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# How to lift positive Ricci curvature

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We show how to lift positive Ricci and almost nonnegative curvatures from an orbit space $M/G$ to the corresponding $G$–manifold, $M$. We apply the results to get new examples of Riemannian manifolds that satisfy both curvature conditions simultaneously.

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## Introduction

Lawson and Yau showed that $M$ admits positive scalar curvature provided $M$ is a compact $G$–manifold, with $G$ a compact, nonabelian, connected Lie group [24]. By Myers' theorem, this result cannot be generalized to positive Ricci curvature; however, one might ask about the case when the fundamental group of $M$ is finite. Towards this end we have the following result.

**Theorem A** *Let $G$ be a compact, connected Lie group acting isometrically and effectively on a compact Riemannian manifold $M$. Suppose the fundamental group of a principal orbit is finite and the orbital distance metric on $M/G$ has Ricci curvature greater than or equal to $1$. Then $M$ admits a $G$–invariant metric with positive Ricci curvature.*

**Remark** Various definitions of lower Ricci curvature bounds on metric spaces are proposed in Kuwae and Shioya [23], Lott and Villani [25], Ohta [28], Sturm [40; 41] and Zhang and Zhu [50]. Our proof only requires that the quotient space of the principal orbits, $M^{\mathrm{reg}}/G$, has Ricci curvature greater than or equal to $1$, and since $M^{\mathrm{reg}}/G$ is a Riemannian manifold, it does not matter which definition we choose.

The analogous result for positive sectional curvature is false. Let $\mathrm{SO}(3)$ act transitively on the second factor of $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$. By Synge's theorem, the positively curved metric on the quotient $\mathbb{R}\mathrm{P}^2$ cannot be lifted to a positively curved metric on $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$. Similarly, the examples of Grove, Verdiani, Wilking and Ziller in [18] and He in [22] show that the analog of Theorem A is also false for nonnegative curvature.

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On the other hand, we can lift almost nonnegative curvature, and we do not even need the hypothesis on the fundamental group of the principal orbits.

**Theorem B** *Let $G$ be a compact, connected Lie group acting smoothly and effectively on a compact smooth $n$–manifold $M$. Let $\{g_\alpha\}_{a=1}^{\infty}$ be a sequence of Riemannian metrics on $M$ for which the $G$–action is isometric.*

*Suppose $\{(M/G, \mathrm{dist}_\alpha)\}_{a=1}^{\infty}$ has almost nonnegative curvature, where each $\mathrm{dist}_\alpha$ is the induced orbital distance metric. Then $M$ admits a $G$–invariant family of metrics with almost nonnegative sectional curvature.*

As both $M$ and $M/G$ are Alexandrov spaces, the following definition of almost nonnegative curvature is valid for both spaces.

**Definition** We say that a sequence of Alexandrov spaces $\{(X, \mathrm{dist}_\alpha)\}_{\alpha=1}^{\infty}$ is almost nonnegatively curved if and only if there is a $D > 0$ so that

$$\mathrm{Diam}(X, \mathrm{dist}_\alpha) \leq D,$$
$$\mathrm{curv}(X, \mathrm{dist}_\alpha) \geq -\frac{1}{\alpha}.$$

Together Theorems A and B are more interesting than either result is separately since their proofs yield the following.

**Theorem C** *If $\{(M, G, g_\alpha)\}_{\alpha=1}^{\infty}$ satisfies the hypotheses of both Theorems A and B, then $M$ admits a family of metrics that simultaneously has positive Ricci curvature and is almost nonnegatively curved.*

We believe that Theorems A, B and C will ultimately lead to many new examples with positive Ricci and almost nonnegative sectional curvatures. To apply these theorems, one needs an orbit space with positive Ricci and/or almost nonnegative sectional curvature. Unfortunately, there does not seem to be an extensive catalog of such orbit spaces, leading us to ask the following two questions.

