Geometry & Topology 19 (2015) 61–111

G_2–instantons over asymptotically cylindrical manifolds

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A concrete model for a 7–dimensional gauge theory under special holonomy is proposed, within the paradigm of Donaldson and Thomas, over the asymptotically cylindrical G_2–manifolds provided by Kovalev's solution to a noncompact version of the Calabi conjecture.

One obtains a solution to the G_2–instanton equation from the associated Hermitian Yang–Mills problem, to which the methods of Simpson et al are applied, subject to a crucial *asymptotic stability* assumption over the "boundary at infinity".

53C07; 58J35, 53C29

Introduction

This article inaugurates a concrete path for the study of 7–dimensional gauge theory in the context of G_2–manifolds. Its core is devoted to the solution of the Hermitian Yang–Mills (HYM) problem for holomorphic bundles over a noncompact Calabi–Yau 3–fold under certain stability assumptions.

This initiative fits in the wider context of gauge theory in higher dimensions, following the seminal works of S Donaldson, R Thomas, G Tian and others. The common thread to such generalisations is the presence of a closed differential form on the base manifold M, inducing an analogous notion of anti-self-dual connections, or *instantons*, on bundles over M. In the case at hand, G_2–manifolds are 7–dimensional Riemannian manifolds with holonomy in the exceptional Lie group G_2, which translates exactly into the presence of such a closed structure. This allows one to make sense of G_2–*instantons* as the energy-minimising gauge classes of connections, solutions to the corresponding Yang–Mills equation.

While similar theories in dimensions four (see Donaldson and Kronheimer [11]) and six (see Thomas [37]) have led to the remarkable invariants associated to instanton moduli spaces, little is currently known about the 7–dimensional case. Indeed, no G_2–instanton has yet been constructed,[1] much less a Casson-type invariant rigorously defined. This

[1] In the course of submission of this article, Thomas Walpuski [38] made significant progress in obtaining instantons over the G_2–manifolds constructed by Dominic Joyce [22].

Published: 27 February 2015 DOI: 10.2140/gt.2015.19.61

is due not least to the success and attractiveness of the previous theories themselves, but partly also to the relative scarcity of working examples of G_2–manifolds; see Bryant [4], Bryant and Salamon [5] and Joyce [22].

In 2003, A Kovalev provided an original construction of compact manifolds M with holonomy G_2 [23; 24]. These are obtained by gluing two smooth asymptotically cylindrical Calabi–Yau 3–folds W' and W'', truncated sufficiently far along the noncompact end, via an additional "twisted" circle component S^1, to obtain a compact Riemannian 7–manifold $M = (W' \times S^1)\,\widetilde{\#}\,(W'' \times S^1)$ with possibly "long neck" and $\mathrm{Hol}(M) = G_2$. This opened a clear, three step path in the theory of G_2–instantons:

(1) Obtain a HYM connection over each $W^{(i)}$, which pulls back to an instanton $A^{(i)}$ over the product $W^{(i)} \times S^1$.

(2) Extend the twisted sum to bundles $\mathcal{E}^{(i)} \to W^{(i)}$, in order to glue instantons A' and A'' and obtain a G_2–instanton as $A = A'\,\widetilde{\#}\,A''$, say, over the compact base M (see Donaldson [10] and Taubes [36]).

(3) Study the moduli space of such instantons and eventually compute invariants in particular cases of interest.

The present work is devoted to the proof of Theorem 58, which guarantees the existence of HYM metrics on suitable holomorphic bundles over such a noncompact 3–fold W, and an explicit construction satisfying the relevant hypotheses is also provided, as an example, in Section 4.2. Thus it completes step (1) of the above strategy, while postponing the gluing theory for a sequel, provisionally cited as [32] and briefly outlined in Section 4.3.

Acknowledgements This paper derives from the research leading to my PhD thesis and it would not be conceivable without the guidance and support of Simon Donaldson. I am also fundamentally indebted to Richard Thomas, Alexei Kovalev and Simon Salamon for many important contributions and to the referee, for a very thorough review of the first version. This project was fully funded by the Capes Scholarship (2327-03-1), from the Ministry of Education of the Federal Republic of Brazil.

The explicit construction of asymptotically stable bundles is joint work with Marcos Jardim, under a postdoctoral grant (2009/10067-0) by Fapesp, São Paulo State Research Council. I would finally like to thank Marcos Jardim and Amar Henni for some useful last-minute suggestions.

Summary

I begin Section 1 by recalling basic properties of G_2–manifolds, then move on to adapt gauge-theoretical notions such as self-duality of 2–forms under a G_2–structure (Definition 3), topological energy bounds for the Yang–Mills functional and the relationship between Hermitian Yang–Mills (HYM) connections over a Calabi–Yau 3–fold (W, ω), satisfying $\widehat{F} := F \cdot \omega = 0$, and so-called G_2–instantons over $W \times S^1$ (Sections 1.1–1.3). Crucially, the question of finding G_2–instantons reduces to the existence of HYM metrics over W:

Proposition 8 *Given a holomorphic vector bundle $\mathcal{E} \to W$ over a Calabi–Yau 3–fold, the canonical projection $p_1\colon W \times S^1 \to W$ gives a one-to-one correspondence between HYM connections on $\mathcal{E}$ and S^1–invariant G_2–instantons on the pullback bundle $p_1^*\mathcal{E}$.*

A quick introduction to Kovalev's manifolds [23; 24] is provided in Section 1.4, featuring the statement of his noncompact Calabi–Yau–Tian theorem (Theorem 11) and a discussion of its ingredients. In a nutshell, given a suitable compact Fano 3–fold $\overline{W}$, one deletes a $K3$ surface $D \in |-K_{\overline{W}}|$ to obtain a noncompact, asymptotically cylindrical Calabi–Yau manifold $W = \overline{W} \setminus D$. Topologically we interpret $W = W_0 \cup W_\infty$ as a compact manifold with boundary W_0, with a tubular end W_∞ attached along ∂W_0 and the divisor D situated "at infinity". This sets the scene for a natural evolution problem over W, given by the "gradient flow" towards a HYM solution.

Sections 2 and 3 form the technical core of the paper. We consider the HYM problem on a holomorphic bundle $\mathcal{E} \to W$ satisfying the *asymptotic stability* assumption that the restriction $\mathcal{E}|_D$ is stable (Definition 12), which allows the existence of a *reference metric* H_0 with "finite energy" and suitable asymptotic behaviour (Definition 13). This amounts to studying a parabolic equation on the space of Hermitian metrics (see Donaldson [7; 8; 9]) over the asymptotically cylindrical base manifold (Section 2.1), and it follows a standard pattern (see Simpson [35], Guo [17] and Buttler [6]), leading to the following existence result:

Theorem 36 *Let $\mathcal{E} \to W$ be asymptotically stable, with reference metric H_0, over an asymptotically cylindrical $\mathrm{SU}(3)$–manifold W; then, for any $0 < T < \infty$, $\mathcal{E}$ admits a 1–parameter family $\{H_t\}$ of smooth Hermitian metrics solving*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\,\widehat{F}_H \\ H(0) = H_0 \end{cases} \qquad \text{on } W \times [0, T].$$

Moreover, each H_t approaches H_0 exponentially in all derivatives along the tubular end.

One begins by solving the associated Dirichlet problem on W_S, an arbitrary finite truncation of W at "length S" along the tube, first obtaining short-time solutions $H_S(t)$ (Section 2.3), then extending them for all time. Fixing arbitrary finite time, one obtains

$$H_S(t) \xrightarrow[S\to\infty]{} H(t)$$

on compact subsets of W (Section 2.4). Moreover, every metric in the 1–parameter family $H(t)$ approaches exponentially the reference metric H_0, in a suitable sense, along the cylindrical end (Section 2.5). Hence one has solved the original parabolic equation and its solution has convenient asymptotia.

From Section 3 onwards I tackle the issue of controlling $\lim_{t\to\infty} H(t)$. It is clear from the evolution problem that such a limit H, if it exists, will indeed be an HYM metric. Moreover, H has the property of C^∞–exponential decay, along cylindrical sections of fixed length, to the reference metric along the tubular end (Notation 34). The process will culminate in this article's main theorem:

Theorem 58 *In the terms of Theorem 36, the limit $H = \lim_{t\to\infty} H_t$ exists and is a smooth Hermitian Yang–Mills metric on $\mathcal{E}$, exponentially asymptotic in all derivatives to H_0 along the tubular end:*

$$\widehat{F}_H = 0, \quad H \xrightarrow[S\to\infty]{C^\infty} H_0.$$

Adapting the "determinant line norm" functionals introduced by Donaldson [7; 8], I predict in Claim 44 a time-uniform lower bound on the "energy density" $\widehat{F}$ over a finite piece down the tube of size roughly proportional to $\|H(t)\|_{C^0(W)}$ (Section 3.2). The proof of this fact is quite intricate and it is carried out in detail in Section 3.3. That, in turn, is a sufficient condition for time-uniform C^0–convergence of $H(t)$ over the whole of W, in view of a negative energy bound derived by the Chern–Weil method (Section 4.1). Such uniform bound then cascades back through the estimates behind Theorem 36 and yields a smooth solution to the HYM problem in the $t\to\infty$ limit, as stated in Theorem 58.

Finally, Section 4.2 brings the illustrative example of a setting $\mathcal{E}\to W$ satisfying the analytical assumptions of Theorem 58, based on a monad construction originally by Jardim [20] and further studied by Jardim and the author [21].

In view of Proposition 8, the Chern connection A_H of the HYM metric H obtained in Theorem 58 pulls back to a G_2–instanton over $W\times S^1$, which effectively completes step (1) of the programme outlined in the introduction.

This article is based on the author's thesis [33].

1 G_2–instantons and Kovalev's tubular construction

This section is devoted to the background language for the subsequent analytical investigation. The main references are Salamon [34] and [4; 22; 23; 24].

1.1 Background on G_2–manifolds

Ultimately we want to consider 7–dimensional Riemannian manifolds with holonomy group G_2. We adopt the conventions of [34, page 155] for the definition of G_2; denoting by $\{e^i\}_{i=1,\dots,7}$ the standard basis of $(\mathbb{R}^7)^*$, $e^{ij} := e^i \wedge e^j$ etc.

Definition 1 The *group* G_2 is the subgroup of $\mathrm{GL}(7)$ preserving the 3–form

(1) $$\varphi_0 = (e^{12} - e^{34}) \wedge e^5 + (e^{13} - e^{42}) \wedge e^6 + (e^{14} - e^{23}) \wedge e^7 + e^{567}$$

under the standard (pullback) action on $\Lambda^3(\mathbb{R}^7)^*$, ie

$$G_2 := \{g \in \mathrm{GL}(7) \mid g^*\varphi_0 = \varphi_0\}.$$

This encodes various interesting geometrical facts, which are discussed in some detail in the author's thesis [33] or this article's preprint version [31], and are thoroughly explored in the above references. In particular, φ_0 defines a Euclidean metric

(2) $$\langle a, b\rangle e^{1\dots7} = (a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0,$$

and the group G_2 has several distinctive properties [4] that we review here for later use.

Theorem 2 *The subgroup* $G_2 \subset \mathrm{SO}(7) \subset \mathrm{GL}(7)$ *is compact, connected, simple and simply connected and* $\dim(G_2) = 14$. *Moreover,* G_2 *acts irreducibly on* $\mathbb{R}^7$ *and transitively on* S^6.

Let M be an oriented simply connected smooth 7–manifold.

Definition 3 A G_2*–structure* on the 7–manifold M is a 3–form $\varphi \in \Omega^3(M)$ such that, at every point $p \in M$, $\varphi_p = f_p^*(\varphi_0)$ for some frame $f_p\colon T_pM \to \mathbb{R}^7$.

Since $G_2 \subset \mathrm{SO}(7)$ (Theorem 2), φ fixes the orientation given by some (and consequently any) such frame f and also the metric $g = g(\varphi)$ given pointwise by (2). We may refer indiscriminately to φ or g as the G_2–structure. The *torsion of* φ is the covariant derivative $\nabla\varphi$ by the induced Levi–Civita connection, and φ is *torsion-free* if $\nabla\varphi = 0$.

Definition 4 A G_2*–manifold* is a pair (M, φ) where M is a 7–manifold and φ is a torsion-free G_2–structure on M.

The following theorem (see [22, 10.1.3] and Fernández and Gray [13]) characterises the holonomy reduction on G_2–manifolds:

Theorem 5 *Let M be a 7–manifold with G_2–structure φ and associated metric g; then the following are equivalent:*

(1) $\nabla\varphi = 0$, *ie M is a G_2–manifold.*

(2) $\mathrm{Hol}(g) \subset G_2$.

(3) *Denoting by $*_\varphi$ the Hodge star from $g(\varphi)$, one has $d\varphi = 0$ and $d *_\varphi \varphi = 0$.*

Finally, on *compact* G_2–manifolds, the holonomy group happens to be *exactly* G_2 if and only if $\pi_1(M)$ is finite [22, 10.2.2]. The purpose of the "twisted" gluing in A Kovalev's construction of asymptotically cylindrical G_2–manifolds is precisely to secure this topological condition, hence strict holonomy G_2.

1.2 Yang–Mills theory in dimension 7

The G_2–structure allows for a 7–dimensional analogue of conventional Yang–Mills theory. The crucial fact is that φ_0 yields a notion of (anti-)self-duality for 2–forms, as $\Lambda^2 = \Lambda^2(\mathbb{R}^7)^*$ splits into irreducible representations.

Since $G_2 \subset \mathrm{SO}(7)$, we have $\mathfrak{g}_2 \subset \mathfrak{so}(7) \simeq \Lambda^2$ under the standard identification of 2–forms with antisymmetric matrices. Denote $\Lambda^2_- := \mathfrak{g}_2$ and Λ^2_+ its orthogonal complement in Λ^2:

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-. \tag{3}$$

Then $\dim \Lambda^2_+ = 7$, and we identify $\Lambda^2_+ \simeq \mathbb{R}^7$ as the linear span of the contractions $\alpha_i := v_i \lrcorner \varphi_0$. Indeed the G_2–action on Λ^2_+ translates to the standard one on $\mathbb{R}^7$:

$$\begin{aligned}(g.\alpha_i)(u_1, u_2) &= \alpha_i(g.u_1, g.u_2) = \varphi_0(v_i, g.u_1, g.u_2) \\ &= \varphi_0(g^{-1}.v_i, u_1, u_2) = ((g^{-1}.v_i)\lrcorner\varphi_0)(u_1, u_2).\end{aligned}$$

One checks easily that $\alpha_i \in (\mathfrak{g}_2)^\perp \subset \mathfrak{so}(7)$; moreover [4, page 541]:

Claim 6 *The space $\Lambda^2_\pm$ has the following properties:*

(1) *$\Lambda^2_\pm$ is an irreducible representation of G_2.*

(2) *$\Lambda^2_\pm$ is the $^{+2}_{-1}$–eigenspace of the G_2–equivariant linear map*

$$\begin{aligned}T\colon \Lambda^2 &\to \Lambda^2, \\ \eta &\mapsto T\eta = *(\eta \wedge \varphi_0).\end{aligned}$$

By analogy with the 4–dimensional case, we will call Λ^2_+ (resp. Λ^2_-) the space of *self-dual* or *SD* (resp. *anti-self-dual* or *ASD*) forms. Still, in the light of Claim 6, there is a convenient characterisation of the "positive" projection in (3). The 4–form dual to the G_2–structure (Definition 1) in our convention is

$$*\varphi_0 = (e^{34} - e^{12}) \wedge e^{67} + (e^{42} - e^{13}) \wedge e^{75} + (e^{23} - e^{14}) \wedge e^{56} + e^{1234} \tag{4}$$

and we consider the G_2–equivariant map (between representations of G_2)

$$L_{*\varphi_0} : \Lambda^2 \to \Lambda^6, \tag{5}$$

$$\eta \mapsto \eta \wedge *\varphi_0. \tag{6}$$

Since $\Lambda^2_\pm$ and Λ^6 are irreducible representations and $\dim \Lambda^2_+ = \dim \Lambda^6$, Schur's lemma gives the following.

Proposition 7 *The above map restricts to $\Lambda^2_\pm$ as*

$$L_{*\varphi_0}|_{\Lambda^2_+} : \Lambda^2_+ \xrightarrow{\sim} \Lambda^6 \quad \textit{and} \quad L_{*\varphi_0}|_{\Lambda^2_-} = 0.$$

Proof It only remains to check that the restriction $L_{*\varphi_0}|_{\Lambda^2_+}$ is nonzero. Using Definition 1 and (4) we find, for instance,

$$L_{*\varphi_0}\alpha_1 = (v_1 \lrcorner \varphi_0) \wedge *\varphi_0 = (e^{25} + e^{36} + e^{47}) \wedge *\varphi_0 = e^{234567}.$$

Not only does this prove the statement, but it also suggests carrying out the full inspection of the elements $L_{*\varphi_0}\alpha_i$, which yields

$$L_{*\varphi_0}\alpha_i = e^{1\dots\hat{\imath}\dots 7}$$

as a somewhat aesthetical fact. □

Hence we may think of $L_{*\varphi_0}$, the "wedge product with $*\varphi_0$", as the orthogonal projection of 2–forms into the subspace $\Lambda^2_+ \simeq \Lambda^6$.

