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Affine unfoldings of convex polyhedra

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We show that every convex polyhedron admits a simple edge unfolding after an affine transformation. In particular, there exists no combinatorial obstruction to a positive resolution of Dürer's unfoldability problem, which answers a question of Croft, Falconer and Guy. Among other techniques, the proof employs a topological characterization of embeddings among the planar immersions of the disk.

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1 Introduction

A well-known problem in geometry (see Demaine and O'Rourke [6], O'Rourke [16], Pak [17] and Ziegler [23]), which may be traced back to the Renaissance artist Albrecht Dürer [7], is concerned with cutting a convex polyhedral surface along some spanning tree of its edges so that it may be isometrically embedded, or unfolded without overlaps, into the plane. Here we show that this is always possible after an affine transformation of the surface. In particular, unfoldability of a convex polyhedron does not depend on its combinatorial structure, which settles a problem of Croft, Falconer and Guy [4, Section B21].

In this work a (compact) *convex polyhedron* P is the boundary of the convex hull of a finite number of affinely independent points of Euclidean space $\mathbb{R}^3$. A *cut tree* $T \subset P$ is a (polygonal) tree which includes all the vertices of P, and each of its leaves is a vertex of P. Cutting P along T yields a compact surface P_T which admits an isometric immersion $P_T \to \mathbb{R}^2$ (see Section 4), called an *unfolding* of P. This unfolding is *simple*, or an *embedding*, if it is one-to-one. We say P is in *general position* with respect to a unit vector or *direction* u provided that the *height function* $h(\cdot) := \langle \cdot, u\rangle$ has a unique maximizer and a unique minimizer on vertices of P. Then T is *monotone* with respect to u provided that h is (strictly) decreasing on every simple path in T which connects a leaf of T to the vertex minimizing h. For $\lambda > 0$, we define the (normalized) *affine stretching* parallel to u as the linear transformation $A_\lambda\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A_\lambda(p) := \tfrac{1}{\lambda}(p + (\lambda - 1)\langle p, u\rangle u),$$

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and set $X^\lambda := A_\lambda(X)$ for any $X \subset \mathbb{R}^3$. Note that if $u = (0,0,1)$, then $A_\lambda(x,y,z) = (x/\lambda, y/\lambda, z)$. Thus A_λ makes any convex polyhedron arbitrarily "thin" or "needle-shaped" for large λ. Our main result is as follows:

Theorem 1.1 *Let P be a convex polyhedron, u a direction with respect to which P is in general position, and $T \subset P$ a cut tree which is monotone with respect to u. Then the unfolding of P^λ generated by T^λ is simple for sufficiently large λ.*

When a cut tree is composed of the edges of P, or is a spanning tree of the edge graph of P, the corresponding unfolding is called an *edge unfolding*. If P admits a simple edge unfolding, then we say P is *unfoldable*. Note that there exists an open and dense set of directions u in the sphere $\mathbb{S}^2$ with respect to which P is in general position. Furthermore, it is easy to construct monotone spanning edge trees for every such direction. They may be generated, for instance, via the well-studied "steepest edge" algorithm (see Schlickenrieder [19], Lucier [13] and [6]), or a general procedure described in Note 1.6. Thus Theorem 1.1 quickly yields:

Corollary 1.2 *An affine stretching of a convex polyhedron, in almost any direction, is unfoldable.* □

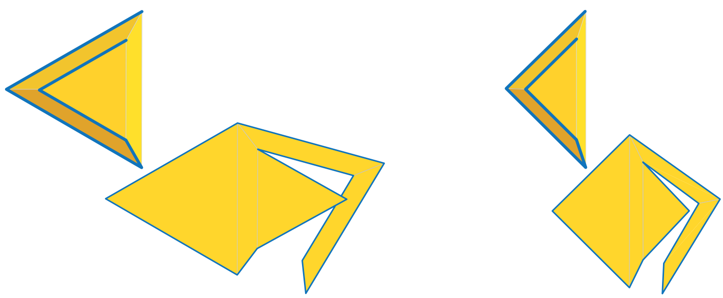

Figure 1

An example of this phenomenon is illustrated in Figure 1. The left side of this figure shows a truncated tetrahedron (viewed from "above") together with an overlapping unfolding of it generated by a monotone edge tree. As we see on the right side, however, the same edge tree generates a simple unfolding once the polyhedron has been stretched.

The rest of this work will be devoted to proving Theorem 1.1. We will start in Sections 2 and 3 by recording some basic definitions and observations concerning the composition of paths in convex polyhedra and their developments in the plane. In particular, we discuss the notion of "mixed developments" which arises naturally in this context and constitutes a useful technical tool. Then, in Section 4, we will show that to each cut tree there is associated a path whose development coincides with the boundary of the corresponding unfolding. Thus, Dürer's problem may be viewed as the search for spanning edge trees with simple developments. To this end, we will obtain in Section 5 a topological criterion for deciding when a closed planar curve which bounds an immersed disk is simple. This will be the principal tool for proving Theorem 1.1, which will be utilized by means of an induction on the number of leaves of the cut tree. To facilitate this approach we will study the structure of monotone cut trees in Section 6, and the effect of affine stretchings on their developments in Section 7. Finally, these observations will be synthesized in Section 8 to complete the proof.

The earliest known examples of simple edge unfoldings for convex polyhedra are due to Dürer [7], although the problem which bears his name was first formulated by Shephard [20]. Furthermore, the assertion that a solution can always be found, which has been dubbed Dürer's conjecture, appears to have been first published by Grünbaum [10; 11]. There is empirical evidence both for and against this supposition. On the one hand, computers have found simple edge unfoldings for countless convex polyhedra through an exhaustive search of their spanning edge trees. On the other hand, there is still no algorithm for finding the right tree [13; 19], and computer experiments suggest that the probability that a random edge unfolding of a generic polyhedron overlaps itself approaches 1 as the number of vertices grow; see Schevon [18]. General cut trees have been studied at least as far back as Alexandrov [1] who first established the existence of simple unfoldings (not necessarily simple *edge* unfoldings) for all convex polyhedra; see also Itoh, O'Rourke and Vîlcu [12], Miller and Pak [14] and Demaine, Demaine, Hart, Iacono, Langerman and O'Rourke [5] for recent related results. Other references and background may be found in [6].

Note 1.3 A chief difficulty in assailing Dürer's problem is the lack of any intrinsic characterization for an edge of a convex polyhedron P. Indeed the edge graph of P is not the unique graph in P whose vertices coincide with those of P, whose edges are geodesics and whose faces are convex. It seems reasonable to expect that Dürer's conjecture should be true if and only if it holds for this wider class of generalized edge graphs. This approach has been studied by Tarasov [22], who has announced some negative results in this direction.

Note 1.4 As we mentioned above, one way to generate some monotone trees in a convex polyhedron is via the "the steepest edge" algorithm which has been well studied due to its relative effectiveness in finding simple unfoldings. Indeed Schlickenrieder [19] had conjectured that every convex polyhedron contains at least one steepest edge tree which generates a simple unfolding. He had successfully tested this conjecture in thousands of cases, after a thorough examination of various kinds of spanning edge trees and cataloguing their failure to produce simple unfoldings. Subsequently, however, Lucier [13] produced a counterexample to Schlickenrieder's conjecture. Although it is not clear whether all monotone trees in Lucier's example fail to produce simple unfoldings.

Note 1.5 Dürer's problem is usually phrased in somewhat broader terms than described above: *Can every convex polyhedral surface be cut along some collection T of its edges so that the resulting surface P_T is connected and admits an isometric embedding into the plane?* In other words, it is not a priori assumed that T is a spanning tree. Assuming that this is the case, however, does not cause loss of generality. Indeed, it is obvious that the cut set T must contain every vertex of P (for otherwise P_T will not be locally isometric to the plane), and T may not contain any cycles (for then P_T will not be connected). Furthermore, it follows fairly quickly from the Gauss–Bonnet theorem that T must be connected [6, Lemma 22.1.2]. So T is indeed a spanning tree.

Note 1.6 A general procedure for constructing monotone spanning edge trees T in a convex polyhedron P may be described as follows. The only requirement here is that P be positioned so that it has a unique bottom vertex r. Then, since P is convex, every vertex v of P other than r will be adjacent to a vertex which lies below it, ie has smaller z–coordinate. Thus, by moving down through a sequence of adjacent vertices, we may connect v to r by means of a monotone edge path (with respect to $u = (0, 0, 1)$). Let v_0 be a top vertex of P, and B_0 be a monotone edge path which connects v_0 to r. If B_0 covers all vertices of P, then we set $T := B_0$ and we are done. Otherwise, from the remaining set of vertices choose an element v_1 which maximizes the z–coordinate on that set. Then we generate a monotone edge path B_1 by connecting v_1 to an adjacent vertex which lies below it and continue to go down through adjacent vertices until we reach a vertex of B_0 (including r). If B_0 and B_1 cover all the vertices of P, then we set $T := B_0 \cup B_1$ and we are done. Otherwise we repeat the above procedure, until all vertices of P have been covered.

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2 Preliminaries

For easy reference, we begin by recording here the definitions and notation which will be used most frequently in the following pages.

2.1 Basic terminology

Throughout this work $\mathbb{R}^n$ is the n–dimensional Euclidean space with standard inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Further $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. The *height function* is the mapping $h\colon \mathbb{R}^n \to \mathbb{R}$ given by $h(x_1, \dots, x_n) = x_n$, and P denotes (the boundary of) a (compact) convex polyhedron in $\mathbb{R}^3$ which is oriented by the outward unit normals to its faces. We assume that P is positioned so that it has a single *top vertex* ℓ_0 and a single *bottom vertex* r, ie h has a unique maximizer and a unique minimizer on P. Furthermore, T is a cut tree of P which is rooted at r. The *leaves* of T are the vertices of T of degree 1 which are different from r. The simple paths in T which connect its leaves to r will be called the *branches* of T. We will assume, unless stated otherwise, that T is *monotone*, by which we will always mean monotone with respect to $u = (0, 0, 1)$. So h will be (strictly) decreasing on each branch of T. We let P_T be the surface obtained by cutting P along T, and $\pi\colon P_T \to P$ be the corresponding projection (as will be defined in Section 4). Further $\overline{P}_T$ will denote the image of P_T under an unfolding $P_T \to \mathbb{R}^2$. We say $\overline{P}_T$ is *simple* if the unfolding map is one-to-one. More generally, for any mapping $f\colon X \to \mathbb{R}^2$ and subset $X_0 \subset X$, we set $\overline{X}_0 := f(X_0)$ and say $\overline{X}_0$ is simple if f is one-to-one on X_0. Finally, by *sufficiently large* we mean for all values bigger than some constant.

2.2 Paths and their compositions

A line segment in $\mathbb{R}^n$ is *oriented* if one of its endpoints, say a, is designated as the "initial point" and the other, say b, as the "final point". Then the segment will be denoted by ab. A *path* Γ is a sequence of oriented line segments in $\mathbb{R}^n$ such that the final point of each segment coincides with the initial point of the succeeding segment. These segments are called the *edges* of Γ, and their endpoints constitute its *vertices*. The vertices of Γ inherit a natural ordering $\gamma_0, \dots, \gamma_k$, where γ_0 is the initial point of the first edge, γ_k is the final point of the last edge, and successive elements share a common edge. Conversely, any sequence of points $\gamma_0, \dots, \gamma_k$ of $\mathbb{R}^n$ with distinct successive elements determines a path denoted by

$$\Gamma = [\gamma_0, \dots, \gamma_k] := (\gamma_0\gamma_1, \dots, \gamma_{k-1}\gamma_k).$$

Then γ_0, γ_k are the *initial* and *final* vertices of Γ respectively. Any other vertex of Γ will be called an *interior vertex*. If only consecutive edges of Γ intersect, and do so only at their common vertex, then Γ is *simple*. We say that Γ is *closed* if $\gamma_k = \gamma_0$, in which case we set $\gamma_{i+k} := \gamma_k$, and consider all vertices of Γ to be interior vertices. An interior vertex is *simple* if its adjacent vertices are distinct. The *trace* of Γ is the union of the edges of Γ, which will again be denoted by Γ. For any pair of vertices v, w

of Γ we let $vw = (vw)_\Gamma$ denote the subpath of Γ with initial point v and final point w. The trace of this path will also be denoted by vw.

We utilize two different notions for combining a pair of paths $\Gamma = [\gamma_0, \dots, \gamma_k]$ and $\Omega = [\omega_0, \dots, \omega_\ell]$, when $\gamma_k = \omega_0$. The *concatenation* of these paths is given by

$$\Gamma \bullet \Omega := [\gamma_0, \dots, \gamma_k, \omega_1, \dots, \omega_\ell],$$

while their *composition* is defined as

$$\Gamma \circ \Omega := [\gamma_0, \dots, \gamma_{k-m}, \omega_{m+1}, \dots, \omega_\ell],$$

where m is the largest integer such that $\gamma_{k-i} = \omega_i$ for $0 \le i \le m$. One may think of $\Gamma \circ \Omega$ as the path obtained from $\Gamma \bullet \Omega$ by excising its largest subpath centered at γ_k which double backs on itself; see Figure 2. This notion has also been studied by Berestovskiĭ and Plaut in [2, page 1770]. Finally we set $\Gamma^{-1} := [\gamma_k, \dots, \gamma_0]$. Note that we have the equality $\Gamma \circ \Gamma^{-1} = [\gamma_0]$ which may be considered a *trivial path*.