**Question 1** *Let $\mathcal{M}$ be the class of compact smooth, manifolds $M$ admitting a smooth, effective action by a compact, connected Lie group, $G$, with $\pi_1$(principal orbit) finite. Which $M \in \mathcal{M}$ admit a $G$–invariant metric with $\mathrm{Ric}(M^{\mathrm{reg}}/G) \geq 1$?*

**Question 2** *Let $\mathcal{M}$ be the class of compact smooth, manifolds $M$ admitting a smooth, effective action by a compact, connected Lie group, $G$. Which $M \in \mathcal{M}$ admit a family of $G$–invariant metrics $\{g_\alpha\}_{a=1}^{\infty}$ for which $\{(M/G, \mathrm{dist}_\alpha)\}_{a=1}^{\infty}$ is almost nonnegatively curved?*

Let $G$ be a Lie group with a bi-invariant metric. Let $H \subset G \times G$ act on $G$ from the left and right, and not freely. The biquotient $G /\!\!/ H$ is nonnegatively curved, and it seems likely that the technique of Schwachhöfer and Tuschmann [38] could yield that the Ricci curvature of $G^{\mathrm{reg}} /\!\!/ H$ is also positive, if $\pi_1(G /\!\!/ H) < \infty$. One could then search for smooth $H_1$–manifolds with $M/H_1 = G /\!\!/ H$, and find a solution to Question 2 and possibly a solution to Question 1.

We have yet to pursue this line of inquiry, but we have proven the following theorem.

**Theorem D** *Let $\mathcal{Y}$ be the class of compact, smooth manifolds consisting of*

$$\begin{aligned}
\Sigma^7 &\equiv \{\textit{all exotic } 7\textit{–spheres}\},\\
\Sigma^{15}_{\mathrm{BP}} &\equiv \{\textit{all exotic } 15\textit{–spheres that bound parallelizable manifolds}\},\\
\mathbf{F}\mathbb{H}\mathbf{P2} &\equiv \{\textit{all double mapping cylinders on } S^3\textit{–bundles over } S^4\\
&\qquad\qquad \textit{whose total spaces are homeomorphic to } S^7\},\\
\mathbf{F}\mathbb{O}\mathbf{P2} &\equiv \{\textit{all double mapping cylinders on } S^7\textit{–bundles over } S^8\\
&\qquad\qquad \textit{whose total spaces are homeomorphic to } S^{15}\}.
\end{aligned}$$

*Any $M \in \mathcal{Y}$ admits a family of metrics $\{g_a\}_{\alpha=1}^{\infty}$ that is simultaneously almost nonnegatively curved and has positive Ricci curvature.*

For the purpose of Theorem D, the double mapping cylinder on a map $p\colon E \to B$ is obtained from the disjoint union

$$B_- \amalg E \times [-1,1] \amalg B_+$$

of $E \times [-1,1]$ and two copies of $B$, denoted $B_-$ and $B_+$, by making only the following identifications. For each $e$ in $E$, let

$$\begin{aligned}
(e,-1) &\sim p(e) \in B_-,\\
(e,1) &\sim p(e) \in B_+.
\end{aligned}$$

Next we give a brief history of prior work related to these results. It is not meant to be comprehensive, rather we limit our attention to results that are specifically relevant to Theorems A, B, C and D.

Combining work of Nash [27] or Poor [33] and Fukaya and Yamaguchi [9], gives us a family of almost nonnegatively curved metrics that are also Ricci positive on the exotic spheres that are bundles.

Wraith showed that all exotic spheres bounding parallelizable manifolds admit metrics with $\mathrm{Ric} > 0$ [47]. In [5], Boyer, Galicki and Nakamaye showed that such exotic spheres admit $\mathrm{Ric} > 0$ metrics that are also Sasaki, provided the dimension is odd.

By combining results of Grove and Ziller [19] and Guijarro [21] one gets nonnegatively curved metrics on the class $\mathbf{F}\mathbb{H}\mathbf{P2}$. These metrics might make a good starting point for an alternative argument that the class $\mathbf{F}\mathbb{H}\mathbf{P2}$ has an almost nonnegatively curved family with positive Ricci curvature.

The exotic spheres in $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$ that are not bundles and the entire class $\mathbf{F}\mathbb{O}\mathbf{P2}$ were not previously known to admit almost nonnegative curvature. To the best of our knowledge, the class $\mathbf{F}\mathbb{O}\mathbf{P2}$ was not previously known to admit positive Ricci curvature.

Theorems A and B were already known in the case when the action is free. Theorem A was established for free actions by Nash in [27] (cf also Berestovskiĭ [3], Bérard-Bergery [2], Gilkey, Park and Tuschmann [10] and [33]). Theorem B was proven by Wei for free actions, with the additional assumption that the base is nonnegatively curved [44]. For actions with only principal orbits, Theorem B follows from work of Fukaya and Yamaguchi (see [9, Theorem 0.18]).

