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Ozsváth–Szabó invariants of contact surgeries

MARCO GOLLA

We give new tightness criteria for positive surgeries along knots in the 3–sphere, generalising results of Lisca and Stipsicz, and Sahamie. The main tools will be Honda, Kazez and Matić's, and Ozsváth and Szabó's Floer-theoretic contact invariants. We compute Ozsváth–Szabó contact invariant of positive contact surgeries along Legendrian knots in the 3–sphere in terms of the classical invariants of the knot. We also combine a Legendrian cabling construction with contact surgeries to get results about rational contact surgeries.

57R17; 57R57

1 Introduction

Every contact manifold falls in one of two families: overtwisted or tight. Eliashberg [8] classified overtwisted contact structures on 3–manifolds according to the homotopy type of the underlying plane field, showing that overtwisted structures are in some sense simple. The classification of tight contact structures, on the other hand, provides us with a much harder task, and many questions remain open: Which 3–manifolds do support tight contact structures? If they do, how many up to isotopy? And how can we describe them?

Many tools have been developed to detect tightness; among them, Ozsváth and Szabó's Floer-theoretic invariant c, living in the 'hat' flavour of Heegaard–Floer (co)homology of the underlying manifold: the set of contact structures with $c \neq 0$ sits in a chain of inclusions between the set of Stein fillable and the set of tight contact structures [38].

Lisca and Stipsicz [28; 29; 30] extensively used this tool and the surgery exact sequences in Heegaard–Floer homology to produce examples of tight contact structures on several manifolds, chiefly obtained by surgery on S^3 along a knot. The twisted version of c has been used, for example, by Ghiggini and Van Horn-Morris [14] to classify tight contact structures on some Brieskorn spheres. Vanishing results for these invariants have been given by Sahamie [47].

Recall that for any knot $K \subset S^3$, Ozsváth and Szabó defined two concordance invariants $\tau(K), \nu(K) \in \mathbb{Z}$ such that $\tau(K) \leq \nu(K) \leq \tau(K)+1$. On the other hand, to any

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Legendrian knot $L \subset (S^3, \xi)$, we can associate two other integers, $\mathrm{tb}(L)$ and $r(L)$, the Thurston–Bennequin and the rotation number respectively: these two and the topological type of L are collectively called the *classical invariants* of L.

Finally, recall that there are two possible contact structures that are obtained as contact n–surgery on a given Legendrian L, and we will denote them with $\xi_n^{\pm}(L)$. We can now state our main theorem:

Theorem 1.1 *Let L be an oriented Legendrian knot in the standard contact structure ξ_{st} on S^3 and let K be the topological type of L.*

For positive n, $\xi_n^-(L)$ has nonvanishing contact invariant if and only if the following hold:

(SL) $\mathrm{tb}(L) - r(L) = 2\tau(K) - 1$.

(SC) $n + \mathrm{tb}(L) \geq 2\tau(K)$.

(TN) $\tau(K) = \nu(K)$.

Moreover, if L' is another Legendrian knot with the same classical invariants (whether or not the three conditions hold), then $c(\xi_n^-(L')) = c(\xi_n^-(L))$.

Remark 1.2 There is an action of $M(Y) := \mathrm{MCG}(Y \setminus B, \partial B)$, the mapping class group of Y with a ball removed, relative to the boundary, on $\widehat{HF}(-Y)$; see Juhász and Thurston [23]. The contact invariants $c(\xi) \in \widehat{HF}(-Y)$, $c(\xi') \in \widehat{HF}(-Y')$ of two contact manifolds (Y, ξ), (Y', ξ'), with Y diffeomorphic to Y', can only be compared using a diffeomorphism $Y \setminus B \to Y' \setminus B'$. Any two such diffeomorphisms differ by an element of $M(Y)$.

The equality $c(\xi_n^-(L')) = c(\xi_n^-(L))$ has to be taken as saying that there is a diffeomorphism

$$S^3_{\mathrm{tb}(L)+n}(L) \longrightarrow S^3_{\mathrm{tb}(L)+n}(L')$$

that takes $c(\xi_n^-(L'))$ to $c(\xi_n^-(L))$; this is equivalent to both of them lying in the same orbit of the action of $M(S^3_{\mathrm{tb}(L)+n}(K))$ on $\widehat{HF}(-S^3_{\mathrm{tb}(L)+n}(K))$.

Remark 1.3 As a mnemonic trick, the abbreviations SL, SC and TN stand for “self-linking”, “surgery coefficient” and “tau-nu” respectively.

Since $\mathrm{tb}(L) - r(L)$ is the self-linking number $\mathrm{sl}(T)$ of the transverse pushoff T of L, the first condition can be interpreted as a transverse condition on T.

The second condition is a condition on the pair (Legendrian knot, surgery coefficient) (L, n); it can also be read as $n + \mathrm{tb}(L) \geq \mathrm{sl}(L) + 1$ or $n \geq 1 - r(L)$.

The third condition could be absorbed in the first one if we just replaced τ by ν in (SL) since $\nu(K)$ is either $\tau(K)$ or $\tau(K)+1$; on the other hand, the first condition is of contact-geometric nature, while the third is of Floer-theoretical nature, and we will realise along the proof that they really are two separate conditions rather than one.

In other words, the contact invariant of an integral surgery along $L \subset (S^3, \xi_{\mathrm{st}})$ does not contain more information about L than the classical invariants, and in particular cannot distinguish surgeries along non-Legendrian isotopic knots that share the same classical invariants.

Remark 1.4 As we will see in Section 4, the "positive" contact surgery $\xi_n^+(L)$ is isotopic to $\xi_n^-(-L)$: the only condition that gets affected by orientation reversal of L is (SL), so we get an analogous statement about $c(\xi_n^+(L))$ if we replace it with the condition $\mathrm{tb}(L)+r(L) = 2\tau(K)-1$.

Example 1.5 Let us consider the knot 8_{20}. It has genus $g(8_{20}) = 1$, but its slice genus is $g_*(8_{20}) = 0$ (which in turn implies also $\tau(8_{20}) = \nu(8_{20}) = 0$). On the other hand, its maximal Thurston–Bennequin number is $\overline{\mathrm{tb}}(8_{20}) = -2$ and its maximal self-linking number is $\overline{\mathrm{sl}}(8_{20}) = -1$. In particular, our Theorem 1.1 applies here, whereas neither the main result in [28] or [30] does. We can therefore exhibit new examples of tight contact structures on the manifolds $S^3_q(8_{20})$ for all $q \geq 0$ rational (see Corollary 1.6).

The knot $m(10_{125})$ has $\tau(m(10_{125})) = -g_*(m(10_{125})) = -1$ and $\overline{\mathrm{sl}}(m(10_{125})) = -3$: the first equality implies that $\nu(m(10_{125})) = 0$. In particular, (TN) does not hold for $m(10_{125})$, but it has a Legendrian representative for which (SL) does hold. We are grateful to Lenny Ng for this example.

As a byproduct of the proof of Theorem 1.1, without much effort, we get:

Corollary 1.6 *If $\tau(K) = \nu(K) \geq 1$ (resp. $\tau(K) = \nu(K) = 0$) and there is a Legendrian representative L of K that satisfies (SL), then for all $q > 2\tau(K)-1$ (resp. $q \geq 0$) the manifold $S^3_q(K)$ supports a tight contact structure.*

Notice that the hypothesis $\tau(K) = \nu(K)$ holds, for example, if $\mathrm{tb}(L) - r(L) = 2g_*(K)-1$, and more generally it holds whenever $\tau(K) = g_*(K)$ (see Remark 6.19).

In [30], a new transverse invariant $\widetilde{c}$ was also defined. Given a transverse knot T in (Y, ξ), for sufficiently large f we can define contact surgery along T with framing f, and take the inverse limit of the contact invariants of these objects. Since we have complete control on these contact invariants for $T \subset S^3$, we can draw the following corollary:

Corollary 1.7 *Given T in (S^3,ξ) of topological type K, the transverse invariant $\widetilde{c}(T)$ is nonzero if and only if $\xi=\xi_{\mathrm{st}}$, $\mathrm{sl}(T)=2\tau(K)-1$ and $\tau(K)=\nu(K)$. Moreover, if T' is another transverse knot of the same topological type of T with $\mathrm{sl}(T')=\mathrm{sl}(T)$, then, up to the action of $\mathrm{MCG}(S^3\setminus K)$, $\widetilde{c}(T')=\widetilde{c}(T)$.*

The proof of Theorem 1.1 has an algebraic flavour, with a topological input coming from a Legendrian cabling construction.

Organisation The paper is organised as follows. In Section 2 we introduce some standard background in Heegaard–Floer homology, sutured Floer homology, contact invariants and gluing maps. Section 3 is devoted to the study of some sutured Floer homology groups and some gluing maps between them. In Section 4 we prove some useful lemmas about contact surgeries and stabilisations; in Section 5 we discuss a Legendrian cabling construction and its interactions with contact surgeries. Finally, Section 6 contains the proof of Theorem 1.1 and its corollaries; we defer the proof of some technical lemmas to Section 7.

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2 Sutured Floer homology and gluing maps

2.1 Sutured manifolds

The definition of balanced sutured manifold is due to Juhász [22].

Definition 2.1 A *balanced sutured manifold*, is a pair (M,Γ), where M is an oriented 3–manifold with nonempty boundary ∂M and Γ is a family of oriented curves in ∂M that satisfies:

- $\bigcup\Gamma$ intersects each component of ∂M.
- $\bigcup\Gamma$ disconnects ∂M into R_+ and R_- with $\pm\Gamma=\partial R_\pm$ (as oriented manifolds).
- $\chi(R_+)=\chi(R_-)$.

Remark 2.2 The condition $\chi(R_+) = \chi(R_-)$ is called the *balancing* condition. Since this is the only kind of sutured manifold we are dealing with, we prefer to just drop the adjective "balanced".

Example 2.3 Any oriented 3–manifold M with S^2–boundary can be turned into a sutured manifold $(M, \{\gamma\})$ by choosing any simple closed curve γ in ∂M. We will often write $M = Y(1)$, where $Y = M \cup_\partial D^3$ is the "simplest" closed 3–manifold containing M.

For every $p/q \in \mathbb{Q} \cup \{\infty\}$, we have a sutured manifold $S^3_{K,p/q}$ given by pairs $(S^3 \setminus N(K), \{\gamma_{p/q}, -\gamma_{p/q}\})$, where $\gamma_{p/q}$ is an oriented curve on the boundary torus $\partial N(K)$ of an open small neighbourhood $N(K)$ of K. The slope of $\gamma_{p/q}$ is $q\lambda_S + p\mu$, and $-\gamma_{p/q}$ is a parallel pushoff of $\gamma_{p/q}$, with the opposite orientation. Here λ_S denotes the Seifert longitude of K. We will use the shorthand $\Gamma_{p/q}$ for $\{\gamma_{p/q}, -\gamma_{p/q}\}$.

Example 2.4 To any Legendrian knot $L \subset (Y, \xi)$ in an arbitrary 3–manifold Y one can associate in a natural way a sutured manifold, that we will denote by Y_L, constructed as follows: There is a standard (open) Legendrian neighbourhood $\nu(L)$ for L, with convex boundary. The dividing set Γ_L on the boundary consists of two parallel oppositely oriented curves parallel to the contact framing of L. The manifold Y_L is then defined as the pair $(Y \setminus \nu(L), \Gamma_L)$. In the case we are mainly interested in, where $Y = S^3$ and L is of topological type K, we have $S^3_L = S^3_{K,\mathrm{tb}(L)}$. More generally, the same identification, $\{\text{framings}\} \leftrightarrow \mathbb{Z}$, can be made canonical whenever K is nullhomologous in a rational homology sphere Y (that is $H_2(Y) = 0$), and we then have $Y_L = Y_{K,\mathrm{tb}(L)}$.

We will often use Y_L also to denote the contact manifold with convex boundary $(Y \setminus \nu(L), \xi|_{Y \setminus \nu(L)})$ without creating any confusion.

There is a decomposition/classification theorem for sutured manifolds, completely analogous to the Heegaard decomposition/Reidemeister–Singer theorem for closed 3–manifolds. Consider a compact surface Σ with boundary and a collection of simple closed curves $\boldsymbol{\alpha}, \boldsymbol{\beta} \subset \Sigma$ such that no two α–curves intersect and no two β–curves intersect; suppose moreover that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. We can build a balanced sutured manifold out of this data as follows: Take $\Sigma \times [0,1]$, glue a 2–handle on $\Sigma \times \{0\}$ for each α–curve and a 2–handle on $\Sigma \times \{1\}$ for each β–curve, and let M be the manifold obtained after smoothing corners; declare $\Gamma = \partial\Sigma \times \{\frac{1}{2}\}$. The pair (M, Γ) is a balanced sutured manifold, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is called a *(sutured) Heegaard diagram* of (M, Γ).

Theorem 2.5 [22] *Every balanced sutured manifold admits a Heegaard diagram, and every two such diagrams become diffeomorphic after a finite number of isotopies of the curves, handleslides and stabilisations taking place in the interior of the Heegaard surface.*

There is one further description of a sutured manifold, relying on arc diagrams: An *arc diagram* $\mathcal{H}^a$ is a quintuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^a, \boldsymbol{\beta}^c, D)$, where Σ is a closed surface, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^c$ are sets of simple closed curves in Σ, with $\boldsymbol{\alpha}$ linearly independent in $H_1(\Sigma)$, D is a closed disc disjoint from $(\bigcup \boldsymbol{\alpha}) \cup (\bigcup \boldsymbol{\beta})$ and $\boldsymbol{\beta}^c$ is a set of pairwise disjoint closed arcs in $\Sigma \setminus \mathrm{Int}(D)$ with endpoints on ∂D (and elsewhere disjoint from D), each disjoint from every β–curve. We ask that $|\boldsymbol{\alpha}| = g = g(\Sigma)$ and $|\boldsymbol{\beta}^c| + |\boldsymbol{\beta}^a| = g$.

We build a sutured manifold out of $\mathcal{H}^a$ in the following way: The set of α–curves determines the attaching circles of g 2–handles on $\Sigma \times \{0\} \subset \Sigma \times [0,1]$; we attach a 3–handle (a ball) to fill up the remaining component of the lower boundary; the set $\boldsymbol{\beta}^c$ of *β–curves* determines the attaching circles of 2–handles on $\Sigma \times \{1\}$. We define M to be the manifold obtained by smoothing corners after these handle attachments; notice that D is an embedded disc in ∂M and $\boldsymbol{\beta}^c$ is a set of embedded arcs in ∂M. Let R_+ be a small neighbourhood of $D \cup \boldsymbol{\beta}^c$ that retracts onto it and $\Gamma = \partial R_+$.

Lemma 2.6 (Zarev [50]) *Every sutured manifold with connected R_+ admits an arc diagram.*

Remark 2.7 Our definition of arc diagram is not related to Zarev's notion of arc diagrams (that describe surfaces with dividing curves); our arc diagrams are related to his bordered sutured diagrams.

2.2 The Floer homology packages

This is meant to be just a recollection of facts about the Floer homology theories we will be working with. The standard references for the material in this subsection are Ozsváth and Szabó [36; 37] and Lipshitz [25] for the Heegaard–Floer part and [22] for the sutured Floer part.

In order to avoid sign issues, we will work with $\mathbb{F} = \mathbb{F}_2$ coefficients.

Consider a pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ representing a 3–manifold Y, and form two Heegaard–Floer complexes $\widehat{CF}(Y)$ and $CF^-(Y)$: the underlying modules are freely generated over $\mathbb{F}$ and $\mathbb{F}[U]$ by g–tuples of intersection points in $\bigcup_{i,j}(\alpha_i \cap \beta_j)$, so that there is exactly one point on each curve in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$.

The differentials $\widehat{\partial}, \partial^-$ are harder to define, and count certain pseudoholomorphic discs in a symmetric product $\mathrm{Sym}^g(\Sigma_g)$, or maps from Riemann surfaces with boundary in $\Sigma_g \times \mathbb{R} \times [0,1]$, with the appropriate boundary conditions. The homology groups $\widehat{HF}(Y) = H_*(\widehat{CF}(Y), \widehat{\partial})$ and $HF^-(Y) = H_*(CF^-(Y), \partial^-)$ so defined are called *Heegaard–Floer homologies* of Y and are independent of the (many) choices made along the way [37].

Sutured Floer homology is a variant of this construction for sutured manifolds (M,Γ). The starting point is a sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for (M,Γ). We form a complex $SFC(M,\Gamma)$ in the same way, generated over $\mathbb{F}$ by d–tuples of intersection points as above, where $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. The differential ∂ is defined by counting pseudoholomorphic discs in $\mathrm{Sym}^d(\Sigma)$ or maps from Riemann surfaces to $\Sigma \times \mathbb{R} \times [0,1]$, again with the appropriate boundary conditions.

The homology $SFH(M,\Gamma) = H_*(SFC(M,\Gamma),\partial)$ is called the *sutured Floer homology* of (M,Γ), and is shown to be independent of all the choices made [22]. It naturally corresponds to a "hat" theory.

There is one more description of sutured Floer homology, due to Zarev [50], coming from arc diagram representations: Given a balanced sutured manifold (M,Γ) with R_+ connected, we can form a Floer complex starting from an arc diagram associated to it. The underlying module is free over the g–tuples of intersection points between α–curves and β–curves and arcs as above; the differential counts holomorphic discs in the symmetric product with boundary on these curves, such that the multiplicity at the regions touching the base-disc D are all 0.

If R_+ is not connected, then (M,Γ) is a product disc decomposition of a manifold (M',Γ') with R'_+ connected. Juhász showed that $SFH(M,\Gamma) = SFH(M',\Gamma')$, so we can compute $SFH(M,\Gamma)$ using an arc diagram for (M',Γ').

Proposition 2.8 [22] *For a closed 3–manifold Y, $\widehat{HF}(Y) = SFH(Y(1))$.*

For a knot K in a closed 3–manifold Y, $\widehat{HFK}(Y,K) = SFH(Y_{K,\infty})$ (that is, the sutures are parallel to the meridian of K in Y).

One key feature of Heegaard–Floer homology is a TQFT-like behaviour: given a four-dimensional cobordism $W\colon Y_1 \rightsquigarrow Y_2$, to each Spin^c–structure $\mathfrak{t} \in \mathrm{Spin}^c(W)$ we associate a map $F_{W,\mathfrak{t}}\colon \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$; only a finite number of Spin^c–structures induce a nontrivial map (see Ozsváth and Szabó [39]), so it makes sense to define the total cobordism map $F_W = \sum F_{W,\mathfrak{t}}$. We will be dealing with cobordisms induced by a single (four-dimensional) 2–handle attachment. In this case, the total cobordism map can be described explicitly as follows.

In such a cobordism, Y_2 is obtained from Y_1 as an integral surgery along a knot K, and in particular Y_1 and Y_2 can be represented as two Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ such that the curves $\gamma_2, \dots, \gamma_g$ in $\boldsymbol{\gamma}$ are obtained from $\beta_2, \dots, \beta_g$ respectively by a small Hamiltonian perturbation. The two remaining curves β_1 and γ_1 represent a pair (meridian, longitude) on the boundary of a neighbourhood of K, and in particular they intersect exactly once. There is a canonical intersection point $\boldsymbol{\Theta}$

in $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$, that corresponds to the top Maslov degree element in $\widehat{HF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \widehat{HF}(\#^{g-1}(S^1 \times S^2))$.

The map $CF(Y_1) \to CF(Y_2)$ associated to this handle attachment counts pseudoholomorphic triangles in $\mathrm{Sym}^g(\Sigma)$ or maps from Riemann surfaces to $\Sigma \times \Delta$ (Δ being a standard triangle), again, with appropriate boundary conditions and involving the point $\boldsymbol{\Theta}$.

One can also compute every single $F_{W,\mathfrak{t}}$: the domain associated to each holomorphic triangle has a well-defined Spin^c–structure, and we restrict our sum to the triangles whose structure is $\mathfrak{t}$.

Arguably, one of the most useful features of Heegaard–Floer homology is the *surgery exact triangle*:

Theorem 2.9 [36] *Given a knot K in a 3–manifold Y and three slopes $f, g, h \in H_1(\partial N(K))$ such that $f \cdot g = g \cdot h = h \cdot f = 1$, there are three maps induced by appropriate integral surgeries, such that the triangle*

$$\begin{tikzcd}
\widehat{HF}(Y_f(K)) \arrow[rr] & & \widehat{HF}(Y_g(K)) \arrow[dl] \\
& \widehat{HF}(Y_h(K)) \arrow[ul] &
\end{tikzcd}$$

is exact. In particular, this holds when K is nullhomologous, $f = \infty$, g is integral and $h = g + 1$.

2.3 Floer-theoretic contact invariants

The first contact invariant to be defined in Heegaard–Floer homology was Ozsváth and Szabó's c [38]. The definition that we give here was given by Honda, Kazez and Matić [21], and lead to the fruitful extension to invariants for manifolds with convex boundary, called EH, living in sutured Floer homology.

Since the latter is a strict generalisation of the former, we just give the definition of EH: if ξ is a contact structure on Y, $c(\xi)$ is equivalent to $EH(\xi')$, where ξ' is the restriction of ξ to $Y \setminus B$ and B is a small Darboux ball.

Definition 2.10 A *partial open book* is a triple (S, P, h) where S is a compact open surface, P is a proper subsurface of S which is a union of 1–handles attached to $S \setminus P$ and $h\colon P \to S$ is an embedding that pointwise fixes a neighbourhood of $\partial P \cap \partial S$.

We can build a contact manifold with convex boundary out of these data in a fashion similar to the usual open books: instead of considering a mapping torus, though, we glue two asymmetric halves, quotienting the disjoint union $S\times[0,\frac{1}{2}]\amalg P\times[\frac{1}{2},1]$ by the relations $(x,t)\sim(x,t')$ for $x\in\partial S$, $(y,\frac{1}{2})\sim(y,\frac{1}{2})$, $(h(y),\frac{1}{2})\sim(y,1)$ for $y\in P$. The contact structure is uniquely determined if we require — as we do — tightness and prescribed sutures on each half $S\times[0,\frac{1}{2}]/\sim$ and $P\times[\frac{1}{2},1]/\sim$ (see Honda [19] for details). Moreover, to any contact manifold with convex boundary we can associate a partial open book, unique up to Giroux stabilisations.

We can build a balanced diagram out of a partial open book. The Heegaard surface Σ is obtained by gluing P to $-S$ along the common boundary.

Definition 2.11 A *basis* for (S,P) is a set $\boldsymbol{a}=\{a_1,\dots,a_k\}$ of arcs properly embedded in $(P,\partial P\cap\partial S)$ whose homology classes generate $H_1(P,\partial P\cap\partial S)$.

Given a basis as above, we produce a set $\boldsymbol{b}=\{b_1,\dots,b_k\}$ of curves using a Hamiltonian vector field on P. We require that under this perturbation the endpoints of a_i move in the direction of ∂P, that each a_i intersects b_i in a single point x_i and is disjoint from all the other b_j.

