Lawrence–Krammer–Bigelow representations
and dual Garside length of braids

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We show that the span of the variable q in the Lawrence–Krammer–Bigelow representation matrix of a braid is equal to twice the dual Garside length of the braid, as was conjectured by Krammer. Our proof is close in spirit to Bigelow's geometric approach. The key observation is that the dual Garside length of a braid can be read off a certain labeling of its curve diagram.

1 Introduction

The question whether the braid group B_n is linear or not was a long-standing problem. At the end of the 20th century, the problem was solved affirmatively by Krammer [7] and Bigelow [1] independently. They showed that a certain linear representation of the braid group first constructed by Lawrence [8] and now called the *Lawrence–Krammer–Bigelow representation* (LKB representation for short) is faithful. Interestingly, the two proofs of the faithfulness of the LKB representations are completely different: Bigelow's proof is geometric whereas Krammer's proof is algebraic.

Our main result (Theorem 1.1) is that the span of the variable q in the image of the LKB representation of a braid β is equal to twice the dual Garside length of β, as conjectured by Krammer in [6]. This is an analogue of Krammer's theorem [7] that the span of the variable t in the image of the LKB representation is equal to the classical Garside length. It is remarkable that, despite the analogy with Krammer's result, our proof is rather based on Bigelow's techniques.

One of our main tools is a labeling of curve diagrams of braids called the *wall crossing labeling*. In Theorem 3.3, we will prove that the wall crossing labeling tells us the dual Garside length of braids. On the other hand, in Lemma 4.1 we will observe that the wall crossing labeling is also related to the variable q in the noodle-fork pairing, a homological intersection pairing appearing in the LKB representation. Thus

wall crossing labelings of curve diagrams serve as a bridge connecting two seemingly unrelated objects, namely the LKB representation and the dual Garside structure.

In order to state our main theorem, we set up some notation. In this paper we denote the LKB representation by

$$\mathcal{L}\colon B_n \to \mathrm{GL}\Big(\frac{n(n-1)}{2}; \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\Big).$$

The matrix representative $\mathcal{L}(\beta)$ depends on the choice of the basis, and there are variations of the matrices of the LKB representation: all matrices given in the three papers [1; 6; 7] are different!

In this paper we use Bigelow's expression of the matrices given in [1, Theorem 4.1], with his correction of a sign error from [2]. This expression has geometric origin, and is essentially the same as the matrices in [6], except that the variable t of [6] corresponds to $-t$ of [1].

Let us take the basis $\{F_{i,j}\}_{1 \le i < j \le n}$ of $\mathbb{R}^{n(n-1)/2}$ defined by the *standard forks*; see Section 2 for details. Using this basis, the matrix representative of the LKB representative is given as

$$[\mathcal{L}(\sigma_i)](F_{j,k}) = \begin{cases} F_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qF_{i,k} + (q^2 - q)F_{i,j} + (1-q)F_{j,k} & i = j-1, \\ F_{j+1,k} & i = j \neq k-1, \\ qF_{j,i} + (1-q)F_{j,k} + (q-q^2)tF_{i,k} & i = k-1 \neq j, \\ F_{j,k+1} & i = k, \\ -q^2 t F_{j,k} & i = j = k-1. \end{cases}$$

The image $\mathcal{L}(\beta)$ is an $((n(n-1))/2 \times (n(n-1))/2)$–matrix whose entries are in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. For a nonzero Laurent polynomial $a \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$, let $M_q(a)$ and $m_q(a)$ be the maximal and the minimal degree of the variable q, respectively. As a convention, we define $M_q(0) = -\infty$ and $m_q(0) = +\infty$. For a matrix $A = (a_{\iota,\kappa}) \in \mathrm{GL}((n(n-1))/2; \mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$ we define

$$M_q(A) = \max_{1 \le \iota, \kappa \le \frac{n(n-1)}{2}} \{M_q(a_{\iota,\kappa})\}, \quad m_q(A) = \min_{1 \le \iota, \kappa \le \frac{n(n-1)}{2}} \{m_q(a_{\iota,\kappa})\}.$$

We denote the supremum, the infimum and the length function of a braid β for the dual Garside structure by $\inf_{\Sigma^*}(\beta)$, $\sup_{\Sigma^*}(\beta)$ and $l_{\Sigma^*}(\beta)$, respectively; see Section 2.1 for precise definitions.

The main result in this paper is the following, which was already conjectured by Krammer in [6].

Theorem 1.1 (Variable q in the LKB representation and the dual Garside length) *For $\beta \in B_n$,*

(1) $2 \sup_{\Sigma^*}(\beta) = M_q(\mathcal{L}(\beta))$,

(2) $2 \inf_{\Sigma^*}(\beta) = m_q(\mathcal{L}(\beta))$,

(3) $2l_{\Sigma^*}(\beta) = \max\{0, M_q(\mathcal{L}(\beta))\} - \min\{0, m_q(\mathcal{L}(\beta))\}$.

2 Preliminaries

First of all we set up our notation and conventions. Let $D^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ be the unit disc in the complex plane and $D_n = D^2 - \{p_1, \ldots, p_n\}$ be the n–punctured disc, where each puncture p_i is put on the real line so that $-1 < p_1 < p_2 < \cdots < p_n < 1$ holds. The braid group B_n is identified with the mapping class group of D_n.

Throughout this paper, we adopt the following conventions. Braids acts on the *left*. A positive standard generator σ_i is identified with the *left-handed*, that is, the *clockwise* half Dehn-twist which interchanges the punctures p_i and p_{i+1}. Also, when calculating winding numbers, the positive direction of winding is taken as the *clockwise direction*. This convention is the opposite of Bigelow's.

2.1 The dual Garside length

First we review the definition of the supremum, the infimum and the length functions of the dual Garside structure. We do not need detailed Garside-theory such as normal forms or algorithms. We only need a length formula from Proposition 2.1 below.

Recall that the braid group B_n is presented as

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i = \sigma_i \sigma_j \sigma_i, |i - j| = 1, \; \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| > 1 \rangle.$$

The elements of the generating set $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$ are called *Artin generators* or *Garside generators*.

In the dual Garside structure, we use a slightly different generating set which contains Σ as a subset. For $1 \leqslant i < j \leqslant n$, let $a_{i,j}$ be the braid

$$a_{i,j} = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-2}) \sigma_{j-1} (\sigma_i \sigma_{i+1} \cdots \sigma_{j-2})^{-1}.$$

The generating set $\Sigma^* = \{a_{i,j} \mid 1 \leqslant i < j \leqslant n\}$ was introduced by Birman, Ko and Lee in [4], and its elements are called the *dual Garside generators*, or *band generators*, or *Birman–Ko–Lee generators*. In our conventions, the braid $a_{i,j}$ is represented by the left-handed half Dehn-twist along an arc connecting p_i and p_j in the lower half of the disc $\{z \in D^2 \mid \mathrm{Im}\, z < 0\}$.