Various examples of $G$–manifolds with positive Ricci curvature and isolated singular orbits are given by Bechtluft-Sachs and Wraith in [1] and by Wraith in [48].

When $\dim M/G = 1$ and $\pi_1(M)$ is finite, Grove and Ziller showed that $M$ admits a $G$–invariant metric with positive Ricci curvature [20], and Schwachhöfer and Tuschmann showed that any cohomogeneity one manifold admits a $G$–invariant metric with almost nonnegative curvature, regardless of the fundamental group [37]. The hypothesis $\mathrm{Ric}(M/G) \geq 1$ in Theorem A implies that $\dim M/G \geq 2$, so Theorem A does not generalize the result of [20], but Theorem B does extend the result of [37].

To prove our theorems we employ two different methods to improve the metric: Cheeger deformations and conformal changes. The same methods were combined by Bettiol in [4] to show that $S^2 \times S^2$ admits positive biorthogonal curvature.

In our context, Cheeger deforming a $G$–invariant metric on $M$ will produce a metric with the desired curvature on any compact subset of the regular part, $M^{\mathrm{reg}}$, of $M$. Rather than explicitly elucidating the aforementioned principle, we have organized the paper to make the proofs of the main results as clear as possible. Nevertheless, it is omnipresent and manifested in Proposition 3.2, Theorems 5.1, 6.2 and 7.4, and Corollary 7.2 below.

We obtain the desired metric in a neighborhood of the singular strata by performing the correct $G$–invariant conformal change. There are two key analytic ideas that make our conformal change work.

The first is based on the universal fact, established by Petersen and the second author in [32], that the Hessian of the distance from any compact Riemannian submanifold $S$

has a prescribed asymptotic behavior at nearby points. It is stated formally in Lemma 2.7 below. Informally, let $\Omega$ be a tubular neighborhood of $S$ on which the closest point map $\mathrm{Pr}\colon \Omega \to S$ is defined. If $\Omega$ is small enough we get an estimate for the Hessian of $\mathrm{dist}(S,\cdot)$ on $\Omega$ by exploiting the fact that the intrinsic metrics on the fibers of Pr are asymptotically Euclidean. This generalizes the known asymptotic estimate for the Hessian of the distance function from a point, which, in turn, is based on the fact that a neighborhood of a point in a Riemannian manifold is asymptotically Euclidean.

The second analytic idea is to perform a conformal change of the metric with a function of the form $e^{2\rho(\mathrm{dist}(S,\cdot))}$, where $\rho\colon (0,\infty) \to \mathbb{R}$ is $C^1$–close to $0$, but $\rho''(t) \ll -1$ for $t$ very close to $0$. Our estimates for the Hessian of $\mathrm{dist}(S,\cdot)$ in Lemma 2.7 coupled with our choice of conformal factor give that the new metric $\widetilde{g} = e^{2\rho\circ\mathrm{dist}(S,\cdot)} g$ has a more desirable curvature. Specifically, given any positive constants $K > 0$ and $\varepsilon > 0$, there is a choice of $\rho$ and a neighborhood $\Omega$ of $S$ so that $\widetilde{g}$ has the following property. For any plane that contains a vector tangent to the fibers of the closest point map $\mathrm{Pr}\colon \Omega \to S$ the sectional curvatures of $\widetilde{g}$ are bounded from below by $K$, and, up to symmetries of the curvature tensor, all other components of the curvature tensor of $\widetilde{g}$ differ from the corresponding components of the curvature tensor of $g$ by no more than $\varepsilon$ (see Theorem 2.1).

The union of the singular strata of a compact $G$–manifold need not be a submanifold, but as it is compact and the union of submanifolds, we can push through a generalization of Theorem 2.1 that applies to the singular strata of a $G$–manifold. This result is Theorem 2.16.