Finally define two sets of attaching curves, $\boldsymbol{\alpha}=\{\alpha_i\}$ and $\boldsymbol{\beta}=\{\beta_i\}$, where $\alpha_i=a_i\cup-a_i$ and $\beta_i=h(b_i)\cup-b_i$. The sutured manifold associated to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is (M,Γ). We call $\boldsymbol{x}(S,P,h)$ the generator $\{x_1,\dots,x_k\}$ in $SFC(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$ supported inside P.

Proposition 2.12 [21] *The chain $\boldsymbol{x}(S,P,h)\in SFC(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$ is a cycle and its class in $SFH(-M,-\Gamma)$ is an invariant of the contact manifold (M,ξ) defined by the partial open book (S,P,h).*

Definition 2.13 $EH(M,\xi)$ is the class $[\boldsymbol{x}(S,P,h)]\in SFH(-M,-\Gamma)$ for some partial open book (S,P,h) supporting (M,ξ).

The type of invariants that we are going to deal with are either invariants of (complements of) Legendrian knots or invariants coming from contact structures on closed manifolds. This allows us to consider (except for Section 7) only sutured manifolds with sphere/torus boundary and one/two sutures, as described in Examples 2.3 and 2.4.

Consider a closed contact manifold (Y,ξ), and let $B\subset Y$ be a small, closed Darboux ball with convex boundary. Then consider the manifold $(Y(1),\xi(1))$, where $Y(1)$ is obtained from Y by removing the interior of B, and $\xi(1)$ is $\xi|_{Y(1)}$. We can now state the following proposition, that will be our definition of the contact invariant c in Heegaard–Floer homology.

Proposition 2.14 [21] *The Ozsváth–Szabó class $c(Y,\xi)$ is mapped to the Honda–Kazez–Matić class $EH(Y(1),\xi(1))$ under the isomorphism of Proposition 2.8.*

As a corollary, all properties of c are inherited by EH, and in particular we recall the following:

Corollary 2.15 *If (Y,ξ) is Stein fillable (resp. overtwisted) then the contact invariant $EH(Y(1),\xi(1))$ does not vanish (resp. vanishes).*

The second type of invariants comes from Legendrian knots. Let us suppose that $L \subset Y$ is a Legendrian knot with respect to a contact structure ξ, then the contact manifold Y_L defined in Example 2.4 has a contact invariant $EH(Y_L) \in SFH(-Y_L)$. We will denote this invariant by $EH(L)$, considering it as an invariant of the Legendrian isotopy class of L rather than of its complement.

2.4 Gluing maps

In their paper [20], Honda, Kazez and Matić define maps associated to the gluing of a contact manifold to another one along some of the boundary components, and show that these maps preserve their EH invariant. Consider two sutured manifolds $(M,\Gamma) \subset (M',\Gamma')$, where M is embedded in $\mathrm{Int}(M')$. Let ξ be a contact structure on $N := M' \setminus \mathrm{Int}(M)$ such that ∂N is ξ–convex and has dividing curves $\Gamma \cup \Gamma'$. For simplicity, and since this will be the only case we need, we will restrict to the case when each connected component of N intersects $\partial M'$ (ie gluing N to M does not kill any boundary component).

Theorem 2.16 *The contact structure ξ on N induces a* gluing map *Φ_ξ, that is a linear map $\Phi_\xi\colon SFH(-M,-\Gamma) \to SFH(-M',-\Gamma')$. If ξ_M is a contact structure on M such that ∂M is ξ_M–convex with dividing curves Γ, then $\Phi_\xi(EH(M,\xi_M)) = EH(M',\xi_M \cup \xi)$.*

This theorem has interesting consequences, even in simple cases:

Corollary 2.17 *If (M,Γ) embeds in a Stein fillable contact manifold (Y,ξ), and ∂M is ξ–convex, divided by Γ, then $EH(M,\xi|_M)$ is not trivial.*

Proof We know that $c(Y,\xi)$ does not vanish, and neither does $EH(Y(1),\xi(1))$. Since we allowed ourselves much freedom in the choice of the ball to remove to get $Y(1)$, we can suppose that $M \subset \mathrm{Int}(Y(1))$. Call $N = Y(1) \setminus \mathrm{Int}(M)$ the closure of the complement of M. The map $\Phi_{\xi|_N}$ carries $EH(M,\xi|_M)$ to $EH(Y(1),\xi(1))$ and since the latter is nonzero, so is the former. □

Remark 2.18 In the proof we have been using something less than being Stein fillable, but just that $c(Y,\xi) \neq 0$. This is equivalent to being Stein fillable for the 3–sphere and for lens spaces (by results of Eliashberg [9] and Honda [19] respectively), but in general the second condition is weaker (as shown, for example, by Lisca and Stipsicz in [28]).

There is also a naturality statement, concerning the composition of two gluing maps: Suppose that we have three sutured manifolds $(M,\Gamma) \subset (M',\Gamma') \subset (M'',\Gamma'')$ as at the beginning of the section and suppose that ξ and ξ' are contact structures on $M' \setminus \mathrm{Int}(M)$ and $M'' \setminus \mathrm{Int}(M')$ respectively, that induce sutures Γ, Γ' and Γ'' on ∂M, $\partial M'$ and $\partial M''$ respectively.

Theorem 2.19 *If ξ and ξ' are as above, then $\Phi_{\xi\cup\xi'} = \Phi_{\xi'} \circ \Phi_{\xi}$.*

Much of our interest will be devoted to stabilisations of Legendrian knots and associated maps, whose discussion will occupy Section 3.3. We give a brief summary of the contact side of their story here.

Let us start with a definition due to Honda [19].

Definition 2.20 Let η be a tight contact structure on $T^2 \times I$ with two dividing curves on each boundary component: call γ_i, $-\gamma_i$ the homology class of the two dividing curves on $T^2 \times \{i\}$, and let $s_i \in \mathbb{Q} \cup \{\infty\}$ be their slope. The pair $(T^2 \times I, \eta)$ is a *basic slice* if it is of the form above and also satisfies the following conditions:

- $\{\gamma_0, \gamma_1\}$ is a basis for $H_1(T^2)$.
- ξ is *minimally twisting*, ie if $T_t = T \times \{t\}$ is convex, the slope of the dividing curves on T_t belongs to $[s_0, s_1]$ (where we assume that if $s_0 > s_1$ the interval $[s_0, s_1]$ is $[-\infty, s_1] \cup [s_0, \infty]$).

Honda and Etnyre proved the following:

Proposition 2.21 (Etnyre and Honda [11] and Honda [19]) *For every integer t there exist exactly two basic slices $(T^2 \times I, \xi_j)$ (for $j = 1, 2$) with boundary slopes $(t, 1)$ and $(t-1, 1)$. The sutured complement of a stabilisation L' of L is gotten by attaching one of the two basic slices to Y_L, where the trivialization of T^2 is given by $(0,1) = \mu$ and $(t,1) = c$, where μ and c are a meridian and the contact framing for L respectively.*

These two different layers correspond to the positive and negative stabilisation of L, once we have chosen an orientation for the knot; reversing the orientation swaps the labelling signs. Since we will be considering oriented Legendrian knots, we can label the two slices with a sign.

Definition 2.22 We call *stabilisation maps* the gluing maps associated to the attachment of a stabilisation basic slice. These will be denoted with $\sigma_\pm$.

Remark 2.23 As happens for the Stipsicz–Vértesi map [49], this basic slice attachment also corresponds to a single bypass attachment.

We also collect here the definition of some gluing maps that we will be considering later. The letter ψ will be used to denote gluing maps associated to contact surgeries.

Definition 2.24 Let $L \subset \nu(L) \subset (Y,\xi)$ be a Legendrian knot and $\nu(L)$ be its standard neighbourhood. Let $B \subset \nu(L)$ be a ball with convex boundary. The map ψ_∞ is associated to the layer $(\nu(L) \setminus B, \xi|_{\nu(L)\setminus B})$. This map is a homomorphism

$$\psi_\infty\colon SFH(-Y_{K,n}) \longrightarrow SFH(-Y(1)) = \widehat{HF}(-Y)$$

for every nullhomologous knot $K \subset Y$.

More generally, contact p/q–surgery is an operation that, given an oriented Legendrian knot $L \subset (Y,\xi)$, removes the standard neighbourhood $\nu(L)$ of L and replaces it with a tight solid torus $(\boldsymbol{T}_{p/q}, \xi_{p/q})$. When $p/q = 1$ there is only one such torus, and when $p/q = n > 1$ is an integer, there are two such choices, called $(\boldsymbol{T}_n^\pm, \xi_n^\pm)$. When $q > 1$ there are many choices for the contact structure on $\boldsymbol{T}_{p/q}$; we still have two "preferred" choices, that we denote with $(\boldsymbol{T}^\pm_{p/q}, \xi^\pm_{p/q})$. Notice that, regardless of the value of p/q, the manifold $\boldsymbol{T}_{p/q}$ is simply a solid torus $S^1 \times D^2$; on the other hand, the resulting sutures do change with p/q. We refer the reader to Section 4 for further details.

Definition 2.25 Let $B \subset \boldsymbol{T}_{p/q}$ be a closed ball with convex boundary and define $\boldsymbol{T}_{p/q}(1)$ to be $\boldsymbol{T}_{p/q} \setminus \mathrm{Int}(B)$.

- For a positive integer n, we define $\psi_n^\pm$ as the gluing map associated to the layer $(\boldsymbol{T}_n^\pm(1), \xi_n|_{\boldsymbol{T}_n^\pm(1)})$.
- We define $\psi_{+1} = \psi_{+1}^\pm$ as the gluing map associated to $(\boldsymbol{T}_1(1), \xi_1|_{\boldsymbol{T}_1(1)})$.
- For a positive rational p/q, we define $\psi^\pm_{p/q}$ as the gluing map associated to the layer $(\boldsymbol{T}^\pm_{p/q}(1), \xi_{p/q}|_{\boldsymbol{T}^\pm_{p/q}(1)})$.

Fix a knot $K \subset Y$, together with an open tubular neighbourhood $N(K)$ and a framing f, that we look at as a curve in $\partial N(K)$; as before, denote with $Y_{K,f}$ the sutured manifold $(Y \setminus N(K), \Gamma_f = \{f, -f\})$. The map $\psi^\pm_{p/q}$ is a homomorphism

$$\psi^\pm_{p/q}\colon SFH(-Y_{K,f}) \longrightarrow SFH(-Y_{p/q}(K,f)(1)) = \widehat{HF}(-Y_{p/q}(K,f)),$$

where the notation $Y_{p/q}(K,f)$ stands for the manifold that we obtain by topological p/q–surgery along K with respect to the framing given by a meridian for K in Y and the longitude f. If $K\subset Y$ is nullhomologous and Y is a rational homology sphere, K has a canonical framing, the Seifert framing f_S. In this case, we are going to write $Y_{p/q}(K)$ for $Y_{p/q}(K,f_S)$.

3 A few facts on $SFH(S^3_{K,n})$ and $\sigma_\pm$

Given a topological knot K in S^3, denote with $S^3_m(K)$ the manifold obtained by (topological) m–surgery along K, and let $\widetilde{K}$ be the dual knot in $S^3_m(K)$, that is the core of the solid torus we glue back in. Notice that an orientation on K induces an orientation of $\widetilde{K}$, by imposing that the intersection of the meridian μ_K of K on the boundary of the knot complement has intersection number $+1$ with the meridian $\mu_{\widetilde{K}}$ of $\widetilde{K}$ on the same surface.

Fix a contact structure ξ on S^3 and a Legendrian representative L of K: we will write t for $\mathrm{tb}(L)$. Since t measures the difference between the contact and the Seifert framings of L, $S^3_t(K)_{\widetilde{K},\infty}$ and S^3_L are sutured diffeomorphic: in particular, $EH(L)$ lives in

$$SFH(-S^3_t(K)_{\widetilde{K},\infty})=\widehat{HFK}(-S^3_t(K),\widetilde{K}),$$

the identification depending on the choice of an orientation for K (or $\widetilde{K}$).

We will often write $\widehat{CFK}(Y,K)$ to denote any chain complex computing $\widehat{HFK}(Y,K)$ that comes from a Heegaard diagram, even though the complex itself depends on the choice of the diagram.

3.1 Gradings and concordance invariants

The groups $\widehat{HFK}(S^3,K)$ and $\widehat{HFK}(-S^3_m(K),\widetilde{K})$ come with a grading that we call the *Alexander grading*. A Seifert surface $F\subset S^3$ for K gives a relative homology class

$$[F,\partial F]\in H_2\big(S^3\setminus N(K),\partial N(K)\big)=H_2\big(S^3_m(K)\setminus N(\widetilde{K}),\partial N(\widetilde{K})\big).$$

Given a generator $\boldsymbol{x}\in\widehat{CFK}(S^3,K)$, there is an induced relative Spin^c structure $\mathfrak{s}(\boldsymbol{x})$ in $\underline{\mathrm{Spin}}^c(S^3,K)$ (see Ozsváth and Szabó [41, Section 2]) and the Alexander grading of $\boldsymbol{x}$ is defined as

$$A(\boldsymbol{x})=\tfrac{1}{2}\langle c_1(\mathfrak{s}(\boldsymbol{x})),[F,\partial F]\rangle.$$

On the other hand, given a generator $\boldsymbol{x} \in \widehat{CFK}(-S^3_m(K), \widetilde{K})$, there is an induced relative Spin^c structure $\mathfrak{s}(\boldsymbol{x}) \in \underline{\mathrm{Spin}}^c(S^3_m(K), \widetilde{K})$ and we can define $A(\boldsymbol{x})$ as

$$A(\boldsymbol{x}) = \tfrac{1}{2}\langle c_1(\mathfrak{s}(\boldsymbol{x})), [F, \partial F]\rangle - \tfrac{1}{2}m. \tag{3-1}$$

We recall the definition of $\tau(K)$ and $\nu(K)$, due to Ozsváth and Szabó [35; 41], and of a third concordance invariant $\varepsilon(K)$, defined by Hom [18].

Recall that the Alexander grading induces a filtration on the knot Floer chain complex $(\widehat{CFK}(S^3, K), \partial)$, where the differential ∂ ignores the presence of the second basepoint, that is $H_*(\widehat{CFK}(S^3, K), \partial) = \widehat{HF}(S^3)$. In particular, every sublevel $\widehat{CFK}(S^3, K)_{A\le s}$ is preserved by ∂ and we can take its homology.

Definition 3.1 We denote by $\tau(K)$ the smallest integer s such that the inclusion of the s^{th} filtration sublevel induces a nontrivial map

$$H_*(\widehat{CFK}(S^3, K)_{A\le s}, \partial) \longrightarrow \widehat{HF}(S^3) = \mathbb{F}.$$

This invariant turns out to provide a powerful lower bound for the slice genus of K, in the sense that $|\tau(K)| \le g_*(K)$ [35]. One of the properties it enjoys, and that we will need, is that $\tau(\overline{K}) = -\tau(K)$ for every K.

The definition of ν is somewhat more involved. It comes from the mapping cone construction [40; 41] that Ozsváth and Szabó used to compute the rank of the Heegaard–Floer homology of integer and rational surgeries along K. We just recall the parts of the construction that we need to get to the definition, without giving any motivation or complete explanation of the mapping cone, following Rasmussen [44].

We define a new complex (A_s, ∂_s) for each integer s as follows: The underlying module A_s is just $C = \widehat{CFK}(S^3, K)$. The differential ∂_s takes into account both the differential ∂ and the differential ∂', for which the role of the basepoints is reversed (ie ∂ counts differentials whose domains pass through the basepoint z but not through w, while ∂' counts differential whose domains pass through w but not through z); in the next formula, ∂_K is just the "graded" differential that counts only discs whose domains avoid both basepoints:

$$\partial_s \boldsymbol{x} = \begin{cases} \partial \boldsymbol{x} & \text{if } A(\boldsymbol{x}) < s, \\ \partial \boldsymbol{x} + \partial' \boldsymbol{x} + \partial_K \boldsymbol{x} & \text{if } A(\boldsymbol{x}) = s, \\ \partial' \boldsymbol{x} & \text{if } A(\boldsymbol{x}) > s. \end{cases}$$

The quotient complexes $A_s/C_{>s}$ and $A_s/C_{<s}$ come with natural chain maps into (C, ∂) and (C, ∂') respectively; the composition of the projection with these chain maps gives two maps $v_s, h_s\colon H_*(A_s, \partial_s) \to \widehat{HF}(S^3) = \mathbb{F}$.

In analogy with the definition of τ, we have the following:

Definition 3.2 We denote by $\nu(K)$ the smallest integer s such that the map v_s is nontrivial.

Ozsváth and Szabó proved that ν is a concordance invariant and that the inequalities $\tau(K) \leq \nu(K) \leq \tau(K)+1$ hold for all knots K. We remarked earlier that τ changes sign when taking the mirror of the knot: ν does not have this property and the discrepancy between $\nu(K)$ and $-\nu(\bar{K})$ is measured by Hom's invariant ε:

Definition 3.3 We define $\varepsilon(K)$ to be $(\tau(K)-\nu(K))-(\tau(\bar{K})-\nu(\bar{K}))$.

The value of ε can only be in $\{-1, 0, 1\}$, and manifestly changes sign when we take the mirror of the knot. Hom also proves that:

Proposition 3.4 [18] *The value of $\varepsilon(K)$ controls the relationship between $\tau(K)$ and $\nu(K)$ as follows:*

- *If $\varepsilon(K) = 0$, then $\tau(K) = \nu(K) = \nu(\bar{K}) = 0$.*
- *If $\varepsilon(K) = 1$ then $\nu(K) = \tau(K)$.*
- *If $\varepsilon(K) = -1$ then $\nu(K) = \tau(K) + 1$.*

3.2 Modules

We now turn our attention back to $\widehat{HFK}(-S^3_t(K), \widetilde{K}) \simeq SFH(-S^3_{K,t})$. Recall that this is a graded $\mathbb{F}$–vector space and that we call its grading A.

The group $\widehat{CFK}(S^3, K)$ is a graded vector space that comes with two differentials ∂_K and ∂ such that the complex $(\widehat{CFK}(S^3, K), \partial)$ has homology $\widehat{HF}(S^3) = \mathbb{F}$, while the complex $(\widehat{CFK}(S^3, K), \partial_K)$ has homology $\widehat{HFK}(S^3, K)$. We call the Alexander grading on this group A as well.

Let us set $d = \dim \widehat{HFK}(S^3, K)$ and fix a basis $\mathcal{B} = \{\eta_i, \eta'_j \mid 0 \leq i < d\}$ of $\widehat{CFK}(S^3, K)$ such that the set $\{\eta_i^{\mathrm{top}}, (\eta'_j)^{\mathrm{top}}\}$ of the highest nontrivial Alexander-homogeneous components of the η_i and η'_j is still a basis for $\widehat{CFK}(S^3, K)$, and the following relations hold (see Lipshitz, Ozsváth and Thurston [26, Section 11.5]):

$$\begin{aligned} \partial\eta_0 &= 0, & \partial_K\eta_0 &= 0, \\ \partial\eta_{2i-1} &= \eta_{2i}, & \partial_K\eta_i &= 0, \\ \partial\eta'_{2j-1} &= \eta'_{2j}, & \partial_K\eta'_{2j-1} &= \eta'_{2j}. \end{aligned}$$

Observe that the set of homology classes of the η_i is a basis for $\widehat{HFK}(S^3, K) = H_*(\widehat{CFK}(S^3, K), \partial_K)$. Finally, call $\delta(i) = A(\eta_{2i}) - A(\eta_{2i-1})$. Let us remark that by definition $A(\eta_0) = \tau := \tau(K)$.

Theorem 3.5 [26] *The homology group $\widehat{HFK}(-S^3_m(K), \widetilde{K})$ is an $\mathbb{F}$–vector space with basis $\{d_{i,j}, d^*_{i,j}, u_\ell \mid 1 \le i \le k, 1 \le j \le \delta(i), 1 \le \ell \le |2\tau - m|\}$, where the generators satisfy*

$$A(d_{i,j}) = A(\eta_{2i}) - (j-1) - (m-1)/2 = -A(d^*_{i,j}),$$
$$A(u_\ell) = \tau - (\ell - 1) - (m-1)/2.$$

Generators with a $*$ are to be thought of as symmetric to the generators without it, and each family $\{d_{i,j}\}_j$ can be interpreted as representing the arrow $\partial\colon \eta_{2i-1} \to \eta_{2i}$ (notice that i varies among *positive* integers), counted with a multiplicity equalling its length (ie the distance it covers in Alexander grading).

Remark 3.6 Not any basis of $\widehat{HFK}(-S^3_m(K), \widetilde{K})$ with the same degree properties works for our purposes: we are actually choosing a basis that is compatible with stabilisation maps, as we are going to see in Theorem 3.11 (see also Remark 3.13).

Definition 3.7 Call S_+ the subspace of $\widehat{HFK}(-S^3_m(K), \widetilde{K})$ generated by $\{d_{i,j}\}$, and S_- the one generated by $\{d^*_{i,j}\}$: the subspace $S = S_+ \oplus S_-$ is the *stable complex*, and elements of S are called *stable elements*. The subspace spanned by $\{u_\ell\}$ is called the *unstable complex* and will be denoted by U_m (although the subscript will be often dropped), so that $\widehat{HFK}(-S^3_m(K), \widetilde{K})$ decomposes as $S_+ \oplus U_m \oplus S_-$.

It is worth remarking that the decomposition given in the definition above is *not* canonical: the three stable subspaces $S_\pm$ and S are canonically defined, but the unstable complex is not. This issue will be addressed at the end of the next section.

There is a handy pictorial description when $|m|$ is sufficiently large; we will be mostly dealing with negative values of m, so let us call $m' = -m \gg 0$. Consider a direct sum $\widetilde{C} = \bigoplus_{i=1}^{m'} C_i$ of m' copies of $C = \widehat{CFK}(S^3, K)$ and (temporarily) denote by $\boldsymbol{x}_i$ the copy of the element $\boldsymbol{x} \in C$ in C_i. Endow $\widetilde{C}$ with a shifted Alexander grading

$$\widetilde{A}(\boldsymbol{x}_i) = \begin{cases} A(\boldsymbol{x}) - (i-1) - (m-1)/2 & \text{for } i \le m'/2, \\ -A(\boldsymbol{x}) - (i-1) - (m-1)/2 & \text{for } i > m'/2, \end{cases}$$

for each homogeneous $\boldsymbol{x}$ in $\widehat{CFK}(S^3, K)$. We picture this situation by considering each copy of C as a vertical tile of $2g(K)+1$ boxes — each corresponding to a value for the Alexander grading, possibly containing no generators at all or more than one generator — and stacking the m' copies of C in staircase fashion, with C_1 as the top block and $C_{m'}$ as the bottom block. Notice that, by our grading convention, the copies in the bottom part of the picture are turned upside down: for example, if $\boldsymbol{x}^{\max} \in C$ has maximal Alexander degree $A(\boldsymbol{x}) = g(K)$, then $\boldsymbol{x}_1^{\max}$ lies in the top box of C_1, while $\boldsymbol{x}_{m'}^{\max}$ lies in the bottom box of $C_{m'}$. Likewise, an element $\boldsymbol{x}^\tau \in C$ has Alexander degree $A(\boldsymbol{x}) = \tau$, then $\boldsymbol{x}_1^\tau$ lies in the $(g(K)-\tau+1)^{\text{th}}$ box from the top in C_1 and $\boldsymbol{x}_{m'}^\tau$ lies in the $(g(K)-\tau+1)^{\text{th}}$ box from the bottom in $C_{m'}$.