A *dual-positive braid* is a braid which is written by a product of positive dual Garside generator Σ^*. The set of dual-positive braids is denoted by B_n^{+*}. The *dual Garside element* is a braid δ given by

$$\delta = a_{1,2} a_{2,3} \cdots a_{n-1,n}.$$

Let $\preccurlyeq_{\Sigma^*}$ be the subword partial ordering with respect to the dual Garside generating set Σ^*: $\beta_1 \preccurlyeq_{\Sigma^*} \beta_2$ if and only if $\beta_1^{-1}\beta_2 \in B_n^{+*}$. For a given braid β, the supremum $\sup_{\Sigma^*}(\beta)$ and the infimum $\inf_{\Sigma^*}(\beta)$ are defined by

$$\sup_{\Sigma^*}(\beta) = \min\{m \in \mathbb{Z} \mid \beta \preccurlyeq_{\Sigma^*} \delta^m\},$$

$$\inf_{\Sigma^*}(\beta) = \max\{M \in \mathbb{Z} \mid \delta^M \preccurlyeq_{\Sigma^*} \beta\},$$

respectively. A *dual-simple* element is a dual-positive braid s which satisfies $1 \preccurlyeq_{\Sigma^*} s \preccurlyeq_{\Sigma^*} \Delta$. The set of dual-simple elements is denoted by $[1, \delta]$. The Garside length l_{Σ^*} is the length function with respect to the generating set $[1, \delta]$.

The next formula relates the supremum, infimum and the length.

Proposition 2.1 (Dual Garside length) *For a braid $\beta \in B_n$ we have*

$$l_{\Sigma^*}(\beta) = \max\{0, \sup_{\Sigma^*}(\beta)\} - \min\{\inf_{\Sigma^*}(\beta), 0\}.$$

See [4] for a proof.

2.2 The Lawrence–Krammer–Bigelow representation of the braid groups

We review a definition of the Lawrence–Krammer–Bigelow representation. Our description of the LKB representation is homological and mainly follows Bigelow, but it is slightly modified so as to agree with our conventions. For details, see [1; 2].

Let $C = \{(z_1, z_2) \in D_n \times D_n \mid z_1 \neq z_2\}/S_2$ be the configuration space of two unordered points of D_n. We take base points $d_1 = \exp(\frac{3}{2}\pi i - \varepsilon)$ and $d_2 = \exp(\frac{3}{2}\pi i + \varepsilon)$ in ∂D_n so that d_2 lies on the right side of d_1; see Figure 1. We take $\{d_1, d_2\}$ as a base point of C.

Let $\langle q,t \rangle$ be the free abelian group of rank two generated by t and q, and define a homomorphism $\phi\colon \pi_1(C) \to \langle q,t \rangle$ as follows: Let $\{\gamma, \gamma'\}\colon [0,1] \to C$ be a loop representing an element $x \in \pi_1(C)$. We define the number a as the sum of the winding numbers along each puncture p_j:

$$a = -\frac{1}{2\pi i} \sum_{j=1}^{n}\left(\int_\gamma \frac{dz}{z-p_j} + \int_{\gamma'} \frac{dz}{z-p_j} \right).$$

We define the number b as twice the winding number of the path $\gamma - \gamma'$ (ie as twice the relative winding number of the points):

$$b = -\frac{1}{\pi i} \int_{\gamma - \gamma'} \frac{dz}{z}.$$

Now we define $\phi(x) = q^a t^b$. (Here we add the minus signs since we adopted the convention that the positive winding direction is the clockwise direction.)

Let $\pi\colon \widetilde{C} \to C$ be the covering of C associated to $\ker \phi$. Fix the lift of the base point $\{d_1, d_2\}$. By abuse of notation, we use the same symbol $\{d_1, d_2\}$ to represent the base point both in C and $\widetilde{C}$. Then q and t are regarded as deck translations and the second homology group $H_2(\widetilde{C}; \mathbb{Z})$ is a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$–module.

To treat homology classes geometrically, we need to consider a slightly technical modification since $\widetilde{C}$ is not compact. For $\varepsilon > 0$, let v_ε be the set of points $\{x, y\} \in C$ such that the distance of x and y is at most ε, or such that x or y are contained in a ε–neighborhood of a puncture. Let $\widetilde{v}_\varepsilon = \pi^{-1}(v_\varepsilon)$ and put $H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z}) = \lim_{\varepsilon \to 0} H_2(\widetilde{C}, \widetilde{v}_\varepsilon; \mathbb{Z})$. This homology group $H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z})$ is often called a Borel–Moore homology (or homology of locally finite chains).

Let Y be the Y–shaped graph shown in Figure 1, having one distinguished external vertex r, two other external vertices v_1 and v_2, and one internal vertex c. We orient the edges of Y as shown in Figure 1.

A *fork* F is an embedded image of Y into D^2 such that

- all points of $Y \setminus \{r, v_1, v_2\}$ are mapped to the interior of D_n;
- the distinguished vertex r is mapped to the base point d_1;
- the other two external vertices v_1 and v_2 are mapped to two different puncture points;
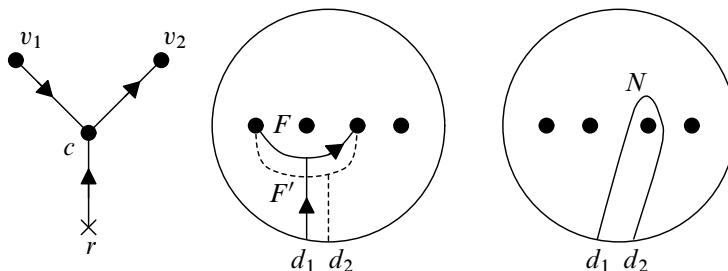- the edge $[r,c]$ and the arc $[v_1, v_2] = [v_1, c] \cup [c, v_2]$ are both mapped smoothly.

Figure 1: Fork and standard fork F_i

The image of the edge $[r, c]$ is called the *handle* of F. The image of $[v_1, v_2] = [v_1, c] \cup [c, v_2]$, regarded as a single oriented arc, is called the *tine*, denoted $T(F)$. The image of c is called the *branch point* of F.

A *parallel fork* F' is a fork as depicted by a dotted line in Figure 1: F' is parallel to F and the distinguished vertex is mapped to d_2.

A *noodle* is an oriented smooth embedded arc which begins at d_1 and ends at d_2.

Let $\gamma, \gamma' \colon [0, 1] \to D_n$ be the handles of the forks F and of its parallel F', respectively. Let $\{\widetilde{\gamma, \gamma'}\} \colon [0, 1] \to \widetilde{C}$ be the lift of the path $\{\gamma, \gamma'\} \colon [0, 1] \to C$ taken so that $\{\widetilde{\gamma, \gamma'}\}(0) = \{d_1, d_2\}$.

Consider the surface $\Sigma(F) = \{\{x, y\} \in C \mid x \in T(F), y \in T(F')\}$ in C. Let $\widetilde{\Sigma}(F) \subset \widetilde{C}$ be the component of $\pi^{-1}\Sigma(F)$ which contains the point $\{\widetilde{\gamma, \gamma'}\}(1)$. The surface $\widetilde{\Sigma}(F)$ in $\widetilde{C}$ defines an element of $H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z})$. By abuse of notation, we will use the same symbol F to represent the 2nd homology class defined by the surface $\widetilde{\Sigma}(F)$. In a similar way, a noodle N defines a surface $\widetilde{\Sigma}(N)$ which defines an element of $H_2(\widetilde{C}, \partial\widetilde{C}; \mathbb{Z})$. Again, by abuse of notation we denote the homology class $[\widetilde{\Sigma}(N)]$ by N.

For $1 \leqslant i < j \leqslant n$, let us take a fork $F_{i,j}$ as shown in Figure 2. These forks are called *standard forks*. We call a standard fork of the form $F = F_{i,i+1}$ a *straight fork*. Similarly, let N_i be the noodle which encloses the ith puncture point p_i as shown in Figure 2. We call such a noodle a *standard noodle*.

The braid group B_n acts on the covering $\widetilde{C}$ so that it commutes with the deck translations t and q. Hence we get a representation $B_n \to \mathrm{GL}(H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z}))$. Bigelow showed in [3, Section 3.1] (cf [2, Conjecture 5.7]) that this representation agrees with the *Lawrence–Krammer–Bigelow (LKB)* representation, given in Section 1.

Remark 2.2 As shown by Paoluzzi and Paris in [9], the LKB representation does not coincide with the action on $H_2(\widetilde{C}; \mathbb{Z})$, although the natural map $H_2(\widetilde{C}; \mathbb{Z}) \to$
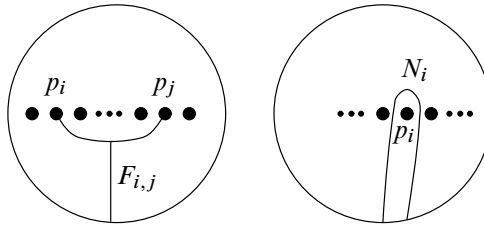
Figure 2: Standard forks and standard noodles

$H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z})$ induces an isomorphism of the representations $H_2(\widetilde{C}; \mathbb{Z}) \otimes \mathbb{Q}(q, t)$ and $H_2(\widetilde{C}, \widetilde{v}; \mathbb{Z}) \otimes \mathbb{Q}(q, t)$, where $\mathbb{Q}(q, t)$ denotes the quotient field of $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$.

2.3 Noodle-fork pairings

The noodle-fork pairing is the natural homology intersection pairing $H_2(\widetilde{C}, \partial\widetilde{C}) \times H_2(\widetilde{C}, \widetilde{v}) \to \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ (see [2, Section 4]). As Bigelow showed in his so-called basic lemma [1, Lemma 2.3], the pairing between an element of $H_2(\widetilde{C}, \partial\widetilde{C})$ represented by a noodle N and an element of $H_2(\widetilde{C}, \widetilde{v})$ represented by a fork F is calculated as follows.

Let $z_1, \ldots, z_m$ be the intersection points of $T(F)$ with N, and let z'_i be the intersection of $T(F')$ with N which corresponds to z_i.

Observe that a pair of intersection points $\{z_i, z'_j\} \in \widetilde{C}$ corresponds to an intersection point of the surfaces $\widetilde{\Sigma}(N)$ and $\widetilde{\Sigma}(F)$. Hence it contributes to the total intersection pairing of N and F as a monomial $\varepsilon_{i,j} m_{i,j} = \varepsilon_{i,j} q^{a_{i,j}} t^{b_{i,j}}$, where $\varepsilon_{i,j}$ denotes the sign of the intersection at $\{z_i, z'_j\}$.

The monomial $\varepsilon_{i,j} m_{i,j}$ is computed as follows. First we define $c_{i,j}$ by

$$c_{i,j} = \begin{cases} +1 & \text{if } d_2 \text{ and } z'_j \text{ belong to the same component of } N - z_i, \\ -1 & \text{if } d_1 \text{ and } z'_j \text{ belong to the same component of } N - z_i. \end{cases}$$

Take three paths A, B and C in D_n as follows:

- A is a path from d_1 to the branch point of F along the handle of F.

- B is a path from the branch point to z_i along the tine $T(F)$.

- C is a path from z_i to d_k along the noodle N. Here $k = 1$ if $c_{i,j} = +1$ and $k = 2$ if $c_{i,j} = -1$. In other words, C goes along N starting from z_i, choosing the direction so as to avoid z'_j.

Similarly, we take three paths A', B' and C' as follows:

- A' is a path from d_1' to the branch point of F' along the handle of F'.

- B' is a path from the branch point to z_j' along the tine $T(F')$.

- C' is a path from z_j' to $d_{k'}$ along the noodle N. Here $k' = 2$ if $c_{i,j} = +1$ and $k' = 1$ if $c_{i,j} = -1$.

Now the concatenation of the three paths $\{C, C'\}\{B, B'\}\{A, A'\}$ defines a loop l in C. The monomial $m_{i,j}$ is given by $\phi(l)$. Then $c_{i,j} = -1$ if and only if the loop l exchanges the position of two points. The sign of the intersection $\varepsilon_{i,j}$ is given by

$$\varepsilon_{i,j} = -ss'c_{i,j},$$

where s (respectively s') is the sign of the intersection of N and $T(F)$ (respectively $T(F')$) at z_i (respectively z_j').

In summary, the noodle-fork pairing is given by the following sum, where we recall that m denotes the number of intersections of $T(F)$ with N:

$$\langle N, F \rangle = \sum_{1 \leqslant i, j \leqslant m} \varepsilon_{i,j} m_{i,j} \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}].$$

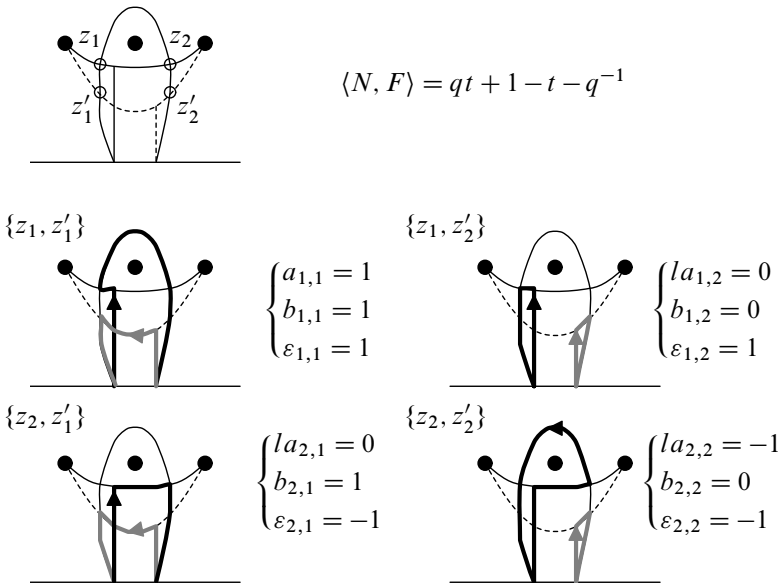$$\langle N, F \rangle = qt + 1 - t - q^{-1}$$

Figure 3: Calculation of the noodle-fork pairing: an example

By direct computations we observe the following, which will play an important role in the proof of main theorems.

Lemma 2.3 *Let N be a standard noodle and let F be a standard fork. Then*

$$M_q(\langle N, F \rangle) \leqslant 1, \quad m_q(\langle N, F \rangle) \geqslant -1.$$

Proof Here we give the calculation of the pairing $\langle N, F \rangle$, for the most complicated case; the other cases are treated similarly. Let $F = F_{i,j}$ and $N = N_k$ be a standard fork and noodle and assume that $i < k < j$. Then $T(F)$ and N intersect at two points, z_1 and z_2, hence the two surfaces $\tilde{\Sigma}(N)$ and $\tilde{\Sigma}(F)$ intersect at four points. Now $a_{i,j}, b_{i,j}$ and $\varepsilon_{i,j}$ are calculated as shown in Figure 3. The paths CBA and $C'B'A'$ are depicted by a black and gray line, respectively. Thus, we conclude $\langle N, F \rangle = qt + 1 - t - q^{-1}$. □

3 The wall crossing labeling of curve diagrams

In this section we introduce the wall crossing labeling on curve diagrams and show that this labeling reflects the dual Garside length of braids. This result is interesting in its own right.

3.1 Curve diagrams

Let $\bar{E}$ be the diagram in D_n consisting of the real line segment between the point -1 (the leftmost point of ∂D^2) and p_n (the rightmost puncture). Similarly, let E be the diagram in D_n consisting of the real line segment between p_1 (the leftmost puncture) and p_n (the rightmost puncture). Both line segments $\bar{E}$ and E are oriented from left to right. Let W_i be a vertical line segment in D_n, oriented upwards, which connects the puncture p_i and the boundary of D_n in the upper half-disk $\{z \in D^2 \mid \operatorname{Im} z > 0\}$; see Figure 4. The lines W_i are called the *walls*, and their union $\bigcup W_i$ is denoted W.

For $\beta \in B_n$, the *total curve diagram* and the *curve diagram* of β is the image of the diagrams $\bar{E}$ and E, respectively, under a diffeomorphism ϕ representing β which satisfies the following conditions.

(1) The number of intersections of $\phi(\bar{E})$ with the walls is as small as possible within the diffeotopy class of ϕ relative to the boundary and punctures.

(2) Near the puncture points, the image $\phi(\bar{E})$ coincides with the real line.

We denote the curve diagram of β by D_β and the total curve diagram by $\bar{D}_\beta$. See the right side of Figure 4 for an example.
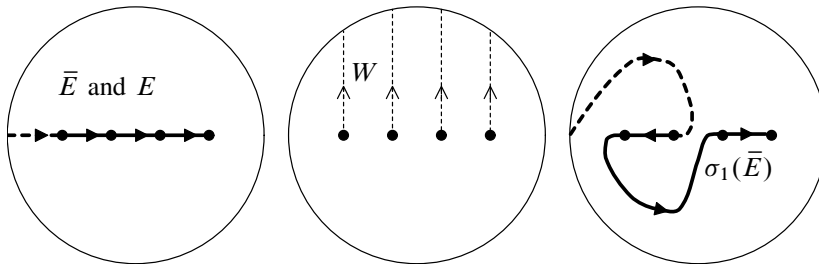
Figure 4: Curve diagram and walls

To introduce the wall-crossing labeling and make the correspondence between fork and curve diagram explicit, we use a modified version of the curve diagrams. For each puncture p_i other than $\beta(p_n)$, we take a small disc neighborhood B_i of p_i, and let $B = \bigcup_i B_i$. Around each puncture p_i, we modify the curve diagram D_β as shown in Figure 5. We denote the resulting (total) curve diagram by MD_β $(\overline{MD}_\beta)$, and call it the *(total) modified curve diagram*. The right side of Figure 5 shows the total modified curve diagram of σ_1.
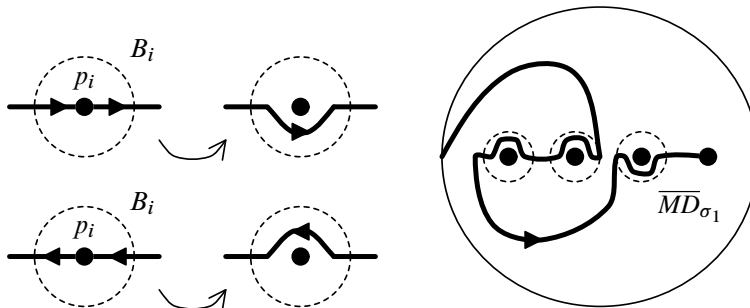

Figure 5: Modified curve diagrams

Take a smooth parametrization of $\overline{MD}_\beta$, viewed as an image of $\gamma\colon [0,1] \to D^n$. Then we define the *wall crossing labeling* as follows.

Definition 3.1 To each connected component α of $\overline{MD}_\beta - (W \cup B)$, we assign the algebraic intersection number of W and the arc $\gamma([0,s])$, where $s \in [0,1]$ is taken so that $\gamma(s) \in \alpha$. We call this integer-valued labeling $\mathrm{Wcr}(\alpha)$ the *wall crossing labeling*.

An *arc segment* of the curve diagram D_β (or the total curve diagram $\overline{D}_\beta$) is a component of $D_\beta - (W \cup B))$ (or of $\overline{D}_\beta - (W \cup B)$, respectively). Since $\overline{MD}_\beta$ and $\overline{D}_\beta$

coincide except on B, an arc segment is identified with the subarc of $\bar{D}_\beta$. Using this correspondence, we assign the wall crossing labeling for each arc segment of the curve diagram; see Figure 6.
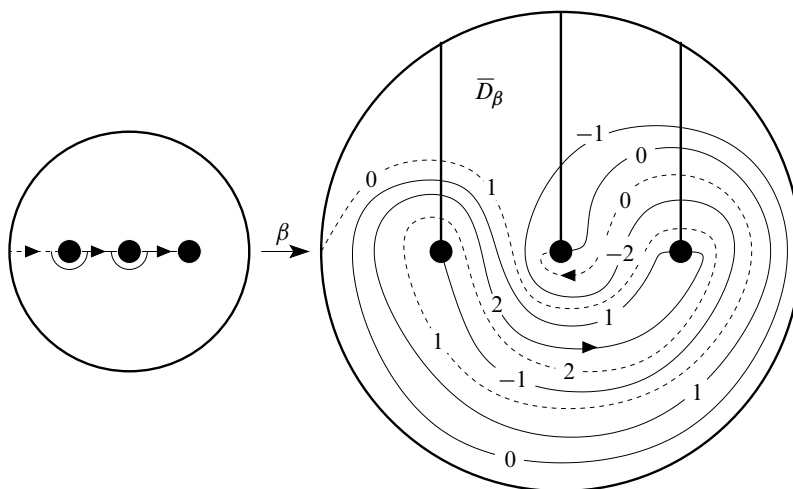

Figure 6: The total curve diagram of the braid $\beta = (\sigma_2^{-1}\sigma_1)^2$, and its wall crossing labeling: among the labels of the solid (not dashed) arcs, the smallest one is -2 and the largest one is 2.

Definition 3.2 For a braid β, we define $\mathsf{LWcr}(\beta)$ and $\mathsf{SWcr}(\beta)$ as the largest and the smallest wall crossing number labelings occurring in the curve diagram D_β.

Notice that in Definition 3.2 we used the largest and smallest labels only of the curve diagram D_β, not the total curve digram $\bar{D}_\beta$. However, in order to determine the wall crossing labelings we need to consider the total curve diagram.

We now show that the dual Garside (Birman–Ko–Lee) length of a braid can be read off the wall crossing labeling of its curve diagram.

Theorem 3.3 *For a braid $\beta \in B_n$ we have the following equalities:*

(1) $\sup_{\Sigma^*}(\beta) = \mathsf{LWcr}(\beta)$.

(2) $\inf_{\Sigma^*}(\beta) = \mathsf{SWcr}(\beta)$.

(3) $l_{\Sigma^*}(\beta) = \max(\mathsf{LWcr}(\beta), 0) - \min(\mathsf{SWcr}(\beta), 0)$.

Example 3.4 Let us consider the braid $\beta = (\sigma_2^{-1}\sigma_1)^2 = (a_{2,3}^{-1}a_{1,2})^2$; see Figure 6. The dual normal form of β is $\delta^{-2}a_{2,3}.a_{2,3}.a_{1,2}.a_{1,2}$ so $\inf_{\Sigma^*}(\beta) = -2$ and $\sup_{\Sigma^*}(\beta) = 2$.

Now Theorem 3.3 asserts that the smallest and largest wall crossing labelings occurring in the curve diagram of β should be -2 and 2, respectively, and Figure 6 shows that this indeed the case.

Proof of Theorem 3.3 First of all, we show that for a dual-positive braid β the equality $\mathsf{LWcr}(\beta) = l_{\Sigma*}(\beta)$ holds.

To treat the dual Garside structure, we temporarily isotope our curve diagram and walls so that the all punctures sit on the circle $|z| = \frac{1}{2}$ and walls are disjoint from the subdisc $|z| \leqslant \frac{1}{2}$ as shown on the left side of Figure 7. This isotopy does not affect the wall crossing labelings, since the wall crossing labeling is defined by using algebraic intersection of arcs and walls.

As shown in [4], the set $[1, \delta]$ of dual-simple braids is in bijection with the set of disjoint collections of convex polygons in D_n whose vertices are punctures. This bijection is given as follows: to any such collection of polygons we can associate a dance of the puncture points, moving each puncture which belongs to some polygon P in the clockwise direction along the boundary of P, to the position of the adjacent vertex. In this way, each dual-simple element can be represented by some disjoint convex polygons in D_n; see Figure 7.

We remark that polygons may be degenerate, having only two vertices; in the associated braids, the two corresponding punctures are interchanged by a clockwise half Dehn-twist.
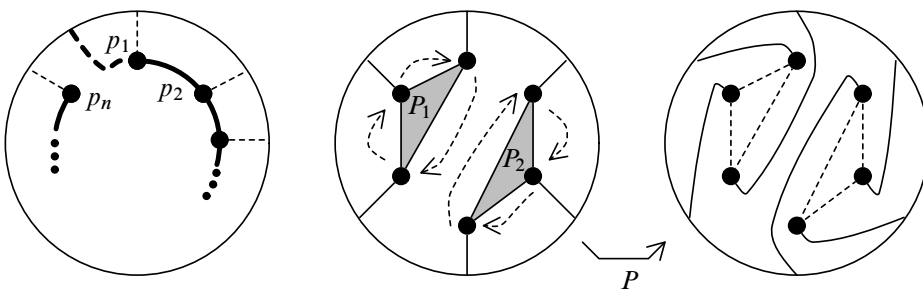

Figure 7: The left picture shows the basic curve diagram (bold lines) and the walls (thin dashed lines). The pictures in the middle and on the left illustrate the action of a dual-simple braid P given by two disjoint polygons P_1 and P_2.

Given a curve diagram D_β, equipped with the wall crossing labeling, and given a collection $P = \{P_1, \ldots, P_k\}$ of disjoint polygons with vertices in the punctures representing some dual-simple braid β^+, let us describe in detail how to obtain the

curve diagram $D_{\beta+\beta}$ and its wall crossing labeling. Consider the disjoint collection of annuli $A_1, \ldots, A_k$ in D_n as follows (see Figure 8):

(1) The outer and inner boundary component of A_i are both homotopic in D_n to the boundary of a regular neighborhood of P_i.

(2) The boundary components of A_i are in reduced relative position (no bigons) with respect to the walls W and also with respect to the curve diagram D_β.

(3) Among all collections of annuli satisfying (1) and (2), we choose the one where the annuli contain as many intersection points $D_\beta \cap W$ as possible. Roughly speaking, we push as much twisting of the curve diagram around the polygons as possible into the annuli.

Let us denote the component of $D_n \setminus A_i$ containing the polygon P_i by $N(P_i)$. Note that the intersection of D_β with A_i is simply a spiral, and that the labels on this spiral interpolate linearly between the labels on the outer and inner boundary component.

Now the β^+–action on D_β is simple to describe: On $D_n \setminus (\bigcup_{i=1}^{k} (A_i \cup N(P_i)))$, the diagram and its labeling is unchanged. On $N(P_i)$ the diagram is turned one notch in the clockwise sense, and all labels are increased by one. On the annuli, we have some twisting, but the labels still just interpolate; see Figure 8.
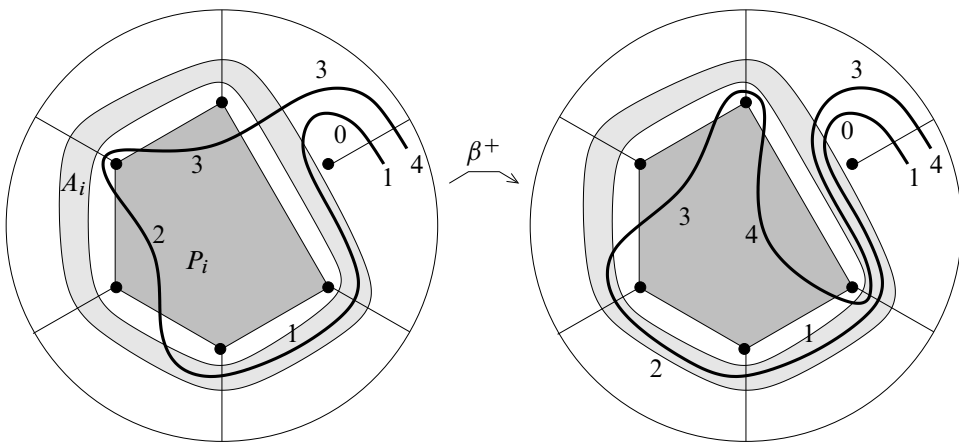

Figure 8: Action of dual-simple elements

The action of a *negative* dual-simple braid β^- is similar, the only difference being that the twisting is in the counterclockwise direction and labels are decreased by one.

From the above description for a dual-positive braid β and a dual-simple element β^+, we have the inequality $\mathsf{LWcr}(\beta^+\beta) \leqslant \mathsf{LWcr}(\beta) + 1$. This implies the inequality

$$\mathsf{LWcr}(\beta) \leqslant l_{\Sigma^*}(\beta).$$

The converse inequality is now implied by the following lemma:

Lemma 3.5 *Every dual-positive braid β can be written as a product of $\mathsf{LWcr}(\beta)$ positive dual-simple braids.*

Proof of Lemma 3.5   From the above description of the action of a dual-positive braid, we see that $\mathsf{SWcr}(\beta) \geqslant 0$, as no negative labels can ever be created from nonnegative ones.

In order to prove the lemma, we distinguish two cases. Firstly, if $\mathsf{SWcr}(\beta) = \mathsf{LWcr}(\beta)$, then no arc of the modified curve diagram MD_β crosses a wall. This implies that β is a power of δ, more precisely, $\beta = \delta^{\mathsf{SWcr}(\beta)}$, and the lemma is true.

Secondly, if $0 \leqslant \mathsf{SWcr}(\beta) < \mathsf{LWcr}(\beta)$, then we proceed inductively. We shall construct a negative dual-simple braid β^- such that $\mathsf{LWcr}(\beta^-\beta) < \mathsf{LWcr}(\beta)$ and $\mathsf{SWcr}(\beta^-\beta) \geqslant \mathsf{SWcr}(\beta)$.

Consider the arc segments having the maximal wall crossing labeling. Each such segment connects two walls. Let $I = \{(i, j)\}$ be the pairs of walls which are connected by some maximally labeled arcs. Let P be the minimal (with respect to inclusion) collection of convex polygons which contains all straight lines connecting p_i and p_j for $(i, j) \in I$. Let β^- be the inverse of the dual-simple braid that corresponds to P. Garside-theoretically speaking, $\beta^- = (\bigvee_{(i,j) \in I} a_{i,j})^{-1}$.

According to our description above of the β^-–action on the curve diagram D_β, the action of β^- decreases all the maximal wall crossing labelings by one, so $\mathsf{LWcr}(\beta^-\beta) = \mathsf{LWcr}(\beta) - 1$.

On the other hand, we claim that $\mathsf{SWcr}(\beta^-\beta) \geqslant \mathsf{SWcr}(\beta)$, ie contrary to the largest label, the smallest label does not decrease during the β^-–action.

Let us prove this claim. First we observe that, roughly speaking, minimally labeled arcs are S–shaped, whereas maximally labeled arcs are ƨ–shaped. More precisely, when both endpoints of a minimally labeled arc lie in the interior of walls, then the initial and terminal segment of the arc lie on the counterclockwise sides of the wall, whereas maximally labeled arcs begin and terminate on clockwise sides of the respective walls (see Figure 9(a)).

Now, in order to prove the claim, we have to rule out the existence of minimally labeled arcs (ie arcs labeled $\mathsf{SWcr}(\beta)$) which, under the β^-–action, give rise to arcs with an even smaller label.

First we observe that a minimally labeled arc α cannot intersect the interior of any of the polygons of P. Indeed, assume that an arc segment α enters into the interior of

one of these polygons, say P_1. Let us assume in addition that both endpoints of α lie in the interior of walls, not in punctures. Then α cuts P_1 into two components, and thus separates the vertex punctures of P_1 into two families (drawn white and black in Figure 9(a)). Now we observe that no maximally labeled arc can connect a wall belonging to a black puncture to a wall belonging to a white puncture. But, by construction of P, that means that the white and black punctures do not belong to the same polygon P_1, which is a contradiction.
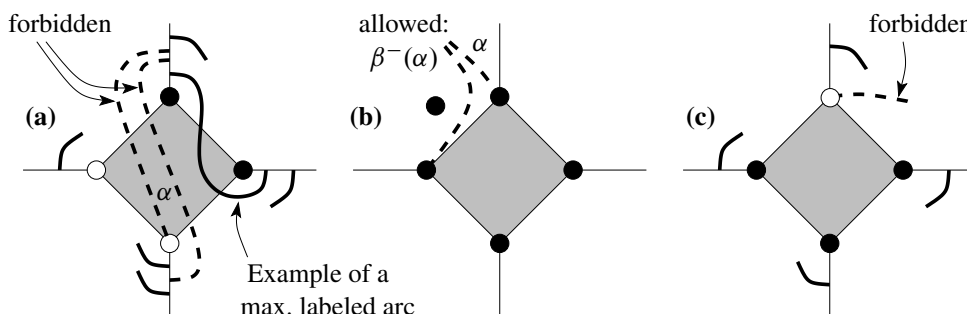

Figure 9: Fat dashed lines indicate minimally Wcr–labeled arcs, and fat solid lines indicate maximally Wcr–labeled arcs

Similarly, when a minimally labeled arc α intersects the interior of P but has one or both endpoints in punctures, then the same argument applies: we only have to decide in which color to paint a puncture at the extremity of α. The choice which works is to group such a puncture with the following punctures in the clockwise direction; see Figure 9(a).

Finally, we have to consider a minimally labeled arc segment α which ends in a vertex of P_1 without intersecting the interior of P_1. Here we have to distinguish two cases. If an extremal segment of α lies on the *counterclockwise* side of the wall corresponding to its terminal puncture, as in Figure 9(b), then there is nothing to worry about since the β^-–action does not decrease its Wcr–label. If, on the other hand, a terminal segment of α lies on the *clockwise* side of the wall corresponding to its terminal puncture, as in Figure 9(c), then this wall cannot be connected to a wall of any of the other punctures of P_1 by a maximally labeled arc. Again, this contradicts the construction of P_1.

In summary, no minimally labeled arc can generate an even smaller label under the β^-–action. This proves the claim, and thus Lemma 3.5. □

To summarize, we have now proved that for a dual-positive braid β, the following equality holds:

$$\mathsf{LWcr}(\beta) = l_{\Sigma^*}(\beta).$$

For a dual-positive braid β, $\inf_{\Sigma^*}(\beta) \geqslant 0$ so by Proposition 2.1 we conclude that

$$\mathsf{LWcr}(\beta) = \sup_{\Sigma^*}(\beta).$$

In a similar way, we prove the equality $\mathsf{SWcr}(\beta) = \inf_{\Sigma^*}(\beta)$ for any dual-*negative* braid β.

In order to prove the same results for arbitrary braids, we recall that the dual Garside element δ acts as the clockwise $\frac{2\pi}{n}$–rotation of the n–gon with vertices in all punctures. Thus the curve diagram $D_{\delta\beta}$ is obtained from D_β simply by a $\frac{2\pi}{n}$–rotation, and the wall crossing labeling on each arc segment of $D_{\delta\beta}$ is obtained from the label of the corresponding arc of D_β by adding one. On the other hand, by definition of inf and sup, left multiplication by δ increases both inf and sup by one. Therefore for a general braid β, we get an equality

$$\begin{aligned}
\mathsf{LWcr}(\beta) &= \mathsf{LWcr}(\delta^{-\inf_{\Sigma^*}(\beta)}\beta) + \inf_{\Sigma^*}(\beta) \\
&= \sup_{\Sigma^*}(\delta^{-\inf_{\Sigma^*}(\beta)}\beta) + \inf_{\Sigma^*}(\beta) = \sup_{\Sigma^*}(\beta).
\end{aligned}$$

A similar calculation also yields

$$\mathsf{SWcr}(\beta) = \inf_{\Sigma^*}(\beta)$$

for a general braid β. Finally, Proposition 2.1 implies the third equality claimed in Theorem 3.3. This completes the proof of Theorem 3.3. $\qquad\square$

Remark 3.6 In [10] the second author defined another labeling on the curve diagram called the *winding number labeling*, and proved the similar formula for the winding number labeling and the usual Garside length (see [10, Theorem 2.1] and the authors [5]). The proof of Theorem 3.3 given here is a direct generalization of the proof of [10, Theorem 2.1].

4 Noodle-fork pairing and wall crossing labeling

In this section we make the observation (in Lemma 4.1) that the wall crossing labeling reflects the exponents of the variable q in the noodle-fork pairing.

We need some technical preparation for this. We first show that for any straight fork $F_{i,i+1}$, and for any braid β, the tine $T(\beta(F_{i,i+1}))$ can in a natural way be equipped with the wall crossing labeling. For this, we will have to relate curve diagrams and forks.

Let us consider the part of the curve diagram D_β that is the image of the line segment between the ith and $(i+1)^{\text{st}}$ punctures. We identify this part of the curve diagram

with $\beta(T(F_{i,i+1}))$, the image of the tine of the straight fork $F_{i,i+1}$. Moreover, up to moving the footpoint from -1 to d_1, part of the modified curve diagram can naturally be regarded as the handle of $\beta(F_{i,i+1})$, as shown in Figure 10. This identification also induces the desired wall crossing labeling on each arc segment of $\beta(F_{i,i+1})$.
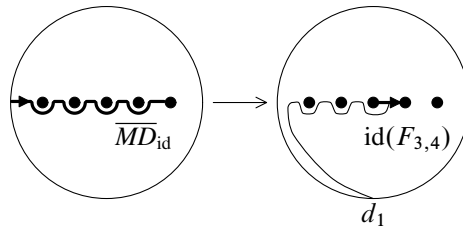

Figure 10: Viewing a curve diagram as a union of tines of forks, and viewing initial segments of modified curve diagrams as tines

Let N be a standard noodle, and let F be a straight fork with tine $T(F)$. Consider the image $\beta(F)$ of a straight fork F. From now on, we always assume that $\beta(T(F))$ is isotoped so that it intersects N minimally. For an intersection point $z_i \in \beta(T(F)) \cap N$, we denote the wall crossing labeling of the arc segment of $\beta(T(F))$ containing z_i by $\mathrm{Wcr}(z_i)$. We recall that the union of the walls is denoted W. We also recall that the intersection pairing of N and $\beta(F)$ is $\langle N, \beta(F) \rangle = \sum_{1 \leqslant i,j \leqslant m} \varepsilon_{i,j} q^{a_{i,j}} t^{b_{i,j}}$ where the sum ranges over all pairs of intersection points $N \cap \beta(T(F))$.

Lemma 4.1 *Let F be a straight fork and let N be a standard noodle. Let $\{z_i\}$ be the set of intersection points of $\beta(T(F))$ and N. Then $a_{i,i}$, the exponent of q at the intersection point $\{z_i, z_i'\}$ is given as follows:*

$$a_{i,i} = \begin{cases} 2\mathrm{Wcr}(z_i) + 1 & \text{if } z_i \text{ and } d_1 \text{ belong to the same component of } N - W, \\ 2\mathrm{Wcr}(z_i) - 1 & \text{if } z_i \text{ and } d_2 \text{ belong to the same component of } N - W. \end{cases}$$

Proof Recall that $a_{i,i}$ is defined as the sum of the winding numbers of the paths CBA and $C'B'A'$ around each puncture,

$$a_{i,i} = -\frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{CBA} \frac{dz}{z - p_j} + \int_{C'B'A'} \frac{dz}{z - p_j} \right),$$

where A, A', B, B', C, C' are the paths defined in Section 2.3. See Figure 11.

For a closed loop γ in D_n, the winding number around the puncture p_j,

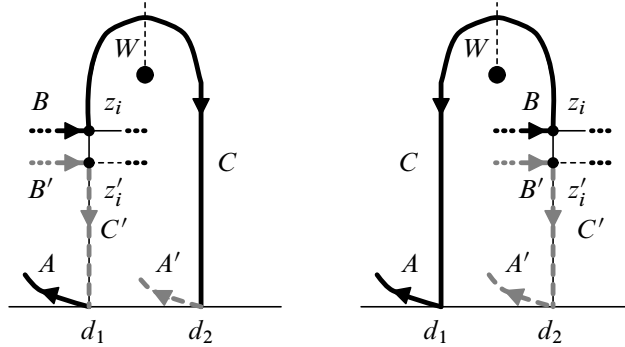$$-\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - p_j},$$

Figure 11: Contribution of the noodle part

is equal to the algebraic intersection number of γ and W_j. Thus, the fork part of the path $\{CBA, C'B'A'\}$, namely the subarc $\{BA, B'A'\}$, contributes to the total winding number $a_{i,i}$ by $2\mathrm{Wcr}(z_i)$. Finally, we observe that the rest of the loop $\{C, C'\}$ (the noodle N part of the path $\{CBA, C'B'A'\}$) contributes to the total winding number $a_{i,i}$ by $+1$ (respectively -1) if z_i and d_1 belongs to the same (respectively to different) components of $N - (Z \cup W)$; see Figure 11. This completes the proof. $\square$

Remark 4.2 For the winding number labeling introduced in [10], and for $b_{i,i}$, the exponent of t at the intersection $\{z_i, z'_i\}$, one can get a similar formula by similar arguments. Thus, schematically speaking, we have the following correspondence among three objects in braid groups:

LKB representation	Curve diagram	Garside structure
Variable t	Winding number labeling	Usual Garside structure
Variable q	Wall crossing labeling	Dual Garside structure

Unfortunately, we did not manage to reprove Krammers result from [7], ie the analogue of Theorem 1.1 for the variable t and the usual Garside length, using our geometric techniques.

5 The dual Garside length formula

In this section we prove Theorem 1.1. For monomials $q^a t^b$ and $q^{a'} t^{b'}$ we define the lexicographical ordering $\leqslant_{q,t}$ by

$$q^a t^b \leqslant_{q,t} q^{a'} t^{b'} \quad \text{if } a < a', \text{ or if } a = a' \text{ and } b \leqslant b'.$$

The next lemma is the crucial result in Bigelow's proof of faithfulness.

Lemma 5.1 [1, Key lemma 3.2 and Claim 3.4] *Assume that a noodle N and a fork F have the minimal geometric intersection. Then all intersection points $\{z_i, z_j'\}$ of N with F which attain the $<_{q,t}$–maximal monomial $m_{i,j}$ in $\langle N, F \rangle$ have the same sign $\varepsilon_{i,j}$.*

Roughly speaking, Lemma 5.1 states that the $<_{q,t}$–maximal contributions to $\langle N, F \rangle$ do not cancel.

Lemma 5.2 *Let $\beta \in B_n$ be a braid. Let F be a straight fork F such that $\beta(T(F))$ contains an arc segment having the largest wall crossing labeling $\mathsf{LWcr}(\beta)$. Then there exists a standard noodle N such that*

$$M_q(\langle N, \beta(F) \rangle) \geqslant 2\mathsf{LWcr}(\beta) + 1.$$

Proof Throughout the proof, we assume that $\beta(T(F))$ intersects each standard noodle N_i minimally. It is sufficient to show that there is a standard noodle N_i and an intersection point $z \in \beta(T(F)) \cap N_i$ such that $\mathsf{Wcr}(z) = \mathsf{LWcr}(\beta)$ and such that the two points z and d_1 lie in the same component of $N_i - (W_i \cap N_i)$: by Lemma 4.1, the intersection point $\{z, z'\}$ contributes to the pairing $\langle N_i, \beta(F) \rangle$ by $\pm q^{2\mathsf{LWcr}(\beta)+1}t^b$ for some $b \in \mathbb{Z}$, and then Lemma 5.1 completes the proof.
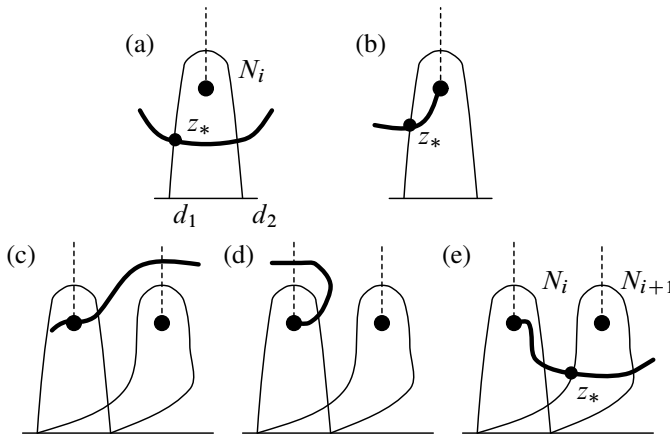

Figure 12: Intersections achieving $q^{2\mathsf{LWcr}(\beta)+1}$ exist

Let γ be an arc segment of $\beta(T(F))$ whose wall crossing labeling is $\mathsf{LWcr}(\beta)$. Take a standard noodle N_i which intersects γ in a point z. Assume that z and d_1 lie on the two different components of $N_i - (W_i \cap N_i)$. If γ does not fall into the puncture p_i, then we can find another intersection point z_* of γ with N_i that lies on the same component of $N_i - (W_i \cap N_i)$ as d_1 (Figure 12(a)) and the proof is complete. Assume that γ falls

into the puncture p_i. If γ falls into the puncture from the left, the intersection point z_* nearest to the puncture provides the desired intersection (Figure 12(b)), so we assume that γ falls into the puncture from the right. Then γ must pass under the adjacent puncture p_{i+1} because otherwise we either find another arc segment having strictly larger wall crossing labeling (Figure 12(c)) or contradict the hypothesis that N_i and $\beta(T(F))$ have the minimal intersection (Figure 12(d)). Then the standard noodle N_{i+1} and γ have an intersection z_* with the desired property (Figure 12(e)). □

Proof of Theorem 1.1 For a braid β, let us take a straight fork F and a standard noodle N as in Lemma 5.2. Thus we have $M_q(\langle N, \beta(F) \rangle) \geqslant 2\mathrm{LWcr}(\beta) + 1$. Let us write the fork $\beta(F)$ as the linear combination of the standard forks

$$\beta(F) = \sum_{1 \leqslant i < j \leqslant n} x_{i,j} F_{i,j} \quad (x_{i,j} \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]).$$

Since F is a straight fork, $x_{i,j}$ is an entry of the matrix $\mathcal{L}(\beta)$. Now we have an equality

$$\langle N, \beta(F) \rangle = \sum_{1 \leqslant i < j \leqslant n} x_{i,j} \langle N, F_{i,j} \rangle.$$

By taking M_q of both sides, we get

$$M_q(\langle N, \beta(F) \rangle) = M_q\left(\sum_{1 \leqslant i < j \leqslant n} x_{i,j} \langle N, F_{i,j} \rangle\right) \leqslant \max_{1 \leqslant i < j \leqslant n} \{M_q(x_{i,j}) + M_q(\langle N, F_{i,j} \rangle)\}.$$

Since in Lemma 2.3 we observed that $M_q(\langle N, F_{i,j} \rangle) \leqslant 1$,

$$M_q(\langle N, \beta(F) \rangle) \leqslant \max_{1 \leqslant i < j \leqslant n} \{M_q(x_{i,j})\} + 1.$$

Hence we conclude

$$\max_{1 \leqslant i < j \leqslant n} \{M_q(x_{i,j})\} \geqslant M_q(\langle N, \beta(F) \rangle) - 1 \geqslant 2\mathrm{LWcr}(\beta) = 2\sup_{\Sigma^*}(\beta),$$

where the second inequality follows from Lemma 5.2 and the last equality from Theorem 3.3. Thus we get an inequality

$$M_q(\mathcal{L}(\beta)) \geqslant 2\sup_{\Sigma^*}(\beta).$$

On the other hand, for each dual-simple element $s \in [1, \delta]$,

$$M_q(\mathcal{L}(s)) \leqslant 2$$

holds by direct calculations. Hence we get the converse inequality

$$M_q(\mathcal{L}(\beta)) \leqslant 2\sup_{\Sigma^*}(\beta).$$

We conclude that

$$M_q(\mathcal{L}(\beta)) = 2\sup_{\Sigma^*}(\beta).$$

The proof of (2) is similar, and (3) follows from Proposition 2.1. □

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