Our conformal change technique will also allow us to show:

**Theorem E** (1) *Given $K, \varepsilon > 0$, $(M, g)$ a Riemannian $n$–manifold such that $\mathrm{Ric}_{(M,g)} \geq n-1$, and $p \in M$, there is a metric $\widetilde{g}$ on $M$ with*

$$\mathrm{Ric}_{(M,\widetilde{g})} \geq n-1-\varepsilon \quad \text{and} \quad \sec_{(M,\widetilde{g})}|_p \geq K.$$

(2) *Given $K > 0$ and $\{(M, g_\alpha)\}_{\alpha=1}^{\infty}$ a family of almost nonnegatively curved Riemannian $n$–manifolds and $p \in M$ there is a sequence of almost nonnegatively curved metrics $\widetilde{g}_\alpha$ on $M$ with*

$$\sec_{(M,\widetilde{g}_\alpha)}|_p > K.$$

(3) *If $\{(M, g_\alpha)\}_{\alpha=1}^{\infty}$ satisfies the hypotheses of parts (1) and (2), then there is a sequence of metrics $\widetilde{g}_\alpha$ on $M$ that satisfies the conclusions of parts (1) and (2).*

(4) *If, in addition, $p$ is a fixed point of an isometric $G$–action for $g$ or $g_\alpha$, then the metrics $\widetilde{g}$ and $\widetilde{g}_\alpha$ can be chosen to be $G$–invariant.*

Recall that $M$ is said to have quasipositive curvature if it is nonnegatively curved and has positive curvature at a point. Just as the set of almost nonnegatively curved metrics is an open neighborhood of the set of nonnegatively curved metrics, so too, the condition in the conclusion of part (2) defines an open neighborhood of the set of metrics with quasipositive curvature.

Since the metrics in part (2) satisfy $\sec_{(M,\tilde{g}_\alpha)}|_p > K$, they also have $\sec_{(M,\tilde{g}_\alpha)} > K$ in a neighborhood of $p$. However, our construction does not allow us to conclude that this neighborhood is independent of the metric $\tilde{g}_\alpha$. Part (2) suggests that a more interesting neighborhood of the quasipositively curved family is the set of almost nonnegatively curved metrics with $\sec_{(M,\tilde{g}_\alpha)} > K$ on an open subset of $M$ that is independent of $\alpha$. The metrics on the Milnor spheres constructed by the second author in [45] are in such a neighborhood.

The paper is organized as follows. In Section 1 we fix notation and review the structure of $G$–manifolds. The discussion of the conformal change occurs in Section 2, where we also prove Theorem E. Cheeger deformations are discussed in Sections 3 and 4. Section 3 reviews the generalities and also discusses the $A$–tensor of the Cheeger submersion on compact subsets of $M^{\mathrm{reg}}$. Section 4 covers the infinitesimal geometry near the singular orbits, especially as it relates to Cheeger deformations. In Section 5 we analyze the curvature of a general $G$–manifold after performing a long term Cheeger deformation, followed by the conformal change of Theorem 2.16. Section 6 concludes the proof of Theorem A. Section 7 finishes the proof of Theorem B, and Section 8 contains the proof of Theorem D.

The sequence of metric deformations used to prove Theorem B can also be used to prove Theorem A, and hence yields a proof of Theorem C. However, the reader who is only interested in the proof of Theorem A can skip Sections 4, 5, 7 and 8. Similarly, the reader only interested in the proof of Theorem B can skip Sections 6 and 8.

**Remark** If $\pi_1(G)$ is finite, then the hypothesis of Theorem A that the principal orbits $G/H$ have finite fundamental group is satisfied, but the converse is false. For example, the principal orbits could be Berger spheres represented as $(S^3 \times S^1)/\Delta(S^1)$. So this is a case where Theorem A is applicable even though $\pi_1(G)$ is infinite.

On the other hand, it would be desirable if the hypothesis that $\pi_1(G/H)$ is finite could be replaced by $\pi_1(M)$ is finite. For example, the round three-sphere admits an isometric torus action with trivial principal isotropy, hence our method does not apply to this simple example.

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Many of the foundational ideas that allowed us to understand the effect of both of our metric deformations were developed in the course of the second author's work with Peter Petersen in [32]; so we are especially indebted to him for numerous conversations with the second author on curvature calculations over the years.

## 1 Notation, conventions and background

In this section we will establish notation and review some background material that we will use in the rest of this paper.

We assume the reader is familiar with the basics of Riemannian submersions as discussed in Gray [12] or O'Neill [29]. We adopt the notation of [29] for the $A$ and $T$ tensors.