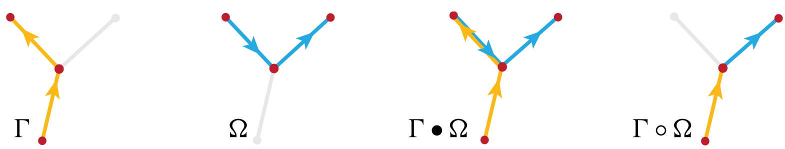

Figure 2

2.3 Sides and angles

In this section P need not be compact; in particular, it may stand for $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ with "outward normal" $(0,0,1)$. A *side* of a simple closed path Γ in P is the closure of a component of $P - \Gamma$. We may distinguish these sides as follows. Choose a point x in the interior of an edge $\gamma_i \gamma_{i+1}$ of Γ, pick a side S of Γ, let F be the face of S which contains x, n be the outward unit normal to F, and ν be a unit normal to $\gamma_i \gamma_{i+1}$ which points inside S. Then S lies to the *left* of Γ provided that $(\gamma_{i+1} - \gamma_i, n, \nu)$ has positive determinant; otherwise, S lies to the *right* of Γ. If Γ is not simple or closed, one may still define a local notion of sides near its interior vertices as we describe below.

For any point $o \in P$, let st_o denote the *star* of o, ie the union of faces of P which contain o. Orient the boundary curve $\partial\, st_o$ by choosing a cyclical ordering for its vertices, so that st_o lies to the left of it. Any point $x \in st_o - \{o\}$ generates a ray $R_x \subset \mathbb{R}^3$ which emanates from o and passes through x. Let $\widehat{st}_o$ denote the intersection of these rays with the unit ball in $\mathbb{R}^3$ centered at o. Then the *total angle* of P at o, denoted by $\angle_P(o)$, is the length of $\partial \widehat{st}_o$. Next, for any pair of points a, b in $st_o - \{o\}$, we define the (left) *angle* $\angle(a,o,b)$ of the path $[a,o,b]$. Consider the projection $st_o - \{o\} \to \partial\widehat{st}_o$ given by $x \mapsto \widehat{x} := R_x \cap \partial\widehat{st}_o$. This establishes a bijection $\partial\, st_o \to \partial\widehat{st}_o$ which orients $\partial\widehat{st}_o$. Let $|\cdot|$ denote the length of oriented segments of $\partial\widehat{st}_o$ and set

$$\angle(a,o,b) := \begin{cases} |\widehat{b}\widehat{a}|, & \widehat{a} \neq \widehat{b}, \\ \angle_P(o), & \widehat{a} = \widehat{b}. \end{cases}$$

In particular note that if $\widehat{a} \neq \widehat{b}$, then

$$\angle(a,o,b) + \angle(b,o,a) = \angle_P(o). \tag{1}$$

If $\widehat{a} = \widehat{b}$, then we define the entire st_o as the *left side* of $[a,o,b]$. Otherwise, $R_a \cup R_b$ divides st_o into a pair of components. The closure of each of these components will be called a *side* of $[a,o,b]$. The projection $st_o - \{o\} \to \partial\widehat{st}_o$ maps one of these regions to the (oriented) segment $\widehat{a}\widehat{b}$ and the other to $\widehat{b}\widehat{a}$, which will be called the *right* and *left* sides of $[a,o,b]$ respectively. Finally, $c \in st_o$ lies *strictly* to the left (resp. right) of $[a,o,b]$ if c lies in the left (resp. right) side of $[a,o,b]$ and is disjoint from $R_a \cup R_b$. The following elementary observations will be useful throughout this work.

Lemma 2.1 *Let $o \in P$, and a, b, $c \in st_o - \{o\}$. Then we have:*

(i) *c lies strictly to the left of $[a,o,b]$ if and only if $\angle(a,o,c) < \angle(a,o,b)$.*

(ii) *If c lies strictly to the left of $[a,o,b]$, then $\angle(a,o,c) + \angle(c,o,b) = \angle(a,o,b)$.*

Proof To see (i) first assume that $\widehat{a} = \widehat{b}$; see the left diagram in Figure 3. Then c lies strictly to the left of $[a,o,b]$ if and only if $\widehat{c} \neq \widehat{a}, \widehat{b}$. Furthermore $\angle(a,o,c) < \angle_P(o) = \angle(a,o,b)$ if and only if $\widehat{c} \neq \widehat{a}, \widehat{b}$. Next we establish (i) when $\widehat{a} \neq \widehat{b}$; see the right diagram in Figure 3.

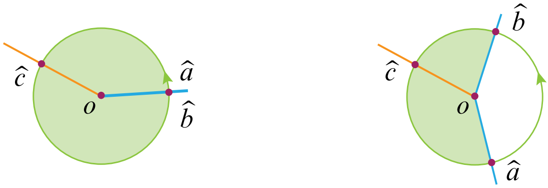

Figure 3

In this case, if c lies strictly to the left of $[a,o,b]$, then $\widehat{c} \in \operatorname{int}(\widehat{b}\widehat{a})$, the interior of $\widehat{b}\widehat{a}$ in $\partial\widehat{st}_o$. Thus $\angle(a,o,c) = |\widehat{c}\widehat{a}| < |\widehat{b}\widehat{a}| = \angle(a,o,b)$. Conversely, if $\angle(a,o,c) < \angle(a,o,b)$ (and $\widehat{a} \neq \widehat{b}$), then $\widehat{c} \neq \widehat{a}, \widehat{b}$. Consequently $|\widehat{c}\widehat{a}| < |\widehat{b}\widehat{a}|$ which yields $\widehat{c} \in \widehat{b}\widehat{a}$. So, since $\widehat{c} \neq \widehat{a}, \widehat{b}$, it follows that c lies strictly to the left of $[a,o,b]$. To see (ii) note that if $\widehat{a} = \widehat{b}$ and $\widehat{c} \neq \widehat{a}, \widehat{b}$, then $\angle(a,o,c) + \angle(c,o,b) = \angle(a,o,c) + \angle(c,o,a) = \angle_P(o) = \angle(a,o,b)$.

If, on the other hand, we have $\widehat{a} \neq \widehat{b}$, and c lies strictly to left of $[a, o, b]$, then $\widehat{c} \in \mathrm{int}(\widehat{b}\widehat{a})$. Thus $\angle(a, o, b) = |\widehat{b}\widehat{a}| < |\widehat{b}\widehat{c}| + |\widehat{c}\widehat{a}| = \angle(a, o, c) + \angle(c, o, b)$. □

3 Mixed developments of paths

In this section we describe a general notion for developing a path $\Gamma = [\gamma_0, \dots, \gamma_k]$ of P into the plane, and show (Proposition 3.1) how this concept interacts with that of composition of paths discussed in the last section. First we define the *left angle* of Γ at an interior vertex γ_i by

$$\theta_i = \theta_i[\Gamma] = \theta_{\gamma_i}[\Gamma] := \angle(\gamma_{i-1}, \gamma_i, \gamma_{i+1}).$$

Further the corresponding *right angle* is given by

$$\theta_i{}' := \angle(\gamma_{i+1}, \gamma_i, \gamma_{i-1}) = \angle_P(\gamma_i) - \theta_i,$$

where the last equality follows from (1). In particular we have

$$\theta_i + \theta_i{}' = \angle_P(\gamma_i) \leq 2\pi, \tag{2}$$

due to the convexity of P. It will also be useful to note that $\theta_i{}'[\Gamma] = \theta_{k-i}[\Gamma^{-1}]$. A *mixed development* of Γ is a path $\overline{\Gamma} = [\overline{\gamma}_0, \dots, \overline{\gamma}_k]$ in $\mathbb{R}^2$ with left angles $\overline{\theta}_i$, and right angles $\overline{\theta}_i'$, such that:

(i) $\|\gamma_i - \gamma_{i-1}\| = \|\overline{\gamma}_i - \overline{\gamma}_{i-1}\|$ for $1 \leq i \leq k$.
(ii) $\overline{\theta}_i = \theta_i$ or $\overline{\theta}_i' = \theta_i{}'$ for $1 \leq i \leq k-1$.

If $\overline{\theta}_i = \theta_i$ for all i, then $\overline{\Gamma}$ is a *(left) development*. We say $\overline{\Gamma}$ is a *mixed development* based at an interior vertex γ_ℓ if $\overline{\theta}_i' = \theta_i{}'$ for $i \leq \ell$, and $\overline{\theta}_i = \theta_i$ for all $i > \ell$. This path will be denoted by $(\overline{\Gamma})_{\gamma_\ell}$, and unless noted otherwise, the term $\overline{\Gamma}$ will be reserved to indicate a (left) development. We also set $(\overline{\Gamma})_{\gamma_0} := \overline{\Gamma}$. Note that $\overline{\Gamma}$ is uniquely determined once its *initial condition* $(\overline{\gamma}_0, \overline{u}_0) \in \mathbb{R}^2 \times \mathbb{S}^1$ has been prescribed, where $\overline{u}_0 := (\overline{\gamma}_1 - \overline{\gamma}_0)/\|\overline{\gamma}_1 - \overline{\gamma}_0\|$ is the direction of the first edge. A pair of paths Γ, Ω in $\mathbb{R}^2$ are *congruent* if they coincide up to a (proper) rigid motion, in which case we write $\Gamma \equiv \Omega$.

Proposition 3.1 *Let $\Gamma = [\gamma_0, \dots, \gamma_k]$, $\Omega = [\omega_0, \dots, \omega_\ell]$ be a pair of paths in P such that $\gamma_i = \omega_i$ for $i = 0, \dots, m < \ell$. Further suppose that either $m = k$, or else ω_{m+1} lies strictly to the left of $[\gamma_{m-1}, \gamma_m, \gamma_{m+1}]$. Then*

$$(\overline{\Gamma})^{-1} \circ \overline{\Omega} \equiv \overline{(\Gamma^{-1} \circ \Omega)}_{\gamma_m},$$

provided that $\overline{\Gamma}$ and $\overline{\Omega}$ have the same initial conditions.

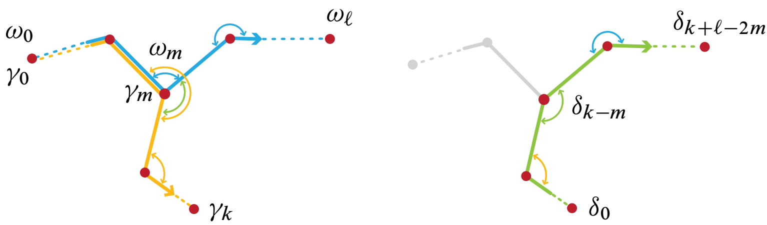

Figure 4

Proof Let $\Delta := \Gamma^{-1} \circ \Omega$ and $\widetilde{\Delta} := (\overline{\Gamma})^{-1} \circ \overline{\Omega}$. Then

$$\Delta = [\gamma_k, \dots, \gamma_0] \circ [\omega_0, \dots, \omega_\ell] = [\gamma_k, \dots, \gamma_m, \omega_{m+1}, \dots, \omega_\ell],$$
$$\widetilde{\Delta} = [\overline{\gamma}_k, \dots, \overline{\gamma}_0] \circ [\overline{\omega}_0, \dots, \overline{\omega}_\ell] = [\overline{\gamma}_k, \dots, \overline{\gamma}_m, \overline{\omega}_{m+1}, \dots, \overline{\omega}_\ell].$$

In particular note that Δ, $\widetilde{\Delta}$ each have $n := k + \ell - 2m + 1$ vertices. If we denote these vertices by δ_i, $\widetilde{\delta}_i$, where $0 \le i \le n-1$, then we have

$$\delta_i = \begin{cases} \gamma_{k-i}, & i \le k-m, \\ \omega_{i-k+2m}, & i \ge k-m, \end{cases} \qquad \widetilde{\delta}_i = \begin{cases} \overline{\gamma}_{k-i}, & i \le k-m, \\ \overline{\omega}_{i-k+2m}, & i \ge k-m; \end{cases}$$

see Figure 4. In particular, $\gamma_m = \delta_{k-m}$. So we have to show that $\widetilde{\Delta} \equiv (\overline{\Delta})_{\delta_{k-m}}$, which means we need to check that:

(i) $\|\delta_i - \delta_{i-1}\| = \|\widetilde{\delta}_i - \widetilde{\delta}_{i-1}\|$ for $1 \le i \le n-1$.

(ii) $\theta_i{}'[\widetilde{\Delta}] = \theta_i{}'[\Delta]$ for $1 \le i \le k-m$, and $\theta_i[\widetilde{\Delta}] = \theta_i[\Delta]$ for $k-m < i < n-1$.

To establish (i) note that, for $1 \le i \le k-m$,

$$\|\delta_i - \delta_{i-1}\| = \|\gamma_{k-i} - \gamma_{k-i+1}\| = \|\overline{\gamma}_{k-i} - \overline{\gamma}_{k-i+1}\| = \|\widetilde{\delta}_i - \widetilde{\delta}_{i-1}\|.$$

Furthermore, for $k-m \le i \le n-1$,

$$\|\delta_i - \delta_{i-1}\| = \|\omega_{i-k+2m} - \omega_{i-k+2m-1}\| = \|\overline{\omega}_{i-k+2m} - \overline{\omega}_{i-k+2m-1}\| = \|\widetilde{\delta}_i - \widetilde{\delta}_{i-1}\|.$$

Next we check (ii). For $1 \le i < k-m$,

$$\theta_i{}'[\Delta] = \theta_i{}'[\Gamma^{-1}] = \theta_{k-i}[\Gamma] = \theta_{k-i}[\overline{\Gamma}] = \theta_i{}'[(\overline{\Gamma})^{-1}] = \theta_i{}'[\widetilde{\Delta}].$$

Furthermore, for $k-m < i < n-1$,

$$\theta_i[\Delta] = \theta_{i-k+2m}[\Omega] = \theta_{i-k+2m}[\overline{\Omega}] = \theta_i[\widetilde{\Delta}].$$

It remains to check that $\theta'_{k-m}[\Delta] = \theta'_{k-m}[\widetilde{\Delta}]$, and to this end it suffices to show

$$\theta'_{k-m}[\Delta] = \theta_m[\Gamma] - \theta_m[\Omega] \quad \text{and} \quad \theta'_{k-m}[\widetilde{\Delta}] = \theta_m[\overline{\Gamma}] - \theta_m[\overline{\Omega}]. \tag{3}$$

To establish the first equation in (3) note that

$$\begin{aligned}\theta_m[\Omega]+\theta'_{k-m}[\Delta] &= \angle(\omega_{m-1},\omega_m,\omega_{m+1})+\angle(\delta_{k-m+1},\delta_{k-m},\delta_{k-m-1})\\ &= \angle(\gamma_{m-1},\gamma_m,\omega_{m+1})+\angle(\omega_{m+1},\gamma_m,\gamma_{m+1}).\end{aligned}$$

Further, since ω_{m+1} lies strictly on the left of $[\gamma_{m-1},\gamma_m,\gamma_{m+1}]$, Lemma 2.1(ii) yields

$$\angle(\gamma_{m-1},\gamma_m,\omega_{m+1})+\angle(\omega_{m+1},\gamma_m,\gamma_{m+1})=\angle(\gamma_{m-1},\gamma_m,\gamma_{m+1})=\theta_m[\Gamma].$$

The second equation in (3) follows from a similar calculation, once we check that $\overline{\omega}_{m+1}$ lies strictly to the left of $[\overline{\gamma}_{m-1},\overline{\gamma}_m,\overline{\gamma}_{m+1}]$. Indeed, since ω_{m+1} lies strictly on left of $[\gamma_{m-1},\gamma_m,\gamma_{m+1}]$, Lemma 2.1(i) yields

$$\theta_m[\Omega]=\angle(\omega_{m-1},\omega_m,\omega_{m+1})=\angle(\gamma_{m-1},\gamma_m,\omega_{m+1})<\angle(\gamma_{m-1},\gamma_m,\gamma_{m+1})=\theta_m[\Gamma].$$

So $\theta_m[\overline{\Omega}]=\theta_m[\Omega]<\theta_m[\Gamma]=\theta_m[\overline{\Gamma}]$. Consequently,

$$\angle(\overline{\gamma}_{m-1},\overline{\gamma}_m,\overline{\omega}_{m+1})=\angle(\overline{\omega}_{m-1},\overline{\omega}_m,\overline{\omega}_{m+1})=\theta_m[\overline{\Omega}]<\theta_m[\overline{\Gamma}]=\angle(\overline{\gamma}_{m-1},\overline{\gamma}_m,\overline{\gamma}_{m+1}).$$

So, by Lemma 2.1(i), $\overline{\omega}_{m+1}$ lies strictly to the left of $[\overline{\gamma}_{m-1},\overline{\gamma}_m,\overline{\gamma}_{m+1}]$ as claimed. □

4 The tracing path of a cut tree

Here we describe precisely how a cut tree T determines an unfolding of P. Further we show that the boundary of this unfolding coincides with a development of a certain path Γ_T which traces T. This leads to the main result of this section, Proposition 4.4 below, which shows that an unfolding of P generated by T is simple if and only if the development of Γ_T is simple. We start by recording some basic lemmas. In this section T need not be monotone.

Since leaves of T are vertices of P, T partitions each face of P into a finite number of polygons. Let $F_T(P):=\{F_i\}$ be the disjoint union of these polygons, and $\pi\colon F_T(P)\to P$ be the projection generated by the inclusion maps $F_i\hookrightarrow P$. Glue each pair F_i, $F_j\in F_T(P)$ along a pair E_{in}, E_{jm} of their edges if and only if $\pi(E_{in})$, $\pi(E_{jm})\notin T$ and $\pi(E_{in})=\pi(E_{jm})$. This yields a compact surface P_T (which we may think of as having resulted from "cutting" P along T). The inclusion maps $F_i\hookrightarrow P$ again define a natural projection $\pi\colon P_T\to P$, which is the identity map on $\operatorname{int}(P_T):=P_T-\partial P_T=P-T$. So, since T is contractible, P_T is a topological disk. Also note that P_T inherits an orientation from P, which in turn induces a cyclical ordering $\widetilde{v}_0,\dots,\widetilde{v}_n$ on the vertices of ∂P_T so that P_T lies to the left of ∂P_T, ie every $\widetilde{v}_i$ has an open neighborhood U_i in P_T such that $\pi(U_i)$ lies to the left of $[\pi(\widetilde{v}_{i-1}),\pi(\widetilde{v}_i),\pi(\widetilde{v}_{i+1})]$ in P. Since P_T contains no vertices in its interior, and all the interior angles of ∂P_T are less than 2π, it is locally isometric to the plane. Therefore, since P_T is simply connected, it may be isometrically immersed in the plane; see eg the author [9, Lemma 2.2]. An *immersion* is a locally one-to-one continuous map, and is *isometric* if it preserves distances. So we have established this:

Lemma 4.1 *P_T is simply connected and is locally isometric to $\mathbb{R}^2$. In particular, there exists an isometric immersion $P_T \to \mathbb{R}^2$.* □

Any such immersion will be called an *unfolding* of P (generated by T) provided that it is *orientation preserving*, ie $\overline{P}_T$ lies locally on the left of $\overline{\partial P_T}$, with respect to the orientation that $\overline{\partial P_T}$ inherits from ∂P_T. Recall that for any set $X \subset P_T$, we let $\overline{X}$ denote the image of X under the unfolding $P_T \to \mathbb{R}^2$, and say $\overline{X}$ is simple provided that $X \to \mathbb{R}^2$ is one-to-one.

Lemma 4.2 *$\overline{P}_T$ is simple if and only if $\overline{\partial P_T}$ is simple.*

Proof This is a special case of the following general fact (see the author [8]): if M is a connected compact surface with boundary components ∂M_i, and $M \to \mathbb{R}^2$ is an immersion, then $\overline{M}$ is simple if and only if each $\overline{\partial M_i}$ is simple. □

So, as far as Dürer's problem is concerned, we just need to decide when $\overline{\partial P_T}$ is simple. To this end it would be useful to think of $\overline{\partial P_T}$ not as the restriction of the unfolding of P_T to ∂P_T, but rather as the development of a path of P. This path is given by

$$\Gamma_T = [v_0, \dots, v_n] := [\pi(\widetilde{v}_0), \dots, \pi(\widetilde{v}_n)],$$

where $\widetilde{v}_0, \dots, \widetilde{v}_n$ is the cyclical ordering of the vertices of ∂P_T mentioned above. Thus, Γ_T traces $\pi(\partial P_T) = T$, and π establishes a bijection $v_i \leftrightarrow \widetilde{v}_i$ between the vertices of Γ_T and ∂P_T. For each $\widetilde{v}_i$ let $\widetilde{st}_{\widetilde{v}_i}$ denote the star of $\widetilde{v}_i$ in P_T. Then, since P_T lies to the left of ∂P_T, we have the following:

Lemma 4.3 *For every vertex v_i of Γ_T, the left side of $[v_{i-1}, v_i, v_{i+1}]$ in P coincides with $\pi(\widetilde{st}_{\widetilde{v}_i})$.* □

In particular, the left angles of Γ_T in P are the same as the interior angles of P_T. So, since the unfolding $P_T \to \mathbb{R}^2$ is orientation preserving, it follows that the left angles of $\overline{\partial P_T}$ are the same as those of a development $\overline{\Gamma}_T$ of Γ_T. Thus $\overline{\partial P_T}$ is congruent to $\overline{\Gamma}_T$, and Lemma 4.2 yields:

Proposition 4.4 *$\overline{P}_T$ is simple if and only if $\overline{\Gamma}_T$ is simple.* □

5 Criteria for embeddedness of immersed disks

As we discussed in the last section, an immersed disk in the plane is simple (or embedded) if and only if its boundary is simple. Here we generalize that observation. Let $D \subset \mathbb{R}^2$ be the unit disk centered at the origin, with oriented boundary ∂D. For p, $q \in \partial D$, let $pq \subset \partial D$ denote the segment with initial point p and final point q. Recall that an *immersion* is a continuous locally one-to-one map. Further recall that for any $X \subset D$, and mapping $f \colon D \to \mathbb{R}^2$, we set $\overline{X} := f(X)$, and say $\overline{X}$ is simple if f is one-to-one on X.

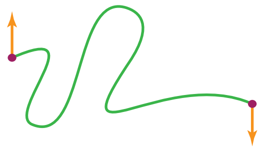

Figure 5

A simple curve segment in $\mathbb{R}^2$, whose endpoints do not have the same height, is *weakly monotone* (with respect to the direction $(0,1)$) if it may be extended to an unbounded simple curve by attaching a vertical ray to its top endpoint which extends upward, and a vertical ray to its bottom endpoint which extends downward; see Figure 5. The main result of this section is that an immersed disk is embedded whenever its boundary admits a decomposition into weakly monotone paths:

Proposition 5.1 *Let $D \xrightarrow{f} \mathbb{R}^2$ be an immersion with polygonal boundary. Suppose there is a pair of points p_0, p_1 in ∂D such that $\overline{p_0 p_1}$ and $\overline{p_1 p_0}$ are weakly monotone. Then $\overline{D}$ is simple.*

The basic strategy for proving the above proposition is to extend f to an immersion of a larger disk which has simple boundary and thus is one-to-one. To this end first note that by *polygonal boundary* here we mean that there are points v_i, $i \in \mathbb{Z}_k$, cyclically arranged along ∂D so that f maps each oriented segment $v_i v_{i+1}$ to a line segment. Then we obtain a closed polygonal path $[\overline{v}_0, \dots, \overline{v}_k]$ in $\mathbb{R}^2$. Since f is locally one-to-one, each v_i has a neighborhood U_i in D such that $\overline{U}_i$ lies one side of $[\overline{v}_{i-1}, \overline{v}_i, \overline{v}_{i+1}]$ as defined in Section 2.3, and it is easy to see that this side must be the same for all i. Thus we may say that $\overline{D}$ *lies locally on one side* of $\overline{\partial D}$.

Proof of Proposition 5.1 As discussed above, $\overline{D}$ lies locally on one side of the path $\overline{\partial D}$. We may assume that this is the left-hand side after composing f with a reflection of $\mathbb{R}^2$. Since $\overline{p_0 p_1}$, $\overline{p_1 p_0}$ are weakly monotone, they are simple and their endpoints have different heights. Suppose that $\overline{p}_0$ is the endpoint with the lower height, and let R_0 be the vertical ray which emanates from $\overline{p}_0$ and extends downward. Similarly, let R_1 be the vertical ray which emanates from $\overline{p}_1$ and extends upward. Let C be a circle so large which contains $\overline{D}$ in the interior of the region it bounds. Then R_0, R_1 intersect C at precisely one point each, say at x_0 and x_1 respectively; see Figure 6.

Figure 6

Now consider the oriented composite path $x_0x_1 := x_0\overline{p}_0 \cup \overline{p_0 p_1} \cup \overline{p}_1 x_1$ shown on the right diagram in Figure 6. Since $\overline{p_0 p_1}$ is weakly monotone, x_0x_1 is simple, and thus it divides the region bounded by C into a pair of disks. Let D_0 be the disk which lies to the right of x_0x_1. Similarly, let $x_1x_0 := x_1\overline{p}_1 \cup \overline{p_1 p_0} \cup \overline{p}_0 x_0$, and D_1 be the disk which lies on the right of the oriented path x_1x_0, as shown in the left diagram in Figure 6. Now gluing D_0 and D_1 along the segments $x_0\overline{p}_0$ and $x_1\overline{p}_1$ yields an immersed annulus A. Note that by construction A lies locally on the right of $\overline{\partial D}$. Thus, gluing A to $\overline{D}$ along $\overline{\partial D}$ yields an immersed disk, say D'. Note that $\partial D' = C$ which is simple. Thus it follows (via [8, Lemma 1.1], or Lemma A.2) that D' and consequently $\overline{D}$ is simple as claimed. □

The criteria which were proved above were the precise conditions we need in the proof of Theorem 1.1. See Appendix A for more general criteria concerned with embeddedness of immersed disks.

6 Structure of monotone cut trees

Here we describe how the leaves of T inherit a cyclical ordering from P, which in turn orders the branches of T. We use this to define a sequence of paths Γ_i in T, together with a class of related paths Γ_i'. The paths Γ_i join the top and bottom vertices of P, while Γ_i' are closed paths which correspond to the boundary of certain disks $D_i \subset P_T$.

6.1 Leaves ℓ_i and junctures j_i

Let $\Gamma := \Gamma_T$ be the path tracing T defined in Section 4. Note that each edge E of T appears precisely twice in Γ, because there are precisely two faces F_1, F_2 of P_T such that $\pi(F_1)$ and $\pi(F_2)$ are adjacent to E. This quickly yields:

Lemma 6.1 *Let v be a vertex of T which has degree n in T. Then there are precisely n vertices of Γ which coincide with v.* □

In particular each leaf of T occurs only once in Γ. Consequently, Γ determines a unique ordering $\ell_0, \ell_1, \dots, \ell_{k-1}$ of the leaves of T. Further we set $\ell_{i+k} := \ell_i$, and designate ℓ_0 (the top vertex of P) as the initial vertex of Γ. Recall that a vertex of Γ is simple if its adjacent vertices are distinct.