Our construction is slightly different from the construction described by Hedden [16, Section 4], and in general it gives a different chain complex for $\widehat{HFK}(S^3_m(K), \widetilde{K})$, but their homologies agree.

The situation is depicted in Figure 3.1. In this concrete example we have $g(K) = 2$ and $\tau(K) = -1$; accordingly, there are $2g(K)+1 = 5$ boxes in each vertical column and $\boldsymbol{x}_1^\tau$ lies in the fourth box from the top in C_1.

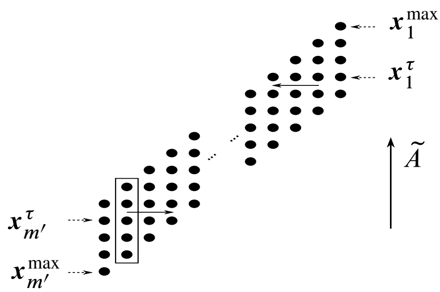

Figure 3.1: We represent here the top (on the right) and bottom (on the left) parts of $\widehat{HFK}(S^3_m(K), \widetilde{K})$ for $m \ll 0$. Each vertical tile is a copy of $\widehat{CFK}(S^3, K)$ and the arrows show the direction of the differentials.

Now define a differential $\widetilde{\partial}$ on $\widetilde{C}$ by

$$\widetilde{\partial} : \begin{cases} (\eta_0)_i \mapsto 0 & \text{for small and large } i, \\ (\eta_{2j-1})_i \mapsto (\eta_{2j})_{i+\delta(j)} \mapsto 0 & \text{for small } i, \\ (\eta_{2j-1})_i \mapsto (\eta_{2j})_{i-\delta(j)} \mapsto 0 & \text{for large } i, \\ (\eta'_{2j-1})_i \mapsto (\eta'_{2j})_i \mapsto 0 & \text{for every } i. \end{cases}$$

We have not yet defined what the differential does to nonprimed generators for intermediate values of i: in the picture we have, the differentials are horizontal and point "inwards" (see Figure 3.1); in particular, every horizontal (ie Alexander-homogeneous) block of boxes is a subcomplex. We extend the differential to be any map $\widetilde{\partial}$ such that the level $\{\widetilde{A} = j\}$ is a subcomplex for every j, with homology $\widehat{HF}(S^3) = \mathbb{F}$ for intermediate values of the j (we can do this as $\{\widetilde{A} = j\}$ has odd rank for every intermediate value of j).

We are now going to analyse what happens on the top and bottom part of the complex (ie when i is small or large, in what follows), when we take the homology.

Pairs $(\eta'_{2j-1})_i, (\eta'_{2j})_i$ cancel out in homology. The element $(\eta_{2j})_i$ is a cycle for each i, j, and it is a boundary only when $j > 0$ and either $i > \delta(j)$ or $i < m' - \delta(j)$: so there are $2\delta(j)$ surviving copies of η_{2j}, in degrees $A(\eta_{2j}) - k - (m-1)/2$ and $-A(\eta_{2j}) + k + (m-1)/2$ for $k = 0, \dots, \delta(j) - 1$. We can declare $d_{i,j} = [(\eta_{2j})_i]$ and $d^*_{i,j} = [(\eta_{2j})_{m'-i}]$.

The element $(\eta_0)_i$ is a cycle for every i, and it is never cancelled out, so it survives when taking homology. Given our grading convention, for *small* values of i, $\widetilde{A}((\eta_0)_i) = A(\eta_0) - (i-1) - (m-1)/2 = \tau(K) - (i-1) - (m-1)/2$, and in particular we have a nonvanishing class $[(\eta_0)_i] = u_i$ in degrees $\tau(K) - (m-1)/2, \tau(K) - (m-1)/2 - 1, \dots$. On the other hand, when i is *large*, $[(\eta_0)_i]$ lies in degree $-\tau(K) - (i-1) - (m-1)/2$, and we get a nonvanishing class $[(\eta_0)_i] = u_{2\tau(K)+i+(m-1)/2}$ in degrees $-\tau(K) + (m-1)/2, -\tau(K) + (m-1)/2 + 1, \dots$.

We also have a string of $\mathbb{F}$ summands in between, giving us a strip of unstable elements of length $2\tau(K) - m$ as in Theorem 3.5.

Remark 3.8 Something more can be said about Spin^c structures: when $m \neq 0$, $\widehat{HFK}(S^3_m(K), \widetilde{K})$ splits as a sum of subcomplexes $\widehat{HFK}(S^3_m(K), \widetilde{K}; \mathfrak{s}_i)$ corresponding to the $|m|$ different Spin^c structures on $S^3_m(K)$. The Alexander grading $\widetilde{A}$ tells us when two horizontal subcomplexes fall into the same Spin^c structure: as one could expect, if $\widetilde{A}(x) \equiv \widetilde{A}(y) \pmod{m}$, then x and y belong to the same summand $\widehat{HFK}(S^3_m(K), \widetilde{K}; \mathfrak{s})$.

3.3 Stabilisation maps

We are going to study the action of the two stabilisation maps $\sigma_\pm$ of Definition 2.22 on the sutured Floer homology groups $SFH(-S^3_L)$. It is worth stressing that these maps do not depend on the contact structure on the knot complement or on the particular Legendrian representative, but just on its Thurston–Bennequin number (which determines domain and codomain).

Notice that if L is a Legendrian knot in S^3 with $\mathrm{tb}(L) = n$, then, as a sutured manifold, S^3_L is just $S^3_{K,n}$. Moreover, if L' is a stabilisation of L, then $S^3_{L'}$ is isomorphic to $S^3_{K,n-1}$ as a sutured manifold.

Recall that we have two families (indexed by the integer n) of stabilisation maps, $\sigma_\pm\colon SFH(-S^3_{K,n}) \to SFH(-S^3_{K,n-1})$, corresponding to the gluing of the negative and positive stabilisation layer: if the knot K is oriented, these maps can be labelled as σ_- or σ_+. With a slight abuse of notation, we are going to ignore the dependence of these maps on the framing.

Remark 3.9 Notice that orientation reversal of L or K is not seen by the sutured groups nor by $EH(L)$, but it swaps the roles of σ_- and σ_+.

Remark 3.10 Let us recall that for an *oriented* Legendrian knot L of topological type K in S^3 the Bennequin inequality holds:

$$\mathrm{tb}(L) + r(L) \leq 2g(K) - 1.$$

In [43], Plamenevskaya proved a sharper result:

$$\mathrm{tb}(L) + r(L) \leq 2\tau(K) - 1. \tag{3-2}$$

This last form of the Bennequin inequality, together with Theorem 3.5, tells us that whenever we are considering knots in the standard S^3, the unstable complex is never trivial in $SFH(-S^3_{K,n})$. More precisely we are always (strictly) below the threshold $2\tau := 2\tau(K)$, so that $2\tau - m$ is always positive; in particular, the dimension of the unstable complex is always positive and *increases* under stabilisations. We will state the theorem in its full generality anyway, even though this remark tells us we need just half of it when working in (S^3, ξ_{st}).

We are going to prove the following result:

Theorem 3.11 *The maps $\sigma_-, \sigma_+ \colon SFH(-S^3_{K,n}) \to SFH(-S^3_{K,n-1})$ act as follows:*

$$\sigma_-\colon \begin{cases} d_{i,j} \mapsto d_{i,j}, \\ u_\ell \mapsto u_\ell, \\ d^*_{i,j} \mapsto d^*_{i,j+1}, \end{cases} \qquad \sigma_+\colon \begin{cases} d_{i,j} \mapsto d_{i,j+1}, \\ u_\ell \mapsto u_{\ell+1}, \\ d^*_{i,j} \mapsto d^*_{i,j}, \end{cases} \qquad \textit{for } n \leq 2\tau;$$

$$\sigma_-\colon \begin{cases} d_{i,j} \mapsto d_{i,j}, \\ u_\ell \mapsto u_\ell, \\ u_{n-2\tau} \mapsto 0, \\ d^*_{i,j} \mapsto d^*_{i,j+1}, \end{cases} \qquad \sigma_+\colon \begin{cases} d_{i,j} \mapsto d_{i,j+1}, \\ u_\ell \mapsto u_{\ell-1}, \\ u_1 \mapsto 0, \\ d^*_{i,j} \mapsto d^*_{i,j}, \end{cases} \qquad \textit{for } n > 2\tau.$$

Notice that we are implicitly choosing an appropriate isomorphism between the group $SFH(-S^3_{K,n})$ and the vector space generated by the $d_{i,j}$ and the u_i (see Theorem 3.5 and Remark 3.13).

There is an interpretation of the maps

$$\sigma_\pm\colon SFH(-S^3_{K,n}) \longrightarrow SFH(-S^3_{K,n-1})$$

in terms of Figure 3.1 when $n \ll 0$: Fix a chain complex C computing $\widehat{HFK}(S^3, K)$ and call $(\widetilde{C}_n, \widetilde{\partial})$ and $(\widetilde{C}_{n-1}, \widetilde{\partial})$ the two complexes defined in the previous section, computing $SFH(-S^3_{K,n})$ and $SFH(-S^3_{K,n-1})$ starting from C. We have two "obvious" chain maps $s_\pm\colon \widetilde{C}_n \to \widetilde{C}_{n-1}$: s_- sends $\boldsymbol{x}_i \in \widetilde{C}_n$ to $\boldsymbol{x}_i \in \widetilde{C}_{n-1}$, while s_+ sends $\boldsymbol{x}_i \in \widetilde{C}_n$ to $\boldsymbol{x}_{i+1} \in \widetilde{C}_{n-1}$. The maps $s_\pm$ induce the two stabilisation maps $\sigma_\pm$ at the homology level.

The map s_- is the inclusion $\widetilde{C}_n \hookrightarrow \widetilde{C}_{n-1}$ that misses the leftmost vertical tile (that is, the copy C_{1-n} of C that is in lowest Alexander degree), while s_+ is the inclusion that misses the rightmost vertical tile (the copy C_1 of C that lies in highest Alexander degree).

As a corollary (of the proof), we obtain a graded version of the result:

Corollary 3.12 *The maps $\sigma_\pm$ are Alexander-homogeneous of degree $\mp\frac{1}{2}$.*

Remark 3.13 Let us first consider the case $n \le 2\tau(K)$: notice that Theorem 3.11 shows that $S_\pm$ is $\bigcup_n \ker \sigma^n_\pm$, and we immediately obtain that $S_\pm$ is independent of the basis we have chosen in Theorem 3.5.

The situation for the unstable complex, on the other hand, is completely different: We have a well-defined *unstable quotient* (see Remark 6.9 below) $SFH(-S^3_{K,n})/S$. The unstable complex, as we defined it, is the image of a homogeneous section of the quotient map $SFH(-S^3_{K,n}) \to SFH(-S^3_{K,n})/S$ with some additional requirements. Namely, we need to choose *any* homogeneous section of the quotient map for $n = 2\tau(K) - 1$ and then we take the subcomplexes generated by compositions of σ_+ and σ_- to generate the unstable complexes in $SFH(-S^3_{K,n})$ for smaller values of n. We give below an example to show that there are actually instances where the choice of the section matters.

Notice that $\sigma_\pm$ act on the unstable quotients as well, and the action for $n \le 2\tau$ is given by the unique homogeneous injection of degree $\mp\frac{1}{2}$.

Finally, let us consider the case $n > 2\tau(K)$. Here the situation is reversed: The unstable complex is the intersection $U = \bigcup_n \ker \sigma^n_- \cap \bigcup_n \ker \sigma^n_+$, and is therefore well defined and independent of the isomorphism. The two unstable complexes, on the other hand, depend on the choice of suitable sections of the quotient maps $\bigcup_n \ker \sigma^n_\pm \to U$.

Example 3.14 We can give a concrete example to show that the choice of the unstable complex is not unique. There is a recent result of Baldwin, Vela-Vick and Vértesi [2] that relates the combinatorial Legendrian invariants $\widehat{\lambda}_\pm(L), \lambda^-_\pm(L)$ of Ozsváth, Szabó and Thurston [42] and the invariants $\widehat{\mathcal{L}}(\pm L), \mathcal{L}^-(L)$ of Lisca, Ozsváth, Stipsicz and Szabó [27]: there are two Legendrian representatives L_1, L_2 of the pretzel knot $K = P(-4,-3,3) = m(10_{140})$ in (S^3, ξ_{st}) that have $\mathrm{tb}(L_i) = -1$, $r(L_i) = 0$, but

$\widehat{\mathcal{L}}(L_1) = 0 \neq \widehat{\mathcal{L}}(L_2)$ (the example is found in Ng, Ozsváth and Thurston [31], where they are distinguished by the combinatorial invariants). Notice that $\tau(K) = 0$ (see Cha and Livingston [3]), therefore the unstable complex in $SFH(-S^3_{L_i})$ has length $|\mathrm{tb}(L_i) - 2\tau(K)| = 1$ and $EH(L_i)$ has the same degree as the degree of the only nonzero element of the unstable complex (see Proposition 3.20 below).

Since the mapping class group of $S^3 \setminus N(K)$ relative to the boundary is trivial (see Kodama and Sakuma [24]), the fact that $\widehat{\mathcal{L}}$ distinguishes these two knots for some parametrisation implies that it distinguishes them for all parametrisations (see the discussion preceding Lemma 6.11 below).

Neither $EH(L_1)$ nor $EH(L_2)$ gets killed by $\sigma_+^n \circ \sigma_-^n$, since the trivial filling (ie contact ∞–surgery) yields back the standard contact structure on S^3 (compare with Lemma 3.18 below). In particular, we can define the generator of the unstable complex to be either of $EH(L_1)$ or $EH(L_2)$, and these two elements are distinct by [49]. Here we are using the fact that $\mathrm{tb}(L_i) = 2\tau(K) - 1$: if this were not the case, and $\mathrm{tb}(L_i) < 2\tau(K) - 1$, we would need to check that an element that does not vanish under $\sigma_+^n \circ \sigma_-^n$ is in fact the stabilisation of at least one element in $SFH(-S^3_{K,\mathrm{tb}(L_i)+1})$, because of Theorem 3.11.

3.4 The proof of Theorem 3.11

In this section we are going to give a proof of Theorem 3.11. The main point is the interaction of stabilisation maps with bordered Floer homology. The reader is referred to [26, Chapter 11 and Appendix A] for definitions and properties.

Let $\mathcal{H} = (\Sigma \setminus \{p\}, \{\beta_1^a, \beta_2^a\}, \boldsymbol{\beta}^c, \boldsymbol{\alpha})$ be a bordered diagram for $S^3 \setminus N(K)$ such that the closures of β_1^a, β_2^a represent the curves $\lambda - (n+1)\mu$ and $\lambda - n\mu$ respectively, where $\lambda \subset \partial N(K)$ is the Seifert longitude for K, and $\mu \subset \partial N(K)$ is the meridian.

Let us call $W = \widehat{CFD}(\mathcal{H})$ and $V = \widehat{CFD}(\mathcal{H}')$, where $\mathcal{H}'$ is the bordered diagram $(\Sigma, \{\mu, \beta_2^a\}, \boldsymbol{\beta}^c, \boldsymbol{\alpha})$. As in [26], we will use the notation V^j, W^j to denote the submodules $\iota_j V$, $\iota_j W$ of V and W in the idempotent ι_j for $j = 0, 1$. When talking about coefficient maps, we will use the superscripts V, W to distinguish between the maps acting on V and the ones acting on W.

Finally, recall that we have four isomorphisms

$$H_*(V^0, D^V) \simeq \widehat{HFK}(S^3, K), \qquad H_*(V^1, D^V) \simeq SFH(S^3_{K,-n}),$$
$$H_*(W^0, D^W) \simeq SFH(S^3_{K,-n-1}), \quad H_*(W^1, D^W) \simeq SFH(S^3_{K,-n}).$$

We denote $\widehat{CFK}(S^3, K)$ by V^0 and we can write down explicitly a chain homotopy equivalence between the model for V^1 found in [26] and the model $\widetilde{C}$ for Theorem 3.5.

Since $\widetilde{C}$ computes the sutured Floer group $SFH(-S^3_{K,n})$, which is in turn the cohomology group $SFH(S^3_{K,n})$, we expect V^1 to be chain homotopic equivalent to the *dual* complex of $\widetilde{C}$: in fact, V^1 *is* the dual to $\widetilde{C}$ at the top and at the bottom and is a sum of copies of $\mathbb{F} = \widehat{HF}(S^3)$ for intermediate values of the Alexander grading. The differential D^V is the knot Floer differential ∂_K on V^0 and it is the adjoint of $\widetilde{\partial}$ on V^1. The maps D^V_1 and D^V_3 are adjoint to the projections

$$\widetilde{C} \to C_n \simeq \widehat{CFK}(S^3, K) \quad \text{and} \quad \widetilde{C} \to C_1 \simeq \widehat{CFK}(S^3, K)$$

respectively. The map D^V_2 is adjoint to the inclusion $C \simeq C_1 \hookrightarrow \widetilde{C}$ and the map D^V_{23} is adjoint to the shift map $\boldsymbol{x}_i \mapsto \boldsymbol{x}_{i-1}$.

Legendrian stabilisations are induced by a single bypass attachment; specialising general results of Zarev [51] to this case, one gets the following proposition.

Proposition 3.15 [51] *The* adjoints *of the coefficient maps D^W_1 and D^W_3 induce the two stabilisation maps*

$$SFH(-S^3_{K,-n}) \simeq H^*(W^1, D^W) \xrightarrow{\sigma_\pm} H^*(W^0, D^W) \simeq SFH(-S^3_{K,-n-1}).$$

Remark 3.16 For convenience, we sketch how Proposition 3.15 follows from Zarev's results. Zarev proves that the bordered algebra associated to a surface decomposes as a direct sum of certain sutured Floer homology groups and that the action of the bordered algebra on bordered Floer homology (more specifically, on $\widehat{CFA}$) can be interpreted via Honda–Kazez–Matić gluing maps.

In our case, we are interested in the sutured Floer group associated to the component $\iota_0 \mathcal{A} \iota_1$ of the bordered algebra of the torus, that is, $\langle \rho_1, \rho_3, \rho_{123} \rangle_{\mathbb{F}}$. By tensoring with the bimodule $\widehat{CFDD}(\mathrm{id})$ we reduce to studying the action of the coefficient maps D_1, D_3, D_{123} on $\widehat{CFD}$. We now know that $\sigma_\pm$ is a linear combination of the adjoints of these three maps.

To see D_1, D_3 are the right ones, first observe that the coefficient maps D_1, D_3, D_{123} and stabilisation maps $\sigma_\pm$ are homogeneous, but that D_1, D_3, D_{123} in general have different degrees. Thus the stabilisation maps $\sigma_\pm$ are adjoint to D_1, D_3 or D_{123} (as opposed to a linear combination thereof).

We can rule out D_{123} because it vanishes on any framed unknot (this follows from a small extension of the computations below).

We are now able to prove Theorem 3.11, which is just a computation in light of the previous proposition.

Proof In [26, Appendix A], Lipshitz, Oszváth and Thurston explicitly describe an $\mathcal{A}(T^2)$–bimodule $\widehat{CFDA}(\tau_\lambda)$ such that $W = \widehat{CFDA}(\tau_\lambda) \boxtimes V$. We describe here its structure.

The bimodule $\widehat{CFDA}(\tau_\lambda)$ is generated over $\mathbb{F}$ by three vectors $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{s}$. The idempotents act on $\widehat{CFDA}(\tau_\lambda)$ as follows

$$W^0 = \boldsymbol{p} \boxtimes V^0 \oplus \boldsymbol{s} \boxtimes V^1, \quad W^1 = \boldsymbol{q} \boxtimes V^1.$$

We also have the relations

$$\begin{aligned} m_{0,1,1}(\boldsymbol{p}, \rho_3) &= \rho_3 \otimes \boldsymbol{q}, & m_{0,1,0}(\boldsymbol{s}) &= \rho_1 \otimes \boldsymbol{q}, \\ m_{0,1,1}(\boldsymbol{s}, \rho_2) &= \boldsymbol{p}, & m_{0,1,1}(\boldsymbol{s}, \rho_{23}) &= \rho_3 \otimes \boldsymbol{q}. \end{aligned}$$

We can now compute the action of the coefficient maps D^W, D_1^W, D_3^W on W: for all $\boldsymbol{x} \in V^0$, $\boldsymbol{y} \in V^1$ we have

$$\begin{aligned} &D^W\colon \boldsymbol{p} \boxtimes \boldsymbol{x} + \boldsymbol{s} \boxtimes \boldsymbol{y} \mapsto \boldsymbol{p} \boxtimes (D^V \boldsymbol{x} + D_2^V \boldsymbol{y}) + \boldsymbol{s} \boxtimes D^V \boldsymbol{y}, \\ &D^W\colon \boldsymbol{q} \boxtimes \boldsymbol{y} \mapsto \boldsymbol{q} \boxtimes D \boldsymbol{y}, \\ &D_1^W\colon \boldsymbol{p} \boxtimes \boldsymbol{x} + \boldsymbol{s} \boxtimes \boldsymbol{y} \mapsto \boldsymbol{q} \boxtimes \boldsymbol{y}, \\ &D_3^W\colon \boldsymbol{p} \boxtimes \boldsymbol{x} + \boldsymbol{s} \boxtimes \boldsymbol{y} \mapsto \boldsymbol{q} \boxtimes (D_3^V \boldsymbol{x} + D_{23}^V \boldsymbol{y}). \end{aligned} \tag{3-3}$$

The model for $SFH(-S^3_{K,-n-1})$ given by the dual of W^0 agrees with the model of Theorem 3.5 under the linear isomorphism that identifies the subspace $\boldsymbol{p} \boxtimes V^0$ with the dual of C_1, sitting as the leftmost column in Figure 3.1, and the subspace $\boldsymbol{s} \boxtimes V^1$ with the dual to $\bigoplus_{k \geq 2} C_k$, consisting of the n rightmost columns.

The adjoint of D^W acts on the dual of W^0 so that the dual of $\boldsymbol{s} \boxtimes V^1$ is a subcomplex. By (3-3), $D_1^W(\boldsymbol{p} \boxtimes \boldsymbol{x} + \boldsymbol{s} \boxtimes \boldsymbol{y}) = \boldsymbol{q} \boxtimes \boldsymbol{x}$: in other words, $\boldsymbol{p} \boxtimes V^0 \subset \ker D_1^W$ and D_1^W is the isomorphism $\boldsymbol{s} \boxtimes V^1$ onto $W^1 \simeq V^1$ that is the identity on the second factor. In particular, the adjoint of D_1^W is the inclusion of the dual of W^1 into the dual of W^0 as the subcomplex dual to $\boldsymbol{s} \boxtimes V^1$, that is the subcomplex generated by the C_k with $k \geq 2$.

