Geometry & Topology 19 (2015) 365–411

Quasimorphisms on contactomorphism groups and contact rigidity

MATTHEW STROM BORMAN
FROL ZAPOLSKY

We build homogeneous quasimorphisms on the universal cover of the contactomorphism group for certain prequantizations of monotone symplectic toric manifolds. This is done using Givental's nonlinear Maslov index and a contact reduction technique for quasimorphisms. We show how these quasimorphisms lead to a hierarchy of rigid subsets of contact manifolds. We also show that the nonlinear Maslov index has a vanishing property, which plays a key role in our proofs. Finally we present applications to orderability of contact manifolds and Sandon-type metrics on contactomorphism groups.

53D35; 53D12, 53D20

Dedicated with gratitude to our teacher Leonid Polterovich on his 50^{th} birthday.

1 Introduction and results

1.1 Quasimorphisms on contactomorphism groups

A *quasimorphism* on a group G is a function $\mu\colon G \to \mathbb{R}$ which is a homomorphism up to a bounded error, that is, there is $D > 0$ such that

$$|\mu(ab) - \mu(a) - \mu(b)| \le D \quad \text{for all } a, b \in G, \tag{1-1}$$

and it is *homogeneous* if $\mu(a^k) = k\mu(a)$ for all $a \in G$ and $k \in \mathbb{Z}$. It is straightforward to show that homogeneous quasimorphisms are conjugation-invariant and restrict to homomorphisms on abelian subgroups. See Bavard [11], Calegari [25] and Kotschick [60] for background on quasimorphisms, their connection with bounded cohomology, and their applications to commutator length and other quantitative group-theoretic questions. For the sake of exposition, in this paper by quasimorphism we will mean a nonzero, homogeneous quasimorphism.

The construction and applications of quasimorphisms on infinite-dimensional groups of symmetries have recently been a popular theme of research; see Entov [37], Fukaya,

Published: 27 February 2015 DOI: 10.2140/gt.2015.19.365

Oh, Ohta and Ono [49], Gambaudo and Ghys [50], Ghys [52; 53], Polterovich [76], Shelukhin [83] and Usher [85]. One reason is that groups of diffeomorphisms are often perfect (see Banyaga [8]), thus admit no nonzero homomorphisms to $\mathbb{R}$ and so one is led to study quasimorphisms on them instead. When the group has an interesting metric such as the hydrodynamic metric on the group of volume-preserving diffeomorphisms of a Riemannian manifold (see Brandenbursky [22] and Brandenbursky and Shelukhin [23]) or Hofer's metric on the Hamiltonian group of a symplectic manifold (see Entov and Polterovich [38] and Py [77]), quasimorphisms can be used to understand the coarse geometry of these groups. Another reason is that oftentimes quasimorphisms on the symmetry groups of symplectic and contact manifolds lead to results on the geometry of the underlying manifolds themselves, which is also the case in the present paper.

We will only consider contact manifolds (V,ξ) where V is connected and closed, unless stated otherwise, and ξ is a cooriented contact structure. We will write (V,ξ,α) if we want to specify a choice of a coorienting contact form α such that $\xi=\ker\alpha$. The Reeb vector field associated to a contact form α will be denoted R_α and is uniquely defined by

$$\alpha(R_\alpha)=1 \quad \text{and} \quad \iota_{R_\alpha}d\alpha=0.$$

Let $\mathrm{Cont}_0(V,\xi)$ be the identity component of the group of contactomorphisms and denote by $\widetilde{\mathrm{Cont}}_0(V,\xi)$ its universal cover.

Given a smooth time-dependent function $h\colon [0,1]\times V\to\mathbb{R}$, called a *contact Hamiltonian*, there is a unique time-dependent vector field $\{X_{h_t}\}_{t\in[0,1]}$ satisfying

$$\text{(1-2)} \quad \alpha(X_{h_t})=h_t \quad \text{and} \quad d\alpha(X_{h_t},\cdot)=-dh_t+dh_t(R_\alpha)\alpha, \quad \text{where } h_t=h(t,\cdot).$$

The vector field $\{X_{h_t}\}$ preserves ξ and integrates into a contact isotopy based at the identity and denoted $\{\phi_h^t\}_{t\in[0,1]}$. This establishes a bijection, depending on the contact form α, between smooth functions $h\colon[0,1]\times V\to\mathbb{R}$ and contact isotopies based at the identity id of V. If $h,g\in C^\infty(V)$, then

$$\text{(1-3)} \qquad \{h,g\}_\alpha := -dg(X_h)+dh(R_\alpha)g$$

is the contact Hamiltonian corresponding to the Lie bracket of X_h and X_g. We write $\widetilde{\phi}_h$ for the element of $\widetilde{\mathrm{Cont}}_0(V,\xi)$ represented by the contact isotopy $\{\phi_h^t\}_{t\in[0,1]}$. For the constant function $h=1$, the vector field $X_1=R_\alpha$ is the Reeb vector field and hence $\widetilde{\phi}_1$ is the element generated by the Reeb flow.

Following Eliashberg and Polterovich [36] for $\widetilde{\phi}\in\widetilde{\mathrm{Cont}}_0(V,\xi)$ we will write $\mathrm{id}\preceq\widetilde{\phi}$ if there is a nonnegative contact Hamiltonian h such that $\widetilde{\phi}=\widetilde{\phi}_h$ in $\widetilde{\mathrm{Cont}}_0(V,\xi)$. The nonnegativity of h is equivalent to X_{h_t} being nowhere negatively transverse to ξ, and therefore it is independent of α. This induces a reflexive and transitive relation on $\widetilde{\mathrm{Cont}}_0(V,\xi)$, where

$$\tilde{\phi} \preceq \tilde{\psi} \quad \text{if and only if} \quad \mathrm{id} \preceq \tilde{\phi}^{-1}\tilde{\psi}, \tag{1-4}$$

which is also bi-invariant [36]. The contact manifold (V,ξ) is called *orderable* if $\preceq$ is a partial order on $\widetilde{\mathrm{Cont}}_0(V,\xi)$, that is, $\preceq$ is also antisymmetric.

Definition 1.1 For a quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ we define the following properties:

(i) *Monotone* $\tilde{\phi} \preceq \tilde{\psi}$ implies $\mu(\tilde{\phi}) \le \mu(\tilde{\psi})$.

(ii) *C^0–continuous* If h is a smooth contact Hamiltonian and there is a sequence of smooth contact Hamiltonians $h^{(n)}$ such that $h^{(n)} \to h$ in $C^0([0,1]\times V)$, then $\mu(\tilde{\phi}_{h^{(n)}}) \to \mu(\tilde{\phi}_h)$.

(iii) *Vanishing* If $U \subset V$ is an open subset and there is $\psi \in \mathrm{Cont}_0(V,\xi)$ with $\psi(U)\cap U = \varnothing$, then $\mu(\tilde{\phi}_h) = 0$ if $\mathrm{supp}(h) \subset [0,1]\times U$.

A subset $S \subset V$ is *displaceable* if there is $\psi \in \mathrm{Cont}_0(V,\xi)$ with $\psi(S)\cap \overline{S} = \varnothing$. Note that the vanishing property is independent of the choice of a contact form.

1.1.1 Givental's asymptotic nonlinear Maslov index Besides Poincaré's rotation number on $\widetilde{\mathrm{Cont}}_0(S^1) \equiv \widetilde{\mathrm{Diff}}_0(S^1)$, the only previous construction of quasimorphisms on contactomorphism groups was Givental's *asymptotic nonlinear Maslov index* (see Givental [54, Section 9]):

$$\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}) \to \mathbb{R}, \tag{1-5}$$

with $\mathbb{R}\mathrm{P}^{2d-1}$ being taken with the standard contact structure. Results in [54, Section 9] imply μ_{Giv} is a homogeneous quasimorphism, as Ben Simon [12, Theorem 0.2] proved. In Section 4.3 we will review the definition and relevant properties of Givental's quasimorphism and prove the following proposition.

Proposition 1.2 *Givental's quasimorphism $\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}) \to \mathbb{R}$ is monotone, C^0–continuous and has the vanishing property.*

For time-independent contact Hamiltonians, Givental proved [54, Corollary 3, Section 9] that μ_{Giv} is monotone and C^0–continuous, and as we will explain, his proofs work in general. The vanishing property, which does not appear in [54], together with Theorem 1.19 below, give an alternative proof of Ben Simon's Theorem 0.6 in [12].

1.1.2 Quasimorphisms for prequantizations of even toric manifolds A *prequantization* of a symplectic manifold (M,ω) is a contact manifold (V,ξ,α) with a map $\pi\colon (V,\alpha)\to(M,\omega)$ defining a principal S^1–bundle such that $\pi^*\omega = d\alpha$, and the Reeb vector field R_α induces the free S^1–action on V, where $S^1=\mathbb{R}/\hbar\mathbb{Z}$, $\hbar>0$ being the minimal period of a closed Reeb orbit.

A *toric* symplectic manifold $(M^{2n},\omega,\mathbb{T})$ is a symplectic manifold endowed with an effective Hamiltonian action of a torus $\mathbb{T}$ of dimension n. The action is induced by a moment map $M\to\mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$, and the image of the moment map is called the moment polytope and denoted Δ. If Δ has d facets, then it is given by

$$\Delta=\{x\in\mathfrak{t}^*\mid \langle\nu_j,x\rangle+a_j\geq 0 \text{ for } j=1,\dots,d\}, \tag{1-6}$$

where $a_j\in\mathbb{R}$ and the conormals $\nu_j\in\mathfrak{t}$ are primitive vectors in the integer lattice $\mathfrak{t}_\mathbb{Z}:=\ker(\exp\colon \mathfrak{t}\to\mathbb{T})$.

A symplectic manifold (M,ω) is *monotone* if and only if there is a positive constant $\lambda>0$ so that $[\omega]=\lambda c_1(M)\in H^2(M;\mathbb{R})$, and for toric manifolds this is equivalent to being able to choose the moment map so that $a_1=\cdots=a_d=\lambda$. We call the moment polytope Δ *even* if $\sum_{j=1}^d \nu_j\in 2\mathfrak{t}_\mathbb{Z}$ and we say that a toric manifold is even if its associated moment polytope is even. In Section 1.6 we give examples of closed monotone even symplectic toric manifolds. We can now formulate our main result.

Theorem 1.3 *Every closed monotone even toric symplectic manifold $(M,\omega,\mathbb{T})$ has a prequantization $(\widehat{M},\xi,\alpha)$ for which there is a quasimorphism*

$$\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M},\xi)\to\mathbb{R}$$

that is monotone, has the vanishing property and is C^0–continuous.

In Section 1.3 below we discuss the significance of this theorem in the context of stable Calabi quasimorphisms on the universal cover of the Hamiltonian group of a symplectic manifold.

Theorem 1.8 below shows how a monotone quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$ can induce a monotone quasimorphism on $\widetilde{\mathrm{Cont}}_0(\overline{V})$ if $(\overline{V},\overline{\xi})$ is the result of performing contact reduction on (V,ξ). In Section 2.1 we will show how the even moment polytope of a monotone toric manifold $(M,\omega,\mathbb{T})$ naturally leads to a prequantization $(\widehat{M},\xi,\alpha)$ obtained from $\mathbb{R}\mathrm{P}^{2d-1}$ via contact reduction. The proof of Theorem 1.3 is then given in Section 2.2 where we apply Theorem 1.8 to Givental's quasimorphism μ_{Giv} on $\widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1})$ to build the monotone quasimorphisms $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M},\xi)\to\mathbb{R}$.

Not all prequantizations $\pi\colon (V,\alpha) \to (M,\omega)$ of a monotone even toric symplectic manifold (M,ω) admit nontrivial monotone quasimorphisms on $\widetilde{\mathrm{Cont}}_0(V)$. This is because if V is not orderable, then there is no monotone quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$ (see Theorem 1.28 below). The basic example is the standard contact sphere S^{2d-1} for $d \geq 2$, which is a prequantization of the even toric manifold $\mathbb{C}\mathrm{P}^{d-1}$ but is not orderable. See Section 1.5.1 for further discussion about orderability and quasimorphisms.

Remark 1.4 If $\pi\colon (V,\alpha) \to (M,\omega)$ is a prequantization, then for any cyclic subgroup $\mathbb{Z}_k \leq S^1$ the quotient manifold $V/\mathbb{Z}_k$ is also a prequantization of M. Pulling back contact Hamiltonians via the projection $V \to V/\mathbb{Z}_k$ induces a homomorphism $\widetilde{\mathrm{Cont}}_0(V/\mathbb{Z}_k) \to \widetilde{\mathrm{Cont}}_0(V)$ and therefore the quasimorphisms of Theorem 1.3 give rise to quasimorphisms on $\widetilde{\mathrm{Cont}}_0(\widehat{M}/\mathbb{Z}_k)$.

Remark 1.5 There is work in progress by Karshon, Pabiniak and Sandon to generalize Givental's construction of the asymptotic nonlinear Maslov index, with lens spaces being the first step. If for a prime p there is a monotone quasimorphism with the vanishing property

$$\mu_{\mathrm{KPS}}\colon \widetilde{\mathrm{Cont}}_0(S^{2d-1}/\mathbb{Z}_p) \to \mathbb{R},$$

where $\mathbb{Z}_p$ acts by multiplication by a p^{th} root of unity, then Theorem 1.3 would generalize to the closed monotone toric symplectic manifolds $(M,\omega,\mathbb{T})$ whose sum of conormals in the moment polytope satisfies $\sum_{j=1}^{d} \nu_j \in p \cdot \mathfrak{t}_{\mathbb{Z}}$ (see the proof of Lemma 2.1).

1.1.3 Reduction for quasimorphisms on contactomorphism groups In [20; 21] a procedure for pushing forward quasimorphisms on the universal cover of the Hamiltonian group of a symplectic manifold via symplectic reduction was developed by Borman. In this paper we will streamline this technique and adapt it to the contact setting in Theorem 1.8, which will be used to prove Theorem 1.3. Before we can formulate the reduction theorem for quasimorphisms, we need the following two definitions.

Definition 1.6 For a contact manifold (V,ξ,α), a closed submanifold $Y \subset V$ transverse to ξ is *strictly coisotropic with respect to* α if it is coisotropic, that is, the subbundle $TY \cap \xi$ of the symplectic vector bundle $(\xi|_Y, d\alpha)$ is coisotropic,

$$\{X \in \xi_y \mid \iota_X d\alpha = 0 \text{ on } T_yY \cap \xi_y\} \subset T_yY \cap \xi_y \quad \text{for all } y \in Y, \tag{1-7}$$

and $R_\alpha(y) \in T_yY$ for all $y \in Y$, that is, the Reeb vector field is tangent to Y.

The property of being coisotropic is independent of the contact form and, assuming transversality, being strictly coisotropic with respect to α is equivalent to

(1-8) $$T_yY^{d\alpha} := \{X \in T_yV \mid \iota_X d\alpha = 0 \text{ on } T_yY\} \subset T_yY \quad \text{for all } y \in Y.$$

One can check that $Y \subset (V,\xi)$ is strictly coisotropic with respect to some contact form if and only if Y is the diffeomorphic image of a coisotropic submanifold under the projection $SV \to V$ where SV is the symplectization of V.

Definition 1.7 Let $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ be a monotone quasimorphism. A closed subset[1] $Y \subset V$ is *μ–subheavy* if

$$\mu(\widetilde{\phi}_h) = 0$$

whenever h is an autonomous contact Hamiltonian with $h|_Y = 0$.

Here now is the reduction theorem for quasimorphisms on contactomorphism groups, which we will prove in Section 3. Consider the setting

(1-9) $$(V,\xi,\alpha) \supset (Y,\alpha|_Y) \xrightarrow{\rho} (\overline{V},\overline{\xi},\overline{\alpha}),$$

where (V,ξ,α) and $(\overline{V},\overline{\xi},\overline{\alpha})$ are closed contact manifolds, $Y \subset V$ is a closed submanifold that is strictly coisotropic with respect to α, and $\rho\colon Y \to \overline{V}$ is a fiber bundle such that $\rho^*\overline{\alpha} = \alpha|_Y$.

Theorem 1.8 *In the setting* (1-9) *if $Y \subset V$ is subheavy for a monotone quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$, then it induces a monotone quasimorphism*

(1-10) $$\overline{\mu}\colon \widetilde{\mathrm{Cont}}_0(\overline{V},\overline{\xi}) \to \mathbb{R} \quad \textit{defined by} \quad \overline{\mu}(\widetilde{\phi}_{\overline{h}}) := \mu(\widetilde{\phi}_h),$$

where $h \in C^\infty([0,1]\times V)$ is any contact Hamiltonian such that $h|_{[0,1]\times Y} = \rho^*\overline{h}$. The vanishing property and C^0–continuity passes from μ to $\overline{\mu}$.

An example of the geometric setting seen in (1-9) arises in contact reduction (see Geiges [51, Theorem 6]) where a compact Lie group G acts on V preserving α with moment map $P\colon V \to \mathfrak{g}^*$. In this case $Y = P^{-1}(0)$ is strictly coisotropic with respect to α and $\overline{V} = Y/G$ is a contact manifold assuming G acts freely on Y. When we prove Theorem 1.3 in Section 2.2 it will be in the case of contact reduction for torus actions on $\mathbb{R}\mathrm{P}^{2d-1}$.

It should be noted that, considering more general group actions on $\mathbb{R}\mathrm{P}^{2d-1}$, it is possible to construct monotone quasimorphisms with the vanishing property on prequantizations

[1] See Remark 1.16 regarding the closed assumption, which also applies to the definition of superheavy sets below.

of symplectic manifolds more general than toric ones, however we shall not pursue this direction here.

1.2 Contact rigidity

Nondisplaceability phenomena in contact manifolds is one aspect of contact rigidity and it is much less studied than nondisplaceability in symplectic manifolds by Hamiltonian diffeomorphisms; see Abreu, Borman and McDuff [2], Abreu and Macarini [4], Biran, Entov and Polterovich [18], Cho [28], Entov and Polterovich [39; 42], Fukaya, Oh, Ohta and Ono [47], McDuff [67], Wilson and Woodward [87] and Woodward [88]. As with the symplectic setting, contact nondisplaceability goes back to a conjecture of Arnold that for the standard contact structure on the jet space $J^1N = T^*N \times \mathbb{R}$ of a closed manifold N, the zero section $\{(q,0,0) \mid q \in N\}$ cannot be displaced from the zero wall $\{(q,0,z) \mid q \in N, z \in \mathbb{R})\}$ by a contact isotopy and this was proved by Chekanov [26] using generating functions. Using spectral invariants from generating functions Zapolsky [89] proved contact rigidity for smooth and singular subsets of the standard contact $T^*N \times S^1$. Floer-theoretic methods have also been used by Eliashberg, Hofer and Salamon [34] and Ono [74] to detect nondisplaceable submanifolds in unit cotangent bundles of closed manifolds and in certain prequantizations. Recently sheaf-theoretic methods have also been playing a role in symplectic and contact rigidity; see for example Tamarkin [84] and Guillermou, Kashiwara and Schapira [55].