For $r > 0$ and a subset $A$ of a metric space $X$ we set

$$B(A, r) \equiv \{x \in X \mid \operatorname{dist}(x, A) < r\}.$$

Let $S$ be a compact submanifold of a compact Riemannian manifold $(M, g)$, and let $\operatorname{inj}(S)$ be the normal injectivity radius of $S$. Let $\Omega$ be an open subset of $B(S, \operatorname{inj}(S)/2)$, the $(\operatorname{inj}(S)/2)$–tubular neighborhood of $S$.

We give $\nu(S)$, the normal bundle of $S$, the Sasaki metric [36]. That is, the foot point map $\nu(S) \to S$ is a Riemannian submersion, the metric on the vertical distribution comes from $g$, and the horizontal distribution $\widetilde{\mathcal{H}}$ is determined by normal parallel transport along $S$.

Let

$$\widetilde{X} \oplus \widetilde{\mathcal{V}}$$

be the orthogonal decomposition of the vertical distribution of $\nu(S) \to S$ where $\widetilde{X}$ is the radial, unit field from the $0$–section $\nu_0(S)$ and $\widetilde{\mathcal{V}}$ is the orthogonal complement of $\widetilde{X}$. Set

$$\mathcal{H} \equiv d\exp_S^{\perp}(\widetilde{\mathcal{H}}), \quad \mathcal{V} \equiv d\exp_S^{\perp}(\widetilde{\mathcal{V}}), \quad X = d\exp_S^{\perp}(\widetilde{X}), \tag{1.0.1}$$

where $\exp_S^{\perp}\colon \nu(S) \to M$ is the normal exponential map.

Note that $X \oplus \mathcal{V}$ is the tangent space to the fibers of the closest point map Pr: $\Omega \setminus S \to S$, and on $\Omega \setminus S$,

$$X = \mathrm{grad}(\mathrm{dist}(S, \cdot)).$$

The distribution $\mathcal{H}$ need not be orthogonal to $X \oplus \mathcal{V}$; however, we will show in Proposition 2.8 that it is asymptotically orthogonal to $X \oplus \mathcal{V}$ near $S$, and hence is very close to $\overline{\mathcal{H}}$, the distribution that is orthogonal to $\mathrm{span}\{X, \mathcal{V}\}$.

We use superscripts to denote components of vectors in subspaces. So, for example, $V^{\mathrm{span}\{X\}}$ is the component of $V$ in $\mathrm{span}\{X\}$ and $V^{\mathcal{V}}$ is the component of $V$ in $\mathcal{V}$.

We write conformal metric changes as $\tilde{g} = e^{2f} g$. We let $\tilde{\nabla}$, $\tilde{R}$, $\widetilde{\sec}$ and $\widetilde{\mathrm{Ric}}$ denote the covariant derivative, curvature tensor, sectional curvature and Ricci tensor of $\tilde{g}$. We denote $R(X,Y,Y,X)$ by $\mathrm{curv}(X,Y)$. We write directional derivatives as $D_V f$, and parameterize geodesics by arc length.

Following Otsu, Shiohama and Yamaguchi [30], we let $\tau \colon \mathbb{R}^k \to \mathbb{R}_+$ be any function that satisfies

$$\lim_{x_1,\dots,x_k \to 0} \tau(x_1, \dots, x_k) = 0. \tag{1.0.2}$$

When making an estimate with a function $\tau$, we implicitly assert the existence of such a function for which the estimate holds.

## 1.1 The stratification of $G$–manifolds

Let $G$ act isometrically on $M$ with both $M$ and $G$ compact. For $x \in M$, we let $G(x)$ be the orbit of $x$, $G_x$ be the isotropy subgroup at $x$ and

$$\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}_x$$

be the decomposition of the Lie algebra of $G$ into $\mathfrak{g}_x$, the Lie algebra of $G_x$, and $\mathfrak{m}_x$ the orthogonal complement of $\mathfrak{g}_x$ with respect to a fixed bi-invariant metric on $G$, $g_{\mathrm{bi}}$.

If $G$ acts isometrically on a Riemannian manifold $M$ and $k \in \mathfrak{g}$, we let $k_M$ denote the Killing field on $M$ generated by $k$.