Similarly, the adjoint of D_3^W is seen to act as the inclusion of the dual of W^1 into the dual of W^1 as the subcomplex generated by the C_k with $k \leq n$.

We have given a concrete identification of $H_*(\widetilde{C})$ with the model of Definition 3.7, where the class of $(\eta_{2j})_i$ is identified with $d_{i,j}$ for $i \leq \delta(j)$ and with $d_{m'-i,j}$ for

$i \geq m' - \delta(j)$, and $(\eta_0)_i$ is identified with u_i. The adjoint of D_1^W is just the inclusion map $\boldsymbol{x}_i \mapsto \boldsymbol{x}_i$, whereas the adjoint of D_3^W is the inclusion $\boldsymbol{x}_i \mapsto \boldsymbol{x}_{i+1}$ for all i. In particular, the induced maps act on homology as claimed.

The result for arbitrary framing parameter n follows from [26, Theorem A.11]: they prove that $V^1 = \iota_1 CFD(S^3 \setminus K)$ decomposes of the sum of a stable complex (containing the dual to $S_+ \oplus S_-$) and an unstable chain (containing the dual to U) as follows. We can pick bases $\{\xi_i\}, \{\eta_i\}$ for $\widehat{CFK}(S^3, K)$ playing the roles of the basis $\mathcal{B}$ used in Theorem 3.5,[1] one with respect to the basepoint z and the other with respect to the basepoint w. We also introduce strings of elements $\{\kappa_i^k\}, \{\lambda_i^k\}$ of length $\delta(i)$ associated to each arrow

$$\xi_i \xrightarrow{\partial_z} \xi_{i+1}, \qquad \eta_i \xrightarrow{\partial_w} \eta_{i+1}$$

respectively, both of length $\delta(i)$.

The stable complex in V^1 looks like

$$\xi_i \xrightarrow{D_1^V} \kappa_1^i \xleftarrow{D_{23}^V} \kappa_2^i \xleftarrow{D_{23}^V} \cdots \xleftarrow{D_{23}^V} \kappa_{\delta(i)}^i \xleftarrow{D_{123}^V} \xi_{i+1},$$
$$\eta_i \xrightarrow{D_3^V} \lambda_1^i \xrightarrow{D_{23}^V} \lambda_2^i \xrightarrow{D_{23}^V} \cdots \xrightarrow{D_{23}^V} \lambda_{\delta(i)}^i \xrightarrow{D_2^V} \eta_{i+1},$$

and it is immediate to find an identification of the ι_1 part of the stable complex with the dual of $S_+ \oplus S_-$ in Theorem 3.5, $d_{i,j}$ with the dual of κ_j^{2i-1} and $d_{i,j}^*$ with the dual of λ_j^{2i-1}.

The unstable chain, on the other hand, depends on the framing as follows:

$$\xi_0 \xrightarrow{D_1^V} \mu_1 \xleftarrow{D_{23}^V} \mu_2 \xleftarrow{D_{23}^V} \cdots \xleftarrow{D_{23}^V} \mu_{2\tau(K)-n} \xleftarrow{D_3^V} \eta_0, \qquad \text{for } n < 2\tau(K),$$
$$\xi_0 \xrightarrow{D_{12}^V} \eta_0, \qquad \text{for } n = 2\tau(K),$$
$$\xi_0 \xrightarrow{D_{123}^V} \mu_1 \xrightarrow{D_{23}^V} \mu_2 \xrightarrow{D_{23}^V} \cdots \xrightarrow{D_{23}^V} \mu_{n-2\tau(K)} \xrightarrow{D_2^V} \eta_0, \qquad \text{for } n > 2\tau(K),$$

and we can identify u_k in the unstable complex of $SFH(-S^3_{K,n})$ with the dual of μ_k.

Let us call $W := \widehat{CFDA}(\tau_\lambda) \boxtimes V$. Then, as above,

$$W^0 = \boldsymbol{p} \boxtimes V^0 \oplus \boldsymbol{s} \boxtimes V^1, \quad W^1 = \boldsymbol{q} \boxtimes V^1,$$

and the action of the maps D^W, D_1^W and D_3^W is controlled by (3-3). We have an obvious identification of the dual of W^1 with $SFH(-S^3_{K,n})$ that respects the stable-unstable decomposition.

[1] We are going to forget about primed elements as they do not play any role in homology.

We identify the stable complex $S_+ \subset SFH(-S^3_{K,n-1})$ with the cohomology of the subcomplex of W^0 spanned by $\boldsymbol{s} \boxtimes \kappa_i^j$ via the map $d_{i,j} \leftrightarrow (\boldsymbol{s} \boxtimes \kappa_j^{2i-1})^*$. We identify $S_- \subset SFH(-S^3_{K,n-1})$ with the cohomology of the subcomplex of W^0 spanned by $\boldsymbol{s} \boxtimes \lambda_i^j$ and $\boldsymbol{p} \boxtimes \eta_i$ via the map $d^*_{i,1} \leftrightarrow (\boldsymbol{p} \boxtimes \eta_{2i-1})^*, d^*_{i,j} \leftrightarrow (\boldsymbol{p} \boxtimes \lambda^{2i-1}_{j-1})^*$. Notice that (3-3) imply that for every odd i there is an arrow

$$\boldsymbol{s} \boxtimes \lambda^i_{\delta(i)} \xrightarrow{D^W} \boldsymbol{p} \boxtimes \eta_{i+1},$$

so that the homology of the stable complex in V has constant rank.

The unstable complex of $SFH(-S^3_{K,n})$ is identified with the cohomology of the subspace of W^0 spanned by $\boldsymbol{s} \boxtimes \mu_\ell$ and $\boldsymbol{p} \boxtimes \eta_0$ via one of these two maps: if $n \leq 2\tau(K)$, we identify u_k with the dual of $\boldsymbol{s} \boxtimes \mu_k$ and $u_{2\tau(K)-n+1}$ with the dual of $\boldsymbol{p} \boxtimes \eta_0$; if $n > 2\tau(K)$, we just identify u_k with the dual of $\boldsymbol{s} \boxtimes \mu_k$. Notice that in the latter case there is an arrow

$$\boldsymbol{s} \boxtimes \mu_{n-2\tau(K)} \xrightarrow{D^W} \boldsymbol{p} \boxtimes \eta_0$$

that cancels out the two generators involved, in cohomology, so that both maps are isomorphisms.

Now, the maps D_1^W, D_3^W act on the stable complex as follows:

$$\begin{tikzcd}
& \boldsymbol{s} \boxtimes \kappa_1^i \arrow[dl, "D_3^W"'] \arrow[d, "D_1^W"'] & \boldsymbol{s} \boxtimes \kappa_2^i \arrow[dl, "D_3^W"'] \arrow[d, "D_1^W"'] & \cdots & & \boldsymbol{s} \boxtimes \kappa^i_{\delta(i)} \arrow[dl, "D_3^W"'] \arrow[d, "D_1^W"'] \\
0 & \boldsymbol{q} \boxtimes \kappa_1^i & \boldsymbol{q} \boxtimes \kappa_2^i & \cdots & \boldsymbol{q} \boxtimes \kappa^i_{\delta(i)-1} & \boldsymbol{q} \boxtimes \kappa^i_{\delta(i)}
\end{tikzcd}$$

$$\begin{tikzcd}
\boldsymbol{p} \boxtimes \eta_i \arrow[dr, "D_3^W"] & \boldsymbol{s} \boxtimes \lambda_1^i \arrow[d, "D_1^W"] \arrow[dr, "D_3^W"] & & \cdots & \boldsymbol{s} \boxtimes \lambda^i_{\delta(i)} \arrow[r, "D^W"] \arrow[d, "D_1^W"] & \boldsymbol{p} \boxtimes \eta_{i+1} \\
& \boldsymbol{q} \boxtimes \lambda_1^i & \boldsymbol{q} \boxtimes \lambda_2^i & \cdots & \boldsymbol{q} \boxtimes \kappa^i_{\delta(i)} &
\end{tikzcd}$$

Finally, the action of the maps on the unstable complex depends on the framing: if $n = 2\tau(K)$ or $n = 2\tau(K) + 1$ there is nothing to prove, since either the unstable complex in the domain or the unstable complex in the range of D_1^W, D_3^W is trivial for these framings.

Let $m = |2\tau(K) - n|$. If $n < 2\tau(K)$, the action is as follows:

$$\begin{tikzcd}
& \boldsymbol{s} \boxtimes \mu_1 \arrow[dl, "D_3^W"'] \arrow[d, "D_1^W"'] & \boldsymbol{s} \boxtimes \mu_2 \arrow[dl, "D_3^W"'] \arrow[d, "D_1^W"'] & \cdots & \boldsymbol{s} \boxtimes \mu_m \arrow[d, "D_1^W"'] & \boldsymbol{p} \boxtimes \eta_0 \arrow[dl, "D_3^W"'] \\
0 & \boldsymbol{q} \boxtimes \mu_1 & \boldsymbol{q} \boxtimes \mu_2 & \cdots & \boldsymbol{q} \boxtimes \mu_m &
\end{tikzcd}$$

If $n > 2\tau(K)+1$, the action is:

$$
\begin{tikzcd}
\boldsymbol{s} \boxtimes \mu_1 \arrow[d, "D_1^W"] \arrow[dr, "D_3^W"] & \boldsymbol{s} \boxtimes \mu_2 \arrow[d, "D_1^W"] & \cdots & \boldsymbol{s} \boxtimes \mu_{m-1} \arrow[d, "D_1^W"] \arrow[dr, "D_3^W"] & \boldsymbol{s} \boxtimes \mu_m \arrow[d, "D_1^W"] \arrow[r, "D^W"] & \boldsymbol{p} \boxtimes \eta_0 \\
\boldsymbol{q} \boxtimes \mu_1 & \boldsymbol{q} \boxtimes \mu_2 & \cdots & \boldsymbol{q} \boxtimes \mu_{m-1} & \boldsymbol{q} \boxtimes \mu_m &
\end{tikzcd}
$$

Using the identification discussed above, Theorem 3.5 follows. □

Proof of Corollary 3.12 According to the computations above, D_1^W shifts the degree by $+\frac{1}{2}$ and D_3^W shifts the degree by $-\frac{1}{2}$. Their adjoints, σ_- and σ_+, shift the degrees by $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively. □

3.5 Sutured Legendrian invariants

Let L be a Legendrian knot in (S^3, ξ) of topological type K. Recall that in [15], the author proved the following two facts:

Proposition 3.17 *The contact structure ξ is overtwisted if and only if*

$$\sigma_+^N\big(\sigma_-^N(EH(L))\big) = 0$$

for sufficiently large N, that is, if and only if $EH(L)$ is stable.

Sketch of proof The "if" direction is easy, since the union of a stabilisation basic slice and the ∞–surgery layer is still an ∞–surgery layer.

The "only if" direction follows from the remark that, in the relevant Alexander grading component, there is only one nonvanishing element x_0: for sufficiently large N, x_0 is the contact element of a Legendrian representative of K in the standard S^3. We can now argue by contradiction. □

The following lemma is a part of the proof of the proposition above and turns out to be useful below.

Lemma 3.18 *A homogeneous element $x \in SFH(-S^3_{K,n})$ is stable if and only if $\psi_\infty(x) = 0$.*

Remark 3.19 If $\mathrm{tb}(L) > 2\tau(K)-1$, then L violates Plamenevskaya's inequality, automatically implying that ξ is overtwisted. On the other hand, if $\mathrm{tb}(L) \le 2\tau(K)-1$, the proposition can be rephrased as follows: $EH(L)$ is stable in $SFH(-S^3_L)$ if and only if ξ is overtwisted.

This is basically a rephrasing of [27, Theorem 1.2] telling us that $\mathcal{L}^-(L)$ is mapped to $c(\xi)$ if we set $U=1$.

Proposition 3.20 *Making the identification $SFH(-S^3_L) = \widehat{HFK}(S^3_{\mathrm{tb}(L)}(K), \widetilde{K})$ as in Proposition 2.8, we have that $EH(L)$ is homogenous of Alexander degree $-r(L)/2$.*

This is a reinterpretation of the fact that $\mathcal{L}^-(L) \in HFK^-(-S^3, K)$ has Alexander degree $2A(\mathcal{L}^-(L)) = \mathrm{tb}(L) - r(L) + 1$.

4 Contact surgeries

Suppose now that L is a Legendrian knot in (Y, ξ) of topological type K: contact surgery on L is an operation on $Y_L = Y \setminus \mathrm{Int}(\nu(L))$ that consists of gluing a solid torus $S^1 \times D^2$ with a tight contact structure η such that the boundary of the torus is η–convex.

Such a tight η exists on $S^1 \times D^2$ as long as the η–dividing curves on the boundary are not parallel to $S^1 \times \{*\}$ (see [19]).

We want the gluing to respect the dividing curves on the boundary of Y_L and $S^1 \times D^2$, so that we can glue ξ and η to get a contact structure on the surgered manifold Y'. In particular, this can be done whenever we do not fill in the meridional slope for L.

We have a natural basis for $\partial Y_L = T^2$, given by the meridian μ for $L \subset Y$ and the dividing curve γ that is homologous to L in $\nu(L)$. The slope of the curve $\{*\} \times \partial D^2$ in ∂Y_L is measured with respect to this natural basis, and we will refer to *contact p/q–surgery* along L to indicate any contact structure obtained with this process on $Y_{p\mu+q\gamma}(L)$.

Remark 4.1 Up to isotopy, there is only one contact structure η on $S^1 \times D^2$ that gives contact ± 1–surgery along L. We denote the resulting contact structure on the surgered manifold with $\xi_{+1}(L)$, or simply ξ_{+1} if L is clear from the context. Similarly, there is only one η that gives contact $1/m$–surgery, that we will denote with $\xi_{1/m}(L)$.

Remark 4.2 Whenever K is nullhomologous in Y, eg when $Y = S^3$, there is another natural framing for K, the Seifert framing. One easily checks that doing contact p/q–surgery on a Legendrian knot L with $\mathrm{tb}(L) = t$ produces a contact structure on $Y_{t+p/q}(K)$, where the surgery coefficient here is measured with respect to the Seifert framing, so that the difference between the contact and the topological surgery framings is just a global shift.

Let us recall here Ding and Geiges's algorithm to identify contact p/q–surgery along a Legendrian knot L in (Y,ξ) as a sequence of contact ±1–surgeries when p/q is positive. Pick the minimal integer k such that $q-kp$ is negative and call r the number $1+p/(kp-q)$. Now consider the negative continued fraction expansion $[a_0,\dots,a_\ell]$ of r: Inductively, $a_0=\lceil r\rceil$ is the smallest integer $a_0\geq r$, and $r=a_0-1/[a_1,\dots,a_\ell]$. Notice that $a_i\geq 2$ for each i, by construction.

Define the link $\mathbb{L}=\mathbb{L}_+\cup\mathbb{L}_-$. Here $\mathbb{L}_+$ is the union of k Legendrian pushoffs of L and $\mathbb{L}_-=L_0\cup L_1\cup\cdots\cup L_\ell\subset Y\setminus\mathbb{L}_+$ is constructed as follows: L_0 is any $(a_0-2)^{\text{th}}$ stabilisation of a pushoff of L, L_{j+1} is any $(a_{j+1}-2)^{\text{th}}$ stabilisation of a pushoff of L_j for $0\leq j\leq \ell-1$. If we have more fractions floating around, we will denote the link associated to p/q as $\mathbb{L}(p/q)$, and the two sublinks as $\mathbb{L}^{\pm}(p/q)$.

Notice that $\mathbb{L}_-$ depends on the choice of the signs of the stabilisations along the way. We suppress this dependence from the notation.

Theorem 4.3 (Ding and Geiges [5]) *Contact p/q–surgery is obtained from Y as contact $+1$–surgery along the link $\mathbb{L}_+$ and Legendrian surgery along the link $\mathbb{L}_-$.*

Example 4.4 For $n>1$, the algorithm gives us $k=1$, and the continued fraction expansion $[3,2,\dots,2]$, where 2 appears $n-2$ times. Thus there are exactly two isotopy classes of contact $+n$–surgeries depending on the choice of a positive or negative stabilisation of L. We will denote them by $\xi_n^{\pm}(L)$ or $\xi_n^{\pm}$, sticking to Lisca and Stipsicz's convention [30].

Remark 4.5 When $n=1$, $\xi_n^+=\xi_n^-=\xi_{+1}$, so the distinction between the choice of the two signs disappears.

Remark 4.6 Let us observe here that since $-L^{\pm}=(-L)^{\mp}$, positive contact surgeries on L are dual to contact surgeries on $-L$: doing p/q surgery on L for a given choice of signs and doing p/q surgery on $-L$ with the opposite choice of signs gives isotopic contact structures, since the two links $\mathbb{L}^+$ and $\mathbb{L}^-$ are isotopic.

In particular, as noted in the introduction, $\xi_n^-(L)$ is isotopic to $\xi_n^+(-L)$.

We will denote with $\xi_{p/q}(L)$ any of the contact structures constructed using the algorithm above.

We want to find an open book decomposition compatible with $\xi_{p/q}(L)$, following Ozbagci [32]: Fix an open book (F,h) for (Y,ξ) compatible with L in the sense that L lives in a page F of the open book and is nontrivial in $H_1(F)$. The sequence of stabilisations prescribes a sequence of stabilisations of the open book and a sequence of curves $L, L_0, \dots, L_\ell$ in the resulting page F'. Call h' the monodromy given by the sequence of stabilisations on (F, h).

An open book for $\xi_{p/q}(L)$ is given by composing h' with k negative Dehn twists along L and a positive Dehn twist along L_i for each i.

Proposition 4.7 *Let $m \geq 1, n \geq 0$ be integers and let p/q be a rational number such that*[2] *$n+1/m \leq p/q < n+1/(m-1)$. Then $\xi_{p/q}(L)$ is obtained from $\xi_{n+1/m}(L)$ through Legendrian surgeries, if the first m stabilisation choices for p/q–surgery coincide with the choices for $n+1/m$.*

Proof Let us apply the Ding–Geiges algorithm to $\alpha := p/q$ and $\beta := n+1/m$. If $n=0$, the number k given by the algorithm is m for both coefficients and the continued fraction expansion for p/q has the continued fraction expansion for $1/m$ (which is the empty expansion) as an initial segment. If $n \geq 1$, the number k is 1 for both fractions and the algorithm tells us to expand the two fractions $r_\beta = 1-(mn+1)/(m-mn-1)$ and $r_\alpha = 1-p/(q-p)$.

Lemma 4.8 *The continued fraction expansion for r_α contains the expansion for r_β as an initial segment also when $n \geq 1$.*

Once we have the lemma, together with the previous considerations, we see that the two links $\mathbb{L}_+$ associated to $\xi_\alpha(L)$ and $\xi_\beta(L)$ are equal; if we choose stabilisations carefully, the link $\mathbb{L}_-(\alpha)$ associated to p/q and L contains the link $\mathbb{L}_-(\beta)$ associated to $m+1/n$ and L, so that $\xi_\alpha(L)$ is obtained from $\xi_\beta(L)$ through Legendrian surgery on $\mathbb{L}_-(\alpha) \setminus \mathbb{L}_+(\beta)$. □

Before proving Lemma 4.8, let us analyse what happens with contact $(n+1/m)$–surgery via the Ding–Geiges algorithm.

Remark 4.9 Contact $(1/m)$–surgery is just a sequence of $+1$–surgeries. On the other hand, when $n \geq 1$, the link $\mathbb{L}_+$ consists of L only and $\mathbb{L}_-$ is nonempty. Let us now distinguish between $n=1$ and $n \geq 2$.

The fraction to expand when $n=1$ is just $1+(m+1)$, so the expansion is $[m+2]$. In this case, $\mathbb{L}_-$ consists of an m^{th} stabilisation of a pushoff of L.

For larger values of n, the fraction to expand is $1+(mn+1)/(m-mn-1)$. By induction on n, its continued fraction expansion is $[3, 2, \dots, 2, m+1]$, where the sequence of 2 has length $n-2$ (but if $m=1$ then 2 appears a total of $n-1$ times).

In any case, there are m stabilisations to be chosen if $n \geq 1$.

[2] Here we adopt the convention that $1/0 = +\infty$.

Proof of Lemma 4.8 As before, we let $\alpha := p/q$.

The statement is trivial if $\alpha = p/q = n+1/m$, so we can assume that both inequalities in the statement of the proposition are strict.

We will prove the nontrivial case by induction on n. When $n=1$, the fraction associated to $1+1/m$ is $1+(1+m)/1$, whose continued fraction expansion is $[m+2]$. We need to expand the fraction $1+p/(p-q) = (2p-q)/(p-q) = (2\alpha-1)/(\alpha-1)$: the inequality $1+1/m < \alpha < 1+1/(m-1)$ can be read as $1/m < \alpha-1 < 1/(m-1)$, so that

$$2+m-1 < 2+\frac{1}{\alpha-1} = \frac{2\alpha-1}{\alpha-1} < 2+m.$$

In particular, the first element of the continued fraction expansion we are looking at is $\lceil (2\alpha-1)/(\alpha-1)\rceil = m+2$ as we wanted.

Let us suppose the statement holds for $n \geq 1$ and prove it holds for $n+1$. The fraction associated to $n+1+1/m$ is $(2mn+m+2)/(mn+1) = 3-((mn-m+1)/(mn+1))^{-1}$ and the one associated to $p/q+1$ is $(2p-q)/(p-q) = 3-((p-2q)/(p-q))^{-1}$. In particular, both expansions start with a 3 and the first one continues with the expansion of $(mn+1)/(mn-m+1)$. In order to prove the statement, it is enough to show that the continued fraction expansion of $(mn-m+1)/(mn+1)$ is an initial segment of the expansion of $(p-2q)/(p-q)$.

Now, the algorithm for $n+1/m$ and $p/q-1$ tells us to expand the two fractions $1+(mn+1)/(mn-m-1)$ and $1+(p-q)/(2p-q)$, and by the inductive hypothesis the expansion of the first one is the initial segment of the expansion of the second one. The result follows. $\square$

We can immediately draw two corollaries:

Corollary 4.10 *If n is a positive integer and $c(\xi_n^-(L)) \neq 0$, then for every $p/q \geq n$, $c(\xi_{p/q}(L)) \neq 0$ whenever the first stabilisation for p/q–surgery is a negative.*

Corollary 4.11 *If $c(\xi_{n+1/m}(L)) \neq 0$ for a positive integer n and all positive integers m, then for all $p/q > n$ there is a sign choice for the Ding–Geiges algorithm such that $c(\xi_{p/q}(L)) \neq 0$.*

For the following proposition, let us some notation. For integers $n \geq 0, m \geq 1$, we denote by $\xi_{n+1/m}^-(L)$ any contact $(n+1/m)$–surgery on L such that all the m stabilisations are chosen to be negative. In particular, when $m=1$, this is consistent with Lisca and Stipsicz's notation for integral surgeries; it is understood that $\xi_{1/m}^-(L)$ is just $\xi_{1/m}(L)$ since there are no stabilisations involved.

Proposition 4.12 *For $n \geq 0, m \geq 1$ integers, the two contact structures $\xi^-_{n+1/m}(L)$ and $\xi^-_{n+1+1/m}(L^-)$ are isotopic.*

Remark 4.13 The case $m = 1$ in the proposition is proved by Lisca and Stipsicz [30]. The proof we present here is a refinement of their first proof.