Lemma 6.2 *Any vertex of Γ which is not a leaf or root of T is simple.*

Proof Let v_i be a vertex of Γ which is not simple. We will show that the degree of v_i in T is 1, which is all we need. Since v_i is nonsimple, the left side of $[v_{i-1}, v_i, v_{i+1}]$ is the entire star st_{v_i} by definition. Thus, by Lemma 4.3, $\pi(\widetilde{st}_{\widetilde{v}_i}) = st_{v_i}$. Choose $r > 0$ so small that the metric "circle" $C \subset P_T$ of radius r centered at $\widetilde{v}_i$ lies in the interior of $\widetilde{st}_{\widetilde{v}_i}$. Then $\pi(C) \subset \mathrm{int}(st_{v_i})$ is a simple closed curve enclosing v_i which intersects T only once. Thus $\deg_T(v_i) = 1$ as claimed. □

Using the last lemma, we now show that the leaves of T may be characterized via the height function h as follows:

Lemma 6.3 *A vertex of Γ is a local maximum point of h on the sequence of vertices of Γ if and only if it is a leaf of T.*

Proof Suppose that v is a leaf of T, and let u, w be its adjacent vertices in Γ. Since T is monotone, and $v \neq r$, there exists a vertex v' of T which is adjacent to v and lies below it. Since $\deg_T(v) = 1$, $u = v' = w$. In particular, u, w lie below v. So v is a local maximizer of h. Conversely, suppose that v is a local maximizer of h. Then u, w lie below v, because T does not have horizontal edges. Let Ω_u, Ω_w be the simple monotone paths in T which connect u, w to r respectively. Then $vu \bullet \Omega_u$ and $vw \bullet \Omega_w$ are simple, since they are monotone. Hence $u = w$, by the uniqueness of simple paths in T. So v is not simple and therefore must be either a leaf or the root of T, by Lemma 6.2. The latter is impossible, since v is a local maximizer of h. □

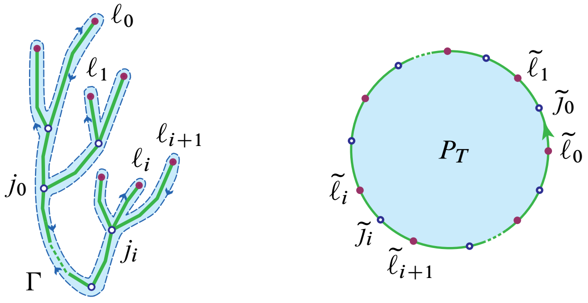

Figure 7

It follows from Lemma 6.3 that between every pair of consecutive leaves ℓ_i, ℓ_{i+1} of Γ there exists a unique vertex j_i, called a *juncture*, which is a local minimizer of h; see Figure 7. Note that some junctures of Γ may coincide with each other, or with the root r of T. For any ordered pair (v, w) of vertices of T let $(vw)_T$ be the (unique) simple path in T joining v to w. Note that the paths $\ell_j j_i$ and $j_i\ell_{i+1}$ of Γ are monotone and therefore simple. Thus

$$(\ell_i j_i)_\Gamma = (\ell_i j_i)_T \quad \text{and} \quad (j_i\ell_{i+1})_\Gamma = (j_i\ell_{i+1})_T. \tag{4}$$

6.2 Branches β_i and the paths Γ_i

For $0 \le i \le k-1$, we define the *branches* of T as the paths $\beta_i := (\ell_i r)_T$, which connect each leaf of T to its root. Note that, by (4), we have

$$\beta_i = (\ell_i r)_T = (\ell_i j_i)_T \bullet (j_i r)_T = (\ell_i j_i)_\Gamma \bullet (j_i r)_T. \tag{5}$$

Having ordered the branches of T, we now describe the first class of paths which are useful for our study of monotone trees:

$$\Gamma_i := (\ell_0\ell_i)_\Gamma \bullet \beta_i = (\ell_0 j_i)_\Gamma \bullet (j_i r)_T \tag{6}$$

for $0 \le i \le k-1$. See Figure 8 for some examples. Next we record how the composition of these paths is related to the branches of T.

Lemma 6.4 *For $0 \le i \le k-2$, $\Gamma_i^{-1} \circ \Gamma_{i+1} = \beta_{i+1}^{-1} \bullet \beta_{i+1}$.*

Proof By (4), (5) and (6), we have

$$\begin{aligned}
\Gamma_i^{-1} \circ \Gamma_{i+1} &= ((\ell_0 j_i)_\Gamma \bullet (j_i r)_T)^{-1} \circ ((\ell_0\ell_{i+1})_\Gamma \bullet \beta_{i+1}) \\
&= (r j_i)_T \bullet (j_i\ell_0)_{\Gamma^{-1}} \circ (\ell_0 j_i)_\Gamma \bullet (j_i\ell_{i+1})_\Gamma \bullet \beta_{i+1} \\
&= (r j_i)_T \bullet (j_i\ell_{i+1})_T \bullet \beta_{i+1} = (r\ell_{i+1})_T \bullet \beta_{i+1} = \beta_{i+1}^{-1} \bullet \beta_{i+1}.
\end{aligned}$$

This concludes the proof. □

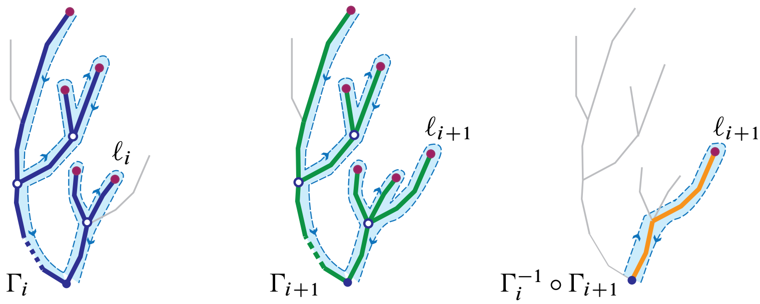

Figure 8

The following observation shows, via Lemma 2.1(i), that Γ_{i+1} lies to left of Γ_i near j_i, if j_i is an interior vertex of Γ_i.

Lemma 6.5 *If $j_i \neq r$, then $\theta_{j_i}[\Gamma_{i+1}] < \theta_{j_i}[\Gamma_i]$, for $0 \leq i \leq k-2$.*

Proof Let v, w be the vertices of Γ_{i+1} which precede and succeed j_i respectively. By Lemma 6.2, $v \neq w$. Next let u denote the vertex of Γ_i which succeeds j_i; see Figure 9. We need to show that $\angle(v, j_i, w) < \angle(v, j_i, u)$. Suppose, towards a contradiction, that $\angle(v, j_i, w) \geq \angle(v, j_i, u)$.

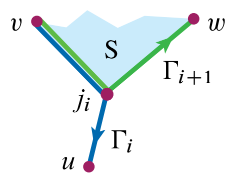

Figure 9

The equality in the last inequality cannot occur, because by definitions of Γ_i and Γ_{i+1}, u lies below j_i while w lies above it (so $u \neq w$). Thus we may assume that $\angle(v, j_i, w) > \angle(v, j_i, u)$. Then, by Lemma 2.1(i), u lies strictly in the left side S of $[v, j_i, w]$. Consequently uj_i intersects the interior of S, which means $\operatorname{int}(S) \cap T \neq \varnothing$. But this is impossible because $S = \pi(\widetilde{st}_{\tilde{j}_i})$ by Lemma 4.3 which yields

$$\operatorname{int}(S) = \operatorname{int}(\pi(\widetilde{st}_{\tilde{j}_i})) = \pi(\operatorname{int}(\widetilde{st}_{\tilde{j}_i})) \subset \pi(\operatorname{int}(P_T)) = P - T. \tag{7}$$

This concludes the proof. □

Now we are ready to prove the main result of this subsection:

Proposition 6.6 *For $0 \leq i \leq k-2$, $(\overline{\Gamma}_i)^{-1} \circ \overline{\Gamma}_{i+1} \equiv \overline{(\beta_{i+1}^{-1} \bullet \beta_{i+1})}_{j_i}$.*

Proof By Lemma 6.4, $\beta_{i+1}^{-1} \bullet \beta_{i+1} = \Gamma_i^{-1} \circ \Gamma_{i+1}$. So we just need to check that

$$\overline{(\Gamma_i^{-1} \circ \Gamma_{i+1})_{j_i}} \equiv (\overline{\Gamma_i})^{-1} \circ \overline{\Gamma}_{i+1},$$

which follows from Proposition 3.1 via Lemma 6.5. More specifically, there are two cases to consider. If $j_i = r$, then Γ_i is a subpath of Γ_{i+1}, which corresponds to the case "$m = k$" in Proposition 3.1. If $j_i \neq r$, then Lemma 6.5 together with Lemma 2.1(i) ensure that Γ_{i+1} lies to the left of Γ_i near j_i, and so the hypothesis of Proposition 3.1 is again satisfied. □

6.3 Dual branches β_i' and the paths Γ_i'

To describe the second class of paths which we may associate to a monotone tree, we first establish the existence of a collection of paths β_i' which are in a sense dual to the branches β_i defined above.

Proposition 6.7 *Each leaf ℓ_i of a monotone cut tree T may be connected to the top leaf ℓ_0 of T via a monotone path β_i' in P, which intersects T only at its endpoints.*

Assume for now that the above proposition holds. Then for each leaf ℓ_i, we fix a path β_i' given by this proposition and set

$$\Gamma_i' := \begin{cases} (\ell_0 \ell_i)_\Gamma \bullet \beta_i' & 1 \le i \le k-1, \\ \Gamma & i = k. \end{cases} \tag{8}$$

Note that since the interior of β_i' lies in $P - T$, it lifts to a unique path $\widetilde{\beta}_i'$ in P_T (see Figure 10) such that $\pi(\widetilde{\beta}_i') = \beta_i'$. Consequently each Γ_i' corresponds to a simple closed curve $\widetilde{\Gamma}_i'$ in P_T, where

$$\widetilde{\Gamma}_i' := (\widetilde{\ell}_0 \widetilde{\ell}_i)_{\partial P_T} \bullet \widetilde{\beta}_i' \quad \text{for } 1 \le i \le k-1 \text{ and } \widetilde{\Gamma}_k' := \partial P_T.$$

Let $D_i \subset P_T$ be the disk bounded by $\widetilde{\Gamma}_i'$ which lies to the left of it, and note that $D_k = P_T$.

The unfolding $P_T \to \overline{P}_T \subset \mathbb{R}^2$ induces unfoldings $D_i \to \overline{D}_i \subset \mathbb{R}^2$. Thus, as was the case for ∂P_T discussed in Section 4, there are two congruent ways to map each boundary curve ∂D_i to $\mathbb{R}^2$: one via the development of $\pi \circ \widetilde{\Gamma}_i' = \Gamma_i'$ and the other via the restriction of the unfolding $\overline{D}_i$ to ∂D_i. So we may record:

Lemma 6.8 *For $1 \le i \le k$, the mappings $\partial D_i \to \mathbb{R}^2$ generated by $\overline{\Gamma}_i'$ and $\overline{\partial D_i}$ coincide, up to a rigid motion. In particular, $\overline{\Gamma}_i'$ bounds an immersed disk.* □

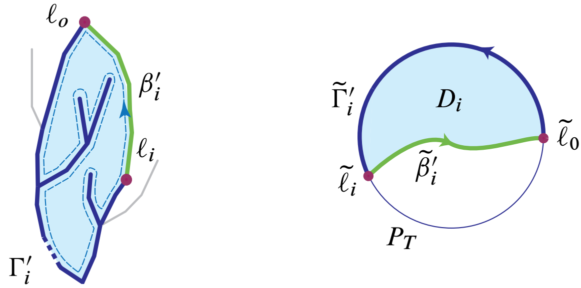

Figure 10

To prove Proposition 6.7, we need the following lemma whose proof is similar to that of Lemma 6.5. Recall that j_i are local minimizers of h on Γ which traces $T = \pi(\partial P_T)$. Thus j_i are local minimizers of $h \circ \pi$ on ∂P_T. The next observation generalizes this fact.

Lemma 6.9 *Each juncture j_i of Γ is a local minimizer of $h \circ \pi$ on P_T.*

Proof Let v, w be vertices of Γ which are adjacent to j_i. If $j_i = r$, then there is nothing to prove, since r is the absolute minimizer of h on P. So assume that $j_i \neq r$. Then $v \neq w$ by Lemma 6.2. Consequently $vw := [v, j_i, w]$ determines a pair of sides in st_{j_i}. Let $X \subset st_{j_i}$ be the set of points whose heights are smaller than $h(j_i)$. Then X is connected and is disjoint from vw. Thus X lies entirely on one side of vw which will be called the *bottom* side, while the other side will be the *top* side. Recall that $S := \pi(\widetilde{st}_{\widetilde{j}_i})$ is one of the sides of vw by Lemma 4.3. We claim that S is the top side, which is all we need. To this end note that the path $j_i r$ of T intersects X. So T intersects the interior of the bottom side. But $\mathrm{int}(S) \cap T = \varnothing$ by (7). Thus S cannot be the bottom side. $\square$

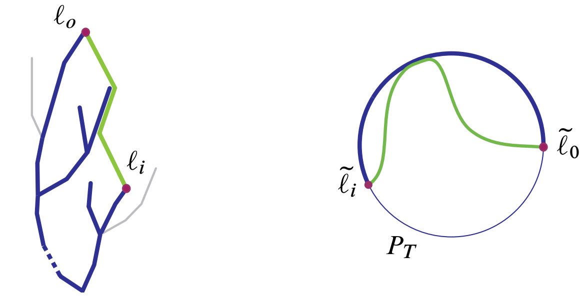

Figure 11

Now we are ready to prove the main result of this subsection:

Proof of Proposition 6.7 Let's say a path in P_T is monotone if its projection into P is monotone. We will connect $\widetilde{\ell}_i$ to $\widetilde{\ell}_0$ with a monotone path $\widetilde{\beta}'_i$ in P_T which intersects ∂P_T only at its endpoints. Then $\beta'_i := \pi(\widetilde{\beta}'_i)$ is the desired path. We will proceed in two stages: first (Part I) we construct a monotone path $\widetilde{\beta}'_i$ in P_T which connects $\widetilde{\ell}_i$ to $\widetilde{\ell}_0$, and then (Part II) perturb $\widetilde{\beta}'_i$ to make sure that its interior is disjoint from ∂P_T. See Figure 11 and compare it to Figure 10.

Part I If $\ell_i = \ell_0$ (ie $i = 0$), we set $\widetilde{\beta}'_i := \widetilde{\ell}_0$ and we are done. So suppose that $\ell_i \neq \ell_0$. Then there is a vertex v of P adjacent to ℓ_i which lies above it. The only edge of T which is adjacent to ℓ_i connects to it from below. Thus $\ell_i v$ is not an edge of T, and therefore corresponds to a unique edge $\widetilde{\ell}_i\widetilde{v}$ of P_T. This will constitute the first edge of $\widetilde{\beta}'_i$. There are three cases to consider:

(i) $v = \ell_0$.

(ii) v is a leaf of T other than ℓ_0.

(iii) v is not a leaf of T.

If (i) holds, we are done. If (ii) holds, then we may connect v to an adjacent vertex v' lying above it to obtain the next edge $\widetilde{v}\widetilde{v}'$ of $\widetilde{\beta}'_i$. If (iii) holds, then, by Lemma 6.9, v cannot be a juncture of Γ, because it is the highest point of $\ell_i v$. Thus v lies in the interior of a subpath $\ell_n j_n$ or $j_n\ell_{n+1}$ of Γ. In particular, there exists a monotone subpath $v\ell_n$ of Γ^{-1} or $v\ell_{n+1}$ of Γ which connects v to a leaf v' of T which lies above it. Lifting this path to ∂P_T will extend our path to $\widetilde{v}'$. Now again there are three cases to consider for v', as listed above, and repeating this process eventually yields the desired path $\widetilde{\beta}'_i$.

Part II After a subdivision, we may assume that all faces of P_T are triangles. If an edge E of $\widetilde{\beta}'_i$ lies on ∂P_T, let F be the face of ∂P_T adjacent to E, choose a point p in the interior of F which has the same height as an interior point of E, and replace E with the pair of line segments which connect the vertices of E to p; see the left diagram in Figure 12.

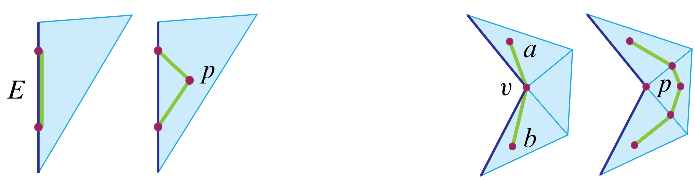

Figure 12

Thus we may perturb each edge of $\widetilde{\beta}'_i$ which lies on ∂P_T so that $\widetilde{\beta}'_i$ intersects ∂P_T only at some of its vertices. Let v be such a vertex. Further let a (resp. b) be a point in the interior of the edge of $\widetilde{\beta}_i'$ adjacent to v which lies above (resp. below) v. We need to replace the segment ab of $\widetilde{\beta}_i'$ with another monotone segment in P_T which avoids v; see the right diagram in Figure 12. Pick a point p in the interior of the star of P_T at v which has the same height as v. It suffices to construct a pair of monotone paths in $\mathrm{int}(P_T) = P - T$ which connect a and b to p. The first path may be constructed as follows, and the other path is constructed similarly. Let R_a, R_b be the rays which emanate from v and pass through a, b respectively. These rays determine a region $\mathcal{R}$ in the star of P_T at v which is contained between them. There exists a face F of P_T which contains a and intersects the interior of $\mathcal{R}$. If $p \in F$, then we connect a to p with a line segment and we are done. If $p \notin F$, then F has a unique edge E which lies in the interior of $\mathcal{R}$ and is adjacent to v. There is a point a' in the interior of E which lies below a (because E is adjacent to v which is below a). Connect a to a' with a line segment. Next consider the face of P_T which is adjacent to E and is different from F. If this face contains p then we connect a' to p with a line segment and we are done. Otherwise we repeat the above procedure until we reach p. □

7 Affine developments of monotone paths

Here we study the effects of the affine stretchings of P on the developments of its piecewise monotone paths. The main results of this section are Propositions 7.5 and 7.6 below. The first proposition shows that affine stretchings of piecewise monotone paths have piecewise monotone developments, and the second proposition states that this development is simple if the original curve double covers a monotone path. First we need to prove the following lemmas. At each interior vertex γ_i of a path Γ in P, let Θ_i denote the angle between $\gamma_{i-1} - \gamma_i$, and $\gamma_{i+1} - \gamma_i$ in $\mathbb{R}^3$. Further recall that θ_i, $\theta_i{}'$ denote the left and right angles of Γ in P.

Lemma 7.1 *At any interior vertex γ_i of a path Γ in P, we have $\theta_i, \theta_i{}' \geq \Theta_i$.*

Proof Let S be a unit sphere in $\mathbb{R}^3$ centered at γ_i, and $\widetilde{\gamma}_{i-1}$, $\widetilde{\gamma}_{i+1}$ be the projections of γ_{i-1} and γ_{i+1} into S as defined in Section 2.3. Then Θ_i is the geodesic distance between $\widetilde{\gamma}_{i+1}$ and $\widetilde{\gamma}_{i-1}$ in S. So it cannot exceed the length of any curve in S connecting $\widetilde{\gamma}_{i+1}$ and $\widetilde{\gamma}_{i-1}$, including those which correspond to θ_i, $\theta_i{}'$. □

Let P^λ denote the image of P under the affine stretching $(x, y, z) \mapsto (x/\lambda, y/\lambda, z)$. For any object X associated to P we also let X^λ denote the corresponding object of P^λ. Further, we let X^∞ denote the limit of X^λ as $\lambda \to \infty$. In particular note that P^∞ lies on the z–axis. A path is *monotone* if the heights h of its vertices form a strictly monotone sequence.

Lemma 7.2 *Let Γ be a monotone path in P. Then $\theta_i^\infty = (\theta_i{}')^\infty = \pi$.*

Proof For each vertex γ_i of Γ, $h(\gamma_i^\lambda)$ is constant. Thus $h(\gamma_i^\infty) = h(\gamma_i)$. Since Γ is monotone, it follows that γ_i^∞ lies in between γ_{i-1}^∞ and γ_{i+1}^∞ on the z–axis. So

$$\gamma_{i-1}^\infty - \gamma_i^\infty \quad \text{and} \quad \gamma_{i+1}^\infty - \gamma_i^\infty$$

are antiparallel vectors, which yields that $\Theta_i^\infty = \pi$. By Lemma 7.1, θ_i^λ, $(\theta_i{}')^\lambda \geq \Theta_i^\lambda$. Thus $\theta_i^\infty, (\theta_i{}')^\infty \geq \pi$. On the other hand, by (2), $\theta_i^\infty + (\theta_i{}')^\infty \leq 2\pi$. So

$$\theta_i^\infty = (\theta_i{}')^\infty = \pi. \qquad \square$$

The last lemma leads to the following observation.

Lemma 7.3 *Let v be a vertex of P. Then, $\angle_P(v)^\infty = 2\pi$ if v is not the top or bottom vertex of P. Otherwise, $\angle_P(v)^\infty = 0$.*

Proof The last statement is obvious. To see the first statement note that if v is not an extremum point of h, then since P is convex there exists a monotone path $[u, v, w]$ in P, where u and w are adjacent vertices of v. Let θ, θ' be the angles of this path at v. Then $\theta^\infty = (\theta')^\infty = \pi$ by Lemma 7.2. So $\angle_P(v)^\infty = 2\pi$ by (2). $\square$

A path is *piecewise monotone* if it is composed of monotone subpaths, or does not contain any horizontal edges. The last two lemmas yield:

Lemma 7.4 *Let Γ be a piecewise monotone path in P, and γ_i be an interior vertex of Γ. If γ_i is a local extremum of h on Γ, then θ_i^∞, $(\theta_i{}')^\infty = 0$ or 2π. Otherwise $\theta_i^\infty = (\theta_i{}')^\infty = \pi$.*

Proof If γ_i is not a local extremum of h (on Γ), then $[\gamma_{i-1}, \gamma_i, \gamma_{i+1}]$ is a monotone path. Consequently, $\theta_i^\infty = (\theta_i{}')^\infty = \pi$ by Lemma 7.2 as claimed. Next suppose that γ_i is a local extremum of h. If γ_i is the top or bottom vertex of P, then $\angle_P(\gamma_i)^\infty = 0$, by Lemma 7.3, which yields that $\theta_i^\infty = (\theta_i{}')^\infty = 0$ by (2), and again we are done. So suppose that γ_i is not an extremum vertex. Then $\angle_P(\gamma_i)^\infty = 2\pi$ by Lemma 7.3, and consequently $\theta_i^\infty + (\theta_i{}')^\infty = 2\pi$ by (2). So we just need to check that $\theta_i^\infty = 0$ or 2π. To see this note that if γ_i is not simple, then $\theta_i^\lambda = \angle_P(\gamma_i)^\lambda$, which yields that $\theta_i^\infty = 2\pi$, by Lemma 7.3. So we may assume that γ_i is simple. If γ_i is a local maximum (resp. local minimum) of h, then there exists a vertex v of P which is adjacent to γ_i and lies above (resp. below) it. Consequently, v lies strictly either to the right or left of $[\gamma_{i-1}, \gamma_i, \gamma_{i+1}]$. Suppose that v lies strictly to left of $[\gamma_{i-1}, \gamma_i, \gamma_{i+1}]$. Then $\theta_i^\lambda = \angle(\gamma_{i-1}, \gamma_i, v)^\lambda + \angle(v, \gamma_i, \gamma_{i+1})^\lambda$, by Lemma 2.1(ii). But $[\gamma_{i-1}, \gamma_i, v]$ and $[v, \gamma_i, \gamma_{i+1}]$

are monotone. Thus by Lemma 7.2, $\angle(\gamma_{i-1},\gamma_i,v)^\infty=\pi=\angle(v,\gamma_i,\gamma_{i+1})^\infty$. So $\theta_i^\infty=\pi+\pi=2\pi$. If, on the other hand, v lies strictly to the right of $[\gamma_{i-1},\gamma_i,\gamma_{i+1}]$, then v lies strictly to the left of $[\gamma_{i+1},\gamma_i,\gamma_{i-1}]$, and a similar reasoning shows that $(\theta_i{}')^\infty=2\pi$, or $\theta_i^\infty=0$. □

We will assume that all developments below have initial condition $((0,0),(0,-1))$, as defined in Section 3. A monotone path is *positively* (resp. *negatively*) *monotone* provided that the heights of its consecutive vertices form an increasing (resp. decreasing) sequence.

Proposition 7.5 *Let Γ be a piecewise monotone path in P and $\overline{\Gamma}$ be a mixed development of Γ. Then $\overline{\Gamma}^\infty$ is a path with vertical edges. Furthermore, each subpath of $\overline{\Gamma}^\infty$ which corresponds to a positively (resp. negatively) monotone subpath of Γ will be positively (resp. negatively) monotone.*

Proof Recall that $h(\gamma_i^\infty)=h(\gamma_i)$. So since Γ is monotone, $\gamma_i^\infty\neq\gamma_{i-1}^\infty$. Then, since $\|\overline{\gamma}_i^\lambda-\overline{\gamma}_{i-1}^\lambda\|=\|\gamma_i^\lambda-\gamma_{i-1}^\lambda\|$, it follows that $\|\overline{\gamma}_i^\infty-\overline{\gamma}_{i-1}^\infty\|=\|\gamma_i^\infty-\gamma_{i-1}^\infty\|\neq 0$. So $\overline{\gamma}_i^\infty\neq\overline{\gamma}_{i-1}^\infty$, which means that $\overline{\Gamma}^\infty$ is a path. In particular $\overline{\theta}_i^\infty$, $(\overline{\theta}_i')^\infty$ are well defined, and are limits of $\overline{\theta}_i^\lambda$, $(\overline{\theta}_i')^\lambda$ respectively. Now Lemma 7.4 quickly completes the argument. □

The *doubling* of a path $\Gamma=[\gamma_0,\dots,\gamma_k]$ is the path

$$D\Gamma:=\Gamma\bullet\Gamma^{-1}=[\gamma_0,\gamma_1,\dots,\gamma_{k-1},\gamma_k,\gamma_{k-1},\dots,\gamma_1,\gamma_0]=:[\gamma_0,\dots,\gamma_{2k}].$$

Our next result shows that doublings of monotone paths which end at vertices of P have simple unfoldings once they get stretched enough.

Proposition 7.6 *Let $\Gamma=[\gamma_0,\dots,\gamma_k]$ be a monotone path in P such that γ_k is a vertex of P different from its top or bottom vertex, and $\overline{D\Gamma}:=(\overline{D\Gamma})_{\gamma_\ell}$ be a mixed development of $D\Gamma$ based at γ_ℓ for some $0\leq\ell<k$. Then, for sufficiently large λ:*

(i) *$\overline{D\Gamma^\lambda}$ is simple.*

(ii) *The line which passes through $\overline{\gamma}_0^\lambda$, $\overline{\gamma}_{2k}^\lambda$ intersects $\overline{D\Gamma^\lambda}$ at no other point.*

(iii) *If α_0^λ, β_0^λ denote the interior angles of $\overline{D\Gamma^\lambda}\bullet[\overline{\gamma}_{2k}^\lambda,\overline{\gamma}_0^\lambda]$ at $\overline{\gamma}_0^\lambda$, $\overline{\gamma}_{2k}^\lambda$, then*

$$\alpha_0^\lambda+\beta_0^\lambda<\pi.$$

Furthermore, α_0^λ, β_0^λ may be arbitrarily close to $\pi/2$.

Proof We proceed by induction on the number of edges of Γ. Clearly the proposition holds when Γ has only one edge. Suppose that it holds for the subpath $\Gamma_1^\lambda := [\gamma_1^\lambda, \dots, \gamma_k^\lambda]$ of Γ^λ. Then we claim that it also holds for Γ^λ. Henceforth we will assume that λ is arbitrarily large and drop the explicit reference to it. Let L_1 be the line passing through the endpoints $\overline{\gamma}_1$, $\overline{\gamma}_{2k-1}$ of $\overline{D\Gamma_1}$, and o be the midpoint of $\overline{\gamma_1\gamma_{2k-1}}$. We may assume, after rigid motions, that o is fixed, L_1 is horizontal, and $\overline{D\Gamma_1}$ lies above L_1; see Figure 13. Furthermore, since by assumption γ_k is not the top or bottom vertex of P, we may assume that the left angle of $D\Gamma$ at γ_k (which coincides with the total angle of P at γ_k) is arbitrarily close to 2π by Lemma 7.3. Then it follows that $\overline{\gamma}_{2k-1}$ lies to the right of $\overline{\gamma}_1$ on L_1, just as depicted in Figure 13.

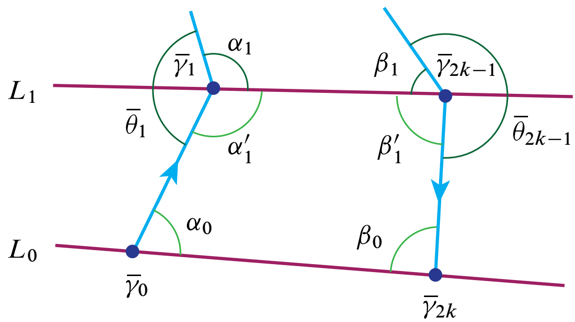

Figure 13

Now we claim that $\overline{\gamma}_0$, $\overline{\gamma}_{2k}$ lie below L_1. To see this, let α_1, β_1 be the interior angles of $\overline{D\Gamma_1} \bullet [\overline{\gamma}_{2k-1}, \overline{\gamma}_1]$ at $\overline{\gamma}_1$, $\overline{\gamma}_{2k-1}$ respectively. Further let $\overline{\theta}_1$, $\overline{\theta}_{2k-1}$ denote respectively the left angles of $\overline{D\Gamma}$ at $\overline{\gamma}_1$ and $\overline{\gamma}_{2k-1}$. We may assume α_1, $\beta_1 \approx \pi/2$ by the inductive hypothesis on Γ_1. By Lemma 7.2, we may also assume that $\overline{\theta}_1, \overline{\theta}_{2k-1} \approx \pi$. So

$$\alpha_1 + \overline{\theta}_1 \approx \tfrac{3\pi}{2} \quad \text{and} \quad \beta_1 + \overline{\theta}_{2k-1} \approx \tfrac{3\pi}{2}, \tag{9}$$

which show that $\overline{\gamma}_0$, $\overline{\gamma}_{2k}$ lie below L_1 as claimed. Next we show that $\overline{\gamma}_1\overline{\gamma}_0$, $\overline{\gamma}_{2k-1}\overline{\gamma}_{2k}$ do not intersect, which will establish (i). It suffices to check that $\alpha_1' + \beta_1' \geq \pi$, where

$$\alpha_1' := 2\pi - \alpha_1 - \overline{\theta}_1 \quad \text{and} \quad \beta_1' := 2\pi - \beta_1 - \overline{\theta}_{2k-1}.$$

There are two cases to consider: either $\overline{\theta}_1 = \theta_1$ or $\overline{\theta}_1{}' = \theta_1{}'$ by the definition of mixed development. If $\overline{\theta}_1 = \theta_1$, then

$$\overline{\theta}_1 + \overline{\theta}_{2k-1} = \theta_1 + \theta_{2k-1} = \theta_1 + \theta_1{}' = \angle_P(\gamma_1) \leq 2\pi,$$

where the identity $\theta_{2k-1} = \theta_1{}'$ used here follows from the definition of $D\Gamma$. If, on the other hand, $\overline{\theta}_1' = \theta_1{}'$, then

$$\overline{\theta}_1 + \overline{\theta}_{2k-1} = 2\pi - \overline{\theta}_1' + \overline{\theta}_{2k-1} = 2\pi - \theta_1{}' + \theta_{2k-1} = 2\pi.$$

So we always have $\overline{\theta}_1 + \overline{\theta}_{2k-1} \leq 2\pi$. Also note that $\alpha_1 + \beta_1 < \pi$ by the inductive hypothesis on Γ_1. Thus it follows that,

$$\alpha'_1 + \beta'_1 = 4\pi - (\alpha_1 + \beta_1) - (\overline{\theta}_1 + \overline{\theta}_{2k-1}) > 4\pi - \pi - 2\pi = \pi, \tag{10}$$

as desired. To establish (ii), let L_0 be the line passing through $\overline{\gamma}_0$, $\overline{\gamma}_{2k}$. By (10) the quadrilateral $Q := \overline{\gamma}_0\overline{\gamma}_1\overline{\gamma}_{2k-1}\overline{\gamma}_{2k}$ is convex. Thus $\overline{\gamma}_1$, $\overline{\gamma}_{2k-1}$ lie on the same side or "above" L_0. It remains to check that $\overline{D\Gamma_1}$ is disjoint from L_0. To this end note that the length of $\overline{D\Gamma_1}$ is bounded from above, since affine stretchings do not increase lengths. So $\overline{D\Gamma_1}$ is contained in a half disk H of some constant radius which lies above L_1 and is centered at o. Further $\overline{\gamma}_1\overline{\gamma}_0$ and $\overline{\gamma}_{2k-1}\overline{\gamma}_{2k}$ are almost orthogonal to L_0 by (9), and they have the same length, which is bounded from below (by $|h(\gamma_1) - h(\gamma_0)|$). Thus L_0 is nearly parallel to L_1 while its distance from o is bounded from below. So L_0 will be disjoint from H. Finally, (iii) follows immediately from (10), since Q is a simple quadrilateral and thus the sum of its interior angles is 2π. $\square$

8 Proof of Theorem 1.1

For convenience, we may assume that $u = (0, 0, 1)$. Let $\Gamma := \Gamma_T$ be the path which traces T as defined in Section 4. Recall that, as we showed in Section 6, Γ admits a decomposition into monotone subpaths:

$$\Gamma = \ell_0 j_0 \bullet j_0 \ell_1 \bullet \cdots \bullet \ell_{k-1} j_{k-1} \bullet j_{k-1} \ell_0.$$

Also recall that $\ell_i j_i$ are negatively monotone, and $j_i \ell_{i+1}$ are positively monotone. By Proposition 4.4 we just need to show that the development $\overline{\Gamma}^\lambda$ is simple for large λ. To this end, we first record how large λ needs to be, and then proceed by induction on the number of leaves of T.

8.1 Fixing the stretching factor λ

Let Γ_i, Γ'_i be the paths defined in Section 6, and recall that these paths also admit decompositions into monotone subpaths:

$$\Gamma_i = \ell_0 j_0 \bullet j_0 \ell_1 \bullet \cdots \bullet \ell_{i-1} j_{i-1} \bullet j_{i-1} \ell_i \bullet \ell_i r, \qquad 0 \leq i \leq k-1,$$
$$\Gamma'_i = \ell_0 j_0 \bullet j_0 \ell_1 \bullet \cdots \bullet \ell_{i-1} j_{i-1} \bullet j_{i-1} \ell_i \bullet \ell_i \ell_0, \quad 1 \leq i \leq k.$$

Let Γ_i^λ, $(\Gamma'_i)^\lambda$ denote the affine stretching of these paths, and $\overline{\Gamma}_i^\lambda$, $(\overline{\Gamma}'_i)^\lambda$ be their corresponding developments with initial condition $((0,0),(0,-1))$, as in Section 7. We need to choose λ so large that:

(C1) For each positively (resp. negatively) monotone subpath of Γ_i or Γ_i' the corresponding subpath of $\overline{\Gamma}_i^\lambda$ or $(\overline{\Gamma}_i')^\lambda$ is positively (resp. negatively) monotone.

(C2) $(\overline{\Gamma}_i^\lambda)^{-1} \circ \overline{\Gamma}_{i+1}^\lambda$ is simple and lies on one side of the line L^λ passing through its endpoints. Furthermore, L^λ is not vertical (see Figure 14).

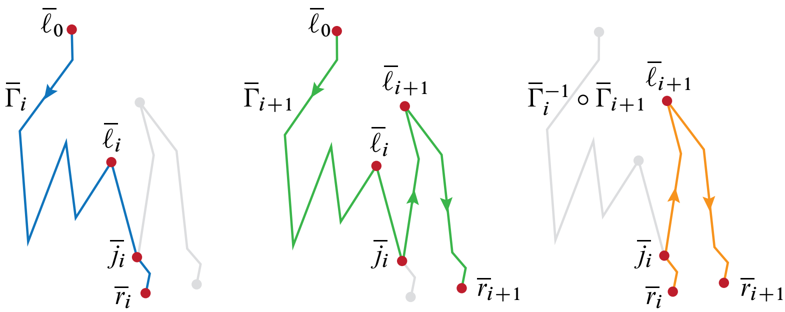

Figure 14

To see that (C1) holds let γ_j, $\overline{\gamma}_j^\infty$ denote the vertices of Γ, $\overline{\Gamma}^\infty$, and set

$$0 < \epsilon < \frac{1}{2} \inf_j \|\overline{\gamma}_j^\infty - \overline{\gamma}_{j-1}^\infty\| = \frac{1}{2} \inf_j |h(\gamma_j) - h(\gamma_{j-1})|.$$

Choose λ so large that $\|\overline{\gamma}_j^\lambda - \overline{\gamma}_j^\infty\| \le \epsilon$. Then $\overline{\gamma}_j^\lambda$ lies below (resp. above) $\overline{\gamma}_{j-1}^\lambda$ if and only if $\overline{\gamma}_j^\infty$ lies below (resp. above) $\overline{\gamma}_{j-1}^\infty$. Thus monotone subpaths of $\overline{\Gamma}_i^\lambda$ correspond to those of $\overline{\Gamma}_i^\infty$, which by Proposition 7.5 correspond to the monotone subpaths of Γ_i. Similarly we may obtain an estimate for λ in $(\overline{\Gamma}_i')^\lambda$. To see that (C2) holds note that, by Proposition 6.6,

$$(\overline{\Gamma}_i^\lambda)^{-1} \circ \overline{\Gamma}_{i+1}^\lambda \equiv (\overline{(\beta_{i+1}^\lambda)^{-1} \bullet \beta_{i+1}^\lambda})_{j_i^\lambda} \equiv (\overline{D\beta_{i+1}^\lambda})_{j_i^\lambda}.$$

So, since β_i are monotone, it follows from Proposition 7.6 that the right-hand side of the above expression is simple and lies on one side of the line L^λ passing through its endpoints. Further, L^λ becomes arbitrarily close to meeting $(\overline{\Gamma}_i^\lambda)^{-1} \circ \overline{\Gamma}_{i+1}^\lambda$ orthogonally, as λ grows large. At the same time, the edges of $(\overline{\Gamma}_i^\lambda)^{-1} \circ \overline{\Gamma}_{i+1}^\lambda$ become arbitrarily close to being vertical, by Proposition 7.5. Thus L^λ cannot be vertical for large λ. For the rest of the proof we fix λ to be so large that (C1), (C2) hold, and drop the explicit reference to it.

8.2 The inductive step

It remains to show that $\overline{\Gamma}$ is simple. To this end recall the definition of weakly monotone from Section 5, and observe that:

Lemma 8.1 *For $0 \le i \le k-1$, if $\overline{\Gamma}_i$ is weakly monotone, then $\overline{\Gamma}'_{i+1}$ is simple.*

Proof By Lemma 6.8, $\overline{\Gamma}'_{i+1}$ bounds an immersed disk. So, by Proposition 5.1, it suffices to show that $\overline{\Gamma}'_{i+1}$ admits a decomposition into a pair of weakly monotone curves. Indeed, $\overline{\Gamma}'_{i+1} = \overline{\ell_0 j_i} \bullet \overline{j_i \ell_0}$; see Figure 15. Note that $\overline{\ell_0 j_i}$ is weakly monotone, because it is a subpath of $\overline{\Gamma}_i$. To show that $\overline{j_i \ell_0}$ is also weakly monotone, via (C1), it suffices to check that $j_i \ell_0$ is monotone. This is the case, since $j_i \ell_0 = j_i \ell_{i+1} \bullet \ell_{i+1} \ell_0 = j_i \ell_{i+1} \bullet \beta'_{i+1}$, and $j_i \ell_{i+1}$, β'_{i+1} are both positively monotone. □

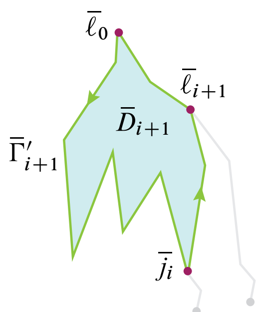

Figure 15

Now recall that $\Gamma'_k = \Gamma$ by (8). Thus, by Lemma 8.1, to complete the proof of Theorem 1.1 it suffices to show that $\overline{\Gamma}_{k-1}$ is weakly monotone. By (C1), $\overline{\Gamma}_0$ is monotone, since $\Gamma_0 = \beta_0$ is monotone. So it remains to show:

Lemma 8.2 *For $0 \le i \le k-2$, if $\overline{\Gamma}_i$ is weakly monotone, then so is $\overline{\Gamma}_{i+1}$.*

To establish this lemma, let a be a point on the y–axis which lies above all paths $\overline{\Gamma}_i$, $\overline{\Gamma}'_i$. Further let $\overline{r}_i$ be the final point of $\overline{\Gamma}_i$ and b_i be a point with the same x–coordinate as $\overline{r}_i$ which lies below all paths $\overline{\Gamma}_j$, $\overline{\Gamma}'_j$. We may also assume that all b_i have the same height. Now set

$$\widehat{\Gamma}_i := a\overline{\ell}_0 \bullet \overline{\Gamma}_i \bullet \overline{r}_i b_i.$$

Then $\overline{\Gamma}_i$ is weakly monotone if and only if $\widehat{\Gamma}_i$ is simple. Thus to prove Lemma 8.2, we need to show that $\widehat{\Gamma}_{i+1}$ is simple, if $\widehat{\Gamma}_i$ is simple. To this end note that $\widehat{\Gamma}_{i+1} = a\overline{\ell}_{i+1} \bullet \overline{\ell}_{i+1} b_{i+1}$. Thus it suffices to check that:

(I) $a\overline{\ell}_{i+1}$ and $\overline{\ell}_{i+1} b_{i+1}$ are each simple.