Consider now a vector bundle $E \to M$ over a compact G_2–manifold (M, φ); the curvature F_A of some connection A conforms to the splitting (3):

$$F_A = F_A^+ \oplus F_A^-, \quad F_A^\pm \in \Omega^2_\pm(\operatorname{End} E).$$

The L^2–norm of F_A, if it is well-defined (eg if M is compact), is the *Yang–Mills functional*

$$\mathrm{YM}(A) := \|F_A\|^2 = \|F_A^+\|^2 + \|F_A^-\|^2. \tag{7}$$

It is well known that the values of $\mathrm{YM}(A)$ can be related to a certain characteristic class of the bundle E,

$$\kappa(E) := \int_M \mathrm{tr}(F_A^2) \wedge \varphi.$$

Using the property $d\varphi = 0$, a standard argument of Chern–Weil theory (see Milnor and Stasheff [25]) shows that $[\mathrm{tr}(F_A^2) \wedge \varphi]^{dR}$ is independent of A, thus the integral is indeed a topological invariant. Using the eigenspace decomposition from Claim 6 we find

$$\kappa(E) = -\int_M \langle F_A \wedge (F_A \wedge \varphi)\rangle_{\mathfrak{g}} = -(F_A, 2F_A^+ - F_A^-) = \|F_A^-\|^2 - 2\|F_A^+\|^2.$$

Comparing with (7) we get

$$\mathrm{YM}(A) = 3\|F_A^+\|^2 + \kappa(E) = \tfrac{1}{2}\big(3\|F_A^-\|^2 - \kappa(E)\big).$$

As expected, $\mathrm{YM}(A)$ attains its absolute minimum at a connection whose curvature is either SD or ASD. Moreover, since $\mathrm{YM} \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other. Fixing $\kappa(E) \geq 0$, these facts motivate our interest in the *G_2–instanton equation*

$$F_A^+ := (L_{*\varphi}|_{\Omega^2_+})^{-1}(F_A \wedge *\varphi) = 0, \tag{8}$$

where $L_{*\varphi}$ is given pointwise by $L_{*\varphi_0}$, as in (5).

1.3 Relation with 3–dimensional Hermitian Yang–Mills

We now switch, for a moment, to the complex-geometric picture and consider a holomorphic vector bundle $\mathcal{E} \to W$ over a Kähler manifold (W, ω). To every Hermitian metric H on $\mathcal{E}$ there corresponds a unique compatible (Chern) connection $A = A_H$, with $F_A \in \Omega^{1,1}(\operatorname{End}\mathcal{E})$. In this context, the *Hermitian Yang–Mills (HYM)* condition is the vanishing of the ω–trace:

$$\widehat{F}_A := F_A \cdot \omega = 0 \in \Omega^0(\operatorname{End}\mathcal{E}). \tag{9}$$

If W is a Calabi–Yau 3–fold, the Riemannian product $M = W \times S^1$ is naturally a real 7–dimensional G_2–manifold; see [4, page 564; 22, 11.1.2]. In this section, we will check the corresponding gauge-theoretic fact that HYM connections on $\mathcal{E} \to W$ pull back to G_2–instantons over the product M.

Indeed, given the Kähler form ω and holomorphic volume form Ω on W, we obtain a natural G_2–structure by

$$\varphi := \omega \wedge d\theta + \operatorname{Im}\Omega, \quad *\varphi = \tfrac{1}{2}\omega \wedge \omega - \operatorname{Re}\Omega \wedge d\theta. \tag{10}$$

Here $d\theta$ is the coordinate 1–form on S^1, and the Hodge star on M is given by the product of the Kähler metric on W and the standard flat metric on S^1.

Now, a connection A on $\mathcal{E} \to W$ pulls back to $p_1^*\mathcal{E} \to M$ via the canonical projection

$$p_1\colon W \times S^1 \to W,$$

and so do the forms ω and Ω (for simplicity I keep the same notation for objects on W and their pullbacks to M). In particular, under the isomorphism $L_{*\varphi}|_{\Omega^2_+}\colon \Omega^2_+ \xrightarrow{\sim} \Omega^6$ (Proposition 7), the SD part of curvature maps to

$$L_{*\varphi}(F_A^+) = F_A \wedge *\varphi = \tfrac{1}{2} F_A \wedge (\omega \wedge \omega - 2\operatorname{Re}\Omega \wedge d\theta). \tag{11}$$

Proposition 8 *Given a holomorphic vector bundle $\mathcal{E} \to W$ over a Calabi–Yau 3–fold, the canonical projection $p_1\colon M = W \times S^1 \to W$ gives a one-to-one correspondence between Hermitian Yang–Mills connections on $\mathcal{E}$ and S^1–invariant G_2–instantons on the pullback bundle $p_1^*\mathcal{E}$.*

Proof An HYM connection A satisfies $F_A \in \Omega^{1,1}(W)$and $\widehat{F}_A = 0$. Taking account of bidegree, the former implies $F_A \wedge \Omega = F_A \wedge \overline{\Omega} = 0$, and hence

$$F_A \wedge 2\operatorname{Re}\Omega = F_A \wedge (\Omega + \overline{\Omega}) = 0.$$

Replacing this in (11), we check that F_A^+ maps isomorphically to the origin,

$$\begin{aligned} F_A^+ &\cong \tfrac{1}{2} F_A \wedge \omega \wedge \omega \ \in \Omega^{3,3}(W) \\ &= (\mathrm{cst})\, \widehat{F}_A \otimes d\mathrm{Vol}_\omega(W) = 0, \end{aligned}$$

using the HYM condition $\widehat{F}_A = 0$ and $\omega \wedge \omega = (\mathrm{cst})/\|\omega\|^2 * \omega$. □

Thus, by solving the HYM equation over CY^3, one obtains G_2–instantons over the product $\mathrm{CY}^3 \times S^1$, which is this article's motivation. For some further discussion of G_2–manifolds of the form $\mathrm{CY}^3 \times S^1$, see Baraglia [1].

1.4 Asymptotically cylindrical Calabi–Yau 3–folds

I will give a brief account of the building blocks in the construction of compact G_2–manifolds by A Kovalev [23; 24]. These are achieved by gluing together, in an ingenious way, a pair of noncompact asymptotically cylindrical 7–manifolds of holonomy SU(3) along their tubular ends. Such components are of the form $W \times S^1$, where (W, ω) is 3–fold given by a noncompact version of the Calabi conjecture, thus they carry G_2–structures as in Section 1.3.

Definition 9 A *base manifold* for our purpose is a compact, simply connected Kähler 3–fold $(\overline{W},\overline{\omega})$ that satisfies the following:

- There is a $K3$–surface $D \in |{-K_{\overline{W}}}|$ that is simply connected, compact, $c_1(D)=0$, with holomorphically trivial normal bundle $\mathcal{N}_{D/\overline{W}}$.
- The complement $W=\overline{W}\setminus D$ has finite fundamental group $\pi_1(W)$.

One wants to think of W as a compact manifold W_0 with boundary $D\times S^1$ and a topologically cylindrical end attached there,

$$W = W_0 \cup W_\infty, \quad W_\infty \simeq (D\times S^1\times\mathbb{R}_+). \tag{12}$$

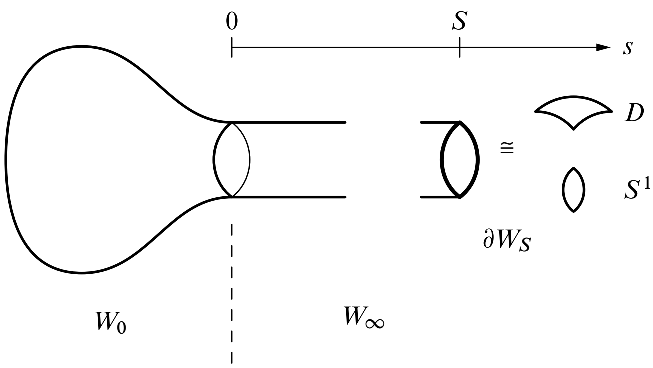

Figure 1: Asymptotically cylindrical Calabi–Yau 3–fold W

Here W_S denotes the truncation of W at "length S" on the $\mathbb{R}_+$ component.

Let $s_0 \in H^0(\overline{W},K_{\overline{W}}^{-1})$ be the defining section of the *divisor at infinity* D; then s_0 defines a holomorphic coordinate z on a neighbourhood $U\subset\overline{W}$ of D. Since $\mathcal{N}_{D/\overline{W}}$ is trivial, we may assume U is a *tubular neighbourhood of infinity*, ie

$$U \simeq D\times\{|z|<1\} \tag{13}$$

as real manifolds. Denoting $s\in\mathbb{R}_+$ and $\alpha\in S^1$, we pass to the asymptotically cylindrical picture (12) via $z=e^{-s-i\alpha}$. For later reference, let us establish a straightforward result on the decay of differential forms along W_∞.

Lemma 10 *With respect to the model cylindrical metric,*

$$|dz|, |d\bar{z}| = O(|e^{-s}|).$$

Proof In holomorphic coordinates (z,ξ^1,ξ^2) around $D\subset U$ with $z=e^{-s-i\alpha}$ and $D=\{z=0\}$, we have $|z|=|\bar z|=e^{-s}$ and

$$dz=-z(ds-i\,d\alpha),\quad d\bar z=-\bar z(ds+i\,d\alpha). \qquad \square$$

Furthermore, the holomorphic coordinate z on U is the same as a local holomorphic function τ, say. The assumptions that $\overline{W}$ is simply connected, compact and Kähler imply the vanishing in Dolbeault cohomology (see Huybrechts [19, Corollary 3.2.12, page 129]),

$$H^{0,1}(\overline{W})\oplus\overline{H^{0,1}(\overline{W})}=H^1(\overline{W},\mathbb{C})=0,$$

in which case one can solve Mittag-Leffler's problem for $1/\tau$ on $\overline{W}$ (see Griffiths and Harris [15, pages 34–35]) and τ extends to a global fibration

$$\tau\colon \overline{W}\xrightarrow{D}\mathbb{C}P^1 \tag{14}$$

with generic fibre a $K3$–surface (diffeomorphic to D) and some singular fibres. In fact, this holomorphic coordinate can be seen as pulled back from $\mathbb{C}P^1$, ie $K_{\overline{W}}^{-1}$ is the pullback of a degree-one line bundle $L\to\mathbb{C}P^1$ and $z=\tau^*s_0$ for some $s_0\in H^0(L)$ [23, Section 3].

Finally, in order to state Kovalev's noncompact version of the Calabi conjecture, notice that the $K3$ divisor D has a complex structure I inherited from $\overline{W}$; by Yau's theorem, it admits a unique Ricci-flat Kähler metric $\kappa_I\in[\bar\omega|_D]$. Ricci-flat Kähler metrics on a complex surface are hyper-Kähler (see Barth, Hulek, Peters and Van de Ven [3, pages 336–338]), which means D admits additional complex structures J and $K=IJ$ satisfying the quaternionic relations, and the metric is also Kähler with respect to any combination $aI+bJ+cK$ with $(a,b,c)\in S^2$. Let us denote their Kähler forms by κ_J and κ_K. In those terms we have the following; see [24, Theorem 2.2].

Theorem 11 (Calabi–Yau–Tian–Kovalev) *For $W=\overline{W}\setminus D$ as in Definition 9:*

1. *W admits a complete Ricci-flat Kähler metric ω.*
2. *Along the cylindrical end $D\times S^1_\alpha\times(\mathbb{R}_+)_s$, the Kähler form ω and its holomorphic volume form Ω are* exponentially asymptotic *to those of the (product) cylindrical metric induced from D:*

$$\omega_\infty=\kappa_I+ds\wedge d\alpha,\quad \Omega_\infty=(ds+i\,d\alpha)\wedge(\kappa_J+i\kappa_K). \tag{15}$$

3. $\mathrm{Hol}(\omega)=\mathrm{SU}(3)$, *ie W is Calabi–Yau.*

By *exponentially asymptotic* one means precisely that the forms can be written along the tubular end as

$$\omega|_{W_\infty} = \omega_\infty + d\psi, \quad \Omega|_{W_\infty} = \Omega_\infty + d\Psi,$$

where the 1–form ψ and the 2–form Ψ are smooth and decay as $O(e^{-\lambda s})$ in all derivatives with respect to ω_∞ for any $\lambda < \min\{1, \sqrt{\lambda_1(D)}\}$, and $\lambda_1(D)$ is the first eigenvalue of the Laplacian on differential forms on D, with the metric κ_I.

2 Hermitian Yang–Mills problem

The interplay between the algebraic and differential geometry of vector bundles is one of the main aspects of gauge theory, dating back at least to Narasimhan and Seshadri's famous correspondence between stability and flatness over Riemann surfaces. In the general Kähler setting, given a bundle $\mathcal{E}$ over (W,ω), Hitchin and Kobayashi suggested a generalisation of the flatness condition for a Hermitian metric H, in terms of the natural contraction of $(1,1)$–forms, expressed by the *Hermite–Einstein (HE)* condition

$$\widehat{F}_H := F_H \cdot \omega = \lambda.\,\mathrm{Id} \in \Omega^0(\operatorname{End}\mathcal{E}), \quad \lambda \in i\mathbb{R},$$

where λ is proportional to the slope of $\mathcal{E}$. The one-to-one correspondence between indecomposable HE connections and stable holomorphic structures was then established by Donaldson over projective algebraic surfaces in [7] and extended to compact Kähler manifolds of any dimension by Uhlenbeck and Yau, with a simpler proof in the projective case again by Donaldson in [8]. The methods organised in those two articles rely on the interpretation of HE metrics as critical points of the Yang–Mills functional (henceforth *Hermitian Yang–Mills* metrics), which allows for the application of PDE techniques, inspired by Eeels and Sampson [12], to prove convergence and regularity of the associated gradient flow. That set of tools forms the theoretical backbone of the present investigation.

Several noncompact variants of the question unfold in the work of numerous authors, but two texts in particular have an immediate, pivotal relevance to my proposed problem. Guo's study of instantons over certain cylindrical 4–manifolds [17] shows that one may obtain uniform bounds on the "heat kernel" of the parabolic flow under suitable asymptotic conditions on the "bundle at infinity", hence prove regularity of solutions by bootstrapping, as in the compact case. On the other hand, Simpson explores the more general, higher-dimensional noncompact case [35], assuming an exhaustion by compact subsets and truncating "sufficiently far" with fixed Dirichlet conditions, then taking the limit in the family of solutions obtained over each compact manifold with boundary. Both insights will be effectively combined in the technical argument to follow.

2.1 The evolution equation on W

Let (W, ω) be an asymptotically cylindrical Calabi–Yau 3–fold as given by Theorem 11 and $\mathcal{E} \to W$ the restriction of a holomorphic vector bundle on $\overline{W}$ under certain stability assumptions. Our guiding thread is the perspective of obtaining a smooth Hermitian metric H on $\mathcal{E}$ satisfying the *Hermitian Yang–Mills* condition

$$\widehat{F}_H = 0 \in \Omega^0(\operatorname{End}\mathcal{E}), \tag{16}$$

which would solve the associated G_2–instanton equation on $W \times S^1$ (Proposition 8). Consider thus the following analytical problem.

Let W_S be the compact manifold (with boundary) obtained by truncating W at length S down the tubular end. On each W_S we consider the nonlinear "heat flow"

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\,\widehat{F}_H \\ H(0) = H_0 \end{cases} \quad \text{on } W_S \times [0, T[, \tag{17}$$

with smooth solution $H_S(t)$, defined for some (short) T, since (17) is parabolic. Here H_0 is a fixed metric on $\mathcal{E} \to W$ extending the pullback "near infinity" of an ASD connection over D (see Definition 13 below), and one imposes the *Dirichlet* boundary condition

$$H|_{\partial W_S} = H_0|_{\partial W_S}. \tag{18}$$

Taking suitable $t \to T$ and $S \to \infty$ limits of solutions to (17) over compact subsets W_{S_0}, we obtain a solution $H(t)$ to the evolution equation defined over W for arbitrary $T < \infty$, with two key properties:

- Each metric $H(t)$ is exponentially asymptotic in all derivatives to H_0 over typical finite cylinders along the tubular end.
- If $H(t)$ converges as $t \le T \to \infty$, then the limit is a HYM metric on $\mathcal{E}$.

Moreover, the infinite-time convergence of $H(t)$ over W can be reduced to establishing a lower "energy bound" on $\widehat{F}_{H(t)}$, over a domain down the tube of length roughly proportional to $\|H(t)\|_{C^0(W)}$. That result is eventually proved, yielding a smooth HYM solution as the $t \to \infty$ limit of the family $H(t)$.