Proof We will prove the result by induction on n.

When $n = 0$, we are comparing $\xi_{1/m}(L)$ with $\xi_{1+1/m}(L^-)$. Suppose we have an open book (F, h, L) for (Y, ξ) compatible with L. According to Ozbagci [32], the open book $(F, D_L^{-m} \circ h)$ supports the contact structure $\xi_{1/m}(L)$.

We can construct an open book supporting $\xi^-_{1+1/m}$ as follows. The Ding–Geiges algorithm tells us that we need to pushoff and stabilise (negatively, according to our choice) L m times and do $+1$–surgery on L and -1 on the pushoff. We can realise L and the pushoff on the page of the same open book by doing m positive stabilisations (using boundary-parallel arcs for the Murasugi sum inside F); the pushoff is represented by a curve L_1 on the page, parallel to L except that it runs once along each of the m handles (see Figure 4.1). Call (F', h') the monodromy for $\xi^-_{1+1/m}(L^-)$, as shown in Figure 4.1.

Claim 4.14 *The contact structure $\xi^-_{1+1/m}(L^-)$ is isotopic to $\xi_{1/m}(L)$.*

Proof We now apply the lantern relation to the monodromy h', where we insert a pair of canceling Dehn twists along the curve labelled with a ± 1 in Figure 4.1. After applying the relation, we see a destabilisation arc (dashed in the figure). After Giroux destabilising, we decrease the number of boundary components and we obtain the monodromy at the bottom right of the figure. If we insert another pair of opposite Dehn twist along the ± 1 curve, we can apply the lantern relation once again and we see another destabilisation arc. The resulting open book looks now exactly like the one we had in the previous step (bottom right in the figure), with one less boundary component. After $m-1$ application of the lantern relation-destabilisation process, we end up with the open book we described for $\xi_{1/m}(L)$. □

Notice how in this process, we always destabilised without any need for conjugation, so we actually proved that the two contact structures are isotopic rather than isomorphic. On the contrary, for the inductive step we will first show:

Claim 4.15 *The contact structure $\xi^-_{n+1+1/m}(L^-)$ is* contactomorphic *to $\xi^-_{n+1/m}(L)$.*

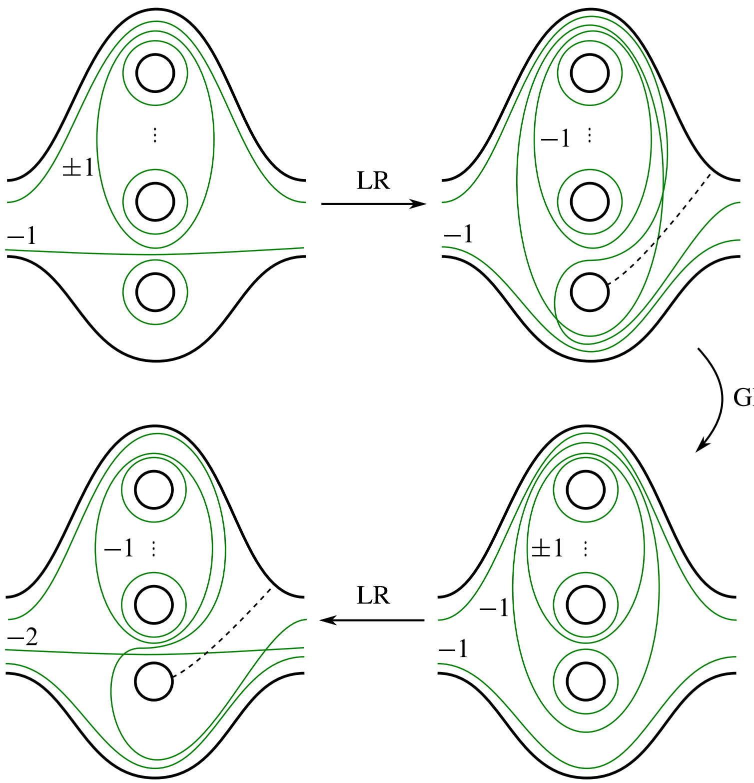

Figure 4.1: The sequence of moves in Claim 4.14: each curve labelled with a ± 1 represents a pair of canceling Dehn twists; curves labelled with -1 represent negative (left-handed) Dehn twists; unlabelled curves represent positive (right-handed) Dehn twists.

Proof We now refer to Figure 4.2. The open book at the top left corresponds to the surgery $\xi^-_{n+1+1/m}(L^-)$. After applying the lantern relation once and conjugating, we find the destabilisation arc (dashed in the figure). The destabilisation arc intersects a single curve d such that the monodromy factorises as $h_1 \circ D_d \circ h_2$. In order to destabilise, we need to have a monodromy of the form $D_d \circ h_3$, so we need to conjugate h with h_1. By conjugating we lose the isotopy result. After Giroux destabilising, we obtain the open book at the bottom that represents $\xi^-_{n+1/m}(L)$. $\square$

Remark 4.16 In the proof of the inductive step, we never used the fact that all the last $m-1$ stabilisations are negative but just that the first one is. In other words, $\xi_{n+1+1/m}(L^-)$ is isomorphic to $\xi_{n+1/m}(L)$ if the first stabilisation is negative for both surgeries and the number of positive stabilisations on the last pushoff is the same.

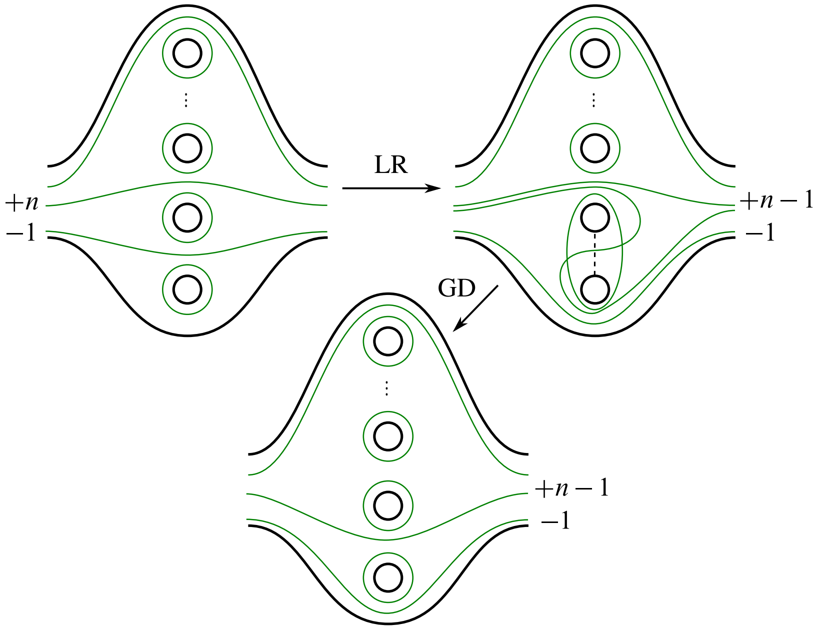

Figure 4.2: The sequence of moves in Claim 4.15. Each curve represents a composition of Dehn twists. The ones that are unlabelled represent a single positive (right-handed) Dehn twists, while the ones labelled with an integer m represent the m^{th} power of a positive Dehn twist. For example, curves labelled with -1 represent a single negative (left-handed) Dehn twist.

Moreover, using Lisca and Stipsicz's trick (see [30, Lemma 2.3]), one proves that $\xi_{n+1+1/m}(L^-)$ is overtwisted if the first (resp. any) stabilisation is positive for all $n \geq 1$ (resp. for $n = 0$). They prove the result by exhibiting an overtwisted disc in the surgered manifold, which is isotopic to the core of the first surgery handle relative to L (not relative to L^-).

Using the remark, we can now complete the proof.

Claim 4.17 *The contact structure* $\xi^-_{n+1+1/m}(L^-)$ *is* isotopic *to* $\xi^-_{n+1/m}(L)$.

Proof We first argue that $\xi^-_{n+1+1/m}(L^-)$ is isotopic to one of the contact $(n+1/m)$–surgeries on L by showing that the relevant solid torus is tight. If L_0 is the Legendrian representative of the positive trefoil with $\mathrm{tb}(L_0) = 1$, by the main result of [28][3] we know that $\xi^-_{n+1+1/m}(L^-_0)$ is tight. This implies that the solid torus we attach to

[3] It also follows from Theorem 1.1 of this paper, and precisely from the implication whose proof is independent of this discussion.

S^3_L (not to $S^3_{L^-}$) to obtain $\xi^-_{n+1+1/m}(L^-)$ is tight. This solid torus is the union of a negative stabilisation layer and a surgery layer and, since it is tight, it is one of the solid tori that we glue in to S^3_L to get one of the contact $(n+1/m)$–surgeries. Let us call this surgery $\overline{\xi}_{n+1/m}(L)$.

The argument above also shows that the choice of the signs of the stabilisations in the algorithm is independent of the Legendrian knot L. To get the claim, it suffices to prove the result in a particular case.

Consider the case $L = L_1$, where L_1 is the n^{th} negative stabilisation of a Legendrian positive torus knot L_0 with maximal Thurston–Bennequin number. By induction, $\overline{\xi}_{n+1/m}(L_1)$ is contactomorphic to $\xi_{1/m}(L_0)$, which in turn has nonvanishing contact invariant, so $c(\overline{\xi}_{n+1/m}(L_1)) \neq 0$. Using the Lisca–Stipsicz trick mentioned above, we immediately see that the first pushoff of L_1 has to be *negatively* stabilised, and this already concludes the proof in the case $m = 1$ (since there are no more stabilisation choices).

If $m > 1$, the algorithm tells us that there are $m-1$ further stabilisations to do, and these latter stabilisations commute. Let us suppose that $p \geq 0$ of them are positive.

Thanks to Remark 4.16 above, $\overline{\xi}_{n+1/m}(L_1)$ is contactomorphic to the $1+1/m$–surgery on L_0^- where the first (only) pushoff of L_0^- has been positively stabilised p times and negatively stabilised $m-p$ times. By the second part of the remark, this latter contact structure is overtwisted if $p > 0$. Since $\overline{\xi}_{n+1/m}(L_1)$ is isotopic to $\overline{\xi}_{n+1+1/m}(L_1^-)$, which in turn is contactomorphic to $\xi_{1/m}(L_0)$, and the latter is tight, we get $p = 0$, ie $\overline{\xi}_{n+1/m}(L_1)$ is isotopic to $\xi^- n+1/m(L_1)$ and is not isotopic to any other $(n+1/m)$–surgery on L_1. □

This also concludes the proof of the proposition. □

Remark 4.18 Doing contact surgery along L corresponds to gluing a solid torus with a tight contact structure to S^3_L. In particular, every contact p/q–surgery induces a map $\psi_{p/q}$ between $SFH(-S^3_{K,t})$ and $SFH(-S^3_{t+p/q}(K))$. When $p/q = n+1/m$, we will denote the map corresponding to $\xi^-_{n+1/m}$ as $\psi^-_{n+1/m}$.

Notice that when $n = 0$, the sign choice is immaterial and the map corresponds to $1/m$–surgery on L.

5 Cables

5.1 Topological cabling

Let K be a nullhomologous knot in a 3–manifold Y. Take a tubular neighbourhood $K \subset N(K) \subset Y$, where we identify $N(K)$ with $\{z \in \mathbb{C} \mid |z| \leq 1\} \times S^1$ in such a way that $K = \{0\} \times S^1$ and $\lambda = \{1\} \times S^1$ is nullhomologous in Y. Together with the meridian $\mu = \{|z| = 1\} \times \{*\}$, λ gives a parametrisation of $\partial N(K)$.

Definition 5.1 Given $p > 0$ and q relatively prime integers, we define the (p,q)–*cable* $K_{p,q}$ of K to be any simple closed curve in $\partial N(K)$, homologous to $p\lambda + q\mu$.

Remark 5.2 Here we adopt the standard convention for the labelling of p and q; this is the convention adopted by Hom and Hedden, while Etnyre and Honda use the opposite convention.

Let us recall the following classical result:

Proposition 5.3 *The manifold $S^3_{pq}(K_{p,q})$ obtained by pq surgery on S^3 along $K_{p,q}$ is diffeomorphic to the connected sum $S^3_{q/p}(K) \# L(p,-q)$.*

Remark 5.4 Here we adopt the convention that the lens space $L(p,-q)$ is obtained by $+p/q$ surgery on the unknot. This choice is quite common, but it is opposite to Ozsváth and Szabó's and Hom's, for which $L(p,-q)$ is obtained by $(-p/q)$– surgery on the unknot.

We are interested in the behaviour of τ and ε under cabling. Hom answered precisely this question:

Theorem 5.5 [18] *The concordance invariants $\tau(K_{p,q})$ and $\varepsilon(K_{p,q})$ are determined by p, q, $\tau(K)$ and $\varepsilon(K)$ in the following way:*

(1) *If $\varepsilon(K) = 0$, then $\tau(K_{p,q}) = \frac{1}{2}(p-1)(q - \operatorname{sgn}(q))$; if $|q| \le 1$, $\varepsilon(K_{p,q}) = 0$, and $\varepsilon(K_{p,q}) = \operatorname{sgn}(q)$ otherwise.*

(2) *If $\varepsilon(K) \neq 0$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p-1)(q-\varepsilon(K))$ and $\varepsilon(K_{p,q}) = \varepsilon(K)$.*

5.2 Legendrian cabling

We want to construct Legendrian cables of Legendrian knots through a "standard" construction. Similar ideas appeared in Rudolph [46] (for Whitehead doubles), Etnyre and Honda [11], Ding and Geiges [6] and more recently in Cochran, Franklin, Hedden and Horn [4].

Consider an oriented Legendrian knot $L \subset (S^3, \xi_{st})$ and its front projection. Take m pushoffs of L under the flow of $\partial/\partial z$, and twist them away from cusps, as in Figure 5.1. Notice that the twists are performed on strands that point *to the right*. When $n \ge 0$ is coprime with m, this is still the front projection of an oriented Legendrian knot so the following definition makes sense:

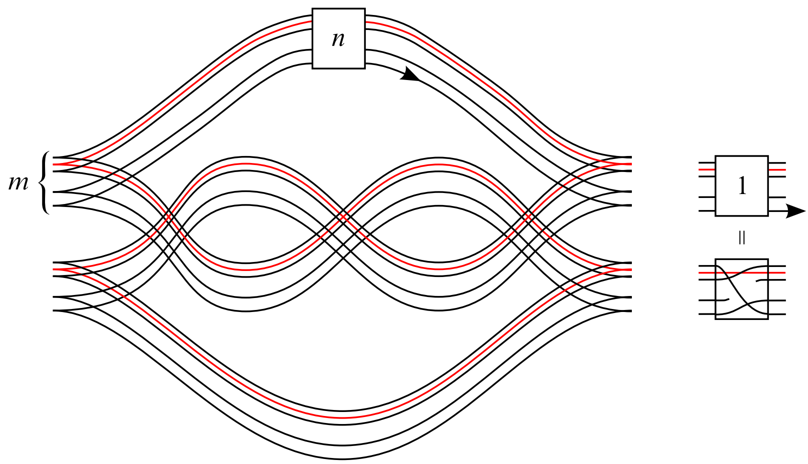

Figure 5.1: A Legendrian cable (in black) of L (in red)

Definition 5.6 We will call the Legendrian knot $L_{m,n}$ obtained by the procedure we just described a *Legendrian* (m,n)*–cable*.

Remark 5.7 We defined $L_{m,n}$ starting from the front projection of L, so *a priori* $L_{m,n}$ depends on the diagram and on the position of the twists. On the other hand, this is a *local* construction, taking place in a standard neighbourhood of L, ie $L_{m,n}$ is obtained by attaching a cabling layer to S^3_L. In particular, the main theorem in [6] shows the well definedness of $L_{m,n}$.

Legendrian representatives of cables have proven to be a remarkable source of examples. For example, some (3,2)-cables of the trefoil (and more generally, cables of positive torus knots) are not Legendrian or transversely simple (see Etnyre and Honda [12] and Etnyre, LaFountain and Tosun [13]).

We want to compute the classical invariants of $L_{m,n}$, given the classical invariants of L. Say that L is of topological type K and has Thurston–Bennequin and rotation numbers $t = \mathrm{tb}(L)$ and $r = r(L)$ respectively. From now on, we denote by L and $L_{m,n}$ both the knots and their front projections.

One can easily show the following two propositions.

Proposition 5.8 *The Legendrian knot* $L_{m,n}$ *is of topological type* $K_{m,mt+n}$*.*

The cable $L_{m,n}$ is clearly an (m,q)–cable of L for some q. This is pinned down by counting the linking number of L and $L_{m,n}$ (see Figure 5.1).

Proposition 5.9 *The classical invariants for $L_{m,n}$ are*

$$\begin{aligned}\mathrm{tb}(L_{m,n}) &= m^2 t + (m-1)n,\\ r(L_{m,n}) &= mr,\\ \mathrm{sl}(L_{m,n}) &= m^2 t - mr + (m-1)n.\end{aligned}$$

This is obtained by computing the writhe and the number of cusps of the front projection of $L_{m,n}$ (Figure 5.1).

One can now compare Hom's formulae for τ and ε with the proposition above. Using Plamenevskaya's inequality (3-2), one checks the following.

Proposition 5.10 *Let L be a Legendrian knot in (S^3, ξ_{st}) of topological type K such that $\varepsilon(K) \neq 0$. Then:*

(i) *(SL) holds for $L_{m,n}$ if and only if (SL) and (TN) hold for L.*

(ii) *Suppose that (SL) holds for $L_{m,n}$. Then (SC) holds for the pair $(L_{m,n}, p)$ if and only if $p \geq 1 - m \cdot r(L)$.*

(iii) *(TN) holds for $L_{m,n}$ if and only if (TN) holds for L.*

On the other hand, if $\varepsilon(K) = 0$ (and therefore $\tau(K) = \nu(K) = 0$):

(i′) *(SL) holds for $L_{m,n}$ if and only if (SL) holds for L and $n \geq 1 - m \cdot \mathrm{tb}(L)$.*

(ii′) *Suppose that (SL) holds for $L_{m,n}$: Then (SC) holds for the pair $(L_{m,n}, p)$ if and only if $p \geq 1 - m \cdot r(L)$.*

(iii′) *(TN) holds for $L_{m,n}$ if and only if $n \geq -1 - m \cdot \mathrm{tb}(L)$.*

Before stating the following lemma, let us introduce some notation. Given a Legendrian knot L, we denote by $L^{(1)} = L^-$ its negative stabilisation and recursively set $L^{(n+1)} = (L^{(n)})^-$. For small values of n, we may use the "differential" notation $L^{(1)} = L'$, $L^{(2)} = L''$, ….

Lemma 5.11 *The two Legendrian knots $(L^{(k)})_{m,km+n}$ and $(L_{m,n})^{(km)}$ are isotopic.*

Remark 5.12 Note that if such an identity exists, then the triple of numbers (m, n, km) defining the second knot is uniquely identified by the classical invariants of the first knot. Also, since the cabling operation commutes with orientation-reversal, the same result holds for positive stabilisations (on both knots).

Proof It is enough to prove the result for $k=1$, which is an easy induction on m. For $m=2$, we can apply the second Legendrian Reidemeister (LR2; see Etnyre [10] for details) move twice as in Figure 5.2.

Let us now suppose we want to prove the result for $m+1$. We can apply LR2 $2m$ times as in the base case and reduce to the inductive assumption. □

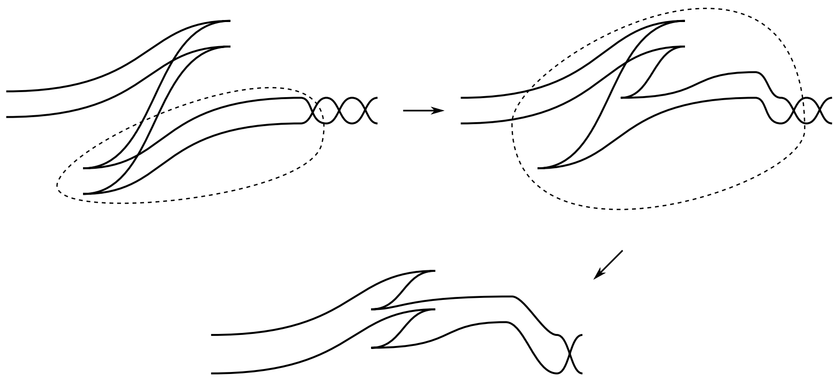

Figure 5.2: A Legendrian isotopy from $(L')_{2,3}$ (top left) to $(L_{2,1})''$ (bottom)

Remark 5.13 The cabling construction, stabilisations and the proof of the lemma above are all *local*, in the sense that they all take place in a neighbourhood of L.

In particular, there exists a *cabling layer* $(\boldsymbol{T}_{m,n}, \xi_{m,n})$ that is topologically a difference of solid tori. The core of the inner torus winds m times around the outer solid torus, and there are two $\xi_{m,n}$–dividing curves on the "outer" (resp. "inner") boundary component that are homologous to the longitude of the bigger (resp. smaller) solid torus.

What the previous lemma says at the level of contact layers is that we have two isotopic layers. One is obtained by gluing a stabilisation layer $(\mathbb{T}^2 \times I, \eta_-)$ (ie a specific basic slice) from the *back* to the *outer* boundary of $\boldsymbol{T}_{m,n}$; the other is obtained by gluing m stabilisation layers $(\mathbb{T}^2 \times I, \eta_-)$ from the *front* to the *inner* boundary of $\boldsymbol{T}_{m,m+n}$.

This also allows us to generalise the notion of Legendrian cabling to Legendrian knots in any contact manifold.

This cabling layer induces a gluing map

$$\kappa_{m,n} := \Psi_{\xi_{m,n}} \colon SFH(-Y_{K,t}) \longrightarrow SFH(-Y^3_{K_{m,mt+n},m^2t+n}).$$

Thanks to the remark, the lemma above can be translated to the sutured world as follows:

Corollary 5.14 *We have* $\kappa_{m,km+n} \circ \sigma_\pm^k = \sigma_\pm^{km} \circ \kappa_{m,n}$.

Remark 5.15 It is clear that $\psi_\infty \circ \kappa_{m,n} = \psi_\infty$, since the corresponding layers are both ∞–surgery layers, so they are isotopic.

We conclude the subsection with a remark on the action of $\kappa_{m,n}$ on sutured Floer homology, when $K \subset S^3$.

Proposition 5.16 *The map* $\kappa_{m,n}$ *sends stable elements to stable elements and therefore descends to a map of unstable complexes, still denoted with* $\kappa_{m,n}$,

$$\kappa_{m,n}\colon SFH(-S^3_{K,t})/S \to SFH(-S^3_{K_{m,mt+n},m^2t+n})/S.$$

Proof Recall from Section 3 that the set of stable elements is $\ker(\sigma_+^N \circ \sigma_-^N)$ for some sufficiently large N, and by Theorem 3.11 it is also spanned by $\ker(\sigma_-^N)$ and $\ker(\sigma_+^N)$. It therefore suffices to show that the proposition holds for an element $x \in SFH(-S^3_{K,t})$ such that $\sigma_\pm^k(x) = 0$ for some k. It follows from Corollary 5.14 that $\sigma_\pm^{km}(\kappa_{m,n}(x)) = \kappa_{m,mk+n}(\sigma_\pm^k(x)) = 0$. □

5.3 Contact surgeries and cabling

Let us start with an easy, general observation.