In the series of papers [18; 38; 39; 42] Entov and Polterovich showed how to use the machinery of their quasimorphisms on the universal cover of the Hamiltonian group of a symplectic manifold (M,ω) and *quasistates* in order to study the rigidity of symplectic intersections. In particular in [42] they showed that there is a hierarchy of rigid subsets in symplectic manifolds for which they introduced the terminology of *heavy* and *superheavy* subsets.

1.2.1 Superheavy and subheavy sets for monotone quasimorphisms on $\widetilde{\mathrm{Cont}}_0$

Inspired by Entov–Polterovich's work, in this paper we will show how monotone quasimorphisms on $\widetilde{\mathrm{Cont}}_0(V)$ can also be used to study the rigidity of intersections in contact manifolds. In analogy to the terms *heavy* and *superheavy* for subsets of symplectic manifolds, we will also show how such monotone quasimorphisms detect a hierarchy of rigid subsets in contact manifolds, namely *subheavy* (defined above) and *superheavy* sets:

Definition 1.9 If $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ is a monotone quasimorphism, then a closed subset $Y \subset V$ is *μ–superheavy* if

$$\mu(\widetilde{\phi}_h) > 0$$

for all autonomous contact Hamiltonians $h \in C^\infty(V)$ such that $h|_Y > 0$.

Given a prequantization $\pi\colon (V,\alpha) \to (M,\omega)$, in Section 1.3 we will discuss how superheavy subsets in the symplectic manifold (M,ω) are related to subheavy and superheavy subsets of the contact manifold (V,α). The basic properties of superheavy sets in contact manifolds are given by the following proposition.

Proposition 1.10 *Let $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ be a monotone quasimorphism.*

(i) *The properties μ–superheavy and μ–subheavy are independent of the choice of contact form α for ξ used to link contact Hamiltonians and contact isotopies.*

(ii) *If Z is μ–superheavy and $Z \subset Y$, then Y is μ–superheavy and likewise for μ–subheavy.*

(iii) *The property of being μ–subheavy is preserved by elements of* $\mathrm{Cont}_0(V,\xi)$, *and likewise for μ–superheavy.*

(iv) *The entire manifold V is μ–superheavy.*

The next theorem and its corollary relates subheavy and superheavy sets with contact rigidity.

Theorem 1.11 *Let $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ be a monotone quasimorphism.*

(i) *All μ–superheavy subsets are μ–subheavy.*

(ii) *If Y is μ–superheavy and Z is μ–subheavy, then $Y \cap Z \neq \varnothing$.*

As an immediate corollary of Proposition 1.10(iii) and Theorem 1.11 we have:

Corollary 1.12 *If $Y \subset V$ is μ–subheavy and $Z \subset V$ is μ–superheavy for a monotone quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$, then the following hold:*

(i) *Y cannot be displaced from Z, that is, $\psi(Y) \cap Z \neq \varnothing$ for all $\psi \in \mathrm{Cont}_0(Y)$.*

(ii) *Z is nondisplaceable, that is, $\psi(Z) \cap Z \neq \varnothing$ for all $\psi \in \mathrm{Cont}_0(Y)$.* □

See Section 4.1 for the proofs of Proposition 1.10 and Theorem 1.11, which together with Corollary 1.12 are analogous to the basic properties of heavy and superheavy subsets of a symplectic manifold [42, Section 1.4]. We also have the following criterion for when a μ–subheavy set is automatically μ–superheavy, which we prove in Section 4.1.

Proposition 1.13 *Let $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ be a monotone quasimorphism and let $Y \subset V$ be a μ–subheavy subset. If Y is preserved by the flow of some positive contact vector field, then Y is μ–superheavy.*

In the context of Theorem 1.8, note that Proposition 1.13 implies that the μ–subheavy subset $Y \subset V$, which is strictly coisotropic, is actually μ–superheavy.

As the next theorem shows, the properties of being subheavy and superheavy are respected by the reduction of the quasimorphisms in Theorem 1.8. Recall in Theorem 1.8 one has contact manifolds (V, ξ, α) and $(\overline{V}, \overline{\xi}, \overline{\alpha})$ and a closed submanifold $Y \subset V$ with a fiber bundle $\rho\colon Y \to \overline{V}$. There are monotone quasimorphisms

$$\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R} \quad \text{and} \quad \overline{\mu}\colon \widetilde{\mathrm{Cont}}_0(\overline{V},\overline{\xi}) \to \mathbb{R} \tag{1-11}$$

where by definition $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) := \mu(\widetilde{\phi}_h)$ for any contact Hamiltonian $h \in C^\infty([0,1] \times V)$ that satisfies $\rho^*\overline{h} = h|_{[0,1]\times Y}$.

Theorem 1.14 *For monotone, C^0–continuous quasimorphisms from* (1-11) *as in Theorem 1.8, if $Z \subset V$ is μ–subheavy, then $\rho(Y \cap Z) \subset \overline{V}$ is $\overline{\mu}$–subheavy and likewise for superheavy sets.*

Proof First note that since Y is μ–superheavy by Proposition 1.13, there is a nontrivial intersection $Y \cap Z \neq \varnothing$ by Theorem 1.11 if Z is μ–subheavy.

Assume $Z \subset V$ is μ–superheavy and let $\overline{h} \in C^\infty(\overline{V})$ be such that $\overline{h}|_{\rho(Y\cap Z)} > 0$. For $\epsilon > 0$ sufficiently small, let $\overline{f} \in C^\infty(\overline{V})$ be such that $\overline{f} = \epsilon$ in a neighborhood of $\rho(Y \cap Z)$ and $\overline{h} \geq \overline{f}$. Now we can pick an extension $f \in C^\infty(V)$ so that $f|_Z = \epsilon$ and $\rho^*\overline{f} = f|_Y$. Since Z is μ–superheavy it follows that $\overline{\mu}(\widetilde{\phi}_{\overline{f}}) = \mu(\widetilde{\phi}_f) > 0$, and hence by monotonicity $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) > 0$. Therefore $\rho(Y \cap Z) \subset \overline{V}$ is $\overline{\mu}$–superheavy.

Assume $Z \subset V$ is μ–subheavy and let $\overline{h} \in C^\infty(\overline{V})$ be such that $\overline{h}|_{\rho(Y\cap Z)} = 0$. Pick a sequence $\overline{f_n} \in C^\infty(\overline{V})$ such that there is C^0–convergence $\overline{f_n} \to \overline{h}$ and there are neighborhoods $\mathcal{N}_n$ of $\rho(Y \cap Z)$ such that $\overline{f_n}|_{\mathcal{N}_n} = 0$. Now pick extensions $f_n \in C^\infty(V)$ so that $f_n|_Z = 0$ and $\rho^*\overline{f_n} = f_n|_Y$. Since Z is μ–subheavy we know

$$\overline{\mu}(\widetilde{\phi}_{\overline{f}_n}) := \mu(\widetilde{\phi}_{f_n}) = 0$$

and since $\overline{\mu}$ is C^0–continuous, it follows that $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) = 0$. Therefore $\rho(Y \cap Z) \subset \overline{V}$ is $\overline{\mu}$–subheavy. □

Remark 1.15 The C^0–continuity assumption was not used to prove that superheaviness descends under reduction. Also the descent for subheaviness holds without the C^0–continuity assumption if Z intersects Y sufficiently nicely, for instance if there is a small tubular neighborhood $\mathrm{pr}\colon U \to Y$ of Y such that $\mathrm{pr}|_{U\cap Z}\colon U \cap Z \to Y \cap Z$ is a fiber bundle. However in general it is not possible to find a smooth extension h of $\rho^*\overline{h}$ with $h|_Z = 0$, which we get around by using the C^0–continuity assumption.

Remark 1.16 We only consider closed subsets in the hierarchy of subheavy and superheavy subsets, and for instance we use this assumption in our proofs of Theorem 1.11 and Proposition 1.13. Of course it is possible to extend the definitions and the theorems to arbitrary subsets via closure, but we have suppressed this for the sake of exposition.

1.2.2 Rigid Legendrians and pre-Lagrangians As demonstrated by previous work in contact rigidity by Eliashberg [33] and [34; 74; 36] two important classes of submanifolds in contact manifolds are Legendrians and pre-Lagrangians. Recall (see [34, Section 2.2]) that a *pre-Lagrangian* submanifold $Y^{n+1} \subset (V^{2n+1}, \xi)$ is one such that Y is transverse to ξ and there is a contact form α such that $d\alpha|_Y = 0$, that is, Y is a strictly coisotropic submanifold of minimal dimension. An equivalent definition from [34, Proposition 2.2.2] is that Y is the diffeomorphic image of a Lagrangian under the projection $SV \to V$, where SV is the symplectization of V. A nice class of examples is as follows: for a prequantization $\pi\colon (V, \alpha) \to (M, \omega)$ and a Lagrangian $L \subset M$, the submanifold $\pi^{-1}(L) \subset V$ is pre-Lagrangian. Note that Proposition 1.13 implies every closed subheavy pre-Lagrangian submanifold is superheavy.

As we will see from our examples of subheavy and superheavy subsets of contact manifolds in Section 1.4, prototypically a subheavy submanifold is a Legendrian and a superheavy submanifold is a pre-Lagrangian. In particular in Corollary 1.26 we explicitly identify a μ–subheavy Legendrian submanifold and a μ–superheavy pre-Lagrangian torus for each of the quasimorphisms in Theorem 1.3. For the case of Givental's quasimorphism on $\mathbb{R}\mathrm{P}^{2d-1}$, a μ_{Giv}–subheavy Legendrian is

$$\mathbb{R}\mathrm{P}^{d-1}_L := \{[z] \in \mathbb{R}\mathrm{P}^{2d-1} \mid z \in \mathbb{R}^d\}$$

and a μ_{Giv}–superheavy pre-Lagrangian torus is

$$T_{\mathbb{R}\mathrm{P}} := \{[z] \in \mathbb{R}\mathrm{P}^{2d-1} \mid |z_1|^2 = \cdots = |z_d|^2 = 1/\pi\},$$

where we are viewing $\mathbb{R}\mathrm{P}^{2d-1}$ as the quotient of the sphere $S^{2d-1} \subset \mathbb{C}^d$ with radius $\sqrt{d/\pi}$. See Lemmas 1.23 and 1.22 for the proofs.

More generally we have the following existence theorem for nondisplaceable pre-Lagrangian tori, analogous to Entov and Polterovich's proof [39, Theorem 2.1] of the existence of nondisplaceable Lagrangians in closed toric symplectic manifolds.

Theorem 1.17 *If $\mu\colon \widetilde{\mathrm{Cont}}_0(V, \xi) \to \mathbb{R}$ is a monotone quasimorphism with the vanishing property and (V, α) is a prequantization of a closed toric manifold (M, ω), then V contains a nondisplaceable pre-Lagrangian torus.*

See Section 4.1 for the proof of Theorem 1.17.

1.3 Quasimorphisms on $\widetilde{\mathrm{Ham}}(M)$ and symplectic quasistates

For a closed symplectic manifold (M,ω), a smooth Hamiltonian $F\colon [0,1]\times M\to\mathbb{R}$ induces a time-dependent vector field $\{X_{F_t}\}_{t\in[0,1]}$ by

$$\iota_{X_{F_t}}\omega=-dF_t,\quad \text{where } F_t=F(t,\cdot). \tag{1-12}$$

Integrating X_{F_t} gives a Hamiltonian isotopy $\{\phi_F^t\}_{t\in[0,1]}$ of M based at id the identity of M and these are in bijection with smooth Hamiltonians $F\colon [0,1]\times M\to\mathbb{R}$ *normalized* so $\int_M F_t\omega^n=0$ for all $t\in[0,1]$. The Hamiltonian group $\mathrm{Ham}(M)$ is the set of time-one maps ϕ_F^1 of such Hamiltonian isotopies and $\widetilde{\mathrm{Ham}}(M)$ is its universal cover. We write $\widetilde{\phi}_H$ for the element of $\widetilde{\mathrm{Ham}}(M)$ represented by the Hamiltonian isotopy $\{\phi_H^t\}_{t\in[0,1]}$. For normalized functions $H,G\in C^\infty(M)$ their Poisson bracket

$$\{H,G\}_\omega:=\omega(X_G,X_H)=-dG(X_H) \tag{1-13}$$

is the Hamiltonian whose vector field is the Lie bracket of X_H and X_G. A subset $S\subset M$ is *displaceable* if there is $\phi\in\mathrm{Ham}(M)$ so that $\phi(S)\cap\overline{S}=\varnothing$.

For a quasimorphism $\mu_M\colon \widetilde{\mathrm{Ham}}(M)\to\mathbb{R}$ one defines the following two properties (see [38] and Entov, Polterovich and Zapolsky [45]):

(i) *Stable* For normalized Hamiltonians $H,G\colon [0,1]\times M\to\mathbb{R}$,

$$\int_0^1\min_M(H_t-G_t)\,dt\le\frac{\mu_M(\widetilde{\phi}_G)-\mu_M(\widetilde{\phi}_H)}{\mathrm{Vol}(M)}\le\int_0^1\max_M(H_t-G_t)\,dt, \tag{1-14}$$

where $\mathrm{Vol}(M)=\int_M\omega^n$.

(ii) *Calabi* If $U\subset M$ is an open displaceable subset and if $H\colon [0,1]\times M\to\mathbb{R}$ has support in $[0,1]\times U$, then

$$\mu_M(\widetilde{\phi}_H)=\mathrm{Cal}_U(\widetilde{\phi}_H):=\int_0^1\int_U H_t\omega^n\,dt.$$

Such quasimorphisms were constructed by Entov and Polterovich in [38] using spectral invariants in Hamiltonian Floer theory and their construction has been refined and extended in Entov and Polterovich [40], Lanzat [64; 62; 63], Monzner, Vichery and Zapolsky [71], Ostrover [75] and [49; 85].

On a closed symplectic manifold (M,ω) a *quasistate* is a functional $\zeta\colon C^\infty(M)\to\mathbb{R}$ satisfying the following properties for all $H,K\in C^\infty(M)$:

(i) *Monotone* If $H\le K$, then $\zeta(H)\le\zeta(K)$.

(ii) *Normalized* $\zeta(1)=1$.

(iii) *Quasilinearity* If $\{H,K\}_\omega=0$, then $\zeta(H+K)=\zeta(H)+\zeta(K)$.

These quasistates are the symplectic version of Aarnes' notion of topological quasistate [1]. As established in [39], every stable quasimorphism $\mu_M\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ induces a quasistate $\zeta_{\mu_M}\colon C^\infty(M) \to \mathbb{R}$ defined by

$$\zeta_{\mu_M}(H) := \frac{\int_M H\omega^n - \mu_M(\widetilde{\phi}_H)}{\mathrm{Vol}(M)}. \tag{1-15}$$

Such quasistates are $\mathrm{Ham}(M)$–invariant. If μ_M also has the Calabi property, then ζ_{μ_M} has the *vanishing property*, that is, $\zeta(H) = 0$ whenever $\mathrm{supp}(H) \subset M$ is displaceable.

Definition 1.18 Let $\zeta\colon C^\infty(M) \to \mathbb{R}$ be a quasistate on a closed symplectic manifold (M,ω). A closed subset $X \subset M$ is *ζ–superheavy* if

$$\min_X H \le \zeta(H) \le \max_X H \tag{1-16}$$

for all $H \in C^\infty(M)$.

This definition was introduced in [42] and ζ–superheavy sets $X \subset M$ are nondisplaceable when ζ is $\mathrm{Ham}(M)$–invariant, by [42, Theorem 1.4]. See [18; 45; 49], Buhovsky, Entov and Polterovich [24], Entov and Polterovich [39; 41; 42; 43], Entov, Polterovich and Py [44] and Khanevsky [58] for various applications of Entov–Polterovich's quasimorphisms and quasistates.

Recall that for a prequantization $\pi\colon (V,\alpha) \to (M,\omega)$ one has the following central extension of Lie algebras:

$$0 \to \mathbb{R} \to (C^\infty(V)^{S^1}, \{\cdot,\cdot\}_\alpha) \to (C^\infty(M)/\mathbb{R}, \{\cdot,\cdot\}_\omega) \to 0.$$

Here $C^\infty(V)^{S^1} \simeq C^\infty(M)$ is the set of S^1–invariant functions on V and $C^\infty(M)/\mathbb{R}$ is canonically the Lie algebra of $\mathrm{Ham}(M)$. When M is closed this sequence has a unique splitting by the Lie algebra homomorphism

$$\sigma\colon C^\infty(M)/\mathbb{R} \to C^\infty(V)^{S^1} \quad \text{given by} \quad H \mapsto \pi^* H - \frac{\int_M H\omega^n}{\mathrm{Vol}(M)}$$

and σ induces a homomorphism

$$\pi^*\colon \widetilde{\mathrm{Ham}}(M) \to \widetilde{\mathrm{Cont}}_0(V), \quad \text{where} \quad \pi^*(\widetilde{\phi}_H) = \widetilde{\phi}_{\sigma(H)}. \tag{1-17}$$

See Ben Simon [13, Section 1.3] for more details on this point and in particular a proof that (1-17) is a homomorphism.

We now have the following result, generalizing Ben Simon [12], which uses the homomorphism (1-17) to relate quasimorphisms on $\widetilde{\mathrm{Cont}}_0$ and $\widetilde{\mathrm{Ham}}$. Recall that we denote $\widetilde{\phi}_1 \in \widetilde{\mathrm{Cont}}_0(V)$ to be the element generated by the Reeb vector field R_α.

Theorem 1.19 *Let $\pi\colon (V,\alpha)\to(M,\omega)$ be a prequantization of a closed symplectic manifold and let $\mu\colon \widetilde{\mathrm{Cont}}_0(V)\to\mathbb{R}$ be a monotone quasimorphism, then*

$$\mu_M := -\frac{\mathrm{Vol}(M)}{\mu(\widetilde{\phi}_1)}(\mu\circ\pi^*)\colon \widetilde{\mathrm{Ham}}(M)\to\mathbb{R} \tag{1-18}$$

is a stable quasimorphism. The quasistate associated to μ_M from (1-15) *has the form*

$$\zeta_{\mu_M}(H) := \frac{\mu(\widetilde{\phi}_{\pi^*H})}{\mu(\widetilde{\phi}_1)}.$$

If μ has the vanishing property, then μ_M has the Calabi property and ζ_{μ_M} has the vanishing property.

A historical remark is in order. While Givental [54] applied his quasimorphism to various contact rigidity phenomena on $\mathbb{R}\mathrm{P}^{2d-1}$, such as the existence of Reeb chords, it was first in the symplectic setting that Entov and Polterovich developed a systematic approach to use their quasimorphisms in order to study symplectic rigidity. However, as Theorem 1.19 shows, for prequantizable symplectic manifolds, quasimorphisms on $\widetilde{\mathrm{Cont}}_0$ are potentially more fundamental objects than quasimorphisms on $\widetilde{\mathrm{Ham}}$. A related question is if it is possible to obtain one of Entov–Polterovich's quasimorphisms on $\widetilde{\mathrm{Ham}}(M)$ from a quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$ via Theorem 1.19, and this is open even for the case of the prequantization $\mathbb{R}\mathrm{P}^3\to\mathbb{C}\mathrm{P}^1$.

The following proposition shows how the Entov–Polterovich notion of superheaviness (1-16) with respect to a symplectic quasistate on (M,ω) is related to sub- and superheaviness with respect to a quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$ when $\pi\colon (V,\alpha)\to(M,\omega)$ is a prequantization.