2.2 The concept of asymptotic stability

We start with a holomorphic vector bundle $\mathcal{E}$ over the original compact Kähler 3–fold $(\overline{W},\overline{\omega})$ (see Section 1.4) and we ask that its restriction $\mathcal{E}|_D$ to the *divisor at infinity* D be (slope-)stable with respect to $[\overline{\omega}]$. As stability is an open condition, it also holds over the nearby $K3$ "slices" in the neighbourhood of infinity $U \supset D$ (see (13)), which we denote

$$D_z := D \times \{z\} \subset U.$$

So there exists $\delta > 0$ such that $\mathcal{E}|_{D_z}$ is also stable for all $|z| < \delta$. In view of this hypothesis, we will say colloquially that $\mathcal{E}|_{\overline{W}\setminus D}$, denoted simply $\mathcal{E} \to W$, is *stable at infinity* over the noncompact Calabi–Yau (W,ω) given by Theorem 11; this is consistent since $[\omega] = [\overline{\omega}|_W]$. Such an asymptotic stability assumption will be crucial in establishing the time-uniform "energy" bounds on the solution to the evolution problem (cf Lemma 45). In summary:

Definition 12 A bundle $\mathcal{E} \to W$ will be called *stable at infinity* (or *asymptotically stable*) if it is the restriction of a holomorphic vector bundle $\mathcal{E} \to \overline{W}$ such that $\mathcal{E}|_D$ is stable, hence also $\mathcal{E}|_{D_z}$ for $|z| < \delta$ (in particular, $\mathcal{E}$ is indecomposable as a direct sum).

Explicit examples satisfying Definition 12 can be obtained for instance by monad techniques, following the ADHM paradigm (see Section 4.2).

The last ingredient is a suitable metric for comparison. Fix a smooth trivialisation of $\mathcal{E}|_U$ "in the z–direction" over the neighbourhood of infinity $U \simeq D\times\{|z|<1\} \subset \overline{W}$, ie an isomorphism

$$\mathcal{E}|_U \simeq \{|z| < 1\} \times \mathcal{E}|_D.$$

Define $H_0|_D$ as the Hermitian Yang–Mills metric on $\mathcal{E}|_D$ and denote by K its pullback to $\mathcal{E}|_U$ under the above identification. For definiteness, let us fix

$$\det H_0|_D = \det K = 1. \tag{19}$$

Indeed, for the purpose of Yang–Mills theory we may always assume $\det H = 1$, since the L^2–norm of $\operatorname{tr} F_H$ is minimised independently by the harmonic representative of $[c_1(\mathcal{E})]^{dR}$.

Then, for each $0 \le |z| < \delta$, the stability assumption gives a self-adjoint element $h_z \in \operatorname{End}(\mathcal{E}|_{D_z})$ such that $H_0|_{D_z} := K.h_z$ is the HYM metric on $\mathcal{E}|_{D_z}$. Supposing, for simplicity, that $\mathcal{E}$ is an $\mathrm{SL}(n,\mathbb{C})$–bundle and fixing $\det h_z$, I claim the family h_z varies smoothly with z across the fibres. This can be read from the HYM condition;

writing $F_{0|z}$ for the curvature of the Chern connection of $H_0|_{D_z}$ and Λ_z for the contraction with the Kähler form on D_z (hence, by definition, $\widehat{F} := \Lambda_z F$), we have

$$\widehat{F_{0|z}} = 0 \Leftrightarrow P_z(h_z) := \Delta_K h_z + i(\widehat{F}_K \cdot h_z + h_z \cdot \widehat{F}_K) + 2i\Lambda_z(\bar{\partial}_K h_z \cdot h_z^{-1} \cdot \partial_K h_z) = 0,$$

where $\{P_z\}$ is a family of nonlinear partial differential operators which depends smoothly on z. Linearising [7, page 14] and using the assumption $\widehat{F_{0|0}} = 0$, we find $(\delta_z P)|_{z=0} = \Delta_0$, the Laplacian of the reference metric, which is invertible on metrics with fixed determinant. This proves the claim by the implicit function theorem; namely, we obtain a smooth bundle metric H_0 over the neighbourhood U that is "slicewise" HYM over each cross-section D_z.

Extending H_0 in any smooth way over the compact end $\overline{W} \setminus U$, we obtain a smooth Hermitian bundle metric on the whole of $\mathcal{E}$. For technical reasons, I also require H_0 to have *finite energy* (see (30) in Section 2.5),

$$\|\widehat{F}_{H_0}\|_{L^2(W,\omega)} < \infty.$$

Definition 13 A *reference metric* H_0 on the asymptotically stable bundle $\mathcal{E} \to W$ is (the restriction of) a smooth Hermitian metric on $\mathcal{E} \to \overline{W}$ such that:

- $H_0|_{D_z}$ are the corresponding HYM metrics on $\mathcal{E}|_{D_z}$, $0 \le |z| < \delta$.
- $\det H_0 \equiv 1$.
- H_0 has finite energy.

Remark 14 Denote by A_0 the Chern connection of H_0, relative to the fixed holomorphic structure of $\mathcal{E}$. Then by assumption each $A_0|_{D_z}$ is ASD [11, page 47]. In particular, $A_0|_D$ induces an elliptic deformation complex

$$\Omega^0(\mathfrak{g}) \xrightarrow{d_{A_0}} \Omega^1(\mathfrak{g}) \xrightarrow{d^+_{A_0}} \Omega^2_+(\mathfrak{g}),$$

where $\mathfrak{g} = \mathrm{Lie}(\mathcal{G}|_D)$ generates the gauge group $\mathcal{G} = \mathrm{End}\,\mathcal{E}$ over D. Thus, the requirement that $\mathcal{E}|_D$ be indecomposable imposes a constraint on cohomology,

$$\boldsymbol{H}^0_{A_0|_D} = 0,$$

as a nonzero horizontal section would otherwise split $\mathcal{E}|_D$.

Furthermore, although this will not be essential here, it is worth observing that one might want to restrict attention to *acyclic* connections [10, page 25], ie those whose gauge class $[A_0]$ is *isolated* in $\mathcal{M}_{\mathcal{E}|_D}$. In other words, such requirement would prohibit infinitesimal deformations of A_0 across gauge orbits, which translates into the vanishing of the next cohomology group,

$$\boldsymbol{H}^1_{A_0|_D} = 0.$$

This nondegeneracy will be central in the discussion of the gluing theory [32].

The underlying heuristic in our definitions is the analogy between looking for finite energy solutions to partial differential equations over a compact manifold, with fixed values on a hypersurface, say, and over a space with cylindrical ends, under exponential decay to a suitable condition at infinity.

2.3 Short-time existence of solutions and C^0–bounds

The short-time existence of a solution to our evolution equation is a standard result:

Proposition 15 *Equation* (17) *admits a smooth solution* $H_S(t)$, $t \in [0,\varepsilon[$, *for* ε *sufficiently small.*

Proof By the Kähler identities, (17) is equivalent to the parabolic equation

$$\frac{\partial h}{\partial t} = -\big\{\Delta_0 h + i(\widehat{F}_0 \cdot h + h \cdot \widehat{F}_0) + 2i\Lambda(\bar{\partial}_0 h \cdot h^{-1} \cdot \partial_0 h)\big\}, \quad h(0) = I, \quad h|_{\partial W_S} = I$$

for a positive self-adjoint endomorphism $h(t) = H_0^{-1} H_S(t)$ of the bundle, with Hermitian metric H_0. Then the claim is an instance of the general theory; see Hamilton [18, Part IV, Section 11, page 122]. □

The task of extending solutions for all time is left to Section 2.4; let us first collect some preliminary results. We begin by recalling the parabolic maximum principle:

Lemma 16 (Maximum principle) *Let* X *be a compact Riemannian manifold with boundary and suppose* $f \in C^\infty(\mathbb{R}_t^+ \times X)$ *is a nonnegative function satisfying*

$$\left(\frac{d}{dt} + \Delta\right) f_t(x) \le 0 \quad \text{for all } (t,x) \in \mathbb{R}_t^+ \times X$$

and the Dirichlet condition

$$f_t|_{\partial X} = 0.$$

Then either $\sup_X f_t$ *is a decreasing function of* t *or* $f \equiv 0$.

The crucial role of the Kähler structure in this type of problem is that it often suffices to control $\sup|\widehat{F}_H|$ in order to obtain uniform bounds on H and its derivatives, hence to take limits in one-parameter families of solutions. Let us then establish such a bound; I denote generally $\hat{e} := |\widehat{F}_H|_H^2$ and, in the immediate sequel, $\hat{e}_t := |\widehat{F}_{H_S(t)}|^2_{H_S(t)}$.

Corollary 17 *Let $\{H_S(t)\}_{0\le t<T}$ be a smooth solution to (17) on W_S. Then $\sup_{W_S}\hat{e}_t$ is nonincreasing with t; in fact, there exists $B>0$ independent of S and T, such that*

$$\sup_{W_S}\left|\hat{F}_{H_S(t)}\right|^2 \le B. \tag{20}$$

Proof Using the Weitzenböck formula [11, page 221], one finds

$$\left(\frac{d}{dt}+\Delta\right)\hat{e_t} = -\left|d^*_{H_S(t)}F_{H_S(t)}\right|^2 \le 0. \tag{21}$$

At the boundary ∂W_S, for $t>0$, the Dirichlet condition (18) means precisely that $H_S|_{\partial W_S}$ is constant, hence the evolution equation gives $\hat{e}_t|_{\partial W_S} = |H_S^{-1}\dot{H}_S|^2|_{\partial W_S} \equiv 0$. Then $B := \sup_{W_S}\hat{e}_0$. □

In order to obtain C^0–bounds and state our first convergence result, let us digress briefly into two ways of measuring metrics, which will be convenient at different stages. First, given two metrics H and K of same determinant we write

$$H = Ke^{\xi},$$

where $\xi\in\Gamma(\operatorname{End}\mathcal{E})$ is traceless and self-adjoint with respect to H and K, and define

$$\bar{\lambda}\colon \operatorname{Dom}(\xi)\subseteq W\to\mathbb{R}_{\ge 0} \tag{22}$$

to be the highest pointwise eigenvalue of ξ. This is a nonnegative Lipschitz function on W, and a transversality argument shows that it is, in fact, smooth away from a set of real codimension 3 [8, pages 240, 244]. Now, clearly

$$|H-K| \le (\mathrm{cst})|K|(e^{\bar{\lambda}}-1), \tag{23}$$

so it is enough to control $\sup\bar{\lambda}$ to get a bound on $\|H\|_{C^0}$ relatively to K. Except where otherwise stated, we will assume $K=H_0$ to be the reference metric.

Remark 18 The space of continuous (bounded) bundle metrics is *complete* with respect to the C^0–norm (see Rudin [30, Theorem 7.15]), so the uniform limit of a family of metrics is itself a well-defined metric.

Second, there is an alternative notion of "distance" [7, Definition 12], which is more natural to our evolution equation, as will become clear in the next few results:

Definition 19 For any two Hermitian metrics H, K on a complex vector bundle $\mathcal{E}$, let

$$\tau(H,K) = \operatorname{tr} H^{-1}K, \quad \sigma(H,K) = \tau(H,K)+\tau(K,H)-2\operatorname{rank}\mathcal{E}.$$

The function σ is symmetric, nonnegative (since $a+a^{-1}\ge 2$ for all $a\ge 0$) and it vanishes if and only if $H=K$.

Although σ is not strictly speaking a distance, it does provide an equivalent criterion for C^0–convergence of metrics with fixed determinant, based on the estimate

$$\sigma(H,K)=\operatorname{tr}(e^{\xi}+e^{-\xi}-2\,\mathrm{Id})\geq e^{\overline{\lambda}}+e^{-\overline{\lambda}}-2=e^{-\overline{\lambda}}(e^{\overline{\lambda}}-1)^2.$$

Remark 20 A sequence $\{H_i\}$ of metrics with fixed determinant converges to H in C^0 if and only if $\sup\sigma(H_i,H)\to 0$. The former because $\sup e^{\overline{\lambda}}\to 1$ and the latter because obviously

$$\limsup|H_i^{-1}H-I|=\limsup|H^{-1}H_i-I|=0.$$

Indeed, σ compares to $\overline{\lambda}$ by increasing functions; from the previous inequality we deduce, in particular,

$$e^{\overline{\lambda}}\leq\sigma+2. \tag{24}$$

Conversely, it is easy to see that

$$\sigma\leq 2r\,e^{(r-1)\overline{\lambda}}\quad\text{with } r:=\operatorname{rank}\mathcal{E}.$$

Furthermore, in the context of our evolution problem, σ lends itself to applications of the maximum principle [7, Proposition 13]:

Lemma 21 *If $H_1(t)$ and $H_2(t)$ are solutions of the evolution equation* (17)*, then $\sigma(t)=\sigma(H_1(t),H_2(t))$ satisfies*

$$\Big(\frac{d}{dt}+\Delta\Big)\sigma\leq 0.$$

Combining Lemmas 16 and 21, we obtain the following straightforward consequences:

Corollary 22 *Any two solutions to* (17) *on W_S, with Dirichlet boundary conditions* (18)*, which are defined for $t\in[0,T[$, coincide for all $t\in[0,T[$.*

Corollary 23 *If a smooth solution $H_S(t)$ to* (17) *on W_S, with Dirichlet boundary conditions* (18)*, is defined for $t\in[0,T[$, then*

$$H_S(t)\xrightarrow[t\to T]{C^0}H_S(T)$$

and $H_S(T)$ is continuous.

Proof The argument is analogous to [7, Corollary 15]. As discussed in Remarks 18 and 20, it suffices to show that $\sup\sigma(H_S(t),H_S(t'))\to 0$ when $t'>t\to T$. Clearly

$$f_t:=\sup_{W_S}\sigma\big(H_S(t),H_S(t+\tau)\big)$$

satisfies the (boundary) conditions of Lemma 16, so it is decreasing and

$$\sup_{W_S} \sigma\big(H_S(t), H_S(t+\tau)\big) < \sup_{W_S} \sigma\big(H_S(0), H_S(\tau)\big)$$

for all $t, \tau, \delta > 0$ such that $0 < T - \delta < t < t + \tau < T$. Taking $\delta < \varepsilon$ in Proposition 15 we ensure continuity of $H_S(t)$ at $t = 0$, so the right-hand side is arbitrarily small for all t sufficiently close to T. Hence the family $\{H_S(t)\}_{t\in[0,T[}$ is uniformly Cauchy as $t \to T$ and the limit is continuous. □

Looking back at (17), we may interpret $\widehat{F}_H$ intuitively as a velocity vector along 1–parameter families $H(t)$ in the space of Hermitian metrics. In this case, Corollary 17 suggests an absolute bound on the variation of H for finite (possibly small) time intervals $0 \le t \le T$ where solutions exist. A straightforward calculation yields

$$\begin{aligned}\dot{\sigma} = \frac{d}{dt}\sigma(H_S(t), H_0) &\le \mathrm{tr}[(H_S^{-1}\dot{H}_S)(e^{\xi} - e^{-\xi})] \\ &\le 2|\mathrm{tr}(i\widehat{F}_{H_S})(e^{\xi} - e^{-\xi})| \le (\mathrm{cst})|\widehat{F}_{H_S}|e^{\bar{\lambda}} \le \widetilde{B}e^{\bar{\lambda}}\end{aligned}$$

with $\widetilde{B} := (\mathrm{cst})\sqrt{B}$, using the evolution equation and Corollary 17. Combining with (24) and integrating, we have

$$e^{\bar{\lambda}} \le \sigma + 2 \le 2e^{\widetilde{B}T} := C_T \quad \text{for all } t \le T. \tag{25}$$

Consequently, for any fixed $S_0 > 0$, the restriction of $H_S(t)$ to W_{S_0} lies in a C^0–ball of radius $\log C_T$ about H_0 in the space of Hermitian metrics for all $S \ge S_0$ and $t \le T$. Since C_T does not depend on S, the next lemma shows that the H_S converge uniformly on compact subsets $W_{S_0} \subset W$ for any fixed interval $[0, T]$ (possibly trivial) where solutions exist for all $S > S_0$:

Lemma 24 *If there exist $C_T > 0$ and $S_0 > 0$ such that for all $S' > S \ge S_0$, the evolution equation*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\widehat{F}_H \\ H(0) = H_0,\ H|_{\partial W_S} = H_0|_{\partial W_S} \end{cases} \qquad \text{on } W_S \times [0, T],$$

admits a smooth solution H_S satisfying

$$\sigma(H_S, H_{S'})|_{W_S} \le C_T.$$

Then the H_S converge uniformly to a (continuous) family H defined on $W_{S_0} \times [0, T]$.

Proof It is of course possible to find a function $\phi\colon W \to \mathbb{R}$ such that

$$\begin{cases} \phi \equiv 0 & \text{on } W_0, \\ \phi(y, \alpha, s) = s & \text{for } s \geq 1, \\ |\Delta\phi| \leq L, \end{cases}$$

thus giving an exhaustion of W by our compact manifolds with boundary $W_S \simeq \{p \in W \mid \phi(p) \leq S\}$, $S \geq S_0$. Taking $S_0 < S < S'$, I claim that

$$\sigma\big(H_S(t)|_{W_S}, H_{S'}(t)|_{W_S}\big)(p) \leq \frac{C_T}{S}(\phi(p) + Lt) \quad \text{for all } (p,t) \in W_S \times [0, T],$$

which yields the statement, since its restriction to W_{S_0} gives

$$\sigma(H_S, H_{S'})|_{W_{S_0}} \leq \frac{C_T(S_0 + LT)}{S} \xrightarrow[S\to\infty]{} 0.$$

The inequality holds trivially at $t = 0$ and on ∂W_S by (25), hence on the whole of $W_S \times [0, T]$ by the maximum principle (Lemma 16) since

$$\Big(\frac{d}{dt} + \Delta\Big)\Big(\sigma(H_S, H_{S'}) - \frac{C_T}{S}(\phi + Lt)\Big) \leq -\frac{C_T}{S}(\Delta\phi + L) \leq 0$$

using $|\Delta\phi| \leq L$ and Lemma 21. □

This defines a 1–parameter family of *continuous* Hermitian bundle metrics over W,

$$H(t) := \lim_{S\to\infty} H_S(t), \quad t \leq T. \tag{26}$$

It remains to show that any H_S can be smoothly extended for all $t \in [0, \infty[$ and that the limit $H(t)$ is itself a smooth solution of the evolution equation on W, with satisfactory asymptotic properties along the tubular end.

2.4 Smooth solutions for all time

Following a standard procedure, we will now check that the bound (20) allows us to smoothly extend solutions H_S up to $t = T$, hence past T for all time; see [35, Corollary 6.5]. More precisely, we can exploit the features of our problem to control a Sobolev norm $\|\Delta H\|$ by $\sup|\hat{F}|$ and weaker norms of H. The fact that one still has "Gaussian" bounds on the norm of the heat kernel in our noncompact case will be central to the argument (Theorem 2, Section A.3).

Lemma 25 *Let H and K be smooth Hermitian metrics on a holomorphic bundle over a Kähler manifold with Kähler form ω. Then for any submultiplicative pointwise norm $\|\cdot\|$,*

$$\|\Delta_K H\| \leq (\mathrm{cst})\big[(\|\hat{F}_H\| + 1)\|H\| + \|\nabla_K H\|^2 \|H^{-1}\|\big], \tag{27}$$

where $\Delta_K := 2i\Lambda_\omega\overline{\partial}\partial_K$ is the Kähler Laplacian and (cst) *depends on K and $\|\cdot\|$ only.*

Proof Write $h = K^{-1}H$ and ∇_K for the Chern connection of K. Since $\nabla_K K = 0$,

$$\Delta_K H = K\Delta_K h.$$

On the other hand, the Laplacian satisfies (see [11, page 46; 7, page 15)

$$\Delta_K h = h(\widehat{F}_H - \widehat{F}_K) + i\Lambda_\omega(\overline{\partial}h \cdot h^{-1} \wedge \partial_K h)$$

so the triangular inequality and again $\nabla_K K = 0$ yield the result. □

That will be the key to the recurrence argument behind Corollary 30, establishing smoothness of H_S as $t \to T$. We will need the following technical facts, adapted from [35, Lemma 6.4].

Lemma 26 *Let $\{H_i\}_{0\le i<I}$ be a one-parameter family of Hermitian metrics on a bundle $\mathcal{E} \to X$ over a compact Kähler manifold with boundary such that:*

(1) $H_i \xrightarrow[i\to I]{C^0(X)} H_I$*, where H_I is a continuous metric.*

(2) $\sup_X |\widehat{F}_{H_i}|$ *is bounded uniformly in i.*

(3) $H_i|_{\partial X} = H_0$.

Then the family $\{H_i\}$ is bounded in $L^p_2(X)$, for all $1 \le p < \infty$, so H_I is of class C^1.

Corollary 27 *If $\{H_S(t)\}_{0\le t<T}$ is a solution of* (17) *with Dirichlet condition* (18) *on ∂W_S, then the $H_S(t)$ are bounded in $L^p_2(W_S)$ uniformly in t for all $1 \le p < \infty$, and $H_S(T)$ is of class C^1.*

Proof By Corollary 23 and (20), $\{H_S(t)\}_{0\le t<T}$ satisfies respectively (1) and (2) in Lemma 26. □

The corollary gives, in particular, a time-uniform bound on $\|F_{H_S}\|_{L^p(W_S)}$. This can actually be improved to a uniform bound on all derivatives of curvature:

Lemma 28 *F_{H_S} is bounded in $C^k(W_S)$, uniformly in $t \in [0, T]$, for each $k \ge 0$.*

Proof By induction on k.

***k* = 0** Following [7, Lemma 18], we obtain a uniform bound on $e_S(t) := |F_{H_S(t)}|^2$, using the fact that

$$\Big(\frac{d}{dt}+\Delta\Big)e_S \le (\mathrm{cst})\big((e_S)^{3/2}+e_S\big)$$

(see [7, Proposition 16, (ii)]), and consequently

$$e_S(t) \le (\mathrm{cst})\bigg(1+\int_0^t \|K_{t-\tau}\|_{L^p(W_S)}\|(e_S)^{3/2}+e_S\|_{L^q(W_S)}\bigg), \tag{28}$$

where K_t is the heat kernel associated to $\frac{d}{dt}+\Delta$ and $1/p+1/q=1$. On the complete (Theorem 11) 6–dimensional Riemannian manifold W_S, K_t satisfies (see [12, Section 9]) the diagonal condition

$$K_t(x,x) \le \frac{(\mathrm{cst})}{t^3} \quad \text{for all } x \in W_S$$

of Theorem 2 in Section A.3, which gives a "Gaussian" bound on the heat kernel. So, fixing $C > 4$ and denoting by $r(\cdot,\cdot)$ the geodesic distance, we have

$$K_t(x,y) \le \frac{(\mathrm{cst})}{t^3}\exp\Big\{-\frac{r(x,y)^2}{Ct}\Big\} \quad \text{for all } x,y \in W_S.$$

Hence, for each $x \in W_S$, we obtain the bound

$$\begin{aligned}\|K_t(x,\cdot)\|_{L^p(W_S)} &\le \frac{(\mathrm{cst})}{t^3}\bigg(\int_{W_S}\exp\{-p\frac{r(x,y)^2}{Ct}\}dy\bigg)^{1/p}\\ &\le \frac{(\mathrm{cst})}{t^3}\bigg(\int_0^\infty (\tfrac{Ct}{p})^3u^5e^{-u^2}du\bigg)^{1/p} \le \widetilde{c}_p t^{(3/p)(1-p)}.\end{aligned}$$

Now, $p < \frac{3}{2} \Leftrightarrow \frac{3}{p}(1-p) > -1$, in which case

$$\int_0^T \|K_t(x,\cdot)\|_{L^p(W_S)}dt \le c_p(T).$$

Inequality (28) gives the desired result provided $(e_S)^{3/2} \in L^q(W_S)$ for some $q > 3$; this means $F_{H_S(t)} \in L^{\widetilde{q}}(W_S)$ for some $\widetilde{q} > 9$, which is guaranteed by Corollary 27.

***k* ⇒ *k* + 1** The general recurrence step is identical to [7, Corollary 17 (ii)], using the maximum principle (Lemma 21) with boundary conditions. □

We are now in shape to put into use the Kähler setting, combining the C^k–bounds on F (hence on $\widehat{F}$) with inequality (27) via elliptic regularity:

Lemma 29 *For $0 \le I \le \infty$, let $\{H_i\}_{0\le i<I}$ be a one-parameter family of Hermitian metrics on a holomorphic vector bundle $\mathcal{E} \to X$ over a compact (real) $2n$–dimensional Kähler manifold with boundary such that:*

(1) *$\{H_i\}$ is bounded in $L^p_2(X)$ for all $1 \le p < \infty$, and*
$$H_i \xrightarrow[i\to I]{L^p_2(X)} H,$$
where H is a continuous metric.

(2) *$\{\widehat{F}_{H_i}\}$ is bounded in $L^p_k(X)$ for each $(p,k) \in [1,\infty[\times\mathbb{N}$.*

(3) *$\{H_i\}$ is bounded in $L^p_k(\partial X)$ for each $(p,k) \in [1,\infty[\times\mathbb{N}$.*

Then $H_i \xrightarrow[i\to I]{C^\infty(X)} H$ and H is smooth.

Proof Fixing $p > 2n$, I will prove the following statement by induction in k:

$\|H_i\|_{L^p_{k+2}(X)}$ and $\|H_i^{-1}\|_{L^p_k(X)}$ are bounded uniformly in i for all $k \ge 0$.

The first hypothesis gives step $k=0$, as well as $\|H_i^{-1}\|_{L^p_1(X)} < \infty$, since the Sobolev embedding implies H_i is in C^1. Now, assuming the statement up to step $k-1 \ge 0$ implies in particular $k \ge 1 > \frac{2n}{p}$, which authorises the multiplication $L^p_k \times L^p_{k-1} \to L^p_{k-1}$ and so

$$\begin{aligned}\|H_i^{-1}\|_{L^p_k} &= \|H_i^{-1}\|_{L^p} + \|\nabla(H_i^{-1})\|_{L^p_{k-1}}\\ &\le \|H_i^{-1}\|_{L^p} + \|H_i^{-1}\|^2_{L^p_{k-1}}\|\nabla H_i\|_{L^p_k}\\ &\le \|H_i^{-1}\|_{L^p_{k-1}}\big(1 + \|H_i^{-1}\|_{L^p_{k-1}}\|H_i\|_{L^p_{k+1}}\big),\end{aligned}$$

so $\|H_i^{-1}\|_{L^p_k}$ is bounded. On the other hand, elliptic regularity on manifolds with boundary and (27), with $K = H_0$, give

$$\begin{aligned}\|H_i\|^2_{L^p_{k+2}} &\le c_0\big(\|\Delta H_i\|^2_{L^p_k} + \|H_i\|^2_{L^p_{k+1}} + \|H_i\|^2_{L^p_{k+3/2}(\partial X)}\big)\\ &\le c_0\big[\|H_i\|^2_{L^p_{k+1}}\big(1 + (1 + \|\widehat{F}_{H_i}\|_{L^p_k} + \|H_i\|_{L^p_{k+1}}\|H_i^{-1}\|_{L^p_k})^2\big)\\ &\qquad\qquad + \|H_i\|^2_{L^p_{k+3/2}(\partial X)}\big],\end{aligned}$$

where c_0 depends on H_0 and X only, and all those terms are bounded by assumption.

Since p could be chosen arbitrarily big, the family $\{H_i\}$ is uniformly bounded in each C^r. But we know it converges to H in C^1, hence in fact it converges in C^∞ and the limit is smooth. □

Corollary 30 *Under the Dirichlet conditions* (18)*, the limit metric $H_S(T)$ is smooth.*

Proof We apply Lemma 29 to the one-parameter family $\{H_S(t)\}_{0\le t\le T}$ of Hermitian metrics on the restriction $\mathcal{E}\to W_S$ given by Lemma 24. Then Corollary 27 gives hypothesis (1), Lemma 28 gives (2) and the Dirichlet condition on ∂W_S gives (3), as H_0 is smooth. □

Since

$$H_S(t) \xrightarrow[t\to T]{C^\infty(W_S)} H_S(T),$$

the solution can be smoothly extended beyond T by short-time existence, hence for all time (see [35, Proposition 6.6]):

Proposition 31 *Given any $T>0$, the family of Hermitian metrics $H(t)$ on $\mathcal{E}\to W$ defined by* (26) *is the unique, smooth solution of the evolution equation*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\,\widehat{F}_H \\ \quad H(0) = H_0 \end{cases} \quad \text{on } W\times[0,T],$$

with $\det H = \det H_0$ *and* $\sup_W |H| < \infty$*. Furthermore,*

$$\sup_W |\widehat{F}_{H(t)}| \le B = \sup_W |\widehat{F}_{H_0}|.$$

Proof Using Lemma 29 on any compact subset $\Omega_{S_0} := W_{S_0}\times[0,T]$, the H_S are C^∞–bounded (uniformly in S). By the evolution equation,

$$\frac{\partial H_S}{\partial t} \xrightarrow[S\to\infty]{C^\infty(\Omega_{S_0})} \frac{\partial H}{\partial t}$$

so H is a solution on Ω_{S_0} satisfying the same bounds, and this is independent of the choice of S_0. The bound on $\sup_W |\widehat{F}_{H(t)}|$ is immediate from Corollary 17 and uniqueness is the statement of Corollary 22. □

2.5 Asymptotic behaviour of the solution

We have a solution $\{H(t)\}$ of the flow on W (Proposition 31), giving a Hermitian metric on $\mathcal{E}\to W$ for each $t\in[0,T]$. Let us study the asymptotic properties of $H(t)$ along the noncompact end. Set

$$\hat{e}_t = |\widehat{F}_{H(t)}|^2.$$

First of all, as a direct consequence of Lemma 10, I claim that

$$\hat{e}_0 \le B\epsilon, \quad \epsilon := \begin{cases} 1 & \text{on } W_0, \\ e^{-s} & \text{on } \partial W_s,\ s \ge 0, \end{cases} \tag{29}$$

where $B = \sup_W \hat{e}_0$ (Corollary 17). In the trivialisation $\mathcal{E}|_U \simeq \{|z| < 1\} \times \mathcal{E}|_D$ over the neighbourhood of infinity U, with coordinates (z, ξ^1, ξ^2) such that $D = \{z = 0\} \subset \overline{W}$, the curvature F_{H_0} is represented by the endomorphism-valued $(1,1)$–form

$$F_{H_0}|_{U \smallsetminus D} = F_{zz} \underbrace{dz \wedge d\bar{z}}_{O(|z|^2)} + \sum_i \Big(F_{zi} \underbrace{dz \wedge d\bar{\xi}^i}_{O(|z|)} + F_{iz} \underbrace{d\xi^i \wedge d\bar{z}}_{O(|z|)} \Big) + \sum_{i,j} F_{ij}\, d\xi^i \wedge d\bar{\xi}^j. \tag{30}$$

The terms involving dz or $d\bar{z}$ decay at least as $O(|z|)$ along the tubular end (Lemma 10), and all the coefficients of F_{H_0} are bounded, so $F_{H_0} \underset{|z|\to 0}{\longrightarrow} \sum F_{ij} d\xi^i \wedge d\xi^j$. Thus,

$$\widehat{F}_{H_0}(z, \xi^1, \xi^2) \underset{|z|\to 0}{\longrightarrow} \widehat{F}_{H_0}|_D(\xi^1, \xi^2) = 0,$$

ie $\widehat{F}_{H_0}$ decays exponentially to zero as $s \to \infty$. From (29) we now obtain the exponential decay of each $\hat{e}_t$ along the cylindrical end:

Proposition 32 *Take B and ϵ as in* (29), *then*

$$\hat{e}_t \le (Be^t)\epsilon \quad \text{on } W.$$

Proof The statement is obvious on W_0. For any $s_0, t_0 \ge 0$, take $T = S > \max\{s_0, t_0\}$, let $\Sigma_S := W_S \smallsetminus W_0$ and consider on $\Sigma_S \times [0, T]$ the comparison function $g(t,s) := Be^{t-s}$. Using the Weitzenböck formula one shows that $(\frac{d}{dt} + \Delta)\hat{e}_S \le 0$ (cf (21)), where $\hat{e}_S = |\widehat{F}_{H_S}|^2$ and H_S is a solution of our flow on W_S as in Lemma 24. For $\psi := \hat{e}_S - g$, one clearly has $(\frac{d}{dt} + \Delta)\psi \le 0$ (recall that our sign convention for the Laplacian is $\Delta \overset{\text{loc}}{=} -\sum \partial^2/\partial x_i^2$) and, by the maximum principle (Lemma 16),

$$\psi \le \max_{\partial([0,T]\times\Sigma_S)} \{\hat{e}_S - Be^{t-s}\} \le 0.$$

To see that the right-hand side is zero, there are four boundary terms to check:

s* = *S The Dirichlet condition means $\hat{e}_S(t, S) = 0$ for all $t > 0$, so $\psi(t, S) \le 0$.

***s* = 0** $\psi(t, 0) \le B(1 - e^t) \le 0$.

***t* = 0** Equation (29) gives $\psi(0, s) \le 0$.

t* = *T Again by Corollary 17 we have $\psi(T, s) \le B(1 - e^{T-s}) \le 0$.

This shows that $\hat{e}_S(t,s) \le Be^{t-s}$ on $\Sigma_S \times [0, T]$. Take $T = S \to \infty$. □

To conclude exponential C^0–convergence of $H(t)$ along the cylindrical end, recall from (25) that the constant $\widetilde{B} = (\mathrm{cst})\sqrt{B}$ is obtained from the uniform bound $\hat{e} \le B$. Now, in the context of Proposition 32, this control is improved to an exponentially decaying pointwise bound along the tube, thus we may replace $\widetilde{B}e^{1/2(T-s)}$ for $\widetilde{B}$ in that expression:

$$\sigma(H(t), H_0)|_{\partial W_S} \le 2\big(e^{\widetilde{B}Te^{1/2(T-S)}} - 1\big) = O(e^{-S}). \tag{31}$$

The next result establishes exponential decay of $H(t)$ in C^1, emulating the proof of [9, Proposition 8]. I state it in rather general terms to highlight the fact that essentially all one needs to control is the Laplacian, hence $\widehat{F}$ in view of (27).

Proposition 33 *Let V be an open set of a Riemannian manifold X, $V' \Subset V$ an interior domain and $Q \to X$ some bundle with connection ∇ and a continuous fibrewise metric. There exist constants $\varepsilon, A > 0$ such that, if a smooth section $\phi \in \Gamma(Q)$ satisfies*

(1) $\|\phi\|_{C^0(V)} \le \varepsilon$,

(2) $|\Delta\phi| \le f(|\nabla\phi|)$ *on V for some nondecreasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$,*

(2′) *assumption (2) remains valid under local rescalings, in the sense that, on every ball $B_r \subset V$, it still holds for some function $\widetilde{f}$ after the radial rescaling $\widetilde{\phi}(\widetilde{x}) := \phi(mx)$, $m > 0$,*

then

$$\|\phi\|_{C^1(V')} \le A\|\phi\|_{C^0(V)}.$$

Proof I first contend that ϕ obeys an *a priori* bound

$$|(\nabla\phi)_x| r(x) \le 1 \quad \text{for all } x \in V,$$

where $r(x)\colon V \to \mathbb{R}$ is the distance to ∂V. Since the term on the left-hand side is zero on ∂V, its supremum is attained at some $\hat{x} \in V$ (possibly not unique). Write

$$m := |(\nabla\phi)_{\hat{x}}|, \quad R = r(\hat{x})$$

and suppose, for contradiction, that $R > 1/m$. If that is the case, then we rescale the ball $B_R(\hat{x})$ by the factor m, obtaining a rescaled local section $\widetilde{\phi}$ defined in $\widetilde{B}_{mR} \supset \widetilde{B}_1$. In this picture, any point in $\widetilde{B} := \widetilde{B}_{1/2}$ is further from ∂V than $R/2$, hence, by definition of $\hat{x}$, $\|\nabla\widetilde{\phi}\|_{C^0(\widetilde{B})} \le 2$. By (2) and (2′), there exists $L > 0$ such that $\|\Delta\widetilde{\phi}\|_{C^0(\widetilde{B})} \le L$, and elliptic regularity gives

$$\|\nabla\widetilde{\phi}\|_{C^{0,\alpha}(\widetilde{B})} \le c_\alpha(L + \varepsilon) := \widetilde{c}_\alpha$$

using assumption (1). Now, the rescaled gradient at $\hat{x}$ has norm $|(\nabla\widetilde{\phi})_{\hat{x}}| = 1$ so taking $\alpha = \frac{1}{2}$ (say) in a smaller ball of radius $\rho = 1/(2\widetilde{c}_{1/2})^2$,

$$|(\nabla\widetilde{\phi})_x| \geq 1 - \widetilde{c}_{1/2}\,\rho^{1/2} \geq \tfrac{1}{2} \quad \text{for all } x \in \widetilde{B}_\rho.$$

This means $|\widetilde{\phi}|$ varies by some definite $\delta > 0$ inside $\widetilde{B}_\rho$ and we reach a contradiction choosing $\varepsilon < \delta$. So

$$|(\nabla\phi)_x| \leq \left(\inf_{\partial U} r\right)^{-1} \quad \text{for all } x \in U \Subset V$$

for some open set $U \Supset V'$. To conclude the proof, it suffices to control the L^2–norm of $\nabla\phi$ on U as

$$\begin{aligned}(\mathrm{cst})\|\nabla\phi\|^2_{C^0(V')} \leq \|\nabla\phi\|^2_{L^2(U)} &= \int_U \langle\nabla\phi, \nabla\phi\rangle = \int_U \langle\phi, \Delta\phi\rangle \\ &\leq \left[f(\|\nabla\phi\|_{C^0(U)})\right]^2 \|\phi\|^2_{L^2(V')}\end{aligned}$$

and the last term is obviously bounded by $(\mathrm{cst})\|\phi\|^2_{C^0(V)}$. □

Now let $\operatorname{End}\mathcal{E} = Q$ in Proposition 33, with a connection ∇_0 induced by H_0.

Notation 34 Given $S > r > 0$, write $\Sigma_r(S)$ for the interior of the cylinder $(W_{S+r} \setminus W_{S-r})$ of "length" $2r$. We denote the C^k*–exponential tubular limit* of an element in $C^k(\Gamma(Q))$ by

$$\phi \xrightarrow[S\to\infty]{C^k} \phi_0 \mathrel{\dot{\Leftrightarrow}} \|\phi - \phi_0\|_{C^k(\Sigma_1(S),\omega)} = O(e^{-S}).$$

For $S \geq 3$, let $V = \Sigma_3(S)$ and $V' = \Sigma_2(S)$ so that the distance of V' to ∂V is always 1. In view of (31), for whatever $\varepsilon > 0$ given by the statement, it is possible to choose $S \gg 0$ so that $\phi = (H(t) - H_0)|_{\Sigma_3(S)}$ satisfies the first condition (for arbitrary fixed t), hence also the second one by (27), with $f(x) = (\mathrm{cst})[(B+1)\varepsilon + x^2]$ and (cst) depending only on H_0 and ε. We conclude, in particular, that $H(t)$ is C^1–exponentially asymptotic to H_0 in the tubular limit,

$$H(t) \xrightarrow[S\to\infty]{C^1} H_0. \tag{32}$$

Furthermore, in our case the bound on the Laplacian (27) holds for any L^p_k–norm, given our control over all derivatives of the curvature (Lemma 28), so the argument above lends itself to the obvious iteration over shrinking tubular segments $\Sigma_{1+(1/k)}(S)$:

Corollary 35 *Let $\{H(t) \mid t \in [0,T]\}$ be the solution to the evolution equation on $\mathcal{E} \to W$ given by Proposition 31. Then*

$$H(t) \xrightarrow[S\to\infty]{C^k} H_0 \quad \textit{for all } k \in \mathbb{N}.$$

Combining existence and uniqueness of the solution for arbitrary time (Proposition 31) and C^∞–exponential decay (Corollary 35), one has the main statement:

Theorem 36 *Let $\mathcal{E} \to W$ be stable at infinity, with reference metric H_0, over an asymptotically cylindrical $\mathrm{SU}(3)$–manifold W as given by Theorem 11. Then, for any $0 < T < \infty$, $\mathcal{E}$ admits a 1–parameter family $\{H_t\}$ of smooth Hermitian metrics solving*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\,\hat{F}_H \\ \quad H(0) = H_0 \end{cases} \quad \text{on } W \times [0,T].$$

Moreover, each H_t approaches H_0 exponentially in all derivatives over tubular segments $\Sigma_1(S)$ along the noncompact end.

3 Time-uniform convergence

There is a standard way (see [7, Section 1.2]) to build a functional on the space of Hermitian bundle metrics over a compact Kähler manifold the critical points of which, if any, are precisely the Hermitian Yang–Mills metrics. This procedure is analogous to the Chern–Simons construction, in that it amounts to integrating along paths a prescribed first-order variation, expressed by a closed 1–form. I will adapt this prescription to W, restricting attention to metrics with suitable asymptotic behaviour, and to the $K3$ divisors $D_z = \tau^{-1}(z)$ along the tubular end. On one hand, the resulting functional $\mathcal{N}_W$ will illustrate the fact that our evolution equation converges to a HYM metric. On the other hand, crucially, the family $\mathcal{N}_{D_z}$ will mediate the role of stability in the time-uniform control of $\{H_t\}$ over W.

3.1 Variational formalism of the functional $\mathcal{N}$

I will set up this analogous framework in some generality at first, defining an *a priori* path-dependent functional $\mathcal{N}_W$ on a suitable set of Hermitian metrics on $\mathcal{E}$. When restricted to the specific 1–parameter family $\{H_t\}$ from our evolution equation, we will see that $\mathcal{N}_W(H_t)$ is in fact decreasing and the study of its derivative will reveal that the $t \to \infty$ limit metric, if it exists, must be HYM on $\mathcal{E}$. Let

$$\mathcal{I}_0 := \left\{ h \in \operatorname{End}\mathcal{E} \;\middle|\; h \text{ is Hermitian, } h \xrightarrow[S\to\infty]{C^\infty} 0 \right\}$$

denote the space of fibrewise Hermitian matrices which decay exponentially along the tube.

Lemma 37 *Let H be a Hermitian bundle metric and h an element of $\mathcal{I}_0$ and denote $\tau := H^{-1}h$. The curvature of the Chern connection of H varies, to first order, by*

$$F_{H+h} = F_H + \bar\partial\partial_H\tau + O(|\tau|^2).$$

Proof Set $g = H^{-1}(H+h) = 1+\tau$, so that (see [11, page 46; 7, page 15])

$$F_{H+h} = F_H + \bar\partial(g^{-1}\partial_H g).$$

Observing that $g^{-1} = 1-\tau+O(|\tau|^2)$, we expand the variation of curvature:

$$\begin{aligned}\bar\partial(g^{-1}\partial_H g) &= -(g^{-1}.\bar\partial g.g^{-1})\partial_H g + g^{-1}\bar\partial\partial_H g\\ &= -(1-\tau)\bar\partial\tau(1-\tau)\partial_H\tau + (1-\tau)\bar\partial\partial_H\tau + O(|\tau|^2)\\ &= \bar\partial\partial_H\tau + O(|\tau|^2).\end{aligned}$$

This completes the proof. □

Definition 38 Let $\mathcal{H}_0$ be the set of smooth Hermitian metrics H on $\mathcal{E}\to W$ such that

$$H \xrightarrow[S\to\infty]{C^\infty} H_0.$$

Remark 39 About the definition:

(1) The exponential decay (29) implies $H_0 \in \mathcal{H}_0$. Indeed, $\mathcal{H}_0$ is a *star domain* in the affine space $H_0 + \mathcal{I}_0$, in the sense that $H_0 + \ell(H - H_0) \in \mathcal{H}_0$ for all $(\ell, H) \in [0,1]\times\mathcal{H}_0$, with $H - H_0 \in \mathcal{I}_0$. Thus $\mathcal{H}_0$ is contractible, hence *connected* and *simply connected.*

(2) There is a well-defined notion of "infinitesimal variation" of a metric H, as an object in the "tangent space"

$$T_H\mathcal{H}_0 \simeq \mathcal{I}_0.$$

(3) We know from (29) that $\hat F_{H_0} \xrightarrow[S\to\infty]{C^0} 0$, hence Lemma 37 implies

$$\|\hat F_H\|_{L^1(W,\omega)} < \infty \quad \text{for all } H \in \mathcal{H}_0.$$

(4) Any "nearby" $H \in \mathcal{H}_0$ for which $\xi = \log H_0^{-1} H$ is well defined (ie $\| H_0^{-1} H - I \|_{C^0(W,\omega)} < 1$), is joined to H_0 by

$$\gamma\colon [0,1] \to \mathcal{H}_0, \quad \gamma(\ell) = H_0 e^{\ell \xi}. \tag{33}$$

Clearly $\ell\xi \xrightarrow[S\to\infty]{C^\infty} 0$, so $\gamma(\ell) \in \mathcal{H}_0$ for all $\ell \in [0,1]$.

(5) Given any $T > 0$, the solutions $\{H_t\}_{t\in[0,T]}$ of our flow form a path in $\mathcal{H}_0$, since $\widehat{F}_{H_t}$ decays exponentially along the tube for each t (Theorem 36).

Following [7, pages 8–11], let $\theta \in \Omega^1(\mathcal{H}_0, \Omega^{1,1}(W))$ be given by

$$\theta_H\colon T_H\mathcal{H}_0 \to \Omega^{1,1}(W), \quad \theta_H(k) = 2i \operatorname{tr}(H^{-1} k F_H). \tag{34}$$

Then we may, at first formally, write

$$(\rho_W)_H(k) = \int_W \theta_H(k) \wedge \omega^2, \tag{35}$$

which will define a smooth 1–form on any domain $H_0 \in \mathcal{U} \subset \mathcal{H}_0$ where the integral converges for all $H \in \mathcal{U}$ and all $k \in T_H\mathcal{H}_0$. The crucial fact is that ρ is identically zero precisely at the HYM metrics:

$$\begin{aligned}(\rho_W)_H = 0 &\Leftrightarrow \int_W \operatorname{tr}(H^{-1} k F_H) \wedge \omega^2 = 0 \quad \text{for all } k \in T_H\mathcal{H}_0 \\ &\Leftrightarrow \widehat{F}_H = (F_H, \omega) = 0.\end{aligned}$$

Following the analogy with Chern–Simons formalism, this suggests integrating ρ_W over a path to obtain a function having the HYM metrics as critical points. Given $H \in \mathcal{H}_0$, let $\gamma(\ell) = H_\ell$ be a path in $\mathcal{H}_0$ connecting H to the reference metric H_0, and form the evaluation of θ along γ:

$$\Phi^\gamma(\ell) := [\theta_\gamma(\dot\gamma)](\ell) = 2i \operatorname{tr}(H_\ell^{-1} \dot H_\ell F_{H_\ell}) \in \Omega^{1,1}(W). \tag{36}$$

For instance, with γ as in (33), we have $H_\ell^{-1} \dot H_\ell = \underbrace{H_\ell^{-1} H_0}_{e^{-\ell\xi}}(\frac{\partial}{\partial \ell} e^{\ell\xi}) = \xi$ and

$$(\rho_W)_{H_\ell}(\dot H_\ell) = \int_W \Phi^\gamma(\ell) \wedge \omega^2 = 2i \int_W \operatorname{tr} \xi\, F_{H_\ell} \wedge \omega^2 = 2i \int_W \operatorname{tr} \xi\, \widehat{F}_{H_\ell}\, d\mathrm{Vol}_\omega$$

is well defined near H_0, since $\xi = \log H_0^{-1} H$ is bounded and $\widehat{F}_{H_\ell}$ is integrable (Remark 39). Thus, in this setting at least, the integral is rigorously defined:

$$\mathcal{N}_W^\gamma(H) := \int_\gamma \rho_W. \tag{37}$$

There is a convenient relation between $\Phi^\gamma(\ell)$ and the rate of change of the "topological" charge density $\operatorname{tr} F^2$ along γ, which will be useful later:

Lemma 40 *Let $\{\gamma(\ell) = H_\ell\} \subset \mathcal{H}_0$ be a 1–parameter family of metrics on $\mathcal{E}$; then the evaluation Φ^γ from* (36) *satisfies*

$$-i\,\overline{\partial}\partial\Phi^\gamma(\ell) = \frac{d}{d\ell}\operatorname{tr} F^2_{H_\ell}.$$

Proof From the first variation of F (Lemma 37) and the Bianchi identity we get

$$\frac{d}{d\ell}\operatorname{tr} F^2_{H_\ell} = 2\operatorname{tr}\Big(\frac{d}{d\ell}F_{H_\ell}\Big)\wedge F_{H_\ell} = 2\operatorname{tr}\overline{\partial}\partial_{H_\ell}(H_\ell^{-1}\dot{H}_\ell)\wedge F_{H_\ell} = -i\,\overline{\partial}\partial\Phi^\gamma(\ell).$$

This completes the proof. □

By the same token, if we restrict attention to our family $\{\gamma(t) = H_t\} \subset \mathcal{H}_0$ satisfying the evolution equation

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2i\,\widehat{F}_H, \\ H(0) = H_0, \end{cases} \tag{38}$$

set $\mathcal{N}_W(H_0) = 0$ and write for short $\Phi_t := \Phi^\gamma(t)$ we obtain a real smooth function

$$\mathcal{N}_W(H_T) = \int_0^T (\rho_W)_{H_t}(\dot{H}_t)\,dt = \int_0^T \Big(\int_W \Phi_t\wedge\omega^2\Big)\,dt. \tag{39}$$

Proposition 41 *The function $\mathcal{N}_W(H_t)$ is well defined for all $t \in [0,\infty[$ and*

$$\frac{d}{dt}\mathcal{N}_W(H_t) = -\tfrac{2}{3}\|\widehat{F}_{H_t}\|^2_{L^2(W)}.$$

Proof Using the evolution equation (38) we get

$$\begin{aligned}\frac{d}{dt}\mathcal{N}_W(H_t) = (\rho_W)_{H_t}(\dot{H}_t) &= \int_W \Phi_t\wedge\omega^2 \\ &= 2\int_W \operatorname{tr}\underbrace{i\,H_t^{-1}\dot{H}_t}_{2\widehat{F}_{H_t}}\ \underbrace{F_{H_t}\wedge\omega^2}_{\frac{1}{6}\widehat{F}_{H_t}\,d\mathrm{Vol}_\omega} = \frac{2}{3}\int_W \operatorname{tr}\widehat{F}^2_{H_t}\,d\mathrm{Vol}_\omega = -\tfrac{2}{3}\|\widehat{F}_{H_t}\|^2_{L^2(W)}\end{aligned}$$

and this is finite, as $\widehat{F}_{H_t}$ decays exponentially along W (Proposition 32). □

The above proposition confirms that we are on the right track: if the $\{H_t\}$ converge to a smooth metric $H = H_\infty$ at all, then H must be HYM.

Finally, our definition of $\mathcal{N}_W$ by integration of ρ_W is *a priori* path dependent and we have briefly examined two examples, (37) and (39), which will be relevant in the ensuing analysis. Let us now eliminate the dependence, so these settings are, in fact, equivalent.