Remark 5.17 Let L be a nullhomologous Legendrian knot in (Y,ξ) of topological type K. The Legendrian knot $L_{m,n}$ is of topological type $K_{m,m\cdot\mathrm{tb}(L)+n}$. If we do contact $(+n)$–surgery on $L_{m,n}$, the topological surgery coefficient is $\mathrm{tb}(L_{m,n}) + n = m(m\cdot\mathrm{tb}(L)+n)$ so the underlying manifold is reducible, more precisely it splits as $Y_{\mathrm{tb}(L)+n/m}(K) \# L(m,-n)$.

It is natural to ask whether there are natural contact structures on the two factors that realise this topological decomposition as a contact connected sum. In fact, this happens to be true (regardless of the homological assumption $[K] = 0 \in H_1(Y)$) when $n = 1$.

Proposition 5.18 *Contact* $+1$*–surgery on* $L_{m,1}$ *is isotopic to a contact connected sum of a contact* $1/m$*–surgery and a tight contact structure on* $L(m,-1)$.

Before getting to the proof, let us find an open book for (S^3,ξ_{st}) compatible with $L_{m,1}$. Start off with an open book (F,h) for (S^3,ξ_{st}) for which L sits on a page and is not nullhomologous in that page. We can assume that $L \subset F$ is a simple closed, not nullhomologous curve. Consider a properly embedded arc $c \in F$ that intersect L in a single point and consider the positive (Giroux) stabilisation (F',h') of (F,h) along c. The situation is depicted in Figure 5.3.

Lemma 5.19 *The curve depicted on the right-hand side of Figure 5.3 represents $L_{m,1}$.*

Before giving the proof of the lemma, let us recall a result of Ozsváth and Stipsicz (see [34, Section 4]).

Proposition 5.20 *If (F, h, L) is an open book decomposition with connected binding for (Y, ξ) compatible with the nullhomologous Legendrian knot L, then:*

(i) *There exists $[Z] = \zeta \in H_1(F)$ such that $[L] = h_*(\zeta) - \zeta$.*
(ii) *$r(L)$ is the Euler class of a 2–chain P such that $\partial P = L + Z - h(Z)$.*
(iii) *$\mathrm{tb}(L)$ is $Z \cdot L$.*

This has some very interesting consequences, whose proofs are straightforward.

Corollary 5.21 *If L, L_1, L_2 are three embedded, homologically nontrivial curves in F such that $[L] = [L_1] + [L_2]$ and Z_i is associated to L_i as in the proposition above, then:*

- $\mathrm{tb}(L) = \mathrm{tb}(L_1) + \mathrm{tb}(L_2) + Z_1 \cdot L_2 + Z_2 \cdot L_1$.
- $r(L) = r(L_1) + r(L_2)$.

Proof of Lemma 5.19 Let us remark that $L_{m,n}$ is Legendrian isotopic to a torus knot in a standard Legendrian neighbourhood $\nu(L)$ of L.

Call L' the Legendrian knot represented by the curve on the right-hand side of Figure 5.3 and suppose that $\partial F'$ is connected. In particular, ∂F has two connected components.

Stabilising along c corresponds to a connected sum of (Y, ξ) with (S^3, ξ_{st}). In particular, we can suppose that a neighbourhood of c in Y is contained in $\nu(L)$. Call b the core of the annulus we do Murasugi sum with (which is the union of c and a core of the new 1–handle of F'). In this case, the connected sum can be performed inside $\nu(L)$. In particular, b is nullhomologous in S^3. Observe now that the curve L' is isotopic in S^3 to a curve on the boundary of $\nu(L)$; L' is homologous to $m\lambda_\xi + b$, where b is given the orientation such that $b \cdot L = 1$, and in particular L' represents a $(m, m \cdot \mathrm{tb}(L) + 1)$–cable of K.

Since L' and $L_{m,1}$ are both local modifications of L and torus knots in the standard solid torus are Legendrian simple [11], the knot represented by L' is isotopic to $L_{m,1}$ provided they have the same classical invariants.

But the corollary above provides us the tools we need to compute $\mathrm{tb}(L')$ and $r(L')$. We know that $[L'] = m[L] + [b]$ and we know that for some $Z_0 \subset F$, $[h(Z_0)] - [Z_0] = [L]$.

Claim 5.22 *If Z_b is parallel to a boundary component of $F \subset F'$, oriented so that $b \cdot Z_b = 1$, then $[h'(Z_b)] - [Z_b] = [b]$.*

Proof The action of D_b on homology is given by $[D_b(\gamma)] = [\gamma] + (b \cdot \gamma)[b]$. Since $[h(Z_b)] = [Z_b]$ and $h' = D_b \circ h$, it follows that $[h'(Z_b)] = [Z_b] + (b \cdot Z_b)[b]$. □

Claim 5.23 *We have $[h'(Z_0)] - [Z_0] = [L] + (b \cdot Z_0 + 1)[b]$.*

Proof We know that $[h(Z_0)] = [Z_0] + [L]$ and that $[D_b(c)] = [c] + (b \cdot c)[b]$, so

$$[h'(Z_0)] = [D_b(h(Z_0))] = [D_b(Z_0)] + [D_b(L)] = [Z_0] + [L] + (b \cdot Z_0 + b \cdot L)[b].$$

That is what we wanted since, by assumption, $b \cdot L = 1$. □

In particular, if we let Z be any curve in the homology class $[Z_0] - (b \cdot Z_0 + 1)[Z_b]$, then $[h(Z)] - [Z] = [L]$.

We can now compute $\mathrm{tb}(L')$. We get

$$\mathrm{tb}(L') = m^2\mathrm{tb}(L) + \mathrm{tb}(b) + m(Z \cdot b + Z_b \cdot L) = m^2\mathrm{tb}(L) - 1 + m,$$

where we used that $b \cdot Z = b \cdot Z_0$, $Z_b \cdot L = 0$ and $\mathrm{tb}(b) = -1$. This last identity comes from the fact that b represents a Legendrian unknot with Thurston–Bennequin number -1 in the (S^3, ξ_{st}) connected summand, corresponding to the stabilisation made on F to get F'.

As we said, the rotation number is linear, so $r(L') = mr(L) + r(b) = mr(L)$ and in particular L' and $L_{m,1}$ have the same classical invariants, and Ding and Geiges' results [6] (see Remark 5.7 above) imply that they are isotopic. □

Recall that b is the core of the annulus we do the Murasugi sum with. Let us also denote by β the positive Dehn twist along b and by λ the positive Dehn twist along $L \subset F'$.

Proof of Proposition 5.18 An open book for $+1$ surgery on $L_{m,1}$ has F' as a page and the monodromy is the composition of h' with a negative Dehn twist along the curve representing $L_{m,1}$. This last curve is isotopic to $\lambda^{-m}(b)$, so the monodromy can be written as $(\lambda^{-m}\beta\lambda^m)^{-1}h' = \lambda^{-m}\beta^{-1}\lambda^m\beta h$.

Fix a contact structure ξ_m on $L(m,-1)$ supported by an open book with an annular page A, and monodromy given by m positive twists along the core of the annulus. Fix also a properly embedded arc d connecting the two boundary components of the annulus.

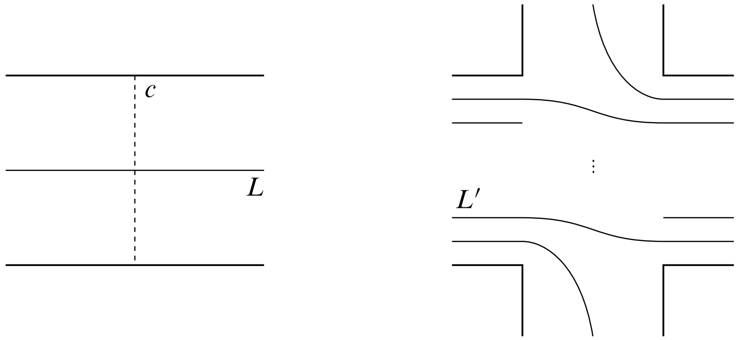

Figure 5.3: The figure on the left shows the open book (F, ϕ, L) and the stabilisation arc c. The figure on the right represents $(F', \phi', L_{m,1})$: $L_{m,1}$ runs m times along L and once along the new 1–handle.

Similarly, an open book for $+1/m$–surgery along L is given by $(F, \lambda^{-m}h)$. An open book for the connected sum of this with ξ_m can be realised via a Murasugi sum of F and A along the arcs $d \subset A$ and an arc $c' \subset F$. Instead of using $c' = c$, though, we will use the arc $c' = \lambda^{1-m}(c)$ to simplify the proof.

The new surface is diffeomorphic to F' and the monodromy, under the obvious identification, is $\lambda^{1-m}\beta^m\lambda^{-(1-m)} \cdot \lambda^m \cdot h = \lambda^{1-m}\beta^m\lambda^{-1}h$.

Notice that b and L intersect exactly once by assumption.

To prove isotopy of the two contact structures, we prove that the two monodromies are isotopic and this is an easy computation in the mapping class group of F':

$$\lambda^{-m}\beta^{-1}\lambda^m\beta h = \lambda^{1-m}\beta^m\lambda^{-1}h \iff \beta^{-1}\lambda^m\beta = \lambda\beta^m\lambda^{-1}.$$

This last equation follows from taking the m^{th} power of the braid relation $\beta^{-1}\lambda\beta = \lambda\beta\lambda^{-1}$. □

Remark 5.24 There is also a less direct proof of Proposition 5.18. The idea is that for some L, eg the right-handed trefoil with maximal Thurston–Bennequin, every Legendrian cable has $\mathrm{tb}(L_{m,1}) = 2g(L_{m,1}) - 1$ and therefore the contact structure $\xi_{+1}(L_{m,1})$ has nonvanishing contact invariant [28]. This implies that the layer $\kappa_{m,1} \cup \boldsymbol{T}_n^-$ has nonvanishing contact invariant and in particular is tight. But $\kappa_{m,1} \cup \boldsymbol{T}_n^-$ *topologically* decomposes as $\boldsymbol{T}_{1/m} \# L(m, -1)$ and the contact structure on the $\boldsymbol{T}_{1/m}$ factor has to be tight, and is the only tight solid torus that gives contact $1/m$–surgery.

Notice that if $m \geq 4$, this does not immediately say anything about the contact structure on the lens space factor. It is not unreasonable that one can extract this information from the Spin^c–structures of the contact invariants $c(\xi_{1/m}(L))$ and $c(\xi_{+1}(L_{m,1}))$.

We now want to turn to the case of $L_{m,n}$ with $n \equiv 1 \pmod m$. We start off with an example.

Example 5.25 Let us consider the case of the Legendrian positive trefoil $L_0 \subset (S^3, \xi_{\mathrm{st}})$ with $\mathrm{tb}(L_0) = 1$ and $r(L_0) = 0$. L_0 satisfies all three conditions in Theorem 1.1. More precisely, since $\varepsilon(T_{3,2}) = 1$, Proposition 5.10 tells us that all of its Legendrian cables $(L_0)_{m,n}$ satisfy the three hypotheses in Theorem 1.1 for any positive surgery coefficient.

Therefore, if we accept the "if" direction of our main theorem (whose proof will not rely on these facts), we know that $\xi_n^-((L_0)_{m,n})$ has nonvanishing contact invariant[4] (in fact, $\xi_n^+((L_0)_{m,n})$ does also).

Thanks to the example, we know that $c(\xi_n^-(L_{m,n})) \neq 0$ for some Legendrian L. In particular, the contact invariant $EH(\boldsymbol{T}_{m,n} \cup \boldsymbol{T}_n^-, \xi_{m,n} \cup \xi_n^-)$ of the union of the cabling layer and the surgery layer has nonvanishing contact invariant (see Corollary 2.17) and therefore is tight.

The layer $\boldsymbol{T}_{m,n} \cup \boldsymbol{T}_n^{\pm}$ splits as a connected sum $\boldsymbol{T}_{n/m} \# L(m,-n)$ of tight manifolds, therefore we proved the following.

Proposition 5.26 *The contact structure $\xi_n^-(L_{m,n})$ splits as the connected sum $\xi_{n/m}(L) \# \eta_{m,n}$ for some choice of n/m surgery along L and a contact structure $\eta_{m,n}$ on $L(m,-n)$. Both the surgery layer and the contact structure on the lens space are independent of L.*

We want to pin down the choice of the contact structures $\xi_{n/m}$ and $\eta_{m,n}$ in the statement above when $n \equiv 1 \pmod m$.

Proposition 5.27 *Suppose $m \geq 1$. Contact n–surgery on $L_{m,n}$ yields the connected sum $\xi_{n/m}^-(L) \# \eta_m$, where η_m is obtained by -1–surgery on the Legendrian unknot with $(\mathrm{tb}, r) = (1-m, 2-m)$.*

This proof is similar in spirit to the proof of Proposition 4.12. The key point of the proof is Remark 4.16. Recall that when $n \equiv 1 \pmod m$ is larger than 1, we have m stabilisations to choose, and the last $m-1$ choices commute (and all choices commute if $n = m+1$). In the following, L_0 will be the Legendrian right-handed trefoil with $\mathrm{tb}(L_0) = 0$.

Proof We first take care of the lens space summand.

Claim 5.28 *The contact structure on the lens space is η_m.*

[4]This follows also from Lisca and Stipsicz's main theorem in [28], since positive cables of the trefoil have $\tau = g$.

Proof Let $k = \lfloor n/m \rfloor = (n-1)/m$; in particular, $mk = n-1$.

Since the contact structure on $L(m,-1)$ does not depend on the particular Legendrian knot, we can pick any L. Let $L = L_0^{(k)}$ be a k^{th} negative stabilisation of the trefoil L_0. Then

$$\xi_n^-(L_{m,n}) = \xi_n^-((L_0)_{m,1}^{(mk)}) = \xi_{+1}((L_0)_{m,1}) = \xi_{1/m}(L_0)\,\#\,\eta_m,$$

where the first equality follows from Corollary 5.14, the second from Proposition 4.12 and the third from Proposition 5.18. □

Remark 5.29 In the proof of the previous claim, we need to use a knot L_0 such that $\xi_{+1}(L_0)_{m,1}$ is tight in order to have uniqueness (up to isomorphism) of the connected sum decomposition (see Ding and Geiges [7]). As a byproduct of the proof, we obtain that $\xi_{n/m}(L)$ has to be tight for a k^{th} stabilisation of L_0.

As in the proof of Proposition 4.12, we now rule out all other possibilities for $\xi_{n/m}(L)$.

Claim 5.30 *We have $\xi_{m+1}^-(L_{m,m+1}) = \xi_{1+1/m}^-(L)\,\#\,\eta_m$.*

Proof Suppose that the contact structure on $\xi_{1+1/m}^-(L)$ was obtained by doing at least one positive stabilisation on the (only) pushoff of L, as dictated by the Ding–Geiges algorithm.

Suppose that $L = L_0'$. By the remark above we know that the contact structure $\xi_{1+1/m}(L)$ is tight and by Remark 4.16 we have that if there is one positive stabilisation, then $\xi_{1+1/m}$ is overtwisted. Therefore, in this particular case, the surgery layer is $\boldsymbol{T}_{1+1/m}^-$.

But this layer is independent of L, concluding the proof. □

Claim 5.31 *We have $\xi_{km+1}^-(L_{m,km+1}) = \xi_{k+1/m}^-(L)\,\#\,\eta_m$.*

Proof As above, let $L = L_0^{(k)}$. On one hand, we know that $\xi_{k+1/m}(L)$ is tight, and on the other hand (see Remark 4.16, and the proof of the claim above) every $\xi_{k+1/m}(L)$ that involves a positive stabilisation in the algorithm is overtwisted.

Again, the surgery layer is independent of the particular choice of L. □

Since we had proved the statement for $n = 1/m$ in the previous section, we have exhausted all cases. □

6 The main theorem

6.1 Technical lemmas

We introduce here three technical lemmas, whose proofs will be given in the next section. The first one is due to Honda (unpublished) and is implicit in [20]. We will give our own proof for convenience.

Proposition 6.1 *The gluing map $\Psi_\xi\colon SFH(-M,-\Gamma)\to SFH(-M',-\Gamma')$ associated to an overtwisted contact structure ξ on $N = M'\setminus \mathrm{Int}(M)$ is trivial.*

Remark 6.2 Proposition 6.1 is easily seen to be true if we restrict Ψ_ξ to the subspace $EH(M,\Gamma)$ of $SFH(-M,-\Gamma)$ generated by contact invariants $EH(M,\xi')$ such that ∂M is ξ'–convex and is divided by Γ.

In general $EH(M,\Gamma)$ is a proper subset of $SFH(-M,-\Gamma)$. Whenever the maximal Thurston–Bennequin number $\overline{\mathrm{tb}}(K)$ of a knot $K\in S^3$ is strictly smaller than $2\tau(K)-1$, no unstable element in $SFH(-S^3_{K,2\tau(K)-1})$ can belong to $EH(S^3_{K,2\tau(K)-1})$.

Recall that a framed knot (K,f) gives a surgery cobordism W_f and the latter induces a map $F_{-W_f}\colon \widehat{HF}(-S^3)\to\widehat{HF}(-S^3_f(K))$. In Definitions 2.24 and 2.25 we introduced the maps ψ_∞ and ψ_{+1} associated to contact ∞– and $(+1)$–surgery. In the last section we will prove the following.

Proposition 6.3 *The following diagram commutes:*

$$\begin{tikzcd}
SFH(-S^3_{K,f}) \arrow[r, "\psi_{+1}"] \arrow[dr, "\psi_\infty"'] & SFH(-S^3_{f+1}(K)(1)) \arrow[r, "\sim"] & \widehat{HF}(-S^3_{f+1}(K)) \\
 & SFH(-S^3(1)) \arrow[r, "\sim"] & \widehat{HF}(-S^3) \arrow[u, "F_{-W_{f+1}}"]
\end{tikzcd}$$

Remark 6.4 As it happens for Proposition 6.1, it is easy to see that this triangle has to be commutative whenever we restrict the domain of ψ_{+1} and ψ_∞ to the subspace $EH(S^3_{K,f})\subset SFH(-S^3_{K,f})$.

There is one more lemma, which is implicit in [41].

Proposition 6.5 [41] *The surgery cobordism map*

$$F_{-W_f}\colon \widehat{HF}(-S^3)\longrightarrow \widehat{HF}(-S^3_f(K))$$

is injective for $f=2\nu(K)$ and is zero for $f=2\nu(K)-2$.

6.2 Algebraic identities

Recall that in Definition 2.22 we introduced the notation $\sigma_\pm$ to denote the two gluing maps associated to the two types of stabilisation of an oriented knot K. In Definition 2.25 we introduced the notation $\psi_n^\pm$ for gluing maps associated to contact n–surgery. Proposition 4.12 (or rather, its proof) has a translation to the sutured world:

Proposition 6.6 *We have the following identities:*

(i) $\psi_{n+1}^\pm \circ \sigma_\pm = \psi_n^\pm$.

(ii) $\psi_n^\pm \circ \sigma_\mp = 0$.

These properties will be used throughout the proof, and will be the tools that allow us to switch from small to very large surgery coefficients and back.

Proof The first part follows directly from Proposition 4.12. The second part is Lisca and Stipsicz's trick of opposite stabilisations (see Remark 4.16) coupled with Proposition 6.1. □

The following two propositions allow us to simplify the proof.

Proposition 6.7 *If x is stable in $SFH(S^3_{K,f})$ then $\psi_n^\pm(x) = 0$ for any $n \geq 1$.*

Proof Suppose first that $x \in S_- = \ker \sigma_-^N$. As a warm up, let us prove the theorem in the case $n = 1$. For some sufficiently large N, we have

$$\psi_1(x) = (\psi_{N+1}^- \circ \sigma_-^N)(x) = \psi_{N+1}^-(0) = 0.$$

Theorem 3.11 tells us that $\sigma_\pm$ is an isomorphism on $S_\mp$, so each $x \in S_- \subset SFH(S^3_{K,f})$ is the image of an element $x' \in S_- \subset SFH(S^3_{K,f-n+1})$, in the sense that we have $x = \sigma_+^{n-1}(x')$.

Using this and the warm-up, we have that for all $x \in S_-$ we can write

$$\psi_n^+(x) = (\psi_n^+ \circ \sigma_+^{n-1})(x') = \psi_1(x'),$$

and the latter vanishes thanks to the warm up.

On the other hand,

$$\psi_n^-(x) = (\psi_{n+N}^- \circ \sigma_-^N)(x) = 0,$$

so we have proven that $S_- \subset \ker \psi_n^\pm$.

We can now exchange the roles of $+$ and $-$ signs in the proof and obtain that the results holds also for $x \in S_+$, which concludes the proof. □

Recall that in Lemma 3.18 we proved an analogous result for ψ_∞. We can therefore draw the following corollary.

Corollary 6.8 *Positive and ∞–surgery maps factor through the unstable complex.*

Remark 6.9 In the following we will often implicitly replace $SFH(-S^3_{K,f})$ with the *unstable quotient* $SFH(-S^3_{K,f})/(S_+ + S_-)$ (see Remark 3.13). This latter group, as said, is a direct sum of copies of $\mathbb{F}$ sitting in different Alexander gradings.

In particular, we will replace the maps ψ_n^- and ψ_∞ with their compositions with the projection $SFH(-S^3_L) \to SFH(-S^3_L)/(S_+ + S_-)$. We will continue to call these maps ψ_n^- and ψ_∞, keeping in mind the domain change. The maps $\sigma_\pm$ will be considered as maps between unstable complexes, too. In what follows, this will be used without further mention.

We find it very convenient to organise all the unstable complexes (as f varies among integers smaller than $2\tau(K)$) in a picture like in Figure 6.1.

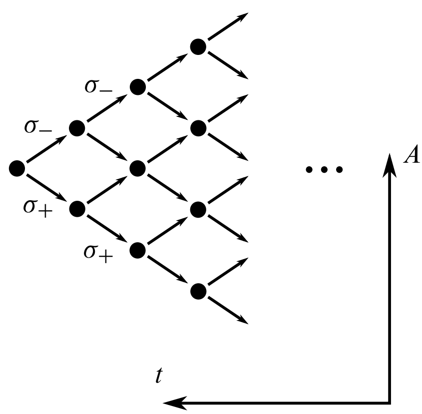

Figure 6.1: Every dot in the picture represents a generator in the unstable complex of some $SFH(-S^3_{K,f})$. Each column is a (Alexander-)homogenous spanning set for the unstable complex for $SFH(-S^3_{K,f})$ for a *fixed* slope f; this slope *decreases* when moving to the right. The vertex represents the generator of $SFH(-S^3_{K,2\tau(K)-1})$. The vertical direction gives the Alexander grading induced by the identification with $\widehat{HFK}(S^3_f(K), \widetilde{K})$. The arrows represent the action of $\sigma_\pm$ on the unstable complexes. The (unstable projection of the) invariant $EH(L)$ lands inside this triangle by Plamenevskaya's inequality (3-2) and Proposition 3.20.