Recall that $G(x)$ is called a *principal* orbit if and only if for all $y \in M$, there is a $g \in G$ with $G_x \subset g G_y g^{-1}$. An orbit $G(x)$ is exceptional if and only if $G_x$ is a finite extension of some principal isotropy subgroup. Otherwise $G(x)$ is called a singular orbit. All of our arguments about singular orbits apply to exceptional orbits, so for this paper we use the term *singular orbit* to mean any *nonprincipal* orbit.

There is a natural stratification of $M$ into smooth submanifolds by orbit type. The stratum of $x \in M$ is defined to be

$$S(G_x) \equiv \{y \in M \mid \text{there exists } g \in G \text{ with } G_x = gG_yg^{-1}\}.$$

We note that $\overline{S(G_x)}$ is

$$\overline{S(G_x)} \equiv \{y \in M \mid \text{there exists } g \in G \text{ with } G_x \subset gG_yg^{-1}\}.$$

Partially order the closed sets $\overline{S(G_x)}$ by inclusion. If $\overline{S(G_x)}$ is minimal with respect to this partial order, then $\overline{S(G_x)} = S(G_x)$ is a closed submanifold.

The union of the principal orbits is called the regular part of $M$, which we denote by $M^{\mathrm{reg}}$. Recall that we have a proper Riemannian submersion

$$\pi^{\mathrm{reg}} = \pi|_{M^{\mathrm{reg}}} \colon M^{\mathrm{reg}} \longrightarrow M^{\mathrm{reg}}/G.$$

Throughout the paper we assume that $G$ is a compact, connected Lie group acting isometrically and effectively on a compact Riemannian $n$–manifold $(M, g)$ with singular strata $S_1, S_2, \dots, S_p$.

**Proposition 1.1** *There is a neighborhood $\Omega \equiv \bigcup \Omega^i$ of the singular strata $\bigcup S_i$ and, for each $i$, a compact subset $\mathcal{C}_i \subset S_i$. For each $i$, $\Omega^i$ and $\mathcal{C}_i$ are related as follows.*

*Let $\operatorname{int}(\mathcal{C}_i)$ be the interior of $\mathcal{C}_i$ when viewed as a subset of $S_i$. Let $\operatorname{inj}(\mathcal{C}_i)$ be the injectivity radius of the normal bundle $\nu(S_i)|_{\mathcal{C}_i}$, and let $\nu_0(S_i)|_{\mathcal{C}_i}$ be the image of the zero section of $\nu(S_i)|_{\mathcal{C}_i} \to \mathcal{C}_i$. Then*

$$\Omega^i = \exp^{\perp}_{S_i}(B(\nu_0(S_i)|_{\operatorname{int}(\mathcal{C}_i)}, r_i)),$$

*where $r_i \in (0, \operatorname{inj}(\mathcal{C}_i)/2)$.*

**Proof** We define the *descendant number* of a stratum $S_i$ to be the integer $\mathcal{D}(S_i)$ if there are precisely $\mathcal{D}(S_i)$ strata contained in $\overline{S}_i$. Call the union of the strata with descendant number $l$, $\mathbb{S}^l$. We denote by $S^l_\alpha$ the strata so that

$$\mathbb{S}^l \setminus \mathbb{S}^{l-1} = \bigcup_{\alpha \in I_l} S^l_\alpha.$$

The first step to prove the proposition is to establish the following induction statement.

**Induction statement** For each $l$, there are compact subsets $\mathcal{C}_{l,\alpha}$ of $S_\alpha^l$ and neighborhoods $U^l$ and $V^l$ of $\mathbb{S}^l$ of the form

$$V^l \equiv \bigcup_{k=1}^{l} \bigcup_{\alpha \in I_k} V_\alpha^k \quad \text{and} \quad U^l \equiv \bigcup_{k=1}^{l} \bigcup_{\alpha \in I_k} U_\alpha^k,$$

where

$$V_\alpha^l = \exp^\perp_{S_\alpha^l}(B(\nu_0(S_\alpha^l)|_{\mathrm{int}(\mathcal{C}_{l,\alpha})}, r_{l,\alpha}))$$

and

$$U_\alpha^l = \exp^\perp_{S_\alpha^l}\Big(B\Big(\nu_0\Big(S_\alpha^l\Big)\Big|_{\mathrm{int}(\mathcal{C}_{l,\alpha})}, \frac{r_{l,\alpha}}{2}\Big)\Big),$$

and $r_{l,\alpha} \in (0, \mathrm{inj}(\mathcal{C}_{l,\alpha})/2)$.

