcal{O}_{T \times \overline{U}}$–modules such that:

(i) $\mathcal{P}_a(E)_{|T \times U} = E$.

(ii) $\mathcal{P}_a(E)/\mathcal{P}_{<a}(E)$ are locally free $\mathcal{O}_{T \times \{\infty\}}$–modules, where $\mathcal{P}_{<a} E = \sum_{b<a} \mathcal{P}_b E$.

(iii) $\mathcal{P}_a(E) = \mathcal{P}_{a+\epsilon}(E)$ for some $\epsilon > 0$.

(iv) $\mathcal{P}_{a+1}(E) = \mathcal{P}_a(E) \otimes \mathcal{O}_{T \times \overline{U}}(T \times \{\infty\})$.

The sheaf $\mathcal{P}_a E$ is obtained as the space of holomorphic sections of E whose norms with respect to h have growth order $O(|w|^{a+\epsilon})$ for any $\epsilon > 0$.

The filtered bundle is useful in the study of the instanton. For example, it turns out that $\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \mathrm{Tr}(F(h)^2)$ is equal to $\int_{T \times \mathbb{P}^1} c_2(\mathcal{P}_a E)$ for any $a \in \mathbb{R}$, where c_2 denotes the second Chern class (Proposition 6.6). In particular, the number $\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \mathrm{Tr}(F(h)^2)$ is an integer. (See Wehrheim [50] for this kind of integrality in a more general situation.[1])

We can also use this filtered bundle to characterise the metric, ie the uniqueness part of the so-called Kobayashi–Hitchin correspondence. The stability condition for this type of filtered bundles is defined in Section 2.4.4 as in [8]. (Note that it is not a standard (slope-)stability condition for filtered bundles.)

Proposition 1.1 (Propositions 6.4 and 6.5) *The associated filtered bundle $\mathcal{P}_* E$ is polystable of degree 0. The metric h is uniquely determined as a Hermitian–Einstein metric of $(E, \overline{\partial}_E)$ adapted to $\mathcal{P}_* E$, up to obvious ambiguity. (See Section 6.1.3 for uniqueness.)*

We observe that we need only a weaker assumption on the curvature decay if we assume the prolongation of the spectral curve.

Theorem 5.17 *Suppose that $F(\nabla) \to 0$ when $|w| \to \infty$ and that the spectral curve $\mathcal{S}p(E)$ extends to a complex subvariety of $T \times \overline{U}$. Then (E, ∇, h) is an L^2 instanton.*

More precisely, we can directly prove the claims of Theorems 5.11 and 5.14 under the assumption without considering the L^2 condition.

[1] The author was informed of this by the referee.

1.1.4 Some remarks In [8], Jardim and Biquard showed that an instanton of rank 2 with quadratic decay is an exponentially small perturbation of a tuple $(V, \overline{\partial}_V, g\,dw, h_V)$ which satisfies the Hitchin equation up to an exponentially small term. Our result could be regarded as a generalization of theirs. However, the methods are rather different. To obtain a decomposition into a T–invariant part and its complement, they started with the construction of a global frame satisfying a nice property, which is an analogue of the Coulomb gauge of Uhlenbeck. Their method seems to require a stronger decay condition than L^2, for example the quadratic decay condition. We use a more natural decomposition induced by a standard method of the Fourier–Mukai transform in complex geometry, which allows us to consider L^2 instantons once we deal with the issue of the prolongation of the spectral curve. (See also Charbonneau [12] for some discussion on the relation between the L^2 property and the quadratic decay property of doubly periodic instantons.)

As mentioned above, we shall establish that an L^2 instanton is an exponentially small perturbation of $(V, \overline{\partial}_V, h_V, \theta_V)$ which satisfies the Hitchin equation up to an exponentially small term. Interestingly to the author, we can obtain a more refined result. Namely, we can naturally construct a harmonic metric h'_V on $(V, \overline{\partial}_V, g\,dw)$ defined on a neighbourhood of ∞ from the L^2 instanton. It is an analogue of the reductions from wild harmonic bundles to tame harmonic bundles studied in [36]. We consider a kind of meromorphic prolongation of the holomorphic vector bundle on the twistor space associated to $T \times \mathbb{C}$ and encounter a kind of infinite-dimensional Stokes phenomenon. By taking the graduation with respect to the Stokes structure, we obtain a wild harmonic bundle. Similarly, in this paper, we consider only the product holomorphic structure of $T \times \mathbb{C}$. From the viewpoint of twistor theory, the holomorphic vector bundle with respect to the other holomorphic structures should also be studied. The prolongation of the twistor family of the holomorphic structure is related to the above construction of harmonic metrics. The author hopes to return to this deeper aspect of the study elsewhere.

Although we do not use it explicitly, we prefer to regard an instanton on $T \times U$ as an infinite-dimensional harmonic bundle on U, which is suggested by Hitchin's reduction. This heuristic is useful in our study of the asymptotic behaviour of L^2 instantons. From this viewpoint, Theorems 5.11 and 5.17 can be naturally regarded as a variant of Simpson's main estimate [45]. (See also the author's [35; 36].)

1.2 Nahm transforms for wild harmonic bundles and L^2 instantons

1.2.1 Nahm transforms and algebraic Nahm transforms As an application of the study of the asymptotic behaviour, we shall establish the equivalence between

L^2 instantons on $T \times \mathbb{C}$ and wild harmonic bundles on $T^\vee$ given by the Nahm transforms, which is a differential geometric variant of the Fourier–Mukai transform. (See Bartocci, Bruzzo and Hernández Ruipérez [5] and [27] for the long history of various versions of the Nahm transforms. See also Braam and Baal [11], Donaldson [13], Hitchin [21], Nakajima [38], Schenk [43] and Szabó [48].)

Once we understand the asymptotic behaviour of an L^2 instanton (E, ∇, h), we can prove the desired property of the associated cohomology groups and harmonic sections. Then the standard L^2 method allows us to construct the Nahm transform $\mathrm{Nahm}(E, \nabla, h)$, which is a wild harmonic bundle on $(T^\vee, \mathcal{S}p_\infty(E))$ (see Section 6.4). Conversely, we may construct the Nahm transform of any wild harmonic bundle $(\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})$ on $(T^\vee, D)$ to L^2 instantons $\mathrm{Nahm}(\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})$ on $T \times \mathbb{C}$, by using the result on wild harmonic bundles on curves (see [36], Sabbah [40] and Zucker [51]), although we need some estimates to establish the L^2 property (see Section 7.1).

To study their more detailed properties, we introduce the algebraic Nahm transforms for filtered Higgs bundles on $(T^\vee, D)$ and filtered bundles on $(T \times \mathbb{C}, T \times \{\infty\})$, which do not necessarily come from wild harmonic bundles or L^2 instantons. The constructions are based on the Higgs interpretation of the Nahm transforms. It could be regarded as a filtered version of the Fourier transform for Higgs bundles studied in Bonsdorff [10], although we restrict ourselves to the case where the base space is an elliptic curve.

As mentioned in Section 1.1.3, we obtain the filtered bundle $\mathcal{P}_*E$ on $(T \times \mathbb{P}^1, T \times \{\infty\})$ associated to any L^2 instanton (E, ∇, h), and the metric h is determined by $\mathcal{P}_*E$ essentially uniquely. We have the good filtered Higgs bundle $(\mathcal{P}_*\mathcal{E}, \theta)$ on $(T^\vee, D)$ associated to any wild harmonic bundle $(\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})$, and the metric $h_{\mathcal{E}}$ is determined by $(\mathcal{P}_*\mathcal{E}, \theta)$ essentially uniquely; see Biquard and Boalch [7] and [36]. So, it is significant to describe the induced transformation between the underlying filtered bundles on $T \times \mathbb{P}^1$ and the underlying good filtered Higgs bundles on $(T^\vee, D)$ that is given by the algebraic Nahm transforms. They allow us to describe how the singular data are transformed. We may also use them to prove that the Nahm transforms are mutually inverse.

1.2.2 Algebraic Nahm transform for filtered Higgs bundles Let us briefly explain how the algebraic Nahm transform is constructed for filtered Higgs bundles $(\mathcal{P}_*\mathcal{E}, \theta)$ on $(T^\vee, D)$. (The details will be given in Section 3 after the preliminaries in Section 2.) We should impose several conditions on the filtered Higgs bundles.

Goodness and admissibility One of the conditions is the compatibility of the filtered bundle $\mathcal{P}_*\mathcal{E}$ and the Higgs field θ at each $P \in D$. Suppose that the filtered Higgs bundle $(\mathcal{P}_*\mathcal{E},\theta)$ comes from a good wild harmonic bundle. Let U_P be a small neighbourhood of P with a coordinate ζ_P with $\zeta_P(P)=0$. If we take a ramified covering $\varphi_p\colon U'_P\to U_P$ given by $\varphi_p(u)=u^p=\zeta_P$ for an appropriate p, then we have a decomposition

$$\varphi_p^*(\mathcal{P}_*\mathcal{E},\theta)=\bigoplus_{\mathfrak{a}\in u^{-1}\mathbb{C}[u^{-1}]}(\mathcal{P}_*\mathcal{E}'_{\mathfrak{a}},\theta'_{\mathfrak{a}}).$$

Here, $\theta'_{\mathfrak{a}}-d\mathfrak{a}$ are logarithmic in the sense that $(\theta'_{\mathfrak{a}}-\mathfrak{a})\mathcal{P}_a\mathcal{E}'_{\mathfrak{a}}\subset\mathcal{P}_a\mathcal{E}'_{\mathfrak{a}}\,du/u$. Such a filtered Higgs bundle is called good. This kind of filtered Higgs bundle is also closely related to L^2 instantons.

But it seems more natural to consider a wider class of filtered Higgs bundles for our algebraic Nahm transform. For $(p,m)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}$ with $\gcd(p,m)=1$, we say that a filtered Higgs bundle has type (p,m) at P if $u^m\varphi_p^*\theta$ gives a morphism of filtered bundles $\varphi_p^*\mathcal{P}_*\mathcal{E}\to\varphi_p^*\mathcal{P}_*\mathcal{E}\,du/u$ on U'_P and, if $(p,m)\neq(1,0)$, the morphism is an isomorphism. We say that $(\mathcal{P}_*\mathcal{E},\theta)$ is admissible at P if it is a direct sum $\bigoplus(\mathcal{P}_*\mathcal{E}_P^{(p,m)},\theta_P^{(p,m)})$ of the filtered bundles of type (p,m), after U_P is shrunk appropriately. We say that its slope is smaller (resp. strictly smaller) than α if $\mathcal{E}_P^{(p,m)}=0$ for $m/p>\alpha$ (resp. $m/p\geq\alpha$).

Each $(\mathcal{P}_*\mathcal{E}_P^{(p,m)},\theta_P^{(p,m)})$ has a refined decomposition as we will explain in Section 2.3.1. In particular, $(\mathcal{P}_*\mathcal{E}_P^{(1,0)},\theta_P^{(1,0)})$ has a decomposition

$$\big(\mathcal{P}_*\mathcal{E}_P^{(1,0)},\theta_P^{(1,0)}\big)=\bigoplus_{\alpha\in\mathbb{C}}\big(\mathcal{P}_*\mathcal{E}_{P,\alpha}^{(1,0)},\theta_{P,\alpha}^{(1,0)}\big).$$

Here, for the expression $\theta_{P,\alpha}^{(1,0)}=f_\alpha^{(1,0)}\,d\zeta_P/\zeta_P$, the eigenvalues of $f_\alpha^{(1,0)}$ go to α when $\zeta_P\to 0$. On U_P, we set

$$\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)_P=\bigoplus_{(p,m)\neq(1,0)}\mathcal{P}_{-1/2-m/p}\mathcal{E}_P^{(p,m)}\oplus\bigoplus_{\alpha\neq 0}\mathcal{P}_{-1/2}\mathcal{E}_{P,\alpha}^{(1,0)}\oplus\mathcal{P}_0\mathcal{E}_{P,0}^{(1,0)},\tag{4}$$

$$\begin{aligned}\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_P=\Big(\bigoplus_{(p,m)\neq(1,0)}\mathcal{P}_{1/2}\mathcal{E}_P^{(p,m)}\oplus\bigoplus_{\alpha\neq 0}\mathcal{P}_{1/2}\mathcal{E}_{P,\alpha}^{(1,0)}\Big)\otimes\Omega^1_{T^\vee}\\ \oplus\big(\mathcal{P}_{<1}\mathcal{E}_{P,0}^{(1,0)}\otimes\Omega^1+\theta_{P,0}^{(1,0)}\mathcal{P}_0\mathcal{E}_{P,0}^{(1,0)}\big).\end{aligned}\tag{5}$$

Here, $(\mathcal{P}_{<1}\mathcal{E}_{P,0}^{(1,0)}\otimes\Omega^1+\theta_{P,0}^{(1,0)}\mathcal{P}_0\mathcal{E}_{P,0}^{(1,0)})$ is the sum taken in $\mathcal{P}_1\mathcal{E}_{P,0}^{(1,0)}\otimes\Omega^1$. The Higgs field θ gives a morphism $\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)_P\to\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_P$. Thus we obtain a complex $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E},\theta)$ on U_P, which is an extension of $\theta\colon\mathcal{E}\to\mathcal{E}\otimes\Omega^1$ on $U_P\setminus P$.

We say that $(\mathcal{P}_*\mathcal{E},\theta)$ on $(T^\vee,D)$ is admissible if its restriction to a neighbourhood of each $P\in D$ is admissible. By considering the extension at each $P\in D$, we obtain a complex $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E},\theta)$ on $T^\vee$ as an extension of $\mathcal{E}\to\mathcal{E}\otimes\Omega^1$ on $T^\vee\setminus D$.

Vanishing conditions on some cohomology groups For each $w \in \mathbb{C}$ and each holomorphic line bundle L of degree 0 on $T^\vee$, we obtain a complex $\mathcal{C}^\bullet_{w,L}(\mathcal{P}_*\mathcal{E},\theta) := \mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}\otimes L, \theta + w\,d\zeta)$. To consider the algebraic Nahm transform for $(\mathcal{P}_*\mathcal{E},\theta)$, it is natural to impose the following vanishing condition:

(A0) $\mathbb{H}^i(T^\vee, \mathcal{C}^\bullet_{w,L}(\mathcal{P}_*\mathcal{E},\theta)) = 0$ unless $i=1$ for any $w \in \mathbb{C}$ and any holomorphic line bundle L of degree 0 on $T^\vee$.

For $I \subset \{1,2,3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. Let $\mathcal{P}oin$ denote the Poincaré bundle on $T^\vee \times T$. We consider the following complex on $T^\vee \times T \times \mathbb{P}^1$:

$$\widetilde{\mathcal{C}}^0 := p_1^*\mathcal{C}^0 \otimes p_{12}^*\mathcal{P}oin \otimes p_3^*\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\theta + w\,d\zeta} \widetilde{\mathcal{C}}^1 := p_1^*\mathcal{C}^1 \otimes p_{12}^*\mathcal{P}oin.$$

It turns out that $\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta) := R^1 p_{23*}\widetilde{\mathcal{C}}^\bullet$ is a locally free $\mathcal{O}_{T\times\mathbb{P}^1}$–module on $T\times\mathbb{P}^1$. In particular, we obtain a locally free $\mathcal{O}_{T\times\mathbb{P}^1}(*(T\times\{\infty\}))$–module

$$\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta) := \boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta) \otimes \mathcal{O}_{T\times\mathbb{P}^1}(*(T\times\{\infty\})).$$

Filtered bundles on $(T\times U, T\times\{\infty\})$ The algebraic Nahm transform of $(\mathcal{P}_*\mathcal{E},\theta)$ is defined to be a filtered bundle over the meromorphic bundle $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$. For the construction of such a filtration, it would be convenient to have a description of any filtered bundle $\mathcal{P}_*E$ on $(T\times U, T\times\{\infty\})$ satisfying the following condition, where U denotes a neighbourhood of ∞ in $\mathbb{P}^1$.

(A1) $\mathrm{Gr}^{\mathcal{P}}_c(E)$ are semistable bundles of degree 0 on T for any $c \in \mathbb{R}$.

By shrinking U, we may assume that $\mathcal{P}_c(E)_{|T\times\{w\}}$ are semistable of degree 0 for any $w \in U$ and for any $c \in \mathbb{R}$. We set $\mathcal{S}p_\infty(E) := \mathcal{S}p(\mathcal{P}_cE_{|T\times\{\infty\}}) \subset T^\vee$, which is independent of $c \in \mathbb{R}$. We fix a lift $\widetilde{\mathcal{S}p}_\infty(E) \subset \mathbb{C}$ of $\mathcal{S}p_\infty(E)$, ie $\widetilde{\mathcal{S}p}_\infty(E)$ is mapped bijectively to $\mathcal{S}p_\infty(E)$ by the projection $\mathbb{C} \to T^\vee$. Then we have a filtered bundle $\mathcal{P}_*V$ on (U,∞) with an endomorphism g such that $\mathcal{S}p(g_{|\infty}) = \widetilde{\mathcal{S}p}_\infty(E)$ corresponding to $\mathcal{P}_*E$. Namely, we have a C^∞ isomorphism $\mathcal{P}_*E \simeq \pi^*\mathcal{P}_*V$, under which $\overline{\partial}_{\mathcal{P}_*E} = \pi^*(\overline{\partial}_{\mathcal{P}_*V}) + g\,d\bar{z}$, where $\pi\colon T\times U \to U$ denotes the projection. The filtered bundle with an endomorphism $(\mathcal{P}_*V, g)$, or equivalently the filtered Higgs bundle $(\mathcal{P}_*V, g\,dw)$, completely determines $\mathcal{P}_*E$. We have the decomposition $(\mathcal{P}_*V, g) = \bigoplus_{\alpha\in\widetilde{\mathcal{S}p}_\infty(E)} (\mathcal{P}_*V_\alpha, g_\alpha)$ with $\mathcal{S}p(g_{\alpha|\infty}) = \{\alpha\}$. The filtered bundle $\mathcal{P}_*E$ satisfying (A1) is called admissible if the following holds:

(A2) The filtered Higgs bundles $(\mathcal{P}_*V_\alpha, (g_\alpha - \alpha)\,dw)$ are admissible for any $\alpha \in \widetilde{\mathcal{S}p}_\infty(E)$.

The slope of $(\mathcal{P}_*V_\alpha, (g_\alpha - \alpha)\,dw)$ is strictly smaller than 1 by construction.

Local algebraic Nahm transform and algebraic Nahm transform The local algebraic Nahm transform $\mathcal{N}^{0,\infty}$ is a transform from the germs of admissible filtered Higgs bundles to the germs of admissible filtered Higgs bundles whose slopes are strictly smaller than 1. It is an analogue of the local Fourier transform $\mathfrak{F}^{0,\infty}$ for meromorphic flat bundles on $\mathbb{P}^1$ in Bloch and Esnault [9] and García López [19]. See also Arinkin [3], Beilison, Bloch, Deligne and Esnault [6], Fang [15], Fu [17], Graham-Squire [20] and Sabbah [41]. (More precisely, it is an analogue of the local Fourier transform of the minimal extension of meromorphic flat bundles.) It gives a procedure to make an admissible filtered bundle $\mathcal{P}_* E_P$ on $(T \times U, T \times \{\infty\})$ such that $\mathcal{S}p_\infty(E_P) = \{P\}$, from an admissible filtered Higgs bundle $(\mathcal{P}_*\mathcal{E}, \theta)_{|U_P}$ on (U_P, P). From the local Nahm transform $\bigoplus_{P\in D} \mathcal{P}_* E_P$ and the meromorphic bundle $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$, we obtain a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$, denoted by $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$, that is the algebraic Nahm transform for admissible filtered Higgs bundles.

1.2.3 Algebraic Nahm transform for admissible filtered bundles Let $\mathcal{P}_* E$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$. To define the algebraic Nahm transform of $\mathcal{P}_* E$, we impose the following vanishing condition.

(A3) $H^0(T \times \mathbb{P}^1, \mathcal{P}_0 E \otimes L) = 0$ and $H^2(T \times \mathbb{P}^1, \mathcal{P}_{<-1} E \otimes L) = 0$ for any holomorphic line bundle L of degree 0 on T.

We set $D := \mathcal{S}p_\infty(E)$. It is easy to observe that condition (A3) implies that $H^i(T \times \mathbb{P}^1, \mathcal{P}_c E \otimes L^\vee) = 0$ $(i \neq 1)$ for any $c \in \mathbb{R}$ unless $L \in D$. For any $I \subset \{1,2,3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. We define

$$
\begin{aligned}
(6) \quad \mathrm{Nahm}(\mathcal{P}_* E) &:= R^1 p_{1*}(p_{12}^* \mathcal{P}oin^\vee \otimes p_{23}^* \mathcal{P}_0 E)(*D) \\
&\simeq R^1 p_{1*}(p_{12}^* \mathcal{P}oin^\vee \otimes p_{23}^* \mathcal{P}_{-1} E)(*D).
\end{aligned}
$$

It is a locally free $\mathcal{O}_{T^\vee}(*D)$–module. The multiplication of $-w\, d\zeta$ gives a Higgs field θ of $\mathrm{Nahm}(\mathcal{P}_* E)$. Thus, we obtain a meromorphic Higgs bundle on $(T^\vee, D)$.

Let U be a small neighbourhood of ∞ in $\mathbb{P}^1$. On $T \times U$, we have a decomposition $\mathcal{P}_* E_{|T\times U} = \bigoplus_{P\in D} \mathcal{P}_* E_P$ with $\mathcal{S}p_\infty(E_P) = \{P\}$. We fix a lift $\widetilde{D} \subset \mathbb{C}$. We have the corresponding filtered bundles $(\mathcal{P}_* V_P, g_P)$. We have the decomposition

$$(\mathcal{P}_* V_P, g_P - \widetilde{P}) = \bigoplus \bigl(\mathcal{P}_* V_P^{(p,m)}, g_P^{(p,m)}\bigr),$$

where $(\mathcal{P}_* V_P^{(p,m)}, g_P^{(p,m)}\, dw)$ has slope (p,m) with $m/p < 1$, and $\widetilde{P} \in \widetilde{D}$ is a lift of P. Moreover, we have the decomposition

$$\bigl(\mathcal{P}_* V_P^{(1,0)}, g_P^{(1,0)}\bigr) = \bigoplus_{\alpha\in\mathbb{C}} \bigl(\mathcal{P}_* V_{P,\alpha}^{(1,0)}, g_{P,\alpha}^{(1,0)}\bigr).$$

It turns out that we have a decomposition of the meromorphic bundle

$$\begin{aligned}\mathrm{Nahm}(\mathcal{P}_*E)_{|U_P} \\ = \mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,0} \oplus \bigoplus_{\alpha\neq\mathbb{C}} \mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,\alpha} \oplus \bigoplus_{(p,m)\neq(1,0)} \mathrm{Nahm}(\mathcal{P}_*E)^{(p,m)}_{P},\end{aligned}$$

and we also have that $\mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,\alpha}$ $(\alpha\neq 0)$ and $\mathrm{Nahm}(\mathcal{P}_*E)^{(p,m)}_{P}$ $((p,m)\neq(1,0))$ are determined by

$$\big(\mathcal{P}_*V^{(1,0)}_{P,\alpha}, g^{(1,0)}_{P,\alpha}\big) \quad \text{and} \quad \big(\mathcal{P}_*V^{(p,m)}_{P,\alpha}, g^{(p,m)}_{P}\big).$$

We have the local algebraic Nahm transform $\mathcal{N}^{\infty,0}$, which is a transform of admissible Higgs bundles $(\mathcal{P}_*V,\theta)$ such that the slopes are strictly smaller than 0 and $\mathcal{P}_*V^{(1,0)}_0 = 0$. It is an analogue of the local Fourier transform $\mathfrak{F}^{\infty,0}$ in [9; 19]. It is an inverse of $\mathcal{N}^{0,\infty}$ except for the part $(p,m)=(1,0)$ and $\alpha=0$. We may introduce filtrations of $\mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,\alpha}$ $(\alpha\neq 0)$ and $\mathrm{Nahm}(\mathcal{P}_*E)^{(p,m)}_{P}$ $((p,m)\neq(1,0))$ by using the local algebraic Nahm transform $\mathcal{N}^{\infty,0}$. As for the part with $(p,m)=(1,0)$ and $\alpha=0$, we have an injection

$$\mathcal{P}_0V^{(1,0)}_{P,0|\infty} \subset R^1p_{1*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{P}_{-1}E)^{(1,0)}_{P,0|P},$$

by which we can introduce a filtration on $\mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,0}$. Therefore, we obtain a filtered Higgs bundle denoted by $\mathrm{Nahm}_*(\mathcal{P}_*E)$. We obtain the following correspondence.

Theorem 1.2 (Propositions 3.13, 3.22 and 3.25) *The Nahm transforms* Nahm_* *give an equivalence of the following objects, and they are mutually inverse:*

- *Admissible filtered Higgs bundles on $(T^\vee, D)$ satisfying condition (A0).*
- *Admissible filtered bundles $\mathcal{P}_*E$ on $(T\times\mathbb{P}^1, T\times\{\infty\})$ with $\mathcal{S}p_\infty(E)=D$ satisfying condition (A3).*

Nahm transforms also preserve the parabolic degrees (Proposition 3.17).

As already mentioned, the filtered Higgs bundles associated to wild harmonic bundles satisfy a stronger condition called goodness. Similarly, it turns out that the filtered bundles associated to L^2 instantons are also good, in the sense that the corresponding filtered Higgs bundles are good. We can observe that the algebraic Nahm transforms preserve the goodness conditions.

Theorem 3.27 *The Nahm transforms* Nahm_* *give an equivalence of the following objects:*

- *Good filtered Higgs bundles on* $(T^\vee, D)$ *satisfying condition (A0).*
- *Good filtered bundles* $\mathcal{P}_*E$ *on* $(T\times\mathbb{P}^1, T\times\{\infty\})$ *with* $\mathcal{S}p_\infty(E) = D$ *satisfying condition (A3).*

1.2.4 Application of the algebraic Nahm transform We have the following compatibility of the Nahm transform and the algebraic Nahm transform.

Theorem 1.3 (Theorems 7.12 and 7.13)

- *Let* (E,∇,h) *be an* L^2 *instanton on* $T\times\mathbb{C}$. *Let* $\mathcal{P}_*E$ *be the associated filtered bundle on* $(T\times\mathbb{P}^1, T\times\{\infty\})$. *Then the associated filtered Higgs bundle of the wild harmonic bundle* $\mathrm{Nahm}(E,\nabla,h)$ *on* $(T^\vee, \mathcal{S}p_\infty(E))$ *is naturally isomorphic to the algebraic Nahm transform* $\mathrm{Nahm}(\mathcal{P}_*E)$.
- *Let* $(\mathcal{E},\overline{\partial}_{\mathcal{E}},\theta,h_{\mathcal{E}})$ *be a wild harmonic bundle on* $(T^\vee, D)$. *Let* $(\mathcal{P}_*\mathcal{E},\theta)$ *be the associated good filtered Higgs bundle on* $(T^\vee, D)$. *Then the associated filtered bundle of the* L^2 *instanton* $\mathrm{Nahm}(\mathcal{E},\overline{\partial}_{\mathcal{E}},h_{\mathcal{E}},\theta)$ *is naturally isomorphic to the algebraic Nahm transform* $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$. □

As an application, we obtain the inversion property of the Nahm transforms.

Corollary 7.14 *For an* L^2 *instanton* (E,∇,h) *on* $T\times\mathbb{C}$, *we have an isomorphism*

$$\mathrm{Nahm}(\mathrm{Nahm}(E,\nabla,h)) \simeq (E,\nabla,h).$$

For a wild harmonic bundle $(\mathcal{E},\overline{\partial}_{\mathcal{E}},\theta,h_{\mathcal{E}})$ *on* $(T^\vee, D)$, *we have an isomorphism*

$$\mathrm{Nahm}(\mathrm{Nahm}(\mathcal{E},\overline{\partial}_{\mathcal{E}},\theta,h_{\mathcal{E}})) \simeq (\mathcal{E},\overline{\partial}_{\mathcal{E}},\theta,h_{\mathcal{E}}).$$

Indeed, it follows from Theorem 1.3 and the uniqueness of the Hermitian–Einstein metric (resp. the harmonic metric) adapted to the filtered bundle (resp. filtered Higgs bundle).

As another application of the compatibility, we can easily compute the characteristic classes of the bundles obtained by the algebraic Nahm transform, which allows us to describe the rank and the second Chern class of the bundle obtained by the Nahm transform. The local algebraic Nahm transform also gives us a rather complete understanding of the transformation of singularity data by the Nahm transform.

1.2.5 Some remarks Recall that the hyperkähler manifold $T \times \mathbb{C}$ has twistor deformations. Namely, for any complex number λ, we have a moduli space $\mathcal{M}^{\lambda}$ of line bundles of degree 0 with a flat λ–connection on $T^{\vee}$. We have $\mathcal{M}^0 = T \times \mathbb{C}$. The spaces $\mathcal{M}^{\lambda}$ can also be regarded as the deformation associated to the hyperkähler structure of $T \times \mathbb{C}$. An instanton on $T \times \mathbb{C}$ naturally induces a holomorphic vector bundle. If the instanton is L^2, the holomorphic bundle with the metric induces a filtered bundle on $(\overline{\mathcal{M}}^{\lambda}, T^{\lambda}_{\infty})$, where $\overline{\mathcal{M}}^{\lambda}$ is a natural compactification of $\mathcal{M}^{\lambda}$, and $T^{\lambda}_{\infty} \simeq T$ is the infinity. A wild harmonic bundle has the underlying good filtered λ–flat bundle for each complex number λ. It is also natural to study the transformation of the underlying filtered bundles on $(\overline{\mathcal{M}}^{\lambda}, T^{\lambda}_{\infty})$ and the underlying filtered λ–flat bundles. It should be a filtered enhancement of the generalised Fourier–Mukai transform for elliptic curves due to G Laumon and M Rothstein. We would like to study this interesting aspect elsewhere.

If we consider a counterpart of the algebraic Nahm transform for the other nonproduct holomorphic structure of $T \times \mathbb{C}$ underlying the hyperkähler structure, it is essentially a filtered version of the generalised Fourier–Mukai transform in Laumon [29] and Rothstein [39]. Interestingly to the author, we have an analogue of the stationary phase formula even in this case. The details will be given elsewhere.

In this paper, we consider transforms between filtered bundles on $T \times \mathbb{P}^1$ and filtered Higgs bundles on $T^{\vee}$. We may introduce similar transforms for filtered Higgs bundles on $\mathbb{P}^1$ with additional work on the local Nahm transform $\mathcal{N}^{\infty,\infty}$, which is an analogue of the local Fourier transform $\mathfrak{F}^{\infty,\infty}$. It should be the Higgs counterpart of the Nahm transforms between wild harmonic bundles on $\mathbb{P}^1$, which is given by the procedure for wild pure twistor D–modules established in [36].

Similarly, Szabó [48] studied the Nahm transform for an interesting type of harmonic bundles on $\mathbb{P}^1$. He also studied the transform of the underlying parabolic Higgs bundles, which looks closely related to the regular version of ours in Section 3.2. K Aker and Szabó [2] introduced a transformation of more general parabolic Higgs bundles on $\mathbb{P}^1$, which they call the algebraic Nahm transform. Their method to define the transform is different from ours, and the precise relation between them is not clear at this moment.

1.3 Outline of the paper

This paper is roughly divided into three parts: Sections 2–3, 4–5 and 6–7. In the first part, we introduce algebraic Nahm transforms and study their basic properties. In the second part, we study the asymptotic behaviour of L^2 instantons on the product of a torus T and a region $\{w \mid |w| \geq R\}$. Then in the third part, we study the Nahm transforms between L^2 instantons on $T \times \mathbb{C}$ and wild harmonic bundles on the dual torus $T^{\vee}$.

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This paper is dedicated to Professor Shigeru Mukai with highest admiration of his influential ideas, in particular his notion of the Fourier–Mukai transform.

2 Preliminaries on filtered objects

2.1 Semistable bundles of degree 0 on elliptic curves

2.1.1 Elliptic curve and the Fourier–Mukai transform For a variable z, let $\mathbb{C}_z$ denote a complex line with the standard coordinate z. For a $\mathbb{C}$–vector space V and a C^∞ manifold X, let $\underline{V}_X$ denote the product bundle $V \times X$ over X. If X is a complex manifold, the natural holomorphic structure of $\underline{V}_X$ is denoted just by $\overline{\partial}$.

We have a real bilinear map $\mathbb{C}_z \times \mathbb{C}_\zeta \to \mathbb{R}$ given by $(z,\zeta) \mapsto \operatorname{Im}(z\overline{\zeta})$. Let $\tau = \tau_1 + \sqrt{-1}\tau_2$ $(\tau_i \in \mathbb{R})$ be a complex number such that $\tau_2 \neq 0$. Let $L := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}_z$. In this paper, the dual lattice $L^\vee$ means

$$L^\vee := \bigl\{\zeta \in \mathbb{C}_\zeta \bigm| \operatorname{Im}(\chi\overline{\zeta}) \in \pi\mathbb{Z} \text{ for all } \chi \in L\bigr\} = \bigl\{\tfrac{\pi}{\tau_2}(n + m\tau) \bigm| n, m \in \mathbb{Z}\bigr\}.$$

We have the elliptic curves $T := \mathbb{C}_z / L$ and $T^\vee := \mathbb{C}_\zeta / L^\vee$.

For any $\nu \in L^\vee$, we have $\rho_\nu \in C^\infty(T)$ given by $\rho_\nu(z) := \exp(2\sqrt{-1}\,\mathrm{Im}(\nu\bar{z})) = \exp(\nu\bar{z} - \bar{\nu}z)$. We have $\bar{\partial}_z\rho_\nu = \rho_\nu\nu\,d\bar{z}$ and $\partial_z\rho_\nu = -\rho_\nu\bar{\nu}\,dz$.

We can naturally regard $T^\vee$ as the moduli space $\mathrm{Pic}_0(T)$ of holomorphic line bundles of degree 0 on T. Indeed, ζ gives a holomorphic bundle $\mathcal{L}_\zeta = (\underline{\mathbb{C}}_T, \bar{\partial} + \zeta\,d\bar{z})$. It induces an isomorphism $T^\vee \simeq \mathrm{Pic}_0(T)$. We have the isomorphism Φ: $\mathcal{L}_\zeta \simeq \mathcal{L}_{\zeta+\nu}$ given by $\Phi(f) = \rho_{-\nu}\cdot f$.

We have the unitary flat connection associated to $\mathcal{L}_\zeta$ with the trivial metric, $d - \bar{\zeta}\,dz + \zeta\,d\bar{z}$. The monodromy along the segment from 0 to $\chi \in L$ is $\exp(2\sqrt{-1}\,\mathrm{Im}(\zeta\bar{\chi}))$.

We recall a differential-geometric construction of the Poincaré bundle on $T \times T^\vee$, following Donaldson and Kronheimer [14]. On $T \times \mathbb{C}_\zeta$, we have the holomorphic line bundle

$$\widetilde{\mathcal{P}oin} = \big(\underline{\mathbb{C}}_{T\times\mathbb{C}_\zeta}, \bar{\partial} + \zeta\,d\bar{z}\big).$$

The $L^\vee$–action on $T\times\mathbb{C}_\zeta$ is naturally lifted to the action on $\widetilde{\mathcal{P}oin}$ given by $\nu(z,\zeta,v) = (z, \zeta+\nu, \rho_{-\nu}(z)v)$. Thus, a holomorphic line bundle is induced on $T \times T^\vee$, which is the Poincaré bundle denoted by $\mathcal{P}oin$. The dual bundle $\mathcal{P}oin^\vee$ is induced by $\widetilde{\mathcal{P}oin}^\vee = (\underline{\mathbb{C}}_{T\times\mathbb{C}_\zeta}, \bar{\partial} - \zeta\,d\bar{z})$ with the action $\nu(z,\zeta,v) = (z, \zeta+\nu, \rho_\nu(z)v)$.

Let S be any complex analytic space. For $I \subset \{1,2,3\}$, let p_I denote the projection of $T \times T^\vee \times S$ onto the product of the i^{th} components $(i \in I)$. For any object $\mathcal{M} \in D^b(\mathcal{O}_{T\times S})$, we set

$$\mathrm{RFM}_\pm(\mathcal{M}) := Rp_{23*}(p_{13}^*(\mathcal{M}) \otimes p_{12}^*\mathcal{P}oin^{\pm1})[1] \in D^b(\mathcal{O}_{T^\vee\times S}).$$

For any object $\mathcal{N} \in D^b(\mathcal{O}_{T^\vee\times S})$, we set

$$\widehat{\mathrm{RFM}}_\pm(\mathcal{N}) := Rp_{13*}(p_{23}^*(\mathcal{N}) \otimes p_{12}^*\mathcal{P}oin^{\pm1}) \in D^b(\mathcal{O}_{T\times S}).$$

Recall that we have a natural isomorphism $\widehat{\mathrm{RFM}}_+ \circ \mathrm{RFM}_-(\mathcal{M}) \simeq \mathcal{M}$ Mukai [37].

2.1.2 Semistable bundles of degree 0 For a holomorphic vector bundle $(E, \bar{\partial}_E)$ on T, we have the degree given by $\deg(E) := \int_T c_1(E)$ and the slope given by $\mu(E) := \deg(E)/\operatorname{rank}(E)$. A holomorphic vector bundle E on T is called semistable if $\mu(F) \le \mu(E)$ holds for any nontrivial holomorphic subbundle $F \subset E$. Semistable bundles on elliptic curves were thoroughly studied by Atiyah in [4]. In the following, we shall not distinguish a holomorphic vector bundle and the associated sheaf of holomorphic sections.

Let E be a semistable bundle of degree 0 on T. It is well known that the support $\mathcal{S}p(E)$ of $\mathrm{RFM}_-(E)$ consists of finite points. Indeed, E is obtained as an extension of the line bundles $\mathcal{L}_\zeta$ $(\zeta \in \mathcal{S}p(E))$. We call $\mathcal{S}p(E)$ the spectrum of E. We have the spectral decomposition $E=\bigoplus_{\alpha\in\mathcal{S}p(E)}E_\alpha$, where the support of $\mathrm{RFM}_-(E_\alpha)$ is $\{\alpha\}$. We say a subset $\widetilde{\mathcal{S}p}(E)\subset\mathbb{C}$ is a lift of $\mathcal{S}p(E)$ if the projection $\Phi\colon\mathbb{C}\to T^\vee$ induces a bijection $\widetilde{\mathcal{S}p}(E)\simeq\mathcal{S}p(E)$. If we fix a lift, an $\mathcal{O}_{\mathbb{C}}$–module $\mathcal{M}(E)$ is determined (up to canonical isomorphisms) by the following conditions: the support of $\mathcal{M}(E)$ is $\widetilde{\mathcal{S}p}(E)$, and $\Phi_*\mathcal{M}(E)\simeq\mathrm{RFM}_-(E)$. Such $\mathcal{M}(E)$ is called a lift of $\mathrm{RFM}_-(E)$. The multiplication of ζ on $\mathcal{M}(E)$ induces endomorphisms of $\mathrm{RFM}_-(E)$ and E. The endomorphism of E is denoted by f_ζ.

Let S be any complex analytic space. Let E be a holomorphic vector bundle on $T\times S$. It is called semistable of degree 0 relative to S if $E_{|T\times\{s\}}$ is semistable of degree 0 for any $s\in S$. The support of $\mathrm{RFM}_-(E)$ is relatively 0–dimensional over S. It is denoted by $\mathcal{S}p(E)$, and called the spectrum of E. If we have a hypersurface $\widetilde{\mathcal{S}p}(E)\subset\mathbb{C}_\zeta\times S$ such that the projection $\Phi\colon\mathbb{C}_\zeta\times S\to T^\vee\times S$ induces $\widetilde{\mathcal{S}p}(E)\simeq\mathcal{S}p(E)$, then we call $\widetilde{\mathcal{S}p}(E)$ a lift of $\mathcal{S}p(E)$. If we have a lift of $\mathcal{S}p(E)$, we obtain a lift $\mathcal{M}(E)$ of $\mathrm{RFM}_-(E)$ as in the case when S is a point. We also obtain an endomorphism f_ζ of E induced by the multiplication of ζ on $\mathcal{M}(E)$.

2.1.3 Equivalence of categories For a vector space V, let $\underline{V}$ denote the product bundle $T\times V$ over T, and let $\overline{\partial}_0$ denote the natural holomorphic structure of $\underline{V}$. For any $f\in\mathrm{End}(V)$, we have the associated holomorphic vector bundle $\mathfrak{G}(V,f):=(\underline{V},\overline{\partial}_0+f\,d\bar{z})$. We have a natural isomorphism $\mathfrak{G}(V,f)\simeq\mathfrak{G}(V,f+\nu\,\mathrm{id}_V)$ for each $\nu\in L^\vee$, induced by the multiplication of $\rho_{-\nu}$. Let $\mathcal{S}p(f)$ denote the set of the eigenvalues of f.

Lemma 2.1 *$\mathfrak{G}(V,f)$ is semistable of degree 0 and $\mathcal{S}p(\mathfrak{G}(V,f))=\Phi(\mathcal{S}p(f))$ in $T^\vee$, where $\Phi\colon\mathbb{C}\to T^\vee$ denotes the projection.*

Proof We have only to consider the case where f has a unique eigenvalue α. In that case, $\mathfrak{G}(V,f)$ is an extension of the line bundle $\mathcal{L}_\alpha$. Then the claim is clear. □

Let VS^* denote the category of finite-dimensional $\mathbb{C}$–vector spaces with an endomorphism, ie an object in VS^* is a finite-dimensional vector space V with an endomorphism f, and a morphism $(V,f)\to(W,g)$ in VS^* is a linear map $\varphi\colon V\to W$ such that $g\circ\varphi-\varphi\circ f=0$. For a given subset $\tilde{\mathfrak{s}}\subset\mathbb{C}$, let $\mathrm{VS}^*(\tilde{\mathfrak{s}})\subset\mathrm{VS}^*$ denote the full subcategory of objects (V,f) such that $\mathcal{S}p(f)\subset\tilde{\mathfrak{s}}$.

Let $\mathrm{VB}_0^{ss}(T)$ denote the category of semistable bundles of degree 0 on T, ie an object in $\mathrm{VB}_0^{ss}(T)$ is a semistable vector bundle of degree 0 on T, and a morphism $V_1\to V_2$ in $\mathrm{VB}_0^{ss}(T)$ is a morphism of coherent sheaves. For a given subset $\mathfrak{s}\subset T^\vee$, let $\mathrm{VB}_0^{ss}(T,\mathfrak{s})\subset\mathrm{VB}_0^{ss}(T)$ denote the full subcategory of semistable bundles of degree 0 whose spectrum are contained in $\mathfrak{s}$.

We have the functor $\mathfrak{G}$: $\mathrm{VS}^* \to \mathrm{VB}_0^{ss}(T)$ given by the above construction. If $\widetilde{\mathfrak{s}}$ is mapped to $\mathfrak{s}$ by the projection Φ: $\mathbb{C}_\xi \to T^\vee$, it induces a functor $\mathfrak{G}$: $\mathrm{VS}^*(\widetilde{\mathfrak{s}}) \to \mathrm{VB}_0^{ss}(T, \mathfrak{s})$.

Proposition 2.2 *If Φ: $\mathbb{C} \to T^\vee$ induces a bijection $\widetilde{\mathfrak{s}} \simeq \mathfrak{s}$, then $\mathfrak{G}$ gives an equivalence of the categories $\mathrm{VS}^*(\widetilde{\mathfrak{s}}) \simeq \mathrm{VB}_0^{ss}(T, \mathfrak{s})$.*

Proof Let us prove that it is fully faithful. We set $E_f := \mathfrak{G}(V, f)$. We will not distinguish between E_f and the associated sheaf of holomorphic sections. Suppose that f has a unique eigenvalue α such that $\alpha \not\equiv 0$ modulo $L^\vee$. Because E_f is obtained as an extension of the holomorphic line bundle $\mathcal{L}_\alpha$, we have $H^0(T, E_f) = H^1(T, E_f) = 0$. In particular, we obtain the following.

Lemma 2.3 *Assume that $f_i \in \mathrm{End}(V)$ has a unique eigenvalue α_i for $i = 1, 2$. If $\alpha_1 \not\equiv \alpha_2$ modulo $L^\vee$, any morphism $E_{f_1} \to E_{f_2}$ is 0.* □

Suppose that f is nilpotent. We have the natural inclusion $V \to C^\infty(T, E_f)$ as constant functions. We have a linear map $V \to C^\infty(T, E_f \otimes \Omega^{0,1})$ given by $s \mapsto s\, d\bar{z}$. They induce a chain map ι from $\mathcal{C}_1 = (f\colon V \to V)$ to the Dolbeault complex $C^\infty(T, E_f \otimes \Omega_T^{0,\bullet})$ of E_f.

Lemma 2.4 *ι is a quasi-isomorphism.*

Proof Let W be the monodromy weight filtration of f. It induces filtrations of $\mathcal{C}_1$ and $C^\infty(T, E_f \otimes \Omega_T^{0,*})$, and ι gives a morphism of filtered chain complex. It induces a quasi-isomorphism of the associated graded complexes. Hence, ι is a quasi-isomorphism. □

We obtain the following lemma as an immediate consequence.

Lemma 2.5 *Assume that $f_i \in \mathrm{End}(V)$ are nilpotent $(i = 1, 2)$. Then holomorphic morphisms $E_{f_1} \to E_{f_2}$ naturally correspond to holomorphic morphisms ϕ: $E_0 \to E_0$ such that $f_2 \circ \phi - \phi \circ f_1 = 0$.*

In particular, if f is nilpotent, holomorphic sections of $\mathrm{End}(E_f)$ bijectively corresponds to holomorphic sections g of $\mathrm{End}(E_0)$ such that $[f, g] = 0$. □

The full faithfulness of the functor $\mathfrak{G}$ follows from Lemmas 2.3 and 2.5. Let us prove the essential surjectivity of $\mathfrak{G}$. Let $E \in \mathrm{VB}_0^{ss}(T, \mathfrak{s})$. We have the $\mathcal{O}_{\mathbb{C}_\xi}$–module $\mathcal{M}(E)$ and the endomorphism f_ξ of E as in Section 2.1.2. We have a natural isomorphism

$\widehat{\mathrm{RFM}}_{+}\circ\mathrm{RFM}_{-}(E)\simeq E$. The functor $\widehat{\mathrm{RFM}}_{+}$ is induced by the holomorphic line bundle on $T\times T^{\vee}$, obtained as the descent of $\widetilde{\mathcal{P}oin}=(\underline{\mathbb{C}},\overline{\partial}_0+\zeta\,d\bar{z})$. Let p and q denote the projections $T\times\mathbb{C}_{\zeta}\to T$ and $T\times\mathbb{C}_{\zeta}\to\mathbb{C}_{\zeta}$. We have $E\simeq p_*(q^*(\mathcal{M}(E))\otimes\widetilde{\mathcal{P}oin})$, and the latter is naturally isomorphic to

$$\big(\underline{H^0(\mathbb{C}_{\zeta},\mathcal{M}(E))},\overline{\partial}_0+f_{\zeta}\,d\bar{z}\big).$$

We obtained the essential surjectivity of $\mathfrak{G}$. The proof of Proposition 2.2 is finished. □

As appeared in the proof of Proposition 2.2, we have another equivalent construction of $\mathfrak{G}$. Let $\mathcal{N}'(V,f)$ denote the cokernel of the endomorphism $\zeta\,\mathrm{id}-f$ on $V\otimes\mathcal{O}_{\mathbb{C}_{\zeta}}$. It naturally induces an $\mathcal{O}_{T^{\vee}}$–module $\mathcal{N}(V,f)$. We obtain $\widehat{\mathrm{RFM}}_{+}(\mathcal{N}(V,f))$, which is naturally isomorphic to $\mathfrak{G}(V,f)$. We obtain a quasi-inverse of $\mathfrak{G}$ as follows. Let E be a semistable bundle of degree 0 on T. We obtain a vector space $H^0(T^{\vee},\mathrm{RFM}_{-}(E))$. If we fix a lift of $\mathcal{S}p(E)$ to $\widetilde{\mathcal{S}p}(E)\subset\mathbb{C}$, then the multiplication of ζ induces an endomorphism g_{ζ} of $H^0(T^{\vee},\mathrm{RFM}_{-}(E))$. The construction of $(H^0(T^{\vee},\mathrm{RFM}_{-}(E)),g_{\zeta})$ from E gives a quasi-inverse of $\mathfrak{G}$.

Let $(E,\overline{\partial}_E)$ be a semistable bundle of degree 0 on T. Let $\tilde{\mathfrak{s}}\subset\mathbb{C}$ be a lift of $\mathcal{S}p(E)$.

Corollary 2.6 *We have a unique decomposition $\overline{\partial}_E=\overline{\partial}_{E,0}+f\,d\bar{z}$ with the following properties:*

- *$(E,\overline{\partial}_{E,0})$ is holomorphically trivial, ie it is isomorphic to a direct sum of copies of $\mathcal{O}_T$.*
- *f is a holomorphic endomorphism of $(E,\overline{\partial}_{E,0})$. We impose the condition that $\mathcal{S}p(H^0(f))\subset\tilde{\mathfrak{s}}$, where $H^0(f)$ is the induced endomorphism of the space of the global sections of $(E,\overline{\partial}_{E,0})$.*

Proof The existence of such a decomposition follows from the essential surjectivity of $\mathfrak{G}$. Let us prove the uniqueness. By considering the spectral decomposition, we have only to consider the case $\tilde{\mathfrak{s}}=\{0\}$. Suppose that $\overline{\partial}_E=\overline{\partial}'_{E,0}+g\,d\bar{z}$ is another decomposition with the desired property. Because f is holomorphic with respect to $\overline{\partial}_E$, we have $\overline{\partial}'_{E,0}f=0$ and $[f,g]=0$ by Lemma 2.5. We put $h=f-g$, which is also nilpotent. The identity induces an isomorphism $(E,\overline{\partial}_{E,0}+h)\simeq(E,\overline{\partial}'_{E,0})$. Because $\mathfrak{G}$ is fully faithful, we obtain $h=0$. □

The family version We have a family version of the equivalence. Let S be any complex manifold. Let $\pi_S\colon T\times S\to S$ denote the projection. Let $\mathrm{VB}^*(S)$ denote the category of pairs (V,f) of a coherent locally free $\mathcal{O}_S$–module V and its endomorphism f. A morphism $(V,f)\to(V',f')$ in $\mathrm{VB}^*(S)$ is a morphism of $\mathcal{O}_S$–modules $g\colon V\to V'$ such that $f'\circ g=g\circ f$. Such (V,f) naturally induces an $\mathcal{O}_{\mathbb{C}_\zeta\times S}$–module $\mathcal{M}(V,f)$. The support is denoted by $\mathcal{S}p(f)$. When we are given a divisor $\widetilde{\mathfrak{s}}\subset\mathbb{C}_\zeta\times S$ which is finite over S, then $\mathrm{VB}^*(S,\widetilde{\mathfrak{s}})$ denote the full subcategory of $(V,f)\in\mathrm{VB}^*(S)$ such that $\mathcal{S}p(f)\subset\widetilde{\mathfrak{s}}$.

Let $\mathrm{VB}_0^{ss}(T\times S/S)$ denote the full subcategory of $\mathcal{O}_{T\times S}$–modules, whose objects are semistable of degree 0 relative to S. When we are given a divisor $\mathfrak{s}\subset T^\vee\times S$ which is finite over S, then let $\mathrm{VB}_0^{ss}(T\times S/S,\mathfrak{s})$ denote the full subcategory of $E\in\mathrm{VB}_0^{ss}(T\times S/S)$ such that $\mathcal{S}p(E)\subset\mathfrak{s}$.

Let V be a holomorphic vector bundle on S with a holomorphic endomorphism f. The C^∞ vector bundle $\pi_S^{-1}V$ is equipped with a naturally induced holomorphic structure obtained as the pullback, denoted by $\overline{\partial}_0$. We obtain a holomorphic vector bundle $\mathfrak{G}(V,f):=(\pi_S^{-1}V,\overline{\partial}_0+f\,d\bar{z})$. By Lemma 2.1, $\mathfrak{G}$ gives a functor $\mathrm{VB}^*(S)\to\mathrm{VB}_0^{ss}(T\times S/S)$. If we are given $\mathfrak{s}\subset T^\vee\times S$ and its lift $\widetilde{\mathfrak{s}}\subset\mathbb{C}_\zeta\times S$, it gives an equivalence of the categories $\mathrm{VB}^*(S,\mathfrak{s})\to\mathrm{VB}_0^{ss}(T\times S/S,\widetilde{\mathfrak{s}})$.

We have another equivalent description of $\mathfrak{G}$. Let $(V,f)\in\mathrm{VB}^*(S)$. We have the naturally induced $\mathcal{O}_{\mathbb{C}_\zeta\times S}$–module $\mathcal{M}(V,f)$, which induces an $\mathcal{O}_{T^\vee\times S}$–module $\mathcal{N}(V,f)$. We have a natural isomorphism $\mathfrak{G}(V,f)\simeq\mathrm{RFM}_-(\mathcal{N}(V,f))$.

Suppose that we are given $\mathfrak{s}\subset T^\vee\times S$ with a lift $\widetilde{\mathfrak{s}}\subset\mathbb{C}_\zeta\times S$. For an object $E\in\mathrm{VB}_0^{ss}(T\times S/S,\mathfrak{s})$, we obtain an $\mathcal{O}_{\mathbb{C}_\zeta\times S}$–module $\mathcal{M}(E)$ such that the support of $\mathcal{M}(E)$ is contained in $\widetilde{\mathfrak{s}}$ and $\Phi_*\mathcal{M}(E)\simeq\mathrm{RFM}_-(E)$. The multiplication of ζ induces an endomorphism of $\mathrm{RFM}_-(E)$, and hence an endomorphism of $\pi_{S*}(\mathrm{RFM}_-(E))$, denoted by g_ζ, where $\pi_S\colon T^\vee\times S\to S$. The construction of $(\pi_{S*}\mathrm{RFM}_-(E),g_\zeta)$ from E gives a quasi-inverse of $\mathfrak{G}$.

2.1.4 Differential-geometric criterion We recall a differential-geometric criterion in terms of the curvature for a metrized holomorphic vector bundle to be semistable of degree 0. Let $(E,\overline{\partial}_E)$ be a holomorphic vector bundle on T with a Hermitian metric h. Let $F(h)$ denote the curvature of the Chern connection. We use the standard metric $dz\,d\bar{z}$ of T.

Lemma 2.7 *There exists a constant $\epsilon>0$, depending only on T and* rank E, *with the following property:*

- *If $|F(h)|_h\le\epsilon$, then $(E,\overline{\partial}_E,h)$ is semistable of degree 0.*

Proof The number $\deg(E) = \int \operatorname{Tr} F(h)$ is an integer, and we have $\int |\operatorname{Tr} F(h)| \leq |T| \operatorname{rank} E\epsilon$, where $|T|$ is the volume of T. Hence, we have $\int \operatorname{Tr} F(E_w) = 0$ if ϵ is sufficiently small. For any subbundle $E' \subset E$, by using the decreasing property of the curvature of subbundles, we also obtain $\deg(E') < 1$ and hence $\deg(E') \leq 0$. □

2.2 Filtered bundles

2.2.1 Filtered sheaves Let us recall the notion of filtered sheaves and filtered bundles. Let X be a complex manifold with a smooth hypersurface D. (We restrict ourselves to the case that D is smooth, because we are interested in only the case in this paper.) Let $\mathcal{E}$ be a coherent $\mathcal{O}_X(*D)$–module. Let $D = \coprod_{i\in\Lambda} D_i$ be the decomposition into the connected components. A filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$ is a sequence of coherent $\mathcal{O}_X$–submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{E}$ indexed by $\mathbb{R}^\Lambda$ satisfying the following.

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_{|X\setminus D} = \mathcal{E}_{|X\setminus D}$: we have $\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$ if $a_i \leq a'_i$ $(i \in \Lambda)$, where $\boldsymbol{a} = (a_i \mid i \in \Lambda)$ and $\boldsymbol{a}' = (a'_i \mid i \in \Lambda)$.
- On a small neighbourhood U of D_i $(i \in \Lambda)$, $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_{|U}$ depends only on a_i, which we denote by $\mathcal{P}_{a_i}(\mathcal{E}_{|U})$, or ${}^i\mathcal{P}_{a_i}(\mathcal{E}_{|U})$ when we emphasise i.
- For each $i \in \Lambda$ and $c \in \mathbb{R}$, there exists $\epsilon > 0$ such that ${}^i\mathcal{P}_c(\mathcal{E}_{|U}) = {}^i\mathcal{P}_{c+\epsilon}(\mathcal{E}_{|U})$.
- We have $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{n}}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E}\bigl(\sum n_i D_i\bigr)$, where $\boldsymbol{n} = (n_i) \in \mathbb{Z}^\Lambda$.

The tuple $(\mathcal{E}, \{\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \mid \boldsymbol{a} \in \mathbb{R}^\Lambda\})$ is denoted by $\mathcal{P}_*\mathcal{E}$. The filtration $\{\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \mid \boldsymbol{a} \in \mathbb{R}^\Lambda\}$ is also denoted by $\mathcal{P}_*\mathcal{E}$. We say that $\mathcal{E}$ is the $\mathcal{O}_X(*D)$–module underlying $\mathcal{P}_*\mathcal{E}$.

For a small neighbourhood U of D_i, we set ${}^i\mathcal{P}_{<a}(\mathcal{E}_{|U}) := \sum_{b<a} \mathcal{P}_b(\mathcal{E}_{|U})$. We also put ${}^i\mathcal{P}_a(\mathcal{E})_{|D_i} := \mathcal{P}_a(\mathcal{E}_{|U})_{|D_i}$, and ${}^i\operatorname{Gr}^{\mathcal{P}}_a(\mathcal{E}) := {}^i\mathcal{P}_a(\mathcal{E}_{|U})/{}^i\mathcal{P}_{<a}(\mathcal{E}_{|U})$, which are coherent $\mathcal{O}_{D_i}$–modules. We set

$$\mathcal{P}ar(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}, i) := \{b \in \,]a_i - 1, a_i] \mid {}^i\operatorname{Gr}^{\mathcal{P}}_b(\mathcal{E}) \neq 0\}, \quad \mathcal{P}ar(\mathcal{P}_*\mathcal{E}, i) := \bigcup_{\boldsymbol{a}\in\mathbb{R}^\Lambda} \mathcal{P}ar(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}, i).$$

A morphism of filtered sheaves $\mathcal{P}_*\mathcal{E}_1 \to \mathcal{P}_*\mathcal{E}_2$ is a morphism of $\mathcal{O}_X$–modules $\mathcal{E}_1 \to \mathcal{E}_2$ compatible with the filtrations. A subobject $\mathcal{P}_*\mathcal{E}_1 \subset \mathcal{P}_*\mathcal{E}$ is a subsheaf $\mathcal{E}_1 \subset \mathcal{E}$ satisfying $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E}_1) \subset \mathcal{P}_{\boldsymbol{a}}(\mathcal{E})$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$. It is called strict if $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E}_1) = \mathcal{E}_1 \cap \mathcal{P}_{\boldsymbol{a}}(\mathcal{E})$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$.

2.2.2 Filtered bundles and basic operations A filtered sheaf $\mathcal{P}_*\mathcal{E}$ is called a filtered bundle if $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ are locally free $\mathcal{O}_X$–modules and ${}^i\operatorname{Gr}^{\mathcal{P}}_a(\mathcal{E})$ are locally free $\mathcal{O}_{D_i}$–modules for any $i \in \Lambda$ and $a \in \mathbb{R}$. In that case, for any $b \in \,]a-1, a]$, we set

$$F_b({}^i\mathcal{P}_a(\mathcal{E})_{|D_i}) := \operatorname{Im}\bigl({}^i\mathcal{P}_b(\mathcal{E})_{|D_i} \to {}^i\mathcal{P}_a(\mathcal{E})_{|D_i}\bigr).$$

This is called the parabolic filtration.

The direct sum of filtered bundles $\mathcal{P}_*\mathcal{E}_i$ $(i=1,2)$ is defined to be the locally free $\mathcal{O}_X(*D)$–module $\mathcal{E}_1\oplus\mathcal{E}_2$ with the $\mathcal{O}_X$–submodules $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E}_1\oplus\mathcal{E}_2)=\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_1\oplus\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_2$ $(\boldsymbol{a}\in\mathbb{R}^{\Lambda})$. The tensor product of filtered bundles $\mathcal{P}_*\mathcal{E}_i$ $(i=1,2)$ is defined as the $\mathcal{O}_X(*D)$–module $\mathcal{E}_1\otimes\mathcal{E}_2$ with the $\mathcal{O}_X$–submodules $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E}_1\otimes\mathcal{E}_2)=\sum_{\boldsymbol{b}+\boldsymbol{c}\le\boldsymbol{a}}\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_1)\otimes\mathcal{P}_{\boldsymbol{c}}(\mathcal{E}_2)$. The inner homomorphism is defined as the $\mathcal{O}_X(*D)$–module $\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{E}_1,\mathcal{E}_2)$ with the $\mathcal{O}_X$–submodules

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{H}om(\mathcal{E}_1,\mathcal{E}_2)=\{f\in\mathcal{H}om(\mathcal{E}_1,\mathcal{E}_2)\mid f(\mathcal{P}_{\boldsymbol{b}}\mathcal{E}_1)\subset\mathcal{P}_{\boldsymbol{b}+\boldsymbol{a}}\mathcal{E}_2\}.$$

The most typical example is the $\mathcal{O}_X(*D)$–module $\mathcal{O}_X(*D)$ with the $\mathcal{O}_X$–submodules $\mathcal{P}_{\boldsymbol{a}}(\mathcal{O}_X(*D)):=\mathcal{O}(\sum[a_i]D_i)$, where $[a]:=\max\{n\in\mathbb{Z}\mid n\le a\}$. The filtered bundle is denoted just by $\mathcal{O}_X(*D)$. For any filtered bundle $\mathcal{P}_*\mathcal{E}$, the dual $\mathcal{P}_*(\mathcal{E}^\vee)$ is defined as $\mathcal{H}om(\mathcal{P}_*\mathcal{E},\mathcal{O}_X(*D))$. We have a natural isomorphism $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E}^\vee)\simeq\mathcal{P}_{<-\boldsymbol{a}+\boldsymbol{\delta}}(\mathcal{E})^\vee$, where $\boldsymbol{\delta}=(1,\dots,1)$.

Let $\varphi\colon(X',D')\to(X,D)$ be a ramified covering with $D'=\coprod_{i\in\Lambda}D_i'$ and $D=\coprod_{i\in\Lambda}D_i$. Let e_i be the degree of the ramification along D_i. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle over $\mathcal{E}$. The pullback of a filtered bundle is defined as the $\mathcal{O}_{X'}(*D')$–module $\varphi^*(\mathcal{E})$ with the $\mathcal{O}_{X'}$–submodules $\mathcal{P}_{\boldsymbol{a}}\varphi^*\mathcal{E}=\sum_{\boldsymbol{e}\boldsymbol{b}+\boldsymbol{n}\le\boldsymbol{a}}\varphi^*(\mathcal{P}_{\boldsymbol{b}}\mathcal{E})\otimes\mathcal{O}_{X'}(\sum n_iD_i')$, where $\boldsymbol{e}\boldsymbol{b}=(e_ib_i\mid i\in\Lambda)$. The filtered bundle is denoted by $\varphi^*(\mathcal{P}_*\mathcal{E})$.

Let $\mathcal{P}_*\mathcal{E}'$ be a filtered bundle on (X',D'). We obtain a locally free $\mathcal{O}_X(*D)$–module $\varphi_*\mathcal{E}'$ with the $\mathcal{O}_X$–submodules $\mathcal{P}_{\boldsymbol{c}}(\varphi_*\mathcal{E}')$ such that $\mathcal{P}_{\boldsymbol{c}}(\varphi_*\mathcal{E}')_{|U}=\varphi_*(\mathcal{P}_{e_ic_i}\mathcal{E}'_{|\varphi^{-1}(U_i)})$. The filtered bundle is denoted by $\varphi_*(\mathcal{P}_*\mathcal{E})$. Suppose that $\varphi\colon(X',D')\to(X,D)$ is a Galois covering with the Galois group $\mathrm{Gal}(\varphi)$, and that $\mathcal{P}_*\mathcal{E}'$ be a $\mathrm{Gal}(\varphi)$–equivariant filtered bundle. Then $\varphi_*(\mathcal{P}_*\mathcal{E})$ is equipped with an induced $\mathrm{Gal}(\varphi)$–action. The $\mathrm{Gal}(\varphi)$–invariant part is called the descent of $\mathcal{P}_*\mathcal{E}'$ with respect to φ.

2.2.3 The parabolic first Chern class Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on (X,D). Suppose that $\mathcal{E}$ is torsion-free. The parabolic first Chern class of $\mathcal{P}_*\mathcal{E}$ is defined as

$$\text{par-c}_1(\mathcal{P}_*\mathcal{E})=c_1(\mathcal{P}_{\boldsymbol{a}}\mathcal{E})-\sum_{i\in\Lambda}\sum_{b\in\mathcal{P}ar(\mathcal{P}_{\boldsymbol{a}}\mathcal{E},i)}b\dim{}^i\mathrm{Gr}_b^{\mathcal{P}}(\mathcal{E})[D_i].$$

Here, $[D_i]$ is the cohomology class of D_i. It is independent of the choice of $\boldsymbol{a}$.

Let U_i be a small neighbourhood of D_i. Suppose that we are given a decomposition $\mathcal{P}_*\mathcal{E}_{|U_i}=\bigoplus_{k\in I(i)}\mathcal{P}_*\mathcal{E}_{i,k}$ for each $i\in\Lambda$. Let $\mathcal{U}$ be a locally free $\mathcal{O}_X$–submodule of $\mathcal{E}$ such that $\mathcal{U}_{|U_i}=\bigoplus_{k\in I(i)}\mathcal{P}_{a(i,k)}\mathcal{E}_{i,k}$, where $a(i,k)\in\mathbb{R}$. It is easy to check the equalities

$$\text{par-c}_1(\mathcal{P}_*\mathcal{E}) = c_1(\mathcal{U}) - \sum_{i\in\Lambda}\sum_{k\in I(i)} \delta(\mathcal{P}_*\mathcal{E}_{i,k}, a(i,k)),$$

$$\delta(\mathcal{P}_*\mathcal{E}_{i,k}, a(i,k)) := \sum_{b\in \mathcal{P}ar(\mathcal{P}_{a(i,k)}\mathcal{E}_{i,k})} b \operatorname{rank} \mathrm{Gr}^{\mathcal{P}}_b(\mathcal{E}_{i,k})[D_i].$$

2.2.4 Compatible frames For simplicity, we consider the case where X is a neighbourhood of 0 in $\mathbb{C}$ and $D = \{0\}$. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle on (X, D). For any section f of $\mathcal{E}$, we set $\deg^{\mathcal{P}}(f) := \min\{a \in \mathbb{R} \mid f \in \mathcal{P}_a\mathcal{E}\}$. Let $\boldsymbol{v} = (v_1, \dots, v_r)$ be a frame of $\mathcal{P}_a\mathcal{E}$. We say that it is compatible with the parabolic structure if for any $b \in \mathcal{P}ar(\mathcal{P}_a\mathcal{E})$, the set $\{v_i \mid \deg(v_i) = b\}$ induces a base of $\mathrm{Gr}^{\mathcal{P}}_b(\mathcal{E})$.

Let φ: $(X', D') \to (X, D)$ be a ramified covering given by $\varphi(u) = u^p$. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle on (X, D). Let $\boldsymbol{v}$ be a compatible frame of $\mathcal{P}_a\mathcal{E}$. Let $c_i := \deg^{\mathcal{P}}(v_i)$. We set $n_i := \max\{n \in \mathbb{Z} \mid n + pc_i \le pa\}$, and $w_i := u^{-n_i}\varphi^* v_i$. Then $\boldsymbol{w} = (w_1, \dots, w_r)$ is a compatible frame of $\varphi^*(\mathcal{P}_*\mathcal{E})$ such that $\deg^{\mathcal{P}}(w_i) = n_i + pc_i$.

Let $\mathcal{P}_*\mathcal{E}'$ be a filtered bundle on (X', D'). Let $\boldsymbol{v}'$ be a compatible frame of $\mathcal{P}_a\mathcal{E}'$. Let $c_i := \deg^{\mathcal{P}}(v'_i)$. For $0 \le j < p$, we set $w'_{ij} := u^j v'_i$. They naturally induce sections of $\mathcal{P}_{a/p}(\varphi_*\mathcal{E})$, denoted by $\widetilde{w}'_{ij}$. Then we have that $\widetilde{\boldsymbol{w}}' := (\widetilde{w}'_{ij} \mid 1 \le i \le \operatorname{rank}\mathcal{E},\ 0 \le j < p)$ gives a compatible frame of $\mathcal{P}_{a/p}(\varphi_*\mathcal{E})$ such that $\deg^{\mathcal{P}}(\widetilde{w}'_{ij}) = (c_i - j)/p$.

2.2.5 Adapted metric Let us return to the setting in Section 2.2.1. Let V be a holomorphic vector bundle on $X \setminus D$ with a Hermitian metric h. Recall that, for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, we obtain a natural $\mathcal{O}_X$–module $\mathcal{P}^h_{\boldsymbol{a}}V$ on X as follows. Let U be any open subset of X. For any $P \in U$, we take a holomorphic coordinate neighbourhood $(X_P, z_1, \dots, z_n)$ around P such that X_P is relatively compact in U, $X_P \cap D = X_P \cap D_i$ for some $i \in \Lambda$, and $X_P \cap D = \{z_1 = 0\}$. Then let $\mathcal{P}^h_{\boldsymbol{a}}(U)$ denote the space of holomorphic sections f of $V_{|U\setminus D}$ such that $|f_{|X_P\setminus D}|_h = O(|z_1|^{-a_i-\epsilon})$ for any $\epsilon > 0$ and any $P \in U$. In general, $\mathcal{P}^h_{\boldsymbol{a}}V$ are not $\mathcal{O}_X$–coherent.

Suppose that we are given a filtered bundle $\mathcal{P}_*V$ on (X, D), and that $V := \mathcal{P}_*V_{|X\setminus D}$ is equipped with a Hermitian metric h such that $\mathcal{P}^h_*V = \mathcal{P}_*V$. In that case, we say that h is adapted to $\mathcal{P}_*V$.

2.3 Filtered Higgs bundles

Let us recall the notion of filtered Higgs bundles on curves. Let X be a complex curve with a discrete subset D. Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on (X, D). Let θ be a Higgs field of $\mathcal{E}$, ie θ is an $\mathcal{O}_X$–homomorphism $\mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$. Then $(\mathcal{P}_*\mathcal{E}, \theta)$ is called a filtered Higgs bundle.

We shall consider two conditions on the compatibility of θ and the filtration $\mathcal{P}_*\mathcal{E}$. One is the admissibility, and the other is goodness. The latter is what we are really interested in, because it is closely related to wild harmonic bundles and L^2 instantons. The former is easier to handle, and more natural when we consider algebraic Nahm transforms. We shall explain the easier one first.

The conditions are given locally around each point of D. So, we shall explain them in the case $X := \{z \in \mathbb{C} \mid |z| < \rho_0\}$ and $D := \{0\}$.

2.3.1 Admissible filtered Higgs bundles For each positive integer p, let $\varphi_p\colon X^{\langle p\rangle} = \{|z_p| < \rho_0^{1/p}\} \to X$ be given by $\varphi_p(z_p) = z_p^p$. Let $\mathcal{P}_*V$ be a filtered bundle on (X, D) with a Higgs field θ. Let $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ such that $\gcd(p, m) = 1$. We say that $(\mathcal{P}_*V, \theta)$ has slope (p, m) if the following hold:

- Let $(\mathcal{P}_*V^{\langle p\rangle}, \theta^{\langle p\rangle})$ be a filtered Higgs bundle obtained as the pullback of $(\mathcal{P}_*V, \theta)$ by φ_p. Then we have $z_p^m\theta^{\langle p\rangle}(\mathcal{P}_cV^{\langle p\rangle}) \subset \mathcal{P}_cV^{\langle p\rangle}dz_p/z_p$ for any $c \in \mathbb{R}$.
- Let $\operatorname{Res}(z_p^m\theta^{\langle p\rangle})$ denote the endomorphism of $\operatorname{Gr}^{\mathcal{P}}_c(V^{\langle p\rangle})$ obtained as the residue of $z_p^m\theta^{\langle p\rangle}$. If $(p, m) \neq (1, 0)$, we impose that $\operatorname{Res}(z_p^m\theta^{\langle p\rangle})$ is invertible for any c.

Although $\operatorname{Res}(z_p^m\theta^{\langle p\rangle})$ may depend on the choice of a coordinate, the above condition is independent. Let $\mathcal{I}(\theta)$ denote the set of the eigenvalues of $\operatorname{Res}(z_p^m\theta^{\langle p\rangle})$. We have $\operatorname{Gal}(\varphi_p)$–action on $(\mathcal{P}_*V^{\langle p\rangle}, \theta^{\langle p\rangle})$ and $\mathcal{I}(\theta)$. The quotient set $\mathcal{I}(\theta)/\operatorname{Gal}(\varphi)$ is denoted by $\boldsymbol{\mathcal{I}}(\theta)$. We have the orbit decomposition $\mathcal{I}(\theta) = \coprod_{\boldsymbol{o} \in \boldsymbol{\mathcal{I}}(\theta)} \boldsymbol{o}$. We say that $(\mathcal{P}_*V, \theta)$ has type $(p, m, \boldsymbol{o})$ if moreover $\boldsymbol{\mathcal{I}}(\theta) = \{\boldsymbol{o}\}$.

If $m \neq 0$, then $\boldsymbol{o}$ is naturally an element of $\mathcal{J}(p, m) := \mathbb{C}^*/\operatorname{Gal}(\varphi_p)$, where the action is given by $(t, \alpha) \mapsto t^m\alpha$. If $m = 0$, then $\boldsymbol{o}$ is an element of $\mathcal{J}(1, 0) := \mathbb{C}$. When $(\mathcal{P}_*V, \theta)$ has slope (p, m), it has a decomposition $(\mathcal{P}_*V, \theta) = \bigoplus_{\boldsymbol{o} \in \mathcal{J}(p,m)}(\mathcal{P}_*V_{\boldsymbol{o}}, \theta_{\boldsymbol{o}})$ after X is shrunk appropriately, such that $(\mathcal{P}_*V_{\boldsymbol{o}}, \theta_{\boldsymbol{o}})$ has type $(p, m, \boldsymbol{o})$.

The filtered Higgs bundle $(\mathcal{P}_*V, \theta)$ is called admissible if it has a decomposition

$$(\mathcal{P}_*V, \theta) = \bigoplus_{(p,m)} \big(\mathcal{P}_*V^{(p,m)}, \theta^{(p,m)}\big)$$

after X is shrunk appropriately, such that each $(\mathcal{P}_*V^{(p,m)}, \theta^{(p,m)})$ has slope (p, m). Here, the decomposition is called the slope decomposition. It is refined to a decomposition $(\mathcal{P}_*V, \theta) = \bigoplus_{(p,m,\boldsymbol{o})} (\mathcal{P}_*V_{\boldsymbol{o}}^{(p,m)}, \theta_{\boldsymbol{o}}^{(p,m)})$ such that $(\mathcal{P}_*V_{\boldsymbol{o}}^{(p,m)}, \theta_{\boldsymbol{o}}^{(p,m)})$ has type

$(p,m,\boldsymbol{o})$. In this paper, the decomposition is called the type decomposition. For $\alpha\in\mathbb{Q}_{\geq 0}$, we say that the slope of $(\mathcal{P}_*V,\theta)$ is smaller (resp. strictly smaller) than α if $\mathcal{P}_*V^{(p,m)}=0$ for $p/m>\alpha$ (resp. $p/m\geq\alpha$) in the slope decomposition.

Suppose that $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$. After X is shrunk appropriately, we have a decomposition

$$\big(\mathcal{P}_*V^{\langle p\rangle},\theta^{\langle p\rangle}\big)=\bigoplus_{\alpha\in\boldsymbol{o}}\big(\mathcal{P}_*V^{\langle p\rangle}_{\alpha},\theta^{\langle p\rangle}_{\alpha}\big)$$

such that $\mathrm{Res}(z_p^m\theta^{\langle p\rangle}_{\alpha})$ has a unique eigenvalue α. We have a natural isomorphism

$$\varphi_{p*}\big(\mathcal{P}_*V^{\langle p\rangle}_{\alpha},\theta^{\langle p\rangle}_{\alpha}\big)\simeq(\mathcal{P}_*V,\theta)$$

for any $\alpha\in\boldsymbol{o}$.

Lemma 2.8 *Let $(\mathcal{P}_*V,\theta)$ be an admissible filtered Higgs bundle on (X,D). Let $\mathcal{P}_*V'$ be a strict filtered Higgs subbundle, ie it is a strict filtered subbundle such that $\theta(V')\subset V'\otimes\Omega^1_X$. The restriction of θ to V' is denoted by θ'. Then $(\mathcal{P}_*V',\theta')$ is admissible.* □

2.3.2 Good filtered Higgs bundles We have a stronger condition. Let X and D be as in Section 2.3.1. We say that a filtered Higgs bundle $(\mathcal{P}_*V,\theta)$ on (X,D) is good if there exists a ramified covering $\varphi_p\colon (X^{\langle p\rangle},D^{\langle p\rangle})\to(X,D)$ given by $\varphi_p(z_p)=z_p^p$ with a decomposition

$$\varphi_p^*(\mathcal{P}_*V,\theta)=\bigoplus_{\mathfrak{a}\in z_p^{-1}\mathbb{C}[z_p^{-1}]}\big(\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{a}},\theta^{\langle p\rangle}_{\mathfrak{a}}\big),\tag{7}$$

such that $\theta^{\langle p\rangle}_{\mathfrak{a}}-d\mathfrak{a}\,\mathrm{id}_{V^{\langle p\rangle}_{\mathfrak{a}}}$ is logarithmic in the sense that it gives a morphism $\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{a}}\to\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{a}}\,dz_p/z_p$. Let $\mathrm{Irr}(\varphi_p^*\theta)$ denote the set of $\mathfrak{a}$ such that $V^{\langle p\rangle}_{\mathfrak{a}}\neq 0$. The Galois group $\mathrm{Gal}(\varphi_p)$ naturally acts on $\varphi_p^*(\mathcal{P}_*V,\theta)$ and $\mathrm{Irr}(\varphi_p^*\theta)$. The quotient set $\mathrm{Irr}(\varphi_p^*\theta)/\mathrm{Gal}(\varphi_p)$ is denoted by $\mathbf{Irr}(\varphi_p^*\theta)$. We have the orbit decomposition

$$\mathrm{Irr}(\varphi_p^*\theta)=\coprod_{\mathfrak{o}\in\mathbf{Irr}(\varphi_p^*\theta)}\mathfrak{o}.$$

We set $(\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{o}},\theta^{\langle p\rangle}_{\mathfrak{o}}):=\bigoplus_{\mathfrak{a}\in\mathfrak{o}}(\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{a}},\theta^{\langle p\rangle}_{\mathfrak{a}})$. We obtain a $\mathrm{Gal}(\varphi_p)$–equivariant decomposition $\varphi_p^*(\mathcal{P}_*V,\theta)=\bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\varphi_p^*\theta)}(\mathcal{P}_*V^{\langle p\rangle}_{\mathfrak{o}},\theta^{\langle p\rangle}_{\mathfrak{o}})$. By the descent, we obtain a decomposition

$$(\mathcal{P}_*V,\theta)=\bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\varphi^*\theta)}(\mathcal{P}_*V_{\mathfrak{o}},\theta_{\mathfrak{o}}).\tag{8}$$

If we have a factorisation $\varphi_p = \varphi_{p_1} \circ \varphi_{p_2}$ such that $\varphi_{p_2}^*(\mathcal{P}_* V, \theta)$ has a decomposition as above, φ_{p_1} gives a bijection $\mathrm{Irr}(\varphi_{p_2}^* \theta) \simeq \mathrm{Irr}(\varphi_p^* \theta)$. It induces a bijection of the quotient sets by the Galois groups. By the identification, we denote them by $\mathrm{Irr}(\theta)$ and $\mathbf{Irr}(\theta)$. The decomposition (8) is independent of the choice of φ_p.

For each $\mathfrak{o} \in \mathbf{Irr}(\theta)$, there exists a minimum $p_{\mathfrak{o}}$ among the numbers p such that $\varphi_p^*(\mathcal{P}_* V_{\mathfrak{o}}, \theta_{\mathfrak{o}})$ has a decomposition such as (7). In this case, we have $|\mathfrak{o}| = p_{\mathfrak{o}}$. We set $X^{\mathfrak{o}} := X^{\langle p_{\mathfrak{o}} \rangle}$, $\varphi_{\mathfrak{o}} := \varphi_{p_{\mathfrak{o}}}$ and $z_{\mathfrak{o}} := z_{p_{\mathfrak{o}}}$. We have the following decomposition on $X^{\mathfrak{o}}$:

$$\varphi_{\mathfrak{o}}^*(\mathcal{P}_* V_{\mathfrak{o}}, \theta_{\mathfrak{o}}) = \bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_* V_{\mathfrak{a}}^{\mathfrak{o}}, \theta_{\mathfrak{a}}^{\mathfrak{o}}). \tag{9}$$

For any $\mathfrak{a} \in \mathfrak{o}$, we have a natural isomorphism $(\mathcal{P}_* V_{\mathfrak{o}}, \theta_{\mathfrak{o}}) \simeq \varphi_{\mathfrak{o}*}(\mathcal{P}_* V_{\mathfrak{a}}^{\mathfrak{o}}, \theta_{\mathfrak{a}}^{\mathfrak{o}})$. We set $m_{\mathfrak{o}} := (\mathrm{ord}_{z_{\mathfrak{o}}^{-1}} \mathfrak{a})$ which is independent of $\mathfrak{a} \in \mathfrak{o}$. In this paper, we say that $(\mathcal{P}_* V, \theta)$ has pure irregularity $\mathfrak{o}$ if $(\mathcal{P}_* V, \theta) = (\mathcal{P}_* V_{\mathfrak{o}}, \theta_{\mathfrak{o}})$.

If X is shrunk appropriately, we have a decomposition (which is a refinement of (7))

$$\varphi_p^*(\mathcal{P}_* V, \theta) = \bigoplus_{\mathfrak{a} \in z_p^{-1}\mathbb{C}[z_p^{-1}]} \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}_* V_{\mathfrak{a},\alpha}^{\langle p \rangle}, \theta_{\mathfrak{a},\alpha}^{\langle p \rangle})$$

such that the eigenvalues of the residues $\mathrm{Res}(\theta_{\mathfrak{a},\alpha}^{\langle p \rangle} - (d\mathfrak{a} + p\alpha\, dz_p/z_p)\, \mathrm{id}_{V_{\mathfrak{a},\alpha}^{\langle p \rangle}})$ are 0. Let $(\mathcal{P}_* V_{\mathfrak{o},\alpha}, \theta_{\mathfrak{o},\alpha})$ be the descent of

$$\bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_* V_{\mathfrak{a},\alpha}^{\langle p \rangle}, \theta_{\mathfrak{a},\alpha}^{\langle p \rangle})$$

to X. We obtain a decomposition

$$(\mathcal{P}_* V, \theta) = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\theta)} \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}_* V_{\mathfrak{o},\alpha}, \theta_{o,\alpha}).$$

On $X^{\mathfrak{o}}$, we have a decomposition

$$\varphi_{\mathfrak{o}}^*(\mathcal{P}_* V_{\mathfrak{o},\alpha}, \theta_{\mathfrak{o},\alpha}) = \bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_* V_{\mathfrak{a},\alpha}^{\mathfrak{o}}, \theta_{\mathfrak{a},\alpha}^{\mathfrak{o}}).$$

Lemma 2.9 *Let $(\mathcal{P}_* V, \theta)$ be a good filtered Higgs bundle. Let $\mathcal{P}_* V'$ be a strict Higgs subbundle. The restriction of θ to V' is denoted by θ'. Then $(\mathcal{P}_* V', \theta')$ is also good.*

Proof Suppose $(\mathcal{P}_* V, \theta)$ is unramifiedly good with the decomposition $(\mathcal{P}_* V, \theta) = \bigoplus (\mathcal{P}_* V_{\mathfrak{a}}, \theta_{\mathfrak{a}})$. Because $\theta(V') \subset V' \otimes \Omega_X^1$, we have $V' = \bigoplus (V' \cap V_{\mathfrak{a}})$. By the strictness, we obtain $\mathcal{P}_* V' = \bigoplus (V' \cap \mathcal{P}_* V_{\mathfrak{a}})$. Hence, $(\mathcal{P}_* V', \theta')$ is good. The ramified case can be reduced to the unramified case by the descent. □

Take $p \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$ with $\gcd(p,m)=1$. Let $\mathbf{Irr}(\theta,p,m) := \{\mathfrak{o} \in \mathbf{Irr}(\theta) \mid p_{\mathfrak{o}}/m_{\mathfrak{o}} = p/m\}$. We have

$$\mathcal{P}_* V^{(p,m)} = \bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\theta,p,m)} \mathcal{P}_* V_{\mathfrak{o}}.$$

For any $\boldsymbol{o} \in \mathcal{J}(p,m)$, we have $\mathbf{Irr}(\theta,p,m,\boldsymbol{o}) \subset \mathbf{Irr}(\theta,p,m)$ such that

$$\mathcal{P}_* V_{\boldsymbol{o}}^{(p,m)} = \bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\theta,p,m,\boldsymbol{o})} \mathcal{P}_* V_{\mathfrak{o}}.$$

Take any $\alpha \in \boldsymbol{o}$. For each $\mathfrak{o} \in \mathbf{Irr}(\theta,p,m,\boldsymbol{o})$, we have $\mathfrak{a} \in \mathfrak{o}$ such that

$$\mathcal{P}_* V_{\alpha}^{\langle p\rangle} = \bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\theta,p,m,\boldsymbol{o})} \varphi_{(p_{\mathfrak{o}}/p)*}(\mathcal{P}_* V_{\mathfrak{a}}^{\mathfrak{o}}).$$

Here, $\varphi_{p_{\mathfrak{o}}/p}$ is the ramified covering $X^{\mathfrak{o}} \to X^{\langle p\rangle}$ given by $\varphi_{p_{\mathfrak{o}}/p}(z_{\mathfrak{o}}) = z_{\mathfrak{o}}^{p_{\mathfrak{o}}/p}$. Let $c \in \mathbb{R}$. We take a frame $\boldsymbol{v}_{\mathfrak{o}} = (v_{\mathfrak{o},i})$ of $\mathcal{P}_{p_{\mathfrak{o}}c} V_{\mathfrak{a}}^{\mathfrak{o}}$ compatible with the parabolic structure. Then the tuple of the sections

$$\bigl\{z_{\mathfrak{o}}^{j} v_{\mathfrak{o},i} \bigm| \mathfrak{o} \in \mathbf{Irr}(\theta,p,m,\boldsymbol{o}),\ 1 \leq i \leq \operatorname{rank} V_{\mathfrak{a}}^{\mathfrak{o}},\ 0 \leq j < p_{\mathfrak{o}}/p\bigr\}$$

gives a frame of $\mathcal{P}_{pc} V_{\alpha}^{\langle p\rangle}$.

2.3.3 Filtered bundles with an endomorphism Let U_τ be a small neighbourhood of 0 in $\mathbb{C}_\tau$. Let $\mathcal{P}_* V$ be a filtered bundle on $(U_\tau, 0)$ with an endomorphism g. We say that $(\mathcal{P}_* V, g)$ has type $(p,m,\boldsymbol{o})$ (slope (p,m)) if $(\mathcal{P}_* V, -\tau^{-2} g\,d\tau)$ has type $(p,m,\boldsymbol{o})$ (resp. slope (p,m)). The condition implies $p \geq m$. We say that $(\mathcal{P}_* V, g)$ is admissible if $(\mathcal{P}_* V, -\tau^{-2} g\,d\tau)$ is admissible. If $(\mathcal{P}_* V, g)$ is admissible, we have the type and slope decompositions

$$(\mathcal{P}_* V, g) = \bigoplus \bigl(\mathcal{P}_* V_{\boldsymbol{o}}^{(p,m)}, g_{\boldsymbol{o}}^{(p,m)}\bigr) \quad \text{and} \quad (\mathcal{P}_* V, g) = \bigoplus \bigl(\mathcal{P}_* V^{(p,m)}, g^{(p,m)}\bigr)$$

respectively, after X is shrunk appropriately.

Similarly, $(\mathcal{P}_* V, g)$ is called good if $(\mathcal{P}_* V, -\tau^{-2} g\,d\tau)$ is a good filtered Higgs bundle.

Remark 2.10 We regard U_τ as a neighbourhood of ∞ in $\mathbb{P}^1$. Of course, $-\tau^{-2} d\tau = dw$ for $w = \tau^{-1}$.

Remark 2.11 We shall be interested in the case that $(\mathcal{P}_* V, g)$ is decomposed into $\bigoplus_{\alpha\in\mathbb{C}} (\mathcal{P}_* V_\alpha, g_\alpha)$ such that $\mathcal{S}p(g_{\alpha|0}) = \{\alpha\}$ and $(\mathcal{P}_* V_\alpha, g_\alpha - \alpha)$ is admissible. In that case, in the slope decomposition

$$(\mathcal{P}_* V_\alpha, g_\alpha - \alpha) = \bigoplus \bigl(\mathcal{P}_* V_\alpha^{(p,m)}, g_\alpha^{(p,m)}\bigr),$$

we have $m/p < 1$ for $V_\alpha^{(p,m)} \neq 0$.

2.4 Filtered bundles on $(T \times \mathbb{P}^1, T \times \{\infty\})$

2.4.1 Local conditions Let $U \subset \mathbb{P}^1$ be a small neighbourhood of ∞. We introduce some conditions on filtered bundles $\mathcal{P}_* E$ on $(T \times U, T \times \{\infty\})$.

(A1) $\mathcal{P}_c E_{|T\times\infty}$ is semistable of degree 0 for any $c \in \mathbb{R}$.

The condition is equivalent to $\mathrm{Gr}_c^{\mathcal{P}}(E)$ being semistable of degree 0 for any $c \in \mathbb{R}$. Let $\mathcal{S}p_\infty(E) \subset T^\vee$ denote the spectrum of $\mathcal{P}_c E_{|T\times\infty}$. It is independent of c. We fix its lift to $\widetilde{\mathcal{S}p}_\infty(E) \subset \mathbb{C}$. Then as observed in Section 2.1, for a small neighbourhood U' of $\infty \in \mathbb{P}^1$, we obtain the corresponding filtered bundle $\mathcal{P}_* V$ with an endomorphism g on (U', ∞) such that $\mathcal{S}p(g_{|\infty}) = \widetilde{\mathcal{S}p}_\infty(E)$. We have the decomposition

$$(\mathcal{P}_* V, g) = \bigoplus_{P \in \mathcal{S}p_\infty(E)} (\mathcal{P}_* V_P, g_P)$$

such that $\mathcal{S}p(g_P) \cap (\mathbb{C} \times \{\infty\}) = \{\widetilde{P}\}$ is the lift of P. A filtered bundle satisfying (A1) is called admissible if it satisfies the following condition.

(A2) $(\mathcal{P}_* V_P, g_P - \widetilde{P}\,\mathrm{id})$ is admissible in the sense of Section 2.3.3 for any $P \in \mathcal{S}p_\infty(E)$. This condition is independent of the choice of $\widetilde{\mathcal{S}p}_\infty(E)$.

We have the type decomposition $(\mathcal{P}_* V_P, g_P - \widetilde{P}\,\mathrm{id}) = \bigoplus_{p,m,o} (\mathcal{P}_* V_{P,o}^{(p,m)}, g_{P,o}^{(p,m)})$, and we have the corresponding decomposition

$$\mathcal{P}_* E = \bigoplus_P \bigoplus_{p,m,o} \mathcal{P}_* E_{P,o}^{(p,m)},$$

which is called the type decomposition of $\mathcal{P}_* E$. The following lemma is clear.

Lemma 2.12 *If $\mathcal{P}_* E$ satisfies condition (A1) (resp. the admissibility), then the dual $\mathcal{P}_*(E^\vee)$ also satisfies condition (A1) (resp. the admissibility).* □

We also have the following condition.

(Good) Let $\mathcal{P}_* E$ be a filtered bundle on $(T \times U, T \times \{\infty\})$ satisfying (A1). Take any lift $\widetilde{\mathcal{S}p}_\infty(E) \subset \mathbb{C}$ of $\mathcal{S}p_\infty(E)$. Then the filtered bundle is called good if the corresponding filtered bundle $\mathcal{P}_* V$ with an endomorphism g is good in the sense of Section 2.3.3.

2.4.2 Some remarks on the cohomology groups Let $\mathcal{P}_*E$ be a filtered bundle on $(T\times\mathbb{P}^1, T\times\{\infty\})$ satisfying (A1). Let $\mathcal{U}$ be any $\mathcal{O}_{T\times\mathbb{P}^1}$–submodule of $\mathcal{P}_cE$ for some $c\in\mathbb{R}$, such that $\mathcal{U}_{|T\times\mathbb{C}_w}=\mathcal{P}_cE_{|T\times\mathbb{C}_w}$ and $\mathcal{U}_{|T\times\{\infty\}}$ is semistable of degree 0. We give some remarks on the cohomology groups of $\mathcal{U}$.

Lemma 2.13 *Suppose that $0\notin\mathcal{S}p_\infty(\mathcal{P}_*E)$. Then we have that $H^j(T\times\mathbb{P}^1,\mathcal{U})=H^j(T\times\mathbb{P}^1,\mathcal{P}_cE)$.*

Proof Let $\pi\colon T\times\mathbb{P}^1\to\mathbb{P}^1$ be the projection. By assumption, we have $R\pi_*(\mathcal{U}\otimes L)\simeq R\pi_*(\mathcal{P}_cE\otimes L)$, because both of them vanish around ∞. Then the claim of the lemma follows. □

Suppose that $\mathcal{P}_*E$ is admissible, and we take some refinement. We have the decomposition $\mathcal{U}=\bigoplus_{\boldsymbol{P}}\bigoplus_{p,m,\boldsymbol{o}}\mathcal{U}^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}$ around $T\times\{\infty\}$, where $\mathcal{U}^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}:=\mathcal{U}\cap\mathcal{P}_cE^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}$. Let $\mathcal{U}'\subset\mathcal{P}_cE$ be a subsheaf satisfying the above conditions. If $0\notin\mathcal{S}p_\infty(\mathcal{P}_*E)$, we have $H^i(T\times\mathbb{P}^1,\mathcal{U})=H^i(T\times\mathbb{P}^1,\mathcal{U}')$ by Lemma 2.13.

Lemma 2.14 *If $0\in\mathcal{S}p_\infty(\mathcal{P}_*E)$ and $\mathcal{U}^{(1,0)}_{0,0}=\mathcal{U}'^{(1,0)}_{0,0}$, we have natural isomorphisms $H^i(T\times\mathbb{P}^1,\mathcal{U})\simeq H^i(T\times\mathbb{P}^1,\mathcal{U}')$ for $i=0,2$.*

Proof We have only to consider the case that $\mathcal{U}\subset\mathcal{U}'$, and we shall prove that the natural morphisms $H^i(T\times\mathbb{P}^1,\mathcal{U})\to H^i(T\times\mathbb{P}^1,\mathcal{U}')$ are isomorphisms. Let $\varphi\in H^0(T\times\mathbb{P}^1,\mathcal{U}')$. Around $T\times\{\infty\}$, we have the decomposition $\varphi=\sum_{\boldsymbol{P},p,m,\boldsymbol{o}}\varphi^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}$. We see that $\varphi^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}=0$ unless $(\boldsymbol{P},p,m,\boldsymbol{o})=(0,1,0,0)$. Hence,

$$H^0(T\times\mathbb{P}^1,\mathcal{U})\to H^0(T\times\mathbb{P}^1,\mathcal{U}')$$

is an isomorphism. The duals $\mathcal{U}^\vee$ and $(\mathcal{U}')^\vee$ are subsheaves of $\mathcal{P}_{c'}(E^\vee)$ for some c', and satisfy the above conditions. Hence, by using the Serre duality, we obtain that $H^2(T\times\mathbb{P}^1,\mathcal{U})\to H^2(T\times\mathbb{P}^1,\mathcal{U}')$ is an isomorphism. □

2.4.3 Vanishing condition Let $(\mathcal{P}_*E,\theta)$ be a filtered bundle on $(T\times\mathbb{P}^1, T\times\{0\})$. We will be concerned with the following condition on the vanishing of the cohomology groups:

(A3) $H^0(T\times\mathbb{P}^1,\mathcal{P}_0E\otimes p^*L)=0$ and $H^2(T\times\mathbb{P}^1,\mathcal{P}_{<-1}E\otimes p^*L)=0$ for any line bundle L of degree 0 on T, where p denotes the projection $T\times\mathbb{P}^1\to T$.

We shall often omit to denote p^* if there is no risk of confusion.

Lemma 2.15 *If $\mathcal{P}_*E$ satisfies condition (A3), the dual $\mathcal{P}_*(E^\vee)$ also satisfies condition (A3).*

Proof Note that $\mathcal{P}_a(E^\vee)^\vee \otimes \Omega^1_{\mathbb{P}^1} \simeq \mathcal{P}_{<-a-1}(E)$. Hence, by Serre duality, we have that $H^0(T\times\mathbb{P}^1, \mathcal{P}_0(E^\vee)\otimes L^\vee)$ is the dual space of $H^2(T\times\mathbb{P}^1, \mathcal{P}_{<-1}E\otimes L)$, and $H^2(T\times\mathbb{P}^1, \mathcal{P}_{<-1}(E^\vee)\otimes L^\vee)$ is the dual space of $H^0(T\times\mathbb{P}^1, \mathcal{P}_0 E\otimes L)$. The claim of the lemma follows. □

Lemma 2.16 *Let L be any holomorphic line bundle on T of degree 0. If $\mathcal{P}_*E$ satisfies (A3), then we have that $H^0(T\times\mathbb{P}^1, \mathcal{P}_c E\otimes L) = 0$ for any $c \le 0$ and $H^2(T\times\mathbb{P}^1, \mathcal{P}_{<c}E\otimes L) = 0$ for any $c \ge -1$.*

Proof We have only to consider the case $L = \mathcal{O}_T$. For $c \le 0$, we have

$$H^0(T\times\mathbb{P}^1, \mathcal{P}_c E) \subset H^0(T\times\mathbb{P}^1, \mathcal{P}_0 E) = 0.$$

For $c \ge -1$, the support of the quotient $\mathcal{P}_{<c}E/\mathcal{P}_{<-1}E$ is one-dimensional. Hence, the morphism $0 = H^2(T\times\mathbb{P}^1, \mathcal{P}_{<-1}E) \to H^2(T\times\mathbb{P}^1, \mathcal{P}_{<c}E)$ is surjective. □

2.4.4 Stability condition We introduce a stability condition for filtered bundles satisfying (A1) on $(T\times\mathbb{P}^1, T\times\{\infty\})$, by following [8]. Note that this is not the same as the standard slope stability condition for filtered bundles on projective varieties; see Maruyama and Yokogawa [33].

Let $\omega_T \in H^2(T\times\mathbb{P}^1, \mathbb{Z})$ denote the pullback of the fundamental class of T by the projection $T\times\mathbb{P}^1 \to T$. For any filtered torsion-free sheaf $\mathcal{P}_*\mathcal{E}$ on $(T\times\mathbb{P}^1, T\times\{\infty\})$, we define the degree of $\mathcal{P}_*\mathcal{E}$ by

$$\deg(\mathcal{P}_*\mathcal{E}) := \int_{T\times\mathbb{P}^1} \text{par-c}_1(\mathcal{P}_*\mathcal{E})\omega_T = \int_{\{z\}\times\mathbb{P}^1} \text{par-c}_1(\mathcal{P}_*\mathcal{E}).$$

We set $\mu(\mathcal{P}_*\mathcal{E}) := \deg(\mathcal{P}_*\mathcal{E})/\operatorname{rank}\mathcal{E}$. We say that a filtered bundle $\mathcal{P}_*E$ is stable (semistable) if $\mu(\mathcal{P}_*\mathcal{E}) < \mu(\mathcal{P}_*E)$ (resp. $\mu(\mathcal{P}_*\mathcal{E}) \le \mu(\mathcal{P}_*E)$) for any $\mathcal{P}_*\mathcal{E} \subset \mathcal{P}_*E$ such that $0 < \operatorname{rank}\mathcal{E} < \operatorname{rank}E$ and $\mathcal{P}_*\mathcal{E}$ also satisfies (A1) around $T\times\{\infty\}$. We say that a semistable filtered bundle $\mathcal{P}_*E$ is polystable if it has a decomposition $\mathcal{P}_*E = \bigoplus \mathcal{P}_*E_i$ such that each $\mathcal{P}_*E_i$ is stable. The following lemma is clear and standard.

Lemma 2.17 *Let $\mathcal{P}_*E$ be a filtered bundle satisfying (A1) on $(T\times\mathbb{P}^1, T\times\{\infty\})$. If $\mathcal{P}_*E$ is stable, then $\mathcal{P}_*E^\vee$ is also stable.* □

It is standard to obtain the vanishing of some cohomology groups under the assumption of the stability and the degree 0.

Lemma 2.18 *Let $\mathcal{P}_*E$ be a filtered bundle satisfying (A1) on $(T\times\mathbb{P}^1, T\times\{\infty\})$. If $\mathcal{P}_*E$ is stable with $\deg(\mathcal{P}_*E) = 0$ and $\operatorname{rank}\mathcal{P}_*E > 1$, it satisfies condition (A3).*

Proof Because $\mathcal{P}_*E$ is stable of degree 0 with $\operatorname{rank} E > 1$, $H^0(T\times\mathbb{P}^1, \mathcal{P}_c E) = 0$ for any $c \le 0$. Indeed, a nonzero section of $\mathcal{P}_c E$ induces a filtered strict subsheaf $\mathcal{P}_*\mathcal{O} \subset \mathcal{P}_*E$ with $\deg(\mathcal{P}_*\mathcal{O}) \ge 0$ and $0 < \operatorname{rank}\mathcal{O} < \operatorname{rank} E$. Because $(\mathcal{P}_*E)^\vee$ is also stable of degree 0, we obtain the vanishing $H^2(T\times\mathbb{P}^1, \mathcal{P}_{<-1}E)$ by using the Serre duality. □

Remark 2.19 Line bundles of degree 0 on T naturally correspond to filtered bundles $\mathcal{P}_*E$ which satisfy (A1) with $\deg(\mathcal{P}_*E) = 0$ and $\operatorname{rank}\mathcal{P}_*E = 1$. Indeed, there exists a line bundle L of degree 0 on T such that $\mathcal{P}_a E \simeq p^*L\otimes\mathcal{O}([a](T\times\{\infty\}))$ for any $a\in\mathbb{R}$, where $[a] := \max\{n\in\mathbb{Z} \mid n \le a\}$. In this case, condition (A3) is not satisfied for L^{-1}.

3 Algebraic Nahm transforms

3.1 Local algebraic Nahm transforms

3.1.1 Complex Let $X := \{z\in\mathbb{C} \mid |z| < \rho_0\}$ and $D := \{0\}$. In the rest of this subsection, we shall shrink X without mentioning it. We shall use the notation of Section 2.3.1. We define a complex of sheaves associated to an admissible filtered Higgs bundle $(\mathcal{P}_*V,\theta)$ on (X,D). First, let us consider the case that $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$. Suppose $(p,m,\boldsymbol{o}) \ne (1,0,0)$. For each $c\in\mathbb{R}$, let $\mathcal{P}_c(V\otimes\Omega_X^\bullet,\theta)$ denote the complex

$$\mathcal{P}_{c-m/p}V \to \mathcal{P}_{c+1}V dz,$$

where the first term sits in the degree 0. Take any $\alpha\in\boldsymbol{o}$. For each $c\in\mathbb{R}$, let $\mathcal{P}_c(V_\alpha^{\langle p\rangle}\otimes\Omega^\bullet_{X^{\langle p\rangle}},\theta_\alpha^{\langle p\rangle})$ denote the following complex on $X^{\langle p\rangle}$:

$$\mathcal{P}_{c-m}V_\alpha^{\langle p\rangle} \xrightarrow{\theta_\alpha^{\langle p\rangle}} \mathcal{P}_c V_\alpha^{\langle p\rangle}\otimes\frac{dz_p}{z_p}.$$

We have a natural isomorphism

$$\mathcal{P}_c(V\otimes\Omega^\bullet,\theta) \simeq \varphi_{p*}\mathcal{P}_{cp}\big(V_\alpha^{\langle p\rangle}\otimes\Omega^\bullet_{X^{\langle p\rangle}},\theta_\alpha^{\langle p\rangle}\big).$$

This is also isomorphic to the descent of $\bigoplus_{\alpha\in\boldsymbol{o}}\mathcal{P}_{cp}(V_\alpha^{\langle p\rangle}\otimes\Omega^\bullet_{X^{\langle p\rangle}},\theta_\alpha^{\langle p\rangle})$. For $c \le c'$, the natural inclusion $\mathcal{P}_c(V\otimes\Omega^\bullet,\theta)\to\mathcal{P}_{c'}(V\otimes\Omega^\bullet,\theta)$ is a quasi-isomorphism. We set $\mathcal{C}^\bullet(\mathcal{P}_*V,\theta) := \mathcal{P}_{-1/2}(V\otimes\Omega^\bullet,\theta)$. In the case $(p,m,\boldsymbol{o}) = (1,0,0)$, we set $\mathcal{C}^0(\mathcal{P}_*V,\theta) := \mathcal{P}_0V$ and

$$\mathcal{C}^1(\mathcal{P}_*V,\theta) := \mathcal{P}_{<1}V\otimes\Omega_X^1 + \theta(\mathcal{P}_0V) \subset \mathcal{P}_1V\otimes\Omega_X^1.$$

Thus, we obtain the complex $\mathcal{C}^\bullet(\mathcal{P}_*V,\theta)$ when $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$.

For a general admissible filtered Higgs bundle $(\mathcal{P}_*V,\theta)$, the complex $\mathcal{C}^\bullet(\mathcal{P}_*V,\theta)$ is defined as the extension of the complex $(V\to V\otimes\Omega^1_X)$ on $X\setminus D$ to a complex on X, such that it is $\bigoplus_{(m,p,\boldsymbol{o})}\mathcal{C}^\bullet(\mathcal{P}_*V_{\boldsymbol{o}}^{(p,m)},\theta_{\boldsymbol{o}}^{(p,m)})$ around D, according to the type decomposition.

Lemma 3.1 *If $(\mathcal{P}_*V,\theta)$ comes from a wild harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on (X,D), then $\mathcal{C}^\bullet(\mathcal{P}_*V,\theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E\otimes\Omega^\bullet$.*

Proof We consider the unramified case. We omit to denote p. We have the naturally defined map $\pi_c\colon\mathcal{P}_cV\to\mathrm{Gr}^{\mathcal{P}}_c(V)$. Let W be the weight filtration of the nilpotent part of the endomorphism $\mathrm{Res}(\theta)$ on $\mathrm{Gr}^{\mathcal{P}}_c(V)$. We set $W_k\mathcal{P}_cV:=\pi_c^{-1}(W_k\,\mathrm{Gr}^{\mathcal{P}}_c(V))$.

We introduce a complex $\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)$. If $(\mathcal{P}_*V,\theta)$ has type $(m,\boldsymbol{o})\neq(0,0)$, let $\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)$ be the complex

$$W_{-2}\mathcal{P}_{-m}V\to W_{-2}\mathcal{P}_0V\otimes\Omega^1_X(\log D).$$

We have a natural inclusion $\mathcal{C}^\bullet(\mathcal{P}_*V,\theta)\to\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)$ which is a quasi-isomorphism. If $(\mathcal{P}_*V,\theta)$ has type $(m,\boldsymbol{o})=(0,0)$, let $\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)$ be the complex

$$W_0\mathcal{P}_0V\to W_{-2}\mathcal{P}_0V\otimes\Omega^1_X(\log D).$$

It is easy to check that the natural inclusion $\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)\to\mathcal{C}^\bullet(\mathcal{P}_*V,\theta)$ is a quasi-isomorphism. In general, we define

$$\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)=\bigoplus\mathcal{C}^\bullet_{L^2}\big(\mathcal{P}_*V_{\boldsymbol{o}}^{(m)},\theta_{\boldsymbol{o}}^{(m)}\big)$$

by using the type decomposition.

According to the result in [36, Section 5.1], $\mathcal{C}^\bullet_{L^2}(\mathcal{P}_*V,\theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E\otimes\Omega^\bullet$. Hence, we are done in the unramified case. The general case can be easily reduced to the unramified case. □

3.1.2 Transform We shall construct some transformations for filtered Higgs bundles, which are analogous to the local Fourier transform in [9; 19]. In the following, for a variable x, let U_x denote a small neighbourhood of 0 in $\mathbb{C}_x$. For two variables x and y, let $U_{x,y}:=U_x\times U_y$, and let $\pi_1\colon U_{x,y}\to U_x$ and $\pi_2\colon U_{x,y}\to U_y$ denote the projections.

Let $(\mathcal{P}_*V,\theta)$ be an admissible filtered Higgs bundle on $(U_\zeta,0)$. Let us define a filtered bundle $\mathcal{N}^{0,\infty}_*(\mathcal{P}_*V,\theta)$ with an endomorphism g on U_τ. We consider the following complex on $U_{\zeta,\tau}$:

$$\pi_1^*\mathcal{C}^0(\mathcal{P}_*V,\theta)\xrightarrow{\tau\theta+d\zeta}\pi_1^*\mathcal{C}^1(\mathcal{P}_*V,\theta).$$

Let $\mathcal{Q}$ be the quotient. We define

$$\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta):=\pi_{2*}\mathcal{Q},\quad \mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta):=\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)(*\tau).$$

Here $(*\tau)$ means the localization with respect to τ. If U_τ is sufficiently small, the support of $\mathcal{Q}$ is proper and relatively 0–dimensional over U_τ. Indeed, $\mathcal{Q}\cap(\{0\}\times U_\zeta)=\{(0,0)\}$. Hence, $\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)$ is coherent. Let us check that $\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)$ is torsion-free. Let v be a section of $\pi_1^*\mathcal{C}^1(\mathcal{P}_*V,\theta)$, such that there exists a section u of $\pi_1^*\mathcal{C}^0(\mathcal{P}_*V,\theta)$ satisfying $\tau v=(\tau\theta+d\zeta)u$. We obtain that $d\zeta\cdot u$ is contained in $\tau\cdot\pi_1^*\mathcal{C}^1(\mathcal{P}_*V,\theta)$. Then we obtain that $u=\tau u'$ for some section u' of $\pi_1^*\mathcal{C}^0(\mathcal{P}_*V,\theta)$, and we have $v=(\tau\theta+d\zeta)u'$. It implies that $\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)$ is torsion-free. Hence, $\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)$ is a locally free $\mathcal{O}_{U_\tau}$–module. In particular, $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$ is a locally free $\mathcal{O}_{U_\tau}(*\tau)$–module. The multiplication of ζ induces the endomorphism g. By setting $\psi:=-g\tau^{-2}d\tau$, we obtain a Higgs field of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$. We shall introduce a filtered bundle $\mathcal{N}^{0,\infty}_*(\mathcal{P}_*V,\theta)=(\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)\mid a\in\mathbb{R})$ over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$.

If $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})\neq(1,0,0)$, we consider the following complexes on $U_{\zeta,\tau}$ for any $c\in\mathbb{R}$:

$$\pi_1^*\mathcal{P}_{c-m/p}(V)\xrightarrow{\tau\theta+d\zeta}\pi_1^*\mathcal{P}_c(V)(d\zeta/\zeta). \tag{10}$$

Let $\mathcal{Q}_c$ denote the quotient. We define

$$\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta):=\pi_{2*}\mathcal{Q}_c,\quad \kappa_1(p,m,c):=\frac{2pc-m}{2(p+m)}.$$

By construction, we have $\mathcal{N}^{0,\infty}_{-1/2}(\mathcal{P}_*V,\theta)=\boldsymbol{\mathcal{N}}(\mathcal{P}_*V,\theta)$ in this case. It is easy to check that $\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)$ are locally free $\mathcal{O}_{U_\tau}$–modules of finite rank. We have a naturally induced map

$$\mathcal{N}^{0,\infty}_{a'}(\mathcal{P}_*V,\theta)\longrightarrow\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)$$

for $a'\leq a$. Its restriction to $\{\tau\neq 0\}$ is an isomorphism, and hence it is injective. We also obtain $\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)(*\tau)=\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$. For $c':=c-(1+m/p)$, the images of $\tau\cdot\pi_1^*\mathcal{P}_cV(d\zeta/\zeta)$ and $\pi_1^*\mathcal{P}_{c'}V(d\zeta/\zeta)$ are the same in the quotient of $\mathcal{Q}_c$. This implies

$$\tau\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)=\mathcal{N}^{0,\infty}_{a-1}(\mathcal{P}_*V,\theta)\quad\text{for any } a\in\mathbb{R}.$$

Hence, $\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)$ $(a\in\mathbb{R})$ gives a filtered bundle over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$.

If $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})=(1,0,0)$, we define $\mathcal{N}^{0,\infty}_0(\mathcal{P}_*V,\theta):=\boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*V,\theta)$. We have natural morphisms

$$\mathcal{N}^{0,\infty}_0(\mathcal{P}_*V,\theta)_{|0}\simeq\mathcal{C}^1(\mathcal{P}_*V,\theta)/\mathcal{C}^0(\mathcal{P}_*V,\theta)d\zeta\to(\mathcal{P}_0V)_{|0}.$$

Here, the subscript "$|0$" means the fibre of the vector bundle over 0, and the latter map is given by the residue, which is injective. Hence, the parabolic filtration of the right-hand side induces a parabolic filtration of $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V,\theta)_{|0}$ indexed by $]-1,0]$. This in turn induces a filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$.

If $(\mathcal{P}_*V,\theta)$ is admissible, we replace U_ζ with smaller neighbourhoods so that it has the type decomposition, and we define

$$\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta) := \bigoplus_{p,m,\boldsymbol{o}} \mathcal{N}_*^{0,\infty}(\mathcal{P}_*V_{\boldsymbol{o}}^{(p,m)},\theta).$$

The construction $\mathcal{N}_*^{0,\infty}$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of filtered Higgs bundles. We set

$$\mathcal{N}_{<a}^{0,\infty}(\mathcal{P}_*V,\theta) := \sum_{b<a} \mathcal{N}_b^{0,\infty}(\mathcal{P}_*V,\theta).$$

Lemma 3.2 *Suppose $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$. The rank of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$ is $(p+m)\operatorname{rank} V/p$ in the case $(p,m,\boldsymbol{o}) \neq (1,0,0)$, or $\operatorname{rank} V - \dim \operatorname{Ker} \operatorname{Gr}_0^{\mathcal{P}}(\operatorname{Res}\theta)$ in the case $(p,m,\boldsymbol{o}) = (1,0,0)$.*

Proof The rank is equal to the dimension of $\mathcal{C}^1(\mathcal{P}_*V,\theta)/\mathcal{C}^0(\mathcal{P}_*V,\theta)d\zeta$ as a $\mathbb{C}$–vector space. Then the claim can be checked by a direct computation. (See also the proof of Proposition 3.3 below for the case $(p,m,\boldsymbol{o}) \neq (1,0,0)$.) □

Proposition 3.3 *$(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta),\psi)$ is admissible. If $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$, then $(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta),\psi)$ has type $(p+m,m,\boldsymbol{o}')$ for some $\boldsymbol{o}'$.*

Proof We have only to consider the case that $(\mathcal{P}_*V,\theta)$ has type $(p,m,\boldsymbol{o})$. Let us consider the case $(p,m,\boldsymbol{o}) = (1,0,0)$. For the expression $\theta = f\,d\zeta/\zeta$, f gives an endomorphism of $\mathcal{P}_cV$ for any c, and $f_{|0}$ is nilpotent. We have $\psi = -\tau^{-1}g(d\tau/\tau)$ and $-\tau^{-1}g$ is induced by f, so it preserves $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V,\theta)$. If we regard $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V,\theta)_{|0}$ as a subspace of $\mathcal{P}_0V_{|0}$ as above, then $(-\tau^{-1}g)_{|0}$ is the restriction of $f_{|0}$. Hence it is nilpotent and preserves the parabolic filtration, ie $(\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta),\psi)$ is admissible of type $(1,0,0)$.

Let us consider the case $(p,m,\boldsymbol{o}) \neq (1,0,0)$. Fix $\alpha \in \boldsymbol{o}$. We consider the following on $U_{\zeta_p,\tau}$:

$$\pi_1^*\mathcal{P}_{pc-m}V_\alpha^{\langle p\rangle} \xrightarrow{\tau\theta_\alpha^{\langle p\rangle}+d\zeta_p^p} \pi_1^*\mathcal{P}_{pc}V_\alpha^{\langle p\rangle}(d\zeta_p/\zeta_p). \tag{11}$$

The quotient is denoted by $\mathcal{Q}'_c$. The pushforward $\pi_{2*}\mathcal{Q}'_c$ is naturally isomorphic to $\mathcal{N}_{\kappa_1(p,m,c)}^{0,\infty}(\mathcal{P}_*V,\theta)$.

The natural map $\mathcal{Q}'_{c'} \to \mathcal{Q}'_c$ $(c' \le c)$ is injective. We set $\mathcal{Q}'_{<c} := \bigcup_{b<c} \mathcal{Q}'_b$. We have an exact sequence

$$0 \longrightarrow \pi_1^* \operatorname{Gr}^{\mathcal{P}}_{pc-m}(V_\alpha^{\langle p\rangle}) \xrightarrow{\tau\theta_\alpha^{\langle p\rangle}} \pi_1^* \operatorname{Gr}^{\mathcal{P}}_{pc}(V_\alpha^{\langle p\rangle})(d\zeta_p/\zeta_p) \longrightarrow \mathcal{Q}'_c/\mathcal{Q}'_{<c} \longrightarrow 0.$$

The sequence induces the following isomorphism of $\mathbb{C}$–vector spaces for any $c \in \mathbb{R}$:

$$\operatorname{Gr}^{\mathcal{P}}_{pc}(V_\alpha^{\langle p\rangle}) \simeq \frac{\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)}{\mathcal{N}^{0,\infty}_{<\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)}. \tag{12}$$

Let $\boldsymbol{v} = (v_i)$ be a frame of $\mathcal{P}_{pc}V_\alpha^{\langle p\rangle}$. Set $c_i := \min\{a \in \mathbb{R} \mid v_i \in \mathcal{P}_a V_\alpha^{\langle p\rangle}\}$. We assume that $\boldsymbol{v}$ is compatible with the parabolic structure in the sense that the induced tuple $\{[v_i] \mid c_i = d\}$ of elements in $\operatorname{Gr}^{\mathcal{P}}_d(V_\alpha^{\langle p\rangle})$ is a basis for any $d \in \,]pc-1, pc]$. We set $v_{ij} := \zeta_p^i v_j\,(d\zeta_p/\zeta_p)$ for $0 \le i \le p+m-1$ and $1 \le j \le \operatorname{rank} V_\alpha^{\langle p\rangle}$. The induced sections of

$$\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$$

are also denoted by the same symbols. Because they induce a basis of

$$\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta) \,\big/\, \tau\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$$

according to (12), these induced sections give a frame of

$$\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$$

on a neighbourhood of 0. (In particular, the rank of $\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$ is $(p+m)\operatorname{rank} V_\alpha^{\langle p\rangle} = p^{-1}(p+m)\operatorname{rank} V$.) Moreover, by the isomorphism (12), the frame is compatible with the parabolic structure of $\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$.

We take a ramified covering $\varphi\colon U_\eta \to U_\tau$ by $\varphi(\eta) = \eta^{p+m}$. Let $\mathcal{P}_*\mathcal{V}$ be the filtered bundle on $(U_\eta, 0)$ obtained as the pullback of $\mathcal{N}^{0,\infty}_*(\mathcal{P}_*V,\theta)$ by φ. The tuple of the sections $\widetilde{v}_{ij} := \eta^{-i}\varphi^* v_{ij}$ gives a frame $\widetilde{\boldsymbol{v}}$ of $\mathcal{P}_{pc-m/2}\mathcal{V}$ which is compatible with the parabolic structure. By the frames $\boldsymbol{v}$ and $\widetilde{\boldsymbol{v}}$, we obtain an isomorphism of $\mathcal{P}_{pc-m/2}\mathcal{V}_{|0}$ to $(\mathcal{P}_{pc}V_\alpha^{\langle p\rangle})_{|0} \otimes \mathbb{C}^{p+m}$.

Let us prove that $\psi = -\tau^{-2} g\, d\tau$ has type $(p+m, m, \boldsymbol{o}')$ for some $\boldsymbol{o}'$. Note that g is induced by the multiplication of $\zeta = \zeta_p^p$. Let g_1 be the endomorphism of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$ which is induced by the multiplication of ζ_p. We have that $g_1(\mathcal{N}^{0,\infty}_a(\mathcal{P}_*V,\theta)) \subset \mathcal{N}^{0,\infty}_{a-1/(p+m)}(\mathcal{P}_*V,\theta)$. Hence, $\eta^{-1}g_1$ gives an endomorphism of $\mathcal{P}_{pc-m/2}\mathcal{V}$. In particular, $\eta^{-p}g$ gives an endomorphism of $\mathcal{P}_{pc-m/2}\mathcal{V}$. Let us prove that the restriction $(\eta^{-p}g)_{|0}$ has a unique nonzero eigenvalue modulo the action of $\operatorname{Gal}(\varphi)$.

We have the parabolic filtration F of $(\mathcal{P}_{pc}V_\alpha)_{|0}$ indexed by $]pc-1, pc]$. Let W denote the monodromy weight filtration of the nilpotent part of $\mathrm{Res}(\zeta_p^m\theta_\alpha^{\langle p\rangle})$ on $\mathrm{Gr}^F(\mathcal{P}_{pc}V_{\alpha|0})$. Let $\pi_a\colon F_a(\mathcal{P}_{pc}V_{\alpha|0}) \to \mathrm{Gr}^F_a(\mathcal{P}_{pc}V_{\alpha|0})$ denote the projection. Let $M := \min\{|a-b| \mid a,b \in \mathcal{P}ar(\mathcal{P}_{pc}V_\alpha), a \neq b\}$. We take a small positive number δ such that $\delta \operatorname{rank} \mathcal{P}_{pc}V_\alpha < M/100$. We set $\widetilde{F}_{a+\delta k} := \pi_a^{-1}(W_k)$. Then we obtain a filtration $\widetilde{F}$ of $\mathcal{P}_{pc}V_{\alpha|0}$ indexed by $]pc-1+\epsilon, pc+\epsilon]$ for some small $\epsilon > 0$. Then $\widetilde{F}$ is preserved by $\mathrm{Res}(\zeta_p^m\theta_\alpha^{\langle p\rangle})$, and the induced endomorphism on the associated graded space $\mathrm{Gr}^{\widetilde{F}}$ is semisimple. We may assume that the frame $\boldsymbol{v}$ is compatible with $\widetilde{F}$.

Let $\widetilde{F}'$ be a filtration of

$$\mathcal{P}_{pc-m/2}\mathcal{V}_{|0} \simeq \mathcal{P}_{pc}V_{\alpha|0}^{\langle p\rangle} \otimes \mathbb{C}^{p+m},$$

indexed by $]pc-1+\delta, pc+\delta]$, determined by the condition that $\deg^{\widetilde{F}'}(\widetilde{v}_{ij}) = \deg^{\widetilde{F}}(v_j)$.

The multiplication of $\eta^{-1}\zeta_p$ induces an endomorphism of $\mathcal{P}_{pc-m/2}\mathcal{V}$. We have $(\eta^{-1}\zeta_p)\widetilde{v}_{ij} = \widetilde{v}_{i+1,j}$ for $i < p+m-1$, and $(\eta^{-1}\zeta_p)\widetilde{v}_{p+m-1,j}$ is equal to the section s induced by

$$-p^{-1}\theta_\alpha^{\langle p\rangle}(\zeta_p^m v_j) = \Bigl(-(\alpha/p)v_j + \sum_{\deg^{\widetilde{F}}(v_k)<\deg^{\widetilde{F}}(v_j)} \gamma_k \cdot v_k + \zeta_p u\Bigr)(d\zeta_p/\zeta_p).$$

Here, γ_k are complex numbers, and u is a section of $\mathcal{P}_{pc}V_\alpha^{\langle p\rangle}$. If $\deg^{\widetilde{F}}(v_{j|0}) = a$, then $s_{|0} + (\alpha/p)\widetilde{v}_{0,j|0} \in \widetilde{F}'_{<a}$.

The endomorphism $\eta^{-1}g$ of $\mathcal{P}_{pc-m/2}\mathcal{V}$ is induced by the multiplication of the p^{th} power of $\eta^{-1}\zeta_p$. Therefore, $(\eta^{-p}g)_{|0}$ is compatible with $\widetilde{F}'$, and the induced endomorphism on $\mathrm{Gr}^{\widetilde{F}'}$ is represented by the matrix

$$\sum_{i=1}^{m} I \otimes E_{p+i,i} + \sum_{i=1}^{p}(-\alpha/p) I \otimes E_{i,m+i}.$$

Here, I is the identity matrix and E_{ij} denote the $(p+m)$–square matrix whose (k,ℓ)–entry is 1 if $(k,\ell) = (i,j)$, and 0 otherwise. Then the set of the eigenvalues is $e^{2\pi\sqrt{-1}j/(p+m)}\alpha^p$ $(j = 0,\dots,p+m-1)$. Thus, we are done. □

Corollary 3.4 *The construction $\mathcal{N}_*^{0,\infty}$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of admissible filtered Higgs bundles whose slopes are strictly less than* 1. □

3.1.3 Inverse transform Let $\mathcal{P}_*V$ be a filtered bundle on $(U_\tau, 0)$ with an endomorphism g, which is admissible in the sense of Section 2.3.3. In this subsection, we impose the following vanishing:

(C0) $V_0^{(1,0)} = 0$ and $V^{(p,m)} = 0$ unless $p > m$.

Note that the eigenvalues of $g(\tau)$ go to 0 when $\tau \to 0$ under the assumption (C0).

If $(\mathcal{P}_*V, g)$ has slope (p,m), we consider the following complex on $U_{\tau,\zeta}$:

$$\pi_1^*\mathcal{P}_cV \xrightarrow{g-\zeta} \pi_1^*\mathcal{P}_cV.$$

The quotient is denoted by $\mathcal{M}_c$. If U_ζ is sufficiently small, the support of $\mathcal{M}_c$ is proper over U_ζ. We define

$$\mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)+1}(\mathcal{P}_*V, g) := \pi_{2*}\mathcal{M}_c, \quad \kappa_2(p,m,c) := \frac{2pc+m}{2(p-m)}.$$

These are locally free $\mathcal{O}_{U_\zeta}$–modules. For $a \le a'$, we naturally have

$$\mathcal{N}^{\infty,0}_a(\mathcal{P}_*V, g) \to \mathcal{N}^{\infty,0}_{a'}(\mathcal{P}_*V, g)$$

which induce $\mathcal{N}^{\infty,0}_a(\mathcal{P}_*V, g)(*\zeta) \simeq \mathcal{N}^{\infty,0}_{a'}(\mathcal{P}_*V, g)(*\zeta)$. We have

$$\mathcal{N}^{\infty,0}_{a-1}(\mathcal{P}_*V, g) = \zeta\mathcal{N}^{\infty,0}_a(\mathcal{P}_*V, g)$$

for any $a \in \mathbb{R}$. Thus, we obtain a filtered bundle $\mathcal{N}^{\infty,0}_*(\mathcal{P}_*V, g)$ on $(U_\zeta, 0)$. In the general case, we define

$$\mathcal{N}^{\infty,0}_*(\mathcal{P}_*V, g) := \bigoplus \mathcal{N}^{\infty,0}_*(\mathcal{P}_*V_{\boldsymbol{o}}^{(p,m)}, g_{\boldsymbol{o}}^{(p,m)})$$

by using the slope decomposition of $(\mathcal{P}_*V, g)$. The multiplication of $-\tau^{-1}$ gives a meromorphic endomorphism f. We put $\theta = f\,d\zeta$. The construction gives a functor from the category of the germs of admissible filtered Higgs bundles satisfying (C0) to the category of the germs of filtered Higgs bundles.

Proposition 3.5 *$(\mathcal{N}^{\infty,0}_*(\mathcal{P}_*V, g), \theta)$ is admissible. If $(\mathcal{P}_*V, g)$ has type $(p,m,\boldsymbol{o})$, then $\mathcal{N}^{\infty,0}_*(\mathcal{P}_*V, g)$ has type $(p-m, m, \boldsymbol{o}')$ for some $\boldsymbol{o}'$, and moreover, the rank is $(p-m)\operatorname{rank} V/p$.*

Proof We have only to consider the case that $(\mathcal{P}_*V, g)$ has type $(p,m,\boldsymbol{o})$. Let $\varphi_p\colon U_\eta \to U_\tau$ be given by $\varphi_p(\eta) = \eta^p$. Let $\varphi\colon U_u \to U_\zeta$ be given by $\varphi(u) = u^{p-m}$. Let $\mathcal{P}_*\mathcal{V}$ be the filtered bundle on U_u obtained as the pullback of $\mathcal{N}^{\infty,0}_*(\mathcal{P}_*V, g)$ by φ.

We use the decomposition $\varphi_p^*(\mathcal{P}_* V, g) = \bigoplus_{\alpha\in\boldsymbol{o}} (\mathcal{P}_* V_\alpha^{\langle p\rangle}, g_\alpha^{\langle p\rangle})$. We consider the following complex on $U_{\eta,\zeta}$:

$$\pi_1^* \mathcal{P}_{pc} V_\alpha^{\langle p\rangle} \xrightarrow{g_\alpha^{\langle p\rangle}-\zeta} \pi_1^* \mathcal{P}_{pc} V_\alpha^{\langle p\rangle}.$$

The quotient is denoted by $\mathcal{M}'_c$. We have $\pi_{2*}\mathcal{M}'_c \simeq \mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)+1}(\mathcal{P}_* V, g)$. Because $g_\alpha^{\langle p\rangle}(\mathcal{P}_a V_\alpha^{\langle p\rangle}) \subset \mathcal{P}_{<a} V_\alpha^{\langle p\rangle}$, we have the following exact sequence, as in the case of $\mathcal{N}^{0,\infty}$ (see the proof of Proposition 3.3):

$$0 \to \pi_1^* \operatorname{Gr}^{\mathcal{P}}_{pc}(V_\alpha^{\langle p\rangle}) \xrightarrow{-\zeta} \pi_1^* \operatorname{Gr}^{\mathcal{P}}_{pc}(V_\alpha^{\langle p\rangle}) \to \mathcal{M}'_c/\mathcal{M}'_{<c} \to 0.$$

It induces the following isomorphism of $\mathbb{C}$–vector spaces:

$$\operatorname{Gr}^{\mathcal{P}}_{pc}(V_\alpha^{\langle p\rangle}) \simeq \frac{\mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)+1}(\mathcal{P}_* V, g)}{\mathcal{N}^{\infty,0}_{<\kappa_2(p,m,c)+1}(\mathcal{P}_* V, g)}. \tag{13}$$

We take a frame $\boldsymbol{v}$ of $\mathcal{P}_{pc} V_\alpha^{\langle p\rangle}$ compatible with the parabolic structure. We set $v_{ij} := \eta^i v_j$. By the isomorphism (13), they induce a frame of

$$\mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)+1}(\mathcal{P}_* V, g)$$

compatible with the parabolic structure. We set $\tilde{v}_{ij} := u^{-i}\eta^i v_j$. The tuple $\tilde{\boldsymbol{v}}$ induces a frame of $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}$ compatible with the parabolic structure.

We consider the endomorphism $h := \eta^{m-p} g_\alpha^{\langle p\rangle}$ on $\mathcal{P}_{pc} V_\alpha^{\langle p\rangle}$, which is invertible. We have $\eta^{-p+m}u^{p-m} = h$ on $\mathcal{V}$. Let k be the integer determined by the condition $0 \le -p + k(p-m) < p-m$. We set $a := -p + k(p-m)$. We have $\eta^{-p}u^p = \eta^a u^{-a} h^k = \eta^{a-(p-m)} u^{-a+(p-m)} h^{k-1}$. We have

$$u^p \eta^{-p} v_{ij} = \begin{cases} \eta^{a+i} u^{-(a+i)} h^k(v_j) & (a+i < p-m), \\ \eta^{a+i-(p-m)} u^{-(a+i)+p-m} h^{k-1}(v_j) & (a+i \ge p-m). \end{cases}$$

Hence $u^p\eta^{-p}$ preserves $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}$.

By the frames $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$, we have an isomorphism $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}_{|0}$ and $\mathcal{P}_{pc} V_{\alpha|0} \otimes \mathbb{C}^{p-m}$. We take a refinement $\tilde{F}$ of the parabolic filtration of $\mathcal{P}_{pc} V_{\alpha|0}$ such that $\tilde{F}$ is preserved by $h_{|0}$ and the induced endomorphism on $\operatorname{Gr}^{\tilde{F}}$ is semisimple with a unique eigenvalue β. It induces a filtration $\tilde{F}'$ of $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}_{|0}$. (See the proof of Proposition 3.3 for a concrete construction.) We express $u^p\eta^{-p}$ by the matrix

$$\sum E_{a+i,i} \otimes \beta^k I + \sum E_{i,i+p-m-a} \otimes \beta^{k-1} I$$

on $\operatorname{Gr}^{\tilde{F}'}$, with respect to an appropriate base. Then $(\mathcal{P}_*\mathcal{N}^{\infty,0}(\mathcal{P}_* V, g), \theta)$ has type $(p-m, m, \boldsymbol{o}')$ for some $\boldsymbol{o}'$. $\square$

Corollary 3.6 *The construction $\mathcal{N}_*^{\infty,0}$ gives a functor from the category of the germs of admissible filtered Higgs bundles satisfying (C0) to the category of the germs of admissible filtered Higgs bundles.* □

We denote $(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta),g)$ in Section 3.1.2 by $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ for simplicity. We also denote $(\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V,g),\theta)$ by $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V,g)$.

Proposition 3.7

- *Suppose that $(\mathcal{P}_*V,\theta)$ is admissible such that $V_0^{(1,0)}=0$ in the type decomposition. Then we have a natural isomorphism of the germs of filtered Higgs bundles $\mathcal{N}_*^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)\simeq(\mathcal{P}_*V,\theta)$.*
- *Suppose that $(\mathcal{P}_*V,g)$ is admissible and satisfies condition (C0). Then we have a natural isomorphism of the germs of filtered bundles with endomorphisms $\mathcal{N}_*^{0,\infty}\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V,g)\simeq(\mathcal{P}_*V,g)$.*

Proof Suppose that $(\mathcal{P}_*V,\theta)$ has type (p,m). Note that, if we set $d:=\kappa_1(p,m,c)$, then we have $\kappa_2(p+m,m,d)=c$. Let p_i be the projection of $U_\zeta\times U_\tau\times U_{\zeta'}$ onto the i^{th} component. We have the following diagram on $U_\zeta\times U_\tau\times U_{\zeta'}$:

$$\begin{tikzcd}
p_1^*\mathcal{P}_{c-m/p}(V) \arrow[r, "\tau\theta+d\zeta"] \arrow[d, "\zeta-\zeta'"'] & p_1^*\mathcal{P}_c(V)d\zeta/\zeta \arrow[d, "\zeta-\zeta'"] \\
p_1^*\mathcal{P}_{c-m/p}(V) \arrow[r, "\tau\theta+d\zeta"'] & p_1^*\mathcal{P}_c(V)\,d\zeta/\zeta
\end{tikzcd}$$

We regard it as a double complex, where the left upper $p_1^*\mathcal{P}_{c-m/p}(V)$ sits in the degree $(0,0)$. Let $C^\bullet$ denote the associated total complex. By construction, we obtain $\mathcal{N}_{c+1}^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ as $p_{3*}\mathcal{H}^2(C^\bullet)$. We can observe that it is isomorphic to the pushforward of $\mathcal{Q}_c$ in Section 3.1.2 by the projection $U_{\zeta,\tau}\to U_\zeta$, which is naturally isomorphic to $\mathcal{P}_cV\,d\zeta/\zeta\simeq\mathcal{P}_{c+1}V$. The action of $-\tau^{-1}$ is equal to f for the expression $\theta=f\,d\zeta$. Hence, we obtain the desired isomorphism $\mathcal{N}_*^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)\simeq(\mathcal{P}_*V,\theta)$.

Suppose that $(\mathcal{P}_*V,g)$ has type (p,m) with $p>m$. Let p_i denote the projection of $U_\tau\times U_\zeta\times U_{\tau'}$ onto the i^{th} component. We have the following commutative diagram of the sheaves on $U_\tau\times U_\zeta\times U_{\tau'}$:

$$\begin{tikzcd}
p_1^*\mathcal{P}_{c-1}V \arrow[r, "g-\zeta"] \arrow[d, "(-\tau'(\tau^{-1})+1)d\zeta"'] & p_1^*\mathcal{P}_{c-1}V \arrow[d, "(-\tau'(\tau^{-1})+1)d\zeta"] \\
p_1^*\mathcal{P}_cV\,d\zeta \arrow[r, "g-\zeta"'] & p_1^*\mathcal{P}_cV\,d\zeta
\end{tikzcd}$$

We regard it as the double complex, where the left upper $p_1^*\mathcal{P}_{c-1}V$ sits in the degree $(0,0)$. Let $C^\bullet$ be the associated total complex. By construction, $\mathcal{N}_c^{0,\infty}\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g)$ is naturally isomorphic to $p_{3*}\mathcal{H}^2(C^\bullet)$. We can observe that it is naturally isomorphic to the pushforward of $\mathcal{M}_c$ in Section 3.1.3 by the projection $U_\tau \times U_\zeta \to U_\tau$, which is naturally isomorphic to $\mathcal{P}_c V$. The action of ζ is given by g. Hence, we obtain the desired isomorphism $\mathcal{N}_*^{0,\infty}\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g) \simeq (\mathcal{P}_*V, g)$. □

3.1.4 Description of the functors Let $(\mathcal{P}_*V, \theta)$ be a filtered Higgs bundle with slope $(p,m) \neq (1,0)$ on U_ζ. Suppose that there exists a ramified covering $\varphi_q\colon U_{\zeta_q} \to U_\zeta$ and a filtered Higgs bundle $(\mathcal{P}_*V', \theta')$ on U_{ζ_q} with an isomorphism $\varphi_{q*}(\mathcal{P}_*V', \theta') \simeq (\mathcal{P}_*V, \theta)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\zeta_q,\tau}$:

$$\mathcal{P}_{q(c-m/p)}V' \xrightarrow{\tau\theta' + d\zeta_q^q} \mathcal{P}_{qc}V'\, d\zeta_q/\zeta_q.$$

The quotient is denoted by $\mathcal{Q}'_c$. The following lemma is clear by construction.

Lemma 3.8 *$\pi_{2*}\mathcal{Q}'_c$ is naturally isomorphic to $\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V, \theta)$.* □

Let $(\mathcal{P}_*V, \psi)$ be a filtered Higgs bundle with slope (p,m) on U_τ, such that $(p,m) \neq (1,0)$ and $p > m$. Suppose that there exist a ramified covering $\varphi_q\colon U_{\tau_q} \to U_\tau$ and a filtered Higgs bundle $(\mathcal{P}_*V', \psi')$ on U_{τ_q} with an isomorphism $\varphi_{q*}(\mathcal{P}_*V', \psi') \simeq (\mathcal{P}_*V, \psi)$. Let $\psi' = g'\varphi^*(-\tau^{-2}d\tau)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\tau_q,\zeta}$:

$$\mathcal{P}_{qc}V' \xrightarrow{g'-\zeta} \mathcal{P}_{qc}V'.$$

Let $\mathcal{M}'_c$ denote the quotient. The following is clear by construction.

Lemma 3.9 *$\pi_{2*}\mathcal{M}'_c$ is naturally isomorphic to $\mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)+1}(\mathcal{P}_*V, g)$.* □

3.2 Algebraic Nahm transform for admissible filtered Higgs bundle

3.2.1 Construction of the transform Let $T^\vee := \mathbb{C}/L^\vee$. Let $D \subset T^\vee$ be an effective reduced divisor. Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a filtered Higgs bundle on $(T^\vee, D)$. Suppose that it is admissible around each point of D in the sense of Section 2.3.1. We shall construct a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ on $(T \times \mathbb{P}^1, T \times \{\infty\})$ from $(\mathcal{P}_*\mathcal{E}, \theta)$. We begin with a construction of an object $N(\mathcal{P}_*\mathcal{E}, \theta)$ in $D^b(\mathcal{O}_{T\times\mathbb{P}^1})$.

For $I \subset \{1,2,3\}$, let p_I be the projections of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. Let $\mathcal{P}oin$ be the Poincaré bundle on $T^\vee \times T$. Applying the construction in Section 3.1.1 around each point of D, we extend $\mathcal{E}$ and $\mathcal{E}\otimes\Omega_X^1$ on $X\setminus D$ to $\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)$ and $\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)$, respectively. We set

$$\widetilde{\mathcal{C}}^0(\mathcal{P}_*\mathcal{E},\theta):=p_1^*\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)\otimes p_{12}^*\mathcal{P}oin\otimes p_3^*\mathcal{O}_{\mathbb{P}^1}(-1),$$
$$\widetilde{\mathcal{C}}^1(\mathcal{P}_*\mathcal{E},\theta):=p_1^*\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)\otimes p_{12}^*\mathcal{P}oin.$$

Let ζ be the standard coordinate of $\mathbb{C}$, which induces local coordinates of $T^\vee$. We have the holomorphic 1–form $d\zeta$ on $T^\vee$. Let w be the standard coordinate of $\mathbb{C}\subset\mathbb{P}^1$, which we can naturally regard as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. Then we have a morphism

$$\theta+w\,d\zeta\colon\widetilde{\mathcal{C}}^0(\mathcal{P}_*\mathcal{E},\theta)\to\widetilde{\mathcal{C}}^1(\mathcal{P}_*\mathcal{E},\theta). \tag{14}$$

Thus we obtain a complex $\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*\mathcal{E},\theta)$ on $T^\vee\times T\times\mathbb{P}^1$. We define

$$\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta):=Rp_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*\mathcal{E},\theta))[1].$$

Lemma 3.10 *There is a neighbourhood U of ∞ in $\mathbb{P}^1$ such that $\mathcal{H}^i(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta))=0$ on $T\times U$ unless $i\neq 0$. Moreover, $\mathcal{H}^0(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta))_{|T\times\{P\}}$ are semistable bundles of degree 0 for any $P\in U$.*

Proof Let π_i denote the projection of $T^\vee\times\mathbb{P}^1$ onto the i^{th} component. We have the following complex $\widetilde{\mathcal{C}}_1^\bullet(\mathcal{P}_*\mathcal{E},\theta)$ on $T^\vee\times\mathbb{P}^1$:

$$\pi_1^*\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1)\xrightarrow{\theta+w\,d\zeta}\pi_1^*\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta).$$

By construction, $\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)$ is isomorphic to $\widehat{\mathrm{RFM}}_+(\widetilde{\mathcal{C}}_1^\bullet(\mathcal{P}_*\mathcal{E},\theta))[1]$. If U is sufficiently small, $\theta+w\,d\zeta$ is injective on $T^\vee\times U$, and the support of the cokernel is relatively 0–dimensional over U. Then the claim of the lemma follows. □

We consider the following vanishing condition.

(A0) $\mathbb{H}^i(T^\vee,\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}\otimes L,\theta+w d\zeta))=0$ unless $i=1$ for any $w\in\mathbb{C}$ and any holomorphic line bundle L of degree 0 on $T^\vee$.

Under the assumption (A0), we naturally identify $\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)$ with the 0^{th} cohomology sheaf $\mathcal{H}^0(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta))$, which is a locally free sheaf on $T\times\mathbb{P}^1$. Indeed, $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}\otimes L,\theta+wd\zeta)$ is naturally identified with the specialization of $\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*\mathcal{E},\theta)$ to $T^\vee\times\{(L,w)\}$. Note that we always have $\mathbb{H}^i(T^\vee,\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}\otimes L,d\zeta))=0$ unless $i=1$ for any L, which corresponds to the specialization at $w=\infty$. We define

$$\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta):=\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)\otimes\mathcal{O}_{T\times\mathbb{P}^1}(*(T\times\{\infty\})).$$

We shall define a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)$ over $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$.

By Lemma 3.10, there is a neighbourhood U of $\infty \in \mathbb{P}^1$ such that $\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)_{|T\times\{\tau_1\}}$ are semistable of degree 0 for any $\tau_1 \in U$. Let $\mathfrak{s} \subset T^\vee \times U$ denote the spectrum. We have $\mathfrak{s} \cap (T^\vee \times \{\infty\}) \subset D$. We fix a lift of D to $\widetilde{D} \subset \mathbb{C}$. Then after shrinking U appropriately, we may have a lift of $\mathfrak{s}$ to $\widetilde{\mathfrak{s}} \subset \mathbb{C} \times U$. We obtain the corresponding holomorphic vector bundle $\boldsymbol{V}$ with an endomorphism g such that $\mathcal{S}p(g) \subset \widetilde{\mathfrak{s}}$. (See Section 2.1.3.) We have the decomposition

$$(\boldsymbol{V}, g) = \bigoplus_{P\in D} (\boldsymbol{V}_P, g_P),$$

where $\mathcal{S}p(g_P) \cap (\mathbb{C} \times \{\infty\})$ is the lift $\widetilde{P}$ of P. We have the induced decomposition on $T \times U$,

$$\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta) = \bigoplus_{P\in D} \mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)_P.$$

Let $U_P \subset T^\vee$ be a small neighbourhood of $P \in D$. We use the coordinate $\zeta_P := \zeta - \widetilde{P}$. By construction, we have a natural isomorphism $\boldsymbol{V}_P \simeq \boldsymbol{\mathcal{N}}^{0,\infty}(\mathcal{P}_*(\mathcal{E},\theta)_{|U_P})$. We have $g_P = g'_P + \widetilde{P}\,\mathrm{id}$, where g'_P is the endomorphism induced by ζ_P. Thus, we obtain a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)_P$ over $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)_P$ by transferring $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*(\mathcal{E},\theta)_{|U_P})$. We obtain a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)$ over $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$ by taking the direct sum.

Remark 3.11 We obtain a different transformation by replacing $\mathcal{P}oin$ and $w\,d\zeta$ with $\mathcal{P}oin^\vee$ and $-w\,d\zeta$, respectively, for which we can argue in a similar way.

Remark 3.12 In [10], the Fourier transform for Higgs bundles on smooth projective curves are studied. The algebraic Nahm transform in this paper may be regarded as a filtered variant, although we consider only the case where the base space is an elliptic curve. We also remark that this construction is an analogue of the Fourier transform of the minimal extension of algebraic meromorphic flat bundles on affine lines.

3.2.2 A property Let $(\mathcal{P}_*\mathcal{E},\theta)$ be a filtered Higgs bundle on $(T^\vee, D)$ which satisfies (A0).

Proposition 3.13 *The filtered bundle* $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)$ *is admissible and satisfies condition (A3).*

Proof of Proposition 3.13 Let $(\mathcal{P}_*\mathcal{E},\theta)$ be an admissible filtered Higgs bundle on $(T^\vee, D)$. Clearly, $\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$ satisfies (A1). It satisfies (A2) by Proposition 3.3.

Let L be any line bundle on T with degree 0. In the following, we also naturally regard it as a point in $T^\vee$. We set $\boldsymbol{N}_L := \boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)\otimes L^\vee$. We have the type decomposition

$$\boldsymbol{N}_L = \bigoplus_P \bigoplus_{p,m,\boldsymbol{o}} (\boldsymbol{N}_L)^{(p,m)}_{P,\boldsymbol{o}}.$$

By construction, we have

$$(\boldsymbol{N}_L)^{(p,m)}_{P,\boldsymbol{o}} = \begin{cases} \mathcal{P}_{-1/2}\,\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)^{(p,m)}_{P\otimes L,\boldsymbol{o}}\otimes L^\vee & \text{if } (p,m,\boldsymbol{o})\neq(1,0,0),\\ \mathcal{P}_0\,\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)^{(1,0)}_{P\otimes L,0}\otimes L^\vee & \text{if } (p,m,\boldsymbol{o})=(1,0,0).\end{cases}$$

Here, $P\otimes L\in T^\vee$ denotes the multiplication of $P, L\in T^\vee$ in the group $T^\vee$. We shall study the cohomology of $\boldsymbol{N}_L$ and its variant. Let us consider the following complex on $T^\vee\times T\times\mathbb{P}^1$:

$$\widetilde{\mathcal{C}}^0_L := \widetilde{\mathcal{C}}^0\otimes p_2^*L^\vee \xrightarrow{\theta+w\,d\zeta} \widetilde{\mathcal{C}}^1_L := \widetilde{\mathcal{C}}^1\otimes p_2^*L^\vee.$$

By construction, we have $\boldsymbol{N}_L\simeq R^1p_{23*}\widetilde{\mathcal{C}}^\bullet_L$. We have $Rp_{12*}\widetilde{\mathcal{C}}^\bullet_L\simeq R^1p_{12*}\widetilde{\mathcal{C}}^\bullet_L[-1]\simeq \mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)\otimes\mathcal{P}oin\otimes L^\vee[-1]$ on $T^\vee\times T$. For the projection $\pi\colon T^\vee\times T\to T^\vee$, we have $R\pi_*(\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)\otimes\mathcal{P}oin\otimes L^\vee)\simeq\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)\otimes R^1\pi_*(\mathcal{P}oin\otimes L^\vee)[-1]$, which is a skyscraper sheaf $\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_{|L}$ at L. Hence, we have

$$\mathbb{H}^i(T^\vee\times T\times\mathbb{P}^1,\widetilde{\mathcal{C}}^\bullet_L)\simeq\begin{cases}0 & (i\neq 2),\\ \mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_{|L} & (i=2).\end{cases} \tag{15}$$

We obtain $H^i(T\times\mathbb{P}^1,\boldsymbol{N}_L)=0$ unless $i=1$, and $H^1(T\times\mathbb{P}^1,\boldsymbol{N}_L)\simeq\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_{|L}$. We have

$$Rp_{12*}(\widetilde{\mathcal{C}}^\bullet_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))\simeq R^1p_{12*}(\widetilde{\mathcal{C}}^\bullet_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))[-1]\simeq\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)\otimes\mathcal{P}oin\otimes L^\vee[-1]$$

on $T\times T^\vee$. Hence, we have

$$\mathbb{H}^i(T^\vee\times T\times\mathbb{P}^1,\widetilde{\mathcal{C}}^\bullet_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))\simeq\begin{cases}0 & (i\neq 2),\\ \mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)_{|L} & (i=2).\end{cases}$$

We obtain

$$\begin{aligned} &H^i(T\times\mathbb{P}^1,\boldsymbol{N}_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))=0 \qquad \text{unless } i=1,\\ &H^1(T\times\mathbb{P}^1,\boldsymbol{N}_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))\simeq\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)_{|L}.\end{aligned}$$

Lemma 3.14 *The map $H^1(T\times\mathbb{P}^1,\boldsymbol{N}_L\otimes\mathcal{O}_{\mathbb{P}^1}(-1))\to H^1(T\times\mathbb{P}^1,\boldsymbol{N}_L)$ induced by the multiplication of w is equal to the map $\mathcal{C}^0(\mathcal{P}_*\mathcal{E},\theta)_{|L}\to\mathcal{C}^1(\mathcal{P}_*\mathcal{E},\theta)_{|L}$ induced by θ, up to signatures.*

Proof Let V_i $(i=0,1)$ be vector spaces with morphisms $f_0, f_\infty \in \mathrm{Hom}(V_0, V_1)$. Let $\alpha_\kappa\colon \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ be morphisms induced as $\mathcal{O}_{\mathbb{P}^1}(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(-\{\kappa\}) \to \mathcal{O}_{\mathbb{P}^1}$. The induced morphisms $\mathcal{O}_{\mathbb{P}^1}(-m-1) \to \mathcal{O}_{\mathbb{P}^1}(-m)$ are also denoted by α_κ.

We consider a complex $C^\bullet$ on $\mathbb{P}^1$ given as $C^0 = V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ and $C^1 = V_1 \otimes \mathcal{O}_{\mathbb{P}^1}$ with $f_0\alpha_0 - f_\infty\alpha_\infty$. We have $\mathbb{H}^1(\mathbb{P}^1, C^\bullet) \simeq V_1$ and $\mathbb{H}^1(\mathbb{P}^1, C^\bullet \otimes \mathcal{O}(-1)) \simeq V_0$. The morphism α_0 induces $C^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to C^\bullet$. Let us prove that the induced map $a\colon \mathbb{H}^1(\mathbb{P}^1, C^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{H}^1(\mathbb{P}^1, C^\bullet)$ is equal to f_∞ up to signatures under the identifications, which implies the claim of the lemma.

We can check this by a direct computation or use the following argument. We consider a double complex given as follows: We set $C^{00} = V_0 \otimes \mathcal{O}(-2)$, $C^{01} = V_1 \otimes \mathcal{O}(-1)$, $C^{10} = V_0 \otimes \mathcal{O}(-1)$ and $C^{11} = V_1 \otimes \mathcal{O}$. The morphisms $C^{0i} \to C^{1i}$ are given by α_0, and the morphisms $C^{i0} \to C^{i1}$ are given by $f_0\alpha_0 - f_\infty\alpha_\infty$.

For $i=0,1$, we set $D_i^{ij} = C^{ij}j$ and $D_i^{kj} = 0$ for $k \neq i$. Then we have an exact sequence of the double complexes $0 \to D_1^{\bullet\bullet} \to C^{\bullet\bullet} \to D_0^{\bullet\bullet} \to 0$. Similarly, we set $E_i^{ji} = C^{ji}$ and $E_i^{jk} = 0$ for $k \neq i$. Then we have an exact sequence $0 \to E_1^{\bullet\bullet} \to C^{\bullet\bullet} \to E_0^{\bullet\bullet} \to 0$. We set $F_0^{00} = C^{00}$ and $F_0^{ij} = 0$ for $(i,j) \neq (0,0)$. We set $F_1^{ij} = C^{ij}$ for $(i,j) \neq (0,0)$ and $F_1^{00} = 0$. Then we have an exact sequence $0 \to F_1^{\bullet\bullet} \to C^{\bullet\bullet} \to F_0^{\bullet\bullet} \to 0$. We have the following commutative diagrams:

$$\begin{array}{ccccc} D_1^{\bullet\bullet} & \longrightarrow & C^{\bullet\bullet} & \longrightarrow & D_0^{\bullet\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ F_1^{\bullet\bullet} & \longrightarrow & C^{\bullet\bullet} & \longrightarrow & F_0^{\bullet\bullet} \end{array} \qquad \begin{array}{ccccc} E_1^{\bullet\bullet} & \longrightarrow & C^{\bullet\bullet} & \longrightarrow & E_0^{\bullet\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ F_1^{\bullet\bullet} & \longrightarrow & C^{\bullet\bullet} & \longrightarrow & F_0^{\bullet\bullet} \end{array}$$

The natural morphisms $\mathbb{H}^*(\mathbb{P}^1, \mathrm{Tot}\, D_i^{\bullet\bullet}) \to \mathbb{H}^*(\mathbb{P}^1, \mathrm{Tot}\, F_i^{\bullet\bullet}) \leftarrow \mathbb{H}^*(\mathbb{P}^1, \mathrm{Tot}\, E_i^{\bullet\bullet})$ are isomorphisms. The map a is regarded as the connecting homomorphism of the long exact sequence associated to $0 \to \mathrm{Tot}\, D_1^{\bullet\bullet} \to \mathrm{Tot}\, C_1^{\bullet\bullet} \to \mathrm{Tot}\, D_0^{\bullet\bullet} \to 0$. The cokernel of $C^{0i} \to C^{1i}$ are the skyscraper sheaf at ∞, whose fibres are V_i. Hence, the connecting homomorphism for $0 \to \mathrm{Tot}\, E_1^{\bullet\bullet} \to \mathrm{Tot}\, C^{\bullet\bullet} \to \mathrm{Tot}\, E_0^{\bullet\bullet} \to 0$ is f_∞ up to signature. Thus, we are done. □

For any Y, let $\iota_\infty\colon Y \times \{\infty\} \to Y \times \mathbb{P}^1$. The morphism $\boldsymbol{N}_L \to \iota_{\infty*}\boldsymbol{N}_{L|T\times\{\infty\}}$ is obtained as the pushforward of $\widetilde{\mathcal{C}}_L^\bullet \to i_{\infty*}(\mathcal{C}^1/\mathcal{C}^0 \otimes \mathcal{P}oin \otimes L^\vee)$. Therefore, $H^1(T\times\mathbb{P}^1, \boldsymbol{N}_L) \to H^1(T, \boldsymbol{N}_{L|T\times\{\infty\}})$ is identified with $\mathcal{C}^1_{|L} \to (\mathcal{C}^1/\mathcal{C}^0)_{|L}$. By construction, the parabolic filtration of $((\boldsymbol{N}_L)_{0,0}^{(1,0)})_{|T\times\{\infty\}}$ is induced by the isomorphism $((\boldsymbol{N}_L)_{0,0}^{(1,0)})_{|T\times\{\infty\}} \simeq (\mathcal{C}^1/\mathcal{C}^0)_{L,0}^{(1,0)} \otimes \mathcal{O}_T$.

We have the following commutative diagram:

$$\begin{tikzcd} H^1(T\times\mathbb{P}^1, \boldsymbol{N}_L\otimes\mathcal{O}(-1)) \arrow[r, "b_1"] \arrow[d, "b_2"'] & H^1\big(T\times\mathbb{P}^1, \operatorname{Gr}^{\mathcal{P}}_{-1}\big((\boldsymbol{N}_L)^{1,0}_{P,0}\big)\big) \arrow[d, "b_3"', "\simeq"] \\ H^1(T\times\mathbb{P}^1, \boldsymbol{N}_L) \arrow[r, "b_4"'] & H^1\big(T\times\mathbb{P}^1, \operatorname{Gr}^{\mathcal{P}}_{0}\big((\boldsymbol{N}_L)^{1,0}_{P,0}\big)\big) \end{tikzcd}$$

Here, b_2 and b_3 are induced by the multiplication of w. By the previous consideration, the composite $b_4\circ b_2$ is identified with

$$\mathcal{C}^1_{|L}\to \operatorname{Gr}^F_1((\mathcal{C}^1/\mathcal{C}^0)_L),$$

which is surjective. Hence, b_1 is surjective. Let $\mathcal{N}_L$ denote the kernel of

$$\boldsymbol{N}_L\otimes\mathcal{O}(-1)\to \operatorname{Gr}^{\mathcal{P}}(\boldsymbol{N}_L)^{(1,0)}_{0,0}.$$

From the surjectivity of b_1 and the fact that $H^2(T\times\mathbb{P}^1, \boldsymbol{N}_L\otimes\mathcal{O}(-1))=0$, we obtain $H^2(T\times\mathbb{P}^1,\mathcal{N}_L)=0$. By the construction, $\mathcal{N}_L\subset\mathcal{P}_0\operatorname{Nahm}(\mathcal{P}_*\mathcal{E},\theta)\otimes L^\vee$ satisfies the conditions in Section 2.4.2, and we have

$$(\mathcal{N}_L)^{(1,0)}_{0,0}=\mathcal{P}_{<-1}(\operatorname{Nahm}(\mathcal{P}_*\mathcal{E},\theta)\otimes L^\vee)^{(1,0)}_{0,0}.$$

Hence, by Lemma 2.14, $\operatorname{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)$ satisfies condition (A3). □

3.2.3 Characteristic number For compact complex manifolds Z_i $(i=1,2)$, let $\omega_{Z_i}\in H^*(Z_1\times Z_2)$ denote the pullback of the fundamental class of Z_i by the projection.

Let $(\mathcal{P}_*\mathcal{E},\theta)$ be a filtered Higgs bundle on $(T^\vee, D)$ satisfying condition (A0). We shall study the characteristic numbers of $\operatorname{Nahm}(\mathcal{P}_*\mathcal{E},\theta)$.

Lemma 3.15 *We have* $\displaystyle\int_{T\times\mathbb{P}^1} c_1(\operatorname{Nahm}_a(\mathcal{P}_*\mathcal{E},\theta))\omega_{\mathbb{P}^1}=0$ *for any* $a\in\mathbb{R}$.

Proof This follows from $\operatorname{Nahm}_a(\mathcal{P}_*\mathcal{E},\theta)/\operatorname{Nahm}_{<a}(\mathcal{P}_*\mathcal{E},\theta)$ being of degree 0 for any $a\in\mathbb{R}$. □

The following lemma can be checked easily.

Lemma 3.16 $c_2(\operatorname{Nahm}_a(\mathcal{P}_*\mathcal{E},\theta))$ *is independent of* $a\in\mathbb{R}$. □

Because of the lemma, we will denote $c_2(\operatorname{Nahm}_a(\mathcal{P}_*\mathcal{E},\theta))$ by $c_2(\operatorname{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta))$.

We have the type decomposition $(\mathcal{P}_*\mathcal{E},\theta)_{|U_P}=\bigoplus_{(p,m,\boldsymbol{o})}(\mathcal{P}_*\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}},\theta^{(p,m)}_{P,\boldsymbol{o}})$ on a small neighbourhood U_P of each $P\in D$. We set

$$\ell_P:=\dim\operatorname{Cok}\bigl(\operatorname{Res}(\theta)\colon \operatorname{Gr}^{\mathcal{P}}_0\bigl(\mathcal{E}^{(1,0)}_{P,0}\bigr)\to \operatorname{Gr}^{\mathcal{P}}_0\bigl(\mathcal{E}^{(1,0)}_{P,0}\bigr)\bigr).$$

We put $r^{(p,m)}_{P,\boldsymbol{o}}=\operatorname{rank}(\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}})/p$ and $r^{(p,m)}_P:=\sum_{\boldsymbol{o}} r^{(p,m)}_{P,\boldsymbol{o}}$. We have $\sum_{p,m} r^{(p,m)}_P p=\operatorname{rank}\mathcal{E}$.

Proposition 3.17 *The following equalities hold:*

$$\operatorname{rank}\operatorname{Nahm}(\mathcal{P}_*\mathcal{E},\theta)=\sum_P\sum_{p,m} r^{(p,m)}_P(p+m)-\sum_P \ell_P. \tag{16}$$

$$\int_{T\times\mathbb{P}^1} c_1(\operatorname{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta))\cdot\omega_T=\deg(\mathcal{P}_*\mathcal{E}). \tag{17}$$

$$\int_{T\times\mathbb{P}^1} c_2(\operatorname{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta))=\operatorname{rank}\mathcal{E}. \tag{18}$$

Proof Let us prove (16) and (17). We have only to consider the rank and the degree of $\operatorname{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)_{|\{0\}\times\mathbb{P}^1}$. Let $\mathcal{V}\subset\mathcal{P}_1\mathcal{E}$ be the subsheaf determined by the following conditions:

- $\mathcal{V}=\mathcal{P}_1\mathcal{E}$ on the complement of D.
- It has a decomposition $\mathcal{V}=\bigoplus_{p,m,\boldsymbol{o}}\mathcal{V}^{(p,m)}_{P,\boldsymbol{o}}$ around each $P\in D$.
- We have $\mathcal{V}^{(p,m)}_{P,\boldsymbol{o}}=\mathcal{P}_{1/2}\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}}$ for $(p,m,\boldsymbol{o})\neq(1,0,0)$, and $\mathcal{V}^{(1,0)}_{P,0}=\mathcal{P}_1\mathcal{E}^{(1,0)}_{P,0}$.

Let π_i denote the projection of $T^\vee\times\mathbb{P}^1$ onto the i^{th} component. We have the K–theoretic description

$$\begin{aligned}&(\widetilde{\mathcal{C}}^1(\mathcal{P}_*\mathcal{E},\theta)-\widetilde{\mathcal{C}}^0(\mathcal{P}_*\mathcal{E},\theta))_{|T^\vee\times\{0\}\times\mathbb{P}^1}\\&=\pi_1^*\Bigl(\mathcal{V}-\sum_{P\in D}\mathcal{O}_P^{\oplus\ell_P}\Bigr)-\pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1)\cdot\pi_1^*\Bigl(\mathcal{V}-\sum_{P\in D}\sum_{p,m,\boldsymbol{o}}\mathcal{O}_P^{\oplus r^{(p,m)}_{P,\boldsymbol{o}}(p+m)}\Bigr).\end{aligned}\tag{19}$$

The Chern character of (19) is equal to

$$\begin{aligned}&\pi_1^*\operatorname{ch}(\mathcal{V})-\sum_{P\in D}\ell_P\omega_{T^\vee}-(1-\omega_{\mathbb{P}^1})\Bigl(\pi_1^*\operatorname{ch}(\mathcal{V})-\sum_P\sum_{p,m} r^{(p,m)}_P(p+m)\omega_{T^\vee}\Bigr)\\&=\Bigl(\sum_P\sum_{p,m} r^{(p,m)}_P(p+m)-\sum_P\ell_P\Bigr)\omega_{T^\vee}\\&\qquad+\omega_{\mathbb{P}^1}\pi_1^*\operatorname{ch}(\mathcal{V})-\omega_{\mathbb{P}^1}\sum_P\sum_{p,m} r^{(p,m)}_P(p+m)\omega_{T^\vee}.\end{aligned}\tag{20}$$

Hence, the Chern character of $N(\mathcal{P}_*\mathcal{E},\theta)_{|\{0\}\times\mathbb{P}^1}$ is

$$\begin{aligned}(21)\quad &\sum_P\sum_{p,m} r_P^{(p,m)}(p+m)-\sum_P \ell_P+\omega_{\mathbb{P}^1}\Bigl(\deg(\mathcal{V})-\sum_P\sum_{p,m} r_P^{(p,m)}(p+m)\Bigr)\\ &=\sum_P\sum_{p,m} r_P^{(p,m)}(p+m)-\sum_P \ell_P+\omega_{\mathbb{P}^1}\Bigl(\deg(\mathcal{V}(-D))-\sum_P\sum_{p,m} r_P^{(p,m)}m\Bigr).\end{aligned}$$

In particular, we obtain (16). We also obtain

$$\deg(N(\mathcal{P}_*\mathcal{E},\theta)_{|\{0\}\times\mathbb{P}^1})=\deg(\mathcal{V}(-D))-\sum_P\sum_{p,m} r_P^{(p,m)}m.$$

We set $a(p,m,\boldsymbol{o})=-\frac{1}{2}$ if $(a,m,\boldsymbol{o})\neq(1,0,0)$, and $a(1,0,0):=0$. For the parabolic characteristic numbers, we have the expressions

$$(22)\qquad \deg(\mathcal{P}_*\mathcal{E})=\deg(\mathcal{V}(-D))-\sum_{P\in D}\sum_{p,m,\boldsymbol{o}}\delta(\mathcal{P}_*\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}},a(p,m,\boldsymbol{o})),$$

$$\begin{aligned}(23)\quad &\deg(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)_{|\{0\}\times\mathbb{P}^1})\\ &=\deg(N(\mathcal{P}_*\mathcal{E},\theta)_{|\{0\}\times\mathbb{P}^1})-\sum_{P\in D}\sum_{p,m,\boldsymbol{o}}\delta(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)^{(p+m,m)}_{P,\boldsymbol{o}},a(p,m,\boldsymbol{o})).\end{aligned}$$

Here, $\delta(B,a(p,m,\boldsymbol{o}))$ denote the contributions of the locally given filtered bundles B to the parabolic degree. (See Section 2.2.3.) In the following, we omit $a(p,m,\boldsymbol{o})$. In the case $(p,m,\boldsymbol{o})=(1,0,0)$, we have

$$\begin{aligned}(24)\qquad \delta(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)^{(1,0)}_{P,0})&=\sum_{-1<c<0} c\dim\mathrm{Gr}^{\mathcal{P}}_c\,\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)^{(1,0)}_{P,0}\\ &=\sum_{-1<c<0} c\dim\mathrm{Gr}^{\mathcal{P}}_c\bigl(\mathcal{E}^{(1,0)}_{P,0}\bigr)=\delta\bigl(\mathcal{P}_*\mathcal{E}^{(1,0)}_{P,0}\bigr).\end{aligned}$$

Let us consider the case $(p,m,\boldsymbol{o})\neq(1,0,0)$. Let $\varphi_p\colon U_u\to U_P$ be given by $\varphi_p(u)=u^p$. We have the decomposition

$$\varphi_p^*\bigl(\mathcal{P}_*\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}},\theta^{(p,m)}_{P,\boldsymbol{o}}\bigr)=\bigoplus_{\alpha\in\boldsymbol{o}}(\mathcal{P}_*V_\alpha,\theta_\alpha).$$

For any $c\in\mathbb{R}$, we put

$$r^{(p,m)}_{P,\boldsymbol{o},c}:=\dim\mathrm{Gr}^{\mathcal{P}}_c V_\alpha.$$

It is independent of the choice of $\alpha\in\boldsymbol{o}$. We have

$$\delta(\mathcal{P}_*\mathcal{E}^{(p,m)}_{P,\boldsymbol{o}})=\sum_{\substack{-p/2-1<c\le -p/2\\ 0\le j\le p-1}} r^{(p,m)}_{P,\boldsymbol{o},c}\frac{c-j}{p}=\sum_{-p/2-1<c\le -p/2} r^{(p,m)}_{P,\boldsymbol{o},c}\bigl(c-\tfrac{1}{2}(p-1)\bigr).$$

We also have the following equality from the expression of the parabolic structure of $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*\mathcal{E},\theta)$ in the proof of Proposition 3.3:

$$\begin{aligned}\text{(25)}\qquad \delta(\mathrm{Nahm}(\mathcal{P}_*\mathcal{E},\theta)^{(p+m,m)}_{\boldsymbol{P},\boldsymbol{o}}) &= \sum_{\substack{-p/2-1<c\le -p/2\\ 0\le j\le p+m-1}} r^{(p,m)}_{\boldsymbol{P},\boldsymbol{o},c}\frac{2c-2j-m}{2(p+m)}\\ &= \sum_{-p/2-1<c\le -p/2} r^{(p,m)}_{\boldsymbol{P},\boldsymbol{o},c}\bigl(c-m-\tfrac{1}{2}(p-1)\bigr).\end{aligned}$$

Then the equality (17) follows from $\sum_{c,\boldsymbol{o}} r^{(p,m)}_{\boldsymbol{P},c,\boldsymbol{o}} = r^{(p,m)}_{\boldsymbol{P}}$.

Let us prove (18). We have $\int_{T\times\mathbb{P}^1} c_2(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)) = \int_{T\times\mathbb{P}^1} c_2(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta))$. We also have

$$\int_{T\times\mathbb{P}^1} c_2(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)) = -\int_{T\times\mathbb{P}^1}\mathrm{ch}_2(\boldsymbol{N}(\mathcal{P}_*\mathcal{E},\theta)) = -\int_{T^\vee\times T\times\mathbb{P}^1}\mathrm{ch}_3(\widetilde{\mathcal{C}}^1-\widetilde{\mathcal{C}}^0).$$

We have $\mathrm{ch}_3(\widetilde{\mathcal{C}}^1)=0$ and $c_1(\mathcal{P}oin)^2=-2\omega_T\omega_{T^\vee}$. We also have

$$\int_{T^\vee\times T\times\mathbb{P}^1}\mathrm{ch}_3(\widetilde{\mathcal{C}}^0) = \int_{T^\vee\times T\times\mathbb{P}^1}\mathrm{rank}(\mathcal{V})\omega_T\omega_{T^\vee}\omega_{\mathbb{P}^1} = \mathrm{rank}(\mathcal{V}).$$

Hence, we obtain (18). □

3.2.4 Stable filtered Higgs bundles of degree 0 We consider the standard stability condition for filtered Higgs bundles on $(T^\vee,D)$. For any filtered bundle $(\mathcal{P}_*\mathcal{E},\theta)$ on a projective curve (X,D), we define the slope $\mu(\mathcal{P}_*\mathcal{E}) := \int_X \text{par-c}_1(\mathcal{P}_*\mathcal{E})/\operatorname{rank}\mathcal{E}$. The bundle is called stable (resp. semistable) if $\mu(\mathcal{P}_*\mathcal{E}')<\mu(\mathcal{P}_*\mathcal{E})$ (resp. $\mu(\mathcal{P}_*\mathcal{E}')\le\mu(\mathcal{P}_*\mathcal{E})$) for any nontrivial filtered subbundle $\mathcal{P}_*\mathcal{E}'\subset\mathcal{P}_*\mathcal{E}$ such that $\theta(\mathcal{E}')\subset\mathcal{E}'\otimes\Omega^1$. A semistable filtered Higgs bundle is called polystable if it is a direct sum of stable ones. The following lemma is easy to see.

Lemma 3.18 *If $(\mathcal{P}_*\mathcal{E},\theta)$ be a stable Higgs bundle on $(T^\vee,D)$, then its dual is also stable.* □

The following proposition is standard.

Proposition 3.19 *Let $(\mathcal{P}_*\mathcal{E},\theta)$ be a stable admissible filtered bundle on $(T^\vee,D)$ with* $\deg(\mathcal{P}_*\mathcal{E})=0$. *If* $\operatorname{rank}\mathcal{E}>1$, *it satisfies condition (A0).*

Proof of Proposition 3.19 Indeed, an element of $\mathbb{H}^0(T^\vee,\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}\otimes L,\theta+w\,d\zeta))$ corresponds to a morphism $(\mathcal{O}_{T^\vee}(*D),0)\to(\mathcal{P}_*\mathcal{E}\otimes L,\theta)$. By the stability with $\deg(\mathcal{P}_*\mathcal{E})=0$ and $\operatorname{rank}\mathcal{E}>1$, we obtain that such a morphism has to be 0. We obtain the vanishing of $\mathbb{H}^2$ from the following lemma.

Lemma 3.20 *$\mathbb{H}^i(T^\vee, \mathcal{C}(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee))$ is naturally isomorphic to the dual space of $\mathbb{H}^{2-i}(T^\vee, \mathcal{C}(\mathcal{P}_*\mathcal{E}, \theta))$.*

Proof We use the natural identification $\Omega^1_{T^\vee} \simeq \mathcal{O}_{T^\vee}$. Let $P \in D$. We have

$$(\mathcal{P}_0\mathcal{E}^{(1,0)}_{P,0})^\vee \simeq \mathcal{P}_{<1}(\mathcal{E}^\vee)^{(1,0)}_{P,0} =: C^1(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0}.$$

Let π denote the projection $\mathcal{P}_0(\mathcal{E}^\vee)^{(1,0)}_{P,0} \to \mathrm{Gr}^{\mathcal{P}}_0((\mathcal{E}^\vee)^{(1,0)}_{P,0})$. We have a subspace

$$\mathrm{Ker}\big(\mathrm{Gr}^{\mathcal{P}}_0\big(\theta^{(1,0)}_{P,0}\big)^\vee\big) \subset \mathrm{Gr}^{\mathcal{P}}_0\big((\mathcal{E}^\vee)^{(1,0)}_{P,0}\big).$$

We have a natural isomorphism

$$C^0\big(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee\big)^{(1,0)}_{P,0} := \pi^{-1}\big(\mathrm{Ker}\big(\mathrm{Gr}^{\mathcal{P}}_0\big(\theta^{(1,0)}_{P,0}\big)^\vee\big)\big) \simeq \big(\mathcal{C}^1\big(\mathcal{P}_*\mathcal{E}, \theta\big)^{(1,0)}_{P,0}\big)^\vee.$$

The Higgs field $-\theta^\vee$ induces

$$C^0(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0} \to C^1(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0}.$$

The complex $C^\bullet(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0}[1]$ is the dual of $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}, -\theta^\vee)^{(1,0)}_{P,0}$. The natural inclusions induce a quasi-isomorphism

$$C^\bullet(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0} \to \mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(1,0)}_{P,0}.$$

For $(p, m, \boldsymbol{o}) \neq (1, 0, 0)$, the dual of the complex $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}, \theta)^{(p,m)}_{P,\boldsymbol{o}}$ is

$$\mathcal{P}_{<1/2}(\mathcal{E}^\vee)^{(p,m)}_{P,-\boldsymbol{o}} \xrightarrow{-\theta^\vee} \mathcal{P}_{<3/2+m/p}(\mathcal{E}^\vee)^{(p,m)}_{P,-\boldsymbol{o}}$$

where the first term sits in the degree -1. It is moreover naturally quasi-isomorphic to $\mathcal{C}^\bullet(\mathcal{P}_*\mathcal{E}^\vee, -\theta^\vee)^{(p,m)}_{P,-\boldsymbol{o}}[1]$. Then the claim of the lemma follows from Serre duality. Thus, we complete the proof of Lemma 3.20 and Proposition 3.19. □

3.2.5 Filtered Higgs bundles of rank 1 on $(T^\vee, D)$ Filtered Higgs bundles of rank 1 are always admissible and stable. Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a filtered Higgs field of rank 1 on $(T^\vee, D)$. For each $P \in D$, we have the complex number $\mathrm{Res}_P(\theta)$. We also have $a(P) \in \mathbb{R}$ such that $\mathcal{P}ar(\mathcal{P}_*\mathcal{E}, P) = \{a(P) + n \mid n \in \mathbb{Z}\}$. Such an $a(P)$ is uniquely determined in $\mathbb{R}/\mathbb{Z}$. We say that P is a nontrivial singularity of $(\mathcal{P}_*\mathcal{E}, \theta)$ if $(\mathrm{Res}_P\,\theta, a(P)) \neq (0, 0)$ in $\mathbb{C} \times (\mathbb{R}/\mathbb{Z})$. If P is a trivial singularity, ie $(\mathrm{Res}_P\,\theta, a(P)) = (0, 0)$, we obtain a filtered Higgs bundle on $(T^\vee, D \setminus \{P\})$ by considering the lattice $\mathcal{P}_0(\mathcal{E})$ around P. The following lemma is clear.

Lemma 3.21 *Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a filtered Higgs bundle of rank 1 on $(T^\vee, D)$.*

- *If each $P \in D$ is a trivial singularity of $(\mathcal{P}_*\mathcal{E}, \theta)$, then $(\mathcal{P}_*\mathcal{E}, \theta) \simeq (L(*D), \alpha\, d\zeta)$ for some $\alpha \in \mathbb{C}$ and some line bundle L of degree 0. Here the parabolic structure of $L(*D)$ is given in a typical way as in Section 2.2.2.*
- *If one of $P \in D$ is a nontrivial singularity of $(\mathcal{P}_*\mathcal{E}, \theta)$, then $(\mathcal{P}_*\mathcal{E}, \theta)$ satisfies (A0).* □

3.3 Algebraic Nahm transform for admissible filtered bundles

3.3.1 Construction of the transform For $I \subset \{1, 2, 3\}$, let p_I be the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. Let $\mathcal{P}oin$ denote the Poincaré bundle on $T^\vee \times T$.

Let $\mathcal{P}_*E$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying condition (A3). We put $D := \mathcal{S}p_\infty(\mathcal{P}_*E)$. We define

$$\mathrm{Nahm}(\mathcal{P}_*E) := R^1 p_{1*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{P}_{-1}E) \otimes \mathcal{O}_{T^\vee}(*D).$$

By (A3), $\mathrm{Nahm}(\mathcal{P}_*E)$ is a locally free $\mathcal{O}_{T^\vee}(*D)$–module. By Lemma 2.13, we have a natural isomorphism

$$\mathrm{Nahm}(\mathcal{P}_*E) \simeq R^1 p_{1*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{P}_0E) \otimes \mathcal{O}_{T^\vee}(*D).$$

Let w be the standard coordinate of $\mathbb{C} \subset \mathbb{P}^1$. It naturally gives a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. The multiplication of $-w$ induces an endomorphism f of $\mathrm{Nahm}(\mathcal{P}_*E)$. We obtain a Higgs field $\theta := f\,d\zeta$ of $\mathrm{Nahm}(\mathcal{P}_*E)$. We shall define a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*E) = (\mathrm{Nahm}_a(\mathcal{P}_*E) \mid a \in \mathbb{R})$ over $\mathrm{Nahm}(\mathcal{P}_*E)$.

We have the type decomposition $\mathcal{P}_*E = \bigoplus_{P \in D} \bigoplus_{p,m,\boldsymbol{o}} \mathcal{P}_*E^{(p,m)}_{P,\boldsymbol{o}}$ on a neighbourhood of $T \times \{\infty\}$. Let $\mathcal{U} \subset \mathcal{P}_cE$ be an $\mathcal{O}_{T \times \mathbb{P}^1}$–submodule for some large $c \in \mathbb{R}$, satisfying the conditions in Section 2.4.2. We suppose

$$\mathcal{P}_{<-1}E^{(1,0)}_{P,0} \subset \mathcal{U}^{(1,0)}_{P,0} \subset \mathcal{P}_0E^{(1,0)}_{P,0}$$

for any $P \in D$. We define $N(\mathcal{U}) := R^1 p_{1*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{U})$. By Lemmas 2.13 and 2.14, we have $R^i p_{1*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{U}) = 0$ unless $i = 1$, and $N(\mathcal{U})$ is a locally free sheaf on $T^\vee$.

We have the following object in $D^b(\mathcal{O}_{T^\vee \times \mathbb{P}^1})$:

$$\mathrm{RFM}_-(\mathcal{U}) := Rp_{13*}(p_{12}^*\mathcal{P}oin^\vee \otimes p_{23}^*\mathcal{U})[1].$$

We can express $\mathrm{RFM}_{-}(\mathcal{U})$ as a two term complex of locally free $\mathcal{O}_{T^\vee\times\mathbb{P}^1}$– modules

$$\mathcal{N}_{-1}\xrightarrow{a}\mathcal{N}_0.$$

Because a is generically isomorphism, it is injective. Hence, we have $\mathrm{RFM}_{-}(\mathcal{U})\simeq\mathcal{H}^0(\mathrm{RFM}_{-}(\mathcal{U}))$. We will not distinguish between the two.

Suppose $0\in D$. Let $U_0\subset T^\vee$ denote a small neighbourhood of 0. Let $W_\infty\subset\mathbb{P}^1$ be a small neighbourhood of ∞. We have the decomposition

$$\mathrm{RFM}_{-}(\mathcal{U})_{|U_0\times W_\infty}=\bigoplus_{p,m,\boldsymbol{o}}\mathrm{RFM}_{-}(\mathcal{U}^{(p,m)}_{0,\boldsymbol{o}}).$$

If $(p,m,\boldsymbol{o})\neq(1,0,0)$, the support of $\mathrm{RFM}_{-}(\mathcal{U}^{(p,m)}_{0,\boldsymbol{o}})$ is proper over U_0. Hence, we have the decomposition

$$\mathrm{RFM}_{-}(\mathcal{U})_{|U_0\times\mathbb{P}^1}=\bigoplus_{(p,m,\boldsymbol{o})\neq(1,0,0)}\mathrm{RFM}_{-}(\mathcal{U}^{(p,m)}_{0,\boldsymbol{o}})\oplus\mathcal{M}(\mathcal{U}).\tag{26}$$

Here, $\mathcal{M}(\mathcal{U})_{|U_0\times W_\infty}=\mathrm{RFM}_{-}(\mathcal{U}^{(1,0)}_{0,0})$. We have similar decompositions for any $P\in D$. We have $N(\mathcal{U})(*D)=\mathrm{Nahm}(\mathcal{P}_*E)$. The following decomposition around any $P\in D$ is induced by the decomposition (26) considered for P:

$$N(\mathcal{U})=\bigoplus_{p,m,\boldsymbol{o}}N(\mathcal{U})^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}.$$

In particular, we have the following decomposition around any $P\in D$:

$$\mathrm{Nahm}(\mathcal{P}_*E)=\bigoplus_{p,m,\boldsymbol{o}}\mathrm{Nahm}(\mathcal{P}_*E)^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}.\tag{27}$$

We fix a lift $\widetilde{P}\in\mathbb{C}$ of any $P\in D$, and we use a local coordinate $\zeta_{\boldsymbol{P}}:=\zeta-\widetilde{P}$ around P. Let W_∞ be a small neighbourhood of ∞. We have the filtered bundles

$$\big(\mathcal{P}_*V^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}},g^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}\big)$$

with an endomorphism on (W_∞,∞), as in Section 2.4.1. If $(p,m,\boldsymbol{o})\neq(1,0,0)$, we have a natural isomorphism

$$\mathrm{Nahm}(\mathcal{P}_*E)^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}\simeq\mathcal{N}^{(\infty,0)}\big(\mathcal{P}_*V^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}},g^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}\big).$$

Under the isomorphism, we define

$$\mathrm{Nahm}_a(\mathcal{P}_*E)^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}:=\mathcal{N}^{\infty,0}_a\big(\mathcal{P}_*V^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}},g^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}\big).$$

Let us consider the case $(p,m,\boldsymbol{o})=(1,0,0)$. First, we define

$$\mathrm{Nahm}_0(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0} := N(\mathcal{P}_{-1}E)^{(1,0)}_{\boldsymbol{P},0}.$$

We set $\mathfrak{C}_{\boldsymbol{P}} := \mathcal{P}_0E^{(1,0)}_{\boldsymbol{P},0}/\mathcal{P}_{-1}E^{(1,0)}_{\boldsymbol{P},0}$. We have the following exact sequence around P:

$$0\to N(\mathcal{P}_{-1}E)^{(1,0)}_{\boldsymbol{P},0}\to N(\mathcal{P}_0E)^{(1,0)}_{\boldsymbol{P},0}\to R^1p_{1*}(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\mathfrak{C}_{\boldsymbol{P}})\to 0.$$

We may regard $\mathfrak{C}_{\boldsymbol{P}}$ as a locally free sheaf on T, and then it is isomorphic to a direct sum of some copies of the line bundle corresponding to P. Hence, the multiplication of $\zeta_{\boldsymbol{P}}$ on $R^1p_{1*}(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\mathfrak{C}_{\boldsymbol{P}})$ is 0. This induces a surjection

$$N(\mathcal{P}_0E)^{(1,0)}_{\boldsymbol{P},0|0} := N(\mathcal{P}_0E)^{(1,0)}_{\boldsymbol{P},0}\otimes\mathcal{O}_{\boldsymbol{P}}\to R^1p_{1*}(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\mathfrak{C}_{\boldsymbol{P}}).$$

Let K denote the kernel. We have the morphisms

$$R^1p_{1*}(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\mathfrak{C}_{\boldsymbol{P}})\simeq N(\mathcal{P}_0E)^{(1,0)}_{\boldsymbol{P},0|0}\,\big/\,K\xrightarrow{h} N(\mathcal{P}_{-1}E)^{(1,0)}_{\boldsymbol{P},0|0}.$$

Here, h is the injection induced by the multiplication of $\zeta_{\boldsymbol{P}}$. We have a natural isomorphism of $\mathbb{C}$–vector spaces

$$R^1p_{1*}(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\mathfrak{C}_{\boldsymbol{P}})\simeq\mathcal{P}_0V^{(1,0)}_{\boldsymbol{P},0|\infty}.$$

Hence, for any $-1<c<0$, we define

$$F_c(\mathrm{Nahm}_0(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}\otimes\mathcal{O}_{\boldsymbol{P}}) := F_c\big(\mathcal{P}_0V^{(1,0)}_{\boldsymbol{P},0|\infty}\big).$$

We also set $F_0(\mathrm{Nahm}_0(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}\otimes\mathcal{O}_{\boldsymbol{P}})=\mathrm{Nahm}_0(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}\otimes\mathcal{O}_{\boldsymbol{P}}$. The filtration of $\mathrm{Nahm}_0(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}\otimes\mathcal{O}_{\boldsymbol{P}}$ indexed by $]-1,0]$ induces a filtered bundle

$$\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}\to\mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{\boldsymbol{P},0}.$$

We obtain a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_*E)$ over $\mathrm{Nahm}(\mathcal{P}_*E)$ by taking the direct sum.

Proposition 3.22 $\mathrm{Nahm}_*(\mathcal{P}_*E)$ *with* θ *is admissible, and satisfies condition (A0). Moreover, the complex*

$$N(\mathcal{V}\otimes\mathcal{O}_{\mathbb{P}^1}(-1))\xrightarrow{-w}N(\mathcal{V})$$

naturally identifies with $\mathcal{C}^\bullet(\mathrm{Nahm}_*(\mathcal{P}_*E))$*. Here,* $\mathcal{V}\subset\mathcal{P}_0E$ *is an* $\mathcal{O}_{T\times\mathbb{P}^1}$*–submodule satisfying the conditions in Section 2.4.2 and*

$$\mathcal{V}^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}=\begin{cases}\mathcal{P}_0E^{(1,0)}_{\boldsymbol{P},0} & (p,m,\boldsymbol{o})=(1,0,0),\\ \mathcal{P}_{-1/2}E^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}} & \text{otherwise.}\end{cases}$$

Proof of Proposition 3.22 If $(p,m,\boldsymbol{o})\neq(1,0,0)$, then by Proposition 3.5, we have $\mathrm{Nahm}_*(\mathcal{P}_*E)^{(p,m)}_{P,\boldsymbol{o}}$ with θ is admissible. Moreover,

$$N(\mathcal{V}\otimes\mathcal{O}_{\mathbb{P}^1}(-1))^{(p,m)}_{P,\boldsymbol{o}}\xrightarrow{-w}N(\mathcal{V})^{(p,m)}_{P,\boldsymbol{o}}$$

is naturally identified with $\mathcal{C}^\bullet(\mathrm{Nahm}_*(\mathcal{P}_*E)^{(p,m)}_{P,\boldsymbol{o}})$ by the construction.

Lemma 3.23 $\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0}$ *with* θ *is admissible of type* $(1,0,0)$ *and the complex*

$$N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}\xrightarrow{-w}N(\mathcal{P}_0E)^{(1,0)}_{P,0}$$

is naturally identified with $\mathcal{C}^\bullet(\mathrm{Nahm}(\mathcal{P}_*E)^{(1,0)}_{P,0})$.

Proof We use the above notation. The morphism f induces

$$f_0\colon N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0|0}\to N(\mathcal{P}_0E)^{(1,0)}_{P,0|0}/K.$$

The endomorphism

$$f_0\circ h\quad\text{on}\quad N(\mathcal{P}_0E)^{(1,0)}_{P,0|0}/K$$

is identified with

$$-wg^{(1,0)}_{P,0}\quad\text{on}\quad \mathcal{P}_0V^{(1,0)}_{P,0|\infty}.$$

It is nilpotent, and it also preserves the parabolic filtration on $\mathcal{P}_0V^{(1,0)}_{P,0|\infty}$. By the construction of the parabolic filtration, $h\circ f_0$ preserves the parabolic filtration on $N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0|0}$. Thus, we obtain that $\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0}$ is admissible of type $(1,0,0)$.

By construction, we have a natural isomorphism

$$N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}\simeq\mathcal{C}^0\big(\mathrm{Nahm}\big(\mathcal{P}_*E\big)^{(1,0)}_{P,0}\big).$$

Because $\zeta_P\cdot N(\mathcal{P}_0E)^{(1,0)}_{P,0}\subset N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}$ and $\Omega^1_{T^\vee}\simeq\mathcal{O}_{T^\vee}$, we have the natural morphism

$$A\colon N(\mathcal{P}_0E)^{(1,0)}_{P,0}\to N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}\otimes\Omega^1(P).$$

Let ρ denote the natural map

$$N(\mathcal{P}_0E)^{(1,0)}_{P,0}\to N(\mathcal{P}_0E)^{(1,0)}_{P,0|0}/K\simeq\mathcal{P}_0V^{(1,0)}_{P,0|\infty}.$$

By the construction of $\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0}$, the image of $\rho^{-1}(F_{<0})$ by A is equal to $\mathcal{P}_{<0}\,\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0}\otimes\Omega^1(P)$. By the construction of θ, $\mathrm{Im}(A)$ also contains $\theta(N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0})$. Hence, $\mathcal{C}^1(\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0})$ is contained in $\mathrm{Im}(A)$. We remark that the following morphism is surjective, because $H^2(T\times\mathbb{P}^1,\mathcal{P}_{<-1}E\otimes L)=0$ for any holomorphic line bundle L of degree 0 on T:

$$N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}\to R^1p_{1*}\big(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\,\mathrm{Gr}^{\mathcal{P}}_{-1}(E^{(1,0)}_{P,0})\big)\overset{w}{\simeq} R^1p_{1*}\big(p_{12}^*\mathcal{P}oin\otimes p_{23}^*\,\mathrm{Gr}^{\mathcal{P}}_{0}(E^{(1,0)}_{P,0})\big).\tag{28}$$

It implies that the morphism

$$N(\mathcal{P}_{-1}E)^{(1,0)}_{P,0}\to\mathcal{P}_0V^{(1,0)}_{P,0|\infty}/F_{<0}$$

which is induced by θ is surjective. Then we obtain that $\mathrm{Im}\,A=\mathcal{C}^1(\mathrm{Nahm}_*(\mathcal{P}_*E)^{(1,0)}_{P,0})$. The proof of Lemma 3.23 is finished. □

Let us prove that $\mathrm{Nahm}_*(\mathcal{P}_*E)$ with θ satisfies condition (A0). For $I\subset\{1,2,3\}$, let p_I denote the projection of $T^\vee\times T\times\mathbb{P}^1$ onto the product of the i^{th} components $(i\in I)$. For any $a\in\mathbb{C}$ and a line bundle L of degree 0 on $T^\vee$, consider the complex

$$\widetilde{\mathcal{C}}:=\Big(p_{23}^*\mathcal{P}_{-1}E\otimes p_{12}^*\mathcal{P}oin\otimes p_1^*L\xrightarrow{-w+a}p_{23}^*\mathcal{P}_0E\otimes p_{12}^*\mathcal{P}oin\otimes p_1^*L\Big)$$

where the first term sits in the degree -1. Since $Rp_{1*}\widetilde{\mathcal{C}}$ is the complex

$$N(\mathcal{P}_{-1}(E))\otimes L\xrightarrow{-w+a}N(\mathcal{P}_0E)\otimes L$$

on $T^\vee$ which is identified with $\mathcal{C}^\bullet(\mathrm{Nahm}_*(E)\otimes L,\theta+a\,d\zeta)$, we have

$$\mathbb{H}^i(T^\vee\times T\times\mathbb{P}^1,\widetilde{\mathcal{C}})\simeq\mathbb{H}^i(T^\vee,\mathcal{C}^\bullet(\mathrm{Nahm}_*(E)\otimes L,\theta+a\,d\zeta))$$

Because $Rp_{23*}\widetilde{\mathcal{C}}$ is quasi-isomorphic to $\mathcal{P}_{-1}E_{|\{L\}\times\mathbb{P}^1}\to\mathcal{P}_0E_{|\{L\}\times\mathbb{P}^1}$, where the first term sits in the degree 0, we have $\mathbb{H}^i(T^\vee\times T\times\mathbb{P}^1,\widetilde{\mathcal{C}})=0$ unless $i=1$. Thus, we obtain that $\mathrm{Nahm}_*(\mathcal{P}_*E)$ with θ satisfies (A0), and the proof of Proposition 3.22 is finished. □

We denote the filtered Higgs bundle $(\mathrm{Nahm}_*(\mathcal{P}_*E),\theta)$ just by $\mathrm{Nahm}_*(\mathcal{P}_*E)$.

Remark 3.24 We obtain a slightly different transformation by replacing $\mathcal{P}oin$ with $\mathcal{P}oin^\vee$, for which we can argue in a similar way.

3.3.2 Inversion

Proposition 3.25
- *Let $(\mathcal{P}_*\mathcal{E},\theta)$ be an admissible filtered Higgs bundle on $(T^\vee,D)$ which satisfies condition (A0). Then we have a natural isomorphism* $\mathrm{Nahm}_*(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta))\simeq(\mathcal{P}_*\mathcal{E},\theta)$.
- *Let $\mathcal{P}_*E$ be an admissible filtered bundle on $(T\times\mathbb{P}^1,T\times\{\infty\})$ satisfying (A3). Then we have a natural isomorphism* $\mathrm{Nahm}_*(\mathrm{Nahm}_*(\mathcal{P}_*E))\simeq\mathcal{P}_*E$.

Proof For any $I \subset \{1,2,3,4\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{P}^1 \times T^\vee$ onto the product of the j^{th} components $(j \in I)$. We set $\mathcal{C}^i := \mathcal{C}^i(\mathcal{P}_*\mathcal{E}, \theta)$. We consider the following complex on $T^\vee \times T \times \mathbb{P}^1 \times T^\vee$:

$$p_1^*\mathcal{C}^0 \otimes p_{12}^*\mathcal{P}oin \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_{24}^*\mathcal{P}oin^\vee \xrightarrow{\theta + w\,d\zeta} p_1^*\mathcal{C}^1 \otimes p_{12}^*\mathcal{P}oin \otimes p_{24}^*\mathcal{P}oin^\vee.$$

We denote the complex by $\overline{\mathcal{C}}^\bullet$. We can observe that $Rp_{14*}\overline{\mathcal{C}}^\bullet$ is quasi-isomorphic to $p_1^*\mathcal{C}^1 \otimes \mathcal{O}_\Delta[-2]$, where $p_1\colon T^\vee \times T^\vee \to T^\vee$ denotes the projection onto the first component, and $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal. Hence, $Rp_{4*}\overline{\mathcal{C}}^\bullet$ is naturally isomorphic to $\mathcal{C}^1[-2]$. We can also observe that $Rp_{234*}\overline{\mathcal{C}}^\bullet$ is quasi-isomorphic to $q_{12}^*\boldsymbol{N}(\mathcal{P}_*\mathcal{E}, \theta) \otimes q_{13}^*\mathcal{P}oin^\vee[-1]$, where q_I denotes the projection of $T \times \mathbb{P}^1 \times T^\vee$ onto the product of the i^{th} components $(i \in I)$. Hence, we have $Rp_{4*}\overline{\mathcal{C}}^\bullet(*D)$ is quasi-isomorphic to $\mathrm{Nahm}(\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta))$. We obtain an isomorphism of meromorphic Higgs bundles

$$\mathrm{Nahm}(\mathrm{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)) \otimes \mathcal{O}(*D) \simeq (\mathcal{E}, \theta).$$

If $(p, m, \boldsymbol{o}) \neq (1, 0, 0)$, then we obtain the comparison of the filtered bundles over $\mathcal{E}^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}$ from Proposition 3.7. We obtain the comparison of the filtered bundles over $\mathcal{E}^{(1,0)}_{\boldsymbol{P},0}$ directly from the construction. Thus, we obtain the first claim.

Let $\mathcal{P}_*E$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying (A3). Let $\mathcal{V} \subset \mathcal{P}_0 E$ be as in Proposition 3.22. By Proposition 3.22, we have $\mathcal{C}^0(\mathrm{Nahm}_*(\mathcal{P}_*E)) = N(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\mathcal{C}^1(\mathrm{Nahm}_*(\mathcal{P}_*E)) = N(\mathcal{V})\,d\zeta$. The differential $\mathcal{C}^0 \to \mathcal{C}^1$ is induced by the multiplication of $-w$. We shall rewrite the complex

$$\widetilde{\mathcal{C}}^0(\mathrm{Nahm}_*(\mathcal{P}_*E)) \xrightarrow{\theta + w\,d\zeta} \widetilde{\mathcal{C}}^1(\mathrm{Nahm}_*(\mathcal{P}_*E)). \tag{29}$$

For $I \subset \{1,2,3,4,5\}$, let p_I denote the projection of $T \times \mathbb{P}^1 \times T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. We set

$$C_0 := p_{12}^*(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes p_{13}^*\mathcal{P}oin^\vee \otimes p_{34}^*\mathcal{P}oin \otimes p_5^*\mathcal{O}_{\mathbb{P}^1}(-1),$$
$$C_1 := p_{12}^*\mathcal{V} \otimes p_{13}^*\mathcal{P}oin^\vee \otimes p_{34}^*\mathcal{P}oin.$$

We regard $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$, and let $\iota\colon \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$ be the natural inclusion. Let $G\colon C_0 \to C_1$ be induced by $-p_2^*w \otimes p_5^*\iota + p_2^*\iota \otimes p_5^*w$. Then (29) is naturally isomorphic to

$$R^1 p_{345*}\Bigl(C_0 \xrightarrow{G} C_1\Bigr).$$

For $I \subset \{1,2,3,4\}$ let q_I denote the projection of $T \times T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components $(i \in I)$. The complex

$$p_{1345*}\Bigl(C_0 \xrightarrow{G} C_1\Bigr)$$

is quasi-isomorphic to

$$q_{14}^*\mathcal{V}\otimes q_{12}^*\mathcal{P}oin^\vee\otimes q_{23}^*\mathcal{P}oin[-1].$$

For $I\subset\{1,2,3\}$, let s_I denote the projection of $T\times T\times\mathbb{P}^1$ onto the product of the i^{th} components $(i\in I)$. We have a natural isomorphism

$$q_{134*}\bigl(q_{14}^*\mathcal{V}\otimes q_{12}^*\mathcal{P}oin^\vee\otimes q_{23}^*\mathcal{P}oin[-1]\bigr)\simeq s_{13}^*\mathcal{V}\otimes s_{12}^*\mathcal{O}_\Delta[-2].$$

Here, $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal in $T\times T$. Then we obtain a natural isomorphism $\mathcal{V}\simeq \boldsymbol{N}(\mathrm{Nahm}(\mathcal{P}_*E,\theta))$ as $\mathcal{O}_{T\times\mathbb{P}^1}$–modules. If $(p,m,\boldsymbol{o})\neq(1,0,0)$, then from Proposition 3.7 we obtain the comparison of the filtered bundles over $\mathcal{V}(*(T\times\{\infty\}))^{(p,m)}_{\boldsymbol{P},\boldsymbol{o}}$ The comparison in the case $(p,m,\boldsymbol{o})=(1,0,0)$ follows directly from the construction. □

Corollary 3.26 *Let $\mathcal{P}_*E$ be an admissible filtered bundle on $(T\times\mathbb{P}^1, T\times\{\infty\})$ satisfying the condition (A3). We have*

$$\deg(\mathcal{P}_*E)=\deg(\mathrm{Nahm}(\mathcal{P}_*E)).$$

Proof It follows from Propositions 3.17 and 3.25. □

3.4 Refinement for good filtered Higgs bundles

3.4.1 A stationary phase formula We have the following type of stationary phase formula for the local Nahm transform, which is analogue of the stationary phase formula for the local Fourier transforms. (See [15; 17; 20; 41], Laumon [30] and Malgrange [32].) We will prove it in Section 3.4.4 after the preliminaries in Sections 3.4.2–3.4.3.

Theorem 3.27 *Let U_ζ be a small neighbourhood of 0 in $\mathbb{C}_\zeta$. Let $(\mathcal{P}_*V,\theta)$ be an admissible filtered Higgs bundle on U_ζ.*

- *$(\mathcal{P}_*V,\theta)$ is good if and only if $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ is good.*
- *Suppose $(\mathcal{P}_*V,\theta)\simeq\varphi_{p*}(\mathcal{P}_*V',\theta')$, where $\theta'-d\mathfrak{a}\,\mathrm{id}$ is logarithmic for some $\mathfrak{a}\in\zeta_p^{-1}\mathbb{C}[\zeta_p^{-1}]$ with $\deg_{\zeta_p^{-1}}\mathfrak{a}=m>0$. Then there exists $(\mathcal{P}_*W',\psi')$ on $U_{\tau_{p+m}}$ such that $\psi'-d\mathfrak{b}$ is logarithmic for some $\mathfrak{b}\in\tau_{p+m}^{-1}\mathbb{C}[\tau_{p+m}^{-1}]$ with $\deg_{\tau_{p+m}^{-1}}\mathfrak{b}=m$, and we have an isomorphism*

  $$\varphi_{p+m*}(\mathcal{P}_*W',\psi')\simeq\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta).$$

  Moreover, we have an isomorphism $\mathrm{Gr}^{\mathcal{P}}_c(V')\simeq\mathrm{Gr}^{\mathcal{P}}_{c-m/2}(W')$ under which $\mathrm{Res}(\varphi_p^*\theta)=\mathrm{Res}(\varphi_{p+m}^*\psi')$. (The choice of $\mathfrak{b}$ will be explained in the proof.)

- *If* $(\mathcal{P}_*V,\theta)$ *is logarithmic,* $(\mathcal{P}_*W,\psi):=\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ *is also logarithmic. Moreover, we have an isomorphism*

$$\mathrm{Gr}_c^{\mathcal{P}}(W)\simeq\begin{cases}\mathrm{Gr}_c^{\mathcal{P}}(V) & (-1<c<0),\\ \mathrm{Im}\big(\mathrm{Res}(\theta)\colon \mathrm{Gr}_0^{\mathcal{P}}(V)\to\mathrm{Gr}_0^{\mathcal{P}}(V)\big) & (c=0).\end{cases}$$

Under the isomorphism, we have $\mathrm{Res}(\psi)=\mathrm{Res}(\theta)$.

We obtain the following corollary from Theorem 3.27. (Recall the notion of good filtered bundle in Section 2.4.1.)

Corollary 3.28
- *Let* $(\mathcal{P}_*\mathcal{E},\theta)$ *be a good filtered Higgs bundle on* $(T^\vee,D)$ *satisfying (A0). Then* $\mathrm{Nahm}_*(\mathcal{P}_*\mathcal{E},\theta)$ *is a good filtered bundle on* $(T\times\mathbb{P}^1,T\times\{\infty\})$.
- *Let* $\mathcal{P}_*E$ *be a good filtered bundle on* $(T\times\mathbb{P}^1,T\times\{\infty\})$ *satisfying (A3) with* $\mathcal{S}p_\infty(E)=D$. *Then* $\mathrm{Nahm}_*(\mathcal{P}_*E)$ *is a good filtered Higgs bundle on* $(T^\vee,D)$.

3.4.2 Description of the parabolic structure of $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$ Let $(\mathcal{P}_*V,\theta)$ be a good filtered Higgs bundle on $(U_\zeta,0)$. For simplicity, we assume that $(\mathcal{P}_*V,\theta)$ has slope (p,m). We take $\mathfrak{a}\in\mathfrak{o}$ for each $\mathfrak{o}\in\mathbf{Irr}(\theta)$. Let $c\in\mathbb{R}$. We take a frame $\boldsymbol{v}_{\mathfrak{o}}=(v_{\mathfrak{o},i})$ of $\mathcal{P}_{cp_{\mathfrak{o}}}V_{\mathfrak{a}}^{\mathfrak{o}}$ that is compatible with the parabolic structure. Each $\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i}dz_{\mathfrak{o}}/z_{\mathfrak{o}}$ induces a section $[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i}dz_{\mathfrak{o}}/z_{\mathfrak{o}}]$ of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V,\theta)$. The following lemma is clear by the construction of the filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V,\theta)$. (See the proof of Proposition 3.3.)

Lemma 3.29 *The tuple*

$$\big\{[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i}d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]\;\big|\;\mathfrak{o}\in\mathbf{Irr}(\theta),\ 0\le j<p_{\mathfrak{o}}+m_{\mathfrak{o}},\ 1\le i\le\operatorname{rank}V_{\mathfrak{a}}^{\mathfrak{o}}\big\}$$

is a frame of $\mathcal{N}^{0,\infty}_{\kappa_1(p,m,c)}(\mathcal{P}_*V,\theta)$, *compatible with the parabolic structure. If the parabolic degree of* $v_{\mathfrak{o},i}$ *is* b, *the parabolic degree of* $[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i}d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]$ *is* $(b-j-m_{\mathfrak{o}}/2)(p_{\mathfrak{o}}+m_{\mathfrak{o}})^{-1}$. □

3.4.3 Description of the parabolic structure of $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V,g)$ Let $(\mathcal{P}_*V,g)$ be a filtered bundle with an endomorphism on $(U_\tau,0)$ such that $\mathcal{P}_*V$ with $\psi:=-\tau^{-2}g\,d\tau$ is a good filtered Higgs bundle. For simplicity, we assume that $(\mathcal{P}_*V,g)$ has a slope (p,m) with $p>m\neq0$.

We take $\mathfrak{a}\in\mathfrak{o}$ for each $\mathfrak{o}\in\mathbf{Irr}(\psi)$. Let $c\in\mathbb{R}$. We take a frame $\boldsymbol{v}_{\mathfrak{o}}=(v_{\mathfrak{o},i})$ of $\mathcal{P}_{cp_{\mathfrak{o}}}V_{\mathfrak{a}}^{\mathfrak{o}}$ that is compatible with the parabolic structure. Each $\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}$ induces a section of $\mathcal{N}^{\infty,0}(\mathcal{P}_*V,g)$, denoted by $[\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}]$. The following lemma is clear by the construction of the filtered bundle $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V,g)$. (See the proof of Proposition 3.5.)

Lemma 3.30 *The tuple*

$$\bigl\{[\tau_{\mathfrak{o}}^{j}\, v_{\mathfrak{o},i}] \bigm| \mathfrak{o} \in \mathbf{Irr}(\psi),\ 0 \le j < p_{\mathfrak{o}} - m_{\mathfrak{o}},\ 1 \le i \le \operatorname{rank} V_{\mathfrak{a}}^{\mathfrak{o}}\bigr\}$$

is a frame of $\mathcal{N}^{\infty,0}_{\kappa_2(p,m,c)}(\mathcal{P}_* V, g)$ that is compatible with the parabolic structure. If the parabolic degree of $v_{\mathfrak{o},i}$ is b, the parabolic degree of $\tau_{\mathfrak{o}}^{j} v_{\mathfrak{o},i}$ is $(b - j + p_{\mathfrak{o}} - m_{\mathfrak{o}}/2) \cdot (p_{\mathfrak{o}} - m_{\mathfrak{o}})^{-1}$. □

3.4.4 Proof of Theorem 3.27 Let us return to the situation in Section 3.4.1. Let us begin with the third claim. We obtain the isomorphism of the associated graded vector spaces by the construction of $\mathcal{P}_* W$. We have the expression $\theta = f\, d\zeta/\zeta$, where f is an endomorphism of $\mathcal{P}_* V$. It naturally induces an endomorphism f' of $\mathcal{P}_* W$, and we have $\psi = f'\, d\tau/\tau$ by the construction. Thus, we obtain the third claim.

Let us consider the second claim. Our argument is close to that in [15]. To simplify the notation, we set $\eta := \tau_{p+m}$ and $u := \zeta_p$. We set $G(u) := u\partial_u \mathfrak{a}(u) = \sum_{j=1}^{m} \alpha_j u^{-j}$. Let $\omega := e^{2\pi\sqrt{-1}/(p+m)}$. We have holomorphic functions $u^{(i)}(\eta)$ $(i = 0, \dots, p+m-1)$ on U_η, satisfying $\partial_\eta u^{(i)}(0) = \partial_\eta u^{(0)}(0)\omega^i \neq 0$ and

$$G(u^{(i)}(\eta)) + p u^{(i)}(\eta)^p / \eta^{p+m} = 0.$$

For any $c \in \mathbb{R}$, we consider $\mathcal{P}_{c-m/2}\mathcal{V} := \mathcal{P}_{c-m/2}\varphi_{p+m}^{*}\mathcal{N}^{0,\infty}(\mathcal{P}_* V, \theta)$. We take a frame $\boldsymbol{v}$ of $\mathcal{P}_c V'$ compatible with the parabolic structure. We put $\widetilde{v}_{ij} := (\eta^{-1}u)^i v_j$ $(0 \le i \le p+m-1,\ 1 \le j \le \operatorname{rank} V')$. They induce a frame of $\mathcal{P}_{c-m/2}\mathcal{V}$, which is compatible with the parabolic structure. By the frames, for $c-1 < d \le c$, we obtain an isomorphism

$$\operatorname{Gr}^{\mathcal{P}}_{d-m/2}(\mathcal{V}) \simeq \operatorname{Gr}^{\mathcal{P}}_{d}(V') \otimes \mathbb{C}^{p+m}. \tag{30}$$

The following lemma can be checked by a direct computation.

Lemma 3.31 *$\eta^{-1}u$ gives an endomorphism F of $\mathcal{P}_*\mathcal{V}$. On $\operatorname{Gr}^{\mathcal{P}}_{d-m/2}(\mathcal{V})$, we have*

$$F(\widetilde{v}_{i,j}) = \begin{cases} \widetilde{v}_{i+1,j} & i < p+m-1, \\ -p^{-1}\alpha_m \widetilde{v}_{0,j} & i = p+m-1. \end{cases}$$

The eigenvalues of F on $\operatorname{Gr}^{\mathcal{P}}$ are $\partial_\eta u^{(i)}(0)$ $(i = 0, \dots, p+m-1)$. □

By the lemma, we obtain the decomposition $(\mathcal{P}_*\mathcal{V}, F) = \bigoplus_{j=0}^{p+m-1} (\mathcal{P}_*\mathcal{V}^{(j)}, F^{(j)})$ such that $F^{(j)}_{|0}$ has a unique eigenvalue $\partial_\eta u^{(j)}(0)$. Note that we have a natural isomorphism

$$\mathcal{N}^{0,\infty}(\mathcal{P}_* V, \theta) \simeq \varphi_{p+m*}\bigl(\mathcal{P}_*\mathcal{V}^{(0)}, \zeta(-\tau^{-2} d\tau)\bigr).$$

We also have an isomorphism $\operatorname{Gr}^{\mathcal{P}}_c(V') \simeq \operatorname{Gr}^{\mathcal{P}}_{c-m/2}(\mathcal{V}^{(0)})$.

We have the expression $\theta_{\mathfrak{a}}=(G(u)+f)\,du/u$, where f is an endomorphism of $\mathcal{P}_*V'$. On $\mathcal{P}_{c-m/2}\mathcal{V}$, we have $\eta^m(G(u)+pu^p/\eta^{p+m})=-\eta^m f$. We have the decomposition

$$\eta^m(G(u)+pu^p/\eta^{p+m}) \\ =\big(\eta^{-1}u-\eta^{-1}u^{(0)}(\eta)\big)\times p\prod_{i=1}^{p+m-1}\big(\eta^{-1}u-\eta^{-1}u^{(i)}(\eta)\big)(\eta^{-1}u)^{-m}.$$

Because $\eta^{-1}u-\eta^{-1}u^{(j)}(\eta)$ $(1\leq j<p-m)$ are invertible on $\mathcal{P}_{c-m/2}\mathcal{V}^{(0)}$, we obtain the following on $\mathcal{P}_{c-m/2}\mathcal{V}^{(0)}$:

$$\eta^{-1}u-\eta^{-1}u^{(0)}(\eta)=-p^{-1}\eta^m\cdot f\cdot\prod_{j=1}^{p+m-1}(\eta^{-1}u-\eta^{-1}u^{(j)}(\eta))^{-1}(\eta^{-1}u)^m$$

Let $Q_k(x,y)=\sum_{i+j=k}x^iy^j$. We have

$$\begin{aligned}\zeta/\tau-\frac{u^{(0)}(\eta)^p}{\eta^{p+m}}&=\eta^{-m}\big(\eta^{-1}u-\eta^{-1}u^{(0)}(\eta)\big)\cdot Q_{p-1}\big(\eta^{-1}u,\eta^{-1}u^{(0)}(\eta)\big)\\ &=-f\prod_{j=1}^{p+m-1}\big(\eta^{-1}u-\eta^{-1}u^{(j)}(\eta)\big)^{-1}(\eta^{-1}u)^m p^{-1}\\ &\qquad\cdot Q_{p-1}\big(\eta^{-1}u,\eta^{-1}u^{(0)}(\eta)\big)\end{aligned}\tag{31}$$

Hence, we obtain that $(\zeta/\tau-u^{(0)}(\eta)^p\eta^{-p-m})\mathcal{P}_*\mathcal{V}^{(0)}\subset\mathcal{P}_*\mathcal{V}^{(0)}$. On $\mathrm{Gr}^{\mathcal{P}}_a(\mathcal{V}^{(0)})$, the endomorphisms u/η and $u^{(0)}(\eta)/\eta$ are the multiplication of $\partial_\eta u^{(0)}(0)$. Hence, $(\zeta/\tau-u^{(0)}(\eta)^p\eta^{-p-m})$ acts as $-(p+m)^{-1}f$ on $\mathrm{Gr}^{\mathcal{P}}(\mathcal{V})$. We set $\mathcal{P}_*W':=\mathcal{P}_*\mathcal{V}^{(0)}$ and $\psi':=-\zeta\tau^{-2}d\tau=-(\zeta/\tau)(p+m)d\eta/\eta$. We have $\mathfrak{b}\in\eta^{-1}\mathbb{C}[\eta^{-1}]$ uniquely determined by the condition that $\eta\partial_\eta\mathfrak{b}$ is equal to the polar part of $-(p+m)u^{(0)}(\eta)^p\eta^{-p-m}$. Then $\psi'-d\mathfrak{b}$ is logarithmic. The residue acts as f. Hence, the second claim of Theorem 3.27 follows. It also implies the "only if" part in the first claim.

Let us prove the "if" part of the first claim. We use the inverse transform. Let $(\mathcal{P}_*W,\psi)$ be a good filtered Higgs bundle on $(U_\tau,0)$ which is isomorphic to $\varphi_{p*}(\mathcal{P}_*W',\psi')$, where $\psi'-d\mathfrak{b}\,\mathrm{id}$ is logarithmic for some $\mathfrak{b}\in\tau_p^{-1}\mathbb{C}[\tau_p^{-1}]$ with $\deg_{\tau_p^{-1}}\mathfrak{b}=m<p$. The claim of Theorem 3.27 follows from the next proposition.

Proposition 3.32 *There exists $(\mathcal{P}_*V',\theta')$ on $U_{\zeta_{p-m}}$ such that $\theta'-d\mathfrak{a}\,\mathrm{id}$ is logarithmic for some $\mathfrak{a}\in\zeta_{p-m}^{-1}\mathbb{C}[\zeta_{p-m}^{-1}]$, and we have an isomorphism $\varphi_{p-m*}(\mathcal{P}_*V',\theta')\simeq\mathcal{N}^{\infty,0}_*(\mathcal{P}_*W,\psi)$.*

Proof To simplify the notation, we set $\eta:=\tau_p$ and $u:=\zeta_{p-m}$. We have the expression

$$\psi'=(G(\eta)\,\mathrm{id}+\eta^p f)\varphi_p^*(-\tau^{-2}d\tau),$$

such that $G(\eta)=\sum_{j=1}^{m}\beta_j\eta^{p-j}$ with $\beta_m\neq 0$, and f is an endomorphism of $\mathcal{P}_*W'$. We fix a holomorphic function $\eta^{(0)}(u)$ such that $G(\eta^{(0)}(u))-u^{p-m}=0$ and that $0<C_1\leq|\eta^{(0)}/u|\leq C_2$ for some constants C_i.

We set $\mathcal{P}_{c+p-m/2}\mathcal{V}:=\mathcal{P}_{c+p-m/2}\varphi_{p-m}^*\mathcal{N}^{\infty,0}(\mathcal{P}_*W,\psi)$. Let $\boldsymbol{v}$ be a frame of $\mathcal{P}_cW'$ compatible with the parabolic structure. We set $\widetilde{v}_{ij}=u^{-i}\eta^iv_j$ $(0\leq i\leq p-m-1$, $1\leq j\leq\operatorname{rank}W')$. They induce a frame of $\mathcal{P}_{c+p-m/2}\mathcal{V}$ compatible with the parabolic structure. By using the frame, for any $c-1<d\leq c$, we obtain an isomorphism $\operatorname{Gr}^{\mathcal{P}}_{d+p-m/2}(\mathcal{V})\simeq\operatorname{Gr}^{\mathcal{P}}_d(W')\otimes\mathbb{C}^{p-m}$. The following lemma can be checked directly.

Lemma 3.33 *$u^{-1}\eta$ gives an endomorphism F of $\mathcal{P}_*\mathcal{V}$, preserving the parabolic structure, and the induced endomorphism on $\operatorname{Gr}^{\mathcal{P}}(\mathcal{V})$ is given by*

$$F(\widetilde{v}_{p-m-1,j})=-\beta_m^{-1}\widetilde{v}_{0,j}\quad\text{and}\quad F(\widetilde{v}_{ij})=\widetilde{v}_{i+1,j}\quad(i=0,\dots,p-m-2).$$

The eigenvalues are $\omega^i\partial_u\eta^{(0)}(0)$ $(i=0,\dots,p-m-1)$, where $\omega=e^{2\pi\sqrt{-1}/(p-m)}$.

We obtain the decomposition $(\mathcal{P}_*\mathcal{V},F)=\bigoplus_{i=0}^{p-m-1}(\mathcal{P}_*\mathcal{V}^{(i)},F^{(i)})$ such that $F^{(i)}_{|0}$ has a unique eigenvalue $\omega^i\partial_u\eta^{(0)}(0)$. We have an isomorphism

$$\varphi_{p-m*}(\mathcal{P}_*\mathcal{V}^{(0)},-\tau^{-1}d\zeta)\simeq\mathcal{N}^{\infty,0}_*(\mathcal{P}_*W,\psi).$$

We also have an isomorphism $\operatorname{Gr}^{\mathcal{P}}_{c+p-m/2}(\mathcal{V}^{(0)})\simeq\operatorname{Gr}^{\mathcal{P}}_c(W')$.

We have $G(\eta)-u^{p-m}=-\eta^pf$ on $\mathcal{V}$. Note that

$$u^{-(p-m-1)}\sum_{j=1}^{m}\beta_jQ_{p-j-1}(\eta^{(0)}(u),\eta)$$

is invertible on $\mathcal{P}_{c+p-m/2}\mathcal{V}^{(0)}$. Hence, we obtain the following on $\mathcal{P}_{c+p-m/2}\mathcal{V}^{(0)}$:

$$u^{p-m-1}(\eta^{(0)}(u)-\eta)=\eta^pf\cdot\Bigl(\sum_{j=1}^{m}\beta_jQ_{p-j-1}(\eta^{(0)}(u),\eta)\Bigr)^{-1}u^{p-m-1}.$$

We have

$$u^{p-m}(\eta^{-p}-\eta^{(0)}(u)^{-p})=f\eta^pQ_{p-1}(\eta^{(0)}(u)^{-1},\eta^{-1})\eta^{(0)}(u)^{-1}\eta^{-1}\cdot\Bigl(\sum_{j=1}^{m}\beta_jQ_{p-j-1}(\eta^{(0)}(u),\eta)\Bigr)^{-1}u^{p-m}.\tag{32}$$

Hence, we obtain that $u^{p-m}(\eta^{-p}-\eta^{(0)}(u)^{-p})$ is an endomorphism of $\mathcal{P}_*\mathcal{V}^{(0)}$. We set $\mathcal{P}_*V':=\mathcal{P}_*\mathcal{V}^{(0)}$ and $\theta':=-\tau^{-1}\varphi_{p-m}^*d\zeta=-\eta^{-p}(p-m)u^{p-m}(du/u)$. We have

$\mathfrak{a}\in u^{-1}\mathbb{C}[u^{-1}]$ uniquely determined by the condition $u\partial_u\mathfrak{a}=-\eta^{(0)}(u)^{-p}(p-m)u^{p-m}$. Then $\theta'-d\mathfrak{a}$ is logarithmic. Thus, the proofs of Proposition 3.32 and Theorem 3.27 are finished. $\square$

4 Family of vector bundles on torus with small curvature

4.1 Small perturbation

We use the notation in Section 2.1. We use the metric $dz\,d\bar{z}$ of T. For any finite-dimensional vector space V, let $L_k^p(V)$ be the space of V–valued L_k^p–functions on T, and let $L_k^p(V\otimes\Omega^{i,j})$ be the space of V–valued L_k^p–differential (i,j)–forms. We have the linear map $\fint_T\colon L_k^p(V)\to V$ given by $\fint_T f := |T|^{-1}\int_T f\,|dz\,d\bar{z}|$, where $|T|$ denotes the volume of T. The kernel is denoted by $L_k^p(V)_0$. We have a natural inclusion $V\to L_k^p(V)$ as constant functions. We have the decomposition $L_k^p(V)=L_k^p(V)_0\oplus V$ as topological vector spaces.

Suppose that V is equipped with a Hermitian metric h_V. Set $r:=\dim V$. Let $p\geq 2$. Let $\mathcal{G}_k^p(V)$ be the space of L_{k+2}^p–maps from T to $\mathrm{GL}(V)$. We set

$$\mathfrak{A}_k^p(V) := \bigl\{\bar{\partial}_0 + A \bigm| A\in L_{k+1}^p(\mathrm{End}(V)\otimes\Omega^{0,1})\bigr\},$$

ie the space of $(0,1)$–type differential operators of the product bundle $\underline{V}$ of the form $\bar{\partial}_0+A$ $(A\in L_{k+1}^p(\mathrm{End}(V)\otimes\Omega^{0,1}))$. Here, $\bar{\partial}_0$ is the trivial holomorphic structure of $\underline{V}$. We have the natural right $\mathcal{G}_k^p$–action on $\mathfrak{A}_k^p(V)$ given by

$$g\bullet\bar{\partial} := g^{-1}\circ\bar{\partial}\circ g = \bar{\partial}+g^{-1}\bar{\partial}g.$$

Let Γ be an endomorphism of V. Let $U_1\subset L_{k+2}^p(\mathrm{End}(V))_0$ be a sufficiently small neighbourhood of 0 such that $1+U_1\subset\mathcal{G}_k^p$. Let U_2 be a neighbourhood of 0 in $\mathrm{End}(V)$. We consider the map $\Psi\colon U_1\times U_2\to\mathfrak{A}_k^p(V)$ given by

$$\Psi(a,b) := (1+a)\bullet(\bar{\partial}_0+(\Gamma+b)\,d\bar{z}).$$

We use the norm on $L_{k+2}^p(\mathrm{End}(V))$ such that

$$L_{k+2}^p(\mathrm{End}(V))\simeq L_{k+2}^p(\mathrm{End}(V))_0\oplus\mathrm{End}(V)$$

is an isometry, and the norm on $L_{k+1}^p(\mathrm{End}(V))$ such that

$$L_{k+2}^p(\mathrm{End}(V))\to L_{k+1}^p(\mathrm{End}(V)),\quad A\mapsto\bar{\partial}_0A+\fint_T A$$

is an isometry.

Proposition 4.1 *Fix $\delta > 0$. Suppose that Γ is decomposed as $\Gamma = \Gamma_0 + \Gamma_1$ where the pieces satisfy the following conditions:*

- *Γ_0 is commutative with its adjoint $\Gamma_0^\dagger$, ie it is diagonalizable and the eigenspaces are orthogonal with respect to h_V. Moreover, there exists $\zeta_0 \in \mathbb{C}$ such that $\mathcal{S}p(\Gamma)$ is contained in*

$$K_1(L, \zeta_0, \delta) := \bigl\{\zeta \in \mathbb{C} \bigm| 0 \le \mathrm{Im}(\zeta - \zeta_0) \le (1-\delta)\pi, 0 \le \mathrm{Im}((\zeta - \zeta_0)\overline{\tau}) \le (1-\delta)\pi\bigr\}.$$

- *$|\Gamma_1|_{h_V} \le \delta/100$.*

Then there exist positive constants C_i $(i = 1, 2)$, independently from Γ and ζ_0, such that the following holds:

- *For $\mathcal{B} \in L^p_{k+1}(\mathrm{End}(V) \otimes \Omega^{0,1})$ with $|\mathcal{B}| \le C_1$, there exists a unique $(a, b) \in U_1 \times U_2$ with $|a| + |b| \le C_2|\mathcal{B}|$ satisfying $\overline{\partial}_0 + \Gamma\, d\overline{z} + \mathcal{B}\, d\overline{z} = \Psi(a, b)$.*

Proof We set $K(L) := \{\zeta \in \mathbb{C} \mid |\mathrm{Im}(\zeta)| \le (1-\delta)\pi,\ |\mathrm{Im}(\zeta\overline{\tau})| \le (1-\delta)\pi\}$. We have $\mathcal{S}p(\mathrm{ad}(\Gamma_0)) \subset K(L)$. In the following, C_i will be positive constants which are independent from Γ and ζ_0.

We have a morphism

$$\Phi_\Gamma\colon L^p_{k+2}(\mathrm{End}(V)) = L^p_{k+2}(\mathrm{End}(V))_0 \oplus \mathrm{End}(V) \to L^p_{k+1}(\mathrm{End}(V) \otimes \Omega^{0,1})$$

given by $\Phi_\Gamma(A, B) = \overline{\partial} A + [\Gamma, A]\, d\overline{z} + B\, d\overline{z}$, where $A \in L^p_{k+2}(\mathrm{End}(V))_0$ and $B \in \mathrm{End}(V)$. We have $\Phi_0(A, B) = \overline{\partial} A + B\, d\overline{z}$, which is an isometry by our choice of the norms.

Lemma 4.2 *Φ_Γ is a homeomorphism.*

Proof Note that Φ_0 is an isomorphism, and that $\Phi_\Gamma - \Phi_0$ is compact. Hence, the index of Φ_Γ is 0. Due to the condition for Γ_0, we have $\|\overline{\partial}_0 A + [\Gamma_0, A]\, d\overline{z}\|_{L^2} \ge \delta\pi |A|_{L^2}$ for any $A \in L^2_1(\mathrm{End}(V))$. By the condition for Γ_1, we obtain that $\overline{\partial}_0 A + [\Gamma, A]\, d\overline{z} \neq 0$ for any $A \in L^2_1(\mathrm{End}(V))$. Then we obtain that Φ_Γ is injective. □

Lemma 4.3 *We have $|\Phi_0^{-1} \circ \Phi_\Gamma| \le C_3$ and $|\Phi_\Gamma^{-1} \circ \Phi_0| \le C_3$, independently from Γ, where $|\cdot|$ denotes the operator norm.*

Proof Let $\mathcal{S}$ be the set of Γ satisfying the conditions of the proposition. It is compact. For any fixed $(A, B) \in L^p_{k+2}(\mathrm{End}(V))_0 \oplus \mathrm{End}(V)$, the map $\Gamma \mapsto \Phi_\Gamma^{-1} \circ \Phi_0(A, B)$ gives a continuous map from $\mathcal{S}$ to $L^p_{k+2}(\mathrm{End}(V)) \oplus \mathrm{End}(V)$, and hence is bounded. Then we obtain the claim for $\Phi_\Gamma^{-1} \circ \Phi_0$ by the uniform boundedness principle. We obtain the claim for $\Phi_0^{-1} \circ \Phi_\Gamma$ similarly. □

We set $\mathcal{A}(a,b) := \Psi(a,b) - \Psi(0,0) \in L^p_{k+1}(\mathrm{End}(V) \otimes \Omega^{0,1})$, ie

$$\mathcal{A}(a,b) = (1+a)^{-1}(\overline{\partial}_0 a + [\Gamma, a]) + \mathrm{Ad}(1+a)b\, d\bar{z}.$$

We have $|\mathcal{A}(a,b)| = O(|a|+|b|)$, independently from Γ. The derivative $T_{(a,b)}\Psi$ of Ψ at any $(a,b) \in U_1 \times U_2$ is given by

$$\begin{aligned}(33)\quad T_{(a,b)}\Psi(X,Y) = \Phi(X,Y) &+ [\mathcal{A}(a,b), (1+a)^{-1}X] \\ &- [\Psi(0,0), (1+a)^{-1}aX] + (\mathrm{Ad}(1+a)-1)Y.\end{aligned}$$

Hence, we obtain an estimate $|\Phi_\Gamma^{-1} \circ T_{(a,b)}\Psi - \mathrm{id}\,| \le C_4(|a|+|b|)$, which is independent from Γ. Then the claim of Proposition 4.1 follows from the classical inverse function theorem (see Lang [28], for example). □

Corollary 4.4 Ψ *gives a diffeomorphism of a neighbourhood of* $(0,0)$ *in* $U_1 \times U_2$ *and a neighbourhood of* $\overline{\partial}_0 + \Gamma\, d\bar{z}$ *in* $\mathfrak{A}^p_k(V)$. □

4.2 Frames

4.2.1 Preliminaries We set $U_1 := \{(x_1, x_2) \mid 0 \le x_i \le 1\}$ and $U_2 := \{(\xi_1, \dots, \xi_{n-2}) \mid |\xi_i| \le 1\}$. Let $T_0 = \mathbb{R}^2/\mathbb{Z}^2$. Let $U_1 \times U_2 \to T_0 \times U_2$ denote the natural projection. We also use the variables $t_i = x_i$ $(i = 1, 2)$ and $t_i = \xi_{i-2}$ $(i = 3, \dots, n)$. We also use $x = x_1$, $y = x_2$.

For any nonnegative integer k, we set $S_1(k) := \{(m_1, m_2) \mid m_1 + m_2 = k, m_i \ge 0\}$. We also set $S_2(k) := \{(m_1, \dots, m_{n-2}) \mid \sum m_i = k, m_i \ge 0\}$. We set $S(k_1, k_2) := S_1(k_1) \times S_2(k_2)$. We put $\partial_{\boldsymbol{x}}^{\boldsymbol{m}} := \prod \partial_{x_i}^{m_i}$ and $\partial_{\boldsymbol{\xi}}^{\boldsymbol{m}} := \prod \partial_{\xi_i}^{m_i}$. We put $N_i(k) := |S_i(k)|$ and $N(k_1, k_2) := N_1(k_1) \times N_2(k_2)$.

Let V be a vector space. For $f \in C^\infty(U_1 \times U_2, V)$, we set

$$D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2}(f) := (\partial_{\boldsymbol{x}}^{\boldsymbol{m}_1} \partial_{\boldsymbol{\xi}}^{\boldsymbol{m}_2} f \mid (\boldsymbol{m}_1, \boldsymbol{m}_2) \in S(k_1, k_2)) \in C^\infty(U_1 \times U_2, V^{N(k_1,k_2)}).$$

Formally, we set $D^0 f := f \in C^\infty(U_1 \times U_2, V)$. We use similar notation for the functions on $T_0 \times U_2$ and $[0,1] \times U_2$.

4.2.2 Orthonormal frame Let E be a topologically trivial C^∞ vector bundle on $T_0 \times U_2$ with a Hermitian metric h and a unitary connection ∇. We set $r := \mathrm{rank}\, E$. Let F denote the curvature of ∇. For any frame $\boldsymbol{v}$ of E, let $A^{\boldsymbol{v}} = \sum_{i=1}^n A_i^{\boldsymbol{v}}\, dt_i$ denote the connection form of ∇ with respect to $\boldsymbol{v}$. We put ${}^1A^{\boldsymbol{v}} := A_1^{\boldsymbol{v}}\, dt_1 + A_2^{\boldsymbol{v}}\, dt_2$ and ${}^2A^{\boldsymbol{v}} := \sum_{i=3}^n A_i^{\boldsymbol{v}} dt_i$. Similarly $F^{\boldsymbol{v}} = \sum F_{ij}^{\boldsymbol{v}}\, dt_i\, dt_j$ denote the curvature form with respect to $\boldsymbol{v}$.

Fix a positive number M. Let ϵ be a small positive number. Assume the following:

- Take any $\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}$ such that $m_1 + m_2 \leq M$ and $\sum_{i=3}^n m_i \leq M$. Then $|\nabla_{t_1}^{m_1} \circ \cdots \circ \nabla_{t_n}^{m_n} F|_h \leq \epsilon$.

Lemma 4.5 *If ϵ is sufficiently small, there exist an orthonormal frame $\boldsymbol{v}$ of (E,h) on $T_0 \times U_2$ and anti-Hermitian matrices $\Lambda^{(x)}$, $\Lambda^{(y)}$ such that the following hold:*

(A1) *For $\kappa = x, y$, there exist $0 \leq \theta_\kappa < 2\pi$ such that any eigenvalue $\sqrt{-1}\alpha$ of $\Lambda^{(\kappa)}$ satisfies $|\alpha - \theta_\kappa| \leq \pi(2r-1)/2r$. They satisfy $\|[\Lambda^{(x)}, \Lambda^{(y)}]\| \leq C\epsilon$.*

(A2) $|{}^1A^{\boldsymbol{v}} - \Lambda| \leq C\epsilon$, *and* $|D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2}({}^1A^{\boldsymbol{v}})| \leq C\epsilon$ *for any* $0 \leq k_1 \leq M$ *and* $0 \leq k_2 \leq M$ *with* $(k_1,k_2) \neq (0,0)$, *where* $\Lambda = \Lambda^{(x)}\,dx + \Lambda^{(y)}\,dy$.

(A3) $|D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2}({}^2A^{\boldsymbol{v}})| \leq C\epsilon$ *for any* $0 \leq k_1, k_2 \leq M$.

Here, the constant C may depend only on r and M.

Proof We shall indicate an outline of the construction, although it is elementary. We say that a quantity $\mathcal{P}$ is $O(\epsilon)$ if $\mathcal{P} \leq C\epsilon$ for some constant C which may depend only on r and M. Let $[a,b]_{\mathbb{Z}}$ denote the set of integers k such that $a \leq k \leq b$. For $j \geq 1$, let H_j be the subset of $U_1 \times U_2$ determined by the condition $t_i = 0$ $(i \in [1,j]_{\mathbb{Z}})$. We set $H_0 := U_1 \times U_2$.

Let $\boldsymbol{u}$ be an orthonormal frame of $\pi^*(E,h)$ on $U_1 \times U_2$ satisfying $\nabla_{t_j}\boldsymbol{u} = 0$ on H_{j-1} for any j. We have $A_p^{\boldsymbol{u}} = 0$ on H_{p-1} by the construction. For $j < p$, we have $\partial_{t_j} A_p^{\boldsymbol{u}} = F_{jp}^{\boldsymbol{u}}$ on H_{j-1}.

For $0 \leq k \leq M$ and $j = 0, \dots, n+1$, we consider the following claim:

$Q(j,k)$ Take $\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}$ such that $m_1 + m_2 \leq k$, $\sum_{i=3}^n m_i \leq k$ and $m_i = 0$ $(i < j)$. Put $P := \prod_{i=1}^n \partial_{t_i}^{m_i}$. Then on H_{j-1}, we have $PA_p^{\boldsymbol{u}} = O(\epsilon)$ for $p \geq j$, and $PF_{pq}^{\boldsymbol{u}} = O(\epsilon)$ for $p,q \geq j$.

If $j = n+1$, the claim holds for any k. We shall prove the claims $Q(j,k)$ for any k by assuming $Q(j+1,k)$ for any k.

The claim $Q(j,0)$ holds by the construction of $A_p^{\boldsymbol{u}}$ and the assumption on F. We have only to prove $Q(j,k)$ by assuming $Q(j,k-1)$. For any section s of $\mathrm{End}(E)$, let $s^{\boldsymbol{u}}$ denote the matrix representation of s with respect to $\boldsymbol{u}$. Suppose that $\nabla_{t_1}^{m_1} \circ \cdots \circ \nabla_{t_n}^{m_n} s = O(\epsilon)$ for any $\boldsymbol{m} \in \mathbb{Z}^n$ such that $m_1 + m_2 \leq k$, $\sum_{i=3}^n m_i \leq k$ and $m_i = 0$ $(i < j)$. Because $(\nabla_{t_1}^{m_1} \circ \cdots \circ \nabla_{t_n}^{m_n} s)^{\boldsymbol{u}} = (\partial_{t_1} + A_1^{\boldsymbol{u}})^{m_1} \circ \cdots \circ (\partial_{t_n} + A_n^{\boldsymbol{u}})^{m_n} s^{\boldsymbol{u}}$, we have $\prod \partial_{t_i}^{m_i} s^{\boldsymbol{u}} = O(\epsilon)$ for any such $\boldsymbol{m}$ if $Q(j,k-1)$ holds. In particular, we obtain $\prod \partial_{t_i}^{m_i} F_{p,q}^{\boldsymbol{u}} = O(\epsilon)$ for any $p,q \geq j$ for such $\boldsymbol{m}$. For a monomial P of $\partial_{t_{j+1}}, \dots, \partial_{t_n}$

and for $p > j$, we have $\partial_{t_j}^{\alpha+1} PA_p^{\boldsymbol{u}} = \partial_{t_j}^{\alpha} PF_{jp}^{\boldsymbol{u}}$ on H_{j-1}. Hence, we obtain the desired estimates for $A_p^{\boldsymbol{u}}$ $(p > j)$ from the estimate on H_j.

By considering the case $j = 0$ and $k = M$, we obtain $D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} A_p^{\boldsymbol{u}} = O(\epsilon)$ and $D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} F^{\boldsymbol{u}} = O(\epsilon)$ for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2$.

Let $G^{(x)}$: $H_1 \to U(r)$ be determined by $\boldsymbol{u}_{|(1,y,\boldsymbol{\xi})} = \boldsymbol{u}_{|(0,y,\boldsymbol{\xi})} G^{(x)}(y, \boldsymbol{\xi})$, where $U(r)$ denotes the r^{th} unitary group. By the equation

$$\partial_{t_i} G^{(x)}(t_2, \dots, t_n) - G^{(x)}(t_2, \dots, t_n) A^{\boldsymbol{u}}_{i|(1,t_2,\dots,t_n)} + A^{\boldsymbol{u}}_{i|(0,t_2,\dots,t_n)} G^{(x)}(t_2, \dots, t_n) = 0,$$

we obtain the equality $|D_{\boldsymbol{x}}^1 G^{(x)}| + |D_{\boldsymbol{\xi}}^1 G^{(x)}| = O(\epsilon)$. By an easy induction, we obtain the equality

$$|D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} G^{(x)}| = O(\epsilon)$$

for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$. We also have $|G^{(x)}(y, \boldsymbol{\xi}) - G^{(x)}(y', \boldsymbol{\xi}')| = O(\epsilon)$.

Let $G^{(y)}(x, \boldsymbol{\xi})$ be determined by $\boldsymbol{u}_{|(x,1,\boldsymbol{\xi})} = \boldsymbol{u}_{|(x,0,\boldsymbol{\xi})} G^{(y)}(x, \boldsymbol{\xi})$. Similarly, we have

$$|D_{\boldsymbol{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} G^{(y)}| = O(\epsilon)$$

for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$, and $|G^{(y)}(x, \boldsymbol{\xi}) - G^{(y)}(x', \boldsymbol{\xi}')| = O(\epsilon)$.

Since $G^{(y)}(0, \boldsymbol{\xi}) G^{(x)}(1, \boldsymbol{\xi}) = G^{(x)}(0, \boldsymbol{\xi}) G^{(y)}(1, \boldsymbol{\xi})$, we have

$$[G^{(y)}(0, 0), G^{(x)}(0, 0)] = O(\epsilon).$$

We set $\widetilde{G}^{(y)} := G^{(y)}(0, 0)$ and $\widetilde{G}^{(x)} := G^{(x)}(0, 0)$.

Let $\mathcal{I}_\kappa$ denote the set of the eigenvalues of $\widetilde{G}^{(\kappa)}$ for $\kappa = x, y$. Let d_{S^1} denote the standard distance on

$$S^1 = \{e^{\sqrt{-1}\theta} \mid \theta \in \mathbb{R}\}$$

which is induced by the metric $d\theta\, d\theta$. There exist $\gamma_\kappa \in S^1$ such that $d_{S^1}(\gamma_\kappa, \gamma) \geq \pi/(2r)$ for any $\gamma \in \mathcal{I}_\kappa$. Let θ_κ be determined by

$$e^{\sqrt{-1}\theta_\kappa} = -\gamma_\kappa \quad \text{and} \quad 0 \leq \theta_\kappa < 2\pi.$$

For any $\gamma \in \mathcal{I}_\kappa$, we can take $\alpha(\gamma)$ satisfying $e^{\sqrt{-1}\alpha(\gamma)} = \gamma$ and $|\theta_\kappa - \alpha(\gamma)| \leq \pi(2r-1)/2r$. We remark that, for any $\gamma_i, \gamma_j \in \mathcal{I}_\kappa$, we have

$$|\alpha(\gamma_i) - \alpha(\gamma_j)| = O(|\gamma_i - \gamma_j|). \tag{34}$$

We have the eigendecompositions $\mathbb{C}^r = \bigoplus_{\gamma \in \mathcal{I}_\kappa} V_\gamma^{(\kappa)}$ for $\widetilde{G}^{(\kappa)}$. We set $\Lambda^{(\kappa)} = \bigoplus_{\gamma \in \mathcal{I}_\kappa} \sqrt{-1}\alpha(\gamma_i)\, \mathrm{id}_{V_\gamma^{(\kappa)}}$. By construction, we have $\exp(\Lambda^{(\kappa)}) = \widetilde{G}^{(\kappa)}$.

Lemma 4.6 *We have* $[\Lambda^{(x)},\Lambda^{(y)}]=O(\epsilon)$.

Proof According to the decomposition $\mathbb{C}^r=\bigoplus_{\gamma\in\mathcal{I}_x}V_\gamma^{(x)}$, we have the decomposition

$$\widetilde{G}^{(y)}=\sum_{\gamma_i,\gamma_j\in\mathcal{I}_x}\widetilde{G}^{(y)}_{\gamma_i,\gamma_j},$$

where $\widetilde{G}^{(y)}_{\gamma_i,\gamma_j}\in\mathrm{Hom}(V^{(x)}_{\gamma_j},V^{(x)}_{\gamma_i})$. Since $[\widetilde{G}^{(x)},\widetilde{G}^{(y)}]=O(\epsilon)$, $(\gamma_i-\gamma_j)\widetilde{G}^{(y)}_{\gamma_i,\gamma_j}=O(\epsilon)$. By using (34), we obtain

$$[\widetilde{G}^{(y)},\Lambda^{(x)}]=O(\epsilon).$$

By using a similar consideration again, we obtain $[\Lambda^{(y)},\Lambda^{(x)}]=O(\epsilon)$. □

Let us return to the proof of Lemma 4.5. We put $g^{(x)}(x):=\exp(-x\Lambda^{(x)})$, $g^{(y)}(y):=\exp(-y\Lambda^{(y)})$, and $g(x,y):=g^{(x)}(x)g^{(y)}(y)$. We obtain an orthonormal frame $\boldsymbol{u}':=\boldsymbol{u}g(x,y)$ of $\pi^*(E,h)$. Let $A':=A^{\boldsymbol{u}'}$. We have $|A'-\Lambda|=O(\epsilon)$ and $|D_{\boldsymbol{x}}^{k_1}D_{\boldsymbol{\xi}}^{k_2}A'|=O(\epsilon)$ for any $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}\setminus\{(0,0)\}$.

Let $G'^{(x)}(y,\boldsymbol{\xi})$ and $G'^{(y)}(x,\boldsymbol{\xi})$ be determined by

$$\boldsymbol{u}'_{|(1,y,\boldsymbol{\xi})}=\boldsymbol{u}'_{|(0,y,\boldsymbol{\xi})}G'^{(x)}(y,\boldsymbol{\xi}),\quad \boldsymbol{u}'_{|(x,1,\boldsymbol{\xi})}=\boldsymbol{u}'_{|(x,0,\boldsymbol{\xi})}G'^{(y)}(x,\boldsymbol{\xi}).$$

We have $G'^{(x)}(y,\boldsymbol{\xi})=g^{(y)}(y)^{-1}G^{(x)}(y,\boldsymbol{\xi})(\widetilde{G}^{(x)})^{-1}g^{(y)}(y)$ and hence $|G'^{(x)}-1|=O(\epsilon)$. We have

$$dG'^{(x)}=g^{(y)}(y)^{-1}\,dG^{(x)}(y,\boldsymbol{\xi})(\widetilde{G}^{(x)})^{-1}g^{(y)}(y)-[g^{(y)}(y)^{-1}dg^{(y)}(y),(G'^{(x)}-1)].$$

Hence, we have $|D^1_yG'^{(x)}|=O(\epsilon)$ and $|D^1_{\boldsymbol{\xi}}G'^{(x)}|=O(\epsilon)$. By an easy induction, we obtain the equality

$$|D_y^{k_1}D_{\boldsymbol{\xi}}^{k_2}G'^{(x)}|=O(\epsilon)$$

for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}\setminus\{(0,0)\}$. We have

$$G'^{(y)}=g^{(x)}(x)^{-1}G^{(y)}(x,\boldsymbol{\xi})g^{(x)}(x)(\widetilde{G}^{(y)})^{-1}$$

and obtain

$$\begin{aligned}(35)\quad G'^{(y)}-1&=g^{(x)}(x)^{-1}(G^{(y)}(x,\boldsymbol{\xi})(\widetilde{G}^{(y)})^{-1}-1)g^{(x)}(x)\\&\quad-g^{(x)}(x)^{-1}G^{(y)}(x,\boldsymbol{\xi})[(\widetilde{G}^{(y)})^{-1},g^{(x)}(x)]=O(\epsilon).\end{aligned}$$

As in the case $\kappa=x$, we also obtain $|D_y^{k_1}D_{\boldsymbol{\xi}}^{k_2}G'^{(x)}|=O(\epsilon)$ for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}\setminus\{(0,0)\}$.

Let $\chi(x)$ be a nonnegative valued C^∞ function on $[0,1]$ such that $\chi(x)=0$ $(x\le 1/3)$ and $\chi(x)=1$ $(x\ge 2/3)$. We put

$$h_2(x,y,\boldsymbol{\xi}):=\chi(x)\exp^{-1}(G'^{(x)}(y,\boldsymbol{\xi})).$$

By construction, we have that $|D_{\boldsymbol{x}}^{k_1}D_{\boldsymbol{\xi}}^{k_2}h_2|=O(\epsilon)$ for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}$.

Let $g_2:=\exp(h_2)$, and we set $\boldsymbol{u}'':=\boldsymbol{u}'g_2$. Let $A''=A^{\boldsymbol{u}''}$. We have

$$A''=g_2^{-1}A'g_2+g_2^{-1}dg_2.$$

Hence, we have $|{}^1A''-\Lambda|=O(\epsilon)$, and $|D_{\boldsymbol{x}}^{k_1}D_{\boldsymbol{\xi}}^{k_2}({}^1A'')|=O(\epsilon)$ for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}\setminus\{(0,0)\}$. We also have $|D_{\boldsymbol{x}}^{k_1}D_{\boldsymbol{\xi}}^{k_2}({}^2A'')|=O(\epsilon)$ for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}$.

We put $G''^{(y)}(x,\boldsymbol{\xi}):=g_2(x,0,\boldsymbol{\xi})^{-1}G'^{(y)}(x,\boldsymbol{\xi})g_2(x,1,\boldsymbol{\xi})$, and then we have

$$\boldsymbol{u}''_{|(x,1,\boldsymbol{\xi})}=\boldsymbol{u}''_{|(x,0,\boldsymbol{\xi})}G''^{(y)}(x,\boldsymbol{\xi}).$$

We have $|G''^{(y)}(x,\boldsymbol{\xi})-1|=O(\epsilon)$, and $|D_{\boldsymbol{x}}^{k_1}D_{\boldsymbol{\xi}}^{k_2}G''^{(y)}|=O(\epsilon)$ for $(k_1,k_2)\in[0,M]^2_{\mathbb{Z}}\setminus\{(0,0)\}$.

We put $g_3:=\exp(\chi(y)\exp^{-1}(G''^{(y)}(x,\boldsymbol{\xi})))$, and $\boldsymbol{v}:=\boldsymbol{u}''g_3$. Then it naturally gives an orthonormal frame of (E,h) on $T_0\times U_2$. By construction, we have the desired estimate for the connection form $A^{\boldsymbol{v}}$. Thus, the proof of Lemma 4.5 is finished. □

4.2.3 Partially almost holomorphic frame We identify the C^∞ manifolds $T_0:=\mathbb{R}^2/\mathbb{Z}^2$ and T by the diffeomorphism $T_0\simeq T$ given by $(x,y)\mapsto x+\tau y=z$. We have the description $\Lambda=\Gamma\,d\bar{z}-{}^t\overline{\Gamma}\,dz$, where Λ is as in Lemma 4.5. Let $\nabla_{\bar{z}}:=\nabla(\partial_{\bar{z}})$ and $\nabla_z:=\nabla(\partial_z)$. For any frame $\boldsymbol{w}$, let $A_z^{\boldsymbol{w}}$ and $A_{\bar{z}}^{\boldsymbol{w}}$ be determined by $\nabla_z\boldsymbol{w}=\boldsymbol{w}A_z^{\boldsymbol{w}}$ and $\nabla_{\bar{z}}\boldsymbol{w}=\boldsymbol{w}A_{\bar{z}}^{\boldsymbol{w}}$, respectively. Let $H(h,\boldsymbol{w})$ denote the function from $T\times U_2$ to the space of r^{th} positive-definite Hermitian matrices, whose (i,j)–entries are $h(w_i,w_j)$. When a function f on $T\times U_2$ is regarded as a function $\widetilde{f}\colon U_2\to L_k^p(T)$, we obtain an $\mathbb{R}_{\ge 0}$–valued function $\|f\|_{L_k^p}(\boldsymbol{\xi}):=\|\widetilde{f}(\boldsymbol{\xi})\|_{L_k^p(T)}$ on U_2.

Proposition 4.7 *If $\epsilon>0$ is sufficiently small, there exists a frame $\boldsymbol{u}$ of E on $T\times U_2$ with the following properties:*

- *$A_{\bar{z}}^{\boldsymbol{u}}$ is constant along the T–direction, and $|A_{\bar{z}}^{\boldsymbol{u}}-\Gamma|=O(\epsilon)$.*
- *$\|A_z^{\boldsymbol{u}}+{}^t\overline{\Gamma}\|_{L_M^p}=O(\epsilon)$ and $\|D_{\boldsymbol{\xi}}^kA_z^{\boldsymbol{u}}\|_{L_M^p}=O(\epsilon)$ for $k\in[1,M]_{\mathbb{Z}}$.*
- *$\|D_{\boldsymbol{\xi}}^k({}^2A^{\boldsymbol{u}})\|_{L_M^p}=O(\epsilon)$ for $k\in[0,M]_{\mathbb{Z}}$.*

Moreover, $\|H(h,\boldsymbol{u})-I\|_{L_{M+1}^p}=O(\epsilon)$ and $\|D_{\boldsymbol{\xi}}^kH(h,\boldsymbol{u})\|_{L_{M+1}^p}=O(\epsilon)$ for $k\in[1,M]_{\mathbb{Z}}$, where I denotes the identity matrix.

Proof Let $\boldsymbol{v}$ be the orthonormal frame as in Lemma 4.5. We have $\nabla_{\bar{z}}\boldsymbol{v} = \boldsymbol{v}(\Gamma + N)$, where $\|D_{\boldsymbol{\xi}}^k N\|_{L^p_M} = O(\epsilon)$.

Lemma 4.8 *We have a decomposition $\Gamma = \Gamma_0 + \Gamma_1$ such that:*

(i) $[\Gamma_0, \Gamma_0^\dagger] = 0$ *and* $\mathcal{S}p(\Gamma_0) = \mathcal{S}p(\Gamma)$.

(ii) $|\Gamma_1| = O(\epsilon^{1/2})$.

Moreover, if $\delta > 0$ is sufficiently much smaller than $1/r$, then there exists $\zeta_0 \in \mathbb{C}$ such that $\mathcal{S}p(\Gamma)$ is contained in $K_1(L, \zeta_0, \delta)$. (See Proposition 4.1 for $K_1(L, \zeta_0, \delta)$.) We may take δ independently from any sufficiently small ϵ. We also have $|\operatorname{Im}\zeta_0| \le \pi$ and $|\operatorname{Im}(\bar{\tau}\zeta_0)| \le \pi$.

Proof We give only an indication of the proof. With an appropriate change of orthonormal basis, we may assume that Γ is upper triangular. By the basis, we identify matrices and endomorphisms. Let Γ_0 be the diagonal part, and we put $\Gamma_1 := \Gamma - \Gamma_0$. By construction, condition (i) is satisfied. Let γ_{ij} denote the (i,j)–entry of Γ. Then the (k,k)–entries of $[\Gamma, \Gamma^\dagger]$ are $\sum_{i>k} |\gamma_{k,i}|^2 - \sum_{i<k} |\gamma_{k,i}|^2$. Then we obtain the desired estimate for Γ_1 from $[\Gamma, \Gamma^\dagger] = O(\epsilon)$, which follows from condition (A1) in Lemma 4.5. Thus, we obtain the first condition.

Let us prove the second condition. Let us observe that there exist decompositions

$$\Lambda^{(\kappa)} = \Lambda_0^{(\kappa)} + \Lambda_1^{(\kappa)} \quad (\kappa = x, y),$$

such that $\Lambda_i^{(\kappa)}$ are anti-Hermitian, $[\Lambda_0^{(x)}, \Lambda_0^{(y)}] = 0$, and $\Lambda_1^{(\kappa)} = O(\epsilon^{1/2})$. We have the eigendecomposition

$$\mathbb{C}^r = \bigoplus_{i \in I} V_i$$

of $\Lambda^{(x)}$, where I denotes the set of eigenvalues of $\Lambda^{(x)}$. We have a decomposition $I = \coprod_{k \in S} I_k$ such that if $\alpha, \beta \in I_k$ then $|\alpha - \beta| \le r^{-1}\pi\epsilon^{1/2}/10$, and if $\alpha \in I_k$ and $\beta \in I_\ell$ with $k \neq \ell$ then $|\alpha - \beta| \ge r^{-2}\pi\epsilon^{1/2}/10$. We set $V_k' := \bigoplus_{i \in I_k} V_i$. We choose $\beta_k \in I_k$ for each $k \in S$, and put

$$\Lambda_0^{(x)} := \bigoplus_{k \in S} \beta_k \operatorname{id}_{V_k'}.$$

We have the decomposition $\Lambda^{(y)} = \sum \Lambda_{k\ell}^{(y)}$ according to $\mathbb{C}^r = \bigoplus V_k'$. Since

$$[\Lambda^{(x)}, \Lambda^{(y)}] = O(\epsilon),$$

we have $\Lambda^{(y)}_{k\ell} = O(\epsilon^{1/2})$ if $k \neq \ell$. We set

$$\Lambda^{(y)}_0 := \sum_k \Lambda^{(y)}_{kk} \quad \text{and} \quad \Lambda^{(\kappa)}_1 := \Lambda^{(\kappa)} - \Lambda^{(\kappa)}_0.$$

Then the decompositions $\Lambda^{(\kappa)} = \Lambda^{(\kappa)}_0 + \Lambda^{(\kappa)}_1$ have the desired properties. We have $\Gamma = (\tau - \overline{\tau})^{-1}(\tau\Lambda^{(x)} - \Lambda^{(y)})$. Because the eigenvalues of Γ are close to those of $(\tau - \overline{\tau})^{-1}(\tau\Lambda^{(x)}_0 - \Lambda^{(y)}_0)$ on the order of $O(\epsilon^{1/2})$, the second condition is satisfied for small $\delta > 0$. $\square$

By Proposition 4.1, if ϵ is sufficiently small, we have that there exist functions $a\colon U_2 \to L^p_{M+1}(M_r(\mathbb{C}))_0$ and $b\colon U_2 \to M_r(\mathbb{C})$ satisfying the following:

- $\|D^k_{\boldsymbol{\xi}} a\|_{L^p_{M+1}} = O(\epsilon)$ for $k \in [0, M]_{\mathbb{Z}}$, and $|D^k_{\boldsymbol{\xi}} b| = O(\epsilon)$ for $k \in [0, M]_{\mathbb{Z}}$.
- $(1 + a) \bullet (\nabla_{\bar{z},0} + (\Gamma + b)\, d\bar{z}) = \nabla_{\bar{z}}$, where $\nabla_{\bar{z},0}$ is given by $\nabla_{\bar{z},0}\boldsymbol{v} = 0$.

Let $\boldsymbol{u} := \boldsymbol{v}(1 + a)$. By construction, we have $\nabla_{\bar{z}}\boldsymbol{u} = \boldsymbol{u}(\Gamma + b)$. The other estimates for $A^{\boldsymbol{u}}_{\bar{z}}$ and ${}^2A^{\boldsymbol{u}}$ are also satisfied. Because $H(h, \boldsymbol{u}) = {}^t(1 + a)\overline{(1 + a)}$, we obtain the estimate for $H(h, \boldsymbol{u})$. $\square$

Remark 4.9 If $A^{\boldsymbol{w}}_{\bar{z}}$ is constant along the T–direction, such a frame $\boldsymbol{w}$ is called a partially almost holomorphic frame, in this paper.

4.2.4 Spectra Let $E_{\boldsymbol{\xi}}$ denote the holomorphic bundle on T which is given by $E_{|T\times\boldsymbol{\xi}}$ with $\nabla_{\bar{z}|T\times\boldsymbol{\xi}}$. According to Lemma 2.7, if ϵ is sufficiently small, $E_{\boldsymbol{\xi}}$ are semistable of degree 0 for any $\boldsymbol{\xi} \in U_2$. We have the spectrum $\mathcal{S}p(E_{\boldsymbol{\xi}}) \subset T^\vee$. We regard it as a point in $\operatorname{Sym}^r T^\vee$. The point is denoted by $[\mathcal{S}p(E_{\boldsymbol{\xi}})]$. Let Γ be as in Section 4.2.3. The eigenvalues of Γ give a point in $\operatorname{Sym}^r \mathbb{C}$, denoted by $[\mathcal{S}p(\Gamma)]$. The quotient map $\Phi\colon \mathbb{C} \to T^\vee$ induces $\operatorname{Sym}^r \mathbb{C} \to \operatorname{Sym}^r T^\vee$, denoted by Φ. Recall that $\operatorname{Sym}^r T^\vee$ is naturally a smooth complex manifold. Let $d_{\operatorname{Sym}^r T^\vee}$ be a distance induced by a C^∞ Riemannian metric.

Corollary 4.10 *There exist $\epsilon_0 > 0$ and $C > 0$, depending only on r, such that the following holds if $\epsilon \le \epsilon_0$:*

$$d_{\operatorname{Sym}^r T^\vee}([\mathcal{S}p(E_{\boldsymbol{\xi}})], \Phi[\mathcal{S}p(\Gamma)]) \le C\epsilon.$$

In particular, for $\boldsymbol{\xi}, \boldsymbol{\xi}' \in U_2$, we have $d_{\operatorname{Sym}^r T^\vee}([\mathcal{S}p(E_{\boldsymbol{\xi}})], [\mathcal{S}p(E_{\boldsymbol{\xi}'})]) \le 2C\epsilon$.

Proof Let $\boldsymbol{u}$ be a frame as in Proposition 4.7. Recall that $\mathrm{Sym}^r\,\mathbb{C}$ is naturally a complex manifold. We take a distance $d_{\mathrm{Sym}^r\,\mathbb{C}}$ induced by a C^∞ Riemannian metric. We have $d_{\mathrm{Sym}^r\,\mathbb{C}}([\mathcal{S}p(\Gamma)],[\mathcal{S}p(A^{\boldsymbol{u}}_{\bar{z}})])\leq C_1\epsilon$. There exists $\zeta_0\in\mathbb{C}$ and $\delta>0$ such that $\mathcal{S}p(\Gamma)$ and $\mathcal{S}p(A^{\boldsymbol{u}}_{\bar{z}})$ are contained in $K_1(L,\zeta_0,\delta)$, and $|\operatorname{Im}\zeta_0|\leq\pi$ and $|\operatorname{Im}(\bar{\tau}\zeta_0)|\leq\pi$. Note that the restriction of Φ to $\mathrm{Sym}^r\,K_1(L,\zeta_0,\delta)$ is Lipschitz continuous, and the Lipschitz constant is uniform for ζ_0. Then the claim of the corollary follows. □

4.3 Estimates

4.3.1 Preliminaries We continue to use the setting in Section 4.2. We impose additional assumptions.

Assumption 4.11 We take ϵ to be sufficiently small so that $E_{\boldsymbol{\xi}}$ is semistable of degree 0 for any $\boldsymbol{\xi}\in U_2$. Moreover, we are given a finite subset $Z\subset\mathbb{C}$ and a positive number $\rho>0$ with the following properties:

- Z is contained in $K_1(L,\zeta_0,\delta)$ for some appropriate ζ_0 and $\delta>0$, where $K_1(L,\zeta_0,\delta)$ is as in Section 4.1.
- For any distinct points $\nu_1,\nu_2\in Z$, $d_{\mathbb{C}}(\nu_1,\nu_2)>100r^2\rho$.
- For any $\kappa\in\mathcal{S}p(E_{\boldsymbol{\xi}})$, there exists $\nu\in Z$ such that $d_{T^\vee}(\Phi(\nu),\kappa)<\rho$, where $\Phi\colon\mathbb{C}\to T^\vee$ denotes the projection.

We also assume that ϵ is sufficiently smaller than ρ^2. □

We have the spectral decomposition $E_{\boldsymbol{\xi}}=\bigoplus_{\nu'\in T^\vee}E_{\boldsymbol{\xi},\nu'}$. Let $E_{\nu,\boldsymbol{\xi}}$ be the direct sum of $E_{\boldsymbol{\xi},\nu'}$, where ν' is contained in a ρ–ball of $\Phi(\nu)$. We obtain a decomposition $E_{\boldsymbol{\xi}}=\bigoplus_{\nu\in Z}E_{\nu,\boldsymbol{\xi}}$. It induces a C^∞ decomposition $E=\bigoplus_{\nu\in Z}E_\nu$, which is compatible with $\nabla_{\bar{z}}$. We may assume that the partially almost holomorphic frame $\boldsymbol{u}$ in Proposition 4.7 is compatible with the decomposition.

We have the decomposition $\nabla_{\bar{z}}=\nabla_{\bar{z},0}+f$ such that $(E,\nabla_{\bar{z},0})_{|T\times\{\boldsymbol{\xi}\}}$ are holomorphically trivial for any $\boldsymbol{\xi}\in U_2$, $\nabla_{\bar{z}}(f)=0$, and $\mathcal{S}p(f)$ is contained in the union of the ρ–balls around $\nu\in Z$. For each $\boldsymbol{\xi}\in U_2$, we obtain the vector space $\mathcal{V}_{\boldsymbol{\xi}}$ of the holomorphic global sections of $(E,\nabla_{\bar{z},0})_{|T\times\{\boldsymbol{\xi}\}}$. It is easy to see that $\mathcal{V}_{\boldsymbol{\xi}}$ ($\boldsymbol{\xi}\in U_2$) naturally gives a C^∞ vector bundle $\mathcal{V}$ on U_2, and that we have a natural isomorphism $p^*\mathcal{V}\simeq E$ as C^∞ bundles. We identify them by the isomorphism. A C^∞ section s of $p^*\mathcal{V}$ is constant along the T–direction if and only if $\nabla_{\bar{z},0}s=0$ under the identification. It can be regarded as a section of $\mathcal{V}$. We have the decomposition $\mathcal{V}=\bigoplus_{\nu\in Z}\mathcal{V}_z$, corresponding to $E=\bigoplus_{\nu\in Z}E_\nu$.

4.3.2 Spaces of functions Let $C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}$ denote the space of C^M–functions

$$U_2 \to L^p_M(T).$$

We use $C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(E)$ denote the sections $f = \sum f_i u_i$ of E such that $f_i \in C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}$, where $\boldsymbol{u} = (u_i)$ is a frame as in Proposition 4.7. It is independent of the choice of $\boldsymbol{u}$. We have the naturally defined integration $\int_T\colon C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(E) \to C^M(U_2, \mathcal{V})$. The kernel is denoted by $C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(E)_0$. Similar spaces are defined for $\mathrm{End}(E)$ and $\mathrm{Hom}(E_i, E_j)$. We set

$$C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))^\circ := \bigoplus_\nu C^M(U_2, \mathrm{End}(\mathcal{V}_\nu)),$$
$$C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))^\perp := \bigoplus_\nu C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E_\nu))_0 \oplus \bigoplus_{\nu \neq \mu} C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{Hom}(E_\nu, E_\mu)).$$

We have a decomposition

$$C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E)) = C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))^\circ \oplus C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))^\perp.$$

For any $s \in C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))$, the corresponding decomposition is denoted by $s = s^\circ + s^\perp$. Any $s \in C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}(\mathrm{End}(E))$ is represented as a matrix valued function $\boldsymbol{s}$ with respect to $\boldsymbol{u}$. We have the decomposition $\boldsymbol{s} = \boldsymbol{s}^\circ + \boldsymbol{s}^\perp$ according to $s = s^\circ + s^\perp$. We use similar notation for sections of $\mathrm{End}(E) \otimes \Omega^{i,j}_T$.

4.3.3 Some estimates Let $\boldsymbol{u}$ be a frame as in Proposition 4.7. We set $H(h, \boldsymbol{u})_{i,j} := h(u_i, u_j)$, and we obtain a function $H(h, \boldsymbol{u})$ from $T \times U_2$ to the space $\mathcal{H}$ of positive definite Hermitian r^{th} matrices. Each entry is of class $C^M_{\boldsymbol{\xi}} L^p_{M,\boldsymbol{x}}$. Let H_1 be a function of U_2 to $\mathcal{H}$ determined by

$$\overline{(H_1)^{-2}} = \int_T H(h, \boldsymbol{u}).$$

Then we have $|H_1 - I| = O(\epsilon)$ and $|D^k_{\boldsymbol{\xi}} H_1| = O(\epsilon)$ for $k \in [1, M]_{\mathbb{Z}}$. Note that $\boldsymbol{u}' := \boldsymbol{u} H_1$ also has the property in Proposition 4.7. So, we may assume that $\int_T H(h, \boldsymbol{u}) = I$ from the beginning.

We set $\widetilde{g} := \overline{H(h, \boldsymbol{u})}$. We have $\|\widetilde{g} - I\|_{L^p_{M+1}} = O(\epsilon)$, $\|D^k_{\boldsymbol{\xi}} \widetilde{g}\|_{L^p_{M+1}} = O(\epsilon)$ ($k \in [1, M]_{\mathbb{Z}}$), and $\int_T \widetilde{g} = I$.

Lemma 4.12 *There exist $C > 0$ and $\epsilon_0 > 0$ such that*

$$\|\widetilde{g} - I\|_{L^p_{M+2}} \leq C \|F^\perp_{z\bar{z}}\|_{L^p_M}$$

holds for any $0 < \epsilon < \epsilon_0$. In particular, $\sup_{T \times \{\boldsymbol{\xi}\}} |\widetilde{g} - I| \leq C' \|F^\perp_{z\bar{z}}\|_{L^2}$ for some $C' > 0$.

Proof We put $B := A^{\boldsymbol{u}}_{\bar{z}}$. Let Γ_2 be the diagonal matrix whose (i,i)–entry ν_i is determined by $u_i \in E_{\nu_i}$. Let Γ be as in Section 4.2.3, which is decomposed $\Gamma = \Gamma_0 + \Gamma_1$ as in Lemma 4.8. We have $|\Gamma_0 - \Gamma_2| \le r\rho$ and $|\Gamma_1| = O(\epsilon^{1/2})$. We have $|A^{\boldsymbol{u}}_{\bar{z}} - \Gamma| = O(\epsilon)$. Hence, if ϵ is sufficiently small, we may have $|A^{\boldsymbol{u}}_{\bar{z}} - \Gamma_2| \le 2r\rho$.

We have $A^{\boldsymbol{u}}_z = -\widetilde{g}^{-1}({}^t\overline{B})\widetilde{g} + \widetilde{g}^{-1}\partial_z\widetilde{g}$. Let $\mathcal{B}_{z\bar{z}}$ be the matrix-valued function determined by $F_{z\bar{z}}\boldsymbol{u} = \boldsymbol{u}\mathcal{B}_{z\bar{z}}$. We have $\mathcal{B}_{z\bar{z}} = \partial_z A^{\boldsymbol{u}}_{\bar{z}} - \partial_{\bar{z}} A^{\boldsymbol{u}}_z + [A^{\boldsymbol{u}}_z, A^{\boldsymbol{u}}_{\bar{z}}]$. Hence, we have

$$\begin{aligned}(36)\quad \mathcal{B}_{z\bar{z}} = [\widetilde{g}^{-1\,t}\overline{B}\widetilde{g}, \widetilde{g}^{-1}\partial_{\bar{z}}\widetilde{g}] - \widetilde{g}^{-1}\partial_{\bar{z}}\partial_z(\widetilde{g}) + (\widetilde{g}^{-1}\partial_{\bar{z}}\widetilde{g})(\widetilde{g}^{-1}\partial_z\widetilde{g})\\ -[\widetilde{g}^{-1\,t}\overline{B}\widetilde{g}, B] - [B, \widetilde{g}^{-1}\partial_z\widetilde{g}].\end{aligned}$$

Let $b := \widetilde{g} - I$. We have a polynomial $Q(t_1,t_2,t_3,t_4,t_5,t_6) = \sum Q_{j_1,\dots,j_m} t_{j_1}t_{j_2}\cdots t_{j_m}$ in noncommutative variables t_i such that if $Q_{j_1,\dots,j_m} \neq 0$ then $m_1 + m_2 + m_3 \ge 2$, where $m_i = \{k \mid j_k = i\}$, and we have

$$\begin{aligned}(37)\quad (\partial_{\bar{z}} + \mathrm{ad}(B)) \circ (\partial_z - \mathrm{ad}({}^t\overline{B}))b = -\widetilde{g}\mathcal{B}_{z\bar{z}} - [{}^t\overline{B}, B]\\ + Q(b, \partial_z b, \partial_{\bar{z}} b, (1+b)^{-1}, B, {}^t\overline{B}).\end{aligned}$$

By taking the $\perp$–part, we find that

$$(38)\quad (\partial_{\bar{z}} + \mathrm{ad}(B)) \circ (\partial_z - \mathrm{ad}({}^t\overline{B}))b = -(\widetilde{g}\mathcal{B}_{z\bar{z}})^{\perp} + Q(b, \partial_z b, \partial_{\bar{z}} b, (1+b)^{-1}, B, {}^t\overline{B})^{\perp}.$$

We then get

$$\|b\|_{L^p_{M+2}} \le C_2 \|F^{\perp}_{z\bar{z}}\|_{L^p_M} + C_2\epsilon\|b\|_{L^p_{M+2}}$$

and hence obtain $\|b\|_{L^p_{M+2}} \le C_3 \|F^{\perp}_{z\bar{z}}\|_{L^p_M}$. □

Lemma 4.13 *Let a_1 and a_2 be sections of $\mathrm{End}(E)_{|T\times\{\boldsymbol{\xi}\}}$. Assume that $a_1 = a_1^{\perp}$ and $a_2 = a_2^{\circ}$. Then we have*

$$\left|\int_T h(a_1, a_2)\right| \le \|a_1\|_{L^2}\,\|a_2\|_{L^2}\,\|(F^{\perp}_{z\bar{z}})_{|T\times\{\boldsymbol{\xi}\}}\|_{L^2}.$$

Proof It follows from Lemma 4.12 and $\overline{H(h,\boldsymbol{u})} = \widetilde{g}$. □

Lemma 4.14 *Let $\mathcal{P}$ be an endomorphism of E, and let $\mathcal{P}^{\dagger}$ denote the adjoint with respect to h. Let $\mathcal{R}$ (resp. $\mathcal{R}^{\dagger}$) be the matrix representing $\mathcal{P}$ (resp. $\mathcal{P}^{\dagger}$) with respect to $\boldsymbol{u}$. Then we have*

$$\begin{aligned}(\mathcal{R}^{\dagger})^{\circ} &= ({}^t\overline{\mathcal{R}})^{\circ} + O(|\mathcal{R}^{\perp}|\,\|F^{\perp}_{z\bar{z}}\|_{L^2}) + O(\|F^{\perp}_{z\bar{z}}\|^2_{L^2}|{}^t\overline{\mathcal{R}}^{\circ}|),\\ (\mathcal{R}^{\dagger})^{\perp} &= ({}^t\overline{\mathcal{R}})^{\perp} + O(|\mathcal{R}^{\perp}|\,\|F^{\perp}_{z\bar{z}}\|_{L^2}) + O(\|F^{\perp}_{z\bar{z}}\|_{L^2}\,|\mathcal{R}^{\circ}|).\end{aligned}$$

In particular, we have $|(\mathcal{R}^{\dagger})^{\perp}| = |\mathcal{R}^{\perp}| + O(|\mathcal{R}|\,\|F^{\perp}_{z\bar{z}}\|_{L^2})$.

Proof Let $H = H(h, \boldsymbol{u})$. We have $\mathcal{R}^{\dagger} = \overline{H^{-1}}({}^t\overline{\mathcal{R}})\overline{H}$. Then the claim follows from the estimate for H. □

Lemma 4.15 *For $k \in [1, M]_{\mathbb{Z}}$, we have* $\|D_{\boldsymbol{\xi}}^k \widetilde{g}\|_{L^p_{m+2}} = O\Bigl(\sum_{j=0}^{k} \|D^j_{\boldsymbol{\xi}} F^{\perp}_{z\bar{z}}\|_{L^p_m}\Bigr)$.

Proof We obtain the estimate from (38) by a standard inductive argument. □

5 Estimates for L^2 instantons

5.1 Preliminaries

Let τ be a complex number such that $\operatorname{Im}\tau > 0$. Let T be a complex torus obtained as the quotient of $\mathbb{C}$ by a lattice $\mathbb{Z} + \mathbb{Z}\tau$. Let z be the standard coordinate of $\mathbb{C}$. It also gives a local coordinate of a small open subset in T, once we fix a lift of the open subset in $\mathbb{C}$. We shall use the metric $dz\, d\bar{z}$ for $\mathbb{C}$ and T unless otherwise specified.

For any open subset $W \subset \mathbb{C}_w$, we use the metric $dw\, d\overline{w}$ on W, and the metric $dz\, d\bar{z} + dw\, d\overline{w}$ on $T \times W$ unless otherwise specified. Let ω denote the associated Kähler form. For $w \in W$, we put $T_w := T \times \{w\} \subset T \times W$.

Let E be a complex C^{∞} vector bundle on $T \times W$ with a Hermitian metric h and a unitary connection ∇. Let $F(\nabla)$ denote the curvature of ∇. We shall often denote it simply by F. The $(0,1)$–part and the $(1,0)$–part of ∇ are denoted by $\overline{\partial}_E$ and ∂_E, respectively. The restrictions of (E, h) to T_w are denoted by (E_w, h_w).

Recall that (E, ∇, h) is called an instanton if $\Lambda_{\omega} F(\nabla) = 0$. For the expression $F(\nabla) = F_{z\bar{z}}\, dz\, d\bar{z} + F_{z\overline{w}}\, dz\, d\overline{w} + F_{w\bar{z}}\, dw\, d\bar{z} + F_{w\overline{w}}\, dw\, d\overline{w}$, the equation is $F_{z\bar{z}} + F_{w\overline{w}} = 0$. We have the following equalities:

$$(\nabla_z\nabla_{\bar{z}} + \nabla_w\nabla_{\overline{w}})F_{w\overline{w}} = -(\nabla_z\nabla_{\bar{z}} + \nabla_w\nabla_{\overline{w}})F_{z\bar{z}} = [F_{z\overline{w}}, F_{w\bar{z}}], \tag{39}$$

$$\begin{aligned}(\nabla_z\nabla_{\bar{z}} + \nabla_w\nabla_{\overline{w}})F_{z\overline{w}} &= 2[F_{w\overline{w}}, F_{z\overline{w}}],\\ (\nabla_z\nabla_{\bar{z}} + \nabla_w\nabla_{\overline{w}})F_{w\bar{z}} &= 2[F_{w\bar{z}}, F_{w\overline{w}}].\end{aligned} \tag{40}$$

5.1.1 Hitchin's equivalence Let us recall the relation between harmonic bundles on an open subset $W \subset \mathbb{C}_w$ and instantons on $T \times W$ due to Hitchin. Let $(E, \overline{\partial}_E, h, \theta)$ be a harmonic bundle on W. Let $\nabla^{(0)} := \overline{\partial}_E + \partial_E$ be the Chern connection. Let $\theta^{\dagger}$ be the adjoint of θ. Let $p\colon T \times W \to W$ be the projection. The pullback $p^*(E, \nabla^{(0)}, h)$ is denoted by $(E_1, \nabla^{(1)}, h_1)$. We set $\nabla := \nabla^{(1)} + f\, d\bar{z} - f^{\dagger} dz$. Then (E_1, ∇, h_1) is an instanton on $T \times W$.

Conversely, let $(E_2, \nabla^{(2)}, h_2)$ be a T–equivariant instanton on $T\times W$. By considering T–equivariant sections, we obtain a vector bundle E on W such that $p^*E \simeq E_2$. It is naturally equipped with a connection $\nabla^{(0)}$ such that $p^*\nabla^{(0)}_v = \nabla^{(2)}_v$, where v denotes the natural horizontal lift of vector fields on W. By using the T–equivariance of $\nabla^{(2)}$, we have the expression $\nabla^{(2)} - p^*\nabla^{(0)} = p^* f\, d\bar{z} - p^* f^\dagger dz$, where f is a section of $\mathrm{End}(E)$. Then $(E, \overline{\partial}_E, h, f\, dz)$ is a harmonic bundle. In summary, we have the following.

Proposition 5.1 (Hitchin) *Harmonic bundles on W naturally correspond to T–equivariant instantons on $T\times W$.* □

5.2 Local estimate

Let U be a closed disc $\{w \mid |w-w_0|\le 1\}$ of $\mathbb{C}$. Let (E,∇,h) be an instanton on $T\times U$.

Assumption 5.2 We assume that $|F(\nabla)|\le\epsilon$ for a given positive small number ϵ. We also impose Assumption 4.11. □

We use the notation from Sections 4.2 and 4.3. Note that $|\nabla_{\bar{z}}^{m_1}\circ\nabla_z^{m_2}\circ\nabla_w^{m_3}\circ\nabla_{\overline{w}}^{m_4} F|_h \le C_{\boldsymbol{m}}\epsilon$, where $C_{\boldsymbol{m}}$ is a constant depending only on $\boldsymbol{m}=(m_1,m_2,m_3,m_4)$.

5.2.1 Estimates of the $\perp$–part of the connection form Let $\boldsymbol{u}$ be a partially almost holomorphic frame as in Proposition 4.7. We assume that $\int_T H(h,\boldsymbol{u}) = I$, as in Section 4.3.3. Let A be the connection form of ∇ with respect to $\boldsymbol{u}$. Let $\mathcal{B}_{z\bar{z}}$ represent $F_{z\bar{z}}$ with respect to $\boldsymbol{u}$. We use $\mathcal{B}_{z\overline{w}}$ and $\mathcal{B}_{w\bar{z}}$ in similar ways.

We prepare notation in a general situation. Let V be any vector bundle with a Hermitian metric h_V on U. Let $\pi\colon T\times U\to U$ be the projection. Let $p\ge 2$. For any section f of π^*V on $T\times U$, let $\|f\|_p$ denote the function on U given by $\|f\|_p(w) = (\int_{T\times\{w\}} |f|_{h_V}^p)^{1/p}$.

Lemma 5.3 *We have $\|A_w^\perp\|_p = O(\|F_{w\bar{z}}^\perp\|_p)$ and $\|\partial_{\overline{w}} A_w^\perp\|_p = O(\|\nabla_{\overline{w}} F_{w\bar{z}}^\perp\|_p) + O(\epsilon\|F_{w\bar{z}}^\perp\|_p)$.*

Proof Because $\partial_w A_{\bar{z}} - \partial_{\bar{z}} A_w + [A_w, A_{\bar{z}}] = \mathcal{B}_{w\bar{z}}$, we have

$$\partial_{\bar{z}} A_w^\perp + [A_{\bar{z}}, A_w^\perp] = -\mathcal{B}_{w\bar{z}}^\perp. \tag{41}$$

Then we obtain the first estimate. We also get

$$\partial_{\bar{z}}\partial_{\overline{w}} A_w^\perp + [A_{\bar{z}}, \partial_{\overline{w}} A_w^\perp] = -\partial_{\overline{w}}\mathcal{B}_{w\bar{z}}^\perp - [\partial_{\overline{w}} A_{\bar{z}}, A_w^\perp].$$

Because $\partial_{\overline{w}} A_{\bar{z}} = O(\epsilon)$, we obtain the second estimate. □

Lemma 5.4 *We have* $\|A^{\perp}_{\overline{w}}\|_p = O(\|F^{\perp}_{w\bar{z}}\|_p + \|F^{\perp}_{z\bar{z}}\|_p + \|\nabla_{\overline{w}} F^{\perp}_{z\bar{z}}\|_p)$. *We also have*

$$\|\partial_w A^{\perp}_{\overline{w}}\|_p = O(\|\nabla_{\overline{w}} F^{\perp}_{w\bar{z}}\|_p + \|F^{\perp}_{w\bar{z}}\|_p + \|F^{\perp}_{z\bar{z}}\|_p + \|\nabla_{\overline{w}} F^{\perp}_{z\bar{z}}\|_p).$$

Proof We set $\widetilde{g} := \overline{H(h,\boldsymbol{u})}$. We have $A_{\overline{w}} = -\widetilde{g}^{-1}\overline{({}^tA_w)}\widetilde{g} + \widetilde{g}^{-1}\partial_{\overline{w}}\widetilde{g}$. Hence, the first claim follows from Lemmas 5.3, 4.12 and 4.15. We have $\partial_w A_{\overline{w}} - \partial_{\overline{w}} A_w + [A_w, A_{\overline{w}}] = \mathcal{B}_{w\overline{w}}$. Hence, we have

$$\|\partial_w A^{\perp}_{\overline{w}}\|_p = O(\|\partial_{\overline{w}} A^{\perp}_w\|_p) + O(\|A^{\perp}_{\overline{w}}\|_p + \|A^{\perp}_w\|_p) + \|F^{\perp}_{w\overline{w}}\|_p.$$

Then the second claim follows. □

5.2.2 Estimate of the ⊥–part of the curvature We prepare notation in a general situation. Let V be any vector bundle with a Hermitian metric h_V on $T\times U$. Let $\pi\colon T\times U\to U$ be the projection. For any section f of V on $T\times U$, let $\|f\|$ denote the function on U given by $(\int_T |f|^2_{h_V})^{1/2}$. For any sections f and g of V, let $((f,g))$ denote the function on U given by $\int_T h_V(f,g)$.

Let $\Delta_w = -\partial_w\partial_{\overline{w}}$.

Proposition 5.5 *We have*

$$\begin{aligned}(42)\quad &\Delta_w\|F^{\perp}_{z\bar{z}}\|^2\\ &\le -\|\nabla_{\bar{z}}F^{\perp}_{z\bar{z}}\|^2 - \|\nabla_z F^{\perp}_{z\bar{z}}\|^2 - \|\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}\|^2 - \|\nabla_w F^{\perp}_{z\bar{z}}\|^2\\ &\quad + O\big(\epsilon\|F^{\perp}_{z\bar{z}}\|^2 + \epsilon\|F^{\perp}_{w\bar{z}}\|\,\|F^{\perp}_{z\bar{z}}\| + \epsilon\|\nabla_{\overline{w}}F^{\perp}_{w\bar{z}}\|\,\|F^{\perp}_{z\bar{z}}\| + \epsilon\|\nabla_{\bar{z}}F^{\perp}_{z\bar{z}}\|\,\|F^{\perp}_{z\bar{z}}\|\big)\\ &\qquad + O\big(\epsilon\|\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}\|\,\|F^{\perp}_{z\bar{z}}\| + \epsilon\|F^{\perp}_{w\bar{z}}\|^2 + \epsilon\|F^{\perp}_{w\bar{z}}\|\,\|\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}\|\big).\end{aligned}$$

Proof We have

$$\begin{aligned}&\Delta_w|F^{\perp}_{z\bar{z}}|^2\\ &\quad= -(\nabla_w\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}, F^{\perp}_{z\bar{z}}) - (F^{\perp}_{z\bar{z}}, \nabla_{\overline{w}}\nabla_w F^{\perp}_{z\bar{z}}) - (\nabla_w F^{\perp}_{z\bar{z}}, \nabla_w F^{\perp}_{z\bar{z}}) - (\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}, \nabla_{\overline{w}}F^{\perp}_{z\bar{z}})\end{aligned}$$

and

$$-(\nabla_w\nabla_{\overline{w}}F^{\perp}_{z\bar{z}}, F^{\perp}_{z\bar{z}}) = -(\nabla_w\nabla_{\overline{w}}F_{z\bar{z}}, F^{\perp}_{z\bar{z}}) + (\nabla_w\nabla_{\overline{w}}F^{\circ}_{z\bar{z}}, F^{\perp}_{z\bar{z}}).$$

Let us consider the estimate of $(\nabla_w\nabla_{\overline{w}}F^{\circ}_{z\bar{z}}, F^{\perp}_{z\bar{z}})$. The endomorphism $\nabla_w\nabla_{\overline{w}}F^{\circ}_{z\bar{z}}$ is represented by the following with respect to $\boldsymbol{u}$:

$$\partial_w\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\bar{z}} + [A_w, \partial_{\overline{w}}\mathcal{B}^{\circ}_{z\bar{z}}] + \partial_w[A_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}] + [A_w, [A_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}]].$$

Recall Lemma 4.13. We have the following estimates:

$$((\partial_w \partial_{\overline{w}} \mathcal{B}^{\circ}_{z\bar{z}}, \mathcal{B}^{\perp}_{z\bar{z}}))_h = O(\|\partial_w \partial_{\overline{w}} \mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h \, \|F^{\perp}_{z\bar{z}}\|_h), \tag{43}$$

$$\begin{aligned}((\lbrack A_w, \partial_{\overline{w}} \mathcal{B}^{\circ}_{z\bar{z}}\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h &= O(\|A^{\perp}_w\|_h \, \|\partial_{\overline{w}} \mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|\lbrack A^{\circ}_w, \partial_{\overline{w}} \mathcal{B}^{\circ}_{z\bar{z}}\rbrack\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h \, \|F^{\perp}_{z\bar{z}}\|_h),\end{aligned} \tag{44}$$

$$\begin{aligned}((\lbrack \partial_w A_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h &= ((\lbrack \partial_w A^{\perp}_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h + ((\lbrack \partial_w A^{\circ}_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h \\ &= O(\|\mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\partial_w A^{\perp}_{\overline{w}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|\lbrack \partial_w A^{\circ}_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}\rbrack\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h \, \|F^{\perp}_{z\bar{z}}\|),\end{aligned} \tag{45}$$

$$\begin{aligned}((\lbrack A_{\overline{w}}, \partial_w \mathcal{B}^{\circ}_{z\bar{z}}\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h &= O(\|\partial_w \mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|A^{\perp}_{\overline{w}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|\lbrack A^{\circ}_{\overline{w}}, \partial_w \mathcal{B}^{\circ}_{z\bar{z}}\rbrack\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h \, \|F^{\perp}_{z\bar{z}}\|_h),\end{aligned} \tag{46}$$

$$\begin{aligned}((\lbrack A_w, \lbrack A_{\overline{w}}, \mathcal{B}^{\circ}_{z\bar{z}}\rbrack\rbrack, \mathcal{B}^{\perp}_{z\bar{z}}))_h &= O(\|A^{\perp}_w\|_h \, \|A^{\perp}_{\overline{w}}\|_h \, \|\mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|A^{\perp}_w\|_h \, \|A^{\circ}_{\overline{w}}\|_h \, \|\mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|A^{\perp}_{\overline{w}}\|_h \, \|A^{\circ}_w\|_h \, \|\mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h) \\ &\quad + O(\|A^{\circ}_w\|_h \, \|A^{\circ}_{\overline{w}}\|_h \, \|\mathcal{B}^{\circ}_{z\bar{z}}\|_h \, \|\mathcal{B}^{\perp}_{z\bar{z}}\|_h \, \|F^{\perp}_{z\bar{z}}\|).\end{aligned} \tag{47}$$

We obtain the following estimate for $(\nabla_w \nabla_{\overline{w}} F^{\circ}_{z\bar{z}}, F^{\perp}_{z\bar{z}})$ from (43)–(47) with Lemma 5.3:

$$\begin{aligned}&((\nabla_w \nabla_{\overline{w}} F^{\circ}_{z\bar{z}}, F^{\perp}_{z\bar{z}})) \\ &\quad = O(\epsilon \|F^{\perp}_{z\bar{z}}\|^2 + \epsilon \|F^{\perp}_{w\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\| + \epsilon \, \|\nabla_{\overline{w}} F^{\perp}_{w\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\| + \epsilon \|\nabla_{\overline{w}} F^{\perp}_{z\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\|).\end{aligned} \tag{48}$$

We also have

$$\begin{aligned}-((\nabla_w \nabla_{\overline{w}} F_{z\bar{z}}, F^{\perp}_{z\bar{z}})) &= ((\nabla_z \nabla_{\bar{z}} F_{z\bar{z}}, F^{\perp}_{z\bar{z}})) + ((\lbrack F_{z\overline{w}}, F_{w\bar{z}}\rbrack, F^{\perp}_{z\bar{z}})) \\ &= -((\nabla_{\bar{z}} F_{z\bar{z}}, \nabla_{\bar{z}} F^{\perp}_{z\bar{z}})) + ((\lbrack F_{z\overline{w}}, F_{w\bar{z}}\rbrack, F^{\perp}_{z\bar{z}})),\end{aligned} \tag{49}$$

$$\begin{aligned}-((\nabla_{\bar{z}} F_{z\bar{z}}, \nabla_{\bar{z}} F^{\perp}_{z\bar{z}})) &= -((\nabla_{\bar{z}} F^{\perp}_{z\bar{z}}, \nabla_{\bar{z}} F^{\perp}_{z\bar{z}})) - ((\nabla_{\bar{z}} F^{\circ}_{z\bar{z}}, \nabla_{\bar{z}} F^{\perp}_{z\bar{z}})) \\ &= -((\nabla_{\bar{z}} F^{\perp}_{z\bar{z}}, \nabla_{\bar{z}} F^{\perp}_{z\bar{z}})) + O(\|\nabla_{\bar{z}} F^{\circ}_{z\bar{z}}\| \, \|\nabla_{\bar{z}} F^{\perp}_{z\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\|),\end{aligned} \tag{50}$$

$$\begin{aligned}((\lbrack F_{z\overline{w}}, F_{w\bar{z}}\rbrack, F^{\perp}_{z\bar{z}})) &= O(\|\lbrack F^{\circ}_{z\overline{w}}, F^{\circ}_{w\bar{z}}\rbrack\| \, \|F^{\perp}_{z\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\|) \\ &\quad + O(\|F^{\circ}_{z\overline{w}}\| \, \|F^{\perp}_{w\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\|) \\ &\quad + O(\|F^{\circ}_{w\bar{z}}\| \, \|F^{\perp}_{z\overline{w}}\| \, \|F^{\perp}_{z\bar{z}}\|) + O(\|F^{\perp}_{z\overline{w}}\| \, \|F^{\perp}_{w\bar{z}}\| \, \|F^{\perp}_{z\bar{z}}\|).\end{aligned} \tag{51}$$

We have a similar estimate for the contribution of $-((F^{\perp}_{z\bar{z}}, \nabla_{\overline{w}} \nabla_w F^{\perp}_{z\bar{z}}))$. In all, we obtain the claim of Proposition 5.5. □

Proposition 5.6 *We have the inequality*

$$\Delta_w \|F^{\perp}_{z\overline{w}}\|^2 \leq -\|\nabla_{\bar z} F^{\perp}_{z\overline{w}}\|^2 - \|\nabla_z F^{\perp}_{z\overline{w}}\|^2 - \|\nabla_w F^{\perp}_{z\overline{w}}\|^2 - \|\nabla_{\overline{w}} F^{\perp}_{z\overline{w}}\|^2 + O\big(\epsilon\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\bar z}\| + \epsilon\|\nabla_{\overline{w}} F^{\perp}_{w\bar z}\|\,\|F^{\perp}_{z\overline{w}}\| + \epsilon\|F^{\perp}_{w\bar z}\|\,\|F^{\perp}_{z\overline{w}}\| + \epsilon\|\nabla_{\overline{w}} F^{\perp}_{z\bar z}\|\,\|F^{\perp}_{z\overline{w}}\|\big) + O\big(\epsilon\|\nabla_{\bar z} F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\bar z}\| + \epsilon\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{w\bar z}\|\big). \tag{52}$$

Proof We have

$$-\partial_w\partial_{\overline{w}}|F^{\perp}_{z\overline{w}}|^2 = -|\nabla_{\overline{w}} F^{\perp}_{z\overline{w}}|^2 - |\nabla_w F^{\perp}_{z\overline{w}}|^2 - (\nabla_w\nabla_{\overline{w}} F^{\perp}_{z\overline{w}}, F^{\perp}_{z\overline{w}}) - (F^{\perp}_{z\overline{w}}, \nabla_{\overline{w}}\nabla_w F^{\perp}_{z\overline{w}}). \tag{53}$$

We also have

$$-(\nabla_w\nabla_{\overline{w}} F^{\perp}_{z\overline{w}}, F^{\perp}_{z\overline{w}}) = -(\nabla_w\nabla_{\overline{w}} F_{z\overline{w}}, F^{\perp}_{z\overline{w}}) + (\nabla_w\nabla_{\overline{w}} F^{\circ}_{z\overline{w}}, F^{\perp}_{z\overline{w}}). \tag{54}$$

Let us look at the contribution of $(\nabla_w\nabla_{\overline{w}} F^{\circ}_{z\overline{w}}, F^{\perp}_{z\overline{w}})$. Let $\mathcal{B}_{z\overline{w}}$ express $F_{z\overline{w}}$ with respect to $\boldsymbol{u}$ as in the proof of Proposition 5.5. Then $\nabla_w\nabla_{\overline{w}} F^{\circ}_{z\overline{w}}$ is represented by

$$\partial_w\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}} + [\partial_w A_{\overline{w}}, \mathcal{B}^{\circ}_{z\overline{w}}] + [A_{\overline{w}}, \partial_w\mathcal{B}^{\circ}_{z\overline{w}}] + [A_w, \partial_{\overline{w}}\mathcal{B}^{\circ}_{z,\overline{w}}] + [A_w, [A_{\overline{w}}, \mathcal{B}^{\circ}_{z\overline{w}}]].$$

We have the estimates

$$-((\partial_w\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}}, \mathcal{B}^{\perp}_{z\overline{w}}))_h = O(\|\partial_w\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h\,\|F^{\perp}_{z\bar z}\|), \tag{55}$$

$$(([\partial_w A_{\overline{w}}, \mathcal{B}^{\circ}_{z\overline{w}}], \mathcal{B}^{\perp}_{z\overline{w}}))_h = O(\|[\partial_w A^{\circ}_{\overline{w}}, \mathcal{B}^{\circ}_{z\overline{w}}]\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h\,\|F^{\perp}_{z\bar z}\|) + O(\|\partial_w A^{\perp}_{\overline{w}}\|_h\,\|\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h), \tag{56}$$

$$(([A_{\overline{w}}, \partial_w\mathcal{B}^{\circ}_{z\overline{w}}], \mathcal{B}^{\perp}_{z\overline{w}}))_h = O(\|A^{\circ}_{\overline{w}}\|_h\,\|\partial_w\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h\,\|F^{\perp}_{z\bar z}\|) + O(\|A^{\perp}_{\overline{w}}\|_h\,\|\partial_w\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h), \tag{57}$$

$$(([A_w, \partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}}], \mathcal{B}^{\perp}_{z\overline{w}}))_h = O(\|A^{\circ}_w\|_h\|\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}}\|_h\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h\,\|F^{\perp}_{z\bar z}\|) + O(\|A^{\perp}_w\|_h\|\partial_{\overline{w}}\mathcal{B}^{\circ}_{z\overline{w}}\|_h\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h), \tag{58}$$

$$(([A_w, [A_{\overline{w}}, \mathcal{B}^{\circ}_{z\overline{w}}]], \mathcal{B}^{\perp}_{z\overline{w}}))_h = O(\|A^{\perp}_w\|_h\,\|A^{\perp}_{\overline{w}}\|_h\,\|\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h) + O(\|A^{\circ}_w\|_h\,\|A^{\perp}_{\overline{w}}\|_h\,\|\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h) + O(\|A^{\perp}_w\|_h\,\|A^{\circ}_{\overline{w}}\|_h\,\|\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h) + O(\|A^{\circ}_w\|_h\,\|A^{\circ}_{\overline{w}}\|_h\,\|\mathcal{B}^{\circ}_{z\overline{w}}\|_h\,\|\mathcal{B}^{\perp}_{z\overline{w}}\|_h\,\|F^{\perp}_{z\bar z}\|_h). \tag{59}$$

Hence, we obtain

$$(60)\quad ((\nabla_w\nabla_{\overline{w}}F^{\circ}_{z\overline{w}},F^{\perp}_{z\overline{w}}))=O(\epsilon\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{z}}\|_h+\epsilon\|\nabla_{\overline{w}}F^{\perp}_{w\overline{z}}\|\,\|F^{\perp}_{z\overline{w}}\|_h$$
$$+\epsilon\|F^{\perp}_{w\overline{z}}|\,|F^{\perp}_{z\overline{w}}\|_h+\epsilon\|\nabla_{\overline{w}}F^{\perp}_{z\overline{z}}\|_h\,\|F^{\perp}_{z\overline{w}}\|_h).$$

We have

$$(61)\quad -((\nabla_w\nabla_{\overline{w}}F_{z\overline{w}},F^{\perp}_{z\overline{w}}))=((\nabla_z\nabla_{\overline{z}}F_{z\overline{w}},F^{\perp}_{z\overline{w}}))-2(([F_{w\overline{w}},F_{z\overline{w}}],F^{\perp}_{z\overline{w}}))$$
$$=-((\nabla_{\overline{z}}F_{z\overline{w}},\nabla_{\overline{z}}F^{\perp}_{z,\overline{w}}))-2(([F_{w,\overline{w}},F_{z\overline{w}}],F^{\perp}_{z\overline{w}})),$$

$$(62)\quad -((\nabla_{\overline{z}}F_{z\overline{w}},\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}))=-((\nabla_{\overline{z}}F^{\perp}_{z\overline{w}},\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}))-((\nabla_{\overline{z}}F^{\circ}_{z\overline{w}},\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}))$$
$$=-((\nabla_{\overline{z}}F^{\perp}_{z\overline{w}},\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}))+O(\|[\nabla_{\overline{z}}F^{\circ}_{z\overline{w}}]\|\,\|\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{z}}\|),$$

and

$$(63)\quad (([F_{w\overline{w}},F_{z\overline{w}}],F^{\perp}_{z\overline{w}}))=O(\|F^{\perp}_{w\overline{w}}\|\,\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{w}}\|)$$
$$+O(\|F^{\circ}_{w\overline{w}}\|\,\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{w}}\|)$$
$$+O(\|F^{\perp}_{w\overline{w}}\|\,\|F^{\circ}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{w}}\|)$$
$$+O(\|[F^{\circ}_{w\overline{w}},F^{\circ}_{z\overline{w}}]\|\,\|F^{\perp}_{z\overline{w}}\|\,\|F^{\perp}_{z\overline{z}}\|).$$

We have a similar estimate for the contribution of $-(F^{\perp}_{z\overline{w}},\nabla_{\overline{w}}\nabla_w F^{\perp}_{z\overline{w}})$. In all, we obtain the desired estimate (52). □

Proposition 5.7 *There exist $C>0$ and $\epsilon_0>0$ such that the following inequality holds if $\epsilon<\epsilon_0$:*

$$(64)\quad \Delta_w(\|F^{\perp}_{z\overline{z}}\|^2+\|F^{\perp}_{z\overline{w}}\|^2)$$
$$\leq -C(\|F^{\perp}_{z\overline{z}}\|^2+\|F^{\perp}_{z\overline{w}}\|^2)$$
$$-C(\|\nabla_z F^{\perp}_{z\overline{z}}\|^2+\|\nabla_{\overline{z}}F^{\perp}_{z\overline{z}}\|^2+\|\nabla_w F^{\perp}_{z\overline{z}}\|^2+\|\nabla_{\overline{w}}F^{\perp}_{z\overline{z}}\|^2)$$
$$-C(\|\nabla_z F^{\perp}_{z\overline{w}}\|^2+\|\nabla_{\overline{z}}F^{\perp}_{z\overline{w}}\|^2+\|\nabla_w F^{\perp}_{z\overline{w}}\|^2+\|\nabla_{\overline{w}}F^{\perp}_{z\overline{w}}\|^2).$$

Proof There exist $C_1>0$ such that $\|\nabla_z s\|\geq C_1\|s\|$ and $\|\nabla_{\overline{z}}s\|\geq C_1\|s\|$ for any section of $\mathrm{End}(E)$ such that $s=s^{\perp}$. Then the claim follows from Propositions 5.5 and 5.6. □

5.2.3 Higher derivative Assume that $\|F^{\perp}_{z\overline{z}}\|^2+\|F^{\perp}_{z\overline{w}}\|^2\leq\delta_1^2$ for some $\delta_1\ll\epsilon$. For $\rho<1$, we set $U(\rho)=\{w\mid |w-w_0|\leq\rho\}\subset U$.

Proposition 5.8 *For any k, p, there exists $C>0$ such that*

$$\|F^{\perp}_{z\overline{z}}\|_{L^p_k(T\times U(\rho))}\leq C\delta_1,\quad \|F^{\perp}_{z\overline{w}}\|_{L^p_k(T\times U(\rho))}\leq C\delta_1.$$

Proof This can be shown by a standard bootstrapping argument. We give only an indication. We take $\rho < \rho' < 1$. In the following, we shall replace ρ' with a smaller one. Let κ denote z, $\bar{z}$, w and $\overline{w}$. By Proposition 5.7, we obtain $\|\nabla_\kappa F^{\perp}_{z\bar{z}}\|_{L^2(T\times U(\rho'))} = O(\delta_1)$ and $\|\nabla_\kappa F^{\perp}_{z\overline{w}}\|_{L^2(T\times U(\rho'))} = O(\delta_1)$.

With respect to the frame $\boldsymbol{u}$, the endomorphism $-\nabla_w\nabla_{\overline{w}}F_{z\bar{z}}$ is represented by

$$-\partial_w\partial_{\overline{w}}\mathcal{B}_{z\bar{z}} - [\partial_w A_{\overline{w}}, \mathcal{B}_{z\bar{z}}] - [A_{\overline{w}}, \partial_w\mathcal{B}_{z\bar{z}}] + [A_w, \partial_{\overline{w}}\mathcal{B}_{z\bar{z}}] + [A_w, [A_{\overline{w}}, \mathcal{B}_{z\bar{z}}]] \tag{65}$$

and the endomorphism $-\nabla_z\nabla_{\bar{z}}F_{z\bar{z}}$ is represented by

$$-\partial_z\partial_{\bar{z}}\mathcal{B}_{z\bar{z}} - [\partial_z A_{\bar{z}}, \mathcal{B}_{z\bar{z}}] - [A_{\bar{z}}, \partial_z\mathcal{B}_{z\bar{z}}] + [A_z, \partial_{\bar{z}}\mathcal{B}_{z\bar{z}}] + [A_z, [A_{\bar{z}}, \mathcal{B}_{z\bar{z}}]]. \tag{66}$$

The sum of (65) and (66) is equal to $[\mathcal{B}_{z\overline{w}}, \mathcal{B}_{w\bar{z}}]$. By looking at the $\perp$–part of the equation, we obtain

$$\text{the } \perp\text{–part of (65)} + \text{the } \perp\text{–part of (66)} = [\mathcal{B}_{z\overline{w}}, \mathcal{B}_{w\bar{z}}]^{\perp}. \tag{67}$$

By using Lemmas 5.3 and 5.4, we obtain $\|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\bar{z}}\|_{L^2(T\times U(\rho'))} = O(\delta_1)$. Similarly, we obtain $\|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\overline{w}}\|_{L^2(T\times U(\rho'))} = O(\delta_1)$. It follows that

$$\|F^{\perp}_{z\bar{z}}\|_{L^4(T\times U(\rho'))} + \|F^{\perp}_{z\overline{w}}\|_{L^4(T\times U(\rho'))} = O(\delta_1),$$
$$\|\nabla_\kappa F^{\perp}_{z\bar{z}}\|_{L^4(T\times U(\rho'))} + \|\nabla_\kappa F^{\perp}_{z\overline{w}}\|_{L^4(T\times U(\rho'))} = O(\delta_1).$$

By using Lemmas 5.3 and 5.4 and (67), we obtain $\|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\bar{z}}\|_{L^4(T\times U(\rho'))} = O(\delta_1)$. Similarly, we obtain $\|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\overline{w}}\|_{L^4(T\times U(\rho'))} = O(\delta_1)$. By the same argument, we obtain the following for any p:

$$\begin{aligned}\|F^{\perp}_{z\bar{z}}\|_{L^p(T\times U(\rho'))} + \|F^{\perp}_{z\overline{w}}\|_{L^p(T\times U(\rho'))} + \|\nabla_\kappa F^{\perp}_{z\bar{z}}\|_{L^p(T\times U(\rho'))}& \\ + \|\nabla_\kappa F^{\perp}_{z\overline{w}}\|_{L^p(T\times U(\rho'))} &= O(\delta_1),\\ \|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\bar{z}}\|_{L^p(T\times U(\rho'))} + \|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\overline{w}}\|_{L^p(T\times U(\rho'))} &= O(\delta_1).\end{aligned}$$

Namely, we obtain $\|F^{\perp}_{z\bar{z}}\|_{L^p_2(T\times U(\rho'))} + \|F^{\perp}_{z\overline{w}}\|_{L^p_2(T\times U(\rho'))} = O(\delta_1)$.

By the argument in Lemma 5.3, we obtain $\|A^{\perp}_{w}\|_{L^p_2} = O(\delta_1)$. By the argument in Lemma 5.4, we obtain $\|A^{\perp}_{\overline{w}}\|_{L^p_1} = O(\delta_1)$. By the relation

$$\partial_w A_{\overline{w}} - \partial_{\overline{w}} A_w + [A_w, A_{\overline{w}}] = \mathcal{B}_{w\overline{w}}$$

we obtain $\|\partial_w A^{\perp}_{\overline{w}}\|_{L^p_1} = O(\delta_1)$. We also have $\|A^{\perp}_{z}\|_{L^p_2} = O(\delta_1)$, which follows from Lemma 4.15. Then we get

$$\|(\partial_w\partial_{\overline{w}} + \partial_z\partial_{\bar{z}})\mathcal{B}^{\perp}_{z\bar{z}}\|_{L^p_1(T\times U(\rho'))} = O(\delta_1).$$

Hence, $\|\mathcal{B}_{z\bar{z}}^{\perp}\|_{L_3^p(T\times U(\rho'))} = O(\delta_1)$. Similarly, we obtain $\|\mathcal{B}_{z\overline{w}}^{\perp}\|_{L_3^p(T\times U(\rho'))} = O(\delta_1)$. By an inductive argument, we obtain

$$\|\mathcal{B}_{z\bar{z}}^{\perp}\|_{L_k^p(T\times U(\rho'))} + \|\mathcal{B}_{z\overline{w}}^{\perp}\|_{L_k^p(T\times U(\rho'))} = O(\delta_1)$$

for any k. □

Corollary 5.9 *For any k, p, there is $C > 0$ such that $\|H(h,\boldsymbol{u})^{\perp}\|_{L_k^p(T\times U(\rho))} \le C\delta_1$.*

Proof This follows from Proposition 5.8 and Lemma 4.15. □

5.3 Global estimate

5.3.1 Preliminaries For $R > 0$, we set $Y_R := \{w \in \mathbb{C} \mid |w| \ge R\}$ and $X_R := T \times Y_R$. An instanton (E,∇,h) is called L^2 if the curvature $F := F(\nabla)$ is L^2. We study the behaviour of L^2 instantons around infinity. We suppose that (E,∇,h) is an L^2 instanton in this subsection. For $w_0 \in Y_R$ and $a > 0$, let $B_{w_0}(a) := \{w \in \mathbb{C} \mid |w - w_0| \le a\}$.

Let $\epsilon > 0$ be sufficiently small. There exists R_1 such that $\|F_{|X_{R_1}}\|_{L^2} < \epsilon$. Let $w_0 \in Y_{2R_1}$. By the theorem of Uhlenbeck [49], for any $(z,w) \in T \times B_{w_0}(1)$, we have $|F(z,w)| = O(\|F_{|T\times B_{w_0}(2)}\|_{L^2}) = O(\epsilon)$. In particular, we may assume that E_w are semistable if $w \in Y_{2R_1}$. Because we are interested in the behaviour around infinity, we may assume that $(E_w, \overline{\partial}_{E_w})$ are semistable of degree 0 for any $w \in Y_R$ from the beginning.

5.3.2 Prolongation of the spectral curve We consider the relative Fourier–Mukai transform $\mathrm{RFM}_-(E,\overline{\partial}_E)$, which is a coherent sheaf on $T^\vee \times Y_R$. Its support is relatively 0–dimensional over Y_R, denoted by $\mathcal{S}p(E)$. It is called the spectral curve of $(E,\overline{\partial}_E)$. Let $\overline{Y}_R$ be the closure of Y_R in $\mathbb{P}^1$, ie $\overline{Y}_R = Y_R \cup \{\infty\}$.

Theorem 5.10 *$\mathcal{S}p(E)$ is extended to a closed subvariety $\overline{\mathcal{S}p}(E)$ in $T^\vee \times \overline{Y}_R$.*

Proof Let ρ denote the rank of E. We have the holomorphic map $\varphi\colon Y_R \to \mathrm{Sym}^\rho T^\vee$ induced by $\mathcal{S}p(E)$. We have only to prove that it extends to a holomorphic map $\overline{Y}_R \to \mathrm{Sym}^\rho T^\vee$. We fix a closed immersion $\mathrm{Sym}^\rho T^\vee \subset \mathbb{P}^N$ for a sufficiently large N, and we regard φ as a holomorphic map $Y_R \to \mathbb{P}^N$. Let $d_{\mathbb{P}^N}$ denote the distance of $\mathbb{P}^N$, induced by the Fubini–Study metric.

Take any $w_0 \in Y_{2R}$. By Corollary 4.10, for any $w_1, w_2 \in B_{w_0}(\frac{1}{2})$, we have

$$d_{\mathbb{P}^N}(\varphi(w_1),\varphi(w_2)) = O(\|F_{|T\times B_{w_0}(2)}\|_{L^2}). \tag{68}$$

Note that φ is holomorphic. We can also regard it as a harmonic map between Kähler manifolds. Let $T_w\varphi$ be the derivative of φ, and $|T_w\varphi|$ denote the norm of $T_w\varphi$ with respect to the Euclidean metric $dw\, d\overline{w}$ and the Fubini–Study metric of $\mathbb{P}^N$. For any $w \in B_{w_0}(\frac{1}{4})$, we obtain the following estimate from (68) by using Cauchy's formula for differentiation in complex analysis:

$$|T_w\varphi| = O(\|F_{|T\times B_{w_0}(2)}\|). \tag{69}$$

Hence, we obtain finiteness of the energy of the harmonic map φ:

$$\int_{Y_{2R_1}} |T_w\varphi|^2\, |dw\, d\overline{w}| < C\|F_{|Y_R}\|_{L^2}^2 < \infty.$$

Then φ is extended on $\overline{Y}_R$, according to [42, Theorem 3.6]. □

The intersection $\overline{\mathcal{S}p}(E) \cap (T^\vee \times \{\infty\})$ is denoted by $\mathcal{S}p_\infty(E)$.

5.3.3 Asymptotic decay By making R larger, we may assume we have a lift of $\overline{\mathcal{S}p}(E)$ to a closed subvariety $\overline{\mathcal{S}p}(E)_1 \subset \overline{Y}_R \times \mathbb{C}_\zeta$, which induces an action of ζ on $\mathrm{RFM}_-(E, \overline{\partial}_E)$. (See Section 2.1.) Let f_ζ be the corresponding holomorphic endomorphism of E. We set $\overline{\partial}_0 := \overline{\partial}_E - f_\zeta d\bar{z}$, which gives a holomorphic structure of E. For each w, the restriction of $\mathcal{E}' = (E, \overline{\partial}_0)$ to $T_z \times \{w\}$ is holomorphically trivial. It is naturally isomorphic to $p^* p_*(\mathcal{E}')$, where $p\colon X_R \to Y_R$ denotes the natural projection. We obtain the decomposition $h = h^\circ + h^\perp$ as in Section 4.3.

Theorem 5.11 *For any polynomial $P(t_1, t_2, t_3, t_4)$ of noncommutative variables, there exists $C > 0$ such that*

$$P(\nabla_z, \nabla_{\bar{z}}, \nabla_w, \nabla_{\overline{w}})h^\perp = O(\exp(-C|w|)).$$

Proof Let $\epsilon > 0$ be any sufficiently small number. We may assume that $\|F_{|X_{R_1}}\| < \epsilon$ for some $R_1 > 0$. By Theorem 5.10, we may assume that Assumption 5.2 is satisfied for the restriction of (E, ∇, h) to any disc contained in X_{R_1}. In particular, we can apply Proposition 5.7 to $(E, \nabla, h)_{|X_{R_1}}$. We obtain

$$\Delta_w(\|F^\perp_{z\bar{z}}\|^2 + \|F^\perp_{z\overline{w}}\|^2) \le -C_1(\|F^\perp_{z\bar{z}}\|^2 + \|F^\perp_{z\overline{w}}\|^2)$$

for some $C_1 > 0$. The following lemma follows from a standard argument.

Lemma 5.12 *We have $\|F^\perp_{z\bar{z}}\|^2 + \|F^\perp_{z\overline{w}}\|^2 = O(\exp(-C_2|w|))$ for some $C_2 > 0$.*

Proof This is a variant of a lemma of Ahlfors [1] and Simpson [45]. We give only an indication. We put $G := \|F^{\perp}_{z\bar{z}}\|^2 + \|F^{\perp}_{z\bar{w}}\|^2$ and $f_\epsilon := C_3 \exp(-2C_1^{1/2}|w|) + \epsilon$, where $\epsilon > 0$ and $C_3 > 0$. We have the inequality $\Delta_w f_\epsilon \geq -C_1 f_\epsilon$. If C_3 is sufficiently large, we have $f_\epsilon > G$ on $\{|w| = R_1\}$. For each $\epsilon > 0$, we have $f_\epsilon > G$ outside a compact subset. We put $U := \{w \mid f_\epsilon(w) < G(w)\}$. Then U is relatively compact, and we have $f_\epsilon = G$ on the boundary of U. On U, we have $\Delta_w(G - f_\epsilon) \leq -C(G - f_\epsilon) < 0$. By the maximum principle, we have $\sup_U(G - f_\epsilon) \leq \max_{\partial U}(G - f_\epsilon) = 0$. Hence, we obtain that U is empty. It means $G \leq f_\epsilon$ on Y_R for any ϵ. We obtain the desired inequality by taking the limit $\epsilon \to 0$. □

Now the claim of Theorem 5.11 follows from Corollary 5.9. □

5.3.4 Reduction to asymptotic harmonic bundles Let $p\colon X_R \to Y_R$ denote the projection. By using the pushforward of $\mathcal{O}$–modules, we obtain a holomorphic vector bundle $V := p_*\mathcal{E}'$ on Y_R. It is equipped with a Higgs field $\theta_V := f_\zeta dw$. For any $s_i \in V_{|w}$ $(i = 1, 2)$, we denote the corresponding holomorphic section of $\mathcal{E}'_{|T_w}$ by $\widetilde{s}_i$. We set $h_V(s_1, s_2) := \int_T h(\widetilde{s}_1, \widetilde{s}_2)$. We have the Chern connection $\overline{\partial}_V + \partial_V$ with respect to h_V. Let $\theta_V^\dagger$ denote the adjoint of θ_V.

Proposition 5.13 *There exists $C > 0$ such that*

$$F(h_V) + [\theta_V, \theta_V^\dagger] = O(\exp(-C|w|)). \tag{70}$$

Proof We identify $p^*V = \mathcal{E}'$. According to Theorem 5.11, the difference $h - p^*h_V$ and its derivatives are $O(\exp(-C_1|w|))$. (The constant C_1 may depend on the order of derivatives.) We also have $\overline{\partial}_E = p^*\overline{\partial}_V + f_\zeta d\bar{z}$. Hence, $(p^*V, p^*\overline{\partial}_V + f_\zeta d\bar{z}, p^*h_V)$ satisfies

$$\Lambda_\omega F(p^*h_V) = O(\exp(-C_2|w|)),$$

which is equivalent to (70). □

5.3.5 Estimate of the curvature

Theorem 5.14 *There exists $\rho > 0$ such that*

$$F(h) = O\left(\frac{dz\, d\bar{z}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw\, d\bar{w}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw\, d\bar{z}}{|w|^{1+\rho}}\right) + O\left(\frac{dz\, d\bar{w}}{|w|^{1+\rho}}\right).$$

Proof We shall use an estimate for asymptotic harmonic bundles which is explained in Section 5.5 below. Let $\varphi\colon \Delta_u = \{|u| < R^{-1/e}\} \to \overline{Y}_R$ be given by $\varphi(u) = u^e$. For the expression $\theta = f_\zeta dw = f_\zeta(-eu^{-e-1}du)$, according to Theorem 5.10, the spectral curve $\mathcal{S}p(f_\zeta) \subset \mathbb{C} \times Y_R$ is contained in $\{|\zeta| \leq R'\} \times Y_R$ for some R', and the closure in $\mathbb{C} \times \overline{Y}_R$ is a complex variety. Hence, we may assume that $\varphi^*(E, \overline{\partial}_E, \theta)$ decomposes into

$$\varphi^* E = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\varphi^*\theta_V)} E_{\mathfrak{a}}$$

as in (71). Moreover, we have $\deg_{u^{-1}} \mathfrak{a} \leq e$ for any $\mathfrak{a} \in \mathrm{Irr}(\varphi^*\theta_V)$.

We set $(V', \overline{\partial}_{V'}, \theta_{V'}, h') := \varphi^{-1}(V, \overline{\partial}_V, \theta_V, h_V)$. According to Proposition 5.13, it satisfies (72). By Corollary 5.19, we have

$$|F(h_V)|_{h_V} = O(|u|^{-2}(\log|u|)^{-2} du\, d\overline{u}).$$

Hence, we have $|F(h)_{w\overline{w}}|_h = |F(h)_{z\overline{z}}|_h = O(|w|^{-2}(\log|w|)^{-2})$.

We take a frame $\boldsymbol{v}$ of $\mathcal{P}_{\boldsymbol{a}} V'$ as in Section 5.5.2. Let Θ be determined by $\varphi^* f_\zeta \boldsymbol{v} = \boldsymbol{v}\Theta$. Let C_w be determined by $\varphi^*(\partial_w)\boldsymbol{v} = \boldsymbol{v} C_w$. We have $\varphi^*(\partial_w f_\zeta)\boldsymbol{v} = \boldsymbol{v}(\varphi^*(\partial_w)\Theta + [C_w, \Theta])$. We have the expression

$$\Theta = \bigoplus \big((\varphi^*\partial_w \mathfrak{a} - e^{-1}\alpha u^e) I_{\mathfrak{a},\alpha} - e^{-1} u^e \Theta_{\mathfrak{a},\alpha}\big),$$

where the entries of $\Theta_{\mathfrak{a},\alpha}$ are holomorphic at $u = 0$. The norm of the endomorphism determined by $\boldsymbol{v}$ and $\Theta_{\mathfrak{a},\alpha}$ is $O((\log|w|)^{-1})$ by Proposition 5.18. Note that $\varphi^*(\partial_w) = -e^{-1} u^{e+1} \partial_u$ and $\varphi^*(\partial_w^2)\mathfrak{a} = O(|\varphi^*(w)|^{-1-\rho})$ for some $\rho > 0$. Hence, the contribution of $\varphi^*(\partial_w)\Theta$ to $\varphi^*(\partial_w f_\zeta)$ is dominated as $O(\varphi^*|w|^{-1-\rho})$ for some $\rho > 0$. Let G_w be the endomorphism determined by $\boldsymbol{v}$ and C_w. By using Lemma 5.21, we obtain $[G_w, \varphi^* f_\zeta] = O(\varphi^*|w|^{-2})$. Hence, we obtain $|\partial_w f_\zeta|_{h_V} = O(|w|^{-1-\rho})$ for some $\rho > 0$. Then we obtain $|F(h)_{z\overline{w}}|_h = |F(h)_{w\overline{z}}|_h = O(|w|^{-1-\rho})$ for some $\rho > 0$. □

Corollary 5.15 *$(E, \overline{\partial}_E, h)$ is acceptable, ie the curvature $F(h)$ is bounded with respect to h and the Poincaré metric $|w|^{-2}(\log|w|)^{-2} dw\, d\overline{w} + dz\, d\overline{z}$ on X_R around $T \times \{\infty\}$.* □

5.3.6 Prolongation to a filtered bundle

We set $\overline{X}_R := T \times \overline{Y}_R$.

Corollary 5.16 *The holomorphic vector bundle $(E, \overline{\partial}_E)$ is naturally extended to a filtered bundle $\mathcal{P}_* E$ on $(\overline{X}_R, T \times \{\infty\})$. (See Section 2.2 for filtered bundles.) Moreover, the filtered bundle is good in the sense of Section 2.4.1.*

Proof Because $(E, \overline{\partial}_E, h)$ is acceptable, we obtain the first claim from [36, Theorem 21.31]. As explained in Section 5.5.2, we obtain a filtered bundle $\mathcal{P}_* V$ on $(\overline{Y}_R, \infty)$ from the Higgs bundle with the Hermitian metric $(V, \overline{\partial}_V, h_V, \theta_V)$. By Proposition 5.18, the filtered Higgs bundle $(\mathcal{P}_* V, \theta_V)$ is good. By construction, $(\mathcal{P}_* V, \theta_V)$ corresponds to $\mathcal{P}_* E$ in the sense of Section 2.4.1. It implies the claim of the corollary. □

We obtain the spectral curve $\mathcal{S}p(\mathcal{P}_a E) \subset T^\vee \times \overline{Y}_{\boldsymbol{R}}$ of $\mathcal{P}_a E$. It is equal to $\overline{\mathcal{S}p}(E)$ in Theorem 5.10, and independent of the choice of $a \in \mathbb{R}$.

5.4 An estimate in a variant case

We continue to use the notation in Section 5.3. Let (E, ∇, h) be an instanton on $X_{\boldsymbol{R}}$. Let $F = F(\nabla)$ be its curvature. We suppose the following:

- $|F(z, w)| \to 0$ when $|w| \to \infty$, ie for any $\delta > 0$, there exists $R_\delta > 0$ such that $|F(z, w)|_h \leq \delta$ for any $|w| \geq R_\delta$. In particular, we obtain the spectral curve $\mathcal{S}p(E, \overline{\partial}_E) \subset T^\vee \times Y_{R_\delta}$ if δ is sufficiently small.
- The closure of $\mathcal{S}p(E)$ in $T^\vee \times \overline{Y}_{\boldsymbol{R}_\delta}$ is a complex subvariety.

We denote the closure by $\overline{\mathcal{S}p}(E)$, and we set $\mathcal{S}p_\infty(E) := \overline{\mathcal{S}p}(E) \cap (T^\vee \times \{\infty\})$. We obtain the following theorem.

Theorem 5.17 *Under the assumption, (E, ∇, h) is an L^2 instanton.*

Proof By the assumption, there exists $R_1 > 0$, such that Assumption 5.2 is satisfied for $(E, \nabla, h)_{|X_{R_1}}$. In particular, we can apply Proposition 5.7 to $(E, \nabla, h)_{|X_{R_1}}$. We obtain the estimate as in Theorem 5.11 by the same argument. Then we obtain estimates as in Proposition 5.13 and Theorem 5.14 by the same arguments. In particular, (E, ∇, h) is an L^2 instanton. □

Theorem 5.17 implies that we can replace the L^2 condition with a weaker one, under the assumption that the spectral curve is extended in a complex analytic way.

5.5 Asymptotic harmonic bundles

In this subsection, we explain that some of the results for the asymptotic behaviour of wild harmonic bundles are naturally extended for Higgs bundles with a Hermitian metric satisfying the Hitchin equation up to an exponentially small term. It is used in the proof of Theorem 5.14.

We put $X := \Delta_z = \{z \in \mathbb{C} \mid |z| < 1\}$, $\overline{X} := \{|z| \leq 1\}$ and $D := \{0\}$. Let $g_{\boldsymbol{p}}$ be the Poincaré metric of $X \setminus D$. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $\overline{X} \setminus D$. We suppose that there exists a decomposition

$$(E, \theta) = \bigoplus_{\substack{\mathfrak{a} \in z^{-1}\mathbb{C}[z^{-1}] \\ \alpha \in \mathbb{C}}} (E_{\mathfrak{a},\alpha}, \theta_{\mathfrak{a},\alpha}) \tag{71}$$

such that, for the expression $\theta_{\mathfrak{a},\alpha} = d\mathfrak{a} + \alpha\, dz/z + f_{\mathfrak{a},\alpha}\, dz/z$, the eigenvalues of $f_{\mathfrak{a},\alpha}(z)$ go to 0 when $z \to 0$. We put $\mathrm{Irr}(\theta) := \{\mathfrak{a} \mid \text{there exists } \alpha \text{ such that } E_{\mathfrak{a},\alpha} \neq 0\}$.

For any $\mathfrak{a}(z) = \sum_{j \geq -N} \mathfrak{a}_j z^j$ with $\mathfrak{a}_{-N} \neq 0$, we set $\mathrm{ord}(\mathfrak{a}) := -N$. We also set $\mathrm{ord}(0) := 0$. We take a negative number p satisfying $p < \min\{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in \mathrm{Irr}(\theta),\ \mathfrak{a} \neq \mathfrak{b}\}$.

Let h be a Hermitian metric of E. Let $\theta^\dagger$ denote the adjoint of θ with respect to h. Let $F(h)$ denote the curvature of $(E, \overline{\partial}_E, h)$. We impose the following condition for some $C_0 > 0$ and $\epsilon_0 > 0$:

$$|F(h) + [\theta, \theta^\dagger]|_{h,g_p} \leq C_0 \exp(-\epsilon_0 |z|^p). \tag{72}$$

5.5.1 Asymptotic orthogonality and acceptability We have the following version of Simpson's main estimate.

Proposition 5.18 *Suppose that* $(E, \overline{\partial}_E, \theta, h)$ *satisfies* (72).

- *If* $\mathfrak{a} \neq \mathfrak{b}$, *there exists* $\epsilon > 0$ *such that* $E_{\mathfrak{a},\alpha}$ *and* $E_{\mathfrak{b},\beta}$ *are* $O(\exp(-\epsilon |z|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})}))$*–asymptotically orthogonal, ie there exists* $C > 0$ *such that, for any* $u, v \in E_{|Q}$, *we have* $|h(u, v)| \leq C_1 \exp(-\epsilon |z(Q)|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})$.
- *If* $\alpha \neq \beta$, *there exists* $\epsilon > 0$ *such that* $E_{\mathfrak{a},\alpha}$ *and* $E_{\mathfrak{a},\beta}$ *are* $O(|z|^\epsilon)$*–asymptotically orthogonal.*
- $\theta_{\mathfrak{a},\alpha} - (d\mathfrak{a} + \alpha\, dz/z)\, \mathrm{id}_{E_{\mathfrak{a},\alpha}}$ *is bounded with respect to* h *and the Poincaré metric* $g_{\boldsymbol{p}}$.

Proof By considering the tensor product with a harmonic bundle of a rank one, we may assume $p < \min\{\mathrm{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \mathrm{Irr}(\theta)\}$. We have a map

$$\eta_\ell \colon z^{-1}\mathbb{C}[z^{-1}] \to \mathcal{I}_\ell := z^{-\ell}\mathbb{C}[z^{-1}]$$

by forgetting the terms $\sum_{j \geq -\ell+1} \mathfrak{a}_j z^j$. For each $\mathfrak{b} \in \mathcal{I}_\ell$, we set

$$E^{(\ell)}_{\mathfrak{b}} := \bigoplus_{\eta_\ell(\mathfrak{a}) = \mathfrak{b}} \bigoplus_{\alpha \in \mathbb{C}} E_{\mathfrak{a},\alpha}.$$

Let $\pi^{(\ell)}_{\mathfrak{a}}$ denote the projection of E onto $E^{(\ell)}_{\mathfrak{b}}$ with respect to the decomposition $E = \bigoplus E^{(\ell)}_{\mathfrak{b}}$. In the case $\ell = 1$, we omit the superscript (1).

Let $\mathrm{Irr}(\theta, \ell)$ be the image of $\mathrm{Irr}(\theta)$ by η_ℓ. We take a total order $\leq'$ on $\mathrm{Irr}(\theta, \ell)$ for each ℓ such that the induced map $\mathrm{Irr}(\theta, 1) \to \mathrm{Irr}(\theta, \ell)$ is order-preserving. Let $E'^{(\ell)}_{\mathfrak{b}}$

be the orthogonal complement of $\bigoplus_{\mathfrak{c}<'\mathfrak{b}} E_{\mathfrak{c}}$ in $\bigoplus_{\mathfrak{c}\leq'\mathfrak{b}} E_{\mathfrak{c}}$. Let $\pi'^{(\ell)}_{\mathfrak{b}}$ be the orthogonal projection onto $E'^{(\ell)}_{\mathfrak{b}}$. In the case $\ell=1$, we omit the superscript (1). We have

$$\pi'^{(\ell)}_{\mathfrak{b}} = \sum_{\eta_\ell(\mathfrak{a})=\mathfrak{b}} \pi'_{\mathfrak{a}}.$$

We put $\zeta_\ell := \eta_\ell - \eta_{\ell+1}$. We have the expression $\theta = f\,dz$. We put

$$f^{(\ell)} := f - \sum_{\mathfrak{a}} \partial_z \eta_{\ell+1}(\mathfrak{a})\pi_{\mathfrak{a}}, \quad \mu^{(\ell)} := f^{(\ell)} - \sum_{\mathfrak{a}} \partial_z \zeta_\ell(\mathfrak{a})\pi'_{\mathfrak{a}}$$

and $\mathcal{R}^{(\ell)}_{\mathfrak{b}} := \pi^{(\ell)}_{\mathfrak{b}} - \pi'^{(\ell)}_{\mathfrak{b}}$. We consider the following claims:

(P_ℓ) $|f^{(\ell')}|_h = O(|z|^{-\ell'-1})$ for $\ell' \geq \ell$.

(Q_ℓ) $|\mu^{(\ell')}|_h = O(|z|^{-\ell'})$ for $\ell' \geq \ell$.

(R_ℓ) $|\mathcal{R}^{(\ell')}_{\mathfrak{b}}|_h = O(\exp(-C|z|^{-\ell'}))$ for $\ell' \geq \ell$ and for $\mathfrak{b} \in \mathrm{Irr}(\theta, \ell')$.

The asymptotic orthogonality of $E_{\mathfrak{a},\alpha}$ and $E_{\mathfrak{b},\beta}$ ($\mathfrak{a} \neq \mathfrak{b}$) follows from ($R_1$).

In the proof of [36, Theorem 7.2.1], we proved the claims for any wild harmonic bundle by using descending induction on ℓ. Essentially the same argument can work. We give an indication for a modification in this situation.

We have the expression $\theta^\dagger = f^\dagger d\bar{z}$. Let $\Delta := -\partial_z\partial_{\bar{z}}$. If a holomorphic section s of $\mathrm{End}(E)$ satisfies $[s, f] = 0$, we obtain the following inequality from (72):

$$\Delta \log |s|_h^2 \leq -\frac{|[f^\dagger, s]|_h^2}{|s|_h^2} + C_0 \exp(-\epsilon_0|z|^p). \tag{73}$$

By applying (73) to f, we obtain the following as in [36, (99)]:

$$\Delta \log |f^{(\ell)}|_h^2 \leq -\frac{|[f^{(\ell)\dagger}, f^{(\ell)}]|_h^2}{|f^{(\ell)}|_h^2} + C_1.$$

By using the argument in the proof of Proposition 2.10 of [34], we obtain $|f|_h = O(|z|^{-p-1})$. Then, we can observe that the claims P_p, Q_p and R_p hold. Then by the same argument as that in [36, Sections 7.3.2–7.3.3], we obtain P_ℓ and Q_ℓ. We put

$$k^{(\ell)}_{\mathfrak{b}} := \log\bigl(|\pi^{(\ell)}_{\mathfrak{b}}|_h^2/|\pi'^{(\ell)}_{\mathfrak{b}}|_h^2\bigr) = \log\bigl(1 + |\mathcal{R}^{(\ell)}_{\mathfrak{b}}|_h^2/|\pi'^{(\ell)}_{\mathfrak{b}}|_h^2\bigr).$$

By applying (73) to $\pi^{(\ell)}_{\mathfrak{b}}$, we obtain

$$\Delta \log k^{(\ell)}_{\mathfrak{b}} \leq -\frac{|[f^\dagger, \pi^{(\ell)}_{\mathfrak{b}}]|_h^2}{|\pi^{(\ell)}_{\mathfrak{b}}|_h^2} + C_0 \exp(-\epsilon_0|z|^p).$$

There exists $C_1 > 0$ and $R_1 > 0$ such that the following holds for any $|z| < R_1$:

$$(74)\quad \Delta \exp(-A|z|^{-\ell}) \geq -\exp(-A|z|^{-\ell})\big(\tfrac{\ell^2}{4}A^2|z|^{-2(\ell+1)}\big)$$
$$\geq -\exp(-A|z|^{-\ell})\tfrac{\ell^2}{4}A^2C_1|z|^{-2(\ell+1)} + C_0\exp(-\epsilon_0|z|^p).$$

Hence we obtain R_ℓ by using the argument in [36, Section 7.3.4]. Similarly, we obtain the asymptotic orthogonality of $E_{\mathfrak{a},\alpha}$ and $E_{\mathfrak{a},\beta}$ $(\alpha \neq \beta)$, and the boundedness of $\theta_{\mathfrak{a},\alpha} - (d\mathfrak{a} + \alpha dz/z)\,\mathrm{id}_{E_{\mathfrak{a},\alpha}}$ by using the argument in [36, Sections 7.3.5–7.3.7] with (73). □

We obtain the following corollary. (See [36, Section 7.2.5] for the argument.)

Corollary 5.19 *$(E, \overline{\partial}_E, h)$ is acceptable, ie the curvature $F(h)$ is bounded with respect to h and $g_{\boldsymbol{p}}$.* □

5.5.2 Prolongation and the norm estimate For any $U \subset X$ and for any $a \in \mathbb{R}$, let $\mathcal{P}_aE(U)$ denote the space of holomorphic sections s of $E_{|U\setminus D}$ such that $|s|_h = O(|z|^{-a-\epsilon})$ (for all ϵ) locally around any point of U. (See Section 2.2.5.) According to a general theory of acceptable bundles, we obtain a locally free $\mathcal{O}_X$–module $\mathcal{P}_aE$, and a filtered bundle $\mathcal{P}_*E = (\mathcal{P}_aE \mid a \in \mathbb{R})$. (See Section 2.2 for a review of filtered bundles.) The decomposition (71) is extended to a decomposition of $\mathcal{P}_aE$:

$$\mathcal{P}_aE = \bigoplus \mathcal{P}_aE_{\mathfrak{a},\alpha}.$$

We set $\mathcal{P}E := \bigcup_{a\in\mathbb{R}} \mathcal{P}_aE$ and $\mathcal{P}E_{\mathfrak{a},\alpha} := \bigcup_{a\in\mathbb{R}} \mathcal{P}_aE_{\mathfrak{a},\alpha}$. Set $\mathrm{Gr}^{\mathcal{P}}_a(E) := \mathcal{P}_aE/\mathcal{P}_{<a}E$, which we naturally regard as $\mathbb{C}$–vector spaces.

By Proposition 5.18, θ gives a section of $\mathrm{End}(\mathcal{P}E) \otimes \Omega^1_X$, which preserves the decomposition $\mathcal{P}E = \bigoplus \mathcal{P}E_{\mathfrak{a},\alpha}$. By the estimate in Proposition 5.18, we have that $\theta_{\mathfrak{a},\alpha} - (d\mathfrak{a} + \alpha dz/z)\,\mathrm{id}_{E_{\mathfrak{a},\alpha}}$ is logarithmic with respect to the lattice $\mathcal{P}_aE_{\mathfrak{a},\alpha}$. Hence, we have the induced endomorphism $\mathrm{Res}(\theta_{\mathfrak{a},\alpha})$ of $\mathrm{Gr}^{\mathcal{P}}_a E_{\mathfrak{a},\alpha}$, which has a unique eigenvalue α. We set $\mathrm{Res}(\theta) = \bigoplus \mathrm{Res}(\theta_{\mathfrak{a},\alpha})$. Let $W\,\mathrm{Gr}^{\mathcal{P}}_a(E)$ be the monodromy weight filtration of the nilpotent part of $\mathrm{Res}(\theta)$.

For each section s of $\mathcal{P}E$, let $\deg^{\mathcal{P}}(s) := \min\{a \mid s \in \mathcal{P}_aE\}$. For any $g \in \mathrm{Gr}^{\mathcal{P}}_a E$, let $\deg^W(g) := \min\{m \mid g \in W_m\}$. Let $\boldsymbol{v} = (v_i)$ be a frame of $\mathcal{P}_aE$ which is compatible with the decomposition $\mathcal{P}_aE = \bigoplus \mathcal{P}_aE_{\mathfrak{a},\alpha}$, the parabolic filtration and the weight filtration, ie each v_i is a section of a direct summand $E_{\mathfrak{a},\alpha}$, the tuple

$$\boldsymbol{v}^{(b)} := \big(v_i \;\big|\; \deg^{\mathcal{P}} v_i = b\big)$$

induces a basis $[\boldsymbol{v}^{(b)}] := ([v_i^{(b)}])$ of $\mathrm{Gr}_b^{\mathcal{P}} E$ for any $a-1 < b \le a$, and the tuple

$$[\boldsymbol{v}^{(b),m}] := \big([v_i^{(b)}] \bigm| \deg^W v_i^{(b)} = m\big)$$

induces a basis of $\mathrm{Gr}_m^W \mathrm{Gr}_b^{\mathcal{P}} E$. We set $a_i := \deg^{\mathcal{P}}(v_i)$ and $k_i := \deg^W(v_i)$. Let h_0 be the metric of E determined by $h_0(v_i, v_i) = |z|^{-2a_i}(-\log|z|)^{k_i}$ and $h_0(v_i, v_j) = 0$ $(i \neq j)$. The following proposition can be proved by the argument in [36, Section 8.1.2].

Proposition 5.20 *h and h_0 are mutually bounded.* □

5.5.3 Connection form Let $\boldsymbol{v}$ be a frame of $\mathcal{P}_a E$, which is compatible with the decomposition $\mathcal{P}_a E = \bigoplus \mathcal{P}_a E_{\mathfrak{a},\alpha}$, the parabolic filtration and the weight filtration. Let G be the endomorphism of E determined by $G(v_i)\,dz = \partial v_i$ for $i = 1, \dots, \operatorname{rank} E$. We can prove the following by the arguments of Lemma 7.5.5, Lemma 10.1.3 and Proposition 10.3.3 of [36].

Lemma 5.21 *We have $|G|_h = O(|z|^{-1})$. For the decomposition $G = \sum G_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}$ according to $E = \bigoplus E_{\mathfrak{a},\alpha}$, we have the estimate*

$$|G_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}|_h = \begin{cases} O(\exp(-\epsilon|z|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})) & \text{if } \mathfrak{a} \neq \mathfrak{b}, \\ O(|z|^{-1+\epsilon}) & \text{if } \mathfrak{a} = \mathfrak{b}, \alpha \neq \beta. \end{cases}$$

for some $\epsilon > 0$. □

We have the expression $\theta = f\,dz$. Let us consider $\partial_h f$. Let Θ be determined by $f\boldsymbol{v} = \boldsymbol{v}\Theta$. Let C be determined by $\partial_h \boldsymbol{v} = \boldsymbol{v} C$. We have $(\partial_h f)\boldsymbol{v} = \boldsymbol{v}(\partial_z \Theta\, dz + [C, \Theta])$ and $[G, f]\boldsymbol{v} = \boldsymbol{v}[C, \Theta]$. We have the decompositions $\partial_h f = \sum (\partial_h f)_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}$ and $\overline{\partial} f^\dagger = \sum (\overline{\partial} f^\dagger)_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}$ according to $E = \bigoplus E_{\mathfrak{a},\alpha}$.

Corollary 5.22 *Let $m := \min\{\operatorname{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \operatorname{Irr}(\theta)\}$. If $m < 0$, we have $\partial_h f = O(|z|^{-2+m} dz)$ with respect to h and $dz\, d\bar{z}$. We have*

$$\big|(\partial_h f)_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}\big|_h = \begin{cases} O(\exp(-\epsilon|z|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})) & \text{if } \mathfrak{a} \neq \mathfrak{b}, \\ O(|z|^{\epsilon-2}) & \text{if } \mathfrak{a} = \mathfrak{b},\ \alpha \neq \beta. \end{cases}$$

We also have

$$\big|(\overline{\partial}_E f^\dagger)_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}\big|_h = \begin{cases} O(\exp(-\epsilon|z|^{\operatorname{ord}(\mathfrak{a}-\mathfrak{b})})) & \text{if } \mathfrak{a} \neq \mathfrak{b}, \\ O(|z|^{\epsilon-2}) & \text{if } \mathfrak{a} = \mathfrak{b},\ \alpha \neq \beta. \end{cases}$$

Proof It follows from Lemma 5.21. □

5.5.4 Some estimates Let t be a C^∞ endomorphism of E. According to the decomposition $E = \bigoplus E_{\mathfrak{a},\alpha}$, we have the decomposition $t = \sum t_{(\mathfrak{a},\alpha)(\mathfrak{b},\beta)}$, where $t_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)} \in \mathrm{Hom}(E_{\mathfrak{b},\beta}, E_{\mathfrak{a},\alpha})$. Let $\mathcal{C}$ be the set of C^∞ endomorphisms t such that the following holds for some $\epsilon > 0$ which may depend on t:

$$|t_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)}|_h = \begin{cases} O(|z|^\epsilon \exp(-\epsilon|z|^{\mathrm{ord}(\mathfrak{a}-\mathfrak{b})})) & \text{if } (\mathfrak{a},\alpha) \neq (\mathfrak{b},\beta), \\ O(1) & \text{if } (\mathfrak{a},\alpha) = (\mathfrak{b},\beta). \end{cases}$$

Note that $\mathcal{C}$ is closed under the addition and the composition.

Proposition 5.23 *Suppose t and $|z|^2\partial_z\partial_{\bar{z}}t$ are contained in $\mathcal{C}$. Then $z\partial_z t$ and $\bar{z}\partial_{\bar{z}}t$ are also contained in $\mathcal{C}$.*

Proof Let $\Psi\colon \mathbb{H} := \{u \in \mathbb{C} \mid \mathrm{Im}\, u > 0\} \to \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ be given by $\Psi(u) = \exp(\sqrt{-1}u)$. Because $\Psi^* t$ and $\partial_u\partial_{\bar{u}}\Psi^*(t)$ are bounded, we obtain that $\partial_u\Psi^* t$ and $\partial_{\bar{u}}\Psi^* t$ are also bounded.

In the following argument, positive constants ϵ can change. We use the notation in the proof of Proposition 5.18. We clearly have $\partial_{\bar{z}}\pi^{(\ell)}_{\mathfrak{b}} = 0$. By Lemma 5.21, we have $\partial_z\pi^{(\ell)}_{\mathfrak{b}} = O(\exp(-\epsilon|z|^{-\ell}))$. We also have

$$\partial_{\bar{z}}\partial_z\pi^{(\ell)}_{\mathfrak{b}} = [F(h), \pi^{(\ell)}_{\mathfrak{b}}] = O(\exp(-\epsilon|z|^{-\ell})).$$

We have the decomposition $t = \sum t^{(\ell)}_{\mathfrak{a},\mathfrak{b}}$ according to the decomposition $E = \bigoplus E^{(\ell)}_{\mathfrak{a}}$. We have $t^{(\ell)}_{\mathfrak{a},\mathfrak{b}} = O(\exp(-\epsilon|z|^{-\ell}))$ if $\mathfrak{a} \neq \mathfrak{b}$. Hence, we have

$$[t, \pi^{(\ell)}_{\mathfrak{b}}] = \sum_{\mathfrak{a}\neq\mathfrak{b}} t^{(\ell)}_{\mathfrak{a},\mathfrak{b}} - \sum_{\mathfrak{a}\neq\mathfrak{b}} t^{(\ell)}_{\mathfrak{b},\mathfrak{a}} = O(\exp(-\epsilon|z|^{-\ell})).$$

We also have $|z|^2\partial_{\bar{z}}\partial_z[t, \pi^{(\ell)}_{\mathfrak{b}}] = [|z|^2\partial_{\bar{z}}\partial_z t, \pi^{(\ell)}_{\mathfrak{b}}] + [\bar{z}\partial_{\bar{z}}t, z\partial_z\pi^{(\ell)}_{\mathfrak{b}}] + [t, |z|^2\partial_{\bar{z}}\partial_z\pi^{(\ell)}_{\mathfrak{b}}] = O(\exp(-\epsilon|z|^{-\ell}))$. Hence, we obtain

$$z\partial_z[t, \pi^{(\ell)}_{\mathfrak{b}}] = O(\exp(-\epsilon|z|^{-\ell})), \quad \bar{z}\partial_{\bar{z}}[t, \pi^{(\ell)}_{\mathfrak{b}}] = O(\exp(-\epsilon|z|^{-\ell})).$$

Therefore, we obtain $z\partial_z t^{(\ell)}_{\mathfrak{a},\mathfrak{b}} = O(\exp(-\epsilon|z|^{-\ell}))$ and $\bar{z}\partial_{\bar{z}} t^{(\ell)}_{\mathfrak{a},\mathfrak{b}} = O(\exp(-\epsilon|z|^{-\ell}))$ for $\mathfrak{a} \neq \mathfrak{b}$.

We have $z\partial_z\pi_{\mathfrak{a},\alpha} = O(|z|^\epsilon)$ and $|z|^2\partial_{\bar{z}}\partial_z\pi_{\mathfrak{a},\alpha} = O(|z|^\epsilon)$ by Lemma 5.21. Then we obtain $z\partial_z t_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)} = |z|^\epsilon$ and $\bar{z}\partial_{\bar{z}} t_{(\mathfrak{a},\alpha),(\mathfrak{a},\beta)} = |z|^\epsilon$ for $\alpha \neq \beta$. If we have that $\mathfrak{a} \neq \mathfrak{b}$ with $\ell = \mathrm{ord}(\mathfrak{a} - \mathfrak{b})$, we obtain the desired estimate by using

$$t_{(\mathfrak{a},\alpha),(\mathfrak{b},\beta)} = \pi_{\mathfrak{a},\alpha} \circ t^{(\ell)}_{\eta_\ell(\mathfrak{a}),\eta_\ell(\mathfrak{b})} \circ \pi_{\mathfrak{b},\beta}.$$

□

5.5.5 Refined asymptotic orthogonality We obtain an asymptotic orthogonality of the derivative by assuming the following with respect to h and $dz\,d\bar{z}$, in addition to (72):

$$\partial_{\bar{z}}\partial_z\big(F(h)+[\theta,\theta^\dagger]\big)=O(\exp(-\epsilon_0|z|^p)). \tag{75}$$

Let $\boldsymbol{v}$ be a holomorphic frame of $\mathcal{P}_0E$, compatible with the decomposition $\mathcal{P}_0E=\bigoplus\mathcal{P}_0E_{\mathfrak{a},\alpha}$, the parabolic filtration and the weight filtration. Let $(\mathfrak{a}_i,\alpha_i)$ be determined by $v_i\in\mathcal{P}_0E_{\mathfrak{a}_i,\alpha_i}$. We say that a matrix valued function $B=(B_{ij})$ satisfies condition $\mathcal{C}_1$ if the following holds for some $\epsilon>0$ which may depend on B:

$$B_{ij}=\begin{cases}O(|z|^{\epsilon}\exp(-\epsilon|z|^{\operatorname{ord}(\mathfrak{a}_i-\mathfrak{a}_j)})) & \text{if } (\mathfrak{a}_i,\alpha_i)\neq(\mathfrak{a}_j,\alpha_j),\\ O(|v_i|_h|v_j|_h) & \text{otherwise.}\end{cases}$$

Let H be the matrix valued function determined by $H_{ij}=h(v_i,v_j)$. Lemma 5.21 implies that $z\partial_zH$ and $\bar{z}\partial_{\bar{z}}H$ satisfy condition $\mathcal{C}_1$.

Proposition 5.24 *$(|z|^2\partial_{\bar{z}}\partial_z)^2H$ satisfies condition $\mathcal{C}_1$.*

Proof Let $G(A)$ denote the endomorphism determined by $\boldsymbol{v}$ and a matrix-valued function A. By Lemma 5.21, we have

$$G(H^{-1}z\partial_zH),G(H^{-1}\bar{z}\partial_{\bar{z}}H),G(\bar{H}^{-1}z\partial_z\bar{H}),G(\bar{H}^{-1}\bar{z}\partial_{\bar{z}}\bar{H})\in\mathcal{C}.$$

Because $G(\bar{z}\partial_{\bar{z}}(\bar{H}^{-1}z\partial_z\bar{H}))=|z|^2F(h)\in\mathcal{C}$, we have $G(\bar{H}^{-1}|z|^2\partial_{\bar{z}}\partial_z\bar{H})\in\mathcal{C}$.

We have the expression $\theta=f\,dz$. We have $\partial_{\bar{z}}\partial_z[f,f^\dagger]=[[F(h)_{\bar{z},z},f],f^\dagger]+[\partial_zf,\overline{\partial}_zf^\dagger]$. It gives an estimate for $\partial_{\bar{z}}\partial_z[f,f^\dagger]$ by Corollary 5.22, from which we can deduce that $|z|^2\partial_z\partial_{\bar{z}}(|z|^2F(h))\in\mathcal{C}$. By Proposition 5.23, $z\partial_z(|z|^2F(h))\in\mathcal{C}$ and $\bar{z}\partial_{\bar{z}}(|z|^2F(h))\in\mathcal{C}$. We have $G(\bar{z}\partial_{\bar{z}}(\bar{z}\partial_{\bar{z}}(\bar{H}^{-1}z\partial_z\bar{H}))),G(z\partial_z(\bar{z}\partial_{\bar{z}}(\bar{H}^{-1}z\partial_z\bar{H})))\in\mathcal{C}$. We also obtain $G(\bar{H}^{-1}(\bar{z}\partial_{\bar{z}})^2z\partial_z\bar{H}),G(\bar{H}^{-1}\bar{z}\partial_{\bar{z}}(z\partial_z)^2\bar{H})\in\mathcal{C}$. Then we obtain $G(\bar{H}^{-1}(z\partial_z)^2(\bar{z}\partial_{\bar{z}})^2\bar{H})\in\mathcal{C}$ from $|z|^2\partial_z\partial_{\bar{z}}(|z|^2F(h))\in\mathcal{C}$. It implies the claim of the lemma. □

Corollary 5.25 *$(z\partial_z)^2H$ satisfies the condition $\mathcal{C}_1$.* □

Remark 5.26 The estimate as in Corollary 5.25 will be used in the study for the extension of the associated twistor family, which will be discussed elsewhere.

6 L^2 instantons on $T \times \mathbb{C}$

6.1 Some standard properties

6.1.1 Instantons of rank one Let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$ with $\operatorname{rank} E = 1$.

Lemma 6.1 *(E, ∇, h) is a unitary flat bundle.*

Proof Because $\operatorname{rank} E = 1$, we have $(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\overline{w}}) F_{z\bar{z}} = 0$ and $(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\overline{w}}) F_{z\overline{w}} = 0$. We obtain the inequalities

$$-(\partial_w \partial_{\overline{w}} + \partial_z \partial_{\bar{z}}) |F_{z\bar{z}}|^2 \le 0, \quad -(\partial_w \partial_{\overline{w}} + \partial_z \partial_{\bar{z}}) |F_{z\overline{w}}|^2 \le 0.$$

We use the notation in Section 5.2.2. By applying the fibre integral for $T \times \mathbb{C} \to \mathbb{C}$, we obtain $-\partial_w \partial_{\overline{w}} \|F_{z\bar{z}}\|^2 \le 0$ and $-\partial_w \partial_{\overline{w}} \|F_{z\overline{w}}\|^2 \le 0$. Because the functions $\|F_{z\bar{z}}\|^2$ and $\|F_{z\overline{w}}\|^2$ are L^1 on $\mathbb{C}_w$, they are 0. □

Corollary 6.2 *Let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$ of an arbitrary rank. Then $\det(E, \nabla, h)$ is a flat unitary bundle, ie we have $\operatorname{Tr} F(\nabla) = 0$.* □

If we do not impose the L^2 property, there exist much more instantons of rank one on $T \times \mathbb{C}$.

(i) Let $\mathfrak{a}$ be any holomorphic function on $\mathbb{C}$. Then the trivial holomorphic line bundle $\mathcal{O}_{\mathbb{C}}$ with the trivial metric and the Higgs field $d\mathfrak{a}$ gives a harmonic bundle $\mathcal{L}(\mathfrak{a})$ on $\mathbb{C}$. By Hitchin's equivalence, we have the associated instanton on $T \times \mathbb{C}$.

(ii) Let ρ be an $\mathbb{R}$–valued harmonic function on $T \times \mathbb{C}$. Then the trivial holomorphic line bundle $\mathcal{O}_{T \times \mathbb{C}} e$ with the metric h_ρ given by $\log h_\rho(e, e) = \rho$ gives an instanton $\mathcal{L}(\rho)$ on $T \times \mathbb{C}$. Note that there exist many harmonic functions which are not the real part of a holomorphic function on $T \times \mathbb{C}$. We can construct such a function by using a Bessel function $I_0(r) = \int_{-1}^{1} \cosh(rt)(t^2 - 1)^{-1/2} dt$ which satisfies $I_0'' + r^{-1} I_0' - I_0 = 0$. It is a C^∞ function on $\mathbb{R}$, satisfying $I_0(r) = I_0(-r)$. In particular, $\kappa(w) := I_0(|w|)$ gives a C^∞ function on $\mathbb{C}$ satisfying $(-\partial_w \partial_{\overline{w}} + 4)\kappa = 0$. We can construct a harmonic function ρ on $T \times \mathbb{C}$ from κ such that ρ is not constant along T, by using Fourier series on $T \times \mathbb{C}$ in a standard way. (See [26].) It is not the real part of any holomorphic function.

In general, any instanton of rank one $(E, \overline{\partial}_E, h)$ can be expressed as the tensor product of instantons of types (i) and (ii). Indeed, by considering the support $\mathrm{RFM}_-(E, \overline{\partial}_E)$, we obtain a holomorphic function $\mathbb{C} \to T^\vee$. Because $\mathbb{C}$ is simply connected, it is lifted to a holomorphic function $\mathfrak{b}\colon \mathbb{C} \to \mathbb{C}$. We have a holomorphic function $\mathfrak{a}$ such that $\partial_w \mathfrak{a} = \mathfrak{b}$. Then we can observe that $(E, \overline{\partial}_E, h)$ is isomorphic to $\mathcal{L}(\mathfrak{a}) \otimes \mathcal{L}(\rho)$ for a harmonic function ρ on $T \times \mathbb{C}$.

6.1.2 Polystability of the associated filtered bundle We let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$. Let $(E, \overline{\partial}_E)$ be the underlying holomorphic vector bundle on $T \times \mathbb{C}$. For a saturated $\mathcal{O}_{T\times\mathbb{C}}$–subsheaf $\mathcal{F} \subset E$, let $h_{\mathcal{F}}$ denote the induced Hermitian metric of the smooth part of $\mathcal{F}$. Let $F(h_{\mathcal{F}})$ denote the curvature. As in [8] and Simpson [44], we set

$$\deg(\mathcal{F}, h) := \sqrt{-1} \int_{T\times\mathbb{C}} \mathrm{Tr}(\Lambda F(h_{\mathcal{F}})) d\mathrm{vol}_{T\times\mathbb{C}} .$$

Let $\pi_{\mathcal{F}}$ denote the orthogonal projection of E to $\mathcal{F}$, where it is considered only on the smooth part of $\mathcal{F}$. By the Chern–Weil formula [44], we have

$$\deg(\mathcal{F}, h) = -\int_{T\times\mathbb{C}} |\overline{\partial}\pi|_h^2 d\mathrm{vol}_{T\times\mathbb{C}} .$$

Lemma 6.3 *The degree* $\deg(\mathcal{F}, h)$ *is finite if and only if:*

(i) *The degree of* $\mathcal{F}_{|T\times\{w\}}$ *is* 0 *for any* $w \in \mathbb{C}$.

(ii) $\mathcal{F}$ *is extended to a saturated subsheaf* $\mathcal{P}_0\mathcal{F}$ *of* $\mathcal{P}_0 E$.

In that case, we have $\deg(\mathcal{F}, h) = 2\pi|T| \int_{\{z\}\times\mathbb{P}^1} \mathrm{par}\text{-}\mathrm{c}_1(\mathcal{P}_*\mathcal{F})$ *for any* z, *where* $\mathcal{P}_*\mathcal{F}$ *denotes* $\mathcal{P}_0\mathcal{F}$ *with the induced parabolic structure, and* $|T|$ *denotes the volume of* T.

Proof This type of claim is standard in the study of Kobayashi–Hitchin correspondence for parabolic objects, and well established by Li and Narasimhan in [31], based on the fundamental results in [44; 45] and Siu [47]. We give only an indication for our situation.

By [44, Lemmas 10.5 and 10.6], $\mathcal{F}_{|\{z\}\times\mathbb{C}}$ is extended to a parabolic subsheaf if and only if $\int_{\mathbb{C}} |\overline{\partial}\pi_{|\{z\}\times\mathbb{C}}|^2 < \infty$. In that case, $(\sqrt{-1}/2\pi) \int_{\mathbb{C}} \mathrm{Tr}(F(h_{\mathcal{F}}))_{|z\times\mathbb{C}}$ is equal to the parabolic degree of the parabolic subsheaf.

If conditions (i) and (ii) are satisfied, then we have

$$\deg(\mathcal{F}, h) = \int_T d\mathrm{vol}_T \left(\int_{z\times\mathbb{C}} \sqrt{-1}\,\mathrm{Tr}(F(h_{\mathcal{F}})) \right) = 2\pi|T| \int_{z\times\mathbb{P}^1} \mathrm{par}\text{-}\mathrm{c}_1(\mathcal{P}_*\mathcal{F}) > -\infty.$$

Conversely, suppose $\deg(\mathcal{F}, h)$ is finite. Because

$$\deg(\mathcal{F}, h) \leq \int_{\mathbb{C}} d\mathrm{vol}_{\mathbb{C}} \left(-\int_T |\nabla_{\bar{z}}\pi|^2 d\mathrm{vol}_T \right) = 2\pi \int_{\mathbb{C}} \deg(\mathcal{F}_{|T\times\{w\}}) d\mathrm{vol}_{\mathbb{C}}$$

we have $\deg(\mathcal{F}_{|T\times\{w\}})=0$ for any w. Because

$$-\deg(\mathcal{F},h)=\int_T d\mathrm{vol}_T\left(\int_{\mathbb{C}}|\overline{\partial}\pi_{|\{z\}\times\mathbb{C}}|^2\right)<\infty,$$

there exists a thick subset $A\subset T^{\vee}$ such that $\mathcal{F}_{|z\times\mathbb{C}}$ is extendable for any $z\in A$. (A subset is called thick if it is not contained in a countable union of complex analytically closed subsets.) Then, $\mathcal{F}$ is extendable according to [47, Theorem 4.5]. □

Proposition 6.4 *$\mathcal{P}_*E$ is polystable. We have $\deg(\mathcal{P}_*E)=0$. (See Section 2.4.1 for the stability condition in this case.)*

Proof The second claim directly follows from Lemma 6.3 and Corollary 6.2. Let $\mathcal{P}_*\mathcal{F}$ be a filtered subsheaf $\mathcal{P}_*E$ satisfying (A1) and (A2) in Section 2.4.1. Let $\mathcal{F}$ be its restriction to $X\times\mathbb{C}$. By Lemma 6.3, we have $\mu(\mathcal{P}_*\mathcal{F})=\mu(\mathcal{F},h)\le 0$. Moreover, if it is 0, the orthogonal projection onto $\mathcal{F}$ is holomorphic. Hence, the orthogonal decomposition $E=\mathcal{F}\oplus\mathcal{F}^{\perp}$ is holomorphic. It is extended to a decomposition $\mathcal{P}_*E=\mathcal{P}_*\mathcal{F}\oplus\mathcal{P}_*\mathcal{F}^{\perp}$. Both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ with the induced metrics are L^2 instantons. Hence, we obtain the first claim of the corollary by an easy induction on the rank. □

6.1.3 Uniqueness of the L^2 instanton adapted to a filtered bundle Let (E,∇,h) be an L^2 instanton on $T\times\mathbb{C}$. We have the associated filtered bundle $\mathcal{P}_*E$ on $(T\times\mathbb{P}^1,T\times\{\infty\})$. Let h' be a Hermitian metric of E, and let $\nabla_{h'}$ be a unitary connection of (E,h') such that $(E,\nabla_{h'},h')$ is an L^2 instanton, the $(0,1)$–parts of $\nabla_{h'}$ and ∇_h are equal, and h' is adapted to $\mathcal{P}_*E$. (See Section 2.2.5 for adaptedness.)

Proposition 6.5 *We have a holomorphic decomposition $(E,\overline{\partial}_E)=\bigoplus_i(E_i,\overline{\partial}_{E_i})$ such that it is orthogonal with respect to both h and h', and for each i, there exists $\alpha_i>0$ such that $h_{|E_i}=\alpha_i h'_{|E_i}$. In particular, we have $\nabla_h=\nabla_{h'}$.*

Proof Let s be the self-adjoint endomorphism determined by $h'=hs$. According to [44], we have the inequality (see [44, page 876])

$$-(\overline{\partial}_z\partial_z+\overline{\partial}_w\partial_w)\operatorname{Tr}(s)+|\overline{\partial}(s)s^{-1/2}|_h^2\le 0.$$

By taking the fiber integral for $T\times\mathbb{C}\to\mathbb{C}$, we obtain

$$-\partial_{\overline{w}}\partial_w\int_T\operatorname{Tr}(s)+\int_T|\overline{\partial}(s)s^{-1/2}|_h^2\le 0.$$

It implies that $\int_T\operatorname{Tr}(s)$ is a subharmonic function on $\mathbb{C}_w$. By using the norm estimate for asymptotically harmonic bundle (Proposition 5.20), we obtain that h and h' are mutually bounded, ie s and s^{-1} are bounded with respect to both of h and h'. Hence, we see that $\int_T \mathrm{Tr}(s)$ is constant. We obtain $\int_T |\overline{\partial}(s)s^{-1/2}|_h^2 = 0$, which implies $\overline{\partial}(s) = 0$. Then the claim of the proposition follows. □

6.1.4 Instanton number Let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$. We have the associated filtered bundle $\mathcal{P}_* E$ on $(T \times \mathbb{P}^1, T \times \{\infty\})$. Note that the second Chern class of $\mathcal{P}_a E$ is independent of $a \in \mathbb{R}$.

Proposition 6.6 *For any $a \in \mathbb{R}$, we have*

$$\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \mathrm{Tr}(F(h)^2) = \int_{T \times \mathbb{P}^1} c_2(\mathcal{P}_a E).$$

Proof Let $U \subset \mathbb{P}^1$ be a small neighbourhood of ∞ such that $\mathcal{P}_a E_{|T \times w}$ is semistable of degree 0 for any $w \in U$. In the following argument, we will shrink U. We fix a lift of $\mathcal{S}p_\infty(\mathcal{P}_* E) \subset T^\vee$ to $\mathbb{C}$. We have the filtered Higgs bundle $(\mathcal{P}_* V, g\, dw)$ on (U, ∞) corresponding to $\mathcal{P}_* E$.

Let p: $T \times U \to U$ be the projection. We have a natural C^∞ isomorphism $\mathcal{P}_a E \simeq p^*(\mathcal{P}_a V)$, and the holomorphic structure of $\mathcal{P}_a E$ is described as $p^*(\overline{\partial}_{\mathcal{P}_a V}) + g\, d\bar{z}$.

We take a holomorphic frame $\boldsymbol{v}$ of $\mathcal{P}_a V$ which is compatible with the parabolic structure. It induces a C^∞ frame $\boldsymbol{u}$ of $\mathcal{P}_a E$ on $T \times U$. We take a C^∞ metric h_0 of $\mathcal{P}_a E$ such that $\boldsymbol{u}$ is orthonormal with respect to $h_{0|T \times U}$. We take a connection $\nabla^{(0)}$ of $\mathcal{P}_a E$ such that $\nabla^{(0)} u_i = 0$ on $T \times U$. We set $A := \nabla - \nabla^{(0)}$.

Let J be the endomorphism of $E_{|T \times (U \setminus \{\infty\})}$ which is determined by $\nabla_w u_i = J(u_i)$ $(i = 1, \dots, \operatorname{rank} E)$. According to Lemma 5.21 and Theorem 5.11, we have $J = O(|w|^{-1})$ with respect to h. On $T \times U$, we have

$$A = J\, dw + g\, d\bar{z} - g_h^\dagger\, dz.$$

Here, $g_h^\dagger$ denotes the adjoint of g with respect to h. We have $|g|_h = |g^\dagger|_h = O(1)$. According to Theorem 5.11 and Proposition 5.18, we have

$$[g, g_h^\dagger] = O(|w|^{-2}(\log|w|)^{-2})$$

with respect to h. According to Lemma 5.21 and Theorem 5.11, we have $[g, J] = O(|w|^{-2})$ and $[g_h^\dagger, J] = O(|w|^{-2})$ with respect to h. Hence, we have

$$A^2 = O(|w|^{-2})\, dw\, d\bar{z} + O(|w|^{-2})\, dw\, dz + O(|w|^{-2})\, dz\, d\bar{z}.$$

We set $\nabla^{(t)} := t\nabla + (1-t)\nabla^{(0)}$ for $0 \le t \le 1$. On $T \times (U \setminus \{\infty\})$, we have the following estimate for some $\rho > 0$, which is uniform for t:

$$F(\nabla^{(t)}) = tF(\nabla) + (t^2 - t)A^2 = O(|w|^{-2})\, dw\, d\overline{w} + O(|w|^{-1-\rho})\, dw\, d\bar{z} + O(|w|^{-1-\rho})\, d\overline{w}\, dz + O(|w|^{-2}) dz\, d\bar{z}. \tag{76}$$

We obtain the following estimate, which is uniform for t:

$$\begin{aligned}(77)\quad \mathrm{Tr}(F(\nabla^{(t)})A) = O(|w|^{-2})\,dw\,d\overline{w}\,dz + O(|w|^{-2})\,dw\,d\overline{w}\,d\bar{z}\\ + O(|w|^{-1-\rho})\,dw\,dz\,d\bar{z} + O(|w|^{-1-\rho})\,d\overline{w}\,dz\,d\bar{z}.\end{aligned}$$

We finally get

$$\begin{aligned}(78)\qquad -\frac{1}{8\pi^2}\int_{T\times\mathbb{C}} \mathrm{Tr}(F(h)^2) &= -\frac{1}{8\pi^2}\int_{T\times\mathbb{P}^1} \mathrm{Tr}(F(\nabla^{(0)})^2)\\ &= \int_{X\times\mathbb{P}^1} \mathrm{ch}_2(\mathcal{P}_a E) = -\int_{X\times\mathbb{P}^1} c_2(\mathcal{P}_a E).\end{aligned}$$

□

6.2 Cohomology

Let (E,∇,h) be an L^2 instanton on $X := T\times\mathbb{C}$. The $(0,1)$–part of ∇ is denoted by $\overline{\partial}_E$. Let $\overline{X} := T\times\mathbb{P}^1$. We put $D := T\times\{\infty\}$. Let $A_c^{0,i}(E)$ denote the space of C^∞ sections of $E\otimes\Omega^{0,i}$ on X with compact supports. Its cohomology group is denoted by $H_c^{0,i}(X,E)$. Let $A^{0,i}(\mathcal{P}_a E)$ denote the space of C^∞ sections of $\mathcal{P}_a E\otimes\Omega^{0,i}$ on $\overline{X}$. Its cohomology group is $H^i(\overline{X},\mathcal{P}_a E)$. In this subsection, we suppose that

$$0\notin \mathcal{S}p_\infty(E).$$

Proposition 6.7 *The natural map $H_c^{0,i}(X,E)\to H^{0,i}(\overline{X},\mathcal{P}_a E)$ is an isomorphism for any $a\in\mathbb{R}$.*

Proof There exists $R>0$ such that, if $|w|>R$, $E_{|T\times\{w\}}$ is semistable of degree 0, and $0\notin \mathcal{S}p(E_{|T\times\{w\}})$. We have two consequences for a C^∞ section s of $\mathcal{P}_a E$ on X_R:

- There exists a C^∞ section t of $\mathcal{P}_a E$ on X_R such that $\nabla_{\bar{z}} t = s$.
- If $\nabla_{\bar{z}} s = 0$, then $s=0$.

Then the claim can be shown easily. □

Let $A_{L^2}^{0,i}(E)$ be the space of L^2 sections s of $E\otimes\Omega^{0,i}$ on X so that $\overline{\partial}_E s$ is also L^2. Here, we consider the L^2 conditions with respect to h and the Euclidean metric of X. The cohomology group of the complex $(A_{L^2}^{0,\bullet}(E),\overline{\partial}_E)$ is denoted by $H_{L^2}^{0,i}(X,E)$.

Proposition 6.8 *The natural map $H_c^{0,i}(X,E)\to H_{L^2}^{0,i}(X,E)$ is an isomorphism.*

Proof Let $A_{L^2,c}^{0,i}(E)\subset A_{L^2}^{0,i}(E)$ be the subspace of the sections with compact supports. It gives a subcomplex, and its cohomology is denoted by $H_{L^2,c}^{0,i}(X,E)$.

Lemma 6.9 *The natural map $H_{L^2,c}^{0,i}(X,E)\to H_{L^2}^{0,i}(X,E)$ is an isomorphism.*

Proof For any L^2 section s of E on X_R, there exists an L^2 section t of E on $X_{R'}$ $(R' > R)$ such that $\nabla_{\bar{z}} t = s$ on $X_{R'}$. If $\nabla_{\overline{w}} s$ is L^2, then $\nabla_{\overline{w}} t$ is also L^2. If an L^2 section s of E on X_R satisfies $\nabla_{\bar{z}} s = 0$, then we have $s = 0$. Then the claim of the lemma can be shown. □

We take a smooth Kähler metric $g_{\overline{X}}$ of $\overline{X}$ and a Hermitian metric $h_{\mathcal{P}_a E}$ of $\mathcal{P}_a E$. Let $B^{0,i}_{L^2}(\mathcal{P}_a E)$ be the space of L^2 sections ω of $\mathcal{P}_a E$ on $\overline{X}$ such that $\overline{\partial}\omega$ is L^2, where we consider the L^2 condition with respect to $g_{\overline{X}}$ and $h_{\mathcal{P}_a E}$. Let

$$B^{0,i}_{L^2,c}(\mathcal{P}_a E) \subset B^{0,i}_{L^2}(\mathcal{P}_a E)$$

denote the subspace of the sections whose support is contained in X. By the same argument, the natural map $B^{0,\bullet}_{L^2,c}(\mathcal{P}_a E) \to B^{0,\bullet}_{L^2}(\mathcal{P}_a E)$ is a quasi-isomorphism. We have a natural identification

$$B^{0,\bullet}_{L^2,c}(\mathcal{P}_a E) = A^{0,\bullet}_{L^2,c}(E)$$

as $\mathbb{C}$–linear spaces. By the L^2 Dolbeault theorem (see Fujiki [18]), the cohomology group of $B^{0,\bullet}_{L^2}(\mathcal{P}_a E)$ is naturally isomorphic to $H^i(\overline{X}, \mathcal{P}_a E)$. Then the claim of Proposition 6.8 follows. □

Corollary 6.10 $H^{0,i}_{L^2}(\overline{X}, E)$ *is finite-dimensional.* □

Proposition 6.11 *We have* $H^0(\overline{X}, \mathcal{P}_a E) = H^2(\overline{X}, \mathcal{P}_a E) = 0$.

Proof Clearly $H^0(\overline{X}, \mathcal{P}_a(E)) = 0$. Let $p\colon \overline{X} \to \mathbb{P}^1$ be the projection. We have $p_* E = 0$, and the support of $R^1 p_* E$ is 0–dimensional. Then, $H^2(\overline{X}, \mathcal{P}_a E) = 0$. □

6.3 Exponential decay of harmonic sections

6.3.1 Statement Let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$. Let $\overline{\partial}_E$ denote the $(0,1)$–part of ∇, and let $\overline{\partial}^*_E$ denote the formal adjoint with respect to h and $dz\, d\bar{z} + dw\, d\overline{w}$. We set

$$\Delta_E := \overline{\partial}^*_E \overline{\partial}_E + \overline{\partial}_E \overline{\partial}^*_E.$$

Proposition 6.12 *Assume that* $0 \notin \mathcal{S}p_\infty(E)$. *Let* ω *be an* L^2 *section of* $E \otimes \Omega^{0,1}$ *on* $T \times \mathbb{C}$ *such that* $\Delta_E \omega = 0$. *Then we have* $|\omega| = O(\exp(-C|w|))$ *for some* $C > 0$.

6.3.2 An estimate Take $R > 0$, and put $Y_R := \{|w| \geq R\}$ and $X_R := T \times Y_R$. Let (E, ∇, h) be an L^2 instanton on X_R.

Lemma 6.13 *Assume that* $0 \notin \mathcal{S}p_\infty(E)$. *Suppose that* ω *is an* L^2 *section of* $E \otimes \Omega^{0,1}_{Y_R}$ *such that* $\overline{\partial}_E \omega = \overline{\partial}^*_E \omega = 0$. *Then there exists* $C > 0$ *such that*

$$|\omega|_h = O(\exp(-C|w|)).$$

Proof Let $\omega = f\,d\bar{z} + g\,d\overline{w}$ be a harmonic section. We have $-\nabla_{\overline{w}} f + \nabla_{\bar{z}} g = 0$ and $\nabla_z f + \nabla_w g = 0$. We have

$$\begin{aligned}(79)\qquad \nabla_w \nabla_{\overline{w}} f &= \nabla_w(\nabla_{\bar{z}} g) = \nabla_{\bar{z}} \nabla_w g + F_{w\bar{z}} g = -\nabla_{\bar{z}} \nabla_z f + F_{w\bar{z}} g\\ &= -\nabla_z \nabla_{\bar{z}} f + F_{z\bar{z}} f + F_{w\bar{z}} g,\\ (80)\qquad \nabla_w \nabla_{\overline{w}} g &= F_{w\overline{w}} g + \nabla_{\overline{w}} \nabla_w g = F_{w\overline{w}} g + \nabla_{\overline{w}}(-\nabla_z f)\\ &= F_{w\overline{w}} g - \nabla_z \nabla_{\overline{w}} f + F_{z\overline{w}} f = F_{w\overline{w}} g - \nabla_z \nabla_{\bar{z}} g + F_{z\overline{w}} f.\end{aligned}$$

We obtain

$$\begin{aligned}(81)\qquad -(\partial_w \partial_{\overline{w}} + \partial_z \partial_{\bar{z}})(f, f) &\le -(\nabla_{\bar{z}} f, \nabla_{\bar{z}} f) - 2\,\mathrm{Re}((\nabla_w \nabla_{\overline{w}} + \nabla_z \nabla_{\bar{z}}) f, f)\\ &= -(\nabla_{\bar{z}} f, \nabla_{\bar{z}} f) - 2\,\mathrm{Re}(F_{z\bar{z}} f + F_{z\overline{w}} g, f).\end{aligned}$$

Using the notation in Section 5.2.2, we obtain

$$-\partial_w \partial_{\overline{w}} \|f\|^2 \le -\|\nabla_{\bar{z}} f\|^2 + O(\|F\|(\|f\|^2 + \|g\|^2)).$$

Similarly, we obtain

$$-\partial_w \partial_{\overline{w}} \|g\|^2 \le -\|\nabla_{\bar{z}} g\|^2 + O(\|F\|(\|f\|^2 + \|g\|^2)).$$

By the assumption $0 \notin \mathcal{S}p_\infty(E)$, there exist $R_1 > R$ and $C_1 > 0$ such that if $|w| \ge R_1$ then we have $\|\partial_{\bar{z}} g\|(w) \ge C_1 \|g\|(w)$ and $\|\partial_{\bar{z}} f\|(w) \ge C_1 \|f\|(w)$. Hence, there exist $\epsilon > 0$ and $R_2 > R$ such that if $|w| > R_2$ then

$$(82)\qquad -\partial_w \partial_{\overline{w}}(\|f\|^2 + \|g\|^2) \le -\epsilon(\|f\|^2 + \|g\|^2).$$

In general, if φ is a positive L^1–subharmonic function on Y_{R_2}, then $\varphi(w) = O(|w|^{-2})$. Indeed, by the mean value property, we have

$$\varphi(w) \le \frac{4}{\pi(|w| - R_2)^2} \int_{|w - w'| \le (|w| - R_2)/2} \varphi(w') \le \frac{C_2}{(|w| - R_2)^2}.$$

Hence, we have $\|f\|^2 + \|g\|^2 = O(|w|^{-2})$. Then by a standard argument with (82), we obtain $\|f\|^2 + \|g\|^2 = O(\exp(-C_3|w|))$. (See the proof of Lemma 5.12.) By a bootstrapping argument, we obtain $|f(z, w)| = O(\exp(-C_4|w|))$ and $|g(z, w)| = O(\exp(-C_4|w|))$. □

6.3.3 Finiteness We continue to use the notation in Section 6.3.2. Let ω be a C^∞ section of $E \otimes \Omega^{0,1}$ on $X_{\underline{R}}$. Suppose that the support of ω is contained in $T \times \{|w| \ge R + 1\}$. We set $\mathcal{D} := \bar{\partial}_E + \bar{\partial}_E^*$. Let $d\mathrm{vol}$ denote the volume form induced by the Euclidean metric.

Lemma 6.14 *Assume that ω and $\Delta_E\omega$ are L^2. Then $\overline{\partial}_E^*\omega$ and $\overline{\partial}_E\omega$ are L^2, and we have*
$$\int h(\omega, \Delta_E\omega)d\mathrm{vol} = \int |\mathcal{D}\omega|_h^2 d\mathrm{vol}.$$

Proof Let $g := dz\, d\bar{z} + dw\, d\overline{w}$. Let $|\cdot|_{h,g}$ denote the norm of sections of $E\otimes\Omega^\bullet$ induced by h and g. Let $\chi(t)$ be a nonnegative valued C^∞ function such that $\chi(t)=1$ $(t\le 0)$ and $\chi(t)=0$ $(t\ge 1)$, and that $\partial_t(\chi)/\chi^{1/2}$ is also C^∞. For a large N, we put $\chi_N(w) := \chi(\log|w|-N)$. Because
$$\chi_N^{-1/2}(w)\partial_w\chi_N(w) = (\chi^{-1/2}\partial_t\chi)(\log|w|-N)\cdot(2w)^{-1},$$
there exists $C_1>0$ such that $|\chi_N^{-1/2}\partial_w\chi_N|\le C_1|w|^{-1}$ and $|\chi_N^{-1/2}\partial_{\overline{w}}\chi_N|\le C_1|w|^{-1}$. We have
$$\begin{aligned}(83)\quad &\left|\int \chi_N h(\omega,\Delta_E\omega)d\mathrm{vol} - \int \chi_N|\mathcal{D}\omega|_h^2 d\mathrm{vol}\right|\\ &\le \left(\int |\overline{\partial}\chi_N|_g^2\chi_N^{-1}|\omega|_{h,g}^2 d\mathrm{vol}\right)^{1/2}\left(\int \chi_N|\overline{\partial}\omega|_{h,g}^2 d\mathrm{vol}\right)^{1/2}\\ &\quad + \left(\int |\partial\chi_N|_g^2\chi_N^{-1}|\omega|_{h,g}^2 d\mathrm{vol}\right)^{1/2}\left(\int \chi_N|\overline{\partial}^*\omega|_{h,g}^2 d\mathrm{vol}\right)^{1/2}.\end{aligned}$$
There exist $C_i>0$ $(i=2,3)$ such that for any N, we have
$$\int \chi_N|\mathcal{D}\omega|_{h,g}^2 d\mathrm{vol} \le C_2\left(\int \chi_N|\mathcal{D}\omega|_{h,g}^2 d\mathrm{vol}\right)^{1/2} + C_3.$$
Then the first claim of Lemma 6.14 follows. We have
$$\begin{aligned}(84)\quad &\left|\int h(\chi_N\omega,\Delta_E\omega)d\mathrm{vol} - \int \chi_N|\mathcal{D}\omega|_{h,g}^2 d\mathrm{vol}\right|\\ &\le C_4\int\left(|\overline{\partial}\chi_N|_g\,|\omega|_{h,g}\,|\overline{\partial}\omega|_{h,g} + |\partial\chi_N|_g\,|\omega|_{h,g}\,|\overline{\partial}^*\omega|_{h,g}\right)d\mathrm{vol}\end{aligned}$$
for some $C_4>0$. By the first claim, the integrands of the right-hand side are dominated by some integrable functions, independently from N. By taking the limit, we obtain the second claim. □

6.3.4 Proof of Proposition 6.12 Let us return to the setting in Section 6.3.1. According to Lemma 6.13, we have only to prove the following lemma to establish Proposition 6.12.

Lemma 6.15 $$\overline{\partial}_E\omega = \overline{\partial}_E^*\omega = 0.$$

Proof By the first claim of Lemma 6.14, $\mathcal{D}\omega$ is L^2. By the argument in the proof of the second claim of the same lemma, we obtain $\int |\mathcal{D}\omega|^2_{h,g} d\mathrm{vol} = 0$, ie $\mathcal{D}\omega = 0$. □

6.4 Nahm transform for L^2 instantons

Let (E, ∇, h) be an L^2 instanton on $T \times \mathbb{C}$ with $\operatorname{rank} E > 1$. Let $D := \mathcal{S}p_\infty(E)$. We shall construct a harmonic bundle on $T^\vee \setminus D$ with the method in [14; 26]. Let $\Phi\colon \mathbb{C} \to T^\vee$ denote the projection. For any $\zeta \in \mathbb{C} \setminus \Phi^{-1}(D)$, let $\mathcal{L}_{-\zeta} = (\underline{\mathbb{C}}, \overline{\partial}_T - \zeta d\bar{z})$ denote the corresponding line bundle on T with the natural Hermitian metric. Let $\mathrm{Nahm}(E,\nabla)_\zeta$ denote the space of L^2 harmonic sections of $E \otimes \mathcal{L}_{-\zeta} \otimes \Omega^{0,1}$. It is finite-dimensional, and naturally isomorphic to

$$H^1(T \times \mathbb{P}^1, \mathcal{P}_{-1}E \otimes \mathcal{L}_{-\zeta}) \simeq H^1(T \times \mathbb{P}^1, \mathcal{P}_0 E \otimes \mathcal{L}_{-\zeta}).$$

(See Section 6.2.) The Euclidean metric $dz\, d\bar{z} + dw\, d\overline{w}$ of $T \times \mathbb{C}$ and the Hermitian metric h of E induce a metric $h_1(\zeta)$ of $\mathrm{Nahm}(E,\nabla)_\zeta$. The multiplication of $-w \in \mathcal{O}_{\mathbb{P}^1}(1)$ induces an endomorphism $F_w(\zeta)$ of $\mathrm{Nahm}(E,\nabla)_\zeta$. It is also described as $-P_\zeta \circ w$, where P_ζ denotes the orthogonal projection of the space of L^2 sections of $E \otimes \mathcal{L}_{-\zeta} \otimes \Omega^{0,1}_{T\times\mathbb{C}}$ onto $\mathrm{Nahm}(E,\nabla)_\zeta$. (Note Proposition 6.12.)

Let $A^{p,q}(E \otimes \mathcal{L}_{-\zeta})$ denote the space of L^2 sections of the bundle $E \otimes \mathcal{L}_{-\zeta} \otimes \Omega^{p,q}_{T\times\mathbb{C}}$. Let $\overline{\partial}_{E,\zeta}$ denote the $\overline{\partial}$–operator of $E \otimes \mathcal{L}_{-\zeta}$ and let $\overline{\partial}^*_{E,\zeta}$ denote its adjoint. Let $\mathcal{D}_\zeta := \overline{\partial}_{E,\zeta} + \overline{\partial}^*_{E,\zeta}$ be a closed operator

$$A^{0,0}(E \otimes \mathcal{L}_{-\zeta}) \oplus A^{0,2}(E \otimes \mathcal{L}_{-\zeta}) \to A^{0,1}(E \otimes \mathcal{L}_{-\zeta}),$$

and let $\mathcal{D}^*_\zeta := \overline{\partial}_{E,\zeta} + \overline{\partial}^*_{E,\zeta}$ denote its adjoint

$$A^{0,1}(E \otimes \mathcal{L}_{-\zeta}) \to A^{0,0}(E \otimes \mathcal{L}_{-\zeta}) \oplus A^{0,2}(E \otimes \mathcal{L}_{-\zeta}).$$

By the results in Section 6.2, we obtain that $\mathcal{D}^*_\zeta$ is surjective. We have

$$\mathrm{Ker}(\mathcal{D}^*_\zeta) = \mathrm{Nahm}(E,\nabla)_\zeta.$$

The family $\bigcup_\zeta \mathrm{Nahm}(E,\nabla)_\zeta$ gives a C^∞ bundle on $\mathbb{C} \setminus \Phi^{-1}(D)$. Because it is naturally $L^\vee$–equivariant, it induces a bundle $\mathrm{Nahm}(E,\nabla)$ on $T^\vee \setminus D$. It is equipped with a C^∞ metric h_1 and a C^∞ endomorphism F_w. It is also equipped with the induced unitary connection ∇_1. The C^∞ bundle $\mathrm{Nahm}(E,\nabla)$ is also constructed as the descent of the family of the cohomology of the complexes of the closed operators $(A^{0,\bullet}(E \otimes \mathcal{L}_{-\zeta}), \overline{\partial}_{E,\zeta})$. It induces a holomorphic structure of $\mathrm{Nahm}(E,\nabla)$ as a bundle on $T^\vee \setminus D$, and F_w is holomorphic. We set $\theta_1 := F_w\, d\overline{\zeta}$. The $(0,1)$–part of ∇_1 is equal to the $\overline{\partial}$–operator of $\mathrm{Nahm}(E,\nabla)$.

Proposition 6.16 *$(E_1, \overline{\partial}_{E_1}, \theta_1, h_1)$ is a wild harmonic bundle.*

Proof Because the argument is rather standard, we give only an indication for the convenience of the readers. For $I \subset \{1,2,3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i^{th} components. By the construction, we have a natural isomorphism $(E_1, \overline{\partial}_{E_1}) \simeq Rp_{1*}(p_{23}^*\mathcal{P}_0 E \otimes p_{12}^*\mathcal{P}oin^{-1})_{|T^\vee\setminus D}$. The endomorphism F_w is equal to the multiplication of

$$-w\colon Rp_{1*}(p_{23}^*\mathcal{P}_{-1} E \otimes p_{12}^*\mathcal{P}oin^{-1})_{|T^\vee\setminus D} \to Rp_{1*}(p_{23}^*\mathcal{P}_0 E \otimes p_{12}^*\mathcal{P}oin^{-1})_{|T^\vee\setminus D}.$$

Hence, we obtain that θ is a wild Higgs field in the sense that, for the local expression $\theta = f\,d\zeta$ around $P \in D$, the coefficients of the characteristic polynomial $\det(t\,\mathrm{id} - f)$ are meromorphic at P.

Let us prove that $(E_1, \overline{\partial}_{E_1}, \theta_1, h_1)$ is a harmonic bundle. We have only to prove that $(E_1, \overline{\partial}_{E_1}, \theta_1, h_1)_{|U}$ is a harmonic bundle for any small open subset $U \subset T^\vee \setminus D$. By fixing a lift of U to $\mathbb{C} \setminus \Phi^{-1}(D)$, we use the holomorphic coordinate ζ on U.

Let Δ_E denote the Laplacian on $A^{0,0}(E)$, ie $\Delta_E = \overline{\partial}_E^*\overline{\partial}_E = -\sqrt{-1}\Lambda\partial_E\overline{\partial}_E$. We have

$$\Delta_E\psi = -2(\nabla_z\nabla_{\bar z} + \nabla_w\nabla_{\overline{w}})\psi.$$

On $A^{0,2}(E)$, the Laplacian is given by $\overline{\partial}_E\overline{\partial}_E^* = (-\sqrt{-1})\overline{\partial}_E\Lambda\partial_E$. We have

$$\overline{\partial}_E\overline{\partial}_E^*(\psi\,d\bar z\,d\overline{w}) = -2(\nabla_{\bar z}\nabla_z + \nabla_{\overline{w}}\nabla_w)\psi\,d\bar z\,d\overline{w}.$$

Because $F_{z\bar z} + F_{w\overline{w}} = 0$, it is equal to $\Delta_E(\psi)\,d\bar z\,d\overline{w}$. Hence, under the natural identification $A^{0,0}(E) \oplus A^{0,2}(E) \simeq A^{0,0}(E) \otimes \langle\langle 1,\ d\bar z\,d\overline{w}\rangle\rangle$, the Laplacian $\mathcal{D}^*\mathcal{D}$ acts as $\Delta_E \otimes \mathrm{id}$, where $\langle\langle a,b\rangle\rangle$ denotes the 2–dimensional vector space generated by a, b. The Green operator of $\mathcal{D}^*\mathcal{D}$ acts as $\mathcal{G}_E \otimes \mathrm{id}$, where $\mathcal{G}_E$ denotes the Green operator for Δ_E on $A^{0,0}(E)$.

We naturally identify $A^{p,q}(E \otimes \mathcal{L}_{-\zeta})$ with $A^{p,q}(E)$. For a differential form τ, let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau \wedge \varphi$. We have

$$\overline{\partial}_{E,\zeta} = \overline{\partial}_E - \zeta\mu(d\bar z) \quad \text{and} \quad \overline{\partial}_{E,\zeta}^* = \overline{\partial}_E^* + \sqrt{-1}\cdot\bar\zeta\Lambda \circ \mu(dz).$$

Let d_U denote the trivial connection of the product vector bundle $A^{0,1}(E) \times U$ over U. For the operators on the space of the sections $U \to A^{0,1}(E) \times U$ we have

$$[d_U, \overline{\partial} + \zeta\,d\bar z] = d\zeta\,\mu(d\bar z), \quad [d_U, (\overline{\partial} + \zeta\,d\bar z)^*] = \sqrt{-1}\,d\bar\zeta\Lambda \circ \mu(dz).$$

We set $\Omega := d\zeta\,\mu(d\bar z) + d\bar\zeta\sqrt{-1}\Lambda\circ\mu(dz)$. Let P_ζ denote the orthogonal projection of $A^{0,1}(E)$ onto the kernel of $\mathcal{D}_\zeta^*$. Let $\Delta_\zeta = \overline{\partial}_{E,\zeta}^*\overline{\partial}_{E,\zeta}$ denote the Laplacian on $A^{0,0}(E)$

for $E\otimes\mathcal{L}_{-\zeta}$. Let $\mathcal{G}_\zeta$ denote the Laplacian for Δ_ζ on $A^{0,0}(E)$, ie $\mathcal{G}_\zeta\Delta_\zeta=\mathrm{id}_{A^{0,0}(E)}$. The Green operator G_ζ for $\mathcal{D}_\zeta^*\mathcal{D}_\zeta$ on $A^{0,0}\oplus A^{0,2}$ is given by $\mathcal{G}_\zeta\otimes\mathrm{id}$. We have

$$P_\zeta=1-\mathcal{D}_\zeta\circ G_\zeta\circ\mathcal{D}_\zeta^*.$$

Let $\mathcal{G}_\zeta\otimes\mathrm{id}$ also denote the naturally induced operator on $A^{0,1}\simeq A^{0,0}\otimes\langle\langle d\bar{z},d\overline{w}\rangle\rangle$.

Let $\langle\cdot,\cdot\rangle$ denote the inner product of $A^{0,\bullet}(E)$ induced by h and $dz\,d\bar{z}+dw\,d\overline{w}$. By a standard computation, the curvature F of the connection ∇_1 is described as follows, for any sections ψ_i $(i=1,2)$ of $\mathrm{Nahm}(E,\nabla)$:

$$\begin{aligned}(85)\quad \langle\psi_1,F\psi_2\rangle&=\langle\psi_1,d_U\circ P_\zeta(d_U\psi_2)\rangle\\ &=-\langle\psi_1,d_U\circ\mathcal{D}_\zeta\circ G_\zeta\circ\mathcal{D}_\zeta^*(d_U\psi_2)\rangle=\langle d_U\psi_1,\mathcal{D}_\zeta\circ G_\zeta\circ\mathcal{D}_\zeta^*(d_U\psi_2)\rangle\\ &=\langle\mathcal{D}_\zeta^*d_U\psi_1,G_\zeta\circ\mathcal{D}_\zeta^*(d_U\psi_2)\rangle=\langle\Omega\psi_1,G_\zeta\Omega\psi_2\rangle\\ &=d\zeta\,d\overline{\zeta}(\langle d\bar{z}\psi_1,d\bar{z}(\mathcal{G}_\zeta\otimes\mathrm{id})\psi_2\rangle-\langle\Lambda(dz\psi_1),\Lambda(dz(\mathcal{G}_\zeta\otimes\mathrm{id})\psi_2)\rangle).\end{aligned}$$

We have $\theta(\psi)=P_\zeta(w\psi)d\zeta$ and $\theta^\dagger(\psi)=P_\zeta(\overline{w}\psi)d\overline{\zeta}$. We have

$$\begin{aligned}(86)\quad &\langle\psi_1,(P_\zeta w\circ P_\zeta\overline{w}-P_\zeta\overline{w}\circ P_\zeta w)\psi_2\rangle\,d\overline{\zeta}\,d\zeta\\ &=-\langle\psi_1,(w(P_\zeta-1)\overline{w}-\overline{w}(P_\zeta-1)w)\psi_2\rangle d\zeta\,d\overline{\zeta}\\ &=(\langle\overline{w}\psi_1,\mathcal{D}_\zeta G_\zeta\mathcal{D}_\zeta^*\overline{w}\psi_2\rangle-\langle w\psi_1,\mathcal{D}_\zeta G_\zeta\mathcal{D}_\zeta^*w\psi_2\rangle)\,d\zeta\,d\overline{\zeta}\\ &=(\langle\mathcal{D}_\zeta^*(\overline{w}\psi_1),G_\zeta\mathcal{D}_\zeta^*(\overline{w}\psi_2)\rangle-\langle\mathcal{D}_\zeta^*(w\psi_1),G_\zeta\mathcal{D}_\zeta^*(w\psi_2)\rangle)d\zeta\,d\overline{\zeta}\\ &=(\langle[\mathcal{D}_\zeta^*,\overline{w}]\psi_1,G_\zeta[\mathcal{D}_\zeta^*,\overline{w}]\psi_2\rangle-\langle[\mathcal{D}_\zeta^*,w]\psi_1,G_\zeta[\mathcal{D}_\zeta^*,w]\psi_2\rangle)d\zeta\,d\overline{\zeta}.\end{aligned}$$

We have $[\mathcal{D}_\zeta^*,\overline{w}]=\mu(d\overline{w})$ and $[\mathcal{D}_\zeta^*,w]=-\sqrt{-1}\Lambda\circ\mu(dw)$. Hence, we obtain

$$\begin{aligned}(87)\quad &\langle\psi_1,(P_\zeta w\circ P_\zeta\overline{w}-P_\zeta\overline{w}\circ P_\zeta w)\psi_2\,d\zeta\,d\overline{\zeta}\rangle\\ &=(\langle d\overline{w}\psi_1,d\overline{w}(\mathcal{G}_\zeta\otimes\mathrm{id})\psi_2\rangle-\langle\Lambda(dw\psi_1),\Lambda(dw(\mathcal{G}_\zeta\otimes\mathrm{id})\psi_2)\rangle)\,d\zeta\,d\overline{\zeta}.\end{aligned}$$

By using $\langle d\bar{z}d\overline{w},d\bar{z}d\overline{w}\rangle=\langle\Lambda\,dwd\overline{w},\Lambda\,dwd\overline{w}\rangle=\langle\Lambda\,dzd\bar{z},\Lambda\,dzd\bar{z}\rangle$ for the metric on $\Omega^\bullet_{T\times\mathbb{C}}$, we get

$$\langle\psi_1,(F+(P\circ w\,d\zeta)\circ(P\circ\overline{w}d\overline{\zeta}))\psi_2\rangle=0.$$

Namely, the Hitchin equation is satisfied. Thus, Proposition 6.16 follows. □

Remark 6.17 We obtain a different transformation by replacing $\mathcal{L}_{-\zeta}$ with $\mathcal{L}_\zeta$, for which we do not need any essential change.

Remark 6.18 We use the operators that are natural the complex geometry, instead of the Dirac operator itself.

7 L^2 instantons and wild harmonic bundles

7.1 Nahm transform for wild harmonic bundles on $T^\vee$

7.1.1 Construction Let D be a nonempty finite subset of $T^\vee$. We fix a Kähler metric $g_{T^\vee\setminus D}$ of $T^\vee \setminus D$, which is Poincaré-like around D. Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on $(T^\vee, D)$. For simplicity, we assume the following:

- $(E, \overline{\partial}_E, \theta, h)$ has a singularity at each point P of D, ie P is a pole of θ, or the parabolic structure at P is nontrivial.
- $(E, \overline{\partial}_E, \theta, h)$ is irreducible, ie it is not a direct sum of harmonic bundles of positive ranks.

We shall construct an L^2 instanton from $(E, \overline{\partial}_E, \theta, h)$ with the method in [14; 24]. Let $H^i_{L^2}(E, \overline{\partial}_E, \theta, h)$ denote the i^{th} L^2 cohomology group of $(E, \overline{\partial}_E, \theta, h)$. By assumption, the associated filtered Higgs bundle $(\mathcal{P}_*E, \theta)$ is stable of degree 0. As recalled in Lemma 3.1, they are isomorphic to the hypercohomology groups of the complex $\mathcal{C}^\bullet(\mathcal{P}_*E \otimes \Omega^\bullet, \theta)$. In particular, they are finite-dimensional, and isomorphic to the space of L^2 harmonic i–forms of $(E, \overline{\partial}_E, \theta, h)$. We also have $H^0_{L^2}(E, \overline{\partial}_E, \theta, h) = H^2_{L^2}(E, \overline{\partial}_E, \theta, h) = 0$ by the above assumptions.

Remark 7.1 If D is empty, $(E, \overline{\partial}_E, \theta, h)$ is isomorphic to $(L, \overline{\partial}_L, \theta, h)$ such that $\operatorname{rank} L = 1$. So we exclude the case $D = \varnothing$.

For any $(z, w) \in \mathbb{C}^2$, let $\mathcal{L}_{z,w}$ denote the harmonic bundle of rank one on $T^\vee$ given by $(\mathbb{C}, \overline{\partial} + z d\overline{\zeta})$ with the trivial metric and the Higgs field $w\, d\zeta$. Let $(E, \overline{\partial}_{E,z}, \theta_w, h)$ denote $(E, \overline{\partial}_E, \theta, h) \otimes \mathcal{L}_{z,w}$. Let $\operatorname{Nahm}(E, \overline{\partial}_E, \theta, h)_{(z,w)}$ be the space of L^2 harmonic 1–forms of $(E, \overline{\partial}_{E,z}, \theta_w, h)$. It is independent of the choice of the Poincaré-like metric $g_{T^\vee\setminus D}$. It is finite-dimensional, and naturally isomorphic to $\operatorname{Nahm}(\mathcal{P}_*E, \theta)_{(z,w)}$. It is naturally equipped with the metric h_1 induced by h.

Let $A^{p,q}(E)$ denote the space of L^2 sections of $E \otimes \Omega^{p,q}_{T^\vee\setminus D}$. Let

$$\overline{\partial}^*_{E,z} \colon A^{p,q} \to A^{p,q-1}$$

denote the adjoint of the closed operator $\overline{\partial}_{E,z} \colon A^{p,q} \to A^{p,q+1}$. Let

$$\theta^\dagger_w \colon A^{p,q} \to A^{p-1,q}$$

denote the adjoint of $\theta_w \colon A^{p,q} \to A^{p+1,q}$. We have

$$\overline{\partial}^*_{E,z} := -\sqrt{-1}[\Lambda, \partial_E - \bar{z}\, d\zeta] \quad \text{and} \quad \theta^*_w = -\sqrt{-1}[\Lambda, \theta^\dagger_w] = -\sqrt{-1}[\Lambda, \theta^\dagger + \overline{w}\, d\overline{\zeta}].$$

We set $S^+ := A^{0,0}(E) \oplus A^{1,1}(E)$ and $S^- := A^1(E) = A^{0,1}(E) \oplus A^{1,0}(E)$. Let

$$\mathcal{D}_{z,w} := \overline{\partial}_{E,z} + \theta_w + \overline{\partial}^*_{E,z} + \theta^*_w$$

be a closed operator $S^+ \to S^-$, and let

$$\mathcal{D}^*_{z,w} := \overline{\partial}_{E,z} + \theta_w + \overline{\partial}^*_{E,z} + \theta^*_w$$

denote its adjoint $S^- \to S^+$. We have $\operatorname{Ker} \mathcal{D}^*_{z,w} = \operatorname{Nahm}(E, \overline{\partial}_E, \theta, h)_{(z,w)}$; see [36]. By the vanishing of $H^i_{L^2}(E, \overline{\partial}_{E,z}, \theta_w, h)$ $(i = 0, 2)$, we obtain that $\mathcal{D}^*$ is surjective. Hence, the family $\bigcup_{(z,w)} \operatorname{Nahm}(E, \overline{\partial}_E, \theta, h)_{(z,w)}$ gives a C^∞ vector bundle on $\mathbb{C}^2$. (See [14].) It is naturally equivariant with respect to the action of L on $\mathbb{C}^2$ by $\chi \bullet (z, w) = (z + \chi, w)$. Hence, we obtain a bundle on $T \times \mathbb{C}$. It is equipped with an induced C^∞ metric h_1 and an induced unitary connection ∇_1. Because the C^∞ bundle is also constructed as a family of the cohomology of the complexes $(A^\bullet(E), \overline{\partial}_{E,z} + \theta_w)$, it is equipped with a naturally induced holomorphic structure, which is equal to the $(0,1)$–part of ∇_1. By the construction, the holomorphic bundle is naturally isomorphic to $\operatorname{Nahm}(\mathcal{P}_*E, \theta)_{|T\times\mathbb{C}}$. (See Section 7.2.2 for more details on this isomorphism.) We shall give the proof of the following theorem in Section 7.1.4 after preliminaries.

Theorem 7.2 $(\operatorname{Nahm}(E, \overline{\partial}_E, \theta, h), h_1, \nabla_1)$ *is an L^2 instanton.*

We give a remark on the proof. It is rather easy and standard to prove that the tuple $(\operatorname{Nahm}(E, \overline{\partial}_E, \theta, h), h_1, \nabla_1)$ is an instanton by using the twistor property of instantons and harmonic bundles. But, we do not give such an argument in the following. Instead, we follow another standard argument to use a description of the curvature $F(\nabla_1)$ in terms of the Green operator. Because we need an estimate for the decay of $F(\nabla_1)$, we need the description, anyway.

7.1.2 Preliminaries We give the preliminary for a general situation. Let X be a torus $\mathbb{C}_\zeta/L$. Let $D \subset X$ be a finite set. Let $g_A = A\,d\zeta\,d\overline{\zeta}$ be a Kähler metric of $X \setminus D$ for some positive valued function A, which is Poincaré-like around D. Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on $X \setminus D$. We set $\mathcal{D} := \overline{\partial}_E + \theta$. Let $\mathcal{D}^*_A$ (resp $\mathcal{D}^*_1$) denote the formal adjoint of $\mathcal{D}$ with respect to h and g_A (resp. $d\zeta\,d\overline{\zeta}$). We set $\Delta_A = \mathcal{D}^*_A\mathcal{D}$ and $\Delta_1 = \mathcal{D}^*_1\mathcal{D}$. We have $\Delta_A = A^{-1}\Delta_1$.

Lemma 7.3 *Let φ be any section of E on $X \setminus D$ such that*

$$\int |\varphi|^2_h A |d\zeta\,d\overline{\zeta}| + \int |\Delta_1\varphi|^2_h |d\zeta\,d\overline{\zeta}| < \infty.$$

Then the following integrals are finite:

$$\int |\varphi|_h^2 |d\zeta\, d\overline{\zeta}| + \int |\Delta_A \varphi|_h^2 A |d\zeta\, d\overline{\zeta}| < \infty, \tag{88}$$

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta\, d\overline{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta\, d\overline{\zeta}| = \int |\mathcal{D}\varphi|_h^2 < \infty. \tag{89}$$

Proof The finiteness (88) is clear. In (89), the first equality is trivial. The second equality and finiteness can be shown by an argument in the proof of Lemma 6.14. □

We set $\mathcal{D}^\dagger := \partial_E + \theta^\dagger$. Let $(\mathcal{D}^\dagger)^*_A$ (resp. $(\mathcal{D}^\dagger)^*_1$) denote the formal adjoint of $\mathcal{D}^\dagger$ with respect to g_A (resp. $d\zeta d\overline{\zeta}$). We have $\Delta_1 = (\mathcal{D}^\dagger)^*_1 \mathcal{D}^\dagger$ and $\Delta_A = (\mathcal{D}^\dagger)^*_A \mathcal{D}^\dagger$.

Lemma 7.4 *Let φ be as in Lemma 7.3. Then we have*

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta\, d\overline{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta\, d\overline{\zeta}| = \int |\mathcal{D}^\dagger \varphi|_h^2 < \infty. \tag{90}$$

Proof The first equality is trivial. For the second, we have only to apply Lemma 7.3 to a harmonic bundle $(E, \partial_E, \theta^\dagger, h)$ on $X \setminus D$. □

7.1.3 Estimate Let X be a torus $\mathbb{C}_\zeta / L$ with a nonempty finite subset D as in Section 7.1.2. We use the Euclidean metric $d\zeta\, d\overline{\zeta}$ of X. Let $d\mathrm{vol}_X = |d\zeta\, d\overline{\zeta}|$ denote the associated volume form. Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on (X, D). Assume that the harmonic bundle has a singularity at each point of D.

We set $\nabla_h^{(z)} := \overline{\partial}_E + \partial_E + z\, d\overline{\zeta} - \bar{z}\, d\zeta$. Let $\mathcal{H}_{z,w}$ be the space of the sections of E on $X \setminus D$ such that

$$\int_X |\varphi|_h^2 d\mathrm{vol}_X + \int_X \bigl(|\nabla_h^{(z)} \varphi|_h^2 + |(\theta + w\, d\zeta)\varphi|_h^2\bigr) < \infty.$$

Proposition 7.5 *There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that if $|w| > R$, then for any $\varphi \in \mathcal{H}_{z,w}$ we have*

$$\int_X \bigl(|\nabla_h^{(z)} \varphi|_h^2 + |(\theta + w\, d\zeta)\varphi|_h^2\bigr) \geq C |w|^\rho \int_X |\varphi|_h^2 d\mathrm{vol}_X$$

(See also a refined estimate in Proposition 7.9 below.)

Proof We use an argument in [48, Section 2.4] with an adjustment to our situation. We use the standard distance on X. We take small neighbourhoods B_P of $P \in D$. There exist $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $|(\theta + w\, d\zeta)\varphi|_h^2 \geq$

$C_1|w|^2|\varphi|_h^2 d\mathrm{vol}_X$ on $X\setminus\bigcup_{P\in D} B_P$. We have only to prove the estimate on each B_P. We may assume $P=0$, and B_P is an ϵ–ball $B_\epsilon=\{|\zeta|\leq\epsilon\}$.

We have a ramified covering $\psi\colon (B'_\epsilon,0)\to(B_\epsilon,0)$ given by $\psi(u)=u^p$ such that $\psi^*(E,\overline{\partial}_E,\theta,h)$ is unramified, ie we have a decomposition

$$\psi^*(E,\overline{\partial}_E,\theta)=\bigoplus_{\mathfrak{a}\in u^{-1}\mathbb{C}[u^{-1}]}(E_{\mathfrak{a}},\overline{\partial}_{E_{\mathfrak{a}}},\theta_{\mathfrak{a}}),$$

where the Higgs fields $\theta_{\mathfrak{a}}-d\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}$ are tame. Let $\ell:=\max\{\deg_{u^{-1}}\mathfrak{a}\mid E_{\mathfrak{a}}\neq 0\}$.

Lemma 7.6 *There exist $R'>0$ and $C'_i>0$ $(i=1,2)$ such that*

$$|\theta\varphi|_h\geq C'_1|w||d\zeta||\varphi|_h$$

on $B_\epsilon\setminus\{|\zeta|<C'_2|w|^{-p/(\ell+p)}\}$ if $|w|\geq R'$.

Proof We have only to estimate each $\theta_{\mathfrak{a}}$ on B'_ϵ. Let us consider the case $\mathfrak{a}\neq 0$. We set $n:=\deg_{u^{-1}}\mathfrak{a}$. For each w, we have the solutions $b_i(w)$ $(i=0,\dots,n+p-1)$ of the equation

$$\partial_u\mathfrak{a}(u)+pwu^{p-1}=0.$$

We have the equality $u^{-p+1}\partial_u\mathfrak{a}(u)+pw=\alpha\prod_{i=0}^{n+p-1}(u^{-1}-b_i(w)^{-1})$ for some $\alpha\in\mathbb{C}\setminus\{0\}$. We have

$$\theta_{\mathfrak{a}}=\partial_u\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}\,du+g_{\mathfrak{a}}\,du,$$

where $|g_{\mathfrak{a}}|_h\leq C_1|u|^{-1}$. We have $R_2>0$ and $C_2>0$ such that if $|w|>R_2$, then

$$C_2^{-1}\leq|b_i(w)||w|^{1/(n+p)}\leq C_2.$$

We take $C_3\gg C_2$. We set $\mathcal{W}_1:=\{|u|\leq C_3|w|^{-1/(n+p)}\}$.

On $B'_\epsilon\setminus\mathcal{W}_1$, we have $|g_{\mathfrak{a}}|_h\leq(C_1/C_3)|w|^{1/(n+p)}$. We also have

$$|u^{-1}-b_i(w)^{-1}|\geq|b_i(w)^{-1}|-|u^{-1}|\geq(C_2^{-1}-C_3^{-1})|w|^{1/(n+p)}$$

for any i, and hence $|u^{-p+1}\partial_u\mathfrak{a}+pw|\geq|\alpha|(C_2^{-1}-C_3^{-1})^{n+p}|w|$. Hence, if C_3 is sufficiently larger than C_2, there exist $R_4>0$ and $C_4>0$ such that if $|w|>R_4$ then

$$|(\partial_u\mathfrak{a}+pwu^{p-1})\,\mathrm{id}_{E_{\mathfrak{a}}}+g_{\mathfrak{a}}|_h\geq C_4|w||u|^{p-1}$$

Hence, we obtain the desired inequality for the integral over $B'_\epsilon\setminus\mathcal{W}_1$ in the case $\mathfrak{a}\neq 0$.

Let us consider the case $\mathfrak{a}=0$. We have the expression $\theta_0=g_0\,du$, and $|g_0|_h\leq C_{10}|u|^{-1}$ for some $C_{10}>0$. We take $C_{11}>C_{10}$, and we consider $\mathcal{W}:=\{|u|\leq C_{11}|w|^{-1/p}\}$. On $B'_\epsilon\setminus\mathcal{W}$, we have $|wu^{p-1}|\geq C_{11}^{p-1}|w|^{1/p}$. We also have $|g_0|_h\leq$

$(C_{10}/C_{11})|w|^{1/p}$. Hence, if C_{11} is sufficiently larger than C_{10}, then for some $C_{12} > 0$ we get

$$|pwu^{p-1}\,\mathrm{id}_{E_0} + g_0|_h \geq C_{12}|wu^{p-1}|.$$

Hence, we obtain the desired inequality in the case $\mathfrak{a} = 0$. □

Let φ be any L^2 section of E on B_ϵ with respect to $d\mathrm{vol}_X$, such that

$$\int_{B_\epsilon} \big(|\nabla_h^{(z)}\varphi|_h^2 + |(\theta + w\,d\zeta)\varphi|_h^2\big) < \infty.$$

We set $\mathcal{W}_1 := \{|\zeta| < 2C_2'|w|^{-p/(\ell+p)}\}$ and $\mathcal{W}_2 := \{|\zeta| < C_2'|w|^{-p/(\ell+p)}\}$. We have a kind of Poincaré inequality, ie there exist $C'' > 0$ and $R'' > 0$ such that if $|w| > R''$, then (see [7; 48, (2.12)])

$$|w|^{2p/(\ell+p)} \int_{\mathcal{W}_1} |\varphi|_h^2 |d\zeta d\overline{\zeta}| \\ \leq C''\bigg(\int_{\mathcal{W}_1} |d|\varphi|_h|^2 + |w|^{2p/(n+p)} \int_{\mathcal{W}_1\setminus\mathcal{W}_2} |\varphi|_h^2 |d\zeta\, d\overline{\zeta}|\bigg).$$

There exists C''' such that the right-hand side is dominated by

$$C'''\bigg(\int_{\mathcal{W}_1} |\nabla_h^{(z)}\varphi|_h^2 + \int_{\mathcal{W}_1\setminus\mathcal{W}_2} |(\theta + w\,d\zeta)\varphi|_h^2\bigg).$$

Thus, the proof of Proposition 7.5 is finished. □

Let $\mathcal{D}(z,w) := \overline{\partial}_E + z\,d\overline{\zeta} + \theta + w\,d\zeta$. Let $\mathcal{D}_1^*(z,w)$ denote the adjoint with respect to the Euclidean metric $d\zeta\, d\overline{\zeta}$. Let $\Delta_1(z,w) := \mathcal{D}_1^*(z,w) \circ \mathcal{D}(z,w)$. Let $g_{X\setminus D}$ be a Kähler metric of $X \setminus D$ which is Poincaré-like around D. Let $d\mathrm{vol}_{X\setminus D}$ be the volume form associated to $g_{X\setminus D}$.

Corollary 7.7 *There exist $\rho > 0$, $C > 0$ and $R > 0$ such that if $|w| > R$, then for any section φ of E such that*

$$\int |\varphi|_h^2 d\mathrm{vol}_{X\setminus D} + \int |\Delta_1(z,w)\varphi|_h^2 d\mathrm{vol}_X < \infty \tag{91}$$

we have

$$C|w|^\rho \bigg(\int |\varphi|_h^2 d\mathrm{vol}_X\bigg)^{1/2} \leq \bigg(\int |\Delta_1(z,w)\varphi|_h^2 d\mathrm{vol}_X\bigg)^{1/2}. \tag{92}$$

(See Corollary 7.11 for a refinement.)

Proof Let $\mathcal{D}^\dagger(z,w) = \partial_E - \bar{z}d\zeta + (\theta^\dagger + \overline{w}d\overline{\zeta})$. From (91) and Lemmas 7.3 and 7.4, we obtain $\int |\mathcal{D}(z,w)\varphi|_h^2 < \infty$ and $\int |\mathcal{D}^\dagger(z,w)\varphi|_h^2 < \infty$. By using the same lemmas and Proposition 7.5, there exist $C_1 > 0$ and $\rho_1 > 0$ such that

$$\begin{aligned} C_1|w|^{\rho_1}\int |\varphi|_h^2 d\mathrm{vol}_X &\le \int |\mathcal{D}(z,w)\varphi|_h^2 + \int |\mathcal{D}^\dagger(z,w)\varphi|_h^2 \\ &= 2\int h(\varphi, \Delta_1(z,w)\varphi)d\mathrm{vol}_X . \end{aligned} \tag{93}$$

Then the claim of the corollary follows. □

7.1.4 Proof of Theorem 7.2 We return to the situation in Section 7.1.1. Let $\omega_{T^\vee\setminus D}$ be the Kähler form associated to the metric $g_{T^\vee\setminus D}$. The multiplication of $\omega_{T^\vee\setminus D}$ induces an isomorphism $A^{0,0}(E)\simeq A^{1,1}(E)$. It gives an identification $S^+\simeq A^{0,0}(E)\otimes\langle\langle 1,\omega_{T^\vee\setminus D}\rangle\rangle$, where $\langle\langle 1,\omega_{T^\vee\setminus D}\rangle\rangle$ denotes the 2–dimensional vector space generated by 1 and $\omega_{T^\vee\setminus D}$. By the general theory of harmonic bundles, the Laplacian $\mathcal{D}_{zw}^*\mathcal{D}_{zw}$ on S^+ is identified with $\Delta_{zw}\otimes\mathrm{id}$ on $A^{0,0}(E)\otimes\langle\langle 1,\omega_{T^\vee\setminus D}\rangle\rangle$, where $\Delta_{zw} := (\overline{\partial}_{E,z}^* + \theta_w^*)\circ(\overline{\partial}_{E,z}+\theta_w)$ on $A^{0,0}(E)$. (See Simpson [46]. In this case, it can be easily checked directly.) The Green operator G_{zw} for $\mathcal{D}_{zw}^*\mathcal{D}_{zw}$ is identified with $\mathcal{G}_{zw}\otimes\mathrm{id}$, where $\mathcal{G}_{zw}$ is the Green operator of Δ_{zw} on $A^{0,0}(E)$.

For any simply connected open subset U_1 of T, we fix its lift U in $\mathbb{C}_z$ with respect to a universal covering $\mathbb{C}_z\to T$. We have only to check the decay condition on $U\times\mathbb{C}_w$.

For a differential form τ on $T^\vee$, let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau\wedge\varphi$. Let $d_{U\times\mathbb{C}}$ denote the trivial connection of the product vector bundle $S^-\times(U\times\mathbb{C})$ over $U\times\mathbb{C}$. We have the following relation for the operators on the space of the sections $U\times\mathbb{C}\to S^-\times(U\times\mathbb{C})$:

$$\begin{aligned} [d_{U\times\mathbb{C}}, \overline{\partial} + z\,d\overline{\zeta}] &= dz\,\mu(d\overline{\zeta}),\\ [d_{U\times\mathbb{C}}, (\overline{\partial} + z\,d\overline{\zeta})^*] &= d\bar{z}(\sqrt{-1}\Lambda\circ\mu(d\zeta)),\\ [d_{U\times\mathbb{C}}, \theta + w\,d\zeta] &= dw\,\mu(d\zeta),\\ [d_{U\times\mathbb{C}}, (\theta + w\,d\zeta)^*] &= d\overline{w}(-\sqrt{-1}\Lambda\mu(d\overline{\zeta})). \end{aligned}$$

We set $\Omega := dz\,\mu(d\overline{\zeta}) + dw\,\mu(d\zeta) + d\bar{z}(\sqrt{-1}\Lambda\mu(d\zeta)) + d\overline{w}(-\sqrt{-1}\Lambda\mu(d\overline{\zeta}))$.

Let $F(\nabla_1)$ be the curvature of the transformed bundle $\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h)$ with the metric and the unitary connection. Let P_{zw} denote the orthogonal projection of S^- onto $\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h)_{(z,w)}$. Let ψ_i be sections of $\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h)$. Let $\langle\cdot,\cdot\rangle$ denote the Hermitian pairing on $A^{p,q}(E)$ induced by h and $\omega_{T^\vee\setminus D}$. The following is a standard computation:

$$\begin{aligned}(94)\quad \langle\psi_1, F(\nabla_1)\psi_2\rangle &= \langle\psi_1, P_{zw}\circ d\circ P_{zw}d\psi_2\rangle = \langle\psi_1, d\circ P_{zw}\circ d\psi_2\rangle\\ &= \langle\psi_1, d\circ(P_{zw}-1)\circ d\psi_2\rangle = -\langle d\psi_1, (P_{zw}-1)\circ d\psi_2\rangle\\ &= \langle d\psi_1, \mathcal{D}_{zw}\circ G_{zw}\circ\mathcal{D}^*_{zw}d\psi_2\rangle = \langle\mathcal{D}^*_{zw}d\psi_1, G_{z,w}\mathcal{D}^*_{zw}d\psi_2\rangle\\ &= \langle[d,\mathcal{D}^*_{zw}]\psi_1, G_{zw}[d,\mathcal{D}^*_{zw}]\psi_2\rangle = \langle\Omega\psi_1, G_{zw}\Omega\psi_2\rangle.\end{aligned}$$

We have the expression $\psi_i = \psi_{i1}\,d\zeta + \psi_{i2}\,d\overline{\zeta}$ and

$$\begin{aligned}\Omega\psi_1 = dz\psi_{11}\,d\overline{\zeta}\,d\zeta + dw\psi_{12}\,d\zeta\,d\overline{\zeta} - \sqrt{-1}d\overline{w}\psi_{11}\Lambda(d\overline{\zeta}\,d\zeta)\\ + \sqrt{-1}\,d\overline{z}\psi_{12}\Lambda(d\overline{\zeta}\,d\zeta).\end{aligned}$$

Let A be determined by $g_{T^\vee\setminus D} = A\,d\zeta d\overline{\zeta}$. Then

$$\begin{aligned}(95)\quad G_{zw}\Omega\psi_2 = dz\mathcal{G}_{zw}(A^{-1}\psi_{21})Ad\overline{\zeta}d\zeta + dw\mathcal{G}_{zw}(A^{-1}\psi_{22})Ad\zeta d\overline{\zeta}\\ - \sqrt{-1}d\overline{w}\mathcal{G}_{zw}(\psi_{21}\Lambda(d\overline{\zeta}d\zeta)) + \sqrt{-1}d\overline{z}\mathcal{G}_{zw}(\psi_{22}\Lambda(d\overline{\zeta}d\zeta)).\end{aligned}$$

We also have

$$\begin{aligned}(96)\quad \langle\psi_{11}d\overline{\zeta}d\zeta, A\mathcal{G}_{zw}(A^{-1}\psi_{21})d\overline{\zeta}d\zeta\rangle &= \langle\psi_{11}\Lambda(d\overline{\zeta}d\zeta), \mathcal{G}_{zw}(\psi_{21}\Lambda(d\overline{\zeta}d\zeta))\rangle\\ &= 4\int(\psi_{11}, \mathcal{G}_{zw}(A^{-1}\psi_{21}))d\mathrm{vol}_{T^\vee},\\ (97)\quad \langle\psi_{12}d\overline{\zeta}d\zeta, A\mathcal{G}_{zw}(A^{-1}\psi_{22})d\overline{\zeta}d\zeta\rangle &= \langle\psi_{12}\Lambda(d\overline{\zeta}d\zeta), \mathcal{G}_{zw}(\psi_{22}\Lambda(d\overline{\zeta}d\zeta))\rangle\\ &= 4\int(\psi_{12}, \mathcal{G}_{zw}(A^{-1}\psi_{22}))d\mathrm{vol}_{T^\vee}.\end{aligned}$$

From these equalities, we obtain $(dz\,d\overline{z}+dw\,d\overline{w})\wedge\langle\psi_1, F(\nabla_1)\psi_2\rangle = 0$, which means that $\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h)$ with the induced metric h_1 and connection ∇_1 is an instanton.

Let us prove that it is an L^2 instanton. Let $(\overline{\partial}_{E,z}+\theta_w)^*_1$ denote the formal adjoint of $\overline{\partial}_{E,z}+\theta_w$ with respect to h and $d\zeta\,d\overline{\zeta}$. We set $\Delta_{zw,1} := (\overline{\partial}_{E,z}+\theta_w)^*_1(\overline{\partial}_{E,z}+\theta_w)$. Because $\Delta_{zw,1} = A\Delta_{zw}$, we have $\Delta_{zw,1}(\mathcal{G}_{zw}(A^{-1}\psi_{21})) = \psi_{21}$. We have

$$\int|\mathcal{G}_{zw}(A^{-1}\psi_{21})|^2_h d\mathrm{vol}_{T^\vee\setminus D} + \int|\psi_{21}|^2_h d\mathrm{vol}_{T^\vee} < \infty.$$

By Corollary 7.7, we have the following for some $\rho > 0$ and $C > 0$:

$$C|w|^{2\rho}\int_{T^\vee}|\mathcal{G}_{zw}(A^{-1}\psi_{21})|^2_h d\mathrm{vol}_{T^\vee} < \int_{T^\vee}|\psi_{21}|^2 d\mathrm{vol}_{T^\vee}.$$

Hence, we obtain

$$\begin{aligned}(98)\quad & \big|\langle \psi_{11} d\overline{\zeta} d\zeta, A\mathcal{G}_{zw}(A^{-1}\psi_{21})\, d\overline{\zeta}\, d\zeta\rangle\big| \\ &= \big|\langle \psi_{11}\Lambda(d\overline{\zeta}d\zeta), \mathcal{G}_{zw}(\psi_{21}\Lambda(d\overline{\zeta}d\zeta))\rangle\big| \\ &< C|w|^{-\rho}\Big(\int |\psi_{11}\, d\zeta|_h^2\Big)^{1/2}\Big(\int |\psi_{21}\, d\zeta|_h^2\Big)^{1/2}.\end{aligned}$$

We have a similar estimate for

$$|\langle \psi_{12} d\overline{\zeta} d\zeta, \mathcal{G}_{zw}(\psi_{22}) d\overline{\zeta} d\zeta\rangle| = |\langle \psi_{12}\Lambda(d\overline{\zeta}d\zeta), \mathcal{G}_{zw}(\psi_{22}\Lambda(d\overline{\zeta}d\zeta))\rangle|.$$

From those estimates, we obtain $|F(\nabla_1)| = O(|w|^{-\rho})$ for some $\rho > 0$. Because $\mathrm{Nahm}(E, \overline{\partial}_E, \theta, h) \simeq \mathrm{Nahm}(\mathcal{P}_* E, \theta)_{|U\times\mathbb{C}}$, we can apply Theorem 5.17, and hence we obtain that $F(\nabla_1)$ is L^2. Thus, the proof of Theorem 7.2 is finished. □

Remark 7.8 Using Corollary 7.11, we can directly prove the curvature is L^2.

7.1.5 Refined estimates (appendix) We refine the estimates in Section 7.1.3, ie we prove that ρ can be replaced with $1+\rho$. Although we do not use it in this paper, this type of argument seems useful in the study of a different type of Nahm transform, and so we would like to keep it.

Proposition 7.9 *There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that, if $|w| > R$, the following holds for any $\varphi \in \mathcal{H}_w$:*

$$\int_X \big(|\nabla_h^{(z)}\varphi|_h^2 + |(\theta + w\, d\zeta)\varphi|_h^2\big) \geq C|w|^{1+\rho}\int_X |\varphi|_h^2 d\mathrm{vol}_X .$$

Proof We again use the argument in [48, Section 2.4] with an adjustment to our case of wild harmonic bundles. We use the standard distance on X. We take small neighbourhoods B_P of $P \in D$. There exists $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $|(\theta + w\, d\zeta)\varphi|_h^2 \geq C_1|w|^2|\varphi|_h^2 d\mathrm{vol}_X$ on $X \setminus \bigcup_{P\in D} B_P$. We have only to prove the estimate on each B_P. We may assume $P = 0$, and B_P is an ϵ–ball $B_\epsilon = \{|\zeta| \leq \epsilon\}$.

We have a ramified covering $\psi\colon (B'_\epsilon, 0) \to (B_\epsilon, 0)$ given by $\psi(u) = u^p$ such that $\psi^*(E, \overline{\partial}_E, \theta, h)$ is unramified, ie we have the decomposition

$$(99)\qquad \psi^*(E, \overline{\partial}_E, \theta) = \bigoplus_{\mathfrak{a}\in u^{-1}\mathbb{C}[u^{-1}]} (E_{\mathfrak{a}}, \overline{\partial}_{E_{\mathfrak{a}}}, \theta_{\mathfrak{a}}),$$

where the Higgs field $\theta_{\mathfrak{a}} - d\,\mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}$ are tame. Let $h' = \bigoplus h_{|E_{\mathfrak{a}}}$, and let $\nabla^{(z)}_{h'}$ denote the unitary connection associated to $\psi^*(E, \overline{\partial}_E)$ with h'. By the asymptotic orthogonality of the decomposition (99) with respect to h (see [36]), we have the inequalities

$$\int_{B'_\epsilon} \big(|\nabla^{(z)}_h \varphi|^2_h + |(\theta + w\,d\zeta)\varphi|^2_h\big) \geq C_2 \int_{B'_\epsilon} \big(|\nabla^{(z)}_{h'} \varphi|^2_{h'} + |(\theta + w\,d\zeta)\varphi|^2_{h'}\big),$$

$$\int_{B'_\epsilon} |\varphi|^2_h \psi^* d\mathrm{vol}_X \leq C_3 \int_{B'_\epsilon} |\varphi|^2_{h'} \psi^* d\mathrm{vol}_X\,.$$

Hence, we need only the estimate with respect to the metric h'.