6.3 The proof: Independence and sufficiency

We start by proving the last statement in the theorem.

Proposition 6.10 *The contact invariant $c(\xi_n^{\pm}(L))$ is independent of the Legendrian isotopy class of L when the classical invariants are fixed.*

Recall (see Remark 1.2) that what we mean by the statement above is that the *orbit* of the contact invariant $c(\xi_n^{\pm}(L))$ is independent of the Legendrian isotopy class.

We need to make a small digression about a similar issue affecting the EH invariants. Suppose that L_1 and L_2 are two Legendrian representatives of K in (S^3, ξ_{st}). Then $EH(L_i)$ is an element in $SFH(S^3_{L_i})$, and if we want to compare $EH(L_1)$ with $EH(L_2)$ we need to give an identification α of the two knot complements. Any two such identifications differ by an element of $\mathrm{MCG}(S^3_{L_i}, \partial S^3_{L_i})$, and this mapping class group acts on $SFH(-S^3_{L_i})$ [23].

We start with a lemma to say how much this identification matters, if L_1 and L_2 also have the same rotation number. Whenever possible, we will implicitly assume that a specific identification has been made without keeping track of it in the notation.

Lemma 6.11 *Regardless of the identification of $S^3_{L_1}$ with $S^3_{L_2}$, $EH(L_1)-EH(L_2)$ is stable.*

Proof Fix a diffeomorphism $\alpha\colon S^3_{L_1} \to S^3_{L_2}$. As a notational shorthand, call ξ_2 the restriction of ξ_{st} to $S^3_{L_2}$.

The diffeomorphism α induces a map $\alpha^*\colon SFH(-S^3_{L_2}) \to SFH(-S^3_{L_1})$ that preserves the Alexander grading and carries $EH(L_2)$ to $EH(\alpha^*\xi_2)$. Since L_1 and L_2 also have the same rotation number, $EH(L_1)$ and $EH(L_2)$ have the same degree, and so does $EH(\alpha^*\xi_2)$. To prove the lemma, it is enough to show that $EH(\alpha^*\xi_2)$ is not stable. If so, it has the same degree as $EH(L_1)$, and $\psi_\infty(EH(\alpha^*\xi_2)) = \psi_\infty(EH(L_1))$, and their difference is stable by Lemma 3.18.

Consider any extension $\widetilde{\alpha}\colon S^3_1 \to S^3_2$ of α from $S^3_1 = S^3_{L_1} \cup \nu(L_1)$ to $S^3_2 = S^3_{L_2} \cup \nu(L_2)$. Both S^3_1 and S^3_2 are diffeomorphic to S^3, but we keep the indices to keep them distinct.

If we do ∞–surgery on L_2, we get the standard contact structure $\xi_{\mathrm{st};2}$ on S^3_2. When we pull back using $\widetilde{\alpha}$ we get a tight contact structure $\alpha^*\xi_{\mathrm{st};2}$ on S^3_1.

Since $\widetilde{\alpha}$ maps $\nu(L_1)$ diffeomorphically to $\nu(L_2)$, the pull back of the contact structure on $\nu(L_2)$ is a tight contact structure on $\nu(L_1)$. Since such a contact structure is unique up to isotopy [19], it means that $\widetilde{\alpha}^*\xi_{\mathrm{st};2}$ is obtained from $\alpha^*\xi_2$ by contact ∞–surgery.

In other words, $\psi_\infty(EH(\alpha^*\xi_2)) = \widetilde{\alpha}^*(c(\xi_{\mathrm{st};2})) \neq 0$, and by Lemma 3.18 $EH(\alpha^*\xi_2)$ is not stable. $\square$

We now turn back to the independence statement.

Proof of Proposition 6.10 If L_1 and L_2 have the same classical invariants, $EH(L_1)$ and $EH(L_2)$ both belong to $SFH(-S^3_{L_1})$ and are homogeneous of the same degree $r(L_1)$. Moreover, since both knots are Legendrian in the standard contact S^3, we have that $\psi_\infty(EH(L_1)) = \psi_\infty(EH(L_2)) = c(\xi_{\mathrm{st}})$. In particular $EH(L_1) - EH(L_2) \in \ker \psi_\infty$ and since this difference is homogeneous, it is stable by Lemma 3.18.

The statement now is a straightforward consequence of Proposition 6.7. Since $\psi_n^\pm$ kills the stable subspace,

$$c(\xi_n^\pm(L_1)) - c(\xi_n^\pm(L_2)) = \psi_n^\pm(EH(L_1) - EH(L_2)) = 0. \qquad \square$$

Remark 6.12 Notice that the previous statement, together with Proposition 3.17, also proves that if (S^3, ξ) is overtwisted and L is ξ–Legendrian, then $c(\xi_n^-(L)) = 0$ for all $n \geq 0$. In particular, this justifies our focus on surgeries on the standard contact S^3.

We now prove the sufficiency of the three conditions.

Proposition 6.13 *If (SL), (SC) and (TN) hold, $c(\xi_n^-(L)) \neq 0$.*

Proof Call $E := EH(L)$, $\tau := \tau(K) = \nu(K)$, $t := \mathrm{tb}(L)$ and $c := c(\xi_n^-(L))$.

Recall that $\xi_{n+1}^-(L)$ is obtained from $\xi_n^-(L)$ through a Legendrian surgery, and if the latter has nonvanishing contact invariant, so does the former. Therefore, it is enough to prove the result when we have equality in (SC), that is, when $n = 2\tau - t$.

Graphically, the condition (SL) means that $EH(L)$ lands on the top edge of the infinite triangle of Figure 6.1. In particular we can find x in $SFH(S^3_{K,2\tau-1})/S$ such that $\sigma_-^{n-1}(x) = E$, and Proposition 6.6 tells us that $c = \psi_n^-(E) = \psi_1(x)$. By Proposition 6.3, $\psi_1(x) = F_{-W_{2\tau}}(\psi_\infty(x))$, and the latter is nonzero by the injectivity of $F_{-W_{2\nu(K)}}$ (Proposition 6.5), the condition (TN) and the nonvanishing of $c(\xi_{\mathrm{st}}) = \psi_\infty(x)$. $\square$

6.4 The proof: Necessity

We now turn to the necessity of the three conditions: first we are going to prove that (SL) is necessary, then we are going to prove that if (SL) holds then (SC) is necessary, and finally we are going to prove that if both (SL) and (SC) hold, then (TN) is also necessary.

In the following, we will call $E := EH(L)$, $\tau := \tau(K)$, $\nu := \nu(K)$, $t := \mathrm{tb}(L)$ and $c := c(\xi_n^-(L))$.

Proposition 6.14 *If (SL) does not hold, then $c(\xi_n^-(L)) = 0$.*

Proof Suppose that (SL) does not hold. In this case, $A(E)=-r(L)$ is not maximal in the unstable complex, since the top generator in the unstable complex for $SFH(-S^3_{K,t})$ has Alexander degree $t-2\tau+1$. Pictorially, the unstable component of E is not the top generator in the relevant column on the left-hand side of Figure 6.2. This implies that E is in the image of σ_+, and we can write $E=\sigma_+(x)$. So by Proposition 6.6,

$$c=\psi_n^-(E)=\psi_n^-(\sigma_+(x))=0. \qquad \square$$

Proposition 6.15 *If (SL) holds but (SC) does not, then $c(\xi_n^-(L))=0$.*

Proof Call s the surgery index $s=t+n<2\tau$.

Claim 6.16 *We can assume $n=1$.*

Proof Since (SL) holds but (SC) does not, the unstable part of E lies in the top (slanted) row of Figure 6.1, and $E=\sigma_-^{n-1}(y)$ for some y in the unstable part of $SFH(S^3_{K,s-1})$. By Proposition 6.6, $\psi_n^-(E)=\psi_n^-(\sigma_-^{n-1}(y))=\psi_1(y)$, and y also fails to satisfy the condition (SC) in the sense that the slope $s-1$ of the sutures (that is, the algebraic counterpart of the Thurston–Bennequin number for L) satisfies $s-1+1=s<2\tau$. $\square$

So let us assume $n=1$, that is $E=y$ in the proof of the claim. Since $s=1+t<2\tau$, the unstable part of E is not the left vertex of the triangle of the right-hand side of Figure 6.2. In particular, $\sigma_+(E)$ is in the image of σ_-, so we can write $\sigma_+(E)=\sigma_-(x)$ for some x in the unstable part of $SFH(S^3_{K,t})$.

By Proposition 6.6, $c=\psi_1(E)=\psi_2^+(\sigma_+(E))=\psi_2^+(\sigma_-(x))=0$. $\square$

For the last part we call into play the Legendrian cabling and surgeries along them.

Proposition 6.17 *Suppose (SL) and (SC) hold but (TN) does not. Then $c(\xi_n^-(L))=0$.*

Proof By Corollary 4.10, it is enough to show that $c(\xi^-_{n+1/2}(L))=0$.

By Proposition 5.27, $c(\xi^-_{n+1/2}(L))\otimes c(\eta_2)=c(\xi^-_{2n+1}(L_{2,2n+1}))$. Since (TN) does not hold, $\varepsilon(K)=-1$ and Proposition 5.10 tells us that $L_{2,2n+1}$ fails to satisfy (SL), so the right-hand side vanishes by Proposition 6.14. Since η_2 is obtained from ξ_{st} through Legendrian surgery, $c(\eta_2)$ is nonvanishing, and therefore $c(\xi^-_{n+1/2}(L))=0$, concluding the proof. $\square$

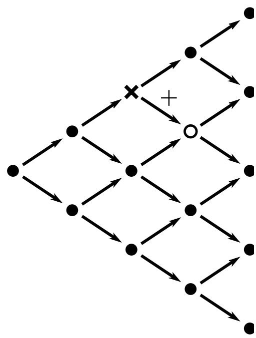

Figure 6.2: The figure on the left represent the situation when (SL) does not hold. The hollow circle is (the stable part of) $EH(L)$ and the $\times$ is the element x in the proof. The figure on the right represent the situation when (SL) holds but (SC) does not. The hollow circle represents the element y and the $\times$ represents the element x in the proof.

6.5 Corollaries

We will prove the following statement, which is slightly stronger than Corollary 1.6.

Proposition 6.18 *If $\varepsilon(K) = 1$ (resp. $\varepsilon(K) = 0$) and there is a Legendrian representative L of K that satisfies (SL), then for all $q > 2\tau(K) - 1$ (resp. $q \geq 0$) the manifold $S^3_q(K)$ supports a tight contact structure.*

Remark 6.19 This is in fact stronger than Corollary 1.6, since Hom [18] proved that $\varepsilon(K) = 0$ implies $\tau(K) = \nu(K) = 0$. Observe also that if $\tau(K) = \nu(K) > 0$, then $\varepsilon(K) = 1$. Hom also proved that $\tau(K) = g(K)$ implies that $\nu(K) = \tau(K)$, and in fact this holds true under the weaker hypothesis $\tau(K) = g_*(K) > 0$. This is obtained as a combination of [28, Proposition 2.1] and Proposition 6.5. In particular, if K is not slice and its maximal self-linking number $\overline{\mathrm{sl}}(K)$ is $2g_*(K)-1$, every manifold $S^3_q(K)$ with $q > 2g_*(K) - 1$ supports a tight contact structure (compare with the main results in [28; 30]).

Proof Suppose that L and K are as in the statement, with L satisfying (SL) and K satisfying (TN). Theorem 1.1 tells us that for all integers $m \geq 2\tau(K)$, we have a contact structure on $S^3_m(K)$ with nonvanishing contact invariant, and therefore tight.

Using Corollary 4.10, we obtain contact structures on $S^3_q(K)$ with nonvanishing contact invariants for all $q \geq 2\tau$.

Now, let us call $t = \mathrm{tb}(L), r = r(L)$, and recall that (SL) implies $r \leq 0$. There is nothing left to prove when $\varepsilon(K) = 0$, so we can suppose that $\varepsilon(K) = 1$.

Proposition 5.10 tells us that for every $n \geq 1-mr \geq 1$ the Legendrian cable $L_{m,n}$ satisfies the three hypotheses in our main theorem, so that $c(\xi^-_{1-mr}(L_{m,1-mr})) \neq 0$. But, by Proposition 5.27, $\xi^-_{1-mr}(L_{m,1-mr})$ splits as a connected sum $\xi^-_{-r+1/m}(L) \# \eta_m$, and in particular $c(\xi^-_{1/m-r}(L)) \neq 0$. This is a contact structure on $S^3_{t-r+1/m}(K) = S^3_{2\tau(K)-1+1/m}(K)$.

Appealing to Corollary 4.11 concludes the proof in the case $\varepsilon(K) = 1$. □

Remark 6.20 Notice that the same trick does not work if $\varepsilon(K) = 0$ because of the odd behaviour of τ and ε for cables when the cabling coefficient q in Theorem 5.5 goes from positive to negative. These values are the "critical" values that allow us to reach slopes below $2\tau(K)$ when $\varepsilon(K) = 1$.

Let us recall now the definition of the transverse invariant $\widetilde{c}$ [30]. Fix a topological knot $K \subset S^3$. The sequence of groups $(\widehat{HF}(-S^3_n(K)))$ comes with a collection of maps $F_{\overline{W}_n}\colon \widehat{HF}(-S^3_n(K)) \to \widehat{HF}(-S^3_{n-1}(K))$ and together they give rise to an inverse system $\{\widehat{HF}(-S^3_n(K)), \phi_{f,g}\}_{g<f}$, where $\phi_{f,g}$ is the composition $F_{\overline{W}_f} \circ \cdots \circ F_{\overline{W}_{g+1}}$. Lisca and Stipsicz call this inverse limit $H(S^3, K)$.

Definition 6.21 Given a transverse knot T, the invariant $\widetilde{c}(T)$ is the class of the sequence $(c(\xi^-_n(L)))_{n\in\mathbb{N}}$ in $H(S^3, K)$, where L is a Legendrian approximation of T.

There is an ambiguity in the definition of $\widetilde{c}$, coming from the ambiguity in the definition of c. Once we fix a Legendrian approximation L of T and an identification of S^3_L with the "abstract" sutured manifold $S^3_{K,\mathrm{tb}(L)}$, though, $\widetilde{c}$ is well defined. The equality in the statement of Corollary 1.7 has to be understood in the sense that the two elements are the same up to fixing the two identifications.

It is proved in [30] that the invariant above is nontrivial (in the sense that it is not identically zero). On the other hand, we prove here that it does not detect more than the classical invariants.

Proof of Corollary 1.7 We know that $c(\xi^-_n(L)) = 0$ if $c(\xi) = 0$ since $S_\pm \subset \ker \psi^\pm_n$, and we know that if $\xi = \xi_{\mathrm{st}}$, $c(\xi^-_n(L)) = 0$ unless $\mathrm{sl}(T) = 2\tau(K)-1$ and $\tau(K) = \nu(K)$.

Suppose therefore that $\mathrm{sl}(T) = 2\tau(K)-1 = 2\nu(K)-1$, and let L' be *any* Legendrian knot of topological type K such that $\mathrm{tb}(L') - r(L') = 2\tau(K)-1$ (L' does not need to be a Legendrian approximation of T). Call d the difference $d = \mathrm{tb}(L) - \mathrm{tb}(L')$, and suppose that $d > 0$. Then for every $n > |d|$, and for every two identifications of $S^3_{L(d)}$ and $S^3_{L'}$ with $S^3_{K,\mathrm{tb}(L)}$ we have

$$c(\xi^-_n(L)) = c(\xi^-_{n+d}(L'))$$

by Theorem 1.1. As a consequence, the classes of the two sequences in $H(S^3, K)$ coincide.

Thus, $\widetilde{c}$ can only see whether the two equalities $\mathrm{sl}(T) = 2\tau(K) - 1$ and $\tau(K) = \nu(K)$ hold, and these are equalities in the classical invariants for T. □

7 Proofs of technical lemmas

This section will be rather dry, and is a detailed account of the various technical ingredients used in the proof.

7.1 The Heegaard–Floer lemma

Recall that we want to prove that the surgery cobordism map F_{-W_f} induced by the surgery cobordism from $-S^3$ to $-S^3_f(K)$ is injective for $f = 2\nu(K)$ and vanishes for $f = 2\nu(K) - 2$.

Similar results appeared in [37, Proposition 3.1] and Hedden [17, Proposition 3.1]; this refined result follows from a computation in [41].

Proof of Proposition 6.5 The map F_{-W_f} fits into an surgery exact triangle:

$$\begin{tikzcd}
\widehat{HF}(-S^3_f) \arrow[rr] & & \widehat{HF}(-S^3_{f-1}) \arrow[dl] \\
& \widehat{HF}(-S^3) \arrow[ul] &
\end{tikzcd}$$

Recall that if in an exact triangle of vector spaces (U, V, W) we have $\dim U + \dim V = \dim W$, then the map between U and V is the zero map.

Having this in mind, we can prove by direct computation, using the "mapping cone" construction of [40] (see also [44]), that

$$\dim \widehat{HF}(-S^3_f(K)) - \dim \widehat{HF}(-S^3_{f-1}(K)) = \pm \dim \widehat{HF}(-S^3),$$

where the sign is a plus if $f = 2\nu(K)$ and is a minus if $f = 2\nu(K) - 2$. In fact, in [41, Proposition 9.1], Ozsváth and Szabó compute the ranks of the two groups on the left-hand side when $\tau(K) \geq 0$:

$$\dim \widehat{HF}(-S^3_f(K)) = |f| + 2\max\{0, 2\nu(K) - 1 - f\} + D,$$

where D is a constant depending only on K.

The condition $\tau(K) \geq 0$ can be always achieved by taking the mirror of the knot if needed. If $\tau(K) = \nu(K) = 0$, this dimension has two minima at $f = \pm 1$, therefore the map F_{-W_f} is injective if $f = 0$ or $f \geq 2$ and zero otherwise. If $\nu(K) \geq 1$, on the other hand, the dimension has a single minimum at $f = 2\nu(K) - 1$ (in fact, the graph of the dimension is a translation of the graph of the absolute value), therefore F_{-W_f} is injective if and only if $f \geq 2\nu(K)$.

We can now use Hom's results [18] to recover what happens when $\tau(K) < 0$. In that case, $\tau(\bar{K}) > 0$, and in particular $\varepsilon(\bar{K}) = -\varepsilon(K) \neq 0$. If $\varepsilon(K) = 1$ then $\varepsilon(\bar{K}) = -1$ and $\nu(K) = \tau(K)$, while $\nu(\bar{K}) = \tau(\bar{K}) + 1$ and

$$\dim \widehat{HF}(-S^3_f(\bar{K})) = \dim \widehat{HF}(S^3_{-f}(K))$$

has a single minimum at $-f = 2\nu(\bar{K}) - 1$ that is exactly $f = 2\nu(K) - 1$. Similarly, if $\varepsilon(K) = -1$, $\nu(K) = \tau(K) + 1$ and $\nu(\bar{K}) = \tau(\bar{K})$, and again $\dim \widehat{HF}(-S^3_f(\bar{K}))$ has a single minimum at $f = 2\nu(\bar{K}) - 1$,

The same argument used in the case $\tau(K) \geq 0$ shows that in either case F_{-W_f} is injective if and only if $f \geq 2\nu(K)$. $\square$

7.2 Sutured Floer lemmas

One of the two key ingredients in the proofs of Propositions 6.1 and 6.3 is the associativity of maps in triple Heegaard diagrams. Recall the following result of Ozsváth and Szabó.

Suppose that we have a quadruple Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$, satisfying some additional admissibility assumption [36]. There are triangle count maps associated to the triple Heegaard diagram. Call them $f_{\alpha\beta\gamma}$, $f_{\alpha\beta\delta}$, $f_{\alpha\gamma\delta}$, $f_{\beta\gamma\delta}$ so that, for example $f_{\alpha\beta\gamma}\colon \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, z)$, and label with the capitalized letters F the induced maps on the homology level.

Proposition 7.1 [39] *These maps satisfy the identity*

$$F_{\alpha\gamma\delta}(F_{\alpha\beta\gamma}(x \otimes y) \otimes v) = F_{\alpha\beta\delta}(x \otimes F_{\beta\gamma\delta}(y \otimes v))$$

for all $x \in \widehat{HF}(Y_{\alpha\beta})$, $y \in \widehat{HF}(Y_{\beta\gamma})$ and $v \in \widehat{HF}(Y_{\gamma\delta})$.

The other key ingredient is given in Rasmussen's paper [45]. The philosophy is that gluing maps can be computed via triangle counts given a handle decomposition of the gluing layer. In this paper, we need three instances of this general fact: bypass attachments (Proposition 7.3 below), ∞– and $(+1)$–surgery maps (Proposition 7.7 below).

When we attach a bypass to a sutured manifold (M,Γ) to obtain (M,Γ'), we change the sutures as in Figure 7.1. Up to a 1–handle attachment (see below), we can assume that both R_+ and R'_+ are connected, so that both (M,Γ) and (M,Γ') are represented by an arc diagram. We can also suppose (see the rightmost picture in Figure 7.1) that the two arc diagrams live on the same Heegaard surface, that they share the α–curves, all β–curves and all but one β–arc. Finally, we can assume that the two β–arcs where they differ intersect at exactly one point. Arguing as in the closed case, this determines a preferred Θ–element in a triple arc diagram, which in turn allows us to define a triangle count. This triangle count is chain-homotopic to the bypass attachment map.

If we have a sutured manifold (M,Γ) with torus boundary and $|\Gamma|=2$, we can attach a $(+1)$–surgery layer to get $(M',\{\gamma\})$ with sphere boundary. As before, we construct an arc diagram for (M,Γ) and an arc diagram for M' on the same Heegaard surface. All α– and β–curves can be chosen to coincide and the new β–curve can be chosen to intersect the β–arc exactly once. This determines a Θ–element in a triple arc diagram, and the resulting triangle count induces ψ_{+1} in homology.

7.2.1 The proof of Proposition 6.1 Recall that we want to prove that gluing maps associated to overtwisted contact structures vanish.

Proof of Proposition 6.1 Suppose that there is an overtwisted disc $D\subset N$ and consider a small neighbourhood B of it with convex boundary. Then join B to a boundary component of N that is going to be glued to M, using a small neighbourhood A of an arc. Call N' the union of A, B and a neighbourhood of the component of the boundary we have joined B to, and suppose that the boundary of N' is convex with respect to ξ. Call N'' the closure of the complement of N' in N. Finally, let ξ',ξ'' be the restrictions of ξ to N' and N'' respectively.

Claim 7.2 *We can suppose $N=N'$.*

Proof By naturality of gluing maps, $\Psi_\xi=\Psi_{\xi''}\circ\Psi_{\xi'}$ and if $\Psi_{\xi'}=0$, then in particular $\Psi_\xi=0$. □

Following Ozbagci [33] (see also Giroux's criterion for overtwistedness of contact structures near a convex surface), we can write the gluing of the overtwisted disc as a double bypass attachment along a curve that makes a small dollar symbol \$ across a single suture as in the top left of Figure 7.2. Unfortunately, there is a small technical detail we need to face: attaching the second bypass disconnects R_+. To overcome this obstacle, we first attach a contact 1–handle H — and this does not affect the sutured Floer homology groups since it is the inverse of a product disc decomposition — and then attach the two bypasses to the new manifold as shown in the second left figure in Figure 7.2.

Suppose that we start off with an arc diagram $\mathcal{H}_0 = (\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0^a, \boldsymbol{\beta}_0^c, D_0)$ for (M, Γ) as in the top left of Figure 7.2. We obtain an arc diagram for $(M \cup H, \Gamma)$ by adding a 1–handle to Σ_0, obtaining a surface $\Sigma = \Sigma_0 \# T^2$. The set of α–curves is the same as before, plus a single α–curve α_0 that is the belt of the (3–dimensional) handle H. The set of β–curves is $\boldsymbol{\beta}_0^c$, and we add a single β–arc β_0 that runs once through the handle as in the top right corner of Figure 7.2. Call this new diagram $\mathcal{H}_\beta$.

Attaching the first bypass we obtain an arc diagram $\mathcal{H}_\gamma$. After attaching the second bypass in the same region, we obtain a third diagram $\mathcal{H}_\delta$ that looks like the bottom right picture in Figure 7.2. Call the four triple Heegaard diagrams we obtain $\mathcal{H}_{\alpha\beta\gamma}$, $\mathcal{H}_{\alpha\beta\delta}$, $\mathcal{H}_{\alpha\gamma\delta}$ and $\mathcal{H}_{\beta\gamma\delta}$.

It is straightforward to check that the admissibility conditions of [36] are satisfied by the arc diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, D)$.

As for the proof of Proposition 3.20, in order to obtain the bypass attachment maps we need to count triangles in the triple Heegaard diagrams $\mathcal{H}_{\alpha\gamma\beta}$ and $\mathcal{H}_{\alpha\delta\gamma}$ and then take the associated *cohomological* maps. More precisely, to the first bypass attachment on $\mathcal{H}_\beta$ we can associate a Θ–element $\boldsymbol{\Theta}_{\beta\gamma}$ constructed as follows. The point on the arc β_0 is the only intersection point of β_0 with the arc γ_0. Every other γ–curve in $\mathcal{H}_\gamma$ is a small perturbation of a β–curve in $\mathcal{H}_\beta$, and therefore there is a preferred choice among the two intersection points as in [37]. We then have the following.

Proposition 7.3 [45] *The map induced in cohomology by the triangle count map $f_{\alpha\gamma\beta}(\cdot \otimes \boldsymbol{\Theta}_{\beta\gamma})$ is the gluing map associated to the bypass attachment.*

Similarly, there is a Θ–element $\boldsymbol{\Theta}_{\gamma\delta}$ in $\mathcal{H}_{\alpha\delta\gamma}$, and the associated triangle count map $f_{\alpha\delta\gamma}(\cdot \otimes \boldsymbol{\Theta}_{\gamma\delta})$ induces the gluing map associated to the second bypass attachment.

Since we are working over the field $\mathbb{F}_2$, studying the maps induced in cohomology is the same as studying the maps associated in homology, which is what we are going to do from now on.

Call (M', Γ') the sutured manifold defined by $\mathcal{H}_\delta$ so that, at the 3–manifold level, $M' = M \cup H \cup N$ and let $\boldsymbol{\Theta}_{\beta\delta}$ be the Θ–element in the triple Heegaard diagram $\mathcal{H}_{\alpha\delta\beta}$. The following claim is a triangle count in $\mathcal{H}_{\delta\gamma\beta}$.

Claim 7.4 *We have $f_{\delta\gamma\beta}(\boldsymbol{\Theta}_{\beta\gamma} \otimes \boldsymbol{\Theta}_{\gamma\delta}) = \boldsymbol{\Theta}_{\beta\delta}$.*

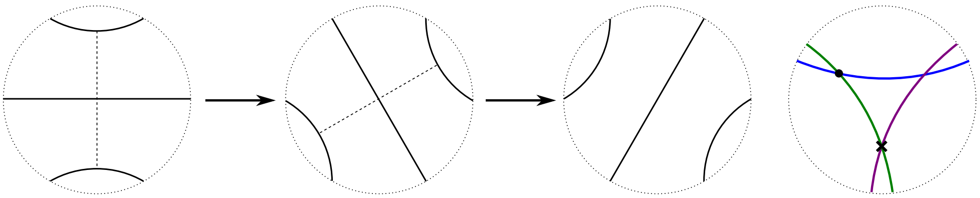

Figure 7.1: The three circles on the left show the (local) effect of a double bypass attachment to the dividing curves of a convex boundary. The figure on the right shows what happens locally to the β–curves of the three arc diagrams coming from the figure on the left: the blue curve is a β–arc for the first diagram, the green curve is a γ–arc for the second diagram, and the purple one is a δ–arc for the third diagram. The two intersection points in evidence are the points in $\mathbf{\Theta}_{\beta\gamma}$ and $\mathbf{\Theta}_{\gamma\delta}$ on the arcs shown.

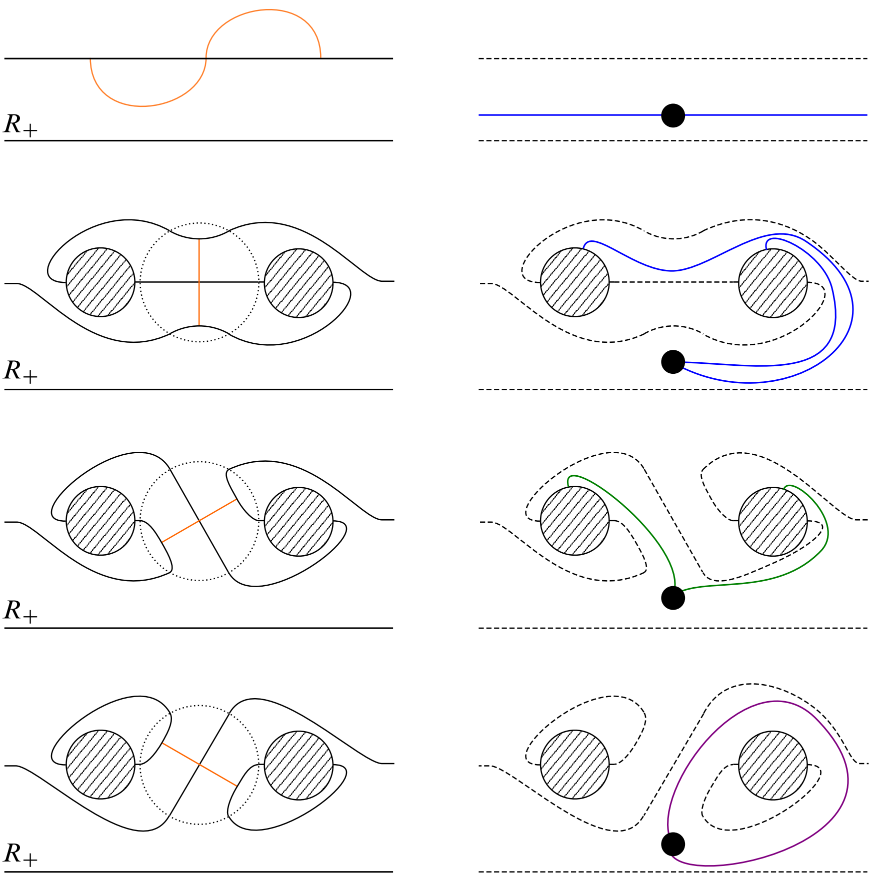

Figure 7.2: On the left column we show ∂M, sutures, bypasses and their effect on the sutures. On the right, we show associated arc diagrams.

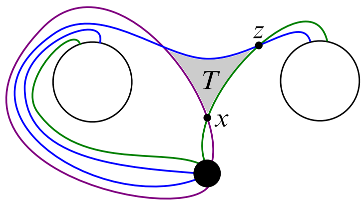

Figure 7.3: The interesting portion of the triple arc diagram of Claim 7.4

Proof We want to count all possible triangular domains $\mathcal{D}$ in $\mathcal{H}_{\delta\gamma\beta}$.

For each index i, β_i intersects γ_i, δ_i and no other curve. Moreover, for $i > 0$, both β_i and δ_i are adjacent to a region touching the base disc D on *both* sides, so $\mathcal{D}$ can have positive multiplicities in this area only. In particular, $\mathcal{D} = \sum \mathcal{D}_i$, with $\mathcal{D}_i$ supported in the spanning region for all $i > 0$, and $\mathcal{D}_0$ supported near β_0.

There is a domain $\overline{\mathcal{D}} \in \pi_2(\mathbf{\Theta}_{\beta\gamma}, \mathbf{\Theta}_{\gamma\delta}, \mathbf{\Theta}_{\beta\delta})$ which is easy to spot: it is the sum of the small triangle T in Figure 7.3 and the small triangles shaded in Figure 7.4. It is well known (see [39]) that this domain has Maslov index 0 and that the associated moduli space of triangles contains one element, thus providing us with a $\mathbf{\Theta}_{\beta\delta}$ summand. We want to show that this is the only positive domain of Maslov index 0 in the triple Heegaard diagram.

Let us suppose that $\mathcal{D} = \sum \mathcal{D}_i$ as before is a positive triangular domain with multiplicity zero at every region touching the base disc.

In what follows, we will call $z_i := (\Theta_{\beta\gamma})_i$, $x_i := (\Theta_{\gamma\delta})_i$, $y_i := (\Theta_{\beta\delta})_i$ and, when $i > 0$, y_i' the other intersection point of β_i and δ_i.

Let us first consider what happens in the region containing β_0, γ_0 and δ_0. Here all pairwise intersections are fixed, and are x_0, y_0 and z_0. The base disc D lies on all three arcs; only one of the two segments into which the three intersection points divide the arcs can be part of $\partial\mathcal{D}_0$. In particular, $\partial\mathcal{D}_0$ has to coincide with $\partial\overline{\mathcal{D}}$. Also, at every intersection, three of the four angles are contained in regions touching the base disc, therefore multiplicities have to be zero outside T and in particular $\mathcal{D}_0 = T$.

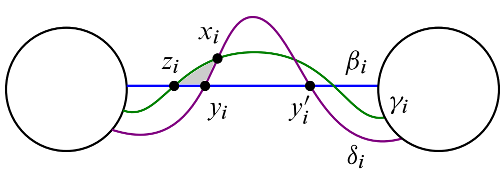

Figure 7.4: The triple Heegaard diagram near β_i for $i > 0$

Suppose $\mathcal{D}_i \in \pi_2(z_i, x_i, y_i)$ when $i > 0$ (see Figure 7.4). Let us follow $\partial\mathcal{D}$ from z_i with the orientation given by $\mathcal{D}_i$. We have to stop at x_i without winding multiple times, because there is a region that touches both sides of β_i (and also of δ_i) and the base disc D, so the multiplicity of $\mathcal{D}_i$ in that region has to be 0.

There are two possible segments: one is contained in the plane in Figure 7.4, the other one runs inside the handle. In the first case, when we arrive at x_i we have to turn *left* (because of orientations) and we have to stop at y_i without running around δ_i multiple times (because now δ_i touches a region containing the basepoint from both sides), and in particular $\mathcal{D}_i$ is the small triangle shaded in Figure 7.4.

In the second case, the domain is an immersed triangle that has multiplicity two on the small triangle region shaded. Using Sarkar's computation [48], we see that this domain gives a contribution to $\mu(\mathcal{D})$ which is strictly bigger than $\frac{1}{2}$.

Suppose now $\mathcal{D}_i \in \pi_2(z_i, x_i, y_i')$. Reasoning as above, we see that there are only two choices for $\mathcal{D}_i$, each obtained by adding one of the bigons in $\pi_2(y_i, y_i')$ to the small shaded triangle. Again, using Sarkar's computation, we see that these domains give a contribution bigger than $\frac{1}{2}$ to $\mu(\mathcal{D})$.

Summing up, if $\mathcal{D} \neq \overline{\mathcal{D}}$, $\mu(\mathcal{D})$ is strictly bigger than $\mu(\overline{\mathcal{D}}) = 0$ and therefore $\mathcal{D}$ is not involved in the triangle count. □

Thanks to the claim and Proposition 7.1, we can consider the single triangle count $f_{\alpha\delta\beta}(\cdot \otimes \mathbf{\Theta}_{\beta\delta})$. In order to achieve admissibility for $\mathcal{H}_{\alpha\delta\beta}$, we need to perturb the new α–curve so that it intersects the new δ–arc in a pair of canceling points as in Figure 7.5.

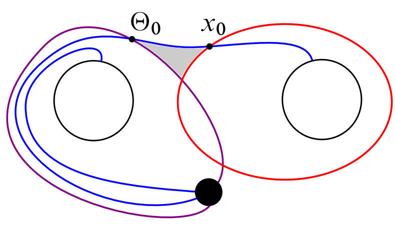

Figure 7.5: The portion of $\mathcal{H}_{\alpha\delta\beta}$ considered in Claim 7.5

Claim 7.5 *There are no positive triangular domains in $\mathcal{H}_{\alpha\delta\beta}$ that appear in the triangle count for $f_{\alpha\delta\beta}$.*

Proof Consider Figure 7.5. This is the same part of the diagram of Figure 7.4, but we are now drawing the α–curve instead of the γ–arc. We will argue by contradiction: let $\mathcal{D}$ be such a domain.

The two points Θ_0 and x_0 are the only two intersection points on the arc β_0. Reasoning as in Claim 7.4, the boundary $\partial\mathcal{D}\cap\beta_0$ is the segment between these two points, oriented from Θ_0 to x_0. But the region above the segment in Figure 7.5 touches the base disc D, therefore the multiplicity there has to be zero, showing that $\partial\mathcal{D}\cap\beta_0=\varnothing$.

In particular, $\pi_2(\cdot,\cdot,\mathbf{\Theta}_{\beta\delta})=\varnothing$. □

This immediately shows that $f_{\alpha\beta\delta}(\cdot\otimes\mathbf{\Theta}_{\beta\delta})=0$, which in turn implies $\Psi_\xi=0$. □

There is an alternative proof of Proposition 6.1, suggested by the referee. More generally, this proves that whenever $EH(\xi)\in SFH(-N,-\Gamma_N)$ vanishes (where we set $\Gamma_N=\Gamma\cup\Gamma'$) the induced gluing map Ψ_ξ is trivial.

Sketch of an alternative proof of Proposition 6.1 We can look at the gluing map Ψ_ξ as the composition of two gluing maps, $\Psi_\xi\circ\Psi_\eta$, where η is an I–invariant contact structure on $\partial M\times I$ such that $M\times\{i\}$ is convex with dividing curves $\pm\Gamma\times\{i\}$ for $i=0,1$. By naturality, $\Psi_\xi\circ\Psi_\eta=\Psi_\xi$.

We can also look at the map Ψ_η as a gluing map

$$\Psi_\eta\colon SFH(-M\amalg -N,-\Gamma\amalg -\Gamma_N)\longrightarrow SFH(-M',-\Gamma')$$

and by naturality we have that $\Psi_\eta(\cdot\otimes EH(\xi))=\Psi_\xi$.

In particular, if $EH(\xi)=0$, then $\Psi_\xi=0$. □

7.2.2 The proof of Proposition 6.3

Recall that Proposition 6.3 says that the diagram

$$\begin{tikzcd}
SFH(-S^3_{K,f}) \arrow[r, "\psi_{+1}"] \arrow[dr, "\psi_\infty"'] & SFH(-S^3_{f+1}(K)(1)) \arrow[r, "\sim"] & \widehat{HF}(-S^3_{f+1}(K)) \\
& SFH(-S^3(1)) \arrow[r, "\sim"] & \widehat{HF}(-S^3) \arrow[u, "F_{-W_{f+1}}"']
\end{tikzcd}$$

commutes. Notice that each of the three maps involved is computed by a triangle count in some triple Heegaard diagram.

Remark 7.6 Ozváth and Szabó [38] proved that the total cobordism map associated to a contact $(+1)$–surgery cobordism W carries $c(\xi)$ to $c(\xi_{+1})$. More recently, Baldwin [1] proved that there exists a Spin^c–structure $\mathfrak{t}_0$ on W such that $F_{W,\mathfrak{t}_0}$ carries $c(\xi)$ to $c(\xi_{+1})$. In Proposition 6.3 we do not worry about Spin^c–structures and consider the total map only.

Proof of Proposition 6.3 Let us fix any Heegaard diagram $\mathcal{H}'$ for $-(S^3, K)$, attach a 1–handle with feet next to the two basepoints z, w and build the three Heegaard diagrams involved in the statement in the usual way. All curves except the ones intersecting the core α_0 of the 1–handle are small Hamiltonian perturbations one of the other, and γ_0 and β_0 are parallel outside a small neighbourhood of β_0. Moreover, any two among β_0, γ_0 and δ_0 intersect in one single point. As for the proof of Proposition 6.1, we can associate a Θ–element $\mathbf{\Theta}_{\beta\gamma}$, $\mathbf{\Theta}_{\gamma\delta}$, $\mathbf{\Theta}_{\beta\delta}$ to each of the three triple diagrams.

It is also easy to check that all three triple diagrams are compatible [36].

The map F_{-W_f} is the map induced in cohomology by the triangle count $f_{\alpha\delta\gamma}(\cdot \otimes \mathbf{\Theta}_{\gamma\delta})$; see [39].

Proposition 7.7 [45] *The map ψ_∞ is the map induced in cohomology by the triangle count $f_{\alpha\gamma\beta}(\cdot \otimes \mathbf{\Theta}_{\beta\gamma})$. The map ψ_{+1} is the map induced in cohomology by the triangle count $f_{\alpha\delta\beta}(\cdot \otimes \mathbf{\Theta}_{\beta\delta})$.*

Since we are working with $\mathbb{F}$ coefficients, proving the cohomological statement is equivalent to proving the dual homological statement. Proposition 6.3 can now be rephrased as

$$F_{\alpha\delta\beta}(\cdot \otimes \mathbf{\Theta}_{\beta\delta}) = F_{\alpha\gamma\beta}(F_{\alpha\delta\gamma}(\cdot \otimes \mathbf{\Theta}_{\gamma\delta}) \otimes \mathbf{\Theta}_{\beta\gamma}).$$

Let us call $\phi_{\gamma\beta}$, $\phi_{\delta\beta}$, $\phi_{\delta\gamma}$ the three triangle counts; namely $\phi_{\gamma\beta} = f_{\alpha\gamma\beta}(\cdot \otimes \mathbf{\Theta}_{\beta\gamma})$, and similarly for the other ϕ–maps.

Lemma 7.8 *We have $f_{\delta\gamma\beta}(\mathbf{\Theta}_{\gamma\delta} \otimes \mathbf{\Theta}_{\beta\gamma}) = \mathbf{\Theta}_{\beta\delta}$.*

Proof Let us consider Figure 7.6. There are small triangles in the region spanned by β_i during the Hamiltonian isotopy that brings β_i to γ_i and δ_i, as shown in the top part of the figure. There is also a "bigger" triangle, shown in the bottom part of the figure, around the three curves β_0, γ_0, δ_0 involved in the surgeries. As before, the domain $\overline{\mathcal{D}}$ obtained by summing these triangular regions gives the summand $\mathbf{\Theta}_{\beta\delta}$.

We claim that there are no other positive domains of Maslov index 0 in the sum $f_{\beta\gamma\delta}(\mathbf{\Theta}_{\beta\gamma} \otimes \mathbf{\Theta}_{\gamma\delta})$. Suppose that $\mathcal{D}$ is one of these triangular domains.

Claim 7.9 *If $\mathcal{D}$ is as above, $\mathcal{D} = \overline{\mathcal{D}}$.*

As before, we will call $z_i := (\Theta_{\beta\gamma})_i, x_i := (\Theta_{\gamma\delta})_i, y_i := (\Theta_{\beta\delta})_i$ and when $i > 0$ let y_i' be the other intersection point of β_i and δ_i.

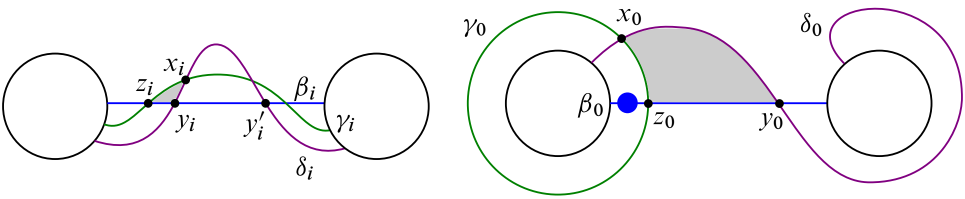

Figure 7.6: The triple arc diagram $(\Sigma, \boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\beta})$ of Lemma 7.8

Proof The situation is very similar to the situation in the proof of Claim 7.4. For each index i, β_i intersects γ_i, δ_i and no other curve. Moreover, for $i > 0$, the boundary of every neighbourhood of the area spanned by β_i under the isotopy lies in a region that touches the base disc D, so $\mathcal{D}$ can have positive multiplicities in this area only. In particular, $\mathcal{D} = \sum \mathcal{D}_i$, with $\mathcal{D}_i$ supported in the spanning region for all $i > 0$, and $\mathcal{D}_0$ supported near β_0.

Let us consider what happens in the region containing β_0, γ_0 and δ_0. Here all pairwise intersections are fixed and are x_0, y_0 and z_0. The base disc D lies on β_0, so one of the two arcs into which z_0 and y_0 divide β_0 cannot be part of $\partial\mathcal{D}_0$. In particular, $\partial\mathcal{D}_0 \cap \beta_0$ has to coincide with $\partial\bar{\mathcal{D}} \cap \beta_0$. Also, the big region (below this arc in the figure) touches the basepoint, so the multiplicity here has to be 0 and the multiplicity above it has to be 1 (we are crossing an arc in $\partial\mathcal{D}_0$), therefore $\mathcal{D}_0$ coincides with $\bar{\mathcal{D}}$ near β_0.

The situation around β_i is exactly the same as in Claim 7.4 and the same argument applies verbatim, showing that $\mathcal{D} = \bar{\mathcal{D}}$. □

In particular, we have that the only summand in the triangle count is $\#\mathcal{M}(\bar{\mathcal{D}}) \cdot \boldsymbol{\Theta}_{\beta\delta}$, concluding the proof of the lemma. □

Let us now get back to the proposition. We have

$$F_{\alpha\delta\beta}(\cdot \otimes \boldsymbol{\Theta}_{\delta\beta}) = F_{\alpha\delta\beta}(\cdot \otimes F_{\delta\gamma\beta}(\boldsymbol{\Theta}_{\gamma\delta} \otimes \boldsymbol{\Theta}_{\beta\gamma})) = F_{\alpha\delta\gamma}(F_{\alpha\gamma\delta}(\cdot \otimes \boldsymbol{\Theta}_{\gamma\delta}) \otimes \boldsymbol{\Theta}_{\gamma\delta}),$$

which is exactly what we wanted to prove. □

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Department of Mathematics, University of Pisa
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

`golla.marco@renyi.mta.hu`

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