Proposition 1.20 *If $\pi\colon (V,\alpha)\to(M,\omega)$ is a prequantization, $\mu\colon \widetilde{\mathrm{Cont}}_0(V)\to\mathbb{R}$ is a monotone quasimorphism and $\mu_M\colon \widetilde{\mathrm{Ham}}(M)\to\mathbb{R}$ is the quasimorphism induced according to Theorem 1.19, then we have the following:*

(i) *If $Y\subset V$ is μ–subheavy, then $\pi(Y)\subset M$ is ζ_{μ_M}–superheavy.*

(ii) *If $X\subset M$ is ζ_{μ_M}–superheavy, then $\pi^{-1}(X)\subset V$ is μ–superheavy.*

Theorem 1.19 and Proposition 1.20 are proved in Section 4.2.

Given a collection $(H_1,\dots,H_k)$ of pairwise Poisson commuting Hamiltonians on M, organized as a map $\Phi\colon M\to\mathbb{R}^k$, in [39] Entov and Polterovich defined a fiber $\Phi^{-1}(p)$ to be a *stem* if every other fiber $\Phi^{-1}(q)\subset M$ was displaceable. They proved in [42, Theorem 1.8] that a stem $X\subset M$ is superheavy with respect to any quasistate with the vanishing property. Using Theorem 1.19 and Proposition 1.20 we now have the following corollary for any prequantization $\pi\colon (\widehat{M},\alpha)\to(M,\omega)$ and monotone quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M})\to\mathbb{R}$ with the vanishing property:

Corollary 1.21 *If $X \subset (M, \omega)$ is a stem, $\pi^{-1}(X) \subset \widehat{M}$ is μ–superheavy.* $\square$

Stems can be very singular subsets, an example being the product of 1–skeletons of fine triangulations of 2–spheres [39, Corollary 2.5].

1.4 Examples of contact rigidity

In this subsection we will present concrete examples of subheavy and superheavy subsets of contact manifolds.

1.4.1 Examples using Givental's quasimorphism We will start with the rigidity results that just use Givental's monotone quasimorphism $\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}) \to \mathbb{R}$. For us it will be convenient to introduce the following models of the standard contact S^{2d-1} and $\mathbb{R}\mathrm{P}^{2d-1}$. For $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$, consider the sphere

$$S^{2d-1}_\gamma = \left\{ z \in \mathbb{C}^d \;\middle|\; \pi \sum_{j=1}^{d} \gamma_j |z_j|^2 = \sum_{j=1}^{d} \gamma_j \right\} \tag{1-19}$$

with the contact form given by the restriction of

$$\alpha_{\mathrm{std}} = \frac{1}{2} \sum_{j=1}^{d} (x_j \, dy_j - y_j \, dx_j) \tag{1-20}$$

to S^{2d-1}_γ with Reeb flow

$$\phi^t_{R_\gamma}(z_1, \dots, z_d) = (e^{2\pi i \gamma_1 t/d} z_1, \dots, e^{2\pi i \gamma_d t/d} z_d). \tag{1-21}$$

For the antipodal $\mathbb{Z}_2$–action on $\mathbb{C}^d$, let

$$(\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma) := (S^{2d-1}_\gamma / \mathbb{Z}_2, \ker \alpha_{\mathrm{std}}). \tag{1-22}$$

Note when $\gamma = (1, \dots, 1)$ that $(\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma)$ is the standard model for $(\mathbb{R}\mathrm{P}^{2d-1}, \xi)$, so we will drop the reference to γ in this case. Via radial projection $z \mapsto (\sqrt{d}/\sqrt{\pi}) \frac{z}{|z|}$, which induces a contactomorphism

$$r\colon (\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma) \to (\mathbb{R}\mathrm{P}^{2d-1}, \xi), \tag{1-23}$$

we have Givental's quasimorphism $\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}_\gamma) \to \mathbb{R}$ for any $\gamma \in \mathbb{N}^d$.

Lemma 1.22 *The torus*

$$T_{\mathbb{R}\mathrm{P}} := \{[z] \in \mathbb{R}\mathrm{P}^{2d-1}_\gamma \mid |z_1|^2 = \cdots = |z_d|^2 = 1/\pi\} \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma \tag{1-24}$$

is μ_{Giv}–superheavy.

Proof of Lemma 1.22 Since the radial projection (1-23) preserves $T_{\mathbb{R}\mathrm{P}}$, it suffices to show $T_{\mathbb{R}\mathrm{P}} \subset \mathbb{R}\mathrm{P}^{2d-1}$ is μ_{Giv}–superheavy. For

$$\mathbb{C}\mathrm{P}^{d-1} = \left\{ [z_1 : \cdots : z_d] \;\middle|\; \pi \sum |z_j|^2 = d \right\},$$

consider the prequantization $\pi \colon \mathbb{R}\mathrm{P}^{2d-1} \to \mathbb{C}\mathrm{P}^{d-1}$. Using the Hamiltonian $U(d)$–action on $\mathbb{C}\mathrm{P}^{d-1}$, the Clifford torus $\mathbb{T}^{d-1}_{\mathrm{Clif}} := \pi(T_{\mathbb{R}\mathrm{P}})$ can be shown to be a stem; see [18, Lemma 5.1]. Since μ_{Giv} has the vanishing property by Proposition 1.2, it follows from Corollary 1.21 that $T_{\mathbb{R}\mathrm{P}}$ is μ_{Giv}–superheavy. □

Lemma 1.22 will play a large role in our proof of Theorem 1.3 for it will ensure we are applying Theorem 1.8 to a μ_{Giv}–superheavy subset.

While by Theorem 1.11 it is impossible for a Legendrian submanifold to be superheavy, since they are always displaceable (for instance by an arbitrarily small positive contact isotopy), it is possible for a Legendrian to be subheavy as the next example shows. The proof is given in Section 4.3.

Lemma 1.23 *The standard Legendrian*

$$\mathbb{R}\mathrm{P}^{d-1}_L := \{[z] \in \mathbb{R}\mathrm{P}^{2d-1}_\gamma \mid z \in \mathbb{R}^d\} \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma \tag{1-25}$$

is μ_{Giv}–subheavy.

Once we take the orbit of $\mathbb{R}\mathrm{P}^{d-1}_L$ under the Reeb flow, which is a closed subset since the Reeb flow is periodic, we get the following immediate corollary of Lemma 1.23 and Proposition 1.13.

Corollary 1.24 *The subset*

$$L_\gamma := \bigcup_{t\in\mathbb{R}} \phi^t_{R_\gamma}(\mathbb{R}\mathrm{P}^{d-1}_L) \subset (\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma) \tag{1-26}$$

is μ_{Giv}–superheavy. □

Corollary 1.24 can be used to prove rigidity in weighted complex projective spaces. Recall for a primitive vector $\gamma \in \mathbb{N}^d$ that the weighted complex projective space $\mathbb{C}\mathrm{P}(\gamma)$ is the symplectic orbifold obtained as the quotient of S^{2d-1}_γ by the Reeb flow (1-21). A Hamiltonian isotopy of $\mathbb{C}\mathrm{P}(\gamma)$ is by definition an isotopy that lifts to a contact isotopy of S^{2d-1}_γ preserving the contact form. If the fixed point set of the involution on $\mathbb{C}\mathrm{P}(\gamma)$ induced by complex conjugation on $\mathbb{C}^d$ is

$$\mathbb{R}\mathrm{P}(\gamma) \subset \mathbb{C}\mathrm{P}(\gamma)$$

and we have

$$T_{\mathbb{C}\mathrm{P}} := \{[z] \in \mathbb{C}\mathrm{P}(\gamma) \mid |z_1|^2 = \cdots = |z_d|^2 = 1/\pi\} \subset \mathbb{C}\mathrm{P}(\gamma),$$

then we have the following proposition.

Proposition 1.25 *If for a primitive vector $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$ each γ_j is odd, then*

$$\mathbb{R}\mathrm{P}(\gamma) \cap \psi(\mathbb{R}\mathrm{P}(\gamma)) \neq \varnothing, \quad \mathbb{R}\mathrm{P}(\gamma) \cap \psi(T_{\mathbb{C}\mathrm{P}}) \neq \varnothing, \quad T_{\mathbb{C}\mathrm{P}} \cap \psi(T_{\mathbb{C}\mathrm{P}}) \neq \varnothing,$$

for all Hamiltonian isotopies ψ of $\mathbb{C}\mathrm{P}(\gamma)$.

Proof The fact that all γ_j are odd is equivalent to the time $t = \frac{1}{2}$ Reeb flow (1-21) being the antipodal map on S_γ^{2d-1}. Therefore if each γ_j is odd, then the quotient map $S_\gamma^{2d-1} \to \mathbb{C}\mathrm{P}(\gamma)$ factors through the projection map

$$\pi\colon \mathbb{R}\mathrm{P}_\gamma^{2d-1} \to \mathbb{C}\mathrm{P}(\gamma)$$

and hence any Hamiltonian isotopy of $\mathbb{C}\mathrm{P}(\gamma)$ lifts to a contact isotopy of $\mathbb{R}\mathrm{P}_\gamma^{2d-1}$.

By the definitions, under the projection map $\pi(L_\gamma) \subset \mathbb{R}\mathrm{P}(\gamma)$ and $\pi(T_{\mathbb{R}\mathrm{P}}) = T_{\mathbb{C}\mathrm{P}}$. Since L_γ and $T_{\mathbb{R}\mathrm{P}}$ are μ_{Giv}–superheavy by Lemmas 1.24 and 1.22 it follows from Theorem 1.11 that both L_γ and $T_{\mathbb{R}\mathrm{P}}$ are nondisplaceable and cannot be displaced from each other by a contact isotopy. Therefore the same holds for $\mathbb{R}\mathrm{P}(\gamma)$ and $T_{\mathbb{C}\mathrm{P}}$ for Hamiltonian isotopies. □

Nondisplaceability of $T_{\mathbb{C}\mathrm{P}} \subset \mathbb{C}\mathrm{P}(\gamma)$ for any primitive $\gamma \in \mathbb{N}^d$ was proved by Woodward [88] and Cho and Poddar [30]. Nondisplaceability of $\mathbb{R}\mathrm{P}(\gamma)$ for an odd primitive vector γ was previously proved by Lu [66].

1.4.2 Examples using the quasimorphisms from Theorem 1.3 In the proof of Theorem 1.3 in Section 2 we will apply Theorem 1.8 to Givental's quasimorphism to build μ. In particular, for an appropriate primitive vector $\gamma \in \mathbb{N}^d$, in Section 2.2 we present the prequantization $(\widehat{M}, \alpha)$ in the setting (1-9) of Theorem 1.8:

$$(\mathbb{R}\mathrm{P}_\gamma^{2d-1}, \xi_\gamma, \alpha_{\mathrm{std}}) \supset (Y, \alpha_{\mathrm{std}}|_Y) \xrightarrow{\rho} (\widehat{M}, \xi, \alpha), \tag{1-27}$$

where Y is a μ_{Giv}–superheavy submanifold containing $T_{\mathbb{R}\mathrm{P}}$ and $\mu := \overline{\mu}_{\mathrm{Giv}}$ is the reduction of Givental's quasimorphism. For the torus $T_{\mathbb{R}\mathrm{P}}$ and standard Legendrian $\mathbb{R}\mathrm{P}_L^{d-1}$ in $\mathbb{R}\mathrm{P}_\gamma^{2d-1}$ from (1-24) and (1-25) define the following two subsets of $\widehat{M}$:

$$T_{\widehat{M}} := \rho(T_{\mathbb{R}\mathrm{P}}) \quad \text{and} \quad \widehat{M}_{\mathbb{R}} := \rho(Y \cap \mathbb{R}\mathrm{P}_L^{d-1}). \tag{1-28}$$

Note that $T_{\widehat{M}}$ is a pre-Lagrangian torus while $\widehat{M}_{\mathbb{R}}$ is Legendrian. We now have the following corollary.

Corollary 1.26 *For a quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M}) \to \mathbb{R}$ from Theorem 1.3, the pre-Lagrangian $T_{\widehat{M}}$ is μ–superheavy and the Legendrian $\widehat{M}_{\mathbb{R}}$ is μ–subheavy.*

Proof This follows directly from Theorem 1.14, and Lemmas 1.22, and 1.23. □

The next result concerns rigidity for the real part $M_{\mathbb{R}} \subset (M,\omega)$ of a symplectic toric manifold, which is characterized as the fixed point set of the antisymplectic involution that preserves the moment map. Using the prequantization $\pi\colon (\widehat{M},\alpha) \to (M,\omega)$ we construct in Section 2.1 for a monotone even toric manifold, the real part of M can be identified with

$$M_{\mathbb{R}} := \pi(\widehat{M}_{\mathbb{R}}),$$

where $\widehat{M}_{\mathbb{R}} \subset \widehat{M}$ is from (1-28). For the quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M}) \to \mathbb{R}$ from Theorem 1.3, let $\zeta_{\mu_M}\colon C^\infty(M) \to \mathbb{R}$ be the induced symplectic quasistate on (M,ω) from Theorem 1.19.

Proposition 1.27 *The real part $M_{\mathbb{R}} \subset (M,\omega)$ of a monotone even toric symplectic manifold is ζ_{μ_M}–superheavy and hence nondisplaceable.*

Proof Using $\widehat{M}_{\mathbb{R}}$ is μ–subheavy by Corollary 1.26 it follows $M_{\mathbb{R}} = \pi(\widehat{M}_{\mathbb{R}})$ is ζ_{μ_M}–superheavy by Proposition 1.20 and therefore is nondisplaceable. □

Haug [56] proved the nondisplaceability part of Proposition 1.27 without the even assumption using Biran and Cornea's Lagrangian quantum homology [16; 17].

Similarly the central toric fiber $T_M \subset (M,\omega)$ of a monotone even toric manifold is nondisplaceable and cannot be displaced from the real part $M_{\mathbb{R}}$. This is because $\pi^{-1}(T_M) = T_{\widehat{M}}$ so Proposition 1.20 and Corollary 1.26 imply T_M is ζ_{μ_M}–superheavy. The nondisplaceability now follows from [42, Theorem 1.4]. These results have been established by various authors; see [4; 39], Alston and Amorim [7], Cho [29], and Fukaya, Oh, Ohta and Ono [48]. In particular Abreu and Macarini [4] showed how simple previous nondisplaceability results in $\mathbb{C}\mathrm{P}^n$ can be combined with symplectic reduction to prove nondisplaceability for T_M and the pair $(T_M, M_{\mathbb{R}})$, but could not prove $M_{\mathbb{R}}$ was nondisplaceable.

1.5 Orderability and metrics on $\widetilde{\mathrm{Cont}}_0$

Recall from Section 1.1 that a contact manifold (V,ξ) is orderable if $\widetilde{\mathrm{Cont}}_0(V,\xi)$ is partially ordered by the relation $\preceq$ from (1-4).

1.5.1 Orderability for contact manifolds and quasimorphisms There has been a fair amount of research concerning orderability of contact manifolds. Since we are mainly dealing with closed contact manifolds, let us give examples of orderable and nonorderable closed contact manifolds. Eliashberg, Kim and Polterovich prove in [35] that the ideal contact boundary of a sufficiently subcritical Weinstein manifold is not orderable. In particular the standard contact spheres S^{2d-1} are not orderable for $d \geq 2$. Cosphere bundles of closed manifolds are known to be orderable (see Albers and Frauenfelder [5], Chernov and Nemirovski [27] and [35; 36]) and more generally Albers and Merry proved in [6] that Liouville-fillable contact manifolds with nonvanishing Rabinowitz Floer homology are orderable. Using the connection between orderability and contact squeezing developed by Eliashberg, Kim and Polterovich [35], Milin [70] and Sandon [81] proved that lens spaces are orderable.

In [36, Section 1.3.E] Eliashberg and Polterovich proved that $\mathbb{R}\mathrm{P}^{2d-1}$ is orderable using Givental's quasimorphism μ_{Giv}. Their argument works in general and implies the following.

Theorem 1.28 [36] *A contact manifold (V,ξ) is orderable if there is a monotone quasimorphism μ on $\widetilde{\mathrm{Cont}}_0(V,\xi)$.*

Proof By [36, Criterion 1.2.C] to prove (V,ξ) is orderable it suffices to prove for any contact Hamiltonian with $h > 0$ on $[0,1]\times V$ that

$$\tilde{\phi}_h \neq \mathrm{id} \quad \text{in } \widetilde{\mathrm{Cont}}_0(V,\xi).$$

Since $\mu(\mathrm{id}) = 0$, we are done because for any such contact Hamiltonian $\mu(\tilde{\phi}_h) > 0$ by Proposition 1.10(iv). □

Corollary 1.29 *The contact manifolds $(\widehat{M},\xi)$ in Theorem 1.3 are orderable.* □

Recall that the contact manifolds $(\widehat{M},\xi)$ are obtained from contact reduction of $\mathbb{R}\mathrm{P}^{2d-1}$, which is of course orderable. It would be interesting to prove Corollary 1.29 directly, that is, to prove orderability persists under contact reduction.

By Theorem 1.28, orderability is a necessary condition for the existence of a nonzero homogeneous monotone quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$. However in general the converse is not well understood and potentially is a delicate question, which we will illustrate with the following examples regarding $\mathbb{R}^{2n}\times S^1$ and its group of compactly supported contactomorphisms $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$, where the contact form is $\alpha_{\mathrm{std}} + dt$ and dt is the angular form on $S^1 = \mathbb{R}/\mathbb{Z}$.

Assumption 1.30 The contactomorphism groups $\mathrm{Cont}_0^c(V)$ and $\widetilde{\mathrm{Cont}}_0^c(V)$ are perfect for every contact manifold (V,ξ).

A proof of Assumption 1.30 appears in Rybicki [78].

Example 1.31 Sandon has proved [79; 80] that $\mathbb{R}^{2n}\times S^1$ is orderable, that is, that the Eliashberg–Polterovich relation (1-4) is indeed a partial order on the group $\widetilde{\mathrm{Cont}}_0^c(\mathbb{R}^{2n}\times S^1)$, and also proved it induces a partial order on $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$. However $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$ and $\widetilde{\mathrm{Cont}}_0^c(\mathbb{R}^{2n}\times S^1)$ do not admit non-zero homogeneous quasi-morphisms, due to a general argument of Kotschick [61, Theorem 4.2] together with Assumption 1.30.

Example 1.32 Consider now the domain $B_R^{2n}\times S^1$, where

$$B_R^{2n} := \{z\in\mathbb{C}^n \mid \pi|z|^2 < R\}.$$

Since $\mathbb{R}^{2n}\times S^1$ is contactomorphic to $B_1^{2n}\times S^1$ by [35, Proposition 1.24], Example 1.31 indicates $\widetilde{\mathrm{Cont}}_0^c(B_1^{2n}\times S^1)$ does not admit a nonzero homogeneous quasimorphism.