We prove this statement by induction on the descendant number. The strata with descendant number 1 contain no strata other than themselves and hence are compact submanifolds. Let $\mathcal{C}_{1,\alpha} = S_\alpha^1$ and for $r_{1,\alpha} \in (0, \mathrm{inj}(S_\alpha^1)/2)$, let $V_\alpha^1 \equiv B(S_\alpha^1, r_{1,\alpha})$ and $U_\alpha^1 \equiv B(S_\alpha^1, r_{1,\alpha}/2)$.

Suppose we have constructed $U^1, \dots, U^l$, $V^1, \dots, V^l$, and $\{\mathcal{C}_{1,\alpha}\}_{\alpha \in I_1}, \dots, \{\mathcal{C}_{l,\alpha}\}_{\alpha \in I_l}$ with the desired properties. Set $\mathcal{C}_{l+1,\alpha} = S_\alpha^{l+1} \setminus \{U^l \cap S_\alpha^{l+1}\}$. Note that $\mathbb{S}^{l+1} \subset V^l \cup \bigcup_{\alpha \in I_{l+1}} \mathrm{int}(\mathcal{C}_{l+1,\alpha})$. For $r_{l+1,\alpha} \in (0, \mathrm{inj}(\mathcal{C}_{l+1,\alpha})/2)$ we set

$$V_\alpha^{l+1} = \exp^\perp_{S_\alpha^{l+1}}(B(\nu_0(S_\alpha^{l+1})|_{\mathrm{int}(\mathcal{C}_{l+1,\alpha})}, r_{l+1,\alpha})),$$

and

$$U_\alpha^{l+1} = \exp^\perp_{S_\alpha^{l+1}}\Big(B\Big(\nu_0\Big(S_\alpha^{l+1}\Big)\Big|_{\mathrm{int}(\mathcal{C}_{l+1,\alpha})}, \frac{r_{l+1,\alpha}}{2}\Big)\Big).$$

Note that

$$V^{l+1} \equiv \bigcup_{k=1}^{l+1} \bigcup_{\alpha \in I_k} V_\alpha^k \quad \text{and} \quad U^{l+1} \equiv \bigcup_{k=1}^{l+1} \bigcup_{\alpha \in I_k} U_\alpha^k$$

are neighborhoods of $\mathbb{S}^{l+1}$, proving the induction statement.

The proposition follows from the induction statement by reindexing so that the $S_\alpha^l$ become the $S_i$ and the $V_\alpha^k$ become the $\Omega^i$. □

For each $\Omega^i$ we have a splitting of $(\Omega^i \setminus \mathcal{S}^i)$ as in (1.0.1). We call this splitting

$$\mathcal{H}^i \oplus \mathcal{V}^i \oplus \mathrm{span}\{X^i\}. \tag{1.1.1}$$

## 2 Conformal change

In this section we establish a universal property of any compact submanifold $S$ of any complete Riemannian manifold $(M, g)$. It is stated formally in Theorem 2.1 below, which we describe briefly here.

Given any positive constants $K$ and $\varepsilon$, there is a conformal change $\tilde{g}$ of $g$ and there are neighborhoods $\Omega_1 \subset \Omega_3 \subset B(S, \mathrm{inj}(S)/2)$ so that the new metric is $C^1$–close to $g$, agrees with $g$ outside of $\Omega_3$ and also has the following property.

For any plane that contains a vector in $\mathrm{span}\{X|_{\Omega_1}\} \oplus \mathcal{V}|_{\Omega_1}$ the sectional curvatures of $\tilde{g}$ are bounded from below by $K$ and, up to symmetries of the curvature tensor, all other components of $\tilde{R}$ differ from the corresponding components of $R$ by no more than $\varepsilon$. To prove this we exploit some universal estimates for the asymptotic behavior of $\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}$ near $S$.

We then generalize the conformal change result to a neighborhood of the union of the singular strata of a $G$–action in Theorem 2.16. Since the singular strata are typically noncompact, we first prove an intermediate result, Theorem 2.13, that generalizes Theorem 2.1 to compact subsets of noncompact manifolds. This will allow us to extend Theorem 2.1 to the union of the singular strata, in part because each stratum has a compact exhaustion.

## