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Residual properties of automorphism groups of (relatively) hyperbolic groups

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We show that $\mathrm{Out}(G)$ is residually finite if G is one-ended and hyperbolic relative to virtually polycyclic subgroups. More generally, if G is one-ended and hyperbolic relative to proper residually finite subgroups, the group of outer automorphisms preserving the peripheral structure is residually finite. We also show that $\mathrm{Out}(G)$ is virtually residually p–finite for every prime p if G is one-ended and toral relatively hyperbolic, or infinitely-ended and virtually residually p–finite.

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1 Introduction

A group G is *residually finite* if, given any $g \neq 1$, there exists a homomorphism φ from G to a finite group such that $\varphi(g) \neq 1$. Residual finiteness is an important property of groups. It is equivalent to the statement that G embeds into its profinite completion. Well-known theorems of Mal'cev show that finitely generated residually finite groups are Hopfian, and finitely presented residually finite groups have solvable word problem. Many groups are known to be residually finite (in particular, finitely generated linear groups), but it is a big open question whether all (Gromov) hyperbolic groups are residually finite.

It is a standard and classical fact (see Baumslag [6]) that the automorphism group $\mathrm{Aut}(G)$ is residually finite if G is finitely generated and residually finite, but this is not true for the outer automorphism group $\mathrm{Out}(G)$. Indeed, any finitely presented group may be represented as $\mathrm{Out}(G)$ with G finitely generated and residually finite; see Bumagin and Wise [8].

A special case of our main theorem is:

Corollary 1.1 *If G is one-ended and hyperbolic relative to a family $\mathcal{P} = \{P_1, \dots, P_k\}$ of virtually polycyclic groups, then $\mathrm{Out}(G)$ is residually finite.*

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This is new even if G is a torsion-free hyperbolic group. Work by Sela implies that a finite-index subgroup of $\mathrm{Out}(G)$ is virtually a central extension of a free abelian group by a direct product of mapping class groups (see the first author [23]). Though mapping class groups are known to be residually finite following work by Grossman [15] based on conjugacy separability (see also Allenby, Kim and Tang [2]), this is not enough to deduce residual finiteness of $\mathrm{Out}(G)$ because the extension may fail to split (ie be a direct product); see the example discussed in the introduction of [23].

To complement Corollary 1.1, recall that $\mathrm{Out}(G)$ is residually finite if G is residually finite and has infinitely many ends; see the second author and Osin [27]. On the other hand, $\mathrm{Out}(G * F_2)$ contains G (with F_2 the free group of rank 2), so it is not residually finite if G is not. Thus one-endedness cannot be dispensed with in Corollary 1.1. This also gives a direct way of proving the following fact, which may otherwise be obtained by combining small cancellation theory over hyperbolic groups with the results from [27].

Proposition 1.2 *The following are equivalent.*

(i) *Every hyperbolic group is residually finite.*

(ii) *For every hyperbolic group G, the group $\mathrm{Out}(G)$ is residually finite.*

Virtual polycyclicity of the P_i is used in two ways in Corollary 1.1 (see Section 7.6). It ensures that P_i is residually finite, and also that automorphisms of G respect the peripheral structure: P_i is mapped to a conjugate of a P_j (this only holds if no P_i is virtually cyclic, but such a restriction causes no loss of generality; see Section 4.1). In fact, the peripheral structure is preserved if every P_i is small (ie it does not contain the free group F_2) or, more generally, NRH (nonrelatively hyperbolic); see the second author and Osin [28] for a proof and a list of examples of NRH groups.

Since the peripheral structure is not always preserved, we restrict to the subgroup $\mathrm{Out}(G;\mathcal{P})$ of $\mathrm{Out}(G)$ defined as the group of classes of automorphisms mapping each P_i to a conjugate.

Theorem 1.3 *Let G be a group hyperbolic relative to a family of proper finitely generated subgroups $\mathcal{P} = \{P_1, \dots, P_k\}$. If G is one-ended relative to $\mathcal{P}$, and every P_i is residually finite, then $\mathrm{Out}(G;\mathcal{P})$ is residually finite.*

Being *one-ended relative to $\mathcal{P}$* means that G does not split over a finite group with each P_i contained in a vertex group (up to conjugacy). This is weaker than having at most one end.

If every P_i is NRH, then $\mathrm{Out}(G;\mathcal{P})$ has finite index in $\mathrm{Out}(G)$, so we deduce residual finiteness of $\mathrm{Out}(G)$.

The following example shows that it is indeed necessary to assume that all peripheral subgroups P_i are residually finite in Theorem 1.3.

Example 1.4 Let H be a torsion-free nonresidually finite group with trivial center (such as the Baumslag–Solitar group $BS(2,3)$). Let K be a one-ended torsion-free hyperbolic group (eg the fundamental group of a closed hyperbolic surface), and let $\langle k\rangle \leqslant K$ be a maximal cyclic subgroup. Let G be the amalgamated product $(H\times\langle k\rangle) *_{\langle k\rangle} K$. Then G is one-ended, torsion-free and hyperbolic relative to $\mathcal{P}=\{H\times\langle k\rangle\}$ (see Osin [29]). Nontrivial elements $h\in H$ define twist automorphisms (they act as conjugation by h on $H\times\langle k\rangle$ and trivially on K), which give rise to nontrivial outer automorphisms in $\mathrm{Out}(G;\mathcal{P})$ because H has trivial center. Thus H can be embedded into $\mathrm{Out}(G;\mathcal{P})$, and so $\mathrm{Out}(G;\mathcal{P})$ is not residually finite.

In the last section of the paper we consider residual p–finiteness. If p is a prime, G is *residually p–finite* if, given any nontrivial element $g\in G$, there exists a homomorphism φ from G to a finite p–group such that $\varphi(g)\neq 1$. A group is *virtually residually p–finite* if some finite-index subgroup is residually p–finite. Evidently residual p–finiteness implies residual nilpotence. And if a group is virtually residually p–finite for at least two distinct primes p, then it is virtually torsion-free.

It is well known that free groups are residually p–finite for any prime p, and it is a classical result that a finitely generated linear group is residually p–finite for all but finitely many p (cf Wehrfritz [42]). Lubotzky [24] proved that for a finitely generated virtually residually p–finite group G, the group $\mathrm{Aut}(G)$ is also virtually residually p–finite, which is a natural analogue of Baumslag's result [6]. Another theorem from [24] states that, if F is a free group of finite rank, then $\mathrm{Out}(F)$ is virtually residually p–finite for any prime p. The latter result was later extended by Paris [32] to fundamental groups of compact oriented surfaces. The next two theorems generalize these results to certain relatively hyperbolic groups:

Theorem 1.5 *If G is a one-ended toral relatively hyperbolic group, then $\mathrm{Out}(G)$ is virtually residually p–finite for every prime number p.*

Recall that G is called *toral relatively hyperbolic* if it is torsion-free and hyperbolic relative to a finite family of finitely generated abelian groups. The theorem also applies to groups containing a one-ended toral relatively hyperbolic group with finite index, in particular to virtually torsion-free hyperbolic groups (see Theorem 8.14).

The following statement is a counterpart of Theorem 1.5 in the case when G has infinitely many ends, giving an "outer" version of Lubotzky's result [24] mentioned above. It is a natural pro-p analogue of [27, Theorem 1.5].

Theorem 1.6 *If G is a finitely generated group with infinitely many ends and G is virtually residually p–finite for some prime number p, then $\mathrm{Out}(G)$ is virtually residually p–finite.*

Recall that limit groups (finitely generated fully residually free groups) are toral relatively hyperbolic; see Alibegović [1] and Dahmani [10]. Residual finiteness of $\mathrm{Out}(G)$ for such a group G was proved by Metaftsis and Sykiotis in [26]. Combining Theorems 1.5 and 1.6 gives the following enhancement:

Corollary 1.7 *If G is a limit group (finitely generated fully residually free group), then $\mathrm{Out}(G)$ is virtually residually p–finite for any prime p.*

We start the paper by giving a rather quick proof of Corollary 1.1 when G is (virtually) torsion-free hyperbolic, using Sela's description of $\mathrm{Out}(G)$ recalled above as a starting point. The proof of Theorem 1.5 (given in Section 8) uses similar arguments, but in order to prove Theorem 1.3, we have to use different techniques.

Say that a subgroup of a relatively hyperbolic group G is *elementary* if it is virtually cyclic or parabolic (conjugate to a subgroup of some P_i). As Guirardel and the first author did in [19], we consider the canonical JSJ decomposition of G over elementary subgroups relative to $\mathcal{P}$. This is a graph of groups decomposition Γ of G such that edge groups are elementary, each P_i is conjugate to a subgroup of a vertex group, and Γ is $\mathrm{Out}(G,\mathcal{P})$–invariant; moreover, vertex groups are either elementary, or quadratically hanging (QH), or rigid.

In the first step of the proof of Theorem 1.3 (Section 5), we replace each rigid vertex group by a new group which is residually finite and has infinitely many ends. In the second step, we make elementary vertex groups, and edge groups, finite (using residual finiteness of the P_i). Apart from simple cases, the new graph of groups represents a residually finite group G'' with infinitely many ends, and so $\mathrm{Out}(G'')$ is residually finite by [27]. Thus we get a homomorphism $\mathrm{Out}(G;\mathcal{P}) \to \mathrm{Out}(G'')$ and we show that such homomorphisms "approximate" $\mathrm{Out}(G;\mathcal{P})$.

The second step is easier when G is torsion-free (see Section 6). Torsion brings technical complications, so in its presence we prefer to give a different argument using Dehn fillings (see Osin [31]) and Grossman's method [15]. Sections 7 and 8 are independent of Section 6.

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2 Notation and residual finiteness

First let us specify some notation.

If G is a group, we denote its center by $Z(G)$. If $H \leqslant G$ is a subgroup, then $Z_G(H)$ is its centralizer in G. We will write $|G:H|$ for the index of a subgroup H in G. For any $g \in G$, we denote by $\tau_g \in \mathrm{Aut}(G)$ the inner automorphism given by $\tau_g\colon x \mapsto gxg^{-1}$ for all $x \in G$.

If $R \subseteq G$, then $\langle\langle R\rangle\rangle^G$ will denote the normal closure of R in G. If A is an abelian group, and $n \geq 1$, we will write $nA = \{g^n \mid g \in A\}$ for the corresponding verbal subgroup of A.

Given $\alpha \in \mathrm{Aut}(G)$, we write $\hat{\alpha}$ for its image in $\mathrm{Out}(G)$. We denote by $\mathrm{Aut}(G;\mathcal{P}) \leqslant \mathrm{Aut}(G)$ the subgroup consisting of automorphisms mapping each P_i to a conjugate, and by $\mathrm{Out}(G;\mathcal{P})$ its image in $\mathrm{Out}(G)$. If every P_i is NRH (eg if P_i is not virtually cyclic and has no nonabelian free subgroups), then $\mathrm{Out}(G;\mathcal{P})$ has finite index in $\mathrm{Out}(G)$ (see [28, Lemma 3.2]).

Given a group G, the cosets of finite-index normal subgroups define a basis of the *profinite topology* on G. This topology is Hausdorff if and only if G is residually finite. A subset S of G is said to be *separable* if S is closed in the profinite topology. Thus, if G is residually finite, then any finite subset of G is separable.

A subgroup $K \leqslant G$ is closed in the profinite topology if and only if K is the intersection of a family of finite-index subgroups. It is easy to see that a normal subgroup $N \lhd G$ is separable if and only if G/N is residually finite. In particular, if G is residually finite and $N \lhd G$ is finite, then G/N is also residually finite.

If $H \leqslant G$ has finite index, then G is residually finite if and only if H is. We will also use the following fact: the fundamental group of a finite graph of groups with residually finite vertex groups and finite edge groups is residually finite (see, for instance, Serre [37, II.2.6.12]).

Recall that in a finitely generated group G every finite-index subgroup $K \leqslant G$ contains a finite-index subgroup N which is characteristic in G, eg one can take $N = \bigcap_{\alpha \in \mathrm{Aut}(G)} \alpha(K)$. Thus, if G finitely generated and residually finite, then for every $g \in G \setminus \{1\}$ there is a characteristic subgroup N of finite index in G such that $g \notin N$.

3 Torsion-free hyperbolic groups

The goal of this section is to give a short proof of the following statement:

Theorem 3.1 *Let G be a one-ended hyperbolic group. If G is virtually torsion-free, then* $\mathrm{Out}(G)$ *is residually finite.*

3.1 Automorphisms with twistors

Let H be a group. Fix finitely many subgroups $C_1, \dots, C_s$ (not necessarily distinct), with $s \geq 1$. We define groups $\mathrm{PMCG}(H)$ and $\mathrm{PMCG}^{\partial}(H)$ as in [23, Section 4].

First, $\mathrm{PMCG}(H)$ is the subgroup of $\mathrm{Out}(H)$ consisting of all elements $\widehat{\alpha}$ represented by automorphisms $\alpha \in \mathrm{Aut}(H)$ acting on each C_i as τ_{a_i} for some $a_i \in H$.

Let $\mathrm{Aut}^{\partial}(H)$ be the subset of $\mathrm{Aut}(H) \times H^s$ given by

$$\mathrm{Aut}^{\partial}(H) = \{(\alpha; a_1, \dots, a_s) \mid \alpha \in \mathrm{Aut}(H),\ a_i \in H,\ \alpha(c) = a_i c a_i^{-1} \text{ for all } c \in C_i \text{ and } i = 1, \dots, s\}.$$

This is a group, with multiplication defined by

$$(\alpha; a_1, \dots, a_s)(\alpha'; a'_1, \dots, a'_s) = (\alpha \circ \alpha'; \alpha(a'_1)a_1, \dots, \alpha(a'_s)a_s),$$

in accordance with the fact that $\alpha \circ \alpha'$ acts on C_i as conjugation by $\alpha(a'_i)a_i$.

One easily checks that $I = \{(\tau_h; h, \dots, h) \in \mathrm{Aut}^{\partial}(H) \mid h \in H\}$ is a normal subgroup of $\mathrm{Aut}^{\partial}(H)$. As in [23], we define $\mathrm{PMCG}^{\partial}(H)$ to be the quotient of $\mathrm{Aut}^{\partial}(H)$ by I. Thus an element of $\mathrm{PMCG}^{\partial}(H)$ is represented by $(\alpha; a_1, \dots, a_s)$, where α is an automorphism of H acting on C_i as τ_{a_i}, with $a_i \in H$. Representatives are defined only up to multiplication by elements of the form $(\tau_h; h, \dots, h)$; in particular, for each i, there is a unique representative with $a_i = 1$. Mapping $(\alpha; a_1, \dots, a_s)$ to $\widehat{\alpha}$ defines a projection $\pi\colon \mathrm{PMCG}^{\partial}(H) \to \mathrm{PMCG}(H)$.

As observed in [23, Lemma 4.1] there is a short exact sequence

$$\{1\} \longrightarrow \mathcal{T}_H \longrightarrow \mathrm{PMCG}^{\partial}(H) \xrightarrow{\pi} \mathrm{PMCG}(H) \longrightarrow \{1\}$$

whose kernel $\mathcal{T}_H$ is the *group of twists*. It fits in an exact sequence

$$\{1\} \longrightarrow Z(H) \longrightarrow \prod_{i=1}^{s} Z_H(C_i) \longrightarrow \mathcal{T}_H \longrightarrow \{1\},$$

where the first map is the diagonal embedding and the second map takes $(z_1, \dots, z_s)$ to the class of $(\mathrm{id}_H; z_1, \dots, z_s)$ in $\mathrm{PMCG}^{\partial}(H)$.

If $N \lhd H$ is a normal subgroup invariant under $\mathrm{PMCG}(H)$, there are natural homomorphisms $\mathrm{PMCG}(H) \to \mathrm{PMCG}(H/N)$ and $\mathrm{PMCG}^{\partial}(H) \to \mathrm{PMCG}^{\partial}(H/N)$, where the target groups are defined with respect to the images of $C_1, \dots, C_s$ in H/N.

Lemma 3.2 *If H is finitely generated and residually finite, then $\mathrm{PMCG}^{\partial}(H)$ is residually finite.*

Proof Given any nontrivial $\Phi \in \mathrm{PMCG}^{\partial}(H)$, we can construct a characteristic finite-index subgroup $N \lhd H$ such that Φ maps nontrivially to the finite group $\mathrm{PMCG}^{\partial}(H/N)$. Indeed, let $(\alpha; 1, \dots, a_s)$ be the representative of Φ with $a_1 = 1$. If $\alpha(h) \neq h$ for some $h \in H$, we choose such an N with $h^{-1}\alpha(h) \notin N$. On the other hand, if $a_i \neq 1$ for some $i \geq 2$, we choose a characteristic subgroup of finite index $N \lhd H$ with $a_i \notin N$. □

Remark 3.3 There are injective homomorphisms

$$\mathrm{Aut}^{\partial}(H) \longrightarrow H^s \rtimes \mathrm{Aut}(H) \quad \text{and} \quad \mathrm{PMCG}^{\partial}(H) \longrightarrow H^{s-1} \rtimes \mathrm{Aut}(H)$$

defined by

$$\begin{aligned} (\alpha; a_1, a_2, \dots, a_s) &\mapsto ((a_1^{-1}, a_2^{-1}, \dots, a_s^{-1}), \alpha), \\ (\alpha; 1, a_2, \dots, a_s) &\mapsto ((a_2^{-1}, \dots, a_s^{-1}), \alpha), \end{aligned}$$

respectively, with $\mathrm{Aut}(H)$ acting on H^s and H^{s-1} diagonally. This yields another way of proving Lemma 3.2, using residual finiteness of the semidirect product of a finitely generated residually finite group and a residually finite group.

3.2 Surfaces

We now specialize to the case when H is the fundamental group of a compact (possibly nonorientable) surface Σ with boundary components $\mathcal{C}_1, \dots, \mathcal{C}_s$; we require $s \geq 1$ and $\chi(\Sigma) < 0$. We fix a representative C_i of $\pi_1(\mathcal{C}_i)$ in G and a generator c_i of C_i. Then $\mathrm{PMCG}^{\partial}(H)$, as defined above, may be identified with the group of isotopy classes of homeomorphisms of Σ equal to the identity on the boundary. In this definition, the isotopy is relative to the boundary, so $\mathrm{PMCG}^{\partial}(H)$ contains Dehn twists near boundary components. If we do not require isotopies to be relative to the boundary, we get $\mathrm{PMCG}(H)$.

There is a central extension

$$\{1\} \longrightarrow \mathcal{T}_H \longrightarrow \mathrm{PMCG}^{\partial}(H) \longrightarrow \mathrm{PMCG}(H) \longrightarrow \{1\}$$

as above, where $\mathcal{T}_H \cong \mathbb{Z}^s$ is generated by Dehn twists near boundary components of Σ. The inclusion from $\mathbb{Z}^s$ to $\mathrm{PMCG}^\partial(H)$ may be written algebraically as

$$(n_1, \ldots, n_s) \mapsto (\mathrm{id}_H; c_1^{n_1}, \ldots, c_s^{n_s}).$$

Lemma 3.4 *Let $n\mathcal{T}_H \lhd \mathcal{T}_H$ be the subgroup generated by the n^{th} powers of the twists. Then $\mathrm{PMCG}^\partial(H)/n\mathcal{T}_H$ is residually finite for all sufficiently large $n \in \mathbb{N}$.*

It is worth noting that residual finiteness does not follow directly from the fact that the group $\mathrm{PMCG}^\partial(H)/n\mathcal{T}_H$ maps onto the residually finite group $\mathrm{PMCG}(H)$ with finite kernel.

Proof Let Σ_n be the closed orbifold obtained by replacing each boundary component of Σ by a conical point of order n, and let $O_n = H/\langle\langle c_1^n, \ldots, c_s^n\rangle\rangle^H$ be its fundamental group.

The Euler characteristic of Σ_n is $\chi(\Sigma_n) = \chi(\Sigma) + \frac{s}{n}$ (see Scott [36] or Thurston [40]). It is negative for n large since $\chi(\Sigma) < 0$, so Σ_n is a hyperbolic orbifold (see [40, Theorem 13.3.6]). It follows that O_n embeds into the group of isometries of the hyperbolic plane as a nonelementary subgroup. In particular, O_n has trivial center and is residually finite.

Defining $\mathcal{T}_{O_n}$ as the kernel of the map $\mathrm{PMCG}^\partial(O_n) \to \mathrm{PMCG}(O_n)$, there is a commutative diagram of short exact sequences

$$\begin{tikzcd}
\{1\} \arrow[r] & \mathcal{T}_H \arrow[r] \arrow[d, "\theta"] & \mathrm{PMCG}^\partial(H) \arrow[r] \arrow[d] & \mathrm{PMCG}(H) \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathcal{T}_{O_n} \arrow[r] & \mathrm{PMCG}^\partial(O_n) \arrow[r] & \mathrm{PMCG}(O_n) \arrow[r] & \{1\}.
\end{tikzcd}$$

Since O_n has trivial center, and the image of C_i in O_n is equal to its centralizer, $\mathcal{T}_{O_n}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^s$, so the kernel of the map θ from $\mathcal{T}_H$ to $\mathcal{T}_{O_n}$ is precisely $n\mathcal{T}_H$ (it is not bigger). The maps from $\mathrm{PMCG}^\partial(H)$ to $\mathrm{PMCG}(H)$ and $\mathrm{PMCG}^\partial(O_n)$ both factor through $\mathrm{PMCG}^\partial(H)/n\mathcal{T}_H$, and the intersection of their kernels is $\ker\theta = n\mathcal{T}_H$. In other words, any nontrivial element of $\mathrm{PMCG}^\partial(H)/n\mathcal{T}_H$ has a nontrivial image in $\mathrm{PMCG}(H)$ or in $\mathrm{PMCG}^\partial(O_n)$. It is well known that $\mathrm{PMCG}(H)$ is residually finite (it is contained in $\mathrm{Out}(H)$, which is residually finite by [15] because H is a finitely generated free group), and $\mathrm{PMCG}^\partial(O_n)$ is residually finite by Lemma 3.2, so $\mathrm{PMCG}^\partial(H)/n\mathcal{T}_H$ is residually finite. □

Remark 3.5 It follows from the classification of nonhyperbolic 2–orbifolds [40] that $n \geq 3$ is always sufficient in Lemma 3.4.

3.3 An algebraic lemma

Lemma 3.6 *Consider a finite set V and groups P_v, $v \in V$, with normal subgroups T_v free abelian of finite rank. Let $P = \prod_{v\in V} P_v$ and $T = \prod_{v\in V} T_v \leqslant P$ be their direct products. Note that nT_v is characteristic in T_v, hence it is normal in P_v.*

If P_v/nT_v is residually finite for every $v \in V$ and for all sufficiently large $n \in \mathbb{N}$, then any subgroup $Z \leqslant T$ is closed in the profinite topology of P. In particular, if Z is normal in P, then P/Z is residually finite.

Proof Let us first prove the result when Z has finite index k in the free abelian group T. It contains nT (with finite index) whenever k divides n. For n large, nT_v is separable in P_v for every $v \in V$ (because P_v/nT_v is residually finite), so $nT = \prod_{v\in V} nT_v$ is separable in P, by the properties of direct products. We deduce that Z is closed in P, because it is equal to a finite union of cosets modulo nT, each of which is separable in P because nT is separable.

The general case follows because T is a free abelian group of finite rank, and therefore every subgroup is the intersection of a collection of finite-index subgroups (because the quotient is clearly residually finite). □

3.4 Proof of Theorem 3.1

First suppose that G is torsion-free. The result is true if $\mathrm{Out}(G)$ is finite, or if G is the fundamental group of a closed surface (in the orientable case this was proved by Grossman [15], and in the nonorientable case by Allenby, Kim and Tang [2]). Otherwise, by [23, Theorem 5.3], the group $\mathrm{Out}(G)$ is virtually a product $\mathbb{Z}^q \times M$, with M a quotient of a finite direct product $\Pi = \prod_v \mathrm{PMCG}^\partial(G_v)$; here G_v is a surface group H as in Section 3.2 (a QH vertex group of the cyclic JSJ decomposition of G), and we denote by $\mathcal{T}_{G_v} \lhd \mathrm{PMCG}^\partial(G_v)$ the corresponding group of twists. Moreover, the kernel Z of the map from Π to M is contained in the free abelian group $\mathcal{T}_\Pi = \prod_v \mathcal{T}_{G_v}$.

Lemma 3.4 implies that $\mathrm{PMCG}^\partial(G_v)/n\mathcal{T}_{G_v}$ is residually finite for all sufficiently large n. It follows that Z is separable in Π by Lemma 3.6. Thus M, and therefore also $\mathrm{Out}(G)$, are residually finite.

Now suppose that G is only virtually torsion-free and let $N \lhd G$ be a torsion-free normal subgroup of finite index. If G is virtually cyclic, $\mathrm{Out}(G)$ is finite (cf [27, Lemma 6.6]). Otherwise, N has trivial center, so some finite-index subgroup of $\mathrm{Out}(G)$ is isomorphic to the quotient of a subgroup of $\mathrm{Out}(N)$ by a finite normal subgroup (see Guirardel and the first author [20, Lemma 5.4] or Lemma 7.15 below). We have shown above that $\mathrm{Out}(N)$ is residually finite, and therefore so is $\mathrm{Out}(G)$.

Remark 3.7 An alternative method to prove Theorem 3.1 could employ Funar's results about residual finiteness of central extensions of mapping class groups [14]. However, writing a complete proof using this approach would still require substantial work, for instance because the surfaces involved may be nonorientable.

4 Relatively hyperbolic groups and trees

In this section we recall basic material about relatively hyperbolic groups and trees.

4.1 Relatively hyperbolic groups

There are many equivalent definitions of relatively hyperbolic groups in the literature. The definition we give below is due to B Bowditch [7]; for its equivalence to the other definitions, see Hruska [22] or Osin [30]. In this paper we will always assume that G and all the groups $P_i \in \mathcal{P}$ are finitely generated. (Note that if G is hyperbolic relative to a finite family of finitely generated subgroups then G is itself finitely generated. This follows, for example, from the equivalence of Definition 4.1 with Osin's definition [30, Definition 1.6].)

Definition 4.1 [7, Definition 2] Consider a group G with a family of subgroups $\mathcal{P} = \{P_1, \dots, P_k\}$. We will say that G is hyperbolic relative to $\mathcal{P}$ if G admits a simplicial action on a connected graph $\mathcal{K}$ such that:

- $\mathcal{K}$ is δ–hyperbolic for some $\delta \geq 0$, and for each $n \in \mathbb{N}$ every edge of $\mathcal{K}$ is contained in finitely many simple circuits of length n.
- The edge stabilizers for this action of G on $\mathcal{K}$ are finite, and there are finitely many orbits of edges.
- $\mathcal{P}$ is a set of representatives of conjugacy classes of the infinite vertex stabilizers.

We usually assume that each P_i is a *proper* subgroup of G, ie $P_i \neq G$ (as any G is hyperbolic relative to itself).

A subgroup $H \leqslant G$ is *elementary* if it is virtually cyclic (possibly finite) or *parabolic* (contained in a conjugate of some P_i). Any infinite elementary subgroup H is contained in a unique maximal elementary subgroup $\widehat{H}$, and $Z_G(H) \subset \widehat{H}$. We say that G itself is *elementary* if it is virtually cyclic or equal to some P_i.

It is well known that, if some P_i is virtually cyclic (or, more generally, hyperbolic), then G is hyperbolic relative to the family $\mathcal{P} \setminus \{P_i\}$; see, for example, [30, Theorem 2.40]. In the context of Corollary 1.1, we may therefore assume that no P_i is virtually cyclic. We do not wish to do so in Theorem 1.3, because it may happen that G is one-ended relative to $\mathcal{P}$, but not relative to $\mathcal{P} \setminus \{P_i\}$ with P_i infinite and virtually cyclic. For simplicity, however, we assume in most of the paper that no P_i is virtually cyclic. The (few) changes necessary to handle the general case are explained in Section 7.5.

4.2 The canonical splitting of a one-ended relatively hyperbolic group

Assume that G is one-ended, or, more generally, one-ended relative to $\mathcal{P}$: it does not split over a finite group relative to $\mathcal{P}$ (ie with every P_i fixing a point in the Bass–Serre tree). Then there is a canonical JSJ tree T over elementary subgroups relative to $\mathcal{P}$ (see Guirardel and the first author [17, Corollary 13.2; 19]). Canonical means, in particular, that T is invariant under the natural action of $\mathrm{Out}(G;\mathcal{P})$.

The tree T is equipped with an action of G. We denote by G_v the stabilizer of a vertex v, by G_e the stabilizer of an edge e (it is elementary). They are infinite and finitely generated [19]. If $e = vw$, we say that G_e is an incident edge stabilizer in G_v and G_w.

We also consider the quotient graph of groups $\Gamma = T/G$. We then denote by G_v the group carried by a vertex v, by G_e the group carried by an edge e. If v is an endpoint of e, we often identify G_e with a subgroup of G_v, and we say that G_e is an incident edge group at v.

Being a tree of cylinders (see [17]), T is bipartite, with vertex set $A_0 \cup A_1$. The stabilizer of a vertex $v_1 \in A_1$ is a maximal elementary subgroup (we also say that v_1 is an elementary vertex). The stabilizer of an edge $\varepsilon = v_0 v_1$ (with $v_i \in A_i$) is a maximal elementary subgroup of G_{v_0} (ie it is maximal among elementary subgroups contained in G_{v_0}), but G_ε is not necessarily maximal elementary in G_{v_1} or in G.

Vertices in A_0 have nonelementary stabilizers. A vertex $v \in A_0$ (or its stabilizer G_v) is either rigid or QH (quadratically hanging). A rigid G_v does not split over an elementary subgroup relative to parabolic subgroups and incident edge stabilizers. Through the Bestvina–Paulin method and Rips theory, this has strong implications on its automorphisms (see [19]). This will be the key point in the proof of Lemma 5.2.

To describe QH vertices, it is more convenient to consider a QH vertex group G_v of the graph of groups Γ. First suppose that G is torsion-free. Then G_v may be identified with the fundamental group of a (possibly nonorientable) compact hyperbolic surface Σ_v whose boundary is nonempty (unless $G_v = G$).

Moreover, each incident edge group G_e is (up to conjugacy) the fundamental group $H_{\mathcal{C}}$ of a boundary component $\mathcal{C}$ of Σ_v. Different edges correspond to different boundary components. Conversely, if no P_i is cyclic, the fundamental group $H_{\mathcal{C}}$ of every boundary component $\mathcal{C}$ is an incident edge group. If cyclic P_i are allowed, it may happen that $H_{\mathcal{C}}$ is not an incident edge group; it is then conjugate to some P_i.

If torsion is allowed, we only have an exact sequence

$$\{1\} \longrightarrow F \longrightarrow G_v \xrightarrow{\xi} P \longrightarrow \{1\},$$

where F is a finite group and P is the fundamental group of a compact hyperbolic 2–orbifold $\mathcal{O}_v$. If $\mathcal{C}$ is a boundary component of $\mathcal{O}_v$, its fundamental group $\pi_1(\mathcal{C}) \subset P$ is infinite cyclic or infinite dihedral; one defines $H_{\mathcal{C}} \subset G_v$ as its full preimage under ξ. It is an incident edge group or is conjugate to a virtually cyclic P_i.

Note that, in all cases, a QH vertex stabilizer G_v of T is virtually free (unless $G_v = G$), and stabilizers of incident edges are virtually cyclic.

The tree T is relative to $\mathcal{P}$: every P_i fixes a point. If P_i is not virtually cyclic, it equals the stabilizer of a vertex $v_1 \in A_1$ or is contained in some G_{v_0} with v_0 rigid (it may happen that $P_i = G_{v_1} \subsetneq G_{v_0}$). In particular, the intersection of P_i with a QH vertex group is virtually cyclic. If P_i is virtually cyclic and infinite, there is the additional possibility that it is contained in a QH vertex stabilizer (and conjugate to an $H_{\mathcal{C}}$ as above).

4.3 The automorphism group of a tree

Let T be any tree with an action of a finitely generated group G. We assume that the action is minimal (there is no proper G–invariant subtree), and T is not a point or a line. Let $\mathrm{Out}(G;T) \subset \mathrm{Out}(G)$ consist of outer automorphisms $\Phi = \hat{\alpha}$ leaving T invariant: in other words, Φ comes from an automorphism $\alpha \in \mathrm{Aut}(G)$ such that there is an isomorphism $H_\alpha\colon T \to T$ satisfying $\alpha(g)H_\alpha = H_\alpha g$ for all $g \in G$. We study $\mathrm{Out}(G;T)$ as in [23, Sections 2–4].

It is more convenient to consider the quotient graph of groups $\Gamma = T/G$. It is finite, and the maps H_α induce an action of $\mathrm{Out}(G;T)$ on Γ. We denote by $\mathrm{Out}_0(G;T) \leqslant \mathrm{Out}(G;T)$ the finite-index subgroup consisting of automorphisms acting trivially on Γ.

We denote by V the vertex set of Γ, by E the set of oriented edges, by E_v the set of oriented edges e with origin $o(e) = v$ (incident edges at v), by $\mathcal{E}$ the set of nonoriented edges. We write G_v or G_e for the group attached to a vertex or an edge, and we view G_e as a subgroup of G_v if $e \in E_v$ (incident edge group).

For $v \in V$, we define groups $\mathrm{PMCG}(G_v) \leqslant \mathrm{Out}(G_v)$ and $\mathrm{PMCG}^\partial(G_v)$ as in Section 3.1, using as C_i the incident edge groups (s is the valence of v in Γ, and there are repetitions if $G_e = G_{e'}$ with $e \neq e'$). We denote by $\pi_v\colon \mathrm{PMCG}^\partial(G_v) \to \mathrm{PMCG}(G_v)$ the natural projection.

There is a natural map (extension by the identity) $\lambda_v\colon \mathrm{PMCG}^\partial(G_v) \to \mathrm{Out}_0(G;T)$ (see [23, Section 2.3]). For instance, if Γ is an amalgam $G = G_v *_{G_e} G_w$, and $\psi \in \mathrm{PMCG}^\partial(G_v)$ is represented by $(\alpha; a_1)$, the image of ψ is represented by the automorphism of G acting as α on G_v and as conjugation by a_1 on G_w. Elements in the image of λ_v act as inner automorphisms of G on G_w for $w \neq v$. The maps λ_v have commuting images and fit together in a map

$$\lambda\colon \prod_{v \in V} \mathrm{PMCG}^\partial(G_v) \longrightarrow \mathrm{Out}_0(G;T).$$

There is also a map

$$\rho = \prod_{v \in V} \rho_v\colon \mathrm{Out}_0(G;T) \longrightarrow \prod_{v \in V} \mathrm{Out}(G_v)$$

recording the action of automorphisms on vertex groups, and the projection

$$\pi = \prod_{v \in V} \pi_v\colon \prod_{v \in V} \mathrm{PMCG}^\partial(G_v) \longrightarrow \prod_{v \in V} \mathrm{PMCG}(G_v) \leqslant \prod_{v \in V} \mathrm{Out}(G_v)$$

factors as $\pi = \rho \circ \lambda$.

We let $\mathrm{Out}_1(G;T) \subset \mathrm{Out}_0(G;T)$ be the image of λ, and

$$\rho_1\colon \mathrm{Out}_1(G;T) \longrightarrow \prod_{v \in V} \mathrm{PMCG}(G_v)$$

the restriction of ρ. In general $\mathrm{Out}_1(G;T)$ is smaller than $\mathrm{Out}_0(G;T)$ because elements of $\mathrm{Out}_1(G;T)$ are required to map into $\mathrm{PMCG}(G_v)$ for all v, and also since $\ker\rho$ may fail to be contained in $\mathrm{Out}_1(G;T)$ because of "bitwists" (which will not concern us here; see the proof of Lemma 5.2).

To sum up, we have written π as the product of two epimorphisms

$$\prod_{v \in V} \mathrm{PMCG}^\partial(G_v) \overset{\lambda}{\twoheadrightarrow} \mathrm{Out}_1(G;T) \overset{\rho_1}{\twoheadrightarrow} \prod_{v \in V} \mathrm{PMCG}(G_v).$$

We now study the *group of twists* $\mathcal{T} = \ker\rho_1 = \ker\rho \cap \mathrm{Out}_1(G;T)$. It is generated by the commuting subgroups $\lambda_v(\mathcal{T}_v)$, where $\mathcal{T}_v$ is the kernel of the projection $\pi_v\colon \mathrm{PMCG}^\partial(G_v) \to \mathrm{PMCG}(G_v)$. As in Section 3.1, $\mathcal{T}_v$ is the quotient of $\prod_{e \in E_v} Z_{G_v}(G_e)$ by $Z(G_v)$ (embedded diagonally), which we call a *vertex relation*. The image of an element $z \in Z_{G_v}(G_e)$ in $\mathrm{Out}(G;T)$ is the *twist by z around e near v*

(note that z does not have to belong to G_e). For instance, in the case of the amalgam considered above, it acts as the identity on G_v and as conjugation by z on G_w.

The group $\mathcal{T}$ is generated by the product $\prod_{e\in E} Z_{G_{o(e)}}(G_e)$. A complete set of relations is given by the vertex relations $Z(G_v)$ (with $Z(G_v)$ embedded diagonally into the factors $\prod Z_{G_{o(e)}}(G_e)$ such that $o(e)=v$) and the *edge relations* $Z(G_e)$ (with $Z(G_e)$ embedded diagonally into the factors $Z_{G_v}(G_e)$ and $Z_{G_w}(G_{\overline{e}})$ if e is an oriented edge vw and $\overline{e}=wv$). In the case of an amalgam $G=G_v *_{G_e} G_w$, the edge relation simply says that conjugating both G_v and G_w by $z\in Z(G_e)$ defines an inner automorphism of G.

In other words, $\mathcal{T}$ is the quotient of

$$\prod_{e\in E} Z_{G_{o(e)}}(G_e)$$

by the image of

$$\prod_{v\in V} Z(G_v)\times\prod_{\varepsilon\in\mathcal{E}} Z(G_\varepsilon),$$

where the products are taken over all oriented edges, all vertices, and all nonoriented edges respectively.

Dividing $\prod_{e\in E} Z_{G_{o(e)}}(G_e)$ by the vertex relations yields $\prod_{v\in V}\mathrm{PMCG}^{\partial}(G_v)$. The edge relations generate the kernel of $\lambda\colon \prod_{v\in V}\mathrm{PMCG}^{\partial}(G_v)\to \mathrm{Out}_0(G;T)$. Note that $\ker\lambda\subset\ker\pi=\prod_{v\in V}\mathcal{T}_v$.

Example In the case of an amalgam, $\mathcal{T}$ is the image of the map $p\colon Z_{G_v}(G_e)\times Z_{G_w}(G_e)\to\mathrm{Out}(G)$ sending (a,b) to the class of the automorphism acting on G_v as conjugation by b and on G_w as conjugation by a. The kernel of p is generated by the elements $(a,1)$ with $a\in Z(G_v)$ and $(1,b)$ with $b\in Z(G_w)$ (vertex relations), together with the elements (c,c) with $c\in Z(G_e)$ (edge relations). We have $\mathcal{T}_v = Z_{G_v}(G_e)/Z(G_v)$ and $\mathcal{T}_w = Z_{G_w}(G_e)/Z(G_w)$. The kernels of $\lambda\colon \mathrm{PMCG}^{\partial}(G_v)\times\mathrm{PMCG}^{\partial}(G_w)$, and of its restriction to $\mathcal{T}_v\times\mathcal{T}_w$, are generated by $Z(G_e)$.

The following lemma will be used in Section 6.

Lemma 4.2 *Let $W\subset V$. Assume that G_v has trivial center if $v\notin W$, and that Γ has no edge with both endpoints in W. Then the map $\lambda_W\colon \prod_{v\in W}\mathrm{PMCG}^{\partial}(G_v)\to\mathrm{Out}(G;T)$ is injective.*

Proof The kernel of λ_W is the intersection of $\ker\lambda$ with $\prod_{v\in W}\mathcal{T}_v\subset\prod_{v\in V}\mathcal{T}_v$. If $v\notin W$, the group G_v has trivial center, so the product $\prod_{e\in E_v} Z(G_e)$, taken over all edges with origin v, injects into $\mathcal{T}_v$. This implies that, if a product $z = \prod_{\varepsilon \in \mathcal{E}} z_\varepsilon$ with $z_\varepsilon \in Z(G_\varepsilon)$ maps to $\prod_{v \in W} \mathcal{T}_v$, it cannot involve edges having an endpoint outside of W. Thus z is trivial since all edges are assumed to have an endpoint not in W. $\square$

Remark 4.3 If edges with both endpoints in W are allowed, the proof shows that the kernel of λ_W is generated by the edge relations associated to these edges.

5 Getting rid of the rigids

Let G be a group hyperbolic relative to a family of finitely generated subgroups $\mathcal{P} = \{P_1, \dots, P_k\}$, and one-ended relative to $\mathcal{P}$. In this section we assume that no P_i is virtually cyclic (see Section 7.5 for a generalization).

We consider the canonical elementary JSJ tree T relative to $\mathcal{P}$ (see Section 4.2). It is invariant under $\mathrm{Out}(G;\mathcal{P})$, so $\mathrm{Out}(G;\mathcal{P}) \leqslant \mathrm{Out}(G;T)$, and bipartite: each edge joins a vertex with elementary stabilizer to a vertex with nonelementary (rigid or QH) stabilizer. In particular, T cannot be a line; we assume that it is not a point.

As above, we consider the quotient graph of groups $\Gamma = T/G$, with vertex set V. Just like those of T, vertices of Γ (and their groups) may be elementary, rigid or QH. We partition V as $V_E \cup V_R \cup V_{QH}$ accordingly; each edge has exactly one endpoint in V_E.

Lemma 5.1 *The group $\mathcal{T}$ is generated by the groups $\lambda_w(\mathcal{T}_w)$ with $w \in V_E$: twists near vertices in V_E generate the whole group of twists of T.*

Proof If $e = vw$ is any edge with $v \in V_{QH} \cup V_R$ (and therefore $w \in V_E$), then G_e is a maximal elementary subgroup of G_v, so $Z_{G_v}(G_e) = Z(G_e)$ since $Z_{G_v}(G_e) \subset G_e$. We can then use edge relations to view twists around e near v as twists near w. $\square$

Lemma 5.2 *Let $\mathrm{Out}^r(G)$ be the image of the restriction*

$$\lambda_{E,QH} \colon \prod_{v \in V_E \cup V_{QH}} \mathrm{PMCG}^\partial(G_v) \longrightarrow \mathrm{Out}(G)$$

of λ (see Section 4.3). Then $\mathrm{Out}^r(G)$ is contained in $\mathrm{Out}(G;\mathcal{P})$ with finite index.

Proof We first prove $\mathrm{Out}^r(G) \subset \mathrm{Out}(G;\mathcal{P})$. Recall that P_i is assumed not to be virtually cyclic, so (up to conjugacy) it is equal to an elementary vertex group or is contained in a rigid vertex group. In either case elements of $\mathrm{Out}^r(G)$ map P_i to a conjugate (trivially if P_i is contained in a rigid group). In fact, $\mathrm{Out}^r(G)$ is contained in $\mathrm{Out}_0(G;\mathcal{P}) = \mathrm{Out}(G;\mathcal{P}) \cap \mathrm{Out}_0(G;T)$, a finite-index subgroup of $\mathrm{Out}(G;\mathcal{P})$. We show that $\mathrm{Out}^r(G)$ has finite index in $\mathrm{Out}_0(G;\mathcal{P})$.

Recall the maps ρ_v: $\mathrm{Out}_0(G;T) \to \mathrm{Out}(G_v)$, and consider their restrictions to the subgroup $\mathrm{Out}_0(G;\mathcal{P})$. It is shown in [19, Proposition 4.1] that the image of such a restriction is finite if $v \in V_R$ (this is a key property of rigid vertices), contains $\mathrm{PMCG}(G_v)$ with finite index if $v \notin V_R$ (note that $\mathrm{PMCG}(G_v)$ is $\mathrm{Out}(G_v;\mathrm{Inc}_v^{(t)})$ in [19]; the assumption that no P_i is virtually cyclic is used to ensure $\mathcal{B}_v = \mathrm{Inc}_v$).

Now consider the homomorphism ρ: $\mathrm{Out}_0(G;T) \to \prod_{v\in V} \mathrm{Out}(G_v)$. The image of $\mathrm{Out}^r(G)$ is $\prod_{v\in V_E \cup V_{QH}} \mathrm{PMCG}(G_v)$, so it has finite index in the image of $\mathrm{Out}_0(G;\mathcal{P})$. We complete the proof by showing that $\mathrm{Out}^r(G)$ contains $\ker\rho$.

It is pointed out in [19, Subsection 3.3] that $\ker\rho$ is equal to the group of twists $\mathcal{T}$ (because, if $v \notin V_E$, then incident edge groups are equal to their normalizer in G_v). By Lemma 5.1, $\mathcal{T}$ is generated by twists near vertices in V_E. These belong to the image of $\lambda_{E,QH}$, so $\mathcal{T} \subset \mathrm{Out}^r(G)$. □

We shall now change the graph of groups Γ into a new graph of groups Γ'. We do not change the underlying graph, or edge groups, or vertex groups G_v for $v \in V_E \cup V_{QH}$, but for $v \in V_R$ we replace G_v by a group G'_v defined as follows.

If $e \in E_v$ is an incident edge, G_e is a maximal elementary subgroup of G_v, in particular it contains the finite group $Z(G_v)$. Consider the groups G_e, for $e \in E_v$, as well as $\mathbb{Z} \times Z(G_v)$. All these groups contain $Z(G_v)$, and we define G'_v as their free amalgam over $Z(G_v)$, ie G'_v is obtained from the free product $(*_{e\in E_v} G_e) * (\mathbb{Z} \times Z(G_v))$ by identifying all copies of $Z(G_v)$.

The inclusion from G_e into the new vertex group G'_v is the obvious one. Note that G'_v is not one-ended relative to incident edge groups (because of the factor $\mathbb{Z} \times Z(G_v)$), and is residually finite if the P_i are (as an amalgam of residually finite groups over a finite subgroup). We denote by G' the fundamental group of Γ'.

Lemma 5.3 *The finite-index subgroup* $\mathrm{Out}^r(G) \subset \mathrm{Out}(G;\mathcal{P})$ *is isomorphic to a subgroup of* $\mathrm{Out}(G')$.

Proof Since nothing changes near vertices in $V_E \cup V_{QH}$, we still have a map

$$\lambda'_{E,QH}\colon \prod_{v\in V_E\cup V_{QH}} \mathrm{PMCG}^{\partial}(G_v) \longrightarrow \mathrm{Out}(G').$$

It suffices to show $\ker\lambda'_{E,QH} = \ker\lambda_{E,QH}$. Recall that the kernel of $\lambda_{E,QH}$ is the same as the kernel of its restriction to $\prod_{v\in V_E\cup V_{QH}} \mathcal{T}_v$, and similarly for $\lambda'_{E,QH}$. For

$v \in V_R$, the group G'_v was defined so that the groups $Z(G_v)$ and $Z_{G_v}(G_e)$ do not change, so $\mathcal{T}_v$ does not change, and neither does the map from $\prod_{v \in V} \mathcal{T}_v$ to $\mathrm{Out}(G)$. The result follows. □

6 Torsion-free relatively hyperbolic groups

This section is devoted to a proof of Theorem 1.3 in the torsion-free case.

Theorem 6.1 *Let G be a torsion-free group hyperbolic relative to a family of proper finitely generated residually finite subgroups $\mathcal{P} = \{P_1, \ldots, P_k\}$, none of which is cyclic. If G is one-ended relative to $\mathcal{P}$, then $\mathrm{Out}(G; \mathcal{P})$ is residually finite.*

See Remark 6.2 for the case when some of the P_i are allowed to be cyclic.

Proof As in the previous section, let Γ be the canonical JSJ decomposition of G. First suppose that Γ is trivial (a single vertex v). If v is rigid, then $\mathrm{Out}(G; \mathcal{P})$ is finite (see [19]). If v is QH, then G is a closed surface group and $\mathrm{Out}(G)$ is residually finite by [15; 2]. The case $v \in V_E$ cannot occur since $P_i \neq G$. From now on, we suppose that Γ is nontrivial.

By Lemma 5.2, it is enough to show that $\mathrm{Out}^r(G)$ is residually finite. Given a nontrivial $\Phi \in \mathrm{Out}^r(G)$, we want to map $\mathrm{Out}^r(G)$ to a finite group without killing Φ. Note that it is enough to map $\mathrm{Out}^r(G)$ to a residually finite group without killing Φ.

Using Lemma 5.3, we view $\mathrm{Out}^r(G)$ as a group of automorphisms of G'. To simplify notation, we will not write the superscripts $'$ unless necessary.

Recall the epimorphisms

$$\prod_{v \in V_E \cup V_{QH}} \mathrm{PMCG}^{\partial}(G_v) \longrightarrow \mathrm{Out}^r(G) \longrightarrow \prod_{v \in V_E \cup V_{QH}} \mathrm{PMCG}(G_v)$$

induced by $\lambda_{E,QH}$ and $\prod_{v \in V_E \cup V_{QH}} \rho_v$. Write Φ as the image of a tuple $(\Phi_v) \in \prod_{v \in V_E \cup V_{QH}} \mathrm{PMCG}^{\partial}(G_v)$ under $\lambda_{E,QH}$. If $v \in V_{QH}$, the group $\mathrm{PMCG}(G_v)$ is residually finite since $\mathrm{PMCG}(G_v) \subset \mathrm{Out}(G_v)$ with G_v free of finite rank, so we may assume that Φ maps trivially to $\mathrm{PMCG}(G_v)$ for $v \in V_{QH}$. This means that $\Phi_v \in \mathcal{T}_v \subset \mathrm{PMCG}^{\partial}(G_v)$ for $v \in V_{QH}$. By Lemma 5.1, we may use edge relations to find a representative of Φ with $\Phi_v = 1$ for $v \in V_{QH}$. We fix such a representative, and we choose $u \in V_E$ with $\Phi_u \neq 1$.

Next we fix a characteristic finite-index subgroup $N_v \lhd G_v$ for each $v \in V_E$, and we denote $H_v = G_v / N_v$. Since G_v is assumed to be residually finite (it is cyclic or conjugate to a P_i), we may also require that, if $e = vw$ is an edge with $w \in V_{QH}$, then the image of $G_e \cong \mathbb{Z}$ in H_v has order $n_e \geq 4$. As in the proof of Lemma 3.2, we also require that Φ_u maps to a nontrivial element under the natural homomorphism from $\mathrm{PMCG}^{\partial}(G_u)$ to $\mathrm{PMCG}^{\partial}(H_u)$.

We now construct a new graph of groups Γ'', with the same underlying graph as Γ and Γ', with vertex groups $G''_v = H_v$ at vertices $v \in V_E$. We describe the edge groups, and the other vertex groups. The inclusions from edge groups to vertex groups will be the obvious ones.

Given an edge $e = vw$ with $v \in V_E$, its group G''_e in Γ'' is the image of G_e in H_v, a cyclic group of order $n_e \geq 4$.

If $v \in V_R$, its group in Γ' is the free product $(* \, G_e) * \mathbb{Z}$, with the first product taken over all $e \in E_v$. We define its new group as $G''_v = (* \, G''_e) * \mathbb{Z}$.

If $v \in V_{QH}$, its group in Γ and Γ' is a surface group $\pi_1(\Sigma_v)$, with boundary components of Σ_v corresponding to incident edges (see Section 4.2). We define G''_v as the fundamental group of the closed orbifold obtained from Σ_v by replacing the boundary component associated to an edge e by a conical point of order n_e. The assumption $n_e \geq 4$ ensures that the orbifold is hyperbolic (see [40, 13.3.6]), so G''_v is a nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. In particular, G''_v has trivial center and is residually finite.

The fundamental group G'' of Γ'' is a quotient of G'. It is residually finite, as the fundamental group of a graph of groups with residually finite vertex groups and finite edge groups. We show that the splitting Γ'' of G'' is nontrivial.

If $v \notin V_E$, the group G''_v is infinite (because of the factor $\mathbb{Z}$ in the rigid case), so triviality would imply that u has valence 1 and the incident edge group in Γ'' is equal to G''_u. This is impossible because $\mathrm{PMCG}^{\partial}(G''_u)$ is nontrivial (it contains the image of Φ_u, which is assumed to be nontrivial).

Since Γ'' is a splitting over finite groups, G'' has infinitely many ends. By [27, Theorem 1.5], $\mathrm{Out}(G'')$ is residually finite. We conclude the proof by constructing a map from $\mathrm{Out}^r(G)$ to $\mathrm{Out}(G'')$ mapping Φ nontrivially.

Let $v \in V_E \cup V_{QH}$. Since the kernel of the map $G_v \to G''_v$ is invariant under $\mathrm{PMCG}(G_v)$, there is an induced map from $\mathrm{PMCG}^{\partial}(G_v)$ to $\mathrm{PMCG}^{\partial}(G''_v)$. Similarly, the kernel of the projection from G' to G'' is invariant under the image of $\mathrm{Out}^r(G)$ in $\mathrm{Out}(G')$ (see Lemma 5.3), so there is an induced map from $\mathrm{Out}^r(G)$ to $\mathrm{Out}(G'')$.

These maps fit in a commutative diagram

$$\begin{tikzcd}
\prod_{v\in V_E\cup V_{QH}} \mathrm{PMCG}^{\partial}(G_v) \arrow[r] \arrow[d] & \prod_{v\in V_E\cup V_{QH}} \mathrm{PMCG}^{\partial}(G''_v) \arrow[d] \\
\mathrm{Out}^r(G) \arrow[r] & \mathrm{Out}(G'').
\end{tikzcd}$$

Since $\Phi_v = 1$ for $v \in V_{QH}$, and Φ_u maps nontrivially to $\mathrm{PMCG}^{\partial}(H_u)$, Lemma 4.2 (applied in Γ'' with $W = V_E$) implies that Φ maps nontrivially to $\mathrm{Out}(G'')$. □

The main difficulty in extending this proof to groups with torsion lies in defining G''_v for $v \in V_{QH}$ when G_v contains a nontrivial finite normal subgroup F. This may be done using small cancellation techniques, but we prefer to give a different proof (using Grossman's method and Dehn fillings) in the general case.

Remark 6.2 Theorem 6.1 holds if the P_i are allowed to be cyclic (recall that an infinitely-ended G may become one-ended relative to $\mathcal{P}$ if cyclic P_i are added). There is a technical complication due to the fact that, if $G_v = \pi_1(\Sigma_v)$ is QH, there may exist boundary components of Σ_v whose fundamental group equals some P_i (up to conjugacy) but is not an incident edge group. Because of this, one must change the definition of $\mathrm{PMCG}^{\partial}(G_v)$ and $\mathrm{PMCG}(G_v)$ slightly. See Section 7.5 for details.

7 Groups with torsion

The goal of this section is to prove Theorem 1.3 (Sections 7.4 and 7.5) and Corollary 1.1 (Section 7.6).

7.1 Finitary fillings of relatively hyperbolic groups

Definition 7.1 Let $\mathcal{I}$ be a property of groups, let G be a group, and let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a nonempty collection of subgroups of G. We will say that *most finitary fillings of* G *with respect to* $\{H_\lambda\}_{\lambda\in\Lambda}$ *have property* $\mathcal{I}$ provided there is a finite subset $S \subset G\setminus\{1\}$ such that, for any family of finite-index normal subgroups $N_\lambda \lhd H_\lambda$ with $N_\lambda \cap S = \varnothing$ for all $\lambda \in \Lambda$, the following two conditions hold:

- For each $\lambda \in \Lambda$ one has $N \cap H_\lambda = N_\lambda$, where $N := \langle\langle N_\lambda \mid \lambda \in \Lambda\rangle\rangle^G$.
- The quotient G/N satisfies $\mathcal{I}$.

Any quotient G/N as above will be called a *finitary filling of* G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. The finite subset S will be called the *obstacle subset*.

In this work we will mainly be concerned with the case when $\mathcal{I}$ is the property of being residually finite, or conjugacy separable.

Remark 7.2 If $G = A *_C B$, where C is finite and A, B are residually finite, then most finitary fillings of G with respect to any of the families $\{A\}$, $\{B\}$ or $\{A, B\}$ are residually finite (it suffices to take $S := C \setminus \{1\}$ and use the universal property of amalgamated free products; G/N will be an amalgam of residually finite groups over C). More generally, if G is the fundamental group of a finite graph of groups with finite edge groups and residually finite vertex groups, then most finitary fillings of G with respect to any collection of vertex groups will be residually finite.

In this subsection, and the next, we will consider a graph of groups Γ, with fundamental group G, satisfying the following assumption (this will be applied to the graph Γ' constructed in Section 5).

Assumption 7.3 We suppose that Γ is a connected finite bipartite graph with vertex set $V = V_1 \sqcup V_2$ (so every vertex from V_1 is only adjacent to vertices from V_2 and vice-versa), and Γ is not a point. Moreover, the following properties hold:

(1) If $u \in V_1$ then the group G_u is residually finite.

(2) If $v \in V_2$ then most finitary fillings of G_v with respect to $\{G_{e_1}, \dots, G_{e_s}\}$ are residually finite, with $e_1, \dots, e_s$ the collection of all oriented edges of Γ starting at v, and $G_{e_1}, \dots, G_{e_s}$ the corresponding edge groups.

(3) For every $u \in V_1$, the group G_u is a proper finitely generated subgroup of $G = \pi_1(\Gamma)$, and G is hyperbolic relative to the family $\{G_u \mid u \in V_1\}$; in particular, G is finitely generated.

The main technical tool for our approach is the theory of Dehn fillings in relatively hyperbolic groups, developed by Osin [31, Theorem 1.1] (in the torsion-free case this was also done independently by Groves and Manning [16, Theorem 7.2]):

Theorem 7.4 *Suppose that a group G is hyperbolic relative to a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then there exists a finite subset $S \subset G \setminus \{1\}$ with the following property. If $\{N_\lambda\}_{\lambda\in\Lambda}$ is any collection of subgroups such that $N_\lambda \lhd H_\lambda$ and $N_\lambda \cap S = \varnothing$ for all $\lambda \in \Lambda$, then:*

(1) *For each $\lambda \in \Lambda$ one has $H_\lambda \cap N = N_\lambda$, where $N := \langle\langle N_\lambda \mid \lambda \in \Lambda\rangle\rangle^G$.*

(2) *The quotient group G/N is hyperbolic relative to the collection $\{H_\lambda/N_\lambda\}_{\lambda\in\Lambda}$.*

Moreover, for any finite subset $M \subset G$, there exists a finite subset $S(M) \subset G \setminus \{1\}$, such that the restriction of the natural homomorphism $G \to G/N$ to M is injective whenever $N_\lambda \cap S(M) = \varnothing$ for all $\lambda \in \Lambda$. □

Suppose that Δ is any graph of groups with fundamental group G, and we are given normal subgroups $N_v \lhd G_v$ for each vertex v. Assume furthermore that $N_v \cap G_e = N_w \cap G_e$ whenever $e = vw$ is an edge of Δ (as usual, we view G_e as a subgroup of both G_v and G_w; to be precise, we want the preimages of N_v and N_w, under the embeddings of G_e into G_v and G_w respectively, to coincide).

Then we can construct a "quotient graph of groups" $\overline{\Delta}$ as follows: the underlying graph is the same as in Δ, the vertex group at a vertex v is G_v/N_v, the group carried by $e = vw$ is $G_e/(N_v \cap G_e)$, and the inclusions are the obvious ones. The fundamental group of $\overline{\Delta}$ is isomorphic to $G/\langle\langle \cup_v N_v \rangle\rangle^G$.

Now suppose that Γ is as in Assumption 7.3. For each $v \in V_2$, there is an obstacle set $S_v \subset G_v \setminus \{1\}$, and we define $S := \bigcup_{v \in V_2} S_v$.

Lemma 7.5 *Consider an arbitrary family of subgroups $\{N_u\}_{u \in V_1}$ such that $N_u \lhd G_u$, $|G_u : N_u| < \infty$ and $N_u \cap S = \varnothing$ for every $u \in V_1$. The group $\overline{G} := G/\langle\langle \bigcup_{u \in V_1} N_u \rangle\rangle^G$ is the fundamental group of a quotient graph of groups $\overline{\Gamma}$ in which every $u \in V_1$ carries G_u/N_u, and every $v \in V_2$ carries a residually finite group. In particular, $\overline{G}$ is residually finite.*

Proof For every edge $e = uv$ of Γ, with $u \in V_1$ and $v \in V_2$, we define a finite-index normal subgroup $L_e \lhd G_e$ by $L_e := G_e \cap N_u$ (as above, we view G_e as a subgroup of both G_u and G_v). Now, for each vertex $v \in V_2$ we let $M_v := \langle\langle L_{e_1} \cup \cdots \cup L_{e_s} \rangle\rangle^{G_v} \lhd G_v$, where $e_1, \dots, e_s$ are the edges of Γ starting at v. Observe that $L_{e_j} \cap S_v = \varnothing$ for $j = 1, \dots, s$ by construction, hence $M_v \cap G_{e_j} = L_{e_j}$ by Definition 7.1.

This shows that $\overline{G}$ is represented by a quotient graph of groups $\overline{\Gamma}$. The group carried by $u \in V_1$ is G_u/N_u, a finite group; in particular, edge groups are finite. The group carried by $v \in V_2$ is G_v/M_v, which is residually finite by Assumption 7.3. Thus $\overline{G}$ is residually finite. □

Remark 7.6 Since every G_u, with $u \in V_1$, is residually finite, a family of normal subgroups $\{N_u\}_{u \in V_1}$ as in Lemma 7.5 exists. We may even require $N_u \cap S' = \varnothing$ if S' is any given finite subset of $G \setminus \{1\}$.

Remark 7.7 Lemma 7.5 only requires the second condition of Assumption 7.3.

Proposition 7.8 *Let G be the fundamental group of a graph of groups Γ as in Assumption 7.3. Then G is residually finite.*

Proof Take any element $x \in G \setminus \{1\}$ and let $M := \{1, x\} \subset G$. Since each G_u, $u \in V_1$, is residually finite, one can find sufficiently small finite-index subgroups $N_u \lhd G_u$ as in Lemma 7.5. We can also assume that each N_u is disjoint from the set $S(M)$ provided by Theorem 7.4 (applied to G relative to the G_u). If $N := \langle\langle N_u \mid u \in V_1 \rangle\rangle^G$, then $\overline{G} = G/N$ is residually finite by Lemma 7.5, and the image of x under the map $\kappa\colon G \to \overline{G}$ is nontrivial by the final claim of Theorem 7.4. Composing κ with a map from $\overline{G}$ to a finite group that does not kill $\kappa(x)$, we get a finite quotient of G in which the image of x is nontrivial. This shows residual finiteness of G. □

The above proposition is true even without the hypothesis that vertex groups from V_1 are finitely generated.

7.2 Using Grossman's method

Let G be hyperbolic relative to a family of proper finitely generated subgroups $\mathcal{P} = \{P_1, \dots, P_k\}$.

Recall that an element $g \in G$ is called *loxodromic* if g has infinite order and is not conjugate to an element of P_i for any i. Two elements $g, h \in G$ are said to be *commensurable* in G if there are $f \in G$ and $m, n \in \mathbb{Z} \setminus \{0\}$ such that $h^n = f g^m f^{-1}$ (we use the terminology of [27], where conjugate elements are considered commensurable). An automorphism $\alpha \in \mathrm{Aut}(G)$ is called *commensurating* if $\alpha(g)$ is commensurable with g for every $g \in G$.

It is known that G contains a unique maximal finite normal subgroup denoted $E(G)$ (see [27, Corollary 2.6]); this subgroup contains the center of G if G is not virtually cyclic.

Lemma 7.9 *Assume that G is not virtually cyclic.*

(1) *If $E(G) = \{1\}$, then every commensurating automorphism of G is inner.*

(2) *Suppose that $\alpha \in \mathrm{Aut}(G)$ is not a commensurating automorphism. Then there exists a loxodromic element $g \in G$ such that $\alpha(g)$ is also loxodromic and $\alpha(g)$ is not commensurable with g in G.*

Proof Statement (1) is proved in [27, Corollary 1.4].

Suppose that $\alpha \in \mathrm{Aut}(G)$ is not commensurating. By [27, Corollary 5.3] there exists a loxodromic element $g_0 \in G$ such that $\alpha(g_0)$ is not commensurable with g_0 in G. Statement (2) now follows after applying [27, Lemma 4.8]. □

Lemma 7.10 [27, Lemma 7.1] *Assume that G is hyperbolic relative to $\{P_1, \dots, P_k\}$, with $k \geq 1$, and $g, h \in G$ are two noncommensurable loxodromic elements. Then g and h are loxodromic and noncommensurable in most finitary fillings of G with respect to $\{P_1, \dots, P_k\}$.* □

Recall that G is *conjugacy separable* if, given any nonconjugate elements $g, h \in G$, there exists a homomorphism φ from G to a finite group such that $\varphi(g)$ and $\varphi(h)$ are not conjugate. Note that this is evidently stronger than residual finiteness of G.

Proposition 7.11 *Let G be the fundamental group of a graph of groups Γ as in Assumption 7.3. Suppose that $E(G) = \{1\}$, and for each $v \in V_2$ most finitary fillings of G_v with respect to $\{G_{e_1}, \dots, G_{e_s}\}$ (where $e_1, \dots, e_s$ is the list of edges of Γ starting at v) are* conjugacy separable. *Then* $\mathrm{Out}(G)$ *is residually finite.*

The proof uses Grossman's method, which is based on the following fact:

Lemma 7.12 *Given a finitely generated group G and $\alpha \in \mathrm{Aut}(G)$, suppose that there is a homomorphism $\psi\colon G \to K$ with K finite, and $g \in G$, such that $\psi(g)$ is not conjugate to $\psi(\alpha(g))$ in K. Then there is a homomorphism $\widehat{\theta}\colon \mathrm{Out}(G) \to L$ with L finite such that $\widehat{\theta}(\widehat{\alpha}) \neq 1$ in L, where $\widehat{\alpha}$ is the image of α in $\mathrm{Out}(G)$.*

Proof Since G is finitely generated, there exists a characteristic finite-index subgroup $N \lhd G$ such that $N \leq \ker \psi$. Let $\varphi\colon G \to G/N$ be the canonical epimorphism. Then ψ factors through φ, hence $\varphi(g)$ is not conjugate to $\varphi(\alpha(g))$ in G/N. Observe that, as N is characteristic in G, there are induced homomorphisms $\theta\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$ and $\widehat{\theta}\colon \mathrm{Out}(G) \to L := \mathrm{Out}(G/N)$. Since $\varphi(g)$ is not conjugate to $\varphi(\alpha(g))$ in G/N, the automorphism $\theta(\alpha)$ is not inner and $\widehat{\theta}(\widehat{\alpha}) \neq 1$. □

Proof of Proposition 7.11 We may assume that G is not virtually cyclic, since $\mathrm{Out}(G)$ is finite if it is (see [27, Lemma 6.6]). Applying Lemma 7.5 and Remark 7.6, we find a quotient graph of groups $\overline{\Gamma}$ where vertices in V_1 carry finite groups and vertices in V_2 carry residually finite groups. We write $\overline{G} = \pi_1(\overline{\Gamma})$, and we let $\kappa\colon G \to \overline{G}$ be the projection.

We shall now enlarge S to ensure that $\overline{\Gamma}$ possesses additional properties. First, we may assume that vertices in V_2 carry a conjugacy separable group. By Dyer [13], $\overline{G}$ is then conjugacy separable, as the fundamental group of a finite graph of groups with conjugacy separable vertex groups and finite edge groups.

Now consider any $\widehat{\alpha} \in \mathrm{Out}(G) \setminus \{1\}$, represented by $\alpha \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$. Since $E(G) = \{1\}$, by Lemma 7.9 there exists a loxodromic element $g \in G$ such that $\alpha(g)$ is a loxodromic element not commensurable with g in G. By Lemma 7.10 (applied to G relative to the G_u), we can enlarge the obstacle set S to assume that the elements $\kappa(g)$ and $\kappa(\alpha(g))$ are noncommensurable in $\overline{G}$.

Since $\overline{G}$ is conjugacy separable, there exists a finite group K and a homomorphism $\eta\colon \overline{G} \to K$ such that $\eta(\kappa(g))$ is not conjugate to $\eta(\kappa(\alpha(g)))$ in K. Thus, setting $\psi := \eta \circ \kappa\colon G \to K$, we can apply Lemma 7.12 to find a finite quotient of $\mathrm{Out}(G)$ separating $\widehat{\alpha}$ from the identity. □

7.3 Quadratically hanging groups

In the next subsection, we will apply Propositions 7.8 and 7.11 to the canonical JSJ decomposition. In order to do this, we need to study finitary fillings of QH vertex groups. We denote such a group by O.

Recall from Section 4.2 that O is an extension

$$\{1\} \longrightarrow F \longrightarrow O \xrightarrow{\xi} P \longrightarrow \{1\},$$

where F is a finite group and P is the fundamental group of a hyperbolic 2–orbifold $\mathcal{O}$ with boundary. Consider full preimages $C_i = \xi^{-1}(B_i)$ of a set of representatives $B_1, \dots, B_s$ of fundamental groups of components of the boundary of $\mathcal{O}$.

The goal of this subsection is the following statement:

Proposition 7.13 *Most finitary fillings of O with respect to $\mathcal{H} = \{C_1, \dots, C_s\}$ are conjugacy separable.*

Proof First assume $F = \{1\}$, so $C_i = B_i$. In this case we shall see that most finitary fillings are fundamental groups of closed hyperbolic orbifolds. Thus they are virtually surface groups, hence conjugacy separable by a result of Martino [25, Theorem 3.7].

We define an obstacle set $S = S_1 \cup \dots \cup S_s$ in $P = \pi_1(\mathcal{O})$ as follows. Let r be a large integer (to be determined later). Recall (see [40]) that each B_i is either infinite cyclic or infinite dihedral. In the cyclic case, B_i is generated by a single element c of infinite order and we let $S_i := \{c, c^2, \dots, c^r\}$. In the dihedral case, B_i is generated by two involutions a, b and we let $S_i := \{a, b, ab, (ab)^2, \dots, (ab)^r\}$.

Any finite-index normal subgroup $K_i \lhd B_i$ with $K_i \cap S_i = \varnothing$ is cyclic, generated by c^{m_i} or $(ab)^{n_i}$ for some m_i or n_i larger than r. It follows that the quotient $\pi_1(\mathcal{O})/\langle\langle K_i \mid i = 1, \dots, s\rangle\rangle^{\pi_1(\mathcal{O})}$ is the fundamental group of a closed orbifold $\mathcal{O}'$, which is obtained from $\mathcal{O}$ by replacing each boundary component by a conical point (elliptic point, in the terminology of [40]) of order m_i in the cyclic case, and by a dihedral point (corner reflector, in the terminology of [40]) of order n_i in the dihedral case.

We claim that $\mathcal{O}'$ is hyperbolic if r is large enough. By [40, Theorem 13.3.6], a 2–orbifold admits a hyperbolic structure if and only if its Euler characteristic is negative. The Euler characteristic $\chi(\mathcal{O}')$ can be computed by the formula (cf [40, 13.3.3; 36])

$$\chi(\mathcal{O}') = \chi(\mathcal{O}) + \sum \frac{1}{m_i} + \sum \frac{1}{2n_i}.$$

Since $\chi(\mathcal{O})$ is negative, so is $\chi(\mathcal{O}')$ for r large, and the claim follows. Defining S using such an r, we deduce that most finitary fillings of O are fundamental groups of closed hyperbolic orbifolds. This proves the proposition when $F = \{1\}$.

In the general case, we have to use Dehn fillings (Theorem 7.4). It is a standard fact [7, Theorem 7.11] that O is hyperbolic relative to the family $\mathcal{H} = \{C_1, \dots, C_s\}$ (these are nonconjugate maximal virtually cyclic subgroups of the hyperbolic group O). Consider the obstacle set $\overline{S} = \xi^{-1}(S) \cup S'$, where S is the set constructed above in $\pi_1(\mathcal{O})$ and S' is provided by Theorem 7.4 (applied to O and $\mathcal{H}$).

Consider any collection of finite-index normal subgroups $N_i \lhd C_i$ such that $N_i \cap \overline{S} = \varnothing$, $i = 1, \dots, s$, and set $N := \langle\langle N_i \mid i = 1, \dots, s \rangle\rangle^O$. By Theorem 7.4 we have $N \cap C_i = N_i$ for each i, so it remains to check that the quotient $O' = O/N$ is conjugacy separable.

Let $\varphi\colon O \to O'$ denote the natural epimorphism. Then O' maps with finite kernel $\varphi(F)$ onto $P/\xi(N) \cong P/\langle\langle \xi(N_i) \mid i = 1, \dots, s \rangle\rangle^P$. Our choice of $\overline{S}$ guarantees that $\xi(N_i)$ does not meet the set S, so $P/\xi(N)$ is the fundamental group of a hyperbolic orbifold $\mathcal{O}'$. The exact sequence $\{1\} \to \varphi(F) \to O' \to \pi_1(\mathcal{O}') \to \{1\}$ implies that O' is virtually a surface group (see [25, Theorem 4.3]), hence it is conjugacy separable as above. □

7.4 Conclusion

We prove Theorem 1.3, starting with a couple of lemmas. We first assume that no P_i is virtually cyclic, postponing the general case to the next subsection.

Lemma 7.14 *Consider the graph of groups Γ' constructed in Section 5, and assume, additionally, that the subgroups $P_1, \dots, P_k$ are residually finite (and not virtually cyclic). Then Γ' satisfies Assumption 7.3. In particular, its fundamental group G' is residually finite (by Proposition 7.8).*

Proof Recall that Γ' is bipartite, with $V_1 = V_E$ and $V_2 = V_R \cup V_{QH}$. Vertex groups in V_1 are elementary, hence residually finite by assumption.

For $v \in V_R$, the vertex group G'_v is obtained by amalgamating $\mathbb{Z} \times C$ and the incident edge groups G_e over a finite group C. Define $S = C \setminus \{1\}$ as the obstacle set. As in Remark 7.2, each finitary filling of G'_v with respect to the incident edge groups is an amalgam over C, with factors being finite or $\mathbb{Z} \times C$.

For QH vertices, residual finiteness (indeed, conjugacy separability) of finitary fillings follows from Proposition 7.13. The assumption that no P_i is virtually cyclic guarantees that $\{C_1, \dots, C_s\}$ is (up to conjugacy) the family of incident edge groups (see Section 4.2).

Relative hyperbolicity follows from standard combination theorems for relatively hyperbolic groups (cf [10; 29]) because vertex groups in V_2 are hyperbolic relative to incident edge groups: this was pointed out in the proof of Proposition 7.13 in the QH case, and in the rigid case this is a consequence of Definition 4.1 (as the graph $\mathcal{K}$ one can take the Bass–Serre tree associated to the splitting of G'_v, $v \in V_R$, as an amalgam over C discussed above). □

Lemma 7.15 *Suppose that G is a finitely generated group, and N is a centerless normal subgroup of finite index in G.*

(1) *Some finite-index subgroup $\mathrm{Out}_0(G) \leqslant \mathrm{Out}(G)$ is isomorphic to a quotient of a subgroup of $\mathrm{Out}(N)$ by a finite normal subgroup L.*

(2) *Let $\mathcal{P}$ be a finite family of subgroups in G and let $\mathcal{Q}$ be a collection of representatives of N–conjugacy classes among $\{N \cap gHg^{-1} \mid H \in \mathcal{P}, g \in G\}$. Then some finite-index subgroup of $\mathrm{Out}(G;\mathcal{P})$ is isomorphic to a quotient of a subgroup of $\mathrm{Out}(N;\mathcal{Q})$ by a finite normal subgroup. In particular, if $\mathrm{Out}(N;\mathcal{Q})$ is residually finite then so is $\mathrm{Out}(G;\mathcal{P})$.*

Proof The first assertion is standard (see for instance [20, Lemma 5.4]). One defines $\mathrm{Aut}_0(G)$ as the set of automorphisms mapping N to itself and acting as the identity on G/N, and $\mathrm{Out}_0(G)$ is its image in $\mathrm{Out}(G)$. Using the fact that N is centerless, one shows that the natural map $\mathrm{Aut}_0(G) \to \mathrm{Aut}(N)$ is injective. The group L comes from the action of inner automorphisms of G.

For (2), observe that automorphisms in $\mathrm{Aut}_0(G)$ preserving the set of conjugacy classes of groups in $\mathcal{P}$ also preserve the (finite) set of N–conjugacy classes of subgroups from $\mathcal{Q}$. □

We can now prove Theorem 1.3 when no peripheral subgroup is virtually cyclic:

Theorem 7.16 *Let G be a group hyperbolic relative to a family of proper finitely generated subgroups $\mathcal{P} = \{P_1, \dots, P_k\}$. If G is one-ended relative to $\mathcal{P}$, no P_i is virtually cyclic, and every P_i is residually finite, then $\mathrm{Out}(G;\mathcal{P})$ is residually finite.*

Proof Consider the canonical elementary JSJ tree T of G relative to the family $\mathcal{P}$, as in Section 4.2. If T is trivial (a single vertex), then either G is rigid or it maps onto the fundamental group of a closed hyperbolic 2–orbifold with finite kernel. In the former case $\mathrm{Out}(G;\mathcal{P})$ is finite (see [19, Theorem 3.9]) and in the latter case G contains a surface subgroup of finite index (see for instance [25, Theorem 4.3]). Therefore in this case $\mathrm{Out}(G)$ is residually finite by a combination of Grossman's theorem [15, Theorem 3] with Lemma 7.15.

Hence we can further assume that the canonical JSJ tree T is nontrivial. Let us apply the construction of Section 5. By Lemma 5.3, a finite-index subgroup $\mathrm{Out}^r(G)$ of $\mathrm{Out}(G;\mathcal{P})$ embeds into $\mathrm{Out}(G')$, where G' is the fundamental group of the bipartite graph of groups Γ'.

If T has at least one rigid vertex, then the group G' has infinitely many ends, because of the way we constructed Γ'. Furthermore, G' is residually finite by Lemma 7.14. Therefore $\mathrm{Out}(G')$ is residually finite by [27, Theorem 1.5], so its subgroup $\mathrm{Out}^r(G)$, and also $\mathrm{Out}(G;\mathcal{P})$, are residually finite.

Hence we can suppose that the JSJ decomposition of G has no rigid vertices. In this case $G' = G$ by construction, and $V_2 = V_{QH}$.

There are two cases. Assume, at first, $E(G) = \{1\}$. Then, according to Proposition 7.13, the graph of groups Γ' satisfies all the assumptions of Proposition 7.11, which allows us to conclude that $\mathrm{Out}(G)$ (and, hence, $\mathrm{Out}(G;\mathcal{P})$) is residually finite.

If $E(G) \neq \{1\}$, we shall deduce residual finiteness of $\mathrm{Out}(G;\mathcal{P})$ from Lemma 7.15. As $E(G)$ is finite and G is residually finite (by Lemma 7.14), there exists a finite-index normal subgroup $N \lhd G$ such that $N \cap E(G) = \{1\}$. It is standard that N is hyperbolic relative to the family $\mathcal{Q}$ described in the second part of Lemma 7.15 (this follows immediately from Definition 4.1). Note that groups in $\mathcal{Q}$ are finitely generated, residually finite, not virtually cyclic, and they are proper subgroups of N since groups in $\mathcal{P}$ have infinite index in G (indeed, each $P_i \in \mathcal{P}$ is almost malnormal in G; see [30, Theorem 1.4]).

The group $E(N)$ is trivial because it is characteristic in N, hence it is contained in $E(G) \cap N = \{1\}$. In particular, N is centerless. Moreover, as pointed out by Guirardel and the first author in [18], it follows from Dicks and Dunwoody [11, Theorem IV.1.3] that N is one-ended relative to $\mathcal{Q}$. Since $E(N) = \{1\}$, we know that $\mathrm{Out}(N;\mathcal{Q})$ is residually finite by the previous case, and Lemma 7.15 implies that $\mathrm{Out}(G;\mathcal{P})$ is residually finite. □

The arguments given above show the following facts, which may be of independent interest:

Corollary 7.17 *Let G be a group hyperbolic relative to a family $\mathcal{P} = \{P_1, \dots, P_k\}$ of proper finitely generated residually finite groups, such that no P_i is virtually cyclic. Suppose that G is one-ended relative to $\mathcal{P}$.*

If the canonical JSJ decomposition of G over elementary subgroups relative to $\mathcal{P}$ has no rigid vertices, then G is residually finite. Otherwise, a finite-index subgroup of $\mathrm{Out}(G;\mathcal{P})$ embeds into $\mathrm{Out}(G')$, where G' is a finitely generated residually finite group with infinitely many ends.

In all cases, $\mathrm{Out}(G;\mathcal{P})$ virtually embeds into $\mathrm{Out}(G')$, where G' is a finitely generated residually finite relatively hyperbolic group. □

7.5 Allowing virtually cyclic P_i

We now prove Theorem 1.3 in general, allowing virtually cyclic peripheral subgroups. We may assume that all P_i are infinite, since removing finite groups from $\mathcal{P}$ does not affect relative one-endedness.

The new phenomenon occurs at QH vertices of the canonical JSJ decomposition Γ. With the notation of Section 7.3, it is still true that incident edge groups of G_v are preimages of fundamental groups of boundary components of $\mathcal{O}$, but there may now be boundary components $\mathcal{C}_j$ such that $C_j = \xi^{-1}(B_j)$ is not an incident edge group, but is conjugate to a group in $\mathcal{P}$.

In Section 5 we used groups $\mathrm{PMCG}(G_v)$ and $\mathrm{PMCG}^{\partial}(G_v)$, defined using incident edge groups of Γ. We must now replace $\mathrm{PMCG}(G_v)$ with a subgroup by requiring that automorphisms act on groups C_j as above as inner automorphisms τ_{a_j} of G_v. The group $\mathrm{PMCG}^{\partial}(G_v)$ is replaced by the preimage of this subgroup under π_v. We do not keep track of the a_j; the corresponding $\mathcal{C}_j$ should be thought of as punctures rather than boundary components, in particular there is no twist near them (in the context of Section 3.2, isotopies are free on the components $\mathcal{C}_j$).

With this modification, all arguments given in Sections 5, 6 and 7 go through. In Proposition 7.13, we define $\mathcal{H}$ using only the groups C_j which are incident edge groups. The hyperbolic orbifold $\mathcal{O}'$ may then have a nonempty boundary. In this case its fundamental group is virtually free, hence conjugacy separable by Dyer [12].

7.6 Proof of Corollary 1.1

Suppose that G is hyperbolic relative to a family $\mathcal{P} = \{P_1, \dots, P_k\}$ of virtually polycyclic groups. Without loss of generality we may assume that no P_i is virtually cyclic (see Section 4.1). The result is true if G is virtually polycyclic; see Wehrfritz [43].

Otherwise, Theorem 1.3 applies since every P_i is residually finite. To conclude, note that $\mathrm{Out}(G;\mathcal{P})$ has finite index in $\mathrm{Out}(G)$ because groups in $\mathcal{P}$ are characterized (up to conjugacy) as maximal virtually polycyclic subgroups which are not virtually cyclic (see [28, Lemma 3.2] for a more general result).

8 Residual p–finiteness for automorphism groups

8.1 Residual p–finiteness

Given a prime p and a group G, we will say that a subgroup $K \leqslant G$ has *p–power index* in G if $|G:K| = p^k$ for some $k \geq 0$.

Remark 8.1 The intersection of two subgroups H_1, H_2 of p–power index is not necessarily of p–power index, but it is if H_1 and H_2 are normal (for then there is an embedding of $G/(H_1 \cap H_2)$ into $G/H_1 \times G/H_2$). In particular, if G is finitely generated, any normal subgroup H of p–power index contains one which is characteristic in G, namely the intersection of all subgroups of G with the same index as H.

The collection of normal subgroups of G of p–power index forms a basis of neighborhoods of the identity in G, giving rise to the *pro-p topology* on G. As in the case of residually finite groups, the pro-p topology on G is Hausdorff if and only if G is *residually p–finite*: given $g \neq 1$, there is a homomorphism φ from G to a finite p–group such that $\varphi(g) \neq 1$ (by Remark 8.1, one may assume that $\ker\varphi$ is characteristic if G is finitely generated).

Residual p–finiteness is a much more delicate condition than residual finiteness. It is still clearly stable under direct products, but in general it is not stable under semidirect products ($\mathbb{Z}$ is residually p–finite for any prime p, but it is easily checked that the Klein bottle group, the nontrivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$, is not residually p–finite if $p > 2$). Even more strikingly, for any given set of prime numbers Π, there exists a 3–generated center-by-metabelian group which is residually p–finite if and only if $p \in \Pi$; see Hartley [21].

Lemma 8.2 *If A has a residually p–finite normal subgroup B of p–power index, then A is residually p–finite.*

Proof Take any $x \in A \setminus \{1\}$. Since B is residually p–finite, there exists a p–power index normal subgroup $N \lhd B$ such that $x \notin N$. The intersection H of all A–conjugates of N is normal in A and has p–power index in B (see Remark 8.1), so $|A:H| = |A:B||B:H|$ is a power of p. Since $x \notin H$, we can conclude that A is residually p–finite. □

Remark 8.3 Combining Lemma 8.2 with an induction on the subnormal index, one can actually prove that any group containing a subnormal residually p–finite subgroup of p–power index is itself residually p–finite.

It is not difficult to see that not every p–power index subgroup of a group G has to be closed in the pro-p topology. In fact, a p–power index subgroup $K \leqslant G$ is closed in the pro-p topology on G if and only if K is subnormal in G (cf Toinet [41, Lemma A.1]).

Lemma 8.4 *If G is residually p–finite, and $N \lhd G$ is a finite normal subgroup, then G/N is also residually p–finite.*

Proof Indeed, since G is residually p–finite, any finite subset of G is closed in the pro-p topology on G. Therefore N is the intersection of p–power index normal subgroups of G, and so G/N is residually p–finite. □

For any prime p and any group H, let $\mathrm{Aut}_p(H)$ be the subgroup of $\mathrm{Aut}(H)$ which consists of automorphisms that act trivially on the first mod p homology of H. Namely, let $K_p := [H, H]H^p$ be the verbal subgroup of H, which is the product of the derived subgroup $[H, H]$ and the subgroup H^p generated by all the p^{th} powers of elements in H. Then

$$\mathrm{Aut}_p(H) = \{\alpha \in \mathrm{Aut}(H) \mid \alpha(hK_p) = hK_p \text{ for all } h \in H\}.$$

If H is finitely generated, then K_p has finite index in H, therefore $\mathrm{Aut}_p(H)$ will have finite index in $\mathrm{Aut}(H)$. Observe also that all inner automorphisms are in $\mathrm{Aut}_p(H)$ because H/K_p is abelian, and the group $\mathrm{Out}_p(H) := \mathrm{Aut}_p(H)/\mathrm{Inn}(H)$ has finite index in $\mathrm{Out}(H)$.

The following classical theorem of P Hall will be useful (cf Robinson [35, 5.3.2, 5.3.3]):

Lemma 8.5 *If H is a finite p–group, then $\mathrm{Aut}_p(H)$ is also a finite p–group.* □

The next statement was originally proved by L Paris in [32, Theorem 2.4]. We present an elementary proof based on Lemma 8.5.

Lemma 8.6 *Let H be a finitely generated residually p–finite group, for some prime p. Then $\mathrm{Aut}_p(H)$ is residually p–finite, hence $\mathrm{Aut}(H)$ is virtually residually p–finite.*

Proof Consider any nontrivial automorphism $\alpha \in \mathrm{Aut}_p(H)$. Then there is $h_0 \in H$ such that $\alpha(h_0) \neq h_0$. Since H is residually p–finite, there exist a finite p–group K and an epimorphism $\psi\colon H \to K$ with $\psi(\alpha(h_0)) \neq \psi(h_0)$ in K. As explained in Remark 8.1, one may assume that $\ker\psi$ is a characteristic subgroup of H. This implies that ψ naturally induces a homomorphism $\varphi\colon \mathrm{Aut}(H) \to \mathrm{Aut}(K)$. Clearly, $\varphi(\mathrm{Aut}_p(H)) \subseteq \mathrm{Aut}_p(K)$, so the restriction φ' of φ to $\mathrm{Aut}_p(H)$ gives a homomorphism from $\mathrm{Aut}_p(H)$ to $\mathrm{Aut}_p(K)$, where the latter is a finite p–group by Lemma 8.5. It remains to observe that $\varphi'(\alpha)$ is nontrivial, because $\varphi'(\alpha)(\psi(h_0)) = \psi(\alpha(h_0)) \neq \psi(h_0)$ by construction. $\square$

8.2 Toral relatively hyperbolic groups

In this section we will prove Theorem 1.5. The method is similar to the one used in Section 3.

Given a group H with a fixed family of peripheral subgroups $C_1, \dots, C_s$, $s \geq 1$, we can define $\mathrm{Aut}^\partial(H)$, $\mathrm{PMCG}^\partial(H)$ and $\mathrm{PMCG}(H)$ as in Section 3.1. For any prime p, let $\mathrm{Aut}^\partial_p(H) \leqslant \mathrm{Aut}^\partial(H)$ consist only of those tuples $(\alpha; a_1, \dots, a_s)$ for which $\alpha \in \mathrm{Aut}_p(H)$. In other words $\mathrm{Aut}^\partial_p(H)$ is the full preimage of $\mathrm{Aut}_p(H)$ under the natural projection $\mathrm{Aut}^\partial(H) \to \mathrm{Aut}(H)$. We also define $\mathrm{PMCG}^\partial_p(H)$ as the image of $\mathrm{Aut}^\partial_p(H)$ in $\mathrm{PMCG}^\partial(H)$, and $\mathrm{PMCG}_p(H)$ will denote its image in $\mathrm{PMCG}(H) \leqslant \mathrm{Out}(H)$.

Remark 8.7 If H is finitely generated, $\mathrm{PMCG}^\partial_p(H)$ has finite index in $\mathrm{PMCG}^\partial(H)$.

Lemma 8.8 *If H is a finite p–group, then so is $\mathrm{PMCG}^\partial_p(H)$. If H is a finitely generated residually p–finite group, then $\mathrm{PMCG}^\partial_p(H)$ is residually p–finite.*

Proof The group $\mathrm{PMCG}^\partial_p(H)$ embeds into $H^{s-1} \rtimes \mathrm{Aut}_p(H)$ (see Remark 3.3); this group is a finite p–group by Lemma 8.5. For the second assertion, argue as in Lemma 3.2, mapping $\mathrm{PMCG}^\partial_p(H)$ to $\mathrm{PMCG}^\partial_p(H/N)$ with H/N a finite p–group. $\square$

Our next goal is Lemma 8.11 below, which is an analogue of Lemma 3.4. We need to prove two auxiliary statements first.

Lemma 8.9 *Fundamental groups of closed hyperbolic surfaces are residually p–finite for all primes p.*

Proof Let Σ be a closed hyperbolic surface. Then $\pi_1(\Sigma)$ is residually free, except for the case when $\Sigma = \Sigma_{-1}$ is the closed nonorientable surface of genus 3 (and Euler characteristic -1); see G Baumslag [5] and B Baumslag [4]. Since free groups are residually p–finite for every prime p (cf [35, 6.1.9]), the lemma follows for all $\Sigma \neq \Sigma_{-1}$. On the other hand, for any prime p, there is a normal cover of degree p of Σ_{-1} (because the abelianization of $\pi_1(\Sigma_{-1})$ is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$). This cover is a surface of higher genus, so its fundamental group is residually p–finite by the previous argument. Hence $\pi_1(\Sigma_{-1})$ is residually p–finite by Lemma 8.2. $\square$

Lemma 8.10 *Let p be a prime, and n be a power of p. Let Σ_n be a closed hyperbolic 2–orbifold whose singularities are cone points of order n. Then $O_n := \pi_1(\Sigma_n)$ is residually p–finite.*

Proof Let Σ be a compact surface obtained by removing a neighborhood of each conical point. We may map $\pi_1(\Sigma)$ to a finite p–group so that the fundamental group of every boundary component has image of order exactly n: if Σ has only one boundary component, this follows from Stebe [39, Lemma 1] (see also [25, Lemma 4.1]); if there are more, the fundamental group of each boundary component is a free generator of $\pi_1(\Sigma)$, and we map $\pi_1(\Sigma)$ to $H_1(\pi_1(\Sigma), \mathbb{Z}/n\mathbb{Z})$, its abelianization mod n.

The corresponding normal covering of Σ extends to a covering of Σ_n by a closed surface, because its restriction to every component of $\partial\Sigma$ has degree exactly n. The fundamental group of this surface is residually p–finite by Lemma 8.9. Its index in O_n is a power of p, so O_n is residually p–finite by Lemma 8.2. □

Now suppose, as in Section 3.2, that H is the fundamental group of a compact surface Σ with negative Euler characteristic and $s \geq 1$ boundary components. Let $C_1, \dots, C_s$ be the fundamental groups of these components, considered as subgroups of H. Let $\mathcal{T}_H \leqslant \mathrm{PMCG}^\partial(H)$ be the corresponding group of twists. Note that $\mathcal{T}_H \subseteq \mathrm{PMCG}^\partial_p(H)$ for any prime p. We have the following analogue of Lemma 3.4:

Lemma 8.11 *Let p be a prime. Then the quotient $\mathrm{PMCG}^\partial_p(H)/n\mathcal{T}_H$ is residually p–finite for every sufficiently large power n of p.*

Proof The proof is similar to that of Lemma 3.4, using PMCG_p instead of PMCG. The kernel of $\theta\colon \mathcal{T}_H \to \mathcal{T}_{O_n}$ is $n\mathcal{T}_H$, and we need to know that $\mathrm{PMCG}_p(H)$ and $\mathrm{PMCG}^\partial_p(O_n)$ are residually p–finite.

Clearly we have $\mathrm{PMCG}_p(H) \leqslant \mathrm{Out}_p(H)$, which is residually p–finite by a result of L Paris [32, Theorem 1.4] since H is a free group. On the other hand, the group $\mathrm{PMCG}^\partial_p(O_n)$ is residually p–finite by Lemmas 8.10 and 8.8. □

We also need to consider abelian groups. Let A be a free abelian group of finite rank with a chosen family of subgroups $C_1, \dots, C_s$. For any prime p, consider the corresponding groups $\mathrm{Aut}^\partial_p(A)$, $\mathrm{PMCG}^\partial_p(A)$, and the normal subgroup $\mathcal{T}_A \lhd \mathrm{PMCG}^\partial_p(A)$, defined as in Section 3.1. Note that $\mathcal{T}_A$ is naturally isomorphic to the quotient of A^s by its diagonal subgroup, hence to A^{s-1}.

Lemma 8.12 *Let p be a prime. The quotient $\mathrm{PMCG}^\partial_p(A)/n\mathcal{T}_A$ is residually p–finite for every power n of p.*

Proof Consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
\{1\} \arrow[r] & \mathcal{T}_A \arrow[r] \arrow[d, "\theta"] & \mathrm{PMCG}^{\partial}_p(A) \arrow[r] \arrow[d] & \mathrm{PMCG}_p(A) \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathcal{T}_{A/nA} \arrow[r] & \mathrm{PMCG}^{\partial}_p(A/nA) \arrow[r] & \mathrm{PMCG}_p(A/nA) \arrow[r] & \{1\}
\end{tikzcd}$$

The map $\theta\colon \mathcal{T}_A \to \mathcal{T}_{A/nA}$ sends A^{s-1} to $(A/nA)^{s-1}$, so its kernel is $n\mathcal{T}_A$ and the proof is reduced to showing that $\mathrm{PMCG}_p(A)$ and $\mathrm{PMCG}^{\partial}_p(A/nA)$ are residually p–finite. Now $\mathrm{PMCG}_p(A) \leqslant \mathrm{Out}_p(A) = \mathrm{Aut}_p(A)$ is residually p–finite by Lemma 8.6, and $\mathrm{PMCG}^{\partial}_p(A/nA)$ is a finite p–group by Lemma 8.8. $\square$

Lemma 8.13 *Consider a finite set V and groups P_v, $v \in V$, with normal subgroups T_v free abelian of finite rank. Let $P = \prod_{v\in V} P_v$ and $T = \prod_{v\in V} T_v \leqslant P$ be their direct products.*

Suppose that p is a prime number and $Z \leqslant T$ is a subgroup such that T/Z contains no q–torsion if $q \neq p$ is a prime. If P_v/nT_v is residually p–finite for all $v \in V$ and for every sufficiently large power n of p, then Z is closed in the pro-p topology of P. In particular, if Z is normal in P, then P/Z is residually p–finite.

Proof This is similar to the proof of Lemma 3.6. One first proves the result when Z has p–power index in T. In the general case, T/Z being residually p–finite guarantees that Z is the intersection of (normal) subgroups of p–power index in T. $\square$

We are now ready to prove the main theorem of this section.

Theorem 8.14 *If some finite-index subgroup of G is a one-ended toral relatively hyperbolic group, then $\mathrm{Out}(G)$ is virtually residually p–finite for any prime p.*

Proof First suppose that G itself is torsion-free and hyperbolic relative to a family $\mathcal{P} = \{P_1, \dots, P_k\}$ of free abelian groups of finite rank. As in the proof of Corollary 1.1, we can assume that no $P_i \in \mathcal{P}$ is cyclic, and restrict to $\mathrm{Out}(G;\mathcal{P})$ because it has finite index in $\mathrm{Out}(G)$. Consider the canonical JSJ tree T relative to $\mathcal{P}$ over abelian groups as in Section 4.2.

If T consists of a single point then either G is rigid, G is the fundamental group of a closed hyperbolic surface Σ, or G is a finitely generated free abelian group. In the first case $\mathrm{Out}(G)$ is finite (see [19] for instance). In the second case, if Σ is orientable then $\mathrm{Out}_p(G)$ is residually p–finite by [32, Theorem 1.4], and if Σ is nonorientable, then it possesses an orientable cover Σ' of degree 2. Since the group $\mathrm{Out}(\pi_1(\Sigma'))$ is virtually residually p–finite by [32, Theorem 1.4], and $\pi_1(\Sigma')$ is a centerless normal subgroup of finite-index of G, we can use Lemmas 7.15 and 8.4 to conclude that $\mathrm{Out}(G)$ is virtually residually p–finite. Finally, if G is a free abelian group of finite rank, then $\mathrm{Out}_p(G) = \mathrm{Aut}_p(G)$ is residually p–finite by Lemma 8.6.

Thus we can suppose that the tree T is nontrivial. In this case we know (cf Lemma 5.2 and Remark 8.7) that $\mathrm{Out}_p^r(G)$, the image by $\lambda_{E,QH}$ of $\prod_{v\in V_E\cup V_{QH}} \mathrm{PMCG}_p^\partial(G_v)$, has finite index in $\mathrm{Out}(G;\mathcal{P})$. We apply Lemma 3.6 to

$$P = \prod_{v\in V_E\cup V_{QH}} \mathrm{PMCG}_p^\partial(G_v),$$

with $T_v = \mathcal{T}_v$.

We know that each $\mathrm{PMCG}_p^\partial(G_v)/n\mathcal{T}_v$ is residually p–finite for n a large power of p, by Lemmas 8.11 and 8.12. The quotient of $\prod_{v\in V_E\cup V_{QH}} \mathcal{T}_v$ by $Z = \ker\lambda_{E,QH}$ is the whole group of twists $\mathcal{T}$ by Lemma 5.1, it is torsion-free (see [19, Corollary 4.4]). Thus $\mathrm{Out}_p^r(G) = P/Z$ is residually p–finite by Lemma 3.6. It has finite index in $\mathrm{Out}(G)$, so $\mathrm{Out}(G)$ is virtually residually p–finite.

Now suppose that G contains a toral relatively hyperbolic group G_0 with finite index. We may assume that G_0 is normal. If G_0 is abelian, then $\mathrm{Out}(G)$ is contained in some $\mathrm{GL}(n,\mathbb{Z})$ by [43], so it is virtually residually p–finite by Lemma 8.6 (as $\mathrm{GL}(n,\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$). Otherwise G_0 has trivial center and we apply Lemmas 7.15 and 8.4. □

8.3 Groups with infinitely many ends

In this subsection we prove Theorem 1.6: *if G is a finitely generated group with infinitely many ends, and G is virtually residually p–finite for some prime number p, then* $\mathrm{Out}(G)$ *is virtually residually p–finite*. The argument will use the following "pro-p" analogue of Lemma 7.12:

Lemma 8.15 *Let p be a prime. Given a finitely generated group G and $\alpha \in \mathrm{Aut}_p(G)$, suppose there is a homomorphism $\psi\colon G \to K$ with K a finite p–group such that $\psi(g)$ is not conjugate to $\psi(\alpha(g))$ in K. Then there is a homomorphism $\phi\colon \mathrm{Out}_p(G) \to L$ with L a finite p–group such that $\phi(\hat\alpha) \neq 1$ in L, where $\hat\alpha$ denotes the image of α in $\mathrm{Out}_p(G)$.*

Proof The proof is almost identical to the proof of Lemma 7.12, except we use Aut_p and Out_p instead of Aut and Out, together with the fact that $\mathrm{Out}_p(H)$ is a finite p–group for any finite p–group H, which immediately follows from Lemma 8.5. □

Proof of Theorem 1.6 Recall that by Stallings' theorem for groups with infinitely many ends [38], the group G splits as an amalgamated product or as an HNN-extension over a finite subgroup $C \leqslant G$. Since G is virtually residually p–finite we can find a finite-index normal subgroup $H \lhd G$ such that $H \cap C = \{1\}$ and H is residually p–finite. It follows from the generalized Kurosh Theorem (cf [11, I.7.7] or Cohen [9, Theorem 8.27]) that $H = A * B$, where A and B are nontrivial finitely generated residually p–finite groups. Note that H has trivial center (as does any nontrivial free product), and so by Lemmas 7.15 and 8.4 it is enough to prove that $\mathrm{Out}(H)$ is virtually residually p–finite.

Observe that H is hyperbolic relative to $\{A, B\}$ and consider any automorphism $\alpha \in \mathrm{Aut}_p(H) \setminus \mathrm{Inn}(H)$. Again, since H splits as a nontrivial free product, H contains no nontrivial finite normal subgroups, hence $E(H) = \{1\}$. Therefore, according to Lemma 7.9, there exists $g \in H$ such that both g and $h := \alpha(g)$ are loxodromic in H and g is not commensurable with h in H. Since A and B are residually p–finite, applying Lemma 7.10, we can find normal subgroups $A' \lhd A$ and $B' \lhd B$ such that $A_1 := A/A'$ and $B_1 := B/B'$ are finite p–groups and the images of g and h are noncommensurable in the free product $H_1 := A_1 * B_1$.

We claim that H_1 is conjugacy p–separable, ie given two nonconjugate $h, h' \in H_1$ there exist a finite p–group K and a homomorphism $\xi\colon H_1 \to K$ such that $\xi(h)$ is not conjugate to $\xi(h')$ in K. Indeed, by the Kurosh subgroup theorem, the kernel of the natural map $A_1 * B_1 \to A_1 \times B_1$ is free, thus H_1 is an extension of a finitely generated free group by the finite p–group $A_1 \times B_1$. Hence, by a theorem of E Toinet [41, Theorem 1.7], H_1 is conjugacy p–separable (in fact, the full strength of Toinet's result is not needed here: conjugacy p–separability of free products of finite p–groups can be derived from the conjugacy p–separability of the free group via a short argument, similar to the one used by V Remeslennikov in [33, Theorem 2]).

Let $\eta\colon H \to H_1$ denote the natural homomorphism with $\ker \eta = \langle\langle A', B' \rangle\rangle^H$. Let

$$\psi := \xi \circ \eta\colon H \to K.$$

Then $\psi(h) = \psi(\alpha(g))$ is not conjugate to $\psi(g)$ in K by construction. Therefore by Lemma 8.15 there is a finite p–group L and a homomorphism $\phi\colon \mathrm{Out}_p(H) \to L$ such that $\phi(\hat{\alpha}) \neq 1$ in L, where $\hat{\alpha}$ is the image of α in $\mathrm{Out}_p(H)$. Thus we have shown that $\mathrm{Out}_p(H)$ is residually p–finite. Since H is finitely generated,

$$|\mathrm{Out}(H) : \mathrm{Out}_p(H)| < \infty,$$

and so $\mathrm{Out}(H)$ is virtually residually p–finite, as required. □

Before concluding let us discuss one application of Theorem 1.6. In a recent paper [3] Aschenbrenner and Friedl proved that the fundamental group of any compact 3–manifold M is residually p–finite for all but finitely many primes p. Recalling Lubotzky's theorem [24, Proposition 2], they derived that $\mathrm{Aut}(\pi_1(M))$ is virtually residually p–finite and mentioned that the similar fact for $\mathrm{Out}(\pi_1(M))$ is not yet known. Theorem 1.6 implies that if a compact orientable 3–manifold M is not irreducible, then $\mathrm{Out}(\pi_1(M))$ is virtually residually p–finite (for all but finitely many primes p). Indeed, since M is not irreducible, either it is $\mathbb{S}^2 \times \mathbb{S}^1$ or it decomposes into a connected sum of prime manifolds. In the former case $\pi_1(M) \cong \mathbb{Z}$, and in the latter case $\pi_1(M)$ splits as a nontrivial free product. Thus either $\mathrm{Out}(\pi_1(M))$ is finite, or $\pi_1(M)$ has infinitely many ends, and so $\mathrm{Out}(\pi_1(M))$ is virtually residually p–finite for all but finitely many p by Theorem 1.6 (using the result of Aschenbrenner and Friedl [3] mentioned above). Therefore, in order to prove that $\mathrm{Out}(\pi_1(M))$ is virtually residually p–finite for all compact orientable 3–manifolds M, it is enough to consider only irreducible manifolds.

As a finishing remark, one can recall the theorem of Rhemtulla [34] stating that if a group is residually p–finite for infinitely many primes p, then it is biorderable. Unfortunately our methods do not allow us to deduce that $\mathrm{Out}(G)$ has a single finite-index subgroup which is residually p–finite for infinitely many p. This is because we rely on Lemma 8.5, requiring one to pass to the subgroup $\mathrm{Out}_p(G)$, the index of which generally depends on the prime p.

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