On the other hand, $\widetilde{\mathrm{Cont}}_0^c(B_R^{2n}\times S^1)$ admits a nonzero homogeneous quasimorphism whenever $2n/(n+1) < R < 2$. When $R<2$ we have the contact embedding

$$\Phi\colon B_R^{2n}\times S^1\to\mathbb{R}\mathrm{P}^{2n+1} \quad \text{given by} \quad (z,t)\mapsto e^{\pi i t}\sqrt{\frac{n+1}{2}}\Big(z,\sqrt{\frac{2}{\pi}-|z|^2}\Big) \tag{1-29}$$

written as a map to $S^{2n+1}=\{z\in\mathbb{C}^{n+1}\mid \pi|z|^2=n+1\}$ from (1-19); where $t\in[0,1)$. When $R>2n/(n+1)$, one can check the image $\Phi(B_R^{2n})$ contains the μ_{Giv}–superheavy torus $T_{\mathbb{R}\mathrm{P}}\subset\mathbb{R}\mathrm{P}^{2n+1}$ from Lemma 1.22 and hence one can use Φ to pullback μ_{Giv} to a nonzero homogeneous quasimorphism on $\widetilde{\mathrm{Cont}}_0^c(B_R^{2n}\times S^1)$.

The reader is also referred to Ben Simon and Hartnick's work [14; 15] regarding a general connection between quasimorphisms and partial orders.

1.5.2 Sandon-type metric In [79] Sandon introduced an unbounded integer-valued conjugation-invariant norm on $\mathrm{Cont}_0^c(\mathbb{R}^{2n}\times S^1)$, the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n}\times S^1$, and such norms have been further studied in [6; 89], by Colin and Sandon [31], and Fraser, Polterovich and Rosen [46]. In what follows we will consider the norm ν defined in [46], whose definition we will now recall.

Consider any orderable contact manifold (V,ξ) for which there is a positive contact Hamiltonian $f>0$ such that $\widetilde{\phi}_f$ is in the center of $\widetilde{\mathrm{Cont}}_0(V)$. Examples of this are given by orderable contact manifolds with a periodic Reeb flow, for instance $\mathbb{R}\mathrm{P}^{2d-1}$ or any of the contact manifolds $\widehat{M}$ from Theorem 1.3. The functionals on $\widetilde{\mathrm{Cont}}_0(V)$,

$$\nu_-(\widetilde{\psi}) := \max\{n \in \mathbb{Z} \mid \widetilde{\phi}_f^n \preceq \widetilde{\psi}\} \quad \text{and} \quad \nu_+(\widetilde{\psi}) := \min\{n \in \mathbb{Z} \mid \widetilde{\psi} \preceq \widetilde{\phi}_f^n\},$$

are conjugation-invariant, since $\widetilde{\phi}_f$ is in the center of $\widetilde{\mathrm{Cont}}_0(V)$, and

$$\nu\colon \widetilde{\mathrm{Cont}}_0(V) \to \mathbb{Z}, \quad \text{where} \quad \nu(\widetilde{\phi}) := \max\{|\nu_+(\widetilde{\phi})|, |\nu_-(\widetilde{\phi})|\},$$

defines a conjugation-invariant norm, by [46, Theorem 2.4]. Using $\widetilde{\phi}_f$ is generated by a strictly positive contact Hamiltonian, it is easy to see from [36, Criterion 1.2.C] that $\nu(\widetilde{\phi}_f^n) = |n|$ for any $n \in \mathbb{Z}$ and hence ν is stably unbounded. This norm is related to monotone quasimorphisms on $\widetilde{\mathrm{Cont}}_0(V)$ as follows:

Lemma 1.33 *If $\mu\colon \widetilde{\mathrm{Cont}}_0(V) \to \mathbb{R}$ is a monotone quasimorphism, then*

$$|\mu(\widetilde{\psi})| \le \mu(\widetilde{\phi}_f)\nu(\widetilde{\psi}).$$

Note that $\mu(\widetilde{\phi}_f) > 0$ since $\mu \neq 0$.

Proof By the definition of $\nu_\pm$ and the fact that μ is monotone and homogeneous,

$$\nu_-(\widetilde{\psi})\mu(\widetilde{\phi}_f) \le \mu(\widetilde{\psi}) \le \nu_+(\widetilde{\psi})\mu(\widetilde{\phi}_f),$$

from which the result follows. □

Next we show the above norm is unbounded on subgroups of $\widetilde{\mathrm{Cont}}_0(V)$ associated to certain open subsets. For an open subset $U \subset V$ let $\widetilde{\mathrm{Cont}}_0(U) \subset \widetilde{\mathrm{Cont}}_0(V)$ be the subgroup consisting of elements $\widetilde{\phi}_h$ where the Hamiltonian h has compact support contained in U.

Theorem 1.34 *If $U \subset V$ is an open subset containing a μ–superheavy subset, then there is $\widetilde{\psi} \in \widetilde{\mathrm{Cont}}_0(U)$ with*

$$\lim_{n\to\infty} \frac{\nu(\widetilde{\psi}^n)}{n} > 0,$$

that is, ν is stably unbounded on $\widetilde{\mathrm{Cont}}_0(U)$.

Proof By the above lemma we have

$$\nu(\widetilde{\psi}) \ge \frac{|\mu(\widetilde{\psi})|}{\mu(\widetilde{\phi}_f)},$$

therefore it suffices to produce an element $\widetilde{\psi} \in \widetilde{\mathrm{Cont}}_0(U)$ with $\mu(\widetilde{\psi}) \neq 0$. If h is such that the restriction of h to the superheavy subset is positive and $\mathrm{supp}(h) \subset U$, then since by definition $\mu(\widetilde{\phi}_h) > 0$, we are done. □

In [31], Colin and Sandon used the notion of a discriminant point to define a nondegenerate bi-invariant metric on $\widetilde{\mathrm{Cont}}_0(V,\xi)$ for any contact manifold, which they called the discriminant metric. Using the relation between Givental's quasimorphism μ_{Giv} with discriminant points (see Section 4.3.1 for more on this), Colin and Sandon were able to show the discriminant metric is stably unbounded on $\widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1})$. It would be interesting to determine if the quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M}) \to \mathbb{R}$ we built in Theorem 1.3 can also be used to show the discriminant metric on $\widetilde{\mathrm{Cont}}_0(\widehat{M})$ is stably unbounded.

1.6 Examples of even monotone polytopes

Moment polytopes corresponding to closed monotone symplectic toric manifolds are known as smooth Fano polytopes. They have been classified by hand up to dimension 4 in Batyrev [9; 10], Sato [82] and Watanabe and Watanabe [86] and there is an algorithm in Øbro [73] for higher dimensions. We give various examples of even smooth Fano polytopes in $\mathbb{R}^n$ and their corresponding symplectic toric manifolds. For the polytopes we just list the interior conormals $\{\nu_j\} \in \mathbb{Z}^n$, where $\{\epsilon_1,\dots,\epsilon_n\}$ is the standard basis.

The first example is $\mathbb{C}\mathrm{P}^n$ with conormals $\{\epsilon_1,\dots,\epsilon_n,-(\epsilon_1+\cdots+\epsilon_n)\}$ and in dimension two there are

$$\mathbb{C}\mathrm{P}^2, \quad \mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1, \quad \mathbb{C}\mathrm{P}^2\,\#\,3\overline{\mathbb{C}\mathrm{P}}^2,$$

where the last one has conormals $\{\pm\epsilon_1,\pm\epsilon_2,\pm(\epsilon_1+\epsilon_2)\}$. In dimension three there are 18 smooth Fano polytopes by the classification [9; 86] and 8 are even. Four are basic,

$$\mathbb{C}\mathrm{P}^3, \quad \mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^2, \quad (\mathbb{C}\mathrm{P}^1)^3, \quad \mathbb{C}\mathrm{P}^1\times(\mathbb{C}\mathrm{P}^2\,\#\,3\overline{\mathbb{C}\mathrm{P}}^2),$$

and the remaining four have the structure of toric bundles (see McDuff and Tolman [69, Definition 3.10]):

(i) The $\mathbb{C}\mathrm{P}^1$–bundle $\mathbb{P}(\mathbb{C}\oplus\mathcal{O}(2))$ over $\mathbb{C}\mathrm{P}^2$ with conormals
$$\{\pm\epsilon_1,\epsilon_2,\epsilon_3,2\epsilon_1-\epsilon_2-\epsilon_3\}.$$

(ii) The $(\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}}^2)$–bundle F_3^4 (in the notation of [86]) over $\mathbb{C}\mathrm{P}^1$ with conormals
$$\{\pm\epsilon_1,\pm\epsilon_2,-\epsilon_1-\epsilon_2,\epsilon_3,-\epsilon_1-\epsilon_2-\epsilon_3\}.$$

(iii) The $\mathbb{C}\mathrm{P}^1$–bundle $\mathbb{P}(\mathbb{C}\oplus\mathcal{O}(1,1))$ over $\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1$ with conormals
$$\{\pm\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_1-\epsilon_2,\epsilon_1-\epsilon_3\}.$$

(iv) The $\mathbb{C}\mathrm{P}^1$–bundle $\mathbb{P}(\mathbb{C}\oplus\mathcal{O}(1,-1))$ over $\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1$ with conormals
$$\{\pm\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_1-\epsilon_2,-\epsilon_1-\epsilon_3\}.$$

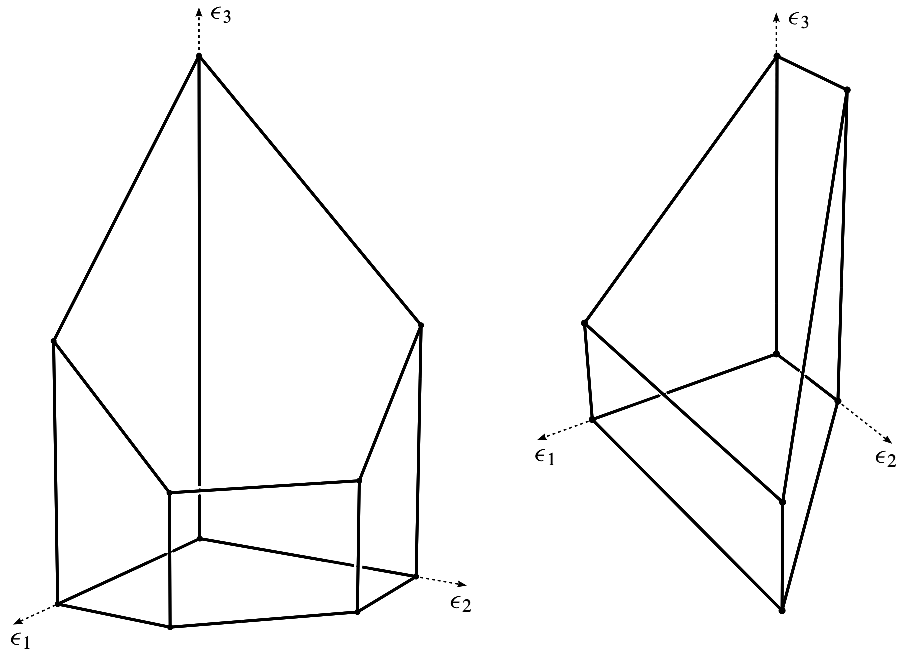

Figure 1: On the left, we have the polytope for the $(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2)$–bundle over $\mathbb{C}P^1$ in (ii). On the right, we have the polytope for the $\mathbb{C}P^1$–bundle $\mathbb{P}(\mathbb{C} \oplus \mathcal{O}(1,-1))$ over $\mathbb{C}P^1 \times \mathbb{C}P^1$ in (iv).

The example in (i) generalizes to the $\mathbb{C}P^1$–bundles $\mathbb{P}(\mathbb{C} \oplus \mathcal{O}(2k))$ over $\mathbb{C}P^n$ where $0 \le 2k \le n$. See Figure 1 for the polytopes from (ii) and (iv).

Acknowledgements We are very grateful to Leonid Polterovich for his wonderful guidance and support with this paper. We thank Miguel Abreu, Alexander Givental and Yael Karshon for helpful conversations and Milena Pabiniak, Sheila Sandon and Egor Shelukhin for their comments on the manuscript. We thank the referee for their helpful comments, suggestions and corrections. We also wish to thank the organizers and participants of the workshop "From Conservative Dynamics to Symplectic and Contact Topology" at the Lorentz Center (Leiden), the Geometry Seminar at Université Lyon 1 (Lyon) and the Geometry and Topology seminars at the University of Haifa and The Technion (Haifa), for the opportunity to present preliminary results of this work. Work on this paper began while visiting the Lorentz Center (Leiden) and our meetings at Tel Aviv University were instrumental in the completion of this project. The first-named author thanks Leonid Polterovich and Yaron Ostrover for their warm hospitality during his stay at Tel Aviv University. Part of the writing was completed while the second-named author was a postdoc at LMU (Munich), and he would like to acknowledge the nourishing research atmosphere and hospitality of this university. The first-named author was partially supported by NSF grants DMS-1006610 and DMS-1304252. The second-named author was partially supported by Deutsche Forschungsgemeinschaft grant DFG/CI 45/5-1 while at LMU (Munich).

2 Proof of the main theorem

In this section we will present the proof of Theorem 1.3. Section 2.1 contains the construction of the prequantization $\pi\colon (\widehat{M},\alpha) \to (M,\omega)$ and Section 2.2 builds the announced quasimorphisms $\mu\colon \widetilde{\mathrm{Cont}}_0(\widehat{M}) \to \mathbb{R}$ using Theorem 1.8.

2.1 Constructing the family of contact manifolds

The goal of this subsection is to present the construction of a prequantization $(\widehat{M},\xi,\alpha)$ for an even closed monotone toric symplectic manifold (M,ω) with moment polytope

$$\Delta = \{x \in \mathfrak{t}^* \mid \langle x, \nu_j\rangle + 1 \geq 0 \text{ for } j = 1,\dots,d\} \tag{2-1}$$

as in (1-6), where $\nu_j \in \mathfrak{t}_{\mathbb{Z}}$ are primitive vectors and each one defines a different facet of the polytope Δ. The polytope Δ is compact and smooth, meaning each k–codimensional face of Δ is the intersection of exactly k facets and the k associated conormals $\{\nu_{l_1},\dots,\nu_{l_k}\}$ can be extended to an integer basis for the lattice $\mathfrak{t}_{\mathbb{Z}}$. In (2-1) we have used the normalization $[\omega] = c_1(M)$ since (M,ω) is monotone and scaling the polytope Δ is equivalent to scaling ω.

2.1.1 The standard toric structure on $\mathbb{C}^d$ and Delzant's construction Let us briefly recall the standard toric structure on $(\mathbb{C}^d, \omega_{\mathrm{std}} = dx \wedge dy)$. The action of $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ on $\mathbb{C}^d$, which rotates each coordinate, is induced by the moment map

$$P\colon \mathbb{C}^d \to \mathbb{R}^{d*}, \quad \text{where} \quad \langle \lambda, P\rangle(z) = \pi \sum_{j=1}^{d} \lambda_j |z_j|^2 \quad \text{for } \lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d.$$

Indeed for $\lambda \in \mathbb{R}^d$, the vector field

$$X_\lambda(z) = 2\pi i(\lambda_1 z_1,\dots,\lambda_d z_d) \in \mathbb{C}^d = T_z\mathbb{C}^d \tag{2-2}$$

is the Hamiltonian vector field for the function $\langle \lambda, P\rangle\colon \mathbb{C}^d \to \mathbb{R}$ and it gives the infinitesimal action of λ on $\mathbb{C}^d$. Observe that for the 1–form

$$\alpha_{\mathrm{std}} = \frac{1}{2}\sum_{j=1}^{d}(x_j\, dy_j - y_j\, dx_j), \tag{2-3}$$

where $d\alpha_{\mathrm{std}} = \omega_{\mathrm{std}}$, one has

$$\alpha_{\mathrm{std}}(X_\lambda) = \langle \lambda, P \rangle \quad \text{and} \quad \iota_{X_\lambda} d\alpha_{\mathrm{std}} = \iota_{X_\lambda} \omega_{\mathrm{std}} = -d\langle \lambda, P \rangle. \tag{2-4}$$

Delzant in [32] gave a way to reconstruct a closed symplectic toric manifold from its moment polytope using symplectic reduction of $\mathbb{C}^d$, which we will now recall in the case of the polytope Δ in (2-1). Define the surjective linear map

$$\beta_\Delta\colon \mathbb{R}^d \to \mathfrak{t} \quad \text{by} \quad \epsilon_j \mapsto \nu_j \quad \text{for } j = 1, \dots, d,$$

where $\{\epsilon_j\}_{j=1}^d$ are the standard basis vectors of $\mathbb{R}^d$ and $\nu_j \in \mathfrak{t}_{\mathbb{Z}}$ are conormals in (2-1). Since Δ is compact and smooth, we know $\beta_\Delta(\mathbb{Z}^d) = \mathfrak{t}_{\mathbb{Z}}$, and so we can define the connected subtorus

$$\mathbb{K} \leq \mathbb{T}^d \tag{2-5}$$

to be the kernel of the induced map $[\beta_\Delta]\colon \mathbb{T}^d \to \mathbb{T}$ with Lie algebra

$$\mathfrak{k} := \ker(\beta_\Delta\colon \mathbb{R}^d \to \mathfrak{t}). \tag{2-6}$$

If $\iota^*\colon \mathbb{R}^{d*} \to \mathfrak{k}^*$ is dual to the inclusion $\mathfrak{k} \subset \mathbb{R}^d$, then the action of $\mathbb{K}$ on $\mathbb{C}^d$ has

$$P_{\mathbb{K}} := \iota^* \circ P\colon \mathbb{C}^d \to \mathfrak{k}^*$$

for its moment map. The torus $\mathbb{K}$ acts freely on the regular level set

$$P_{\mathbb{K}}^{-1}(c) \subset \mathbb{C}^d, \quad \text{where} \quad c := \iota^*(1, \dots, 1) \in \mathfrak{k}^*, \tag{2-7}$$

and for $\lambda \in \mathfrak{k}$ it follows from (2-4) that $(\mathcal{L}_{X_\lambda}\omega_{\mathrm{std}})|_{P_{\mathbb{K}}^{-1}(c)} = 0$. Therefore symplectic reduction gives a symplectic manifold $(M_\Delta, \omega_\Delta)$ where

$$M_\Delta := P_{\mathbb{K}}^{-1}(c)/\mathbb{K}, \tag{2-8}$$

and the symplectic form ω_Δ is induced from $\omega_{\mathrm{std}}|_{P_{\mathbb{K}}^{-1}(c)}$. It follows from Delzant's theorem [32] that $(M_\Delta, \omega_\Delta)$ and (M, ω) are equivariantly symplectomorphic as toric manifolds.

The following lemma that shows the significance of the assumption that Δ is an even moment polytope.