Lemma 42 *Let $H \in \mathcal{H}_0$ and $h,k \in T_H\mathcal{H}_0 \cong \mathcal{I}_0$. In the terms of* (34)*, the difference*

$$\eta_H(h,k) := \tfrac{1}{2i}\big(\theta_{H+h}(k) - \theta_H(k)\big)$$

is antisymmetric to first order, modulo $\operatorname{img}\partial + \operatorname{img}\overline{\partial}$.

Proof In the notation of Lemma 37 and setting $\sigma := hK^{-1}$, the antisymmetrisation of η_H is

$$\begin{aligned}(40)\quad \xi_H(h,k) := \eta_H(h,k) - \eta_H(k,h) = \operatorname{tr}\big((\sigma\tau - \tau\sigma)F_H + \sigma\overline{\partial}\partial_H\tau - \tau\overline{\partial}\partial_H\sigma\big)\\ + O(|\sigma||\tau|^2) + O(|\tau||\sigma|^2).\end{aligned}$$

The curvature of the Chern connection of H obeys $F_H = \overline{\partial}\partial_H + \partial_H\overline{\partial}$, so

$$\begin{aligned}\sigma\overline{\partial}\partial_H\tau &= \sigma F_H\tau - \sigma\partial_H\overline{\partial}\tau\\ &= \sigma F_H\tau - \partial_H(\sigma\overline{\partial}\tau) - \overline{\partial}(\partial_H\sigma\tau) + \tau\overline{\partial}\partial_H\sigma;\end{aligned}$$

mutatis mutandis,

$$\tau\overline{\partial}\partial_H\sigma = \tau F_H\sigma - \partial_H(\tau\overline{\partial}\sigma) - \overline{\partial}(\partial_H\tau\sigma) + \sigma\overline{\partial}\partial_H\tau.$$

Substituting these and using the cyclic property of trace in (40) we find

$$\begin{aligned}\xi_H(h,k) = \tfrac{1}{2}\operatorname{tr}(\partial_H(\tau\overline{\partial}\sigma - \sigma\overline{\partial}\tau) + \overline{\partial}(\partial_H\tau\sigma - \partial_H\sigma\tau)) + O(|\sigma||\tau|^2) + O(|\tau||\sigma|^2)\\ \in \operatorname{img}\partial + \operatorname{img}\overline{\partial} \mod O(|\sigma||\tau|^2) + O(|\tau||\sigma|^2).\end{aligned}$$

This completes the proof. □

Corollary 43 *Let $\mathcal{U} \subset \mathcal{H}_0$ be a subset where the integral defining the* 1*–form ρ_W in* (35) *converges for all $H \in \mathcal{U}$ and all $k \in T_H\mathcal{H}_0$; then $\rho_W|_{\mathcal{U}}$ is closed.*

Proof Recall that a 1–form is closed precisely when its infinitesimal variation is symmetric to first order. In view of the previous lemma, it remains to check that $\xi_H(h,k) \wedge \omega^2$ integrates to zero modulo terms of higher order:

$$\lim_{S\to\infty}\int_{W_S}\xi_H(h,k)\wedge\omega^2 = 0.$$

Taking into account bidegree and de Rham's theorem, modulo terms with decay $O(|\sigma||\tau|^2)+O(|\tau||\sigma|^2)$, one has

$$\begin{aligned}\int_{W_S}\xi_H(h,k)\wedge\omega^2 &= \frac{1}{2}\int_{\partial W_S}\operatorname{tr}\big[(\tau\bar\partial\sigma-\sigma\bar\partial\tau)+(\partial_H\tau\sigma-\partial_H\sigma\tau)\big]\wedge\omega^2\\ &= \frac{1}{2}\int_{\partial W_S}\operatorname{tr}\big[2\tau\bar\partial\sigma+2\partial_H\tau\sigma-\nabla_H(\sigma\tau)\big]\wedge\omega^2\\ &= \int_{\partial W_S}\operatorname{tr}\big(\tau\bar\partial\sigma+\partial_H\tau\sigma\big)\wedge\omega^2\xrightarrow[S\to\infty]{}0.\end{aligned}$$

On the other hand, from Theorem 11 (cf (15)),

$$\omega^2|_{\partial W_S}=\kappa_I^2+O(e^{-S}).$$

Consequently, as $S\to\infty$, the operation "$\cdot\wedge\omega^2$" annihilates all components of the 1–form $\operatorname{tr}(\tau\bar\partial\sigma+\partial_H\tau\sigma)$ except those transversal to D_z ($|z|=e^{-S}$) in $\partial W_S\simeq D_z\times S^1$:

$$\begin{aligned}\int_{\partial W_S}\operatorname{tr}(\tau.\bar\partial\sigma+\partial_H\tau\sigma)\wedge\omega^2 &= \int_{\partial W_S}\operatorname{tr}\Big(\tau\frac{\partial\sigma}{\partial\bar z}d\bar z+((\partial_H\tau)_z\sigma dz)\wedge(\kappa_I^2+O(e^{-S}))\Big)\\ &= \int_{|z|=e^{-S}}O(|z|)\wedge(\kappa_I^2+O(|z|))\xrightarrow[|z|\to0]{}0.\end{aligned}$$

Here we used that $|dz|,|d\bar z|=O(e^{-S})=O(|z|)$ (Lemma 10), while τ and σ also decay exponentially in all derivatives (Definition 38). □

Since $\mathcal{H}_0$ is simply connected, we conclude that

$$\mathcal{N}_W=\int_\gamma\rho_W \tag{41}$$

does not depend on the choice of path γ.

3.2 A lower bound on "energy density" via $\mathcal{N}_{D_z}$

It is easy to adapt this prescription to the $K3$ divisors $D_z=\tau^{-1}(z)$ along the tubular end. Such a family $\mathcal{N}_{D_z}$ will mediate the role of stability in the time-uniform control of $\{H_t\}$ over W. Still following [7, pages 8–11], by analogy with (34) and (35), define $\theta_z\in\Omega^1(\mathcal{H}_0,\Omega^{1,1}(D_z))$ by

$$(\theta_z)_H\colon T_H\mathcal{H}_0\to\Omega^{1,1}(D_z),\quad (\theta_z)_H(k)=2i\operatorname{tr}\big(H|_{D_z}^{-1}k\,F_H|_{D_z}\big), \tag{42}$$

and accordingly

$$(\rho_{D_z})_H(k)=\int_{D_z}\theta_H(k)\wedge\omega. \tag{43}$$

Setting each $\mathcal{N}_{D_z}(H_0)=0$, and choosing, at first, a curve $\gamma(\ell)=H_\ell|_{D_z}$ of Hermitian metrics on $\mathcal{E}|_{D_z}$, we write (cf (35))

$$\mathcal{N}_{D_z} := \int_\gamma \rho_{D_z} = \int_\gamma \int_{D_z} \theta \wedge \omega. \tag{44}$$

Each D_z being a compact complex surface, this definition of $\mathcal{N}_{D_z}$ is in fact path independent.

In view of Proposition 57 below, which underlies this article's main result (Theorem 58), one would like to derive, for small enough $|z|$, a time-uniform *lower* bound on the "energy density" given by the ω–trace of the restriction of curvature $F_{H_t}|_{D_z}$:

$$\widehat{F_{t|z}} := \widehat{F_{H_t}|_{D_z}} = (F_{H_t}|_{D_z}, \omega|_{D_z}).$$

Recalling that $\xi_t \in \Gamma(\operatorname{End}\mathcal{E})$ is defined by $H_t = H_0 e^{\xi_t}$ (hence is self-adjoint with respect to both metrics), write $\overline{\lambda}_t$ for its highest eigenvalue as in (22) and set

$$L_t := \sup_W \overline{\lambda}_t.$$

Claim 44 *There are constants $c, c' > 0$ independent of t and z such that for every $t \in \,]0,\infty[$, there exists an open set $A_t \subset \tau(W_\infty) \subset \mathbb{C}P^1$ of parameters (cf (14)) satisfying:*

1. *For all $z \in A_t$, $|z| < \delta$ as in Definition 12, ie $\mathcal{E}|_{D_z}$ is stable.*
2. *In the cylindrical measure μ_∞ induced on $\tau(W_\infty)$ by $ds^2 + d\alpha^2$, with $z = e^{-s+i\alpha}$ (cf (15)), if L_t is sufficiently large, then the following estimate holds:*
$$\int_{A_t} \|\widehat{F_{t|z}}\|^2_{L^2(D_z)}\, ds \wedge d\alpha \geq \frac{c}{2}\mu_\infty(A_t).$$
3. *Moreover, when $L_t \to \infty$, one has $\mu_\infty(A_t)/\sqrt{L_t} \to c'$.*

Together with a uniform upper "energy bound" on F_{H_t} over W, Claim 44 will suffice to establish the time-uniform C^0–bound on $\overline{\lambda}_t$. In the course of its proof, at last, the asymptotic stability assumption on $\mathcal{E}$ intervenes, by an instance of the following result:

Lemma 45 *Suppose $\mathcal{E}|_{D_z}$ is stable with Hermitian Yang–Mills metric $H_0|_{D_z}$. Then there exists a constant $c_z > 0$ such that*

$$\mathcal{N}_{D_z}(H_t) \geq c_z(\|\xi_t\|_{L^{4/3}(D_z)} - 1).$$

Moreover, one can choose $\delta > 0$ in Definition 12 small enough to obtain a definite infimum

$$\inf_{|z|<\delta} c_z > 0.$$

Proof The estimate for any given D_z is the content of [8, Lemma 24]. In particular, each constant c_z is determined by the Riemannian geometry of D_z, as it comes essentially from an application of Sobolev's embedding theorem, hence it varies continuously with z. As $\mathcal{E}|_D$ is itself an instance of our hypotheses, with metric $H_0|_D$ over $D = D_0$, we have in this case $c_0 > 0$. Now one can certainly pick smaller $\delta > 0$, in Definition 12, so that $|z| < \delta \Rightarrow c_z > c_0/2$, say. □

Intuitively, our argument for Claim 44 will proceed as follows: On a fixed D_z far enough along the tube, the quantity $\mathcal{N}_{D_z}(H_t)$ is controlled, in a certain sense, by the ω–traced restriction of curvature $\widehat{F_{t|z}}$ (Lemma 46, below). On the other hand, the stability assumption implies that the same $\mathcal{N}_{D_z}(H_t)$ controls above the slicewise norm $\|\xi_t\|_{L^{4/3}(D_z)}$ (Lemma 45), so $\xi_t|_{D_z}$ arbitrarily "big" would imply on $\widehat{F_{t|z}}$ being at least "somewhat big". Moreover, if this happens at some z_0 then it must still hold over a "large" set $A_t \subset \mathbb{C}P^1$ of parameters z, roughly proportional to the supremum $L_t = \|\overline{\lambda}_t\|_{C^0(D_{z_0})}$ (Claim 44) at some z_0. Adapting the archetypical Chern–Weil technique (see Section 1.2), I establish an absolute bound on the L^2–norm of F_{H_t} over W [estimate (49), below], so that the set A_t, carrying a "proportional amount of energy", cannot be too large in the measure μ_∞. Hence the supremum L_t, roughly of magnitude $\mu_\infty(A_t)$, can only grow up to a definite, time-uniform value.

I start by proving essentially "half" of Claim 44:

Lemma 46 *There is a constant $c_1 > 0$, independent of t and z, such that*

$$\mathcal{N}_{D_z}(H_t) \leq c_1 L_t \|\widehat{F_{t|z}}\|_{L^2(D_z)} \quad \text{for all } t \in]0, \infty[.$$

Proof Fixing $t > 0$ and $|z| < \delta$, simplify notation by $\xi = \xi_t$ and $\|\cdot\| = \|\cdot\|_{L^2(D_z)}$, and consider the curve

$$\gamma\colon [0,1] \to \mathcal{H}_0|_{D_z}, \quad \gamma(\ell) = H_0 e^{\ell\xi},$$

with $\gamma(1) = H_t$ and $\gamma_\ell^{-1} \cdot \dot{\gamma}_\ell = \xi$. Using the first variation of curvature (Lemma 37) we obtain $\frac{d}{d\ell} F_\ell = \overline{\partial}\partial_\ell \xi$, with $\partial_\ell := \partial_{\gamma_\ell}$ and $F_\ell := F_{\gamma(\ell)}$. Form (cf (36))

$$m(\ell) := \int_0^\ell \int_{D_z} \Phi^\gamma(\ell) \wedge \omega = 2i \int_0^\ell \left(\int_{D_z} \xi \cdot F_\ell \wedge \omega \right) d\ell$$

so that $m(1) = \mathcal{N}_{D_z}(H_t)$ and $m(0) = 0$; differentiating along γ we have

$$m'(\ell) = 2i \int_{D_z} \xi \cdot F_\ell \wedge \omega.$$

The function $m(\ell)$ is in fact convex; we have

$$m''(\ell) = 2i \int_{D_z} \operatorname{tr} \xi \cdot (\bar\partial_\ell \partial_\ell \xi) \wedge \omega = 2\|\partial_\ell \xi\|^2 = \|\nabla_\ell \xi\|^2 \geq 0$$

since ξ is real, and so $|\partial_\ell \xi|^2 = |\bar\partial_\ell \xi|^2 = \frac{1}{2}|\nabla_\ell \xi|^2$. Now, by the mean value theorem, there exists some $\ell \in [0,1]$ such that

$$\mathcal{N}_{D_z}(H_t) = m(1) = m(0) + m'(\ell) \leq m'(1) \leq 2 \int_{D_z} |\xi \cdot F_1 \wedge \omega| \leq c_1 L_t \| \widehat{F_1|_{D_z}} \|$$

using convexity and Cauchy–Schwartz, with $c_1 := 2\sup_{|z|<\delta} \sqrt{\mathrm{Vol}(D_z)}$ and the notation $F_1 = F_{H_t}$. □

Remark 47 Convexity implies that $\mathcal{N}_{D_z}(H_t) = m(1)$ is *positive* for all t, because $m'(0) = 0$ gives an absolute minimum at $\ell = 0$, so $m(1) \geq m(0) = 0$.

Finally, it can be shown that $\Delta\bar\lambda$ is (weakly) uniformly bounded (see [8, page 246]), hence the maximum principle suggests that $|\xi_t|$ cannot "decrease faster" than a certain concave parabola along the cylindrical end. With that in mind, let us assume, for the sake of argument, that the following can be made rigorous:

In the terms of Claim 44, there exists a set A_t, "proportional" to $c'\sqrt{L_t}$, such that

$$\|\xi_t\|_{L^{4/3}(D_z)} \geq c_2 L_t \quad \text{for all } t \in]0,\infty[,$$

where c_2 is independent of t and z.

Then, together with Lemmas 45 and 46, this would prove Claim 44, with

$$c = \frac{2c_2}{c_1}\Big(\inf_{|z|<\delta} c_z\Big).$$

Those are the heuristics underlying the proof in the next section.

3.3 Proof of Claim 44

Recall that one would like to establish, for small enough $|z|$, a time-uniform *lower* bound on the "energy density" given by the ω–trace of the restriction $F_{H_t}|_{D_z}$,

$$\widehat{F_{t|z}} := \widehat{F_{H_t}|_{D_z}} = (F_{H_t}|_{D_z}, \omega|_{D_z}),$$

in the *weak* sense that its L^2–norm over a cylindrical segment Σ far enough down the tubular end is bounded below by a scalar multiple of $\mathrm{Vol}\,\Sigma$. Moreover, the length of such a cylinder Σ can be assumed roughly proportional to $L_t := \sup_W \bar\lambda_t$, so that $L_t \gg 0$ implies a large "energy" contribution. This will yield the C^0–bound in Proposition 57.

The strategy here consists, on one hand, of using the weak control over the Laplacian from Lemma 50 below to show that, around the furthest point down the tube where $L_t = \max_W \overline{\lambda}_t$ is attained, the slicewise supremum of $\overline{\lambda}_t$ is always on top of a certain concave parabola P_t. On the other hand, the integral along the tube of the slicewise norms $\|\overline{\lambda}\|_{L^{4/3}(D_z)}$, which bounds below $\mathcal{N}_{D_z}(H_t)$ (Lemma 45), can be shown to be itself bounded below by those slicewise suprema, using again the weak bound on the Laplacian to apply a Harnack estimate on "balls" of a standard shape, which fill essentially "half" of the corresponding tubular volume.

Since $\mathcal{N}_{D_z}(H_t)$ is controlled above by $\widehat{F}_{H_t}$, in the sense of Lemma 46, this leads to the desired minimal "energy" contribution, "proportional" to the length (roughly $\sqrt{L_t}$) of the tubular segment underneath the parabola P_t.

3.3.1 Preliminary analysis Here I establish some basic wording and technical facts underlying the proof.

Definition 48 Given functions $f, g\colon W \to \mathbb{R}$, we denote

$$\Delta f \overset{w}{\leq} g \quad \dot{\Leftrightarrow} \quad \int_W f\Delta\varphi \leq \int_W g\varphi \quad \text{for all } \varphi \in C_c^\infty(W, \mathbb{R}^+).$$

In particular, a constant $\beta \in \mathbb{R}$ is called a *weak bound on the Laplacian of* f if $\Delta f \overset{w}{\leq} \beta$.

The following lemma is implicitly assumed in [8, page 246], so the proof may also interest careful readers of that reference.

Lemma 49 (Weak maximum principle) *Let $f\colon W \to \mathbb{R}$ be a (nonconstant) nonnegative Lipschitz function, smooth away from a singular set N_0 with* codim $N_0 \geq 3$. *If*

$$\Delta f \overset{w}{\leq} 0$$

(cf Definition 48) over a (bounded open) domain $U \subset W$, then

$$f|_U \leq \max_{\partial U} f.$$

Proof Outside of an ε–neighbourhood N_ε of the singular set N_0, the function f is smooth, so the strong maximum principle on $U \setminus N_\varepsilon$ implies, at the limit $\varepsilon \to 0$, that any interior maximum point $q \in U$ must be in N_0. Take then a (small) coordinate neighbourhood $q \in V \Subset U$ such that the gradient field ∇f "points inwards" to q almost everywhere along ∂V.

I claim one can choose test functions $\varphi \in C_c^\infty(W, \mathbb{R}^+)$, with $\operatorname{supp}(\varphi) \subset V$, such that $(\nabla f, \nabla\varphi)$ is practically nonnegative, in a suitable sense. Since f is Lipschitz and almost everywhere differentiable, $f \in L^2_{1,\mathrm{loc}}(W) \hookrightarrow L^2_1(V)$; by density, for every $\delta > 0$ there exists $\varphi_\delta \in C_c^\infty(V) \hookrightarrow C_c^\infty(W)$ such that

$$\| f - \varphi_\delta \|^2_{L^2_1(V)} < \delta.$$

Moreover, since f is (nonconstant and) nonnegative and q is a maximum, we may assume without loss that $\varphi_\delta \geq 0$, so φ_δ is indeed a test function on V. Then

$$0 \geq \int_W f \Delta\varphi_\delta = \int_V f \Delta\varphi_\delta = \lim_{\varepsilon\to 0} \int_{V\smallsetminus N_\varepsilon} f\Delta\varphi_\delta \overset{(*)}{=} \int_V (\nabla f, \nabla\varphi_\delta) > -\delta \underset{\delta\to 0}{\longrightarrow} 0,$$

which either iterates all over U to imply $f \equiv \max_{\partial U} f$ or contradicts the assumption that q is an interior point. Step $(*)$ is rigorous because N_0 has large enough codimension so that $d\mathrm{Vol}_\omega|_{\partial N_\varepsilon} = O(\varepsilon^2)$. Thus boundary terms in the integration by parts over $V \smallsetminus N_\varepsilon$ vanish when $\varepsilon \to 0$, again since f is Lipschitz; see [8, page 244]. □

Recall that $\xi_t \in \Gamma(\operatorname{End}\mathcal{E})$ is defined by $H_t = H_0 e^{\xi_t}$ (hence is self-adjoint with respect to both metrics), and write $\overline{\lambda}_t$ for its highest eigenvalue.

Lemma 50 *The Laplacian of $\overline{\lambda}$ admits a weak bound $\beta > 0$:*

$$\Delta\overline{\lambda}_t \overset{w}{\leq} \beta \quad \textit{for all } t > 0.$$

Proof We know from [8, Lemma 25] that

$$\Delta\overline{\lambda} \overset{w}{\leq} 2\big(\|\hat{F}_{H_t}\|_{H_t} + \|\hat{F}_{H_0}\|_{H_0}\big),$$

and that the right-hand side is controlled by the time-uniform bound on $\hat{F}_{H_t}$ (see Corollary 17). □

Lemma 51 *Let (D, g) be a compact Riemannian manifold, $f \in L^\infty(D, \mathbb{R}^+)$, $p > 1$ and $x > 0$; then there exists a constant $k_p = k_p(D, g) > 0$ such that*

$$\|f\|_p \geq \frac{k_p}{F^x} \|f^{1+x}\|_1$$

with $\|\cdot\|_q := \|\cdot\|_{L^q(D,g)}$, $1 < q \leq \infty$, and $F := \|f\|_\infty$.

Proof It suffices to write

$$\|f\|_p := \left(\int_D f^p d\mathrm{Vol}_g\right)^{1/p} \geq \left(\int_D f^p \left(\frac{f}{F}\right)^{xp} d\mathrm{Vol}_g\right)^{1/p} = F^{-x}\|f^{1+x}\|_p$$

then apply Hölder's inequality, finding $k_p = \mathrm{Vol}_g(D)^{(1/p)-1}$. □

3.3.2 A concave parabola as lower bound In tubular coordinates $|z| = e^{-s}$, the supremum of $\overline{\lambda}_t$ on a transversal slice along W_∞ defines a smooth function

$$\ell_t\colon \mathbb{R}^+ \to \mathbb{R}^+, \quad \ell_t(s) := \sup_{\partial W_s} \overline{\lambda}_t.$$

Moreover, for each $t > 0$, denote S_t the "furthest length" down the tube at which L_t is attained, ie $S_t = \max\{s \geq 0 \mid L_t = \ell_t(s)\}$, and set

$$I_t := [S_t, S_t + \delta_t^+] \subset \mathbb{R}^+$$

with $\delta_t^+ := \frac{1}{2}\big(\sqrt{1 + (8/\beta)L_t} - 1\big)$ and β as in Lemma 50.

Lemma 52 *For each $t > 0$, the transversal supremum ℓ_t is bounded below over I_t by the concave parabola $P_t(s - S_t) := L_t - \frac{1}{2}\beta(s - S_t)(s - S_t + 1)$, ie*

$$\ell_t(s) \geq P_t(s) \quad \textit{for all } s \in I_t.$$

Proof Fix $t > 0$, $\delta_t^+ \geq \delta > 0$ and set $J_t(\delta) :=]S_t - 1, S_t + \delta[$ (I suppress henceforth the t subscript everywhere, for clarity). The parabola takes the value $P(-1) = P(0) = L$ at the points $S - 1$ and S, and its concavity $-P_t'' = \beta$ is precisely the weak bound on $\Delta\overline{\lambda}$ (Lemma 50), so we have

$$\Delta(\overline{\lambda} - P) \overset{w}{\leq} 0$$

on $\overline{J(\delta)}$ and Lemma 49 gives

$$(\overline{\lambda} - P)|_{J(\delta)} < \max\Big\{\sup_{s=S-1}(\overline{\lambda} - P), \sup_{s=S+\delta}(\overline{\lambda} - P)\Big\}$$
$$\Rightarrow \quad 0 = \ell(S) - P(S) < \ell(S + \delta) - P(S + \delta) \quad \text{for all } 0 < \delta < \delta^+,$$

since by assumption $\ell(S - 1) \leq L = P(-1)$. Hence $\ell \geq P$ for all $s \in I$. □

Remark 53 Fixing $0 < \epsilon < 1$ and setting $\delta_{\epsilon,t}^+ := \frac{1}{2}\big(\sqrt{1 + (8/\beta)(1-\epsilon)L_t} - 1\big)$, one has

$$\ell_t(s) \geq \epsilon L_t \quad \text{for all } s \in I_{\epsilon,t} := [S_t, S_t + \delta_{\epsilon,t}^+]. \tag{45}$$

Clearly, if $L_t \to \infty$, the interval length grows quadratically as fast:

$$\frac{\delta_{\epsilon,t}^+}{\sqrt{L_t}} \to c'_\epsilon := \sqrt{\frac{2(1-\epsilon)}{\beta}}.$$

Moreover, we can use a Harnack-type estimate over transversal slices to establish the following inequality for the finite cylinder under the parabola:

Lemma 54 *Given $t > 0$ and $0 < \epsilon < 1$, let $\Sigma_t(\epsilon) := I_{\epsilon,t} \times S^1 \times D \simeq \overline{W_{S_t+\delta^+_{\epsilon,t}} \setminus W_{S_t}}$ be the finite cylinder along W, under the parabola P_t of Lemma 52, determined by the interval of length $\delta^+_{\epsilon,t}$ on which (45) holds, and suppose $2\pi\delta^+_{\epsilon,t} \in \mathbb{N}$. Then, for each $x > 0$, there exist time-uniform constants $a_{x,\epsilon}, b_{x,\epsilon} > 0$ such that if $L_t > b_{x,\epsilon}$, the following estimate holds:*

$$\int_{\Sigma_t(\epsilon)} \overline{\lambda}_t^{1+x} d\mathrm{Vol}_{\omega_\infty} \geq a_{x,\epsilon} \cdot \delta^+_{\epsilon,t} \cdot (L_t - b_{x,\epsilon})^{1+x}. \tag{46}$$

Proof Again let me suppress the t subscript for tidiness and work all along in the cylindrical metric ω_∞. For each $s \in I_\epsilon$, let $p_s \in \partial W_s \simeq S^1 \times D_s$ be a point on the corresponding transversal slice such that the maximum $\ell(s) = \overline{\lambda}(p_s)$ is attained, and form the "unit" open cylinder $B_s \subset \Sigma_\epsilon$ of length $1/2\pi$ (so that the volume integral over B_s along the $S^1 \times I_\epsilon$ directions is 1), centred on p_s, such that

$$\mathrm{Vol}\, B_s = \mathrm{Vol}(B_s \cap D_s) = \tfrac{1}{2}\,\mathrm{Vol}\, D,$$

where $\mathrm{Vol}\, D \equiv \mathrm{Vol}\, D_s$ denotes the (same) four-dimensional volume of (every) D_s.

By Lemma 50, $\overline{\lambda}$ is a weak subsolution of the elliptic problem $\Delta u = \beta$, so we may apply Moser's iteration method (see [14, Theorem 8.25] and [26, Theorem 2]) over an open set V such that $\partial W_s \subset V \subset B_s$ to obtain local boundedness of $\overline{\lambda}$ in terms of its $L^{1+x}(B_s, \omega_\infty)$–norm. Indeed we can choose V such that

$$\ell(s) \leq C_x\Big[\Big(\frac{1}{\mathrm{Vol}\, B_s}\Big)^{1/(1+x)} \|\overline{\lambda}\|_{1+x} + 1\Big]$$

for some constant $C_x = C_x(B_s, \beta, x) > 0$ which in fact is uniform in s, as all B_s are congruent by translation. Setting $b_{x,\epsilon} := C_x/\epsilon$ and using (45) we have

$$\Big[\frac{1}{b_{x,\epsilon}}(L - b_{x,\epsilon})\Big]^{1+x} \leq \frac{1}{\mathrm{Vol}\, B_s} \int_{B_s} \overline{\lambda}^{1+x}. \tag{47}$$

In particular, one can choose at most $2\pi\delta^+_\epsilon \in \mathbb{N}$ values $s_j \in I_\epsilon$ such that the corresponding B_{s_j} are necessarily disjoint, and form their union

$$B(\epsilon) := \coprod_{j=1}^{2\pi\delta^+_\epsilon} B_{s_j}.$$

Clearly $\mathrm{Vol}\, B(\epsilon) \geq \frac{1}{2}(2\pi\delta^+_\epsilon)\,\mathrm{Vol}\, D$. Now, the statement about averages (47) goes over to the disjoint union, hence

$$\int_{\Sigma(\epsilon)} \overline{\lambda}^{1+x} \geq \int_{B(\epsilon)} \overline{\lambda}^{1+x} \geq \mathrm{Vol}\, B(\epsilon) \cdot \Big[\frac{1}{b_{x,\epsilon}}(L - b_{x,\epsilon})\Big]^{1+x}$$

which proves the lemma, with $a_{x,\epsilon} := (\pi\,\mathrm{Vol}\, D)/(b_{x,\epsilon})^{1+x}$. □

3.3.3 End of proof It is now just a matter of putting together the previous results. Recalling the final remarks of Section 3.2, we know from Lemmas 45 and 46 that

$$L\|\widehat{F}\|_2 \geq k'(\|\overline{\lambda}\|_{4/3} - 1)$$

over each D_z sufficiently far down the tube for a uniform constant $k' > 0$. Choosing $0 < \epsilon < 1$ and $x > 0$, integrating over $A_\epsilon := I_\epsilon \times S^1$ and applying Lemma 51 we have

$$\int_{A_\epsilon} \|\widehat{F}\|_2\, ds \wedge d\alpha \geq \frac{k''}{L^{1+x}}\left(\int_{A_\epsilon} \|\overline{\lambda}^{1+x}\|_1 ds \wedge d\alpha\right) - k'\frac{\delta_\epsilon^+}{L},$$

where $k'' := k' \cdot k_{4/3}$ is still a uniform constant. Moreover, by Lemma 54, the integral term is bounded below by $k'' \cdot a_{x,\epsilon} \cdot \delta_\epsilon^+ \cdot (1-(b_{x,\epsilon}/L))^{1+x}$. By Hölder's inequality,

$$(\mathrm{Vol}\, A_\epsilon)^{1/2}\left(\int_{A_\epsilon} \|\widehat{F}\|_2^2\, ds \wedge d\alpha\right)^{1/2} \geq \delta_\epsilon^+\Big[k'' \cdot a_{x,\epsilon} \cdot \Big(1 - \frac{b_{x,\epsilon}}{L}\Big)^{1+x} - \frac{k'}{L}\Big].$$

Since the interval I_ϵ has length precisely δ_ϵ^+, we have $\mu_\infty(A_\epsilon) = \mathrm{Vol}\, A_\epsilon = 2\pi\delta_\epsilon^+$, so

$$\begin{aligned}\int_{A_\epsilon} \|\widehat{F}\|_2^2\, ds \wedge d\alpha &\geq \frac{\mu_\infty(A_\epsilon)}{4\pi^2}\Big[k'' \cdot a_{x,\epsilon} \cdot \Big(1 - \frac{b_{x,\epsilon}}{L}\Big)^{1+x} - \frac{k'}{L}\Big]^2 \\ &= \frac{\mu_\infty(A_\epsilon)}{2}\Big(\frac{k''\cdot a_{x,\epsilon}}{\sqrt{2}\pi}\Big)^2\Big[\Big(1 - \frac{b_{x,\epsilon}}{L}\Big)^{1+x} - \frac{k'}{k''\cdot a_{x,\epsilon}}\frac{1}{L}\Big]^2.\end{aligned}$$

Now fix eg $\epsilon = \frac{1}{2}$, $x = 1$. Since parts (2) and (3) of the claim only address regimes in which L_t is large, the $O(1/L_t)$ terms in

$$\Big[1 - (2b_{1,1/2} + \frac{k'}{k''\cdot a_{1,1/2}})\frac{1}{L_t} + (b_{1,1/2})^2\frac{1}{L_t^2}\Big]^2$$

on the right-hand side of the above inequality are negligible. Adjusting accordingly the interval length in Remark 53, we obtain the required constants as

$$c := \Big(\frac{k''\cdot a_{1,1/2}}{\sqrt{2}\pi}\Big)^2 = \frac{(k')^2(\mathrm{Vol}\, D)^{(3/2)}}{32(C_1)^4}, \quad c' := c'_{(1/2)} = 2\pi\sqrt{\frac{1}{\beta}}.$$

4 Conclusion

4.1 Solution of the Hermitian Yang–Mills problem

Let $\{H_t\}$ be the family of smooth Hermitian metrics on $\mathcal{E} \to W$ given for arbitrary finite time by Proposition 31. In order to obtain a HYM metric as $H = \lim_{t\to\infty} H_t$ it would suffice to show that $\{H_t\}$ is C^0–bounded, for then it is actually C^∞–bounded on any compact subset of W and the limit H is smooth (Lemmas 24, 26 and 29).

Concretely, this would mean improving the constant C_T in (25) to a time-uniform bound C_∞ or, what is the same, controlling the sequence $\overline{\lambda}_t$ of highest eigenvalues of $\xi_t = \log H_0^{-1} H_t$ (cf (22)):

$$\|\overline{\lambda}_t\|_{C^0(W)} \leq C_\infty \quad \text{for all } t > 0. \tag{48}$$

I will show that this task reduces essentially to Claim 44, as the problem (48) amounts in fact to controlling the size of the set A_t where the "energy density" $\widehat{F_{t|z}}$ is bigger than a definite constant. I begin by stating the announced upper bound:

Proposition 55 *A solution $\{H_t\}$ to the evolution equation given by Theorem 36 satisfies the* negative energy *condition*

$$E(t) := \int_W (|F_{H_t}|^2 - |F_{H_0}|^2)\, d\mathrm{Vol}_\omega \leq 0 \quad \textit{for all } t \in]0, \infty[. \tag{49}$$

Proof The curvature of a Chern connection splits orthogonally as $F = \widehat{F} \cdot \omega \oplus F^\perp$ in $\Omega^{1,1}(\operatorname{End} \mathcal{E})$, so $|F|^2 = |F^\perp|^2 + |\widehat{F}|^2$ (setting $|\omega| = 1$). On the other hand, the Hodge–Riemann equation (1) (see the appendix) reads

$$\operatorname{tr} F^2 \wedge \omega = \big(|F^\perp|^2 - |\widehat{F}|^2\big)\omega^3.$$

Comparing terms, we find $|F|^2\omega^3 = \operatorname{tr} F^2 \wedge \omega + 2|\widehat{F}|^2\omega^3$ (cf Section 1.2), so

$$\begin{aligned} E(t) &= \int_W \big(\operatorname{tr} F_{H_t}^2 - \operatorname{tr} F_{H_0}^2\big) \wedge \omega + 2\int_W (\hat{e}_t - \hat{e}_0)\omega^3, \\ \dot{E}(t) &\leq \int_W (-i\bar{\partial}\partial\Phi_t) \wedge \omega + 2\int_W (-\Delta\hat{e}_t)\omega^3 \\ &\leq \lim_{S\to\infty} \int_{\partial W_S} 2\Big[\partial \operatorname{tr}\big(\widehat{F}_{H_t} \cdot F_{H_t}\big) \wedge \omega + \frac{\partial \hat{e}_t}{\partial \nu}\, d\mathrm{Vol}_\omega|_{\partial W_S}\Big] = 0, \end{aligned}$$

using Lemma 40 along with its proof and $(\frac{d}{dt} + \Delta)\hat{e}_t \leq 0$ as in (21), then complex integration by parts (Lemma 1 in the appendix) and the Gauss–Ostrogradsky theorem, and finally the exponential decay

$$\widehat{F}_{H_t} \xrightarrow[S\to\infty]{C^\infty} 0,$$

a direct consequence from Proposition 32 and Corollary 35. This shows that $E(t)$ is nonincreasing, while obviously $E(0) = 0$. □

From now on I will write, in cylindrical coordinates, D_s for D_z when $|z| = e^{-s}$. Reasoning as above we find, for a Chern connection on $\mathcal{E}|_{D_s}$,

$$|F|^2\omega^2 = \operatorname{tr} F^2 + 2|\widehat{F}|^2\omega^2. \tag{50}$$

Denoting, for convenience, the component of $d\mathrm{Vol}_\omega$ containing the "tubular" factor $ds \wedge d\alpha$ by

$$d\mathrm{V}_{\mathrm{tub}} := \tfrac{1}{2} ds \wedge d\alpha \wedge (\kappa_I + d\psi)^2,$$

the remainder certainly satisfies $\widetilde{d\psi} := d\mathrm{Vol}_\omega - d\mathrm{V}_{\mathrm{tub}} = O(e^{-s})$. Moreover, around each D_s we have

$$d\mathrm{Vol}_\omega|_{D_s} = \tfrac{1}{2}(\omega|_{D_s})^2 = \tfrac{1}{2}(\kappa_I + d\psi)^2|_{D_s},$$

so that

$$d\mathrm{V}_{\mathrm{tub}} = ds \wedge d\alpha \wedge d\mathrm{Vol}_\omega|_{D_s}. \tag{51}$$

Lemma 56 *Let $\Sigma = \tau^{-1}(A) \subset W_\infty$ be any cylindrical domain along the tubular end, parametrised by $A \subset \mathbb{C}P^1$. Then*

$$\int_\Sigma (|F_t|^2 - |F_0|^2 + R_0)\, d\mathrm{V}_{\mathrm{tub}} \geq 2 \int_A \| \widehat{F_{t|s}} \|^2_{L^2(D_s)}\, ds \wedge d\alpha,$$

where R_0 decays exponentially along the tube.

Proof Isolating the component of curvature along each transversal slice D_s in the asymptotia (30) of F_0, we have $|F_0|^2 = |F_{0|s}|^2 + R_0$ over W_∞, where the remainder indeed satisfies $R_0 = O(e^{-s})$. The Hodge–Riemann property (50) and the decomposition (51) then give, by Fubini's theorem,

$$\begin{aligned}\int_\Sigma (|F_0|^2 - R_0)\, d\mathrm{V}_{\mathrm{tub}} &= \int_A \left\{ \int_{D_s} |F_{0|s}|^2\, d\mathrm{Vol}_\omega|_{D_s} \right\} ds \wedge d\alpha \\ &= \int_A \langle c_2(\mathcal{E}|_{D_s}), [D_s] \rangle\, ds \wedge d\alpha.\end{aligned}$$

On the other hand, again by (50),

$$\begin{aligned}\int_\Sigma |F_t|^2\, d\mathrm{V}_{\mathrm{tub}} &\geq \int_\Sigma |F_{t|s}|^2\, d\mathrm{V}_{\mathrm{tub}} \\ &= \int_A \left\{ \int_{D_s} |F_{t|s}|^2\, d\mathrm{Vol}_\omega|_{D_s} \right\} ds \wedge d\alpha \\ &\geq \int_A \{ \langle c_2(\mathcal{E}|_{D_s}), [D_s] \rangle + 2 \int_{D_s} |\widehat{F_{t|s}}|^2\, d\mathrm{Vol}_\omega|_{D_s} \} ds \wedge d\alpha,\end{aligned}$$

using the normalisation $c_1(\mathcal{E}|_{D_s}) = 0$ (cf discussion following (19)) so that only $c_2(\mathcal{E}|_{D_s})$ contributes to the integral of $\operatorname{tr} F_{t|s}^2$. Comparing and cancelling the topological terms yields the result. □

In the terms of Claim 44, use $\Sigma_t := A_t \times D_s$ to denote the finite cylinder $\tau^{-1}(A_t)$ along the tubular end W_∞. Then the lemma gives a lower estimate on the curvature over Σ_t:

$$\int_{\Sigma_t} (|F_t|^2 - |F_0|^2 + R_0)\, d\mathrm{V}_{\mathrm{tub}} \geq 2 \int_{A_t} \| \widehat{F_{t|s}} \|^2_{L^2(D_s)}\, ds \wedge d\alpha. \tag{52}$$

This discussion culminates in the following result.

Proposition 57 *Under the hypotheses of Claim 44, there exists a constant C_∞, independent of t and z, such that*

$$L_t \leq C_\infty \quad \text{for all } t \in \,]0,\infty[.$$

Proof Either L_t is uniformly bounded in t and there is nothing to prove, or

$$\frac{\mu_\infty(A_t)}{\sqrt{L_t}} \underset{t\to\infty}{\longrightarrow} c''$$

and so (a subsequence of) the family of sets $\{A_t\}_{0<t<\infty}$ gets arbitrarily μ_∞–large as $t \to \infty$. In the latter case, part (2) of the claim gives

$$c\mu_\infty(A_t) \leq \int_{\Sigma_t} (|F_t|^2 - |F_0|^2 + R_0)\, d\mathrm{V}_{\mathrm{tub}}.$$

Let us examine the energy contributions of each component of $W = W_0 \cup W_\infty$ (cf (12)). On one hand, the complement $W_\infty \setminus \Sigma_t$ is a cylindrical domain, which by Lemma 56 has nonnegative energy in the model metric $d\mathrm{V}_{\mathrm{tub}}$; so we may extend the above integral over W_∞. On the other hand, in the actual metric $d\mathrm{Vol}_\omega$ we have

$$\begin{aligned} E(t) &= \left(\int_{W_0} + \int_{W_\infty}\right)(|F_t|^2 - |F_0|^2)\, d\mathrm{Vol}_\omega \\ &\geq -\|F_0\|^2_{L^2(W_0,\omega)} + \int_{W_\infty} (|F_t|^2 - |F_0|^2)\, d\mathrm{Vol}_\omega. \end{aligned}$$

To connect both facts, consider the ratio of volume forms over W_∞

$$d\mathrm{V}_{\mathrm{tub}} = f\, d\mathrm{Vol}_\omega$$

and extend it to a bounded positive function $f\colon W \to \mathbb{R}^+$. Then

$$c\mu_\infty(A_t) \leq \left(\sup_W f\right)\left(E(t) + \|F_0\|^2_{L^2(W_0,\omega)} + \|R_0\|_{L^1(W,\omega)}\right)$$

and we know from the negative energy condition (49) that this is bounded above, uniformly in t. Hence the cylinder Σ_t cannot stretch indefinitely and, by contradiction, there must exist a uniform upper bound

$$C_\infty \simeq \Big[\frac{\sup f}{c c''}\big(\|F_0\|^2_{L^2(W_0)} + \|R_0\|_{L^1(W)}\big)\Big]^2. \qquad \square$$

Finally, replacing the uniform bound for C_T in (25), this control cascades into the exponential C^0–decay in (31) and hence the C^∞–decay in Corollary 35. By Proposition 41, the limit metric $H = \lim_{t\to\infty} H_t$ satisfies $\widehat{F}_H = 0$. We have thus proved the following instance of the HYM problem:

Theorem 58 *Let $\mathcal{E} \to W$ be stable at infinity (cf Definition 12), equipped with a reference metric H_0 (cf Definition 13), over an asymptotically cylindrical* $\mathrm{SU}(3)$*–manifold W as given by Theorem 11 and let $\{H_t = H_0 e^{\xi_t}\}$ be the 1–parameter family of Hermitian metrics on $\mathcal{E}$ given by Theorem 36. The limit $H = \lim_{t\to\infty} H_t$ exists and is a smooth Hermitian Yang–Mills metric on $\mathcal{E}$, exponentially asymptotic in all derivatives (cf Notation 34) to H_0 along the tubular end of W:*

$$\widehat{F}_H = 0, \quad H \xrightarrow[S\to\infty]{C^\infty} H_0.$$

4.2 Examples of asymptotically stable bundles

It is fair to ask whether there are any holomorphic bundles at all satisfying the asymptotic stability conditions of Definition 12, thus providing concrete instances for Theorem 58.

We know, on one hand, that Kovalev's base manifolds (cf Definition 9) are Kähler 3–folds coming from blowups $\overline{W} = \mathrm{Bl}_C X$, where X is Fano and the curve $C = D\cdot D$ represents the self-intersection of a $K3$ divisor $D \in |-K_X|$; see [23, Corollary 6.43]. In addition, several of the examples provided (eg $X = \mathbb{C}P^3$, complete intersections etc) are nonsingular, projective and satisfy the *cyclic conditions*: $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{Pic}(D) = \mathbb{Z}$. On the other hand, certain linear monads over projective varieties of the above kind yield stable bundles as their middle cohomology [20]. Those are called *instanton monads* and have the form

$$0 \longrightarrow \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \longrightarrow 0. \tag{53}$$

Then, denoting $K := \ker\beta$, the middle cohomology $\mathcal{E} := K/\operatorname{img}\alpha$ is always a stable bundle with

$$\mathrm{rank}(\mathcal{E}) = 2, \quad c_1(\mathcal{E}) = 0, \quad c_2(\mathcal{E}) = c\cdot h^2,$$

where $h := c_1(\mathcal{O}_X(1))$ is the hyperplane class and $c \geq 1$ is an integer.

In those terms, twist the monad by $\mathcal{O}_X(-d)$, $d = \deg D$, so the relevant data fit in the following *canonical diagram*:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \mathcal{O}_X(-(d+1))^{\oplus c} \arrow[d] & & & \\
0 \arrow[r] & K(-d) \arrow[r] \arrow[d] & \mathcal{O}_X(-d)^{\oplus 2+2c} \arrow[r] & \mathcal{O}_X(-(d-1))^{\oplus c} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}(-d) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & \mathcal{E}|_D \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Computing cohomologies, one checks by Hoppe's criterion (see Okonek, Schneider and Spindler [29, pages 165–166]) that indeed $\mathcal{E}|_D$ is stable, ie the bundle $\mathcal{E}$ is asymptotically stable.

Nota bene Fixing $\overline{W} = \mathrm{Bl}_C(\mathbb{C}P^3)$ and $c = 1$, $d = 4$ as monad parameters, this specialises to the well-known null-correlation bundle; see [29] and Barth [2].

A detailed study with many more examples of asymptotically stable bundles over base manifolds admissible by Kovalev's construction will be published separately in [21].

4.3 Future developments

Kovalev's construction culminates in producing new examples of *compact* 7–manifolds with holonomy strictly equal to G_2. Loosely speaking, the asymptotically cylindrical G_2–structures on certain "matching" pairs $W' \times S^1$ and $W'' \times S^1$ are superposed along a truncated gluing region down the tubular ends, using cutoff functions to yield a global G_2–structure φ on the compact manifold

$$M_S := W' \,\Sigma\, W'',$$

defined by a certain "twisted" gluing procedure. This structure can be chosen to be torsion-free, by a "stretch the neck" argument on the length parameter S. Step (2) in our broad programme, as proposed in the introduction, is the corresponding problem of gluing the G_2–instantons obtained in the present article.

First of all, one needs to extend the twisted sum operation to bundles $\mathcal{E}^{(i)} \to W^{(i)}$ in some natural way, respecting the technical "matching" conditions. More seriously, however, one should expect transversality to play an important role. As mentioned in Remark 14, the easiest way to proceed is probably to restrict attention at first to *acyclic* instantons (see [10, page 25]), ie those whose gauge class “at infinity” is *isolated* in its moduli space $\mathcal{M}_{\mathcal{E}|_D}$. This would require, of course, the existence of asymptotically stable bundles which are also *asymptotically rigid*, in the obvious sense. Fortunately, Kovalev’s construction admits (at least) a certain class of prime Fano 3–folds $X_{22} \hookrightarrow \mathbb{P}^{13}$, thoroughly studied by Mukai in [28, Section 3; 27, Section 3], which admit bundles exhibiting precisely those properties, thus such investigation does not seem void from the outset.

Furthermore, if an asymptotically rigid gluing result does not present any further meanders, one might then consider the full question of transversality towards a gluing theory for families of instantons. A detailed study of this matter will appear in the sequel, provisionally cited as [32].

Appendix: Facts from geometric analysis

I collect in this appendix three analytical results from both complex and Riemannian geometry which, although well known to specialists, should be stated in precise terms and in compatible notation due to their importance in the text.

A.1 Integration by parts on complex manifolds with boundary

Lemma 1 (Integration by parts) *Let $X^n \subseteq W$ be a compact complex (sub)manifold (possibly $n=3$), Φ a $(1,1)$–form, Ω a closed $(n-2,n-2)$–form and f a meromorphic function on X. Then*

$$\int_X \Phi \wedge dd^c f \wedge \Omega = \int_X f \cdot (-i\bar\partial\partial\Phi) \wedge \Omega + i\int_{\partial X} (\Phi \wedge \bar\partial f + f\,\partial\Phi) \wedge \Omega.$$

Proof By the Leibniz rule and Stokes’ theorem, using $d = \partial + \bar\partial$ and taking bidegrees into account, we have

$$\int_X \Phi \wedge dd^c f \wedge \Omega = \int_X \Phi \wedge i\,\partial\bar\partial f \wedge \Omega = i\int_{\partial X} \Phi \wedge \bar\partial f \wedge \Omega \underbrace{- \int_X i\,\partial\Phi \wedge \bar\partial f \wedge \Omega}_{(*)},$$

and again

$$(*) = i\int_{\partial X} f\,\partial\Phi \wedge \Omega + \int_X f \cdot (-i\bar\partial\partial\Phi) \wedge \Omega. \qquad \square$$

A.2 The Hodge–Riemann bilinear relation

The curvature on a Kähler n–fold splits as $F = \widehat{F}\cdot\omega \oplus F^{\perp} \in \Omega^{1,1}(\operatorname{End}\mathcal{E})$, so

$$F^2\wedge\omega^{n-2} = \widehat{F}^2\cdot\omega^n + (F^{\perp})^2\wedge\omega^{n-2}.$$

The Hodge–Riemann pairing $(\alpha,\beta)\longmapsto\alpha\wedge\beta\wedge\omega^{n-2}$ on $\Omega^{1,1}(W)$ is positive-definite along ω and negative-definite on the primitive forms in $\langle\omega\rangle^{\perp}$ [19, pages 39–40] (with respect to the reference Hermitian bundle metric); since the curvature F is real as a bundle-valued 2–form, we have

$$\operatorname{tr} F^2\wedge\omega^{n-2} = \big(|F^{\perp}|^2 - |\widehat{F}|^2\big)\omega^n, \tag{1}$$

using that $\operatorname{tr}\xi^2 = -|\xi|^2$ on the Lie algebra part.

A.3 Gaussian upper bounds for the heat kernel

The following instance of Grigoryan's result [16, Theorem 1.1] stems from a long series, going back to J Nash (1958) and D Aronson (1971), of generalised "Gaussian" upper bounds (ie given by a Gaussian exponential on the geodesic distance r) for the heat kernel K_t of a Riemannian manifold.

Theorem 2 *Let M be an arbitrary connected Riemannian n–manifold, $x,y\in M$ and $0\le T\le\infty$. If there exist suitable (see below) real functions f and g satisfying the "diagonal" conditions*

$$K_t(x,x)\le\frac{1}{f(t)}\quad\text{and}\quad K_t(y,y)\le\frac{1}{g(t)}\quad\text{for all } t\in(0,T),$$

then, for any $C>4$, there exists $\delta=\delta(C)>0$ such that

$$K_t(x,y)\le\frac{(\mathrm{cst})}{\sqrt{f(\delta t)g(\delta t)}}\exp\Big\{-\frac{r(x,y)^2}{Ct}\Big\}\quad\text{for all } t\in(0,T),$$

where (cst) *depends only on the Riemannian metric.*

For the present purposes one may assume simply $f(t)=g(t)=t^{n/2}$, but in fact f and g can be *much* more general; see [16, page 37].

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Received: 6 April 2012
Revised: 9 April 2014