Let us begin with the estimate for sections of $E_{\mathfrak{a}}$ with $\mathfrak{a} \neq 0$. We set $n := \deg_{u^{-1}} \mathfrak{a}$.

Lemma 7.10 *There exist constants $R' > 0$ and $C' > 0$ such that if $|w| \geq R'$, then for any L^2 section φ of $E_{\mathfrak{a}}$ on B'_ϵ with respect to $\psi^* d\mathrm{vol}_X$ such that*

$$\int_{B'_\epsilon} \big(|\nabla^{(z)}_{h'} \varphi|^2_{h'} + |(\theta_{\mathfrak{a}} + w d\zeta)\varphi|^2_{h'}\big) < \infty$$

we have

$$|w|^e \int_{B'_\epsilon} |\varphi|^2_{h'} \psi^* d\mathrm{vol}_X < C' \int_{B'_\epsilon} \big(|\nabla^{(z)}_{h'} \varphi|^2_{h'} + |(\theta_{\mathfrak{a}} + w d\zeta)\varphi|^2_{h'}\big)$$

Here, $e = 1 + p/(n+p) > 1$.

Proof For each w, we have solutions $b_i(w)$ $(i = 0, \dots, n+p-1)$ of the equation

$$\partial_u \mathfrak{a}(u) + p w u^{p-1} = 0.$$

We have the equality $u^{-p+1}\partial_u \mathfrak{a}(u) + pw = \alpha \prod_{i=0}^{n+p-1} (u^{-1} - b_i(w)^{-1})$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$\theta_{\mathfrak{a}} = \partial_u \mathfrak{a}\,\mathrm{id}_{E_{\mathfrak{a}}}\, du + g_{\mathfrak{a}}\, du,$$

where $|g_{\mathfrak{a}}|_{h'} \leq C_1 |u|^{-1}$. We have $R_2 > 0$ and $C_2 > 0$ such that if $|w| > R_2$, then

$$C_2^{-1} \leq |b_i(w)|\,|w|^{1/(n+p)} \leq C_2.$$

We take $C_3 \gg C_2$. We set $\mathcal{U}_1 := \{|u| \leq C_3^{-1}|w|^{-1/(n+p)}\}$ and $\mathcal{U}_2 := \{|u| \leq C_3|w|^{-1/(n+p)}\}$.

Let us consider the estimate on $B'_\epsilon \setminus \mathcal{U}_2$. We have $|g_{\mathfrak{a}}|_{h'} \leq (C_1/C_3)|w|^{1/(n+p)}$. We also have

$$|u^{-1} - b_i(w)^{-1}| \geq |b_i(w)^{-1}| - |u^{-1}| \geq (C_2^{-1} - C_3^{-1})|w|^{1/(n+p)}$$

for any i, and hence $|u^{-p+1}\partial_u\mathfrak{a}+pw|\geq|\alpha|(C_2^{-1}-C_3^{-1})^{n+p}|w|$. Hence, if C_3 is sufficiently larger than C_2, we have

$$|(\partial_u\mathfrak{a}+pwu^{p-1})\varphi+g_{\mathfrak{a}}(\varphi)|_{h'}\geq C_4|w|\,|\varphi|_{h'}|u|^{p-1}$$

for some $C_4>0$. Hence, we obtain the inequality for the integral over $B'_\epsilon\setminus\mathcal{U}_2$.

Let us consider the estimate on $\mathcal{U}_1$. There exist $C_5>0$ and $R_5>0$ such that

$$|(\partial_u\mathfrak{a}+pwu^{p-1})\varphi\,du|_{h'}\geq C_5|u|^{-n-1}|\varphi|_{h'}|du|.$$

We also have $|g_{\mathfrak{a}}\varphi\,du|_{h'}\leq C_1|u^{-1}|\,|\varphi|_{h'}|du|$. Hence, there exists $C_6>0$ such that

$$\begin{aligned}|(\theta_{\mathfrak{a}}+wpu^{p-1}\,du)\varphi|^2_{h'}&\geq C_6|\varphi|^2_{h'}|u|^{-2(n+p)}|u|^{2(p-1)}|du\,d\bar{u}|\\&\geq C_6C_3|\varphi|^2_{h'}|w|^2|u|^{2(p-1)}|du\,d\bar{u}|.\end{aligned}\tag{100}$$

Therefore, we have the desired inequality for the integral over $\mathcal{U}_1$.

We consider the estimate on $\mathcal{U}_2\setminus\mathcal{U}_1$. For each $i=0,\dots,n+p-1$, we set $\widetilde{\mathcal{V}}_i:=\{|u-b_i(w)|\leq\epsilon_1|w|^{-1/(n+p)}\}$ for some $\epsilon_1>0$. Let $u\in\mathcal{U}_2\setminus(\mathcal{U}_1\cup\bigcup_i\widetilde{\mathcal{V}}_i)$. We have

$$\begin{aligned}|u^{-p+1}\partial_u\mathfrak{a}+pw|&=|pw|\,|u|^{-p-n}\prod_{i=0}^{n+p-1}|u-b_i(w)|\\&\geq pC_3^{-1}|w|^2\prod_{i=0}^{n+p-1}|u-b_i(w)|\geq pC_3^{-1}\epsilon_1^{p+n}|w|.\end{aligned}\tag{101}$$

We also have

$$\begin{aligned}|g_{\mathfrak{a}}\varphi|_{h'}\leq C_1|u|^{p-1}|\varphi|_{h'}\cdot|u|^{-p}&\leq C_1|u|^{p-1}|\varphi|_{h'}\cdot C_3|w|^{p/(n+p)}\\&=C_1|u|^{p-1}|\varphi|_{h'}|w|\cdot C_3|w|^{-n/(n+p)}.\end{aligned}\tag{102}$$

Hence, there exists $C_7>0$ and $R_7>0$ such that the following holds on $\mathcal{U}_2\setminus(\mathcal{U}_1\cup\bigcup_i\widetilde{\mathcal{V}}_i)$ if $|w|\geq R_7$:

$$|(\partial_u\mathfrak{a}+pwu^{p-1})\varphi\,du+g_{\mathfrak{a}}\varphi\,du|_{h'}\geq C_7|w|\,|\varphi|_{h'}|u|^{p-1}|du|.$$

We set $a:=(n+2)/2(n+p)$. We put $\mathcal{V}_i:=\{|u-b_i(w)|\leq\epsilon_1|w|^{-a}\}$ and $\mathcal{V}'_i:=\{|u-b_i(w)|\leq\epsilon_1|w|^{-a}/2\}$. On $\widetilde{\mathcal{V}}_i\setminus\mathcal{V}'_i$, we have

$$
\begin{aligned}
(103) \qquad |u^{-p+1}\partial_u\mathfrak{a}+pw| &= pC_3^{-1}|w|^2\prod_{i=0}^{n+p-1}|u-b_i(w)| \\
&\geq pC_3^{-1}|w|^2(\epsilon_1|w|^{-1/(n+p)})^{n+p-1}\times(\epsilon_1|w|^{-a}/2) \\
&\geq pC_3^{-1}\epsilon_1^{p+n}|w|^{1+1/(n+p)-a}.
\end{aligned}
$$

We also have $|g_{\mathfrak{a}}|\leq C_1C_3|w|^{1/(n+p)}$. Because $-(p-1)/(n+p)+1+1/(n+p)-a>1/(n+p)$, there exist $C_8>0$ and $R_8>0$ such that if $|w|\geq R_8$, then on $\widetilde{\mathcal{V}}_i\setminus\mathcal{V}_i'$ we have

$$
\begin{aligned}
(104) \qquad |(\theta_{\mathfrak{a}}+pwu^{p-1}du)\varphi|_{h'} &\geq C_8|w|^{1+1/(n+p)-a}|u|^{p-1}|du||\varphi|_{h'} \\
&= C_8|w|^{(n+2p)/2(n+p)}|u|^{p-1}|du||\varphi|_{h'}.
\end{aligned}
$$

We have the following kind of Poincaré inequality, ie there exist $C_9>0$ and $R_9>0$ such that if $|w|>R_9$, then the following holds on $\mathcal{V}_i$ (see [7; 48, (2.12)]):

$$
\begin{aligned}
&|w|^{(n+2)/(n+p)}\int_{\mathcal{V}_i}|\varphi|_{h'}^2|du\,d\overline{u}| \\
&\qquad\leq C_9\biggl(\int_{\mathcal{V}_i}|d|\varphi|_{h'}|^2+|w|^{(n+2)/(n+p)}\int_{\mathcal{V}_i\setminus\mathcal{V}_i'}|\varphi|_{h'}^2|du\,d\overline{u}|\biggr).
\end{aligned}
$$

We also have

$$
\begin{aligned}
(105) \qquad &|w|^{(n+2p)/(n+p)}\int_{\mathcal{V}_i}|\varphi|_{h'}^2|u|^{2(p-1)}|du\,d\overline{u}| \\
&\qquad\leq C_3^{2(p-1)}|w|^{(n+2)/(n+p)}\int_{\mathcal{V}_i}|\varphi|_{h'}^2|du\,d\overline{u}|,
\end{aligned}
$$

$$
\begin{aligned}
(106) \qquad &|w|^{(n+2)/(n+p)}\int_{\mathcal{V}_i\setminus\mathcal{V}_i'}|\varphi|_{h'}^2|du\,d\overline{u}| \\
&\qquad\leq C_3^{2(p-1)}|w|^{(n+2p)/(n+p)}\int_{\mathcal{V}_i\setminus\mathcal{V}_i'}|\varphi|_{h'}^2|u|^{2(p-1)}|du\,d\overline{u}| \\
&\qquad\leq C_8C_3^{2(p-1)}\int_{\mathcal{V}_i\setminus\mathcal{V}_i'}|(\theta_{\mathfrak{a}}+pwu^{p-1}du)\varphi|_{h'}^2.
\end{aligned}
$$

Then we obtain the desired inequality for the integral over $\mathcal{V}_i$. Thus, the proof of Lemma 7.10 is finished. □

Let us consider the case $\mathfrak{a}=0$. Because this part is essentially contained in [48], we give just an indication. We take a positive number C_{10} which is sufficiently larger than $|\alpha|$ for any eigenvalues α of the residue of θ_0. We may assume $|g_0|\leq(C_{10}/10)|\zeta|^{-1}$

on B_ϵ. Take $R_{10}>0$ sufficiently larger than C_{10}. For $|w|\geq R_{10}$, let $\mathcal{U}:=\{|\zeta|\leq C_{10}|w|^{-1}\}$ and $\mathcal{U}':=\{|\zeta|\leq C_{10}|w|^{-1}/2\}$. On $B_\epsilon\setminus\mathcal{U}'$, we have

$$|(\theta_0+w\,d\zeta)\varphi|_{h'}\geq |w|\,|\varphi|_{h'}|d\zeta|-|g_0|_{h'}|\varphi|_{h'}|d\zeta|\geq \tfrac{4}{5}|w|\,|\varphi|_{h'}|d\zeta|. \tag{107}$$

There exist $C_{11}>0$ and $R_{11}>0$ such that if $|w|\geq R_{11}$, then on $\mathcal{U}$ we have

$$\begin{aligned}|w|^2\int_{\mathcal{U}}|\varphi|_{h'}^2|d\zeta\,d\bar{\zeta}| &\leq C_{11}\int_{\mathcal{U}}|d|\varphi|_{h'}|^2+|w|^2\int_{\mathcal{U}\setminus\mathcal{U}'}|\varphi|^2_{h'}|d\zeta\,d\bar{\zeta}|\\ &\leq \int_{\mathcal{U}}\big(C_{11}\big|\nabla^{(z)}_{h'}\varphi\big|^2_{h'}+4|(\theta_0+w d\zeta)\varphi|^2_{h'}\big).\end{aligned} \tag{108}$$

We obtain the desired inequality for sections of E_0 from (107) and (108). Thus, the proof of Proposition 7.9 is finished. □

The following is a refinement of Corollary 7.7.

Corollary 7.11 *There exist $\rho>0$, $C>0$ and $R>0$ such that if $|w|\geq R$, then for any section φ of E such that*

$$\int|\varphi|_h^2 d\mathrm{vol}_{X\setminus D}+\int|\Delta_1(z,w)\varphi|_h^2 d\mathrm{vol}_X<\infty, \tag{109}$$

we have

$$C|w|^{1+\rho}\Big(\int|\varphi|_h^2 d\mathrm{vol}_X\Big)^{1/2}\leq\Big(\int|\Delta_1(z,w)\varphi|_h^2 d\mathrm{vol}_X\Big)^{1/2}. \tag{110}$$

Proof This is proved by the argument in Corollary 7.7, by using Proposition 7.9, instead of Proposition 7.5. □

7.2 Comparison with the algebraic Nahm transform

7.2.1 Statements Let $(E,\bar{\partial}_E,\theta,h)$ be a wild harmonic bundle on $(T^\vee,D)$. Let $\mathcal{P}_*E$ be the associated filtered bundle on $(T^\vee,D)$. Let (E_1,h_1,∇_1) be the L^2 instanton on $T\times\mathbb{C}$ obtained as the Nahm transform of $(E,\bar{\partial}_E,\theta,h)$ (see Section 7.1). Let $\mathcal{P}_*E_1$ be the associated filtered bundle on $(T\times\mathbb{P}^1,T\times\{\infty\})$.

Theorem 7.12 *$\mathcal{P}_*E_1$ is naturally isomorphic to* $\mathrm{Nahm}_*(\mathcal{P}_*E,\theta)$.

Conversely, let (E_1,∇_1,h_1) be an L^2 instanton on $T\times\mathbb{C}$. Let $\mathcal{P}_*E_1$ be the associated filtered bundle on $(T\times\mathbb{P}^1,T\times\{\infty\})$. Let $(E,\bar{\partial}_E,\theta,h)$ be the wild harmonic bundle on $(T^\vee,D)$ obtained as the Nahm transform of (E_1,∇_1,h_1) (see Section 6.4). Let $(\mathcal{P}_*E,\theta)$ be the associated filtered Higgs bundle.

Theorem 7.13 *$(\mathcal{P}_*E,\theta)$ is naturally isomorphic to* $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$.

We obtain the involutivity of the Nahm transform in the following sense.

Corollary 7.14 *For an L^2 instanton (E_1,∇_1,h_1) on $T\times\mathbb{C}$, we have an isomorphism*

$$\mathrm{Nahm}(\mathrm{Nahm}(E_1,\nabla_1,h_1))\simeq(E_1,\nabla_1,h_1).$$

For a wild harmonic bundle $(E,\overline{\partial}_E,\theta,h)$ on $T^\vee$, we have an isomorphism

$$\mathrm{Nahm}(\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h))\simeq(E,\overline{\partial}_E,\theta,h).$$

Proof It follows from Proposition 3.25, Theorems 7.12 and 7.13, and the uniqueness of the harmonic metric or Hermitian–Einstein metric adapted to the filtered bundle. (See Proposition 6.5 for the uniqueness of Hermitian–Einstein metric; see [7] for the uniqueness of the harmonic metric; see also [36]; see Section 2.2.5 for adaptedness of metrics and filtered bundles.) □

7.2.2 Proof of Theorem 7.12 We begin by constructing an isomorphism $(E_1,\overline{\partial}_{E_1})\simeq\mathrm{Nahm}(\mathcal{P}_*E,\theta)_{|T\times\mathbb{C}}$. We recall the monad construction of $E_1=\mathrm{Nahm}(E,\overline{\partial}_E,\theta,h)$; see [14]. We use the notation in Section 7.1.1. Let $g_{T^\vee\setminus D}$ be a Poincaré-like Kähler metric of $T^\vee\setminus D$. Let $\mathcal{A}^i(E,\overline{\partial}_E,\theta,h)$ denote the space of sections φ of $E\otimes\Omega^i$ on $T^\vee\setminus D$ such that φ and $(\overline{\partial}_E+\theta)\varphi$ are L^2 with respect to h and $g_{T^\vee\setminus D}$. Note that the conditions also imply $(\overline{\partial}_{E,z}+\theta_w)\varphi$ are L^2 for any $(z,w)\in\mathbb{C}^2$. Let $\underline{\mathcal{A}}^i$ denote the sheaf of holomorphic sections of the product bundle $\mathcal{A}^i(E,\overline{\partial}_E,\theta,h)\times\mathbb{C}^2$ over $\mathbb{C}^2$. We have the morphisms $\delta^i\colon\underline{\mathcal{A}}^i\to\underline{\mathcal{A}}^{i+1}$ induced by $\overline{\partial}_{E,z}+\theta_w$. They are naturally equivariant with respect to the action of the lattice L on $\mathbb{C}^2$ given by $\chi(z,w)=(z+\chi,w)$, as in the construction of the Poincaré bundle. The induced bundles with operators on $T\times\mathbb{C}$ are denoted by the same notation. The sheaf of holomorphic sections of E_1 is isomorphic to $\mathrm{Ker}\,\delta^1/\mathrm{Im}\,\delta^0$.

Applying the construction in the proof of Lemma 3.1 around each point of D, we extend E and $E\otimes\Omega^1$ to $\mathcal{C}^0_{L^2}(\mathcal{P}_*E,\theta)$ and $\mathcal{C}^1_{L^2}(\mathcal{P}_*E,\theta)$. We let $\mathcal{C}^{i,\bullet}_{L^2}(\mathcal{P}_*E,\theta)$ denote the Dolbeault resolution of $\mathcal{C}^i_{L^2}(\mathcal{P}_*E,\theta)$.

For $I\subset\{1,2,3\}$, let p_I denote the projection of $T^\vee\times T\times\mathbb{C}$ onto the product of the i^{th} components $(i\in I)$. On $T^\vee\times T\times\mathbb{C}$, we set

$$\widetilde{\mathcal{C}}^i_{L^2}:=\bigoplus_{k+\ell=i}p_1^{-1}\mathcal{C}^{k,\ell}_{L^2}(\mathcal{P}_*E,\theta)\otimes_{p_1^{-1}\mathcal{O}_{T^\vee}}p_{12}^*\mathcal{P}oin.$$

We have $\delta^i\colon \widetilde{\mathcal{C}}^i_{L^2} \to \widetilde{\mathcal{C}}^{i+1}_{L^2}$ induced by $\overline{\partial}_{E,z} + \theta_w$. We have a natural inclusion of complexes

$$\Phi\colon p_{23*}\widetilde{\mathcal{C}}^{\bullet}_{L^2} \to \underline{\mathcal{A}}^{\bullet}$$

on $T \times \mathbb{C}$. According to [36, Section 5.1], Φ is a quasi-isomorphism. We also have the following natural isomorphisms in $D^b(\mathcal{O}_{T\times\mathbb{C}})$:

$$\begin{aligned} p_{23*}\widetilde{\mathcal{C}}^{\bullet}_{L^2} &\simeq Rp_{23*}(p_1^*(\mathcal{C}^{\bullet}_{L^2}(\mathcal{P}_*E,\theta)) \otimes p_{12}^*\mathcal{P}oin) \\ &\overset{\Psi}{\simeq} Rp_{23*}(p_1^*(\mathcal{C}^{\bullet}(\mathcal{P}_*E,\theta)) \otimes p_{12}^*\mathcal{P}oin) \\ &\simeq \mathrm{Nahm}(\mathcal{P}_*E,\theta)_{|T\times\mathbb{C}}[-1]. \end{aligned} \tag{111}$$

(See Lemma 3.1 for Ψ.) We obtain the desired isomorphism $E_1 \simeq \mathrm{Nahm}(\mathcal{P}_*E,\theta)_{|T\times\mathbb{C}}$, by which we shall identity the two.

Lemma 7.15 *To prove Theorem 7.12, we have only to prove* $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta) \subset \mathcal{P}_aE_1$ *for any* a.

Proof By Proposition 3.17, we have $\deg(\mathrm{Nahm}_*(\mathcal{P}_*E,\theta)) = \deg(\mathcal{P}_*E,\theta) = 0$. By Proposition 6.4, we also have $\deg(\mathcal{P}_*E_1) = 0$. Hence, $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta) \subset \mathcal{P}_aE_1$ implies $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta) = \mathcal{P}_aE_1$. □

To prove $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta) \subset \mathcal{P}_aE_1$, we need an estimate of the upper bound of the norms of sections of $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta)$. We use an argument of scaling in [48]. Because we need only the upper bound, we will not consider more precise estimates for harmonic representatives or their approximation.

Let $U_\tau \subset \mathbb{P}^1$ be a neighbourhood of ∞ with the coordinate $\tau = w^{-1}$. If U_τ is sufficiently small, we have the decomposition $\mathrm{Nahm}_*(\mathcal{P}_*E,\theta) = \bigoplus_{P\in D} \mathrm{Nahm}_*(\mathcal{P}_*E,\theta)_P$ by the spectrum on $T \times U_\tau$. We have the refined decomposition

$$\mathrm{Nahm}_*(\mathcal{P}_*E,\theta)_P = \bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\theta,P)} \bigoplus_{\alpha\in\mathbb{C}} \mathrm{Nahm}_*(\mathcal{P}_*E,\theta)_{P,\mathfrak{o},\alpha},$$

according to the decomposition $(\mathcal{P}_*E,\theta) = \bigoplus_{\mathfrak{o}\in\mathbf{Irr}(\theta,P)} \bigoplus_{\alpha\in\mathbb{C}} (\mathcal{P}_*E_{P,\mathfrak{o},\alpha}, \theta_{P,\mathfrak{o},\alpha})$ near each $P \in D$. We have only to prove that $\mathrm{Nahm}_a(\mathcal{P}_*E,\theta)_{P,\mathfrak{o},\alpha} \subset \mathcal{P}_aE_1$. We shall argue the case $P = \{0\}$ in the following. The other case can be established similarly. We omit the subscript P. We take a small neighbourhood $U_\zeta \subset T^\vee$ of $\{0\}$.

Let us consider the case $(\mathfrak{o},\alpha) \neq (0,0)$. Take $\mathfrak{a} \in \mathfrak{o}$. For each $c \in \mathbb{R}$, we have the frame of $\mathrm{Nahm}_c(\mathcal{P}_*E,\theta)_{P,\mathfrak{o},\alpha}$ in Lemma 3.29. We have only to prove that

$$\big|[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i}\, d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]\big|_{h_1} = O(|w|^{(b-j-m_{\mathfrak{o}}/2)(p_{\mathfrak{o}}+m_{\mathfrak{o}})^{-1}}). \tag{112}$$

Here, b is the parabolic degree of $v_{\mathfrak{o},i}$.

We give a preliminary. We have the expression $\zeta_{\mathbf{o}}\partial_{\zeta_{\mathbf{o}}}\mathfrak{a}+p_{\mathbf{o}}\alpha=\sum_{j=0}^{m_{\mathbf{o}}}\alpha_j\zeta_{\mathbf{o}}^{-j} =: G(\zeta_{\mathbf{o}})$. We fix a complex number γ such that $\alpha_{m_{\mathbf{o}}}+p_{\mathbf{o}}\gamma^{p_{\mathbf{o}}+m_{\mathbf{o}}}=0$. Take a covering $U_\eta\to U_\tau$ given by $\eta\mapsto\eta^{p_{\mathbf{o}}+m_{\mathbf{o}}}$. If U_τ is sufficiently small, we can take holomorphic functions $u_0^{(i)}(\eta)$ $(i=1,\dots,p_{\mathbf{o}+m_{\mathbf{o}}})$ satisfying

$$G\big(u_0^{(i)}(\eta)\big)+p_{\mathbf{o}}u_0^{(i)}(\eta)^{p_{\mathbf{o}}}\eta^{-p_{\mathbf{o}}-m_{\mathbf{o}}}=0,\quad \lim_{\eta\to 0}u_0^{(i)}(\eta)/\eta=\gamma\exp\big(2\pi\sqrt{-1}i/(m_{\mathbf{o}}+p_{\mathbf{o}})\big).$$

There exist $C_1>0$ and $\epsilon_1>0$ such that

$$\big|\eta^{-1}u_0^{(i)}(\eta)-\gamma\exp\big(2\pi\sqrt{-1}i/(m_{\mathbf{o}}+p_{\mathbf{o}})\big)\big|\le C_1|\eta|^{\epsilon_1}.$$

Lemma 7.16 *Let Z_η denote the support of $\operatorname{Cok}(\eta^{p_{\mathbf{o}}+m_{\mathbf{o}}}\zeta_{\mathrm{o}}^{m_{\mathbf{o}}}\theta^{\mathbf{o}}_{\mathfrak{a},\alpha}+p_{\mathbf{o}}\zeta_{\mathbf{o}}^{p_{\mathbf{o}}+m_{\mathbf{o}}}d\zeta_{\mathbf{o}}/\zeta_{\mathbf{o}})$ on $U_{\zeta_{\mathbf{o}}}$. If U_ζ and U_τ are sufficiently small, then there exists a decomposition*

$$Z_\eta=\coprod_{i=1}^{p_{\mathbf{o}}+m_{\mathbf{o}}}Z_\eta^{(i)}$$

such that for any $u\in Z_\eta^{(i)}$, we have

$$|u_0^{(i)}(\eta)-u|\le C|\eta|^{1+m_{\mathbf{o}}+\epsilon}.$$

Here C and ϵ are positive constants which are independent of η.

Proof Take $u_1\in Z_\eta$. There exists a possibly multivalued holomorphic 1–form $\nu(\zeta_{\mathbf{o}})\,d\zeta_{\mathbf{o}}/\zeta_{\mathbf{o}}$ obtained as the eigenvalue of $\theta^{\mathbf{o}}_{\mathfrak{a}}$, such that $\nu(u_1)+\eta^{-p_{\mathbf{o}}-m_{\mathbf{o}}}p_{\mathbf{o}}u_1^{p_{\mathbf{o}}}=0$. Because $\nu(\zeta_{\mathbf{o}})-G(\zeta_{\mathbf{o}})=O(\zeta_{\mathbf{o}}^{\epsilon})$, there exist $C_2>0$ and $\epsilon_2>0$, independently from η, such that the for some unique i we have

$$\big|\eta^{-1}u_1-\gamma\exp\big(2\pi\sqrt{-1}i/(p_{\mathbf{o}}+m_{\mathbf{o}})\big)\big|\le C_2|\eta|^{\epsilon_2}. \tag{113}$$

We obtain a decomposition of $Z_\eta=\coprod Z_\eta^{(i)}$ by condition (113).

Let $u_1\in Z_\eta^{(i)}$. We set $Q_q(x,y):=\sum_{i+j=q}x^iy^j$ and have

$$\begin{aligned}\big((u_0^{(i)}(\eta)/\eta)^{-1}-(u_1/\eta)^{-1}\big)&\\ \times\Bigg(\sum_{j=1}^{m_{\mathbf{o}}}\alpha_j\eta^{m_{\mathbf{o}}-j}Q_{j-1}\big((u_0^{(i)}/\eta)^{-1},(u_1/\eta)^{-1}\big)&\\ -(u_0^{(i)}/\eta)(u_1/\eta)p_{\mathbf{o}}Q_{p_{\mathbf{o}}-1}(u_0^{(i)}/\eta,u_1/\eta)\Bigg)&\\ =O(|u_1/\eta|^{\epsilon}|\eta|^{m_{\mathbf{o}}+\epsilon})&\end{aligned}\tag{114}$$

We obtain $|(u_0^{(i)}(\eta)/\eta)^{-1}-(u_1/\eta)^{-1}|=O(|\eta|^{m_{\mathbf{o}}+\epsilon})$. Then we obtain the desired estimate. □

Let ρ be an $\mathbb{R}_{\geq 0}$–valued function on $\mathbb{C}_\eta$ such that $\rho(\eta)=1$ for $|\eta|<\frac{1}{2}$ and $\rho(\eta)=0$ for $|\eta|>1$. We set $u_0:=u_0^{(0)}$. We consider the following C^∞ sections of $E^{\mathfrak{o}}_{\mathfrak{a},\alpha}\otimes\Omega^1_{X^{\mathfrak{o}}}$:

$$\mu_1(v_{\mathfrak{o},i},\xi):=\rho(|\xi|^{1+m_{\mathfrak{o}}/2}(\zeta_{\mathfrak{o}}-u_0(\xi)))v_{\mathfrak{o},i}\,d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}},$$
$$\mu_2(v_{\mathfrak{o},i},\xi):=(\theta^{\mathfrak{o}}_{\mathfrak{a}}+\xi^{p_{\mathfrak{o}}+m_{\mathfrak{o}}}d\zeta_{\mathfrak{o}}^{p_{\mathfrak{o}}})^{-1}(\overline{\partial}_E\mu_1(v_{\mathfrak{o},i},\xi)).$$

By Lemma 7.16, if $|\xi|$ is sufficiently large, $\rho(|\xi|^{1+m_{\mathfrak{o}}/2}(\zeta_{\mathfrak{o}}-u_0(\xi)))$ is constantly 1 around Z_ξ. Hence, the tuple $\boldsymbol{\mu}(v_{\mathfrak{o},i},\xi)=(\mu_1(v_{\mathfrak{o},i},\xi),\mu_2(v_{\mathfrak{o},i},\xi))$ gives a representative of $[v_{\mathfrak{o},i}d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]$.

By an elementary change of variables, for any $\delta>0$ we get

$$\begin{aligned}(115)\quad \int|\mu_1(v_{\mathfrak{o},i},\xi)|_h^2 &\leq C_{1\delta}\int\rho(|\xi|^{1+m_{\mathfrak{o}}/2}(\zeta_{\mathfrak{o}}-u_0(\xi)))^2|\zeta_{\mathfrak{o}}|^{-2(b+\delta)-2}|d\zeta_{\mathfrak{o}}\,d\overline{\zeta}_{\mathfrak{o}}|\\ &\leq C'_{1\delta}|\xi|^{2(b+\delta)-m_{\mathfrak{o}}}.\end{aligned}$$

Note that $|\zeta_{\mathfrak{o}}-u_0(\xi)|\sim|\xi|^{-1-m_{\mathfrak{o}}/2}$ for $\zeta_{\mathfrak{o}}$ such that $\overline{\partial}\rho(|\xi|^{1+m_{\mathfrak{o}}/2}(\zeta_{\mathfrak{o}}-u_0(\xi)))\neq 0$. Hence, we also have

$$\begin{aligned}(116)\quad \int|\mu_2(v_{\mathfrak{o},i},\xi)|_h^2 &\leq C_{2\delta}\int|\overline{\partial}\rho(|\xi|^{1+m_{\mathfrak{o}}/2}(\zeta_{\mathfrak{o}}-u_0(\xi)))|^2\\ &\qquad\cdot|\zeta_{\mathfrak{o}}|^{-2(b+\delta)}\frac{1}{|\zeta_{\mathfrak{o}}\partial_{\zeta_{\mathfrak{o}}}\mathfrak{a}+p_{\mathfrak{o}}\alpha+p_{\mathfrak{o}}\xi^{p_{\mathfrak{o}}+m_{\mathfrak{o}}}\zeta_{\mathfrak{o}}^{p_{\mathfrak{o}}}|^2}\\ &\leq C'_{2\delta}|\xi|^{2(b+\delta)-2(m_{\mathfrak{o}}+1)+2(1+m_{\mathfrak{o}}/2)}=C'_{2\delta}|\xi|^{2(b+\delta)-m_{\mathfrak{o}}}\end{aligned}$$

By the construction of h_1, we have

$$\big|[v_{\mathfrak{o},i}d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]\big|_{h_1}^2\leq\int\big(|\mu_1(v_{\mathfrak{o},i},\xi)|_h^2+|\mu_2(v_{\mathfrak{o},i},\xi)|_h^2\big).$$

Hence, we obtain the desired estimate (112) for $[v_{\mathfrak{o},i}d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]$. We obtain the estimate for $[v_{\mathfrak{o},i}\zeta_{\mathfrak{o}}^j\,d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]$ similarly.

Let us consider the case $(\mathfrak{o},\alpha)=(\{0\},0)$. The following lemma is easy to see.

Lemma 7.17 *Let Z_w denote the support of* $\mathrm{Cok}(\theta_{0,0}+w\,d\zeta)$. *There exist $C>0$ and $\epsilon>0$ such that $|u|\leq C|w|^{-1-\epsilon}$ holds for any $u\in Z_w$.* □

For a holomorphic section s of $\mathcal{C}^1(\mathcal{P}_*E_{P,\{0\},0}\otimes\Omega^\bullet,\theta)$ (see Section 3.1.1), we consider the following C^∞ sections of $E_{P,\{0\},0}\otimes\Omega^1$:

$$\begin{aligned}\mu_1(s,w)&:=(\rho(\zeta)-\rho(w\zeta))s\,d\zeta/\zeta, &\mu_2(s,w)&:=(\theta_{P,\{0\},0}+w\,d\zeta)^{-1}(\overline{\partial}\mu_1(s)),\\ \mu'_1(s,w)&:=\rho(\zeta)s\,d\zeta/\zeta, &\mu'_2(s,w)&:=(\theta_{P,\{0\},0}+w\,d\zeta)^{-1}(\overline{\partial}\mu'_1(s)).\end{aligned}$$

By Lemma 7.17, μ_2, μ_2' are well defined. The tuples $\boldsymbol{\mu}(s,w) = (\mu_1(s,w), \mu_2(s,w))$ and $\boldsymbol{\mu}'(s,w) = (\mu_1'(s,w), \mu_2'(s,w))$ naturally induce the same holomorphic section of $\mathrm{Nahm}(\mathcal{P}_*E)_{\boldsymbol{P}}$. If s is a section of $\mathcal{P}_c E_{\{0\},0}$, then it is elementary to prove that for any $\delta > 0$ we have

$$\int |\mu_i(s,w)|_h^2 \le C_\delta |w|^{2(c+\delta)}.$$

We obtain $|\boldsymbol{\mu}(s,w)|_{h_1} \le C_\delta' |w|^{c+\delta}$ for any $\delta > 0$. Then $\mathrm{Nahm}_*(\mathcal{P}_*E)_{\boldsymbol{P},\{0\},0} \subset \mathcal{P}_*E_1$. Thus, the proof of Theorem 7.12 is finished. □

7.2.3 Proof of Theorem 7.13 Let us construct an isomorphism of the Higgs bundles $(E, \overline{\partial}_E, \theta) \simeq \mathrm{Nahm}(\mathcal{P}_*E_1)_{|T^\vee \setminus D}$. Let us recall the monad construction of $\mathrm{Nahm}(E_1, \nabla_1)$. Let $\mathcal{A}^{0,i}$ denote the space of sections φ of $E_1 \otimes \Omega^{0,i}$ on $T \times \mathbb{C}$, such that φ and $\overline{\partial}_{E_1}\varphi$ are L^2 with respect to h_1 and the Euclidean metric. Let $\Phi\colon \mathbb{C} \to T^\vee$ denote a universal covering. Let $\underline{\mathcal{A}}^{0,i}$ denote the sheaf of holomorphic sections of the product bundle $\mathcal{A}^{0,i} \times (\mathbb{C} \setminus \Phi^{-1}(D))$ over $\mathbb{C} \setminus \Phi^{-1}(D)$. We have a morphism

$$\delta^i\colon \underline{\mathcal{A}}^{0,i} \to \underline{\mathcal{A}}^{0,i+1}$$

induced by $\overline{\partial}_{E_1} - \zeta\, d\bar{z}$. They are naturally equivariant with respect to the action of $L^\vee$ on $\mathbb{C}$ by the translation, as in the construction of the Poincaré bundle. The induced bundles and the operators are denoted by the same notation. The sheaf of holomorphic sections of $(E, \overline{\partial}_E)$ is isomorphic to $\mathrm{Ker}\,\delta^1 / \mathrm{Im}\,\delta^0$.

For $I \subset \{1,2,3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{C}$ onto the product of the i^{th} components. By construction, we have a natural morphism

$$Rp_{1*}(p_{23}^*\mathcal{P}_{<-1}E \otimes p_{12}^*\mathcal{P}oin) \to \underline{\mathcal{A}}^{0,\bullet}.$$

By the results in Section 6.2, it is a quasi-isomorphism. Hence, we obtain a holomorphic isomorphism $E \simeq \mathrm{Nahm}(\mathcal{P}_*E_1)_{|T^\vee \setminus D}$, by which we identify them. The Higgs fields are equal, because they are induced by the multiplication of $-w$.

We give a preliminary. Let $U \subset \mathbb{P}^1$ be a small neighbourhood of ∞. On $T \times U$, we have a decomposition

$$\mathcal{P}_*E_1 = \bigoplus_{P \in \mathcal{S}p_\infty(E_1)} \bigoplus_{\mathfrak{o},\alpha} \mathcal{P}_*(E_1)_{P,\mathfrak{o},\alpha}. \tag{117}$$

Fix a lift of $\mathcal{S}p_\infty(E_1) \subset T^\vee$ to $\widetilde{\mathcal{S}p}_\infty(E_1) \subset \mathbb{C}$. We have the filtered bundles with an endomorphism $(\mathcal{P}_*V, g)$ on U, corresponding to $\mathcal{P}_*E_1$. It has a decomposition $(\mathcal{P}_*V, g) = \bigoplus (\mathcal{P}_*V_{P,\mathfrak{o},\alpha}, g_{P,\mathfrak{o},\alpha})$.

Let $\mathcal{U} \subset \mathcal{P}_{-1}(E_1)$ be the subsheaf such that $\mathcal{U}_{|T\times\mathbb{C}} = \mathcal{P}_{-1}(E_1)_{|T\times\mathbb{C}}$ and

$$\mathcal{U} = \bigoplus_{P}\Big(\mathcal{P}_{-1}(E_1)_{P,\{0\},0} \oplus \bigoplus_{(\mathfrak{o},\alpha)\neq(\{0\},0)} \mathcal{P}_{<-1}(E_1)_{P,\mathfrak{o},\alpha}\Big) \tag{118}$$

around $T\times\{\infty\}$. We use the notation from Section 3.3.1.

Lemma 7.18 *We have* $N(\mathcal{U}) \subset \mathcal{P}_0 E$.

Proof We give an argument around $0 \in T^\vee$, by supposing $0 \in D$. The other case can be proved similarly. We may suppose the lift of $0 \in D$ is $0 \in \mathbb{C}$. Let t be a holomorphic section of $N(\mathcal{U})$ around $0 \in T^\vee$. We have to prove $|t|_h = O(|\zeta|^{-\delta})$ for any $\delta > 0$. It is represented by a family of C^∞ sections $\kappa(\zeta) = \kappa^1(\zeta)\,d\bar z + \kappa^2(\zeta)\,d\overline{w}$ of $\mathcal{P}_{-1}E_1 \otimes \Omega^{0,1}_{T\times\mathbb{P}^1} \otimes L_\zeta^{-1}$. According to the decomposition (118), we have

$$\kappa^i(\zeta) = \sum_{P,\mathfrak{o},\alpha} \kappa^i(\zeta)_{P,\mathfrak{o},\alpha}.$$

If $P \neq 0$, we may assume $\kappa^i(\zeta)_{P,\mathfrak{o},\alpha} = 0$ on U. (See the proof of Proposition 6.7.) Let $d\mathrm{vol} := |dz\,d\bar z\,dw\,d\overline{w}|$.

We take a C^∞ metric h_2 of $\mathcal{U}$. Note $h_1 = O(h_2|w|^{-2+\delta})$ for any $\delta > 0$ on $\mathcal{P}_{-1}(E_1)_{P,\{0\},0}$, and $h_1 = O(h_2|w|^{-2-\epsilon})$ for some $\epsilon > 0$ on $\mathcal{P}_{<-1}(E_1)_{P,\mathfrak{o},\alpha}$ for $(\mathfrak{o},\alpha) \neq (\{0\},0)$.

If $P = 0$ and $(\mathfrak{o},\alpha) \neq (\{0\},0)$, we have the following finiteness uniformly for ζ:

$$\int_{T\times U} |\kappa^i(\zeta)_{0,\mathfrak{o},\alpha}|^2_{h_1} d\mathrm{vol} \le C_0 \int_{T\times U} |\kappa^i(\zeta)_{0,\mathfrak{o},\alpha}|^2_{h_2} |w|^{-2-\epsilon} d\mathrm{vol} < \infty.$$

We consider the contribution from $P = 0$ and $(\mathfrak{o},\alpha) = (\{0\},0)$. We have $|g_{0,\{0\},0}|_{h_1} \le C_1|w|^{-1}$ for some C_1. We take a sufficiently small $C_2 > 0$, and we put $H_\zeta := \{w \mid |w|^{-1} < C_2|\zeta|\}$. We can find a unique family of C^∞ sections $\mu(\zeta)$ of $\mathcal{P}_{-1}E \otimes L_\zeta^{-1}$ on H_ζ such that

$$(\overline{\partial}_E + \zeta\,d\bar z)\mu(\zeta) = (\kappa^1(\zeta)_{0,\{0\},0} d\bar z + \kappa^2(\zeta)_{0,\{0\},0} d\overline{w})_{|H_\zeta}.$$

There exists $C_3 > 0$ such that

$$\int_{T\times\{w\}} |\mu(\zeta)|^2_{h_2} |dz\,d\bar z| \le C_3|\zeta|^{-2} \int_{T\times\{w\}} |\kappa^1(\zeta)_{0,\{0\},0}|^2_{h_2} |dz\,d\bar z|.$$

Let $\chi(w)$ be an $\mathbb{R}_{\geq 0}$–valued C^∞ function such that $\chi(w)=1$ if $|w|^{-1}\leq C_2/4$ and $\chi(w)=0$ if $|w|^{-1}\geq C_2/2$. We set

$$\begin{aligned}\widetilde{\kappa}^1(\zeta)&=\kappa^1(\zeta)_{0,\{0\},0}-\partial_{\bar z}(\chi(w\zeta)\mu(\zeta))=(1-\chi(w\zeta))\kappa^1(\zeta)_{0,\{0\},0},\\ \widetilde{\kappa}^2(\zeta)&=\kappa^2(\zeta)_{0,\{0\},0}-\partial_{\overline{w}}(\chi(w\zeta)\mu(\zeta))\\ &=(1-\chi(w\zeta))\kappa^2(\zeta)_{0,\{0\},0}-(\partial_{\overline{w}}\chi)(w\zeta)\cdot\zeta\cdot\mu(\zeta).\end{aligned}$$

For any $\delta>0$, we have the following finiteness, which is uniform for ζ:

$$\int_{T\times U}\big(|\widetilde{\kappa}^1(\zeta)|^2_{h_2}+|\widetilde{\kappa}^2(\zeta)|^2_{h_2}\big)|dz\,d\bar z|\frac{|dw\,d\overline{w}|}{|w|^{2+\delta}}\leq C_{1,\delta}.$$

For any $\delta>0$, we have

$$\begin{aligned}\int_{T\times U}&\big(|\widetilde{\kappa}^1(\zeta)|^2_{h_1}+|\widetilde{\kappa}^2(\zeta)|^2_{h_1}\big)|dz\,d\bar z\,dw\,d\overline{w}|\\ &\leq C_{2,\delta}\int_{T\times U}\big(|\widetilde{\kappa}^1(\zeta)|^2_{h_2}+|\widetilde{\kappa}^2(\zeta)|^2_{h_2}\big)|dzd\bar z|\frac{|dw\,d\overline{w}|}{|w|^{2+\delta}}|\zeta|^{-2\delta}\leq C_{3,\delta}|\zeta|^{-2\delta}\end{aligned}\tag{119}$$

Hence, we obtain $|t(\zeta)|_h\leq C_{4\epsilon}|\zeta|^{-\delta}$ for any $\delta>0$. Thus, the proof of Lemma 7.18 is finished. □

Let us prove $\mathrm{Nahm}_*(\mathcal{P}_*E_1)=\mathcal{P}_*E$. The following lemma is similar to Lemma 7.15.

Lemma 7.19 *We have only to prove* $\mathrm{Nahm}_a(\mathcal{P}_*E_1)\subset\mathcal{P}_aE$ *for any* a. □

Around each $P\in D$, we have the decomposition

$$\mathrm{Nahm}_*(\mathcal{P}_*E_1)=\bigoplus_{\mathfrak{o},\alpha}\mathrm{Nahm}_*(\mathcal{P}_*E_1)_{P,\mathfrak{o},\alpha},\tag{120}$$

according to the decomposition (117). We have only to prove $\mathrm{Nahm}_a(\mathcal{P}_*E_1)_{P,\mathfrak{o},\alpha}\subset\mathcal{P}_a(E)$. We shall argue the case $P=0$ in the following. The other case can be proved similarly. We shall omit the subscript P. We take a small neighbourhood U_ζ of P.

Let us consider the case $(\mathfrak{o},\alpha)\neq(0,0)$. Let $U_\tau\subset\mathbb{P}^1$ be a small neighbourhood of ∞ with the coordinate $\tau=w^{-1}$. Take $\mathfrak{a}\in\mathfrak{o}$. For each $c\in\mathbb{R}$, we have the frame of Lemma 3.30. We only have to prove that

$$\big|[\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}]\big|_h=O(|\zeta|^{-(b-j+p_{\mathfrak{o}}-m_{\mathfrak{o}}/2)(p_{\mathfrak{o}}-m_{\mathfrak{o}})^{-1}}).\tag{121}$$

Here, b is the parabolic degree of $v_{\mathfrak{o},i}$.

We give a preliminary. We take a ramified covering $U_u \to U_\zeta$ given by $\zeta = u^{p_{\mathbf{o}}-m_{\mathbf{o}}}$. We put

$$G(\tau_{\mathbf{o}}) := \partial_w \mathfrak{a}(\tau_{\mathbf{o}}) - \alpha p_{\mathbf{o}} \tau_{\mathbf{o}}^{p_{\mathbf{o}}} = \sum_{j=0}^{m_{\mathbf{o}}} \beta_j \tau_{\mathbf{o}}^{p_{\mathbf{o}}-j}.$$

Let γ be a complex number such that $\beta_{m_{\mathbf{o}}} \gamma^{p_{\mathbf{o}}-m_{\mathbf{o}}} - 1 = 0$. If U_ζ is sufficiently small, there are holomorphic functions $\eta_0^{(i)}(u)$ $(i = 1, \dots, p_{\mathbf{o}} - m_{\mathbf{o}})$ on U_ζ satisfying

$$G\big(\eta_0^{(i)}(u)\big) - u^{p_{\mathbf{o}}-m_{\mathbf{o}}} = 0, \quad \lim_{u\to 0} u^{-1}\eta_0^{(i)}(u) = \gamma \exp\big(2\pi\sqrt{-1}i/(p_{\mathbf{o}} - m_{\mathbf{o}})\big).$$

There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that

$$\big|u^{-1}\eta_0^{(i)}(u) - \gamma \exp\big(2\pi\sqrt{-1}i/(p_{\mathbf{o}} - m_{\mathbf{o}})\big)\big| \le C_1 |u|^{\epsilon_1}.$$

The following lemma is similar to Lemma 7.16.

Lemma 7.20 *Let Z_u denote the support of $\mathrm{Cok}(g_{\mathfrak{a},\alpha} - u^{p_{\mathbf{o}}-m_{\mathbf{o}}})$ on $U_{\tau_{\mathbf{o}}}$. If U_τ and U_ζ are sufficiently small, then we have a decomposition*

$$Z_u = \coprod_{i=1}^{p_{\mathbf{o}}-m_{\mathbf{o}}} Z_u^{(i)}$$

and we have positive constants C and ϵ such that

$$\big|\eta_0^{(i)}(u) - \eta_1\big| \le C|u|^{1+m_{\mathbf{o}}+\epsilon}$$

for any $\eta_1 \in Z_u^{(i)}$. □

We set $d := 1 + m_{\mathbf{o}}/2$. We consider the following sections of $E_{\mathfrak{a}}^{\mathbf{o}} \otimes \Omega^{0,1}$:

$$\mu_1(v_{\mathbf{o},i}, u) := \rho(|u|^{-d}(\tau_{\mathbf{o}} - \eta_0(u)))v_{\mathbf{o},i}\, d\bar{z},$$
$$\mu_2(v_{\mathbf{o},i}, u) := (g_{\mathfrak{a},\alpha} - u^{p_{\mathbf{o}}-m_{\mathbf{o}}})^{-1}(\bar{\partial}\rho(|u|^{-d}(\tau_{\mathbf{o}} - \eta_0(u))))v_{\mathbf{o},i}.$$

The tuple $\boldsymbol{\mu}(v_{\mathbf{o},i}, u) := (\mu_1(v_{\mathbf{o},i}, u), \mu_2(v_{\mathbf{o},i}, u))$ induces a section of $\mathrm{Nahm}(\mathcal{P}_* E_1)_{P,\mathbf{o},\alpha}$. By Lemma 7.20, $\rho(|u|^{-d}(\tau_{\mathbf{o}} - \eta_0(u)))$ is constantly 1 around Z_u. Hence, $\boldsymbol{\mu}(v_{\mathbf{o},i}, u)$ induces $[v_{\mathbf{o},i}]$.

By an elementary change of variables, for any $\delta > 0$ we get

$$(122) \quad \int |\mu_1(v_{\mathbf{o},i}, u)|_h^2 \le C_{1\delta} \int |\rho(|u|^{-d}(\tau_{\mathbf{o}} - \eta_0(u)))|^2 |\tau_{\mathbf{o}}|^{-2(b+\delta)} |dz\, d\bar{z}||dw\, d\overline{w}|$$
$$\le C'_{1\delta}|u|^{-2(b+\delta)-2p_{\mathbf{o}}-2+2d} = C'_{1\delta}|u|^{-2(b+\delta+p_{\mathbf{o}}-m_{\mathbf{o}}/2)}.$$

We also have

$$\int |\mu_2(v_{\mathfrak{o},i},u)|_h^2 \leq C_{2\delta} \iint |\overline{\partial}\rho(|u|^{-d}(\tau_{\mathfrak{o}}-\eta_0(u)))|^2 |\tau_{\mathfrak{o}}|^{-2(b+\delta)} \cdot \frac{1}{|\partial_w \mathfrak{a}(\tau_{\mathfrak{o}}) - \alpha p_{\mathfrak{o}} \tau_{\mathfrak{o}}^{p_{\mathfrak{o}}} - u^{p_{\mathfrak{o}}-m_{\mathfrak{o}}}|^2} \leq C'_{2\delta}|u|^{-2(b+\delta)-2(p_{\mathfrak{o}}-m_{\mathfrak{o}}-1)-2d} = C'_{2\delta}|u|^{-2(b+\delta+p_{\mathfrak{o}}-m_{\mathfrak{o}}/2)} \tag{123}$$

Hence, we obtain the estimate (121).

Let us consider the case $(\mathfrak{o},\alpha)=(\{0\},0)$. Note that $N(\mathcal{P}_{-1}E_1)_{0,\{0\},0}=N(\mathcal{U})_{0,\{0\},0}\subset \mathrm{Nahm}_0(\mathcal{P}_*E_1)$. Let $\nu\in \mathrm{Nahm}_{1+c}(\mathcal{P}_*E_1)_{0,\{0\},0}/N(\mathcal{P}_{-1}E_1)_{0,\{0\},0}$ for $-1<c\leq 0$. We take $v\in \mathcal{P}_cV_{0,\{0\},0}$ which represents ν. We naturally regard v as a C^∞ section of $\mathcal{P}_c(E_1)_0$. Fix a sufficiently small number $b>0$, and let ρ be a $\mathbb{R}_{\geq 0}$–valued C^∞ function on $\mathbb{C}_\tau$ such that $\rho(\tau)=1$ if $|\tau|\leq b/2$ and $\rho(\tau)=0$ if $|\tau|\geq b$. We obtain a C^∞ section $\overline{\partial}(\rho(\tau)v\,d\bar{z})$ of $\mathcal{P}_{-1}(E_1)_0\otimes\Omega^{0,2}$. By using $H^2(T\times\mathbb{P}^1,\mathcal{U}\otimes L_{-\zeta})=0$ for any ζ, we can take a holomorphic family of C^∞ forms $\kappa(\zeta)=\kappa^1(\zeta)\,d\bar{z}+\kappa^2(\zeta)\,d\overline{w}$ of $\mathcal{U}\otimes\Omega^{0,1}$ such that $\overline{\partial}_{E\otimes L_{-\zeta}}\kappa(\zeta)=\overline{\partial}(\rho(\tau)v\,d\bar{z})$. Then $\rho(\tau)v\,d\bar{z}-\kappa(\zeta)$ induces a holomorphic section $\widetilde{\nu}$ of $\mathrm{Nahm}_{1+c}(\mathcal{P}_*E_1)$ around P which induces ν in $\mathrm{Nahm}_{1+c}(\mathcal{P}_*E_1)/N(\mathcal{U})$.

We consider the following sections:

$$\mu_1(v,\zeta):=(\rho(\tau)-\rho(\zeta^{-1}\tau))v\,d\bar{z},$$
$$\mu_2(v,\zeta):=\overline{\partial}(\rho(\tau)-\rho(\zeta^{-1}\tau))(g_{0,\{0\},0}-\zeta)^{-1}(v).$$

Then $\mu_1(v,\zeta)+\mu_2(v,\zeta)-\kappa(\zeta)$ induces the same section $\widetilde{\nu}$.

For any $\delta>0$, we have

$$\int |\mu_1|_{h_1}^2|dw\,d\overline{w}|\leq C_\delta\int_{|\tau|\geq A|\zeta|}|\tau|^{-2(c+\delta)-4}|d\tau\,d\overline{\tau}|\leq C_\delta|\zeta|^{-2(c+1+\delta)}.$$

We also have

$$\int |\mu_2|_{h_1}^2|dz\,d\bar{z}|\leq C_\delta\int|\overline{\partial}\rho(\zeta^{-1}\tau)|^2|\zeta|^{-2}|\tau|^{-2(c+\delta)}\leq C_\delta|\zeta|^{-2(c+1+\delta)}.$$

Because the support of $\overline{\partial}(\rho(\tau)v\,d\bar{z})$ is compact, we obtain

$$\int |\kappa|_{h_1}^2 d\mathrm{vol}=O(|\zeta|^{-\delta})$$

for any $\delta > 0$, by the argument in the proof of Lemma 7.18. We obtain $|\widetilde{\nu}|_h \le C_\delta|\zeta|^{-(c+1+\delta)}$ for any $\delta > 0$. Thus, we obtain $\mathrm{Nahm}_{1+c}(\mathcal{P}_*E_1)_{0,\{0\},0} \subset \mathcal{P}_{1+c}E$, and the proof of Theorem 7.13 is finished. $\square$

7.3 Kobayashi–Hitchin correspondence for L^2 instantons

7.3.1 Statements Let $\mathcal{P}_*E_1$ be a good filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ of degree 0 satisfying condition (A3). (See Section 2.4.1 for good filtered bundles.)

Proposition 7.21 *$\mathcal{P}_*E_1$ is stable if and only if $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$ is a stable filtered Higgs bundle. (See Section 2.4.4 for the stability condition of $\mathcal{P}_*E_1$.)*

Before going to the proof, we give a consequence.

Theorem 7.22 *Let $\mathcal{P}_*E_2$ be a stable good filtered bundle on $(T\times\mathbb{P}^1, T\times\{\infty\})$ with $\deg(\mathcal{P}_*E_2) = 0$. We set $E_2 := (\mathcal{P}_aE_2)_{|T\times\mathbb{C}}$ which is independent of a. Then there exists a Hermitian–Einstein metric h of E_2 on $T\times\mathbb{C}$ such that its curvature is L^2 with respect to h and the Euclidean metric, and it is adapted to the filtered bundle $\mathcal{P}_*E_2$. (See Section 2.2.5 for adaptedness.) Such a metric is unique up to the multiplication of positive constants.*

Proof If $\operatorname{rank} E_2 = 1$, then E_2 is the pullback of a line bundle L of degree 0 on T by the projection $T \times \mathbb{C} \to T$, and the parabolic structure is the natural one, as in Remark 2.19. A flat metric of L induces a Hermitian–Einstein metric of E_2 adapted to $\mathcal{P}_*E_2$.

Suppose $\operatorname{rank} E_2 > 1$. By Proposition 7.21, $\mathrm{Nahm}_*(\mathcal{P}_*E_2)$ is stable. By Corollary 3.26, we have

$$\deg \mathrm{Nahm}_*(\mathcal{P}_*E_2) = \deg(\mathcal{P}_*E_2) = 0.$$

By Corollary 3.28, we have that $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$ is a good filtered Higgs bundle. Hence, by the Kobayashi–Hitchin correspondence for wild harmonic bundles on curves [7], we obtain an adapted harmonic metric for $\mathrm{Nahm}(\mathcal{P}_*E_1)$. Its Nahm transform induces a Hermitian–Einstein metric of E_1 adapted to the filtered bundle $\mathcal{P}_*E_1$, by Theorem 7.12 and Proposition 3.25. $\square$

Note that the converse is given in Proposition 6.4.

Remark 7.23 This proof of Theorem 7.22 is based on the idea mentioned in [8, Remark 5.13].

7.3.2 Proof of Proposition 7.21 Let us prove the "if" part in Proposition 7.21. Suppose $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$ is stable. By the Kobayashi–Hitchin correspondence for wild harmonic bundles on curves [7] (see also [36] for the case of good filtered flat bundles), we have an adapted harmonic metric for $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$. By Theorem 7.12, its Nahm transform gives an adapted Hermitian–Einstein metric for $\mathcal{P}_*E_1$. By Proposition 6.4, $\mathcal{P}_*E_1$ is polystable. If it is not stable, the decomposition into the stable components induces a decomposition of $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$, which contradicts with the stability of $\mathrm{Nahm}_*(\mathcal{P}_*E_1)$. Hence, $\mathcal{P}_*E_1$ is stable.

Let us prove the "only if" part in Proposition 7.21. Let $(\mathcal{P}_*E,\theta) := \mathrm{Nahm}_*(\mathcal{P}_*E_1)$. Let $(\mathcal{P}_*E',\theta') \subset (\mathcal{P}_*E,\theta)$ be a strict filtered Higgs subbundle with $0 < \operatorname{rank} E' < \operatorname{rank} E$. We obtain a subcomplex $\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta') \subset \widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta)$ on $T^\vee \times T \times \mathbb{P}^1$. Let $\widetilde{\mathcal{Y}}^\bullet = (\widetilde{\mathcal{Y}}^0 \to \widetilde{\mathcal{Y}}^1)$ be the quotient complex.

Lemma 7.24 *The induced morphism* $R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))\to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta))$ *is injective.*

Proof By the construction, $\widetilde{\mathcal{Y}}^0$ is locally free. Hence, we obtain that $R^0p_{23*}\widetilde{\mathcal{Y}}^0$ is torsion-free. Because $R^0p_{23*}(\widetilde{\mathcal{Y}}^\bullet) \to R^0p_{23*}\widetilde{\mathcal{Y}}^0$ is injective, we obtain that $R^0p_{23*}\widetilde{\mathcal{Y}}^\bullet$ is torsion-free.

We take a small neighbourhood U of ∞ in $\mathbb{P}^1$ on which we have the vanishing $R^ip_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta')) = R^ip_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta)) = 0$ unless $i=1$. It is easy to check that

$$R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))_{|T\times\{\infty\}} \to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta))_{|T\times\{\infty\}}$$

is injective. Hence,

$$R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))_{|T\times U} \to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta))_{|T\times U}$$

is injective. Because

$$0 \to R^0p_{23*}\widetilde{\mathcal{Y}}^\bullet \to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta')) \to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta))$$

is exact, $R^0p_{23*}\widetilde{\mathcal{Y}}^\bullet = 0$, and

$$R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta')) \to R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E,\theta))$$

is injective. □

We define the parabolic structure of $R^1p_{23*}(\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))$ as in Section 3.2.1. The filtered sheaf is denoted by $\mathcal{P}_*\mathcal{V}_1$. We have a naturally defined injective morphism

$\mathcal{P}_*\mathcal{V}_1 \to \mathcal{P}_* E_1$. Hence, we have $\deg(\mathcal{P}_*\mathcal{V}_1) \le 0$. By the argument in Section 3.2.3, we obtain

$$\int_{T\times\mathbb{P}^1} c_1(\mathcal{P}_*\mathcal{V}_1)\omega_T - \int_{T\times\mathbb{P}^1} c_1(R^2 p_{23*}\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))\omega_T = \deg(\mathcal{P}_*E').$$

Since $R^2 p_{23*}\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta')$ is a torsion sheaf,

$$\int_{T\times\mathbb{P}^1} c_1(R^2 p_{23*}\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta'))\omega_T \ge 0.$$

Hence, $\deg(\mathcal{P}_*E') \le 0$, ie $(\mathcal{P}_*E,\theta)$ is semistable.

We have $(\mathcal{P}_*E',\theta') \subset (\mathcal{P}_*E,\theta)$ such that $(\mathcal{P}_*E',\theta')$ is stable of degree 0. If $(\mathcal{P}_*E',\theta')$ has no singularity, it is isomorphic to a line bundle on $T^\vee$ with a Higgs field $\alpha\, d\zeta$ $(\alpha\in\mathbb{C})$, and so $R^1 p_{23*}\widetilde{\mathcal{C}}^\bullet(\mathcal{P}_*E',\theta')$ is a nonzero torsion subsheaf of E_1, contradicting Lemma 7.24. Thus, $(\mathcal{P}_*E',\theta')$ has a singularity, and $\mathrm{Nahm}_*(\mathcal{P}_*E',\theta') \neq 0$ is a good filtered subbundle of $\mathcal{P}_*E_1$. By the stability of $\mathcal{P}_*E_1$, we have that the rank of $\mathrm{Nahm}_*(\mathcal{P}_*E',\theta')$ is equal to rank E_1. Because $\deg \mathrm{Nahm}_*(\mathcal{P}_*E',\theta') = \deg(\mathcal{P}_*E)$, we have $\mathrm{Nahm}_*(\mathcal{P}_*E',\theta') = \mathcal{P}_*E_1$ in codimension one. Because both of them are filtered bundles, we have $\mathrm{Nahm}_*(\mathcal{P}_*E',\theta') = \mathcal{P}_*E_1$ on $T\times\mathbb{P}^1$. Then we obtain $(\mathcal{P}_*E',\theta') = (\mathcal{P}_*E,\theta)$ by the involutivity of the algebraic Nahm transforms. □

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