Lemma 2.1 *Let $\tau \in \mathbb{T}^d$ be the element such that $\tau \cdot z = -z$ for $z \in \mathbb{C}^d$. The torus $\mathbb{K}$ from* (2-5) *contains the element τ if and only if Δ is even.*

Proof Note that $\tau = \left[\frac{1}{2}, \dots, \frac{1}{2}\right]$ in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and therefore since $\mathbb{T} = \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$ it is clear from (2-5) that $\tau \in \mathbb{K}$ if and only if $\sum_{j=1}^d \frac{1}{2}\nu_j \in \mathfrak{t}_{\mathbb{Z}}$. □

2.1.2 The contact manifold $(\widehat{M},\xi)$ from Delzant's construction of (M,ω) Using Delzant's construction we will now describe the contact manifold $(\widehat{M},\xi)$ associated to an even monotone symplectic toric manifold with moment polytope (2-1). Define

$$\mathfrak{k}_0 := \ker(c\colon \mathfrak{k} \to \mathbb{R})$$

to be the annihilator of the linear functional $c = \iota^*(1,\dots,1) \in \mathfrak{k}^*$ from (2-7) and

$$\mathbb{K}_0 \le \mathbb{K} \tag{2-9}$$

to be the connected codimension-1 subtorus with $\mathrm{Lie}(\mathbb{K}_0) = \mathfrak{k}_0$. Since Δ in (2-1) is an even moment polytope, by Lemma 2.1 we know $\mathbb{K}_0 + \langle\tau\rangle \le \mathbb{K}$, where $\langle\tau\rangle \le \mathbb{K}$ is the subgroup generated by τ. Therefore $\mathbb{K}_0 + \langle\tau\rangle$ also acts freely on the level set $P_{\mathbb{K}}^{-1}(c)$ from (2-7).

The contact manifold $(\widehat{M},\xi = \ker\alpha)$ is given by

$$\widehat{M} := P_{\mathbb{K}}^{-1}(c)/(\mathbb{K}_0 + \langle\tau\rangle) \tag{2-10}$$

and the contact form α, which is induced from $\alpha_{\mathrm{std}}|_{P_{\mathbb{K}}^{-1}(c)}$, is well-defined because the infinitesimal action of $\mathbb{K}_0$ is tangent to $\ker\alpha_{\mathrm{std}}$ along $P_{\mathbb{K}}^{-1}(c)$, which follows from (2-4). For the circle $S^1 = \mathbb{K}/(\mathbb{K}_0 + \langle\tau\rangle)$, the natural projection map

$$\pi\colon (\widehat{M},\alpha) \to (M_\Delta,\omega_\Delta) \tag{2-11}$$

defines a principal S^1–bundle and satisfies $\pi^*\omega_\Delta = d\alpha$ since $\omega_{\mathrm{std}} = d\alpha_{\mathrm{std}}$. Therefore by using a symplectomorphism $(M_\Delta,\omega_\Delta) \simeq (M,\omega)$, we have that (2-11) is the desired prequantization in Theorem 1.3.

We will now present a formula for the period of the Reeb vector field of $(\widehat{M},\alpha)$ and hence the Euler class $e(\pi) \in H^2(M;\mathbb{Z})$ of the principal S^1–bundle (2-11). For the functional $c\colon \mathfrak{k} \to \mathbb{R}$ from (2-7), let

$$c_{\mathfrak{k}} \in \mathbb{Z}$$

be the positive generator of the image $c(\mathfrak{k}_{\mathbb{Z}}) \subset \mathbb{Z}$ of the integer lattice $\mathfrak{k}_{\mathbb{Z}} := \mathfrak{k} \cap \mathbb{Z}^d$ and let

$$\delta := \begin{cases} 1 & \text{if } \tau \in \mathbb{K}_0, \\ 2 & \text{if } \tau \notin \mathbb{K}_0. \end{cases} \tag{2-12}$$

Proposition 2.2 *The Reeb vector field for $(\widehat{M},\alpha)$ has period $c_{\mathfrak{k}}/\delta$ and the Euler class of $\pi\colon (\widehat{M},\alpha) \to (M,\omega)$ equals*

$$e(\pi) = -\frac{\delta}{c_{\mathfrak{k}}} c_1(M) \in H^2(M;\mathbb{Z}).$$

Proof Recall for a principal S^1–bundle $\pi\colon V \to M$ that if α is a connection 1–form on V and $\omega = \pi_*(d\alpha)$ is the curvature 2–form on M, then the Euler class is given by (see Morita in [72, Section 6.2(d)])

$$e(\pi) := \frac{-1}{\int_{\pi^{-1}(m)} \alpha}[\omega] \in H^2(M;\mathbb{Z}),$$

ie the negative of the curvature form divided by the integral of the connection form over a fiber. For our prequantization, $\int_{\pi^{-1}(m)}\alpha$ is the period of the Reeb vector field and we used the normalization $[\omega] = c_1(M)$, so it suffices to compute that $\int_{\pi^{-1}(m)}\alpha = c_{\mathfrak{k}}/\delta$.

Consider $\widehat{M}_\Delta := P_{\mathbb{K}}^{-1}(c)/\mathbb{K}_0$ with the 1–form α_Δ induced from α_{std}. Under the identification $(M,\omega) \simeq (M_\Delta, \omega_\Delta)$ from (2-8), the projection map defines a prequantization

$$\pi_\Delta\colon (\widehat{M}_\Delta, \alpha_\Delta) \to (M,\omega).$$

We will compute that $\int_{\pi_\Delta^{-1}(\overline{z})}\alpha_\Delta = c_{\mathfrak{k}}$ for any $\overline{z} \in M$, and this will suffice since $\widehat{M}_\Delta \to \widehat{M}$ is a degree-δ cover. By the definition of $(\widehat{M}_\Delta, \alpha_\Delta)$, its Reeb vector field can be represented by the infinitesimal action of X_λ from (2-2) on $P_{\mathbb{K}}^{-1}(c)$ for any

$$\lambda \in \mathfrak{k} \quad \text{such that} \quad \langle \lambda, c\rangle = 1. \tag{2-13}$$

For any such λ, the period of the Reeb orbit can also be characterized as the smallest $T > 0$ so that $\exp(T\lambda) \in \mathbb{K}_0$ for the exponential map $\exp\colon \mathfrak{k} \to \mathbb{K}$. Since $\langle \lambda_0, c\rangle = 0$ for any $\lambda_0 \in \mathfrak{k}_0$ and $\mathbb{K}_0$ is connected, we can choose λ as in (2-13) so that the first return is at $\exp(T\lambda) = 1 \in \mathbb{K}$. In this case $T\lambda \in \mathfrak{k}_{\mathbb{Z}}$ and $T = \langle T\lambda, c\rangle \in c(\mathfrak{k}_{\mathbb{Z}})$, so therefore $T = c_{\mathfrak{k}}$, the minimal positive generator of $c(\mathfrak{k}_{\mathbb{Z}})$. □

Remark 2.3 Both possibilities in (2-12) actually occur. For the case of $\mathbb{C}\mathrm{P}^n$ we have $\tau \notin \mathbb{K}_0$, since $\mathbb{K}_0 = 1$, and for $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ below we have $\tau \in \mathbb{K}_0$.

2.1.3 An example in the case $M = \mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}$

Consider the even toric monotone symplectic manifold

$$(M,\omega) = (\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}, n\sigma \oplus n\sigma),$$

where $\int_{\mathbb{C}\mathrm{P}^1}\sigma = 1$. Its moment polytope is

$$\left\{(x,x') \in (\mathbb{R}^{2n-2})^* \;\middle|\; x_j + 1 \geq 0, -\sum_{j=1}^{n-1} x_j + 1 \geq 0, x'_j + 1 \geq 0, -\sum_{j=1}^{n-1} x'_j + 1 \geq 0\right\},$$

where we have identified $\mathbb{T} = \mathbb{R}^{2n-2}/\mathbb{Z}^{2n-2}$. In this case, $\mathbb{K}_0 \leq \mathbb{K}$ are the subtori of $\mathbb{T}^{2n}$ whose Lie algebras in $\mathbb{R}^{2n}$ have bases

$$\mathfrak{k}_0 = \operatorname{span}\left\{\sum_{j=1}^{n} \epsilon_j - \sum_{j=1}^{n} \epsilon_j'\right\} \quad \text{and} \quad \mathfrak{k} = \operatorname{span}\left\{\sum_{j=1}^{n} \epsilon_j, \sum_{j=1}^{n} \epsilon_j'\right\}.$$

The moment map $P_{\mathbb{K}}\colon \mathbb{C}^{2n} \to \mathfrak{k}^* = (\mathbb{R}^2)^*$ for the action of $\mathbb{K}$ on $\mathbb{C}^{2n}$ is

$$P_{\mathbb{K}}(z, z') = \pi\left(\sum_{j=1}^{n} |z_j|^2, \sum_{j=1}^{n} |z_j'|^2\right)$$

and we have $P_{\mathbb{K}}^{-1}(c)$ is $S^{2n-1} \times S^{2n-1} \subset \mathbb{C}^{2n}$ since

$$P_{\mathbb{K}}^{-1}(c) = \left\{(z, z') \in \mathbb{C}^{2n} \;\middle|\; \pi \sum_{j=1}^{n+1} |z_j|^2 = \pi \sum_{j=1}^{n+1} |z_j'|^2 = n\right\}.$$

The action of the circle $\mathbb{K}_0$ on $\mathbb{C}^{2n}$ is given by $\zeta \cdot (z, z') = (\zeta z, \bar{\zeta} z')$ for ζ in the unit circle S^1, and note that $\tau \in \mathbb{K}_0$. The contact manifold is

$$\widehat{M} = (S^{2n-1} \times S^{2n-1})/\mathbb{K}_0, \tag{2-14}$$

with contact form α induced by $\alpha_{\mathrm{std}}|_{S^{2n-1}} \oplus \alpha_{\mathrm{std}}|_{S^{2n-1}}$. The Reeb vector field R_α is represented by X_λ with $\lambda = \frac{1}{2n}(1, \dots, 1) \in \mathbb{R}^{2n}$ from (2-2) and it has period is $c_{\mathfrak{k}} = n$, so therefore the prequantization is the $\mathbb{R}/n\mathbb{Z}$–bundle

$$\pi\colon (\widehat{M}, \alpha) \to (\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}, n\sigma \oplus n\sigma). \tag{2-15}$$

Since the first Chern class $c_1(\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}) = (n, n) \in H^2(\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z})$,

$$e(\pi) = (-1, -1) \in H^2(\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z})$$

from Proposition 2.2.

Rescaling so that prequantization is a $\mathbb{R}/\mathbb{Z}$–bundle, we see that

$$\pi\colon \widehat{M} \to (\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{n-1}, \sigma \oplus \sigma)$$

is the standard Boothby–Wang prequantization [19]. For the case of $n = 2$, it is known that $\widehat{M}$ is contactomorphic to the unit cotangent bundle UT^*S^3 of S^3; for instance see Abreu and Macarini [3, Section 6.1]. However when $n \geq 3$, it follows from [19, Theorem 8] that $\widehat{M}$ is not even topologically a unit cotangent bundle.

2.2 Applying Theorem 1.8 to prove Theorem 1.3

Proof of Theorem 1.3 Theorem 1.3 will be proved by applying Theorem 1.8 to Givental's quasimorphism $\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}_\gamma) \to \mathbb{R}$ in the setting

$$(\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma, \alpha_{\mathrm{std}}) \supset (Y, \alpha_{\mathrm{std}}|_Y) \xrightarrow{\rho} (\widehat{M}, \xi, \alpha) \tag{2-16}$$

for an appropriate γ and Y that we describe below.

By [4, Lemma 4.8] there is a primitive vector $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathfrak{k}_{\mathbb{Z}} := \mathbb{Z}^d \cap \mathfrak{k}$ where each $\gamma_j \geq 1$. Fix such a γ and consider the sphere S^{2d-1}_γ from (1-19). Note that

$$P_{\mathbb{K}}^{-1}(c) \subset S^{2d-1}_\gamma = \{z \in \mathbb{C}^d \mid \langle \gamma, P\rangle(z) = \langle \gamma, c\rangle\}$$

since $z \in P_{\mathbb{K}}^{-1}(c)$ is equivalent to $\langle \lambda, P\rangle(z) = \langle \lambda, c\rangle$ for all $\lambda \in \mathfrak{k}$. Since Δ is even we know $\tau \in \mathbb{K}$ by Lemma 2.1 and modding out by the antipodal $\mathbb{Z}_2 = \langle \tau \rangle$ action gives the submanifold

$$Y := P_{\mathbb{K}}^{-1}(c)/\langle \tau\rangle \subset (\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma), \tag{2-17}$$

where $(\mathbb{R}\mathrm{P}^{2d-1}_\gamma, \xi_\gamma) = (S^{2d-1}_\gamma/\langle\tau\rangle, \ker \alpha_{\mathrm{std}})$. The natural projection map

$$\rho\colon Y \to \widehat{M} \tag{2-18}$$

is a principal $(\mathbb{K}_0 + \langle\tau\rangle)/\langle\tau\rangle$–bundle and by the construction of the 1–form α from (2-10) it follows that $\rho^*\alpha = \alpha_{\mathrm{std}}|_Y$.

To verify the geometric setting (1-9) of Theorem 1.8 it remains to prove that $Y \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma$ is strictly coisotropic with respect to α_{std}. Note that $P_{\mathbb{K}}^{-1}(c) \subset (\mathbb{C}^d, \omega_{\mathrm{std}})$ is a coisotropic submanifold, meaning that for all $z \in P_{\mathbb{K}}^{-1}(c)$ we have

$$(T_z P_{\mathbb{K}}^{-1}(c))^{\omega_{\mathrm{std}}} := \{X \in T_z\mathbb{C}^d \mid \iota_X\omega_{\mathrm{std}} = 0 \text{ on } T_zP_{\mathbb{K}}^{-1}(c)\} \subset T_zP_{\mathbb{K}}^{-1}(c), \tag{2-19}$$

since $P_{\mathbb{K}}^{-1}(c)$ is the regular level set of a moment map or as can be verified with (2-4). It now follows from (2-19) and $d\alpha_{\mathrm{std}} = \omega_{\mathrm{std}}$ that $P_{\mathbb{K}}^{-1}(c) \subset S^{2d-1}_\gamma$ satisfies the condition (1-8) to be strictly coisotropic with respect to α_{std} and therefore so is $Y \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma$.

Using the definition (2-7) of $P_{\mathbb{K}}^{-1}(c)$ we know that

$$\{|z_1|^2 = \cdots = |z_d|^2 = 1/\pi\} \subset P_{\mathbb{K}}^{-1}(c),$$

since if $z \in \{|z_1|^2 = \cdots = |z_d|^2 = 1/\pi\}$, then for any $\lambda \in \mathfrak{k}$ one has

$$\langle \lambda, P\rangle(z) = \pi \sum_{j=1}^d \lambda_j \frac{1}{\pi} = \sum_{j=1}^d \lambda_j = \langle \lambda, c\rangle, \tag{2-20}$$

where the last equality follows from the fact that $c := \iota^*(1,\dots,1)$. Hence $T_{\mathbb{R}\mathrm{P}} \subset Y$ for the torus $T_{\mathbb{R}\mathrm{P}} \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma$ from (1-24), which is μ_{Giv}–superheavy by Lemma 1.22. Therefore by Theorem 1.11 we know $Y \subset \mathbb{R}\mathrm{P}^{2d-1}_\gamma$ is μ_{Giv}–subheavy. Applying Theorem 1.8 to (2-16) constructs the desired quasimorphism $\overline{\mu}\colon \widetilde{\mathrm{Cont}}_0(\widehat{M},\xi) \to \mathbb{R}$ that is monotone, C^0–continuous and has the vanishing property. □

Remark 2.4 For any closed even symplectic toric manifold $(M,\omega,\mathbb{T})$ the construction in this section can be modified to produce a prequantization $\pi\colon (\widehat{M},\alpha) \to (M,\omega)$ that is constructed by contact reduction of a real projective space. Without the monotonicity assumption however, one needs to replace $c = \iota^*(1,\dots,1)$ with $\iota^*(a_1,\dots,a_d)$ where the a_j are the support constants in the moment polytope (1-6). With this change (2-20) no longer holds so the reduction will not pass through the superheavy torus $T_{\mathbb{R}\mathrm{P}}$. This is similar to the proof of [4, Proposition 4.9].

3 Proof of the reduction theorem for quasimorphisms

In this section we will present the proof of Theorem 1.8.

3.1 Preliminary lemmas

3.1.1 Geometric setting of Theorem 1.8 Let us begin by collecting a few lemmas about the geometric setting of Theorem 1.8,

$$(V,\xi,\alpha) \supset (Y,\alpha|_Y) \xrightarrow{\rho} (\overline{V},\overline{\xi},\overline{\alpha}), \tag{3-1}$$

where (V,ξ,α) and $(\overline{V},\overline{\xi},\overline{\alpha})$ are closed contact manifolds, $Y \subset V$ is a closed submanifold that is strictly coisotropic with respect to α, and ρ is a fiber bundle such that $\rho^*\overline{\alpha} = \alpha|_Y$.

Lemma 3.1 *The map $d\rho\colon TY \to T\overline{V}$ relates the Reeb vector fields:*

$$d\rho \circ R_\alpha|_Y = R_{\overline{\alpha}} \circ \rho.$$

Proof Note that by Definition 1.6 of strictly coisotropic the Reeb vector field $R_\alpha|_Y$ is tangent to Y. To show $d\rho \circ R_\alpha = R_{\overline{\alpha}} \circ \rho$ one computes

$$\overline{\alpha}(d\rho(R_\alpha)) = \rho^*\overline{\alpha}(R_\alpha) = \alpha(R_\alpha) = 1$$

and for any $u \in TY$ one has

$$d\overline{\alpha}(d\rho(R_\alpha), d\rho(u)) = d(\rho^*\overline{\alpha})(R_\alpha, u) = d\alpha(R_\alpha, u) = 0,$$

which proves $\iota_{d\rho(R_\alpha)} d\overline{\alpha} = 0$ since $d\rho\colon TY \to T\overline{V}$ is surjective. By definition of $R_{\overline{\alpha}}$ this proves $d\rho \circ R_\alpha|_Y = R_{\overline{\alpha}} \circ \rho$. □

For $\overline{h} \in C^\infty([0,1] \times \overline{V})$, an *extension* of $\overline{h}$ will be any $h \in C^\infty([0,1] \times V)$ such that

$$h|_{[0,1]\times Y} = \rho^* \overline{h}.$$

Lemma 3.2 *If $h \in C^\infty([0,1] \times V)$ is an extension of $\overline{h} \in C^\infty([0,1] \times \overline{V})$, then:*

(i) *The contact vector field $X_{h_t}|_Y$ is tangent to Y.*

(ii) *The contact vector fields of $h, \overline{h}$ are related by $d\rho$: $d\rho \circ X_{h_t}|_Y = X_{\overline{h}_t} \circ \rho$.*

(iii) *As maps, $\rho \circ \phi^t_h|_Y = \phi^t_{\overline{h}} \circ \rho \colon Y \to \overline{V}$ and $\rho \circ (\phi^t_h)^{-1}|_Y = (\phi^t_{\overline{h}})^{-1} \circ \rho \colon Y \to \overline{V}$ for all $t \in [0,1]$.*

Proof It suffices to prove (i) for autonomous $h \in C^\infty(V)$ and $\overline{h} \in C^\infty(\overline{V})$. Let $u \in TY$, then by the definition of X_h from (1-2) and the relations

$$\rho^* \overline{\alpha} = \alpha|_Y, \quad \rho^* \overline{h} = h|_Y, \quad d\rho \circ R_\alpha|_Y = R_{\overline{\alpha}} \circ \rho$$

we have

$$\begin{aligned} d\alpha(X_h, u) = -dh(u) + dh(R_\alpha)\alpha(u) &= -d\overline{h}(d\rho(u)) + d\overline{h}(R_{\overline{\alpha}})\overline{\alpha}(d\rho(u)) \\ &= d\overline{\alpha}(X_{\overline{h}}, d\rho(u)). \end{aligned}$$

Since $X_{\overline{h}} = d\rho(v)$ for some $v \in TY$, taking any $u \in (TY)^{d\alpha} \subset TY$ (see (1-8)),

$$d\alpha(X_h, u) = d\overline{\alpha}(d\rho(v), d\rho(u)) = d\alpha(v, u) = 0,$$

and hence $X_h|_Y \in ((TY)^{d\alpha})^{d\alpha}$. Since Y is strictly coisotropic, it follows from (1-8) that $((TY)^{d\alpha})^{d\alpha} = TY$ and hence $X_h|_Y \in TY$.

For item (ii), similar considerations as above show

$$\overline{\alpha}(d\rho(X_h)) = \alpha(X_h) = \overline{h} \quad \text{at the point } \rho(y),$$

and likewise for any $u \in TY$,

$$\begin{aligned} d\overline{\alpha}(d\rho(X_h), d\rho(u)) = d\alpha(X_h, u) &= -dh(u) + dh(R_\alpha)\alpha(u) \\ &= -d\overline{h}(d\rho(u)) + d\overline{h}(R_{\overline{\alpha}})\overline{\alpha}(d\rho(u)). \end{aligned}$$

Since $d\rho \colon TY \to T\overline{V}$ is surjective this shows $d\rho \circ X_h|_Y = X_{\overline{h}} \circ \rho$.

The first part of item (iii) immediately follows from item (ii). Via the identity

$$\rho \circ (\phi^t_h)^{-1}|_Y = (\phi^t_{\overline{h}})^{-1} \circ \phi^t_{\overline{h}} \circ \rho \circ (\phi^t_h)^{-1}|_Y = (\phi^t_{\overline{h}})^{-1} \circ \rho$$

the second part of item (iii) follows from the first part of (iii). □

3.1.2 Using the subheavy assumption Recall that in Theorem 1.8 that in addition to the geometric setting in (3-1), $Y \subset V$ is subheavy with respect to a monotone quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(V) \to \mathbb{R}$.

Lemma 3.3 *If $h \in C^\infty([0,1] \times V)$ is such that $X_{h_t}|_Y \in TY$ for all $t \in [0,1]$ and*

$$\rho \circ \phi_h^t|_Y = \rho\colon Y \to \overline{V} \quad \text{for all } t \in [0,1], \tag{3-2}$$

then $\mu(\widetilde{\phi}_h) = 0$.

Proof Differentiating (3-2) with respect to t gives $d\rho(X_{h_t}) = 0$ and hence on Y,

$$h_t = \alpha(X_{h_t}) = \overline{\alpha}(d\rho(X_{h_t})) = 0.$$

Pick autonomous Hamiltonians $g, k \in C^\infty(V)$ so that $g \le h \le k$ and $g|_Y = k|_Y = 0$. Since Y is μ–subheavy it follows that $\mu(\widetilde{\phi}_g) = \mu(\widetilde{\phi}_k) = 0$ and therefore $\mu(\widetilde{\phi}_h) = 0$ since μ is monotone. □

Lemma 3.4 *Let $\overline{h} \in C^\infty([0,1] \times \overline{V})$ be such that $\{\phi_{\overline{h}}^t\}_{t\in[0,1]}$ is a contractible loop in $\mathrm{Cont}_0(\overline{V})$. Then $\mu(\widetilde{\phi}_h) = 0$ for any contact Hamiltonian $h \in C^\infty([0,1] \times V)$ that is an extension of $\overline{h}$.*

Proof Let $\Phi\colon [0,1]^2 \to \mathrm{Cont}_0(\overline{V})$ be a null-homotopy of loops for $\{\phi_{\overline{h}}^t\}$, so that $\{\Phi_t^s\}_{t\in[0,1]}$ is a loop of contactomorphisms of $\overline{V}$ for fixed s, such that

$$\Phi_t^0 = \mathrm{id}_{\overline{V}} \quad \text{and} \quad \Phi_t^1 = \phi_{\overline{h}}^t.$$

Let

$$\overline{H}_\bullet^s\colon [0,1] \times \overline{V} \to \mathbb{R}$$

be the contact Hamiltonian generating the contact isotopy $\{\Phi_t^s\}_{t\in[0,1]}$ for fixed s and note $\overline{H}_t^1 = \overline{h}_t$.

Let $H\colon [0,1]^2 \times V \to \mathbb{R}$ be an extension of $\overline{H}$ so that $H_t^s|_Y = \overline{H}_t^s \circ \rho$ and $H_t^1 = h_t$ is the chosen extension of $\overline{h}$. Let $\{\Psi_t^s\}_{t\in[0,1]}$ be the contact isotopy of V generated by

$$H_\bullet^s\colon [0,1] \times V \to \mathbb{R}$$

for fixed s. It follows from Lemma 3.2 that $\rho \circ \Psi_t^s|_Y = \Phi_t^s \circ \rho\colon Y \to \overline{V}$ for all s and t. In particular the concatenation of paths

$$\{\Psi_0^{1-u}\}_{u\in[0,1]} \# \{\Psi_u^0\}_{u\in[0,1]} \# \{\Psi_1^u\}_{u\in[0,1]}$$

defines an isotopy $\{\psi^u\}_{u\in[0,1]}$ in $\mathrm{Cont}_0(V)$ such that $\psi^u(Y)=Y$ and $\rho\circ\psi^u|_Y=\mathrm{id}_{\overline{V}}\circ\rho$ since

$$\mathrm{id}_{\overline{V}}=\{\Phi_0^{1-u}\}_{u\in[0,1]}\#\{\Phi_u^0\}_{u\in[0,1]}\#\{\Phi_1^u\}_{u\in[0,1]}\quad\text{in }\mathrm{Cont}_0(\overline{V}).$$

Let $\widetilde{\psi}\in\widetilde{\mathrm{Cont}}_0(V)$ be the element represented by $\{\psi^u\}_{u\in[0,1]}$. By Lemma 3.3 we know $\mu(\widetilde{\psi})=0$ and hence $\mu(\widetilde{\phi}_h)=\mu(\widetilde{\psi})=0$ since $\{\psi^u\}_{u\in[0,1]}$ is homotopic with fixed endpoints to the isotopy $\{\Psi_t^1\}_{t\in[0,1]}=\{\phi_h^t\}_{t\in[0,1]}$. □

3.2 Proof of Theorem 1.8

Let $\mathcal{P}\mathrm{Cont}_0(\overline{V})$ denote the group of contact isotopies of $(\overline{V},\overline{\xi})$ based at the identity with time-wise composition as the product: $\{\phi_{\overline{h}}^t\}*\{\phi_{\overline{k}}^t\}=\{\phi_{\overline{h}}^t\circ\phi_{\overline{k}}^t\}$. Passing to homotopy classes of isotopies with fixed endpoints is a group homomorphism $\mathcal{P}\mathrm{Cont}_0(\overline{V})\to\widetilde{\mathrm{Cont}}_0(\overline{V})$ whose kernel consists of contractible loops based at the identity.

Proof of Theorem 1.8 We will break the proof of Theorem 1.8 into a few steps.

Independence of choice of extension We will first prove

$$\overline{\mu}\colon\mathcal{P}\mathrm{Cont}_0(\overline{V})\to\mathbb{R}\quad\text{defined by}\quad\overline{\mu}(\{\phi_{\overline{h}}^t\})=\mu(\widetilde{\phi}_h),\tag{3-3}$$

where $h\in C^\infty([0,1]\times V)$ is any extension of $\overline{h}\in C^\infty([0,1]\times\overline{V})$ is well defined. So let h and k both be extensions of $\overline{h}$. For any positive integer m by Lemma 3.2(iii) it follows that

$$\rho\circ(\phi_h^t)^m\circ(\phi_k^t)^{-m}|_Y=(\phi_{\overline{h}}^t)^m\circ(\phi_{\overline{k}}^t)^{-m}\circ\rho=\rho$$

and hence $\mu(\widetilde{\phi}_h^m\widetilde{\phi}_k^{-m})=0$ by Lemma 3.3. Using that μ is a homogeneous quasimorphism (1-1) we have

$$\big|\mu(\widetilde{\phi}_h)-\mu(\widetilde{\phi}_k)\big|=\frac{1}{m}\big|\mu(\widetilde{\phi}_h^m)-\mu(\widetilde{\phi}_k^m)\big|\le\frac{1}{m}(D+\mu(\widetilde{\phi}_h^m\widetilde{\phi}_k^{-m}))=\frac{D}{m}$$

and taking the limit as $m\to\infty$ shows $\mu(\widetilde{\phi}_h)=\mu(\widetilde{\phi}_k)$.

Homogeneous quasimorphism on $\mathcal{P}\mathrm{Cont}_0(\overline{V})$ We will first show that (3-3) defines a quasimorphism. Let $g,h,k\in C^\infty([0,1]\times V)$ be extensions of $\overline{g},\overline{h},\overline{k}\in C^\infty([0,1]\times\overline{V})$ where $\overline{g}$ generates the product of $\overline{h}$ and $\overline{k}$, that is, $\phi_{\overline{g}}^t=\phi_{\overline{h}}^t\circ\phi_{\overline{k}}^t$. By Lemma 3.2(iii) we have

$$\rho\circ(\phi_g^t)^{-1}\circ\phi_h^t\circ\phi_k^t|_Y=(\phi_{\overline{g}}^t)^{-1}\circ\phi_{\overline{h}}^t\circ\phi_{\overline{k}}^t\circ\rho=\rho$$

so by Lemma 3.3 it follows $\mu(\widetilde{\phi}_g^{-1}\widetilde{\phi}_h\widetilde{\phi}_k)=0$. If D is as in (1-1) for μ, it follows that

$$\big|\overline{\mu}(\{\phi_{\overline{h}}^t\}*\{\phi_{\overline{k}}^t\})-\overline{\mu}(\{\phi_{\overline{h}}^t\})-\overline{\mu}(\{\phi_{\overline{k}}^t\})\big|=\big|\mu(\widetilde{\phi}_g)-\mu(\widetilde{\phi}_h)-\mu(\widetilde{\phi}_k)\big|\le 2D$$

and therefore $\overline{\mu}$ in (3-3) is a quasimorphism.

We will now show that $\overline{\mu}$ is homogeneous. If $h, g^{(m)} \in C^\infty([0,1]\times V)$ are extensions of $\overline{h}, \overline{g} \in C^\infty([0,1]\times \overline{V})$ where $\phi^t_{\overline{g}} = (\phi^t_{\overline{h}})^m$ for an integer $m \in \mathbb{Z}$, then again Lemma 3.3 shows

$$\mu(\widetilde{\phi}^{-1}_{g^{(m)}}\widetilde{\phi}^m_h) = 0.$$

It follows that

$$\left|\overline{\mu}(\{\phi^t_{\overline{h}}\}^m) - m\overline{\mu}(\{\phi^t_{\overline{h}}\})\right| = \left|\mu(\widetilde{\phi}_{g^{(m)}}) - \mu(\widetilde{\phi}^m_h)\right| \le D + \left|\mu(\widetilde{\phi}^{-1}_{g^{(m)}}\widetilde{\phi}^m_h)\right| = D. \tag{3-4}$$

Dividing (3-4) by m and taking $m \to \infty$ shows that

$$\lim_{m\to\infty} \frac{\overline{\mu}(\{\phi^t_{\overline{h}}\}^m)}{m} = \overline{\mu}(\{\phi^t_{\overline{h}}\}), \tag{3-5}$$

and it follows from (3-5) that $\overline{\mu}$ is homogeneous; see for instance [25, Lemma 2.21].

Descent to a quasimorphism on $\widetilde{\mathrm{Cont}}_0(\overline{V})$ Since $\overline{\mu}\colon \mathcal{P}\mathrm{Cont}_0(\overline{V}) \to \mathbb{R}$ vanishes on the kernel of the map $\mathcal{P}\mathrm{Cont}_0(\overline{V}) \to \widetilde{\mathrm{Cont}}_0(\overline{V})$ by Lemma 3.4, it follows that $\overline{\mu}$ in (3-3) descends to a well-defined homogeneous quasimorphism $\overline{\mu}\colon \widetilde{\mathrm{Cont}}_0(\overline{V}) \to \mathbb{R}$ by [20, Lemma 3.2].

Nonzero To see the quasimorphism $\overline{\mu}\colon \widetilde{\mathrm{Cont}}_0(\overline{V}) \to \mathbb{R}$ is not zero, let $\overline{h} \in C^\infty(\overline{V})$ be any positive contact Hamiltonian and pick $h \in C^\infty(V)$ to be a positive extension. Since V is μ–superheavy by Proposition 1.10(iv) it follows that $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) = \mu(\widetilde{\phi}_h) > 0$.

Monotone If $\overline{h} \le \overline{k}$, then one can pick extensions h and k such that $h \le k$. Since μ is monotone it follows $\mu(\widetilde{\phi}_h) \le \mu(\widetilde{\phi}_k)$ and therefore $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) \le \overline{\mu}(\widetilde{\phi}_{\overline{k}})$ by definition.

Vanishing Assume now that μ has the vanishing property. Let $U \subset \overline{V}$ be an open set that is displaceable by an element of $\mathrm{Cont}_0(\overline{V})$, then it follows from Lemma 3.2(iii) that an open neighborhood $N \subset V$ of $\rho^{-1}(U) \subset Y$ is displaceable by an element of $\mathrm{Cont}_0(V)$. Now if $\overline{h} \in C^\infty([0,1]\times\overline{V})$ has $\mathrm{supp}(\overline{h}) \subset [0,1]\times U$, then there is an extension $h \in C^\infty([0,1]\times V)$ of $\overline{h}$ with $\mathrm{supp}(h) \subset [0,1]\times N$. Since μ has the vanishing property it follows $\mu(\widetilde{\phi}_h) = 0$ and so by definition $\overline{\mu}(\widetilde{\phi}_{\overline{h}}) = 0$ as well. Therefore $\overline{\mu}$ also has the vanishing property.

C^0–continuity Assume that μ is C^0–continuous and let $\overline{h}^{(n)} \in C^\infty([0,1]\times\overline{V})$ be a sequence of contact Hamiltonians C^0–converging to $\overline{h} \in C^\infty([0,1]\times\overline{V})$. Then we can pick extensions $h^{(n)}$ and h in $C^\infty([0,1]\times V)$ with C^0–convergence $h^{(n)} \to h$. Since μ is C^0–continuous, we have $\mu(\widetilde{\phi}_{h^{(n)}}) \to \mu(\widetilde{\phi}_h)$ and hence $\overline{\mu}(\widetilde{\phi}_{\overline{h}^{(n)}}) \to \overline{\mu}(\widetilde{\phi}_h)$ as well. Therefore $\overline{\mu}$ is C^0–continuous. □

4 Proof of rigidity and vanishing results

In this section we will present the remaining proofs.

4.1 Proof of rigidity results from Section 1.2

We will first prove the following lemma that shows that there is no difference between positive and negative in terms of defining a subset to be superheavy with respect to a quasimorphism on $\widetilde{\mathrm{Cont}}_0(V)$.

Lemma 4.1 *If $\mu\colon \widetilde{\mathrm{Cont}}_0(V,\xi) \to \mathbb{R}$ is a monotone quasimorphism and $Y \subset V$ is a closed subset, then Y is μ–superheavy if and only if $\mu(\widetilde{\phi}_h) < 0$ for all autonomous contact Hamiltonians where $h|_Y < 0$.*

Proof If h is autonomous, then $\widetilde{\phi}_h^{-1}$ is generated by the contact Hamiltonian $-h$ and therefore $\mu(\widetilde{\phi}_{-h}) = -\mu(\widetilde{\phi}_h)$ since μ is homogeneous. The lemma now follows from the definition of Y being superheavy from Definition 1.9. □

Let us now prove Proposition 1.10 and Theorem 1.11, detailing the basic properties of superheavy and subheavy sets.

Proof of Proposition 1.10 To prove item (i) recall that any two contact forms α and α' for ξ differ by multiplication by a positive function $f\colon V \to \mathbb{R}\colon \alpha' = f\alpha$. If $h \in C^\infty([0,1]\times V)$ is the contact Hamiltonian associated to the contact isotopy $\{\phi^t\}_{t\in[0,1]}$ using the form α, then $f\cdot h$ is the contact Hamiltonian associated to the same isotopy using the form α'. Hence $h|_{[0,1]\times Y} > 0$ if and only if $f\cdot h|_{[0,1]\times Y} > 0$, so μ–superheaviness is independent of contact form and likewise for μ–subheaviness.

Item (ii) is immediate since if $h|_Y > 0$, then $h|_Z > 0$ and hence $\mu(\widetilde{\phi}_h) > 0$ since Z is μ–superheavy. The argument for μ–subheaviness is analogous.

To prove (iii), first note homogeneous quasimorphisms are conjugation-invariant so

$$\mu(\widetilde{\phi}_h) = \mu(\psi^{-1}\widetilde{\phi}_h\psi)$$

for any $h \in C^\infty(V)$ and $\psi \in \mathrm{Cont}_0(V)$, where we use the natural action of $\mathrm{Cont}_0(V)$ on $\widetilde{\mathrm{Cont}}_0(V)$ by conjugation. Furthermore $\psi^{-1}\widetilde{\phi}_h\psi = \widetilde{\phi}_g$, where

$$g := \alpha(d\psi^{-1}(X_h)\circ\psi) = (f\cdot h)\circ\psi$$

for some positive function $f \in C^\infty(V)$, so therefore $g|_Y > 0$ if and only if $h|_{\psi(Y)} > 0$. Hence Y is μ–superheavy if and only if $\psi(Y)$ is μ–superheavy and likewise for μ–subheavy.

For item (iv), recall that we assume all quasimorphisms are homogeneous and nonzero. Suppose there is an $h \in C^\infty(V)$ such that $h > 0$ and $\mu(\widetilde{\phi}_h) = 0$. For any integer m, $\mu(\widetilde{\phi}_{mh}) = \mu(\widetilde{\phi}_h^m) = 0$ since h is autonomous. Since for any $k \in C^\infty([0,1] \times V)$ there is a positive integer m such that $-mh \le k \le mh$, it follows from the monotonicity of μ that $\mu(\widetilde{\phi}_k) = 0$. Therefore $\mu = 0$, which is a contradiction. $\square$

Proof of Theorem 1.11 For item (i), let Y be μ–superheavy and h be an autonomous contact Hamiltonian where $h|_Y = 0$. Recall for any $\phi \in \mathrm{Cont}_0(V)$ that $\phi^*\alpha = k\alpha$, where $k\colon V \to \mathbb{R}$ is a positive function. It follows then for any positive integer m and real number $\epsilon > 0$ that

$$g_t := \alpha(X_{mh} + d\phi^t_{mh}(m\epsilon R_\alpha) \circ (\phi^t_{mh})^{-1}),$$

which is the contact Hamiltonian so that $\phi^t_g = \phi^t_{mh}\phi^t_{m\epsilon}$ for all $t \in [0,1]$, satisfies $g_t|_Y > \delta$ for some $\delta > 0$. Using that Y is μ–superheavy and μ is monotone we have

$$\mu(\widetilde{\phi}_{mh}\widetilde{\phi}_{m\epsilon}) = \mu(\widetilde{\phi}_g) > 0.$$

Since h is autonomous, $\widetilde{\phi}_{mh} = \widetilde{\phi}_h^m$, and using μ is a homogeneous quasimorphism (1-1) we get

$$m\mu(\widetilde{\phi}_h) = \mu(\widetilde{\phi}_{mh}) \ge \mu(\widetilde{\phi}_{mh}\widetilde{\phi}_{m\epsilon}) + \mu(\widetilde{\phi}_{m\epsilon}^{-1}) - D > \mu(\widetilde{\phi}_{m\epsilon}^{-1}) - D = -m\epsilon\mu(\widetilde{\phi}_1) - D.$$

By dividing through by m and taking the limit as $m \to \infty$ gives $\mu(\widetilde{\phi}_h) > -\epsilon\mu(\widetilde{\phi}_1)$ for all $\epsilon > 0$, and therefore taking $\epsilon \to 0$ gives

$$\mu(\widetilde{\phi}_h) \ge 0.$$

One proves $\mu(\widetilde{\phi}_h) \le 0$ similarly using Lemma 4.1. Therefore $\mu(\widetilde{\phi}_h) = 0$ and hence Y is μ–subheavy.

To prove item (ii), suppose that Y and Z are disjoint and pick a contact Hamiltonian h so that $h|_Y > 0$ and $h|_Z = 0$, which is possible since Y and Z are closed subsets. This leads to a contradiction since by the definitions of superheavy and subheavy we have $\mu(\phi_h) > 0$ and $\mu(\phi_h) = 0$. $\square$

Next up is the proof of Theorem 1.17 about the existence of nondisplaceable pre-Lagrangians in prequantizations of toric symplectic manifolds.

Proof of Theorem 1.17 Let $P\colon M^{2n} \to \Delta \subset \mathbb{R}^n$ be a moment map for the toric structure on M, let $\pi\colon (V,\alpha) \to (M,\omega)$ be the prequantization map and let

$$\widehat{P} = P \circ \pi\colon V \to \Delta.$$

Every fiber of $\widehat{P}$ is either a pre-Lagrangian torus or a sits over a strictly isotropic torus in M and the latter are always displaceable (see Laudenbach [65]), so it suffices to show not every fiber of $\widehat{P}$ is displaceable.

Suppose every fiber of $\widehat{P}$ is displaceable. Then we can take an open cover $\{U_j\}_{j=1}^d$ of Δ such that each $\widehat{P}^{-1}(U_j) \subset V$ is displaceable. Since the coordinate functions of P commute, for any two functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ the contactomorphisms $\widetilde{\phi}_{\widehat{P}^* f}$ and $\widetilde{\phi}_{\widehat{P}^* g}$ commute and $\widetilde{\phi}_{\widehat{P}^*(f+g)} = \widetilde{\phi}_{\widehat{P}^* f} \widetilde{\phi}_{\widehat{P}^* g}$. In particular if $\{f_j\}$ is a partition of unity subordinate to $\{U_j\}$, then

$$\mu(\widetilde{\phi}_1) = \mu(\widetilde{\phi}_{\widehat{P}^* f_1} + \cdots + \widetilde{\phi}_{\widehat{P}^* f_d}) = \sum_{j=1}^{d} \mu(\widetilde{\phi}_{\widehat{P}^* f_j}) = 0$$

since homogeneous quasimorphisms are homomorphisms when restricted to abelian subgroups and also that $\mu(\widetilde{\phi}_{\widehat{P}^* f_j}) = 0$ by the vanishing property. However $\mu(\widetilde{\phi}_1) > 0$, so we have a contradiction. □

Remark 4.2 The proof of Theorem 1.17 also shows if there is monotone quasimorphism $\mu\colon \widetilde{\mathrm{Cont}}_0(V, \xi) \to \mathbb{R}$ with the vanishing property and (V, ξ) is completely integrable contact manifold, in the sense of Khesin and Tabachnikov [59], then at least one of the pre-Lagrangian fibers is nondisplaceable.

Let us now prove Proposition 1.13 which states that if a subheavy subset $Y \subset V$ is preserved by a positive contact vector field, then it is μ–superheavy.

Proof of Proposition 1.13 We will assume that Y is invariant under the flow for the Reeb vector field R_α, since any positive contact vector field is the Reeb vector for some contact form; see McDuff and Salamon [68, Chapter 3.4]. Given $h \in C^\infty(V)$ such that $h|_Y > 0$, since Y is closed we have $h|_Y \geq c$ for some positive $c \in \mathbb{R}$. Let m be a positive integer and note that $\phi_g^t = \phi_{-mc}^t \phi_{mh}^t$, where

$$g_t := \alpha(-mcR_\alpha + d\phi_{-mc}^t(mX_h) \circ (\phi_{-mc}^t)^{-1}) = m(-c + h \circ \phi_{mc}^t).$$

Since $\phi_{mc}^t = \phi_{mcR_\alpha}^t$ is a reparametrization of the Reeb flow, which preserves Y, it follows that $g_t|_Y \geq 0$ and hence $\mu(\widetilde{\phi}_{-mc}\widetilde{\phi}_{mh}) = \mu(\widetilde{\phi}_g) \geq 0$, since μ is monotone and Y is μ–subheavy. Since h is autonomous it follows $\widetilde{\phi}_{mh} = \widetilde{\phi}_h^m$ and because μ is a homogeneous quasimorphism we have

$$m\mu(\widetilde{\phi}_h) = \mu(\widetilde{\phi}_{mh}) \geq \mu(\widetilde{\phi}_{-mc}\widetilde{\phi}_{mh}) + \mu(\widetilde{\phi}_{mc}) - D \geq \mu(\widetilde{\phi}_{mc}) - D = m\mu(\widetilde{\phi}_c) - D.$$

By dividing through by m and taking the limit as $m \to \infty$ gives $\mu(\widetilde{\phi}_h) \geq \mu(\widetilde{\phi}_c)$ and $\mu(\widetilde{\phi}_c) > 0$ since V is μ–superheavy by Proposition 1.10(iv). □

4.2 Proof of results from Section 1.3

Here we will prove the results in Section 1.3 about the relation between quasimorphisms on $\widetilde{\mathrm{Cont}}_0(V)$ and $\widetilde{\mathrm{Ham}}(M)$ when $\pi\colon (V,\alpha) \to (M,\omega)$ is a prequantization. Before proving Theorem 1.19 we need the following lemma.

Lemma 4.3 *If $\pi\colon (V,\alpha)\to(M,\omega)$ is a prequantization and $\mu\colon \widetilde{\mathrm{Cont}}_0(V)\to\mathbb{R}$ is a monotone quasimorphism, then*

$$\mu(\widetilde{\phi}_{c+\pi^*H}) = \left(\int_0^1 c(t)\,dt\right)\mu(\widetilde{\phi}_1) + \mu(\widetilde{\phi}_{\pi^*H})$$

for all smooth functions $H\colon [0,1]\times M\to\mathbb{R}$ and $c\colon [0,1]\to\mathbb{R}$.

Proof By using the contact Poisson bracket (1-3), or just the definitions, one sees that $\widetilde{\phi}_c$ and $\widetilde{\phi}_{\pi^*H}$ commute in $\widetilde{\mathrm{Cont}}_0(V)$ and $\widetilde{\phi}_{c+\pi^*H} = \widetilde{\phi}_c\widetilde{\phi}_{\pi^*H}$. Therefore since homogeneous quasimorphisms are homomorphisms on abelian subgroups,

$$\mu(\widetilde{\phi}_{c+\pi^*H}) = \mu(\widetilde{\phi}_c) + \mu(\widetilde{\phi}_{\pi^*H}),$$

and hence it suffices to prove that $\mu(\widetilde{\phi}_c) = \left(\int_0^1 c(t)\,dt\right)\mu(\widetilde{\phi}_1)$.

Since $\widetilde{\phi}_\kappa = \widetilde{\phi}_c$ via a time-reparametrization where $\kappa = \int_0^1 c(t)\,dt$, this reduces to proving $\mu(\widetilde{\phi}_\kappa) = \kappa\mu(\widetilde{\phi}_1)$ for all real numbers $\kappa\in\mathbb{R}$. For any integer $m\in\mathbb{Z}$, this holds since μ is homogeneous and $\widetilde{\phi}_m = \widetilde{\phi}_1^m$. This extends to rational numbers and since μ is monotone it then holds for all real scalars. □

Proof of Theorem 1.19 Since $\pi^*\colon \widetilde{\mathrm{Ham}}(M)\to\widetilde{\mathrm{Cont}}_0(V)$ from (1-17) is a homomorphism it is clear that μ_M is a quasimorphism. For stability let $c(t) := \min_M(H_t - G_t)$, then by monotonicity and Lemma 4.3 we have

$$\left(\int_0^1 c(t)\,dt\right)\mu(\widetilde{\phi}_1) + \mu(\widetilde{\phi}_{\pi^*G}) = \mu(\widetilde{\phi}_{c+\pi^*G}) \le \mu(\widetilde{\phi}_{\pi^*H})$$

and hence

$$\left(\int_0^1 \min_M(H_t - G_t)\,dt\right)\mu(\widetilde{\phi}_1) \le \mu(\widetilde{\phi}_{\pi^*H}) - \mu(\widetilde{\phi}_{\pi^*G}).$$

After translating to the definition of μ_M in (1-18) this is the left-hand part of the stability condition (1-14). The right-hand side is proved analogously.

Lemma 4.3 shows that the formulas for ζ_{μ_M} in (1-15) and Theorem 1.19 are equal. It follows from the formula in Theorem 1.19 that if μ has the vanishing property, then so does ζ_{μ_M}. This is because if $X\subset M$ is displaceable by an element of $\mathrm{Ham}(M)$, then $\pi^{-1}(X) \subset V$ is displaceable by an element of $\mathrm{Cont}_0(V)$. Going back to the quasimorphism μ_M, it follows from [21, Proposition 1.7] that μ_M has the Calabi property if the associated quasistate ζ_{μ_M} has the vanishing property. □

Proof of Proposition 1.20 For item (i), it is enough to show that if $H \in C^\infty(M)$ is such that $H|_{\pi(Y)} = 0$, then $\zeta_{\mu_M}(H) = 0$. If $H|_{\pi(Y)} = 0$, then $\pi^* H|_Y = 0$ and $\mu(\widetilde{\phi}_{\pi^* H}) = 0$ by the definition of Y being μ–subheavy. It then follows from Theorem 1.19 that $\zeta_{\mu_M}(H) = 0$.

For item (ii), let $Y = \pi^{-1}(X)$ and let $h \in C^\infty(V)$ be such that $h|_Y > 0$. There is $H \in C^\infty(M)$ with $\pi^* H \le h$ and $H|_X > 0$. From the monotonicity of μ and Theorem 1.19 we have

$$\mu(\widetilde{\phi}_h) \ge \mu(\widetilde{\phi}_{\pi^* H}) = \mu(\widetilde{\phi}_1)\zeta_{\mu_M}(H)$$

and therefore we are done since $\zeta_{\mu_M}(H) \ge \min_X H > 0$ by the definition of ζ_{μ_M}–superheavy and since $\mu(\widetilde{\phi}_1) > 0$ by Proposition 1.10(iv). □

4.3 Proofs about Givental's quasimorphism

4.3.1 A brief summary of Givental's quasimorphism Recall that a point $v \in (V, \xi)$ in a contact manifold is a *discriminant point* for a contactomorphism $\phi \in \mathrm{Cont}(V, \xi)$ if

$$\phi(v) = v \quad \text{and} \quad (\phi^* \alpha)_v = \alpha_v \tag{4-1}$$

for some (and hence every) contact form α and the *discriminant* of $\mathrm{Cont}_0(V, \xi)$ is

$$\Sigma(V, \xi) := \{\phi \in \mathrm{Cont}_0(V, \xi) \mid \phi \text{ has at least one discriminant point}\}.$$

A C^∞–generic contactomorphism has no discriminant points. Indeed if v is a discriminant point of ϕ, then the image of $d\phi_v - \mathrm{id}_{T_v V}$ is contained in ξ_v and hence $d\phi_v - \mathrm{id}_{T_v V}$ has a nontrivial kernel. This means v is a degenerate fixed point and it is a standard fact that C^∞–generic contactomorphisms do not have degenerate fixed points (see Hofer and Salamon [57, Theorem 3.1] for a proof in the Hamiltonian case). In fact any $\phi \in \mathrm{Cont}_0(V)$ on the discriminant $\Sigma(V)$ can be perturbed off $\Sigma(V)$ via the Reeb flow, but we will not include the proof since this is not necessary for what follows.

In [54] Givental showed how to coorient the discriminant $\Sigma \subset \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ using generating functions. Given a smooth path $\gamma\colon [0, \tau] \to \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ with endpoints not on Σ, the coorientation gives a well-defined intersection index between γ and Σ, denoted

$$\mu^G(\gamma) \in \mathbb{Z}$$

which Givental called the *nonlinear Maslov index*. From the intersection viewpoint, Givental specified [54, Section 9] conventions so that μ^G is defined for all paths of contactomorphisms. Alternatively, as noted by Colin and Sandon [31, Section 7], the nonlinear Maslov index can be defined purely in terms of generating families, leading to a uniform definition of the nonlinear Maslov index for any smooth path of contactomorphisms of $\mathbb{R}\mathrm{P}^{2d-1}$. Here are some key properties of the nonlinear Maslov index:

(i) Given two paths $\gamma_i\colon [0,\tau_i] \to \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ with $\gamma_0(\tau_0) = \gamma_1(0)$, one has

$$\mu^G(\gamma_0) + \mu^G(\gamma_1) = \mu^G(\gamma_0 * \gamma_1), \tag{4-2}$$

where $\gamma_0 * \gamma_1\colon [0, \tau_0 + \tau_1] \to \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ is their concatenation.

(ii) For any path γ in $\mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ and element $\phi \in \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$,

$$|\mu^G(\gamma\phi) - \mu^G(\gamma)| \le 2d, \tag{4-3}$$

where $\gamma\phi$ is the path defined by $t \mapsto \gamma(t)\phi$.

(iii) If a path γ in $\mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ is disjoint from the discriminant, then

$$\mu^G(\gamma) = 0. \tag{4-4}$$

(iv) The nonlinear Maslov index $\mu^G(\gamma)$ is invariant under homotopies of γ with fixed endpoints.

The first item follows from the construction as an intersection index, the second item is [54, Theorem 9.1(a)], and the final two properties are established by both Givental [54, Section 9] and Colin and Sandon [31, Section 7].

If $\mathcal{P}\mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ denotes the space of contact isotopies $\{\phi^t\}_{t\in[0,1]}$ with $\phi^0 = \mathrm{id}$, then one defines the *asymptotic nonlinear Maslov index* to be

$$\mu_{\mathrm{Giv}}(\{\phi^t\}_{t\in[0,1]}) := \lim_{\tau\to\infty} \frac{\mu^G(\{\phi^t\}_{t\in[0,\tau]})}{\tau}, \tag{4-5}$$

where $\{\phi^t\}_{t\in[0,\tau]}$ is given by concatenation so $\phi^{k+s} := \phi^s(\phi^1)^k$ for $s \in [0,1]$ and $k \in \mathbb{N}$. Since μ^G is invariant under homotopies with fixed endpoints, the map in (4-5) descends to a map

$$\mu_{\mathrm{Giv}}\colon \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1}) \to \mathbb{R}$$

and this is the definition of Givental's quasimorphism from (1-5). As a special case of (4-5) we have

$$\mu_{\mathrm{Giv}}(\widetilde{\phi}) = \lim_{m\to\infty} \frac{\mu^G(\widetilde{\phi}^m)}{m} \tag{4-6}$$

for $\widetilde{\phi} \in \widetilde{\mathrm{Cont}}_0(\mathbb{R}\mathrm{P}^{2d-1})$ and hence μ_{Giv} is homogeneous: $\mu_{\mathrm{Giv}}(\widetilde{\phi}^m) = m\mu_{\mathrm{Giv}}(\widetilde{\phi})$.

4.3.2 A subheavy Legendrian

Proof of Lemma 1.23 It suffices to prove $\mathbb{R}\mathrm{P}^{d-1}_L \subset \mathbb{R}\mathrm{P}^{2d-1}$ is μ_{Giv}–subheavy since it is preserved by radial projection (1-23).

If h is an autonomous contact Hamiltonian that vanishes on $\mathbb{R}\mathrm{P}^{d-1}_L$, then X_h is always tangent to $\mathbb{R}\mathrm{P}^{d-1}_L$ since it is Legendrian. Therefore the Legendrian nonlinear Maslov index $\mu(\lambda)$ from [54, Section 9] of the constant path of Legendrians $\lambda := \{\phi^t_h(\mathbb{R}\mathrm{P}^{d-1}_L)\}_{t\in[0,\tau]}$ vanishes. By the definition of μ_{Giv} in (4-5) and [54, Section 9, Corollary 2] we know that

$$\mu_{\mathrm{Giv}}(\widetilde{\phi}_h) = \lim_{\tau\to\infty} \frac{\mu^G(\{\phi^t_h\}_{t\in[0,\tau]})}{\tau} = \lim_{\tau\to\infty} \frac{\mu(\{\phi^t_h(\mathbb{R}\mathrm{P}^{d-1}_L)\}_{t\in[0,\tau]})}{\tau}$$

so $\mu_{\mathrm{Giv}}(\widetilde{\phi}_h) = 0$ and therefore $\mathbb{R}\mathrm{P}^{d-1}_L$ is μ_{Giv}–subheavy. □

4.3.3 Proving properties of Givental's quasimorphism in Proposition 1.2

Proof of Proposition 1.2 (Monotonicity) By [54, Theorem 9.1(b)], or equivalently, by [31, Lemma 7.6], we know that $\mu^G(\widetilde{\phi}) \geq 0$ if $\widetilde{\phi} \succeq \mathrm{id}$, so it follows from (4-6) that

$$0 \leq \mu_{\mathrm{Giv}}(\widetilde{\phi}) \quad \text{if } \mathrm{id} \preceq \widetilde{\phi}.$$

Now if $\widetilde{\phi} \preceq \widetilde{\psi}$, then $\mathrm{id} \preceq \widetilde{\psi}^m \circ \widetilde{\phi}^{-m}$ and hence $\mu_{\mathrm{Giv}}(\widetilde{\psi}^m \circ \widetilde{\phi}^{-m}) \geq 0$. Using this and that μ_{Giv} is a homogeneous quasimorphism, we get

$$m\mu_{\mathrm{Giv}}(\widetilde{\psi}) - m\mu_{\mathrm{Giv}}(\widetilde{\phi}) = \mu_{\mathrm{Giv}}(\widetilde{\psi}^m) - \mu_{\mathrm{Giv}}(\widetilde{\phi}^m) \geq \mu_{\mathrm{Giv}}(\widetilde{\psi}^m \circ \widetilde{\phi}^{-m}) - D \geq -D.$$

Dividing by m and taking the limit $m \to \infty$, gives $\mu_{\mathrm{Giv}}(\widetilde{\phi}) \leq \mu_{\mathrm{Giv}}(\widetilde{\psi})$ and hence μ_{Giv} is monotone. □

Proof of Proposition 1.2 (C^0–continuity) Givental proved in [54, Corollary 3, Section 9] that μ_{Giv} is C^0–continuous for time-independent contact Hamiltonians and as he explained to us the proof generalizes to time-dependent contact Hamiltonians in the following way.