(II) $a\overline{\ell}_{i+1} \cap \overline{\ell}_{i+1} b_{i+1} = \{\overline{\ell}_{i+1}\}$.

8.3 Proof of the inductive step

It remains to establish items (I) and (II) above subject to the assumption that $\widehat{\Gamma}_i$ is simple, or $\overline{\Gamma}_i$ is weakly monotone, in which case $\overline{\Gamma}'_{i+1}$ is also simple by Lemma 8.1.

8.3.1 Verifying (I)

First we check that $\overline{\ell}_{i+1} b_{i+1}$ is simple. Note that

$$\overline{\ell}_{i+1} b_{i+1} = \overline{\ell}_{i+1}\overline{r}_{i+1} \bullet \overline{r}_{i+1} b_{i+1} = \overline{\ell_{i+1} r} \bullet \overline{r}_{i+1} b_{i+1};$$

see the right diagram in Figure 16. Recall that $\overline{r}_{i+1} b_{i+1}$ is negatively monotone by the definition of b_{i+1}. Further, by (C1), $\overline{\ell_{i+1} r}$ is negatively monotone as well, since $\ell_{i+1} r$ is negatively monotone by the definition of Γ_{i+1}. So $\overline{\ell}_{i+1} b_{i+1}$ is monotone and therefore simple.

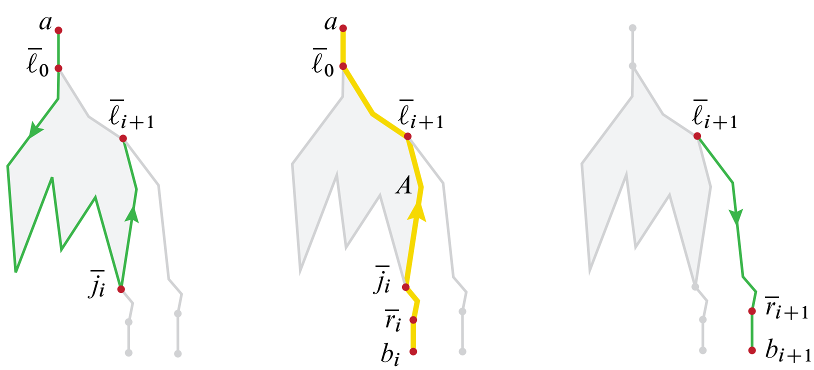

Figure 16

Next we establish the simplicity of $a\overline{\ell}_{i+1}$. Note that $a\overline{\ell}_{i+1} = a\overline{j}_i \bullet \overline{j}_i\overline{\ell}_{i+1}$, and $a\overline{j}_i$ is simple because it is a subpath of $\widehat{\Gamma}_i$; see the left diagram in Figure 16. Furthermore, $\overline{j}_i\overline{\ell}_{i+1}$ is simple as well, because it is a subpath of $\overline{\Gamma}'_{i+1}$. It remains to check that

$$a\overline{j}_i \cap \overline{j}_i\overline{\ell}_{i+1} = \{\overline{j}_i\}.$$

To see this note that $a\overline{j}_i = a\overline{\ell}_0 \bullet \overline{\ell}_0\overline{j}_i$. Thus it suffices to show that

$$\overline{\ell}_0\overline{j}_i \cap \overline{j}_i\overline{\ell}_{i+1} = \{\overline{j}_i\} \quad \text{and} \quad a\overline{\ell}_0 \cap \overline{j}_i\overline{\ell}_{i+1} = \varnothing.$$

The first equality holds because $\overline{\ell}_0\overline{j}_i$ and $\overline{j}_i\overline{\ell}_{i+1}$ are both subpaths of $\overline{\Gamma}'_{i+1}$. To see the second equality note that $\overline{j}_i\overline{\ell}_{i+1} = \overline{j_i\ell_{i+1}}$ is positively monotone by (C1) while $a\overline{\ell}_0$ is negatively monotone by definition of a. So it suffices to check that $\overline{\ell}_{i+1}$ lies below $\overline{\ell}_0$. This is the case because $\overline{\ell}_{i+1}\overline{\ell}_0 = \overline{\ell_{i+1}\ell_0} = \overline{\beta}'_{i+1}$, and β'_{i+1} is positively monotone. Thus $\overline{\ell}_{i+1}\overline{\ell}_0$ is positively monotone by (C1).

8.3.2 Verifying (II) Let

$$A := b_i\overline{r}_i \bullet (\overline{r_i\, j_i})_{\overline{\Gamma}_i^{-1}} \bullet (\overline{j_i\ell_0})_{\overline{\Gamma}'_{i+1}} \bullet \overline{\ell}_0 a;$$

see the middle diagram in Figure 16. Since each of the paths in this composition is positively monotone, A is simple. Let $S \subset \mathbb{R}^2$ be the slab contained between the horizontal line passing through a and the horizontal line on which all b_i lie. Then $S - A$ will have precisely two components, whose closures will be called the *sides* of A, and may be distinguished as the left and the right side in the obvious way. To establish claim (II) above it suffices to show:

(i) $a\overline{\ell}_{i+1}$ lies to the left of A.

(ii) One point of $\overline{\ell}_{i+1}b_{i+1}$ lies strictly to the right of A.

(iii) $\overline{\ell}_{i+1}b_{i+1} \cap A = \{\overline{\ell}_{i+1}\}$.

Indeed, (ii) and (iii) show that all of $\overline{\ell}_{i+1}b_{i+1}$ lies to the right of A, because $\overline{\ell}_{i+1}b_{i+1}$ is connected and lies in the slab S. This together with (i) show that $a\overline{\ell}_{i+1}$ and $\overline{\ell}_{i+1}b_{i+1}$ may intersect only along A, and then (iii) ensures that the intersection is $\overline{\ell}_{i+1}$. It remains to establish each of the three items listed above:

(i) We have $a\overline{\ell}_{i+1} = a\overline{\ell}_0 \bullet \overline{\ell}_0\overline{j}_i \bullet \overline{j}_i\overline{\ell}_{i+1}$. Note that $a\overline{\ell}_0$ and $\overline{j}_i\overline{\ell}_{i+1}$ lie on A. Thus it remains to check that $\overline{\ell}_0\overline{j}_i$ lies to the left of A. We have

$$A = b_i\overline{j}_i \bullet \overline{j}_i\overline{\ell}_0 \bullet \overline{\ell}_0 a.$$

Note that $\overline{\ell}_0\overline{j}_i$ meets $\overline{\ell}_0 a$ and $b_i\overline{j}_i$ only at its endpoints, since all these paths lie on $\widehat{\Gamma}_i$. Further $\overline{\ell}_0\overline{j}_i$ meets $\overline{j}_i\overline{\ell}_0$ again only at its endpoints, since these paths lie on $\overline{\Gamma}'_{i+1}$. So A meets $\overline{\ell}_0\overline{j}_i$ only at its endpoints. It suffices to show then that a point in the interior of $\overline{\ell}_0\overline{j}_i$ lies on the left of A. This is so, because $\overline{D}_{i+1}$ lies on the left of $\overline{\Gamma}'_{i+1}$ and the orientations of A and $\overline{\Gamma}'_{i+1}$ agree where they meet.

(ii) Near $\overline{\ell}_{i+1}$, A coincides with $\overline{\Gamma}'_{i+1}$. Let C be a circle centered at $\overline{\ell}_{i+1}$ whose radius is so small that it intersects $\overline{\Gamma}'_{i+1}$ and A only at two points; see the right diagram in Figure 17.

Then there exists a neighborhood U of $\widetilde{\ell}_{i+1}$ in D_{i+1} whose image $\overline{U}$ coincides with the left side of $\overline{\Gamma}'_{i+1}$ in C. Consider the edge $\overline{E}$ of $\overline{\ell}_{i+1}b_{i+1}$ which is adjacent to $\overline{\ell}_{i+1}$, and let E be the corresponding edge of $\widetilde{\Gamma}_{i+1}$ in ∂P_T; see the left diagram in Figure 17. We claim that there is a point of $\overline{E}$ inside C which lies strictly to the right of $\overline{\Gamma}'_{i+1}$, or is disjoint from $\overline{U}$, which is all we need. To see this recall that the unfolding $P_T \to \mathbb{R}^2$ is locally one-to-one. Thus it suffices to note that the interior of E is disjoint from D_{i+1}. Indeed, since $E \subset \partial P_T$, $E \cap D_{i+1} = E \cap D_{i+1} \cap \partial P_T = E \cap \widetilde{\ell}_0\widetilde{\ell}_{i+1} = \{\widetilde{\ell}_{i+1}\}$.

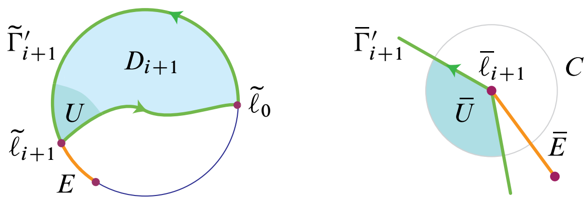

Figure 17

(iii) Since $\overline{\ell}_{i+1}b_{i+1}$ is negatively monotone, it may intersect A only along its subpath which lies below $\overline{\ell}_{i+1}$, that is $b_i\overline{\ell}_{i+1}$. So it suffices to check that $b_i\overline{\ell}_{i+1} \cap \overline{\ell}_{i+1}b_{i+1} = \{\ell_{i+1}\}$, or $b_ib_{i+1} := b_i\overline{\ell}_{i+1} \bullet \overline{\ell}_{i+1}b_{i+1}$ is simple. To see this, note that

$$b_ib_{i+1} = b_i\overline{r}_i \bullet \overline{r_ir_{i+1}} \bullet \overline{r}_{i+1}b_{i+1};$$

see Figure 18. The first and third paths in this decomposition are simple. Further, $\overline{r_ir_{i+1}} = \overline{\Gamma}_i^{-1} \circ \overline{\Gamma}_{i+1}$ which is also simple by (C2). Furthermore, again by (C2), $\overline{r_ir_{i+1}}$ lies above the line L passing through its endpoints, while $b_i\overline{r}_i$ and $\overline{r}_{i+1}b_{i+1}$ lie below L ("above" and "below" here are all well defined, since L is not vertical by (C2)). So b_ib_{i+1} is simple, as claimed.

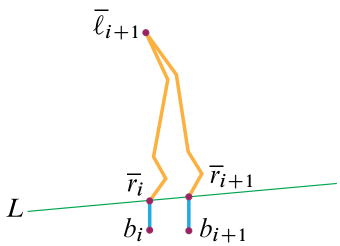

Figure 18

Appendix A: More on embeddedness of immersed disks

Here we generalize Proposition 5.1, in case it might be useful in making further progress on Dürer's problem. We say $R \subset \mathbb{R}^2$ is a *ray* emanating from p if there exists a continuous one-to-one map $r\colon [0,\infty) \to \mathbb{R}^2$ such that $r([0,\infty)) = R$, $r(0) = p$ and $\|r(t) - p\| \to \infty$ as $t \to \infty$. Also, as before, for any $X \subset D$, and mapping $f\colon D \to \mathbb{R}^2$, we set $\overline{X} := f(X)$, and say $\overline{X}$ is simple if f is one-to-one on X.

Theorem A.1 *Let $D \xrightarrow{f} \mathbb{R}^2$ be an immersion. Suppose there are $k \geq 2$ distinct points p_i, $i \in \mathbb{Z}_k$, cyclically arranged in ∂D such that $\overline{p_i p_{i+1}}$ is simple. Further suppose that there are rays $R_i \subset \mathbb{R}^2$ emanating from $\overline{p}_i$ such that:*

(i) *$R_i \cap R_{i+1} = \varnothing$.*

(ii) *$R_i \cap \overline{p_{i-1}p_{i+1}} = \{\overline{p}_i\}$.*

(iii) *There is an open neighborhood U_i of p_i in D and a point $r_i \in R_i - \{\overline{p}_i\}$ such that $\overline{U}_i \cap \overline{p}_i r_i = \{\overline{p}_i\}$.*

Then $\overline{D}$ is simple.

To prove this theorem we need a pair of lemmas, which follow from the theorem on the invariance of domain (if M and N are manifolds of the same dimension and without boundary, $U \subset M$ is open, and $f\colon U \to N$ is a one-to-one continuous map, then $f(U)$ is open in N). The first lemma also uses the fact that a simply connected manifold admits only trivial coverings.

Lemma A.2 *Let M be a compact connected surface, and $M \xrightarrow{f} \mathbb{R}^2$ be an immersion. Suppose that $\overline{\partial M}$ lies on a simple closed curve $C \subset \mathbb{R}^2$. Then $\overline{M}$ is simple.*

Proof There is a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps C to $\mathbb{S}^1$, by the theorem of Schoenflies; see Moise [15]. So we may assume that $\overline{\partial M} \subset \mathbb{S}^1$, after replacing f with $\phi \circ f$. Since M is compact, it contains a point x which maximizes $\|f\|\colon M \to \mathbb{R}$. By invariance of domain, $\overline{\mathrm{int}(M)}$ is open in $\mathbb{R}^2$. Thus it follows that $x \in \partial M$, or $\overline{x} \in \mathbb{S}^1$, which in turn implies that $\|f\| \le 1$, or $\overline{M} \subset D$. Now since $\overline{\partial M} \subset \mathbb{S}^1 = \partial D$, $f\colon M \to D$ is a local homeomorphism. To see this let U be an open neighborhood in M such that f is one-to-one on the closure $\mathrm{cl}\, U$ of U. Then $\mathrm{cl}\, U$ is homeomorphic to $\overline{\mathrm{cl}\, U}$ (any one-to-one continuous map from a compact space into a Hausdorff space is a homeomorphism onto its image). So U is homeomorphic to $\overline{U}$. Further since $\overline{(U \cap \partial M)} \subset \partial D$ it follows that $\overline{U}$ is open in D, as claimed. Now since M is compact and D is connected, f is a covering map (this is a basic topological fact; see do Carmo [3, page 375]). But D is simply connected, and M is connected; therefore, f is one-to-one. □

For every $x \in \mathbb{R}^2$ let $B_r(x)$ denote the (closed) disk of radius r centered at x. Then for any $X \subset \mathbb{R}^2$, we set $B_r(X) := \bigcup_{x \in X} B_r(x)$.

Lemma A.3 *Let $D \to \mathbb{R}^2$ be an immersion, and $A \subset \partial D$ be a closed set such that $\overline{A}$ is simple. Then for every closed connected set $X \subset \mathrm{int}(A)$ and $\epsilon > 0$, there exists a connected open neighborhood U of X in D such that $\overline{U}$ is simple and lies in $B_\epsilon(\overline{X})$. Furthermore, $\overline{U - A}$ is open, connected, and $\overline{U - A} \cap \overline{A} = \varnothing$.*

Proof Let $U := \mathrm{int}(B_\delta(X)) \cap D$; see Figure 19. We claim that if $\delta > 0$ is sufficiently small, then U is the desired set. Indeed (for small δ) $\bar{U} \subset B_\epsilon(\bar{X})$, since $D \to \mathbb{R}^2$ is continuous and X is compact. Further, since $D \to \mathbb{R}^2$ is locally one-to-one, X is compact, and $\bar{X}$ is simple, it follows that $\bar{U}$ is simple (this is a basic fact; see, eg, Spivak [21, page 345]).

Figure 19

Next note that since $X \subset \mathrm{int}(A)$, X is disjoint from $\partial D - \mathrm{int}(A)$, which is compact. Thus U will be disjoint from $\partial D - A$ as well. Consequently $U - A = U - \partial D$ which is open in $\mathbb{R}^2$. So, since $\overline{U - A}$ is simple, it follows from the invariance of domain that $\overline{U - A}$ is open, and it is connected as well since $U - A$ is connected. Finally note that if we set $V := \mathrm{int}(B_\delta(A)) \cap D$, then $\bar{V}$ will be simple, just as we had argued earlier for $\bar{U}$. So, since U, $A \subset V$, we have $\overline{(U - A)} \cap \bar{A} = \overline{(U - A) \cap A} = \varnothing$. □

Now we are ready to prove the main result of this section:

Proof of Theorem A.1 We will extend f to an immersion $\tilde{f}\colon M \to \mathbb{R}^2$ where M is a compact connected surface containing D, $\tilde{f} = f$ on D, and $\tilde{f}(\partial M)$ lies on a simple closed curve. Then $\tilde{f}$ is one-to-one by Lemma A.2, and hence so is f.

Part I: Constructions of M and $\tilde{f}$ Let $C \subset \mathbb{R}^2$ be a circle which encloses $\bar{D}$ and is disjoint from it. Then each ray R_i must intersect C at some point. Let $q_i \in R_i$ be the first such point, assuming that R_i is oriented so that $\bar{p}_i$ is its initial point; see Figure 20.

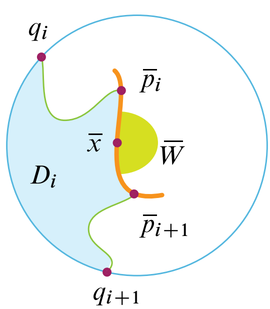

Figure 20

Now set $A_i := \overline{p}_i q_i \cup \overline{p_i p_{i+1}} \cup \overline{p}_{i+1} q_{i+1}$. Then, by conditions (i) and (ii) of the theorem, A_i is a simple curve. Consequently it divides the disk bounded by C into a pair of closed subdisks, which we call the sides of A_i. Let $x \in \operatorname{int}(p_i p_{i+1})$. By Lemma A.3, there is an open neighborhood W of x in D such that $\overline{W} - \overline{p_i p_{i+1}}$ is connected and is disjoint from $\overline{p_i p_{i+1}}$. Further, choosing ϵ sufficiently small in Lemma A.3, we can make sure that $\overline{W}$ is disjoint from $\overline{p}_i q_i$ and $\overline{p}_{i+1} q_{i+1}$. So it follows that $\overline{W} - A_i$ is connected and is disjoint from A_i. Consequently, it lies in the interior of one of the sides of A_i. Let D_i be the opposite side. Glue each D_i to D_{i+1} along $\overline{p}_{i+1} q_{i+1}$. Further, glue each D_i to D by identifying $p_i p_{i+1}$ with $\overline{p_i p_{i+1}}$ via f. This yields a compact connected surface M which contains D. Define

$$\widetilde{f}\colon M \to \mathbb{R}^2$$

by letting $\widetilde{f} = f$ on D, and $\widetilde{f}$ be the inclusion map $D_i \hookrightarrow \mathbb{R}^2$ on D_i. Then $\widetilde{f}$ is continuous and $\widetilde{f}(\partial M) \subset C$ as desired.

Part II: Local injectivity of $\widetilde{f}$ Recall that $\widetilde{f}$ is locally one-to-one on the interiors of D and each D_i by definition. Also note that $\widetilde{f}$ is one-to-one near every point of C different from q_i. So it remains to check that $\widetilde{f}$ is one-to-one near every point of $\overline{p_i p_{i+1}}$ and $\overline{p}_i q_i$. There are four cases to consider:

(i) First we check the points $\overline{x'} \in \operatorname{int}(\overline{p_i p_{i+1}})$. It suffices to show that there exists an open neighborhood W' of x' in D such that $\overline{W'} - A_i$ is disjoint from D_i (this would show that D_i and $\overline{D}$ lie on different sides of A_i near $\overline{x'}$). To see this let X be the segment xx' of A_i, and W' be a small open neighborhood of X in D given by Lemma A.3. Then, just as we had argued earlier, $\overline{W'} - A_i$ will be disjoint from A_i, and thus will lie on one side of it. Since $x \in X$, $\overline{W'} - A_i$ intersects $\overline{W} - A_i$, which by definition lies outside D_i. Thus $\overline{W'} - A_i$ also lies outside D_i, as claimed.

(ii) Next we check p_i. Let $B := B_\epsilon(\overline{p}_i)$, where $\epsilon > 0$ is so small that $\overline{p}_{i-1}$, $\overline{p}_{i+1}$ and q_i lie outside B. Let a, b, c be the first points where the (oriented curves) $\overline{p_i p_{i-1}}$, $\overline{p_i p_{i+1}}$, $\overline{p}_i q_i$ intersect ∂B respectively. Assuming ϵ is small, ab will be simple, since $\overline{\partial D}$ is locally simple. Also note that $\overline{p}_i c$ is simple, since R_i is simple. Furthermore,

$$\overline{p}_i c \cap ab \subset R_i \cap \overline{p_{i-1} p_{i+1}} = \{p_i\}$$

by the second condition of the theorem. So $ab \cup \overline{p}_i c$ divides B into 3 closed sectors; see the left diagram in Figure 21.

Let S_1 be the sector which contains a and b, S_2 be the sector which contains a and c, and S_3 be the sector which contains c and b. Next note that an open neighborhood of p_i in M consists of three components: a neighborhood V_i of p in D,

Figure 21

and neighborhoods U_i, U_{i-1}, of $\overline{p}_i$ in D_i, D_{i-1} respectively. We claim that when these neighborhoods are small, each lies in a different sector of B. Then, since $\widetilde{f}$ is one-to-one on each of these neighborhoods, it will follow that $\widetilde{f}$ is one-to-one near p_i, as desired. To establish the claim note that by the third condition of the theorem there is an open neighborhood V_i of p_i in D such that $\overline{V}_i$ is disjoint from the interior of $\overline{p}_i c$ (assuming ϵ is small). Further, we may assume that V_i is connected and is so small that $\overline{V}_i$ fits inside B. Then $\overline{V}_i$ must lie in S_1. Next, by Lemma A.3, we may choose a connected open neighborhood U_i of $\overline{p}_i$ in D_i such that $\overline{U}_i = U_i$ fits in B, and $\overline{U}_i - \overline{bc} = U_i - bc$ is connected, where $bc := \overline{p}_i b \cup \overline{p}_i c$. Note that U_i contains some interior points of $\overline{p}_i c$ and $\overline{p_i p_{i+1}}$. So $U_i - bc$ cannot lie entirely in S_1 or S_2, and therefore intersects S_3. Consequently $U_i - bc \subset S_3$, because $U_i - bc$ is connected and disjoint from the boundary of S_3. So $U_i \subset S_3$. A similar argument shows $U_{i-1} \subset S_2$.

(iii) Now we check the points $x' \in \operatorname{int}(\overline{p}_i q_i)$. Let $X \subset \operatorname{int}(\overline{p}_i q_i)$ be a connected compact set which contains x' and a point of the neighborhood U_{i-1} of $\overline{p}_i$ discussed in part (ii). Then again by Lemma A.3, there exists a connected open neighborhood W' of X in D_{i-1} such that $W' - A_i$ lies entirely on one side of A_i. By design $W' - A_i$ intersects $U_{i-1} - A_i$, which lies outside D_i as we showed in part (ii). Thus $W' - A_i$ also lies outside D_i. So D_i, D_{i-1} lie on opposite sides of A_i near x', which shows that $\widetilde{f}$ is one-to-one near x'.

(iv) It remains to check q_i. Again, we have to show that there exists an open neighborhood of q_i in D_{i-1} which lies outside D_i. The argument is similar to that of part (ii), and uses part (iii). Let $B := B_\epsilon(q_i)$, where $\epsilon > 0$ is so small that B intersects C in precisely two points and $\overline{p}_i$ lies outside B; see the right diagram in Figure 21. Then the segment of C in B together with the smallest segment of $q_i \overline{p}_i$ in B determine three sectors. Only two of these sectors border both C and a neighborhood of q_i in $q_i \overline{p}_i$, and these are where D_i and D_{i-1} lie near q_i. We have to show that, near q_i, D_i and D_{i-1} lie in different sectors. To this end it suffices to note that every open neighborhood of q_i in D_{i-1}, given by Lemma A.3, intersects a neighborhood of the type W' discussed in part (iii), which lies outside D_i. □

Appendix B: Index of symbols

Symbol	Principal use	Section
P	a convex polyhedron	1
T	a cut tree of P	1
P_T	the compact disk obtained by cutting P along T	4
h	the height function	2.1
λ	the stretching factor	1
$\pi\colon P_T \to P$	the natural projection	4
$\overline{P}_T$	image of P_T under an unfolding	2.1
$\Gamma = \Gamma_T$	the tracing path of T	4
$\overline{\Gamma}$	image of Γ under a (left) development	3
$(\overline{\Gamma})_{\gamma_i}$	a mixed development of Γ based at the vertex γ_i	3
st_o	star of P at a point o,	2.3
$\widetilde{st}_{\widetilde{o}}$	star of P_T at a point $\widetilde{o}$	4
$[\gamma_0, \dots, \gamma_k]$	a path with vertices γ_i	2.2
$\bullet$	the operation for concatenation of two paths	2.2
$\circ$	the operation for composition of two paths	2.2
Γ^{-1}	inverse of a path Γ	2.2
$\angle_P(o)$	total angle of P at a point o	2.3
$\angle(a, o, b)$	(left) angle of the path $[a, o, b]$ at o	2.3
θ_i	left angles of Γ	3
θ_i'	right angles of Γ	3
v_i	vertices of Γ_T	4
$\overline{v}_i$	vertices of $\overline{\Gamma}_T$ which correspond to v_i	4
$\widetilde{v}_i$	vertices of P_T which correspond to v_i	4
ℓ_i	leaves of T as ordered by Γ_T	6
ℓ_0	the top leaf of T	2.1
r	the root of T	2.1
j_i	junctures of Γ_T	6
β_i	branches of T	6
β_i'	dual branches of T	6
Γ_i	concatenation of the subpath $\ell_0\ell_i$ of Γ_T with β_i	6
Γ_i'	concatenation of the subpath $\ell_0\ell_i$ of Γ_T with β_i'	6
$\widetilde{\Gamma}_i'$	the closed path in P_T corresponding to Γ_i'	6
D_i	the subdisk of P_T bounded by $\widetilde{\Gamma}_i'$	6
$D\Gamma$	doubling of a path Γ	7

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