Suppose for smooth contact Hamiltonians $h^{(n)}, h \in C^\infty([0,1] \times \mathbb{R}\mathrm{P}^{2d-1})$ we have C^0–convergence $h^{(n)} \to h$. For a given $\epsilon > 0$, pick an integer $m > 0$ such that $6d/m < \epsilon$ and by [54, Theorem 9.1(c)] we know that if n is sufficiently large, then

$$|\mu^G(\{\phi^t_{h^{(n)}}\}_{t\in[0,m]}) - \mu^G(\{\phi^t_h\}_{t\in[0,m]})| \leq 2d.$$

By (4-2) and (4-3), for any two integers $m, N > 0$,

$$|\mu^G(\{\phi^t_k\}_{t\in[0,Nm]}) - N\mu^G(\{\phi^t_k\}_{t\in[0,m]})| \leq 2dN$$

for every $k \in C^\infty([0,1] \times \mathbb{R}\mathrm{P}^{2d-1})$, which applied to the previous inequality gives

$$|\mu^G(\{\phi^t_{h^{(n)}}\}_{t\in[0,Nm]}) - \mu^G(\{\phi^t_h\}_{t\in[0,Nm]})| \le 6dN$$

if n is sufficiently large. Dividing by Nm and taking the limit as $N \to \infty$ gives

$$|\mu_{\mathrm{Giv}}(\widetilde{\phi}_{h^{(n)}}) - \mu_{\mathrm{Giv}}(\widetilde{\phi}_h)| \le \frac{6d}{m} < \epsilon$$

if n is sufficiently large and therefore $\lim_{n\to\infty} \mu_{\mathrm{Giv}}(\widetilde{\phi}_{h^{(n)}}) = \mu_{\mathrm{Giv}}(\widetilde{\phi})$. □

Proof of Proposition 1.2 (Vanishing property) For an open $U \subset \mathbb{R}\mathrm{P}^{2d-1}$ suppose there is a $\psi \in \mathrm{Cont}_0(\mathbb{R}\mathrm{P}^{2d-1})$ such that $\psi(U) \cap U = \varnothing$ and without loss of generality we may assume ψ has no discriminant points. By (4-2) we know that

$$|\mu^G(\{\phi^t_h\psi\}_{t\in[0,\tau]}) - \mu^G(\{\phi^t_h\}_{t\in[0,\tau]})| \le 2d,$$

so if $\mu^G(\{\phi^t_h\psi\}_{t\in[0,\tau]}) = 0$ for all $\tau \ge 0$, then it will follow from (4-5) that

$$\mu_{\mathrm{Giv}}(\widetilde{\phi}_h) = \lim_{\tau\to\infty} \frac{\mu^G(\{\phi^t_h\}_{t\in[0,\tau]})}{\tau} = 0.$$

Therefore by (4-4) it remains to prove that $\phi^t_h\psi$ has no discriminant points for all $t \ge 0$. Assume p is a discriminant point for some $\phi^t_h\psi$. If $p \in U$, then $\psi(p) = (\phi^t_h)^{-1}(p) \in U$ but this contradicts that $\psi(U) \cap \overline{U} = \varnothing$. If $p \notin U$, then $\psi(p) = (\phi^t_h)^{-1}(p) = p$ so p is a fixed point of ψ and also a discriminant point of ψ, but we assumed they did not exist. □

References

[1] **J F Aarnes**, *Quasistates and quasimeasures*, Adv. Math. 86 (1991) 41–67 MR1097027

[2] **M Abreu**, **M S Borman**, **D McDuff**, *Displacing Lagrangian toric fibers by extended probes*, Algebr. Geom. Topol. 14 (2014) 687–752 MR3159967

[3] **M Abreu**, **L Macarini**, *Contact homology of good toric contact manifolds*, Compos. Math. 148 (2012) 304–334 MR2881318

[4] **M Abreu**, **L Macarini**, *Remarks on Lagrangian intersections in toric manifolds*, Trans. Amer. Math. Soc. 365 (2013) 3851–3875 MR3042606

[5] **P Albers**, **U Frauenfelder**, *Erratum to: A variational approach to Givental's nonlinear Maslov index*, Geom. Funct. Anal. 23 (2013) 482–499 MR3037906

[6] **P Albers**, **W J Merry**, *Orderability, contact nonsqueezing and Rabinowitz Floer homology* `arXiv:1302.6576`

[7] **G Alston**, **L Amorim**, *Floer cohomology of torus fibers and real Lagrangians in Fano toric manifolds*, Int. Math. Res. Not. 2012 (2012) 2751–2793 MR2942709

[8] **A Banyaga**, *The structure of classical diffeomorphism groups*, Math. and its Applications 400, Kluwer Academic Publ. Group (1997) MR1445290

[9] **V V Batyrev**, *Toric Fano 3–folds*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981) 704–717, 927 MR631434

[10] **V V Batyrev**, *On the classification of toric Fano 4–folds*, J. Math. Sci. (New York) 94 (1999) 1021–1050 MR1703904

[11] **C Bavard**, *Longueur stable des commutateurs*, Enseign. Math. 37 (1991) 109–150 MR1115747

[12] **G Ben Simon**, *The nonlinear Maslov index and the Calabi homomorphism*, Commun. Contemp. Math. 9 (2007) 769–780 MR2372458

[13] **G Ben Simon**, *The geometry of partial order on contact transformations of prequantization manifolds*, from: "Arithmetic and geometry around quantization", (Ö Ceyhan, Y I Manin, M Marcolli, editors), Progr. Math. 279, Birkhäuser, Basel (2010) 37–64 MR2656942

[14] **G Ben Simon**, **T Hartnick**, *Quasitotal orders and translation numbers* `arXiv:1106.6307`

[15] **G Ben Simon**, **T Hartnick**, *Reconstructing quasimorphisms from associated partial orders and a question of Polterovich*, Comment. Math. Helv. 87 (2012) 705–725 MR2980524

[16] **P Biran**, **O Cornea**, *A Lagrangian quantum homology*, from: "New perspectives and challenges in symplectic field theory", (M Abreu, F Lalonde, L Polterovich, editors), CRM Proc. Lecture Notes 49, Amer. Math. Soc. (2009) 1–44 MR2555932

[17] **P Biran**, **O Cornea**, *Rigidity and uniruling for Lagrangian submanifolds*, Geom. Topol. 13 (2009) 2881–2989 MR2546618

[18] **P Biran**, **M Entov**, **L Polterovich**, *Calabi quasimorphisms for the symplectic ball*, Commun. Contemp. Math. 6 (2004) 793–802 MR2100764

[19] **W M Boothby**, **H C Wang**, *On contact manifolds*, Ann. of Math. 68 (1958) 721–734 MR0112160

[20] **M S Borman**, *Symplectic reduction of quasimorphisms and quasistates*, J. Symplectic Geom. 10 (2012) 225–246 MR2926996

[21] **M S Borman**, *Quasistates, quasimorphisms and the moment map*, Int. Math. Res. Not. 2013 (2013) 2497–2533 MR3065086

[22] **M Brandenbursky**, *Quasimorphisms and L^p–metrics on groups of volume-preserving diffeomorphisms*, J. Topol. Anal. 4 (2012) 255–270 MR2949242

[23] **M Brandenbursky**, **E Shelukhin**, *On the large-scale geometry of the L^p–metric on the symplectomorphism group of the two-sphere* `arXiv:1304.7037`

[24] **L Buhovsky**, **M Entov**, **L Polterovich**, *Poisson brackets and symplectic invariants*, Selecta Math. 18 (2012) 89–157 MR2891862

[25] **D Calegari**, *scl*, MSJ Memoirs 20, Math. Soc. Japan, Tokyo (2009) MR2527432

[26] **Y V Chekanov**, *Critical points of quasifunctions, and generating families of Legendrian manifolds*, Funktsional. Anal. i Prilozhen. 30 (1996) 56–69, 96 MR1402081 In Russian; translated in Fun. Anal. Appl. 30 (1996) 118–128

[27] **V Chernov**, **S Nemirovski**, *Nonnegative Legendrian isotopy in* ST^*M, Geom. Topol. 14 (2010) 611–626 MR2602847

[28] **C-H Cho**, *Holomorphic discs, spin structures, and Floer cohomology of the Clifford torus*, Int. Math. Res. Not. 2004 (2004) 1803–1843 MR2057871

[29] **C-H Cho**, *Nondisplaceable Lagrangian submanifolds and Floer cohomology with nonunitary line bundle*, J. Geom. Phys. 58 (2008) 1465–1476 MR2463805

[30] **C-H Cho**, **M Poddar**, *Holomorphic orbidiscs and Lagrangian Floer cohomology of symplectic toric orbifolds* (2014) `arXiv:1206.3994v4`

[31] **V Colin**, **S Sandon**, *The discriminant and oscillation lengths for contact and Legendrian isotopies*, to appear in J. Eur. Math. Soc.

[32] **T Delzant**, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988) 315–339 MR984900

[33] **Y Eliashberg**, *New invariants of open symplectic and contact manifolds*, J. Amer. Math. Soc. 4 (1991) 513–520 MR1102580

[34] **Y Eliashberg**, **H Hofer**, **D Salamon**, *Lagrangian intersections in contact geometry*, Geom. Funct. Anal. 5 (1995) 244–269 MR1334868

[35] **Y Eliashberg**, **S S Kim**, **L Polterovich**, *Geometry of contact transformations and domains: orderability versus squeezing*, Geom. Topol. 10 (2006) 1635–1747 MR2284048

[36] **Y Eliashberg**, **L Polterovich**, *Partially ordered groups and geometry of contact transformations*, Geom. Funct. Anal. 10 (2000) 1448–1476 MR1810748

[37] **M Entov**, *Commutator length of symplectomorphisms*, Comment. Math. Helv. 79 (2004) 58–104 MR2031300

[38] **M Entov**, **L Polterovich**, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. 2003 (2003) 1635–1676 MR1979584

[39] **M Entov**, **L Polterovich**, *Quasistates and symplectic intersections*, Comment. Math. Helv. 81 (2006) 75–99 MR2208798

[40] **M Entov**, **L Polterovich**, *Symplectic quasistates and semisimplicity of quantum homology*, from: "Toric topology", (M Harada, Y Karshon, M Masuda, T Panov, editors), Contemp. Math. 460, Amer. Math. Soc. (2008) 47–70 MR2428348

[41] **M Entov**, **L Polterovich**, C^0*–rigidity of the double Poisson bracket*, Int. Math. Res. Not. 2009 (2009) 1134–1158 MR2487493

[42] **M Entov**, **L Polterovich**, *Rigid subsets of symplectic manifolds*, Compos. Math. 145 (2009) 773–826 MR2507748

[43] **M Entov**, **L Polterovich**, *C^0–rigidity of Poisson brackets*, from: "Symplectic topology and measure preserving dynamical systems", (A Fathi, Y-G Oh, C Viterbo, editors), Contemp. Math. 512, Amer. Math. Soc. (2010) 25–32 MR2605312

[44] **M Entov**, **L Polterovich**, **P Py**, *On continuity of quasimorphisms for symplectic maps*, from: "Perspectives in analysis, geometry, and topology", (I Itenberg, B Jöricke, M Passare, editors), Progr. Math. 296, Birkhäuser, Basel (2012) 169–197 MR2884036

[45] **M Entov**, **L Polterovich**, **F Zapolsky**, *Quasimorphisms and the Poisson bracket*, Pure Appl. Math. Q. 3 (2007) 1037–1055 MR2402596

[46] **M Fraser**, **L Polterovich**, **D Rosen**, *On Sandon-type metrics for contactomorphism groups* arXiv:1207.3151

[47] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Lagrangian intersection Floer theory: Anomaly and obstruction, Parts I–II*, AMS/IP Studies in Adv. Math. 46, Amer. Math. Soc. (2009) MR2553465

[48] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Lagrangian Floer theory on compact toric manifolds II: Bulk deformations*, Selecta Math. 17 (2011) 609–711 MR2827178

[49] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Spectral invariants with bulk quasimorphisms and Lagrangian Floer theory* (2011) arXiv:1105.5123v2

[50] **J-M Gambaudo**, **É Ghys**, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems 24 (2004) 1591–1617 MR2104597

[51] **H Geiges**, *Constructions of contact manifolds*, Math. Proc. Cambridge Philos. Soc. 121 (1997) 455–464 MR1434654

[52] **É Ghys**, *Groups acting on the circle*, Enseign. Math. 47 (2001) 329–407 MR1876932

[53] **É Ghys**, *Knots and dynamics*, from: "International Congress of Mathematicians, I", (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), Eur. Math. Soc. (2007) 247–277 MR2334193

[54] **A B Givental**, *Nonlinear generalization of the Maslov index*, from: "Theory of singularities and its applications", (V I Arnold, editor), Adv. Soviet Math. 1, Amer. Math. Soc. (1990) 71–103 MR1089671

[55] **S Guillermou**, **M Kashiwara**, **P Schapira**, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math. J. 161 (2012) 201–245 MR2876930

[56] **L Haug**, *On the quantum homology of real Lagrangians in Fano toric manifolds*, Int. Math. Res. Not. 2013 (2013) 3171–3220 MR3085757

[57] **H Hofer**, **D A Salamon**, *Floer homology and Novikov rings*, from: "The Floer memorial volume", (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 483–524 MR1362838

[58] **M Khanevsky**, *Hofer's metric on the space of diameters*, J. Topol. Anal. 1 (2009) 407–416 MR2597651

[59] **B Khesin**, **S Tabachnikov**, *Contact complete integrability*, Regul. Chaotic Dyn. 15 (2010) 504–520 MR2679761

[60] **D Kotschick**, *What is ... a quasimorphism?*, Notices Amer. Math. Soc. 51 (2004) 208–209 MR2026941

[61] **D Kotschick**, *Stable length in stable groups*, from: "Groups of diffeomorphisms", (R Penner, D Kotschick, T Tsuboi, N Kawazumi, T Kitano, Y Mitsumatsu, editors), Adv. Stud. Pure Math. 52, Math. Soc. Japan, Tokyo (2008) 401–413 MR2509718

[62] **S Lanzat**, *Hamiltonian Floer homology for compact convex symplectic manifolds* `arXiv:1302.1025`

[63] **S Lanzat**, *Quantum homology of compact convex symplectic manifolds* `arXiv:1302.1021`

[64] **S Lanzat**, *Quasimorphisms and symplectic quasistates for convex symplectic manifolds*, Int. Math. Res. Not. 2013 (2013) 5321–5365 MR3142258

[65] **F Laudenbach**, *Homotopie régulière inactive et engouffrement symplectique*, Ann. Inst. Fourier (Grenoble) 36 (1986) 93–111 MR850746

[66] **G Lu**, *Symplectic fixed points and Lagrangian intersections on weighted projective spaces*, Houston J. Math. 34 (2008) 301–316 MR2383710

[67] **D McDuff**, *Displacing Lagrangian toric fibers via probes*, from: "Low-dimensional and symplectic topology", (M Usher, editor), Proc. Sympos. Pure Math. 82, Amer. Math. Soc. (2011) 131–160 MR2768658

[68] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, 2nd edition, Oxford Univ. Press (1998) MR1698616

[69] **D McDuff**, **S Tolman**, *Polytopes with mass linear functions, I*, Int. Math. Res. Not. 2010 (2010) 1506–1574 MR2628835

[70] **I Milin**, *Orderability of contactomorphism groups of lens spaces*, PhD thesis, Stanford University (2008) Available at `http://search.proquest.com/docview/304469611`

[71] **A Monzner**, **N Vichery**, **F Zapolsky**, *Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization*, J. Mod. Dyn. 6 (2012) 205–249 MR2968955

[72] **S Morita**, *Geometry of characteristic classes*, Translations of Math. Monographs 199, Amer. Math. Soc. (2001) MR1826571

[73] **M Øbro**, *An algorithm for the classification of smooth Fano polytopes* `arXiv:0704.0049`

[74] **K Ono**, *Lagrangian intersection under Legendrian deformations*, Duke Math. J. 85 (1996) 209–225 MR1412444

[75] **Y Ostrover**, *Calabi quasimorphisms for some nonmonotone symplectic manifolds*, Algebr. Geom. Topol. 6 (2006) 405–434 MR2220683

[76] **L Polterovich**, *Floer homology, dynamics and groups*, from: "Morse-theoretic methods in nonlinear analysis and in symplectic topology", (P Biran, O Cornea, F Lalonde, editors), NATO Sci. Ser. II Math. Phys. Chem. 217, Springer, Dordrecht (2006) 417–438 MR2276956

[77] **P Py**, *Quasimorphismes et invariant de Calabi*, Ann. Sci. École Norm. Sup. 39 (2006) 177–195 MR2224660

[78] **T Rybicki**, *Commutators of contactomorphisms*, Adv. Math. 225 (2010) 3291–3326 MR2729009

[79] **S Sandon**, *An integer-valued bi-invariant metric on the group of contactomorphisms of* $\mathbb{R}^{2n} \times S^1$, J. Topol. Anal. 2 (2010) 327–339 MR2718127

[80] **S Sandon**, *Contact homology, capacity and nonsqueezing in* $\mathbb{R}^{2n} \times S^1$ *via generating functions*, Ann. Inst. Fourier (Grenoble) 61 (2011) 145–185 MR2828129

[81] **S Sandon**, *Equivariant homology for generating functions and orderability of lens spaces*, J. Symplectic Geom. 9 (2011) 123–146 MR2811649

[82] **H Sato**, *Toward the classification of higher-dimensional toric Fano varieties*, Tohoku Math. J. 52 (2000) 383–413 MR1772804

[83] **E Shelukhin**, *The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures*, Comment. Math. Helv. 89 (2014) 69–123 MR3177909

[84] **D Tamarkin**, *Microlocal condition for non-displaceablility* `arXiv:0809.1584`

[85] **M Usher**, *Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasimorphisms*, Geom. Topol. 15 (2011) 1313–1417 MR2825315

[86] **K Watanabe**, **M Watanabe**, *The classification of Fano* 3*–folds with torus embeddings*, Tokyo J. Math. 5 (1982) 37–48 MR670903

[87] **G Wilson**, **C T Woodward**, *Quasimap Floer cohomology for varying symplectic quotients*, Canad. J. Math. 65 (2013) 467–480 MR3028569

[88] **C T Woodward**, *Gauged Floer theory of toric moment fibers*, Geom. Funct. Anal. 21 (2011) 680–749 MR2810861

[89] **F Zapolsky**, *Geometry of contactomorphism groups, contact rigidity and contact dynamics in jet spaces*, Int. Math. Res. Not. 2013 (2013) 4687–4711

Department of Mathematics, Stanford University
Building 380, Stanford, CA 94305, USA

Department of Mathematics, University of Haifa
Mount Carmel, Haifa 31905, Israel

borman@stanford.edu, frol.zapolsky@gmail.com

Proposed: Yasha Eliashberg | Received: 15 August 2013
Seconded: Danny Calegari, Walter Neumann | Revised: 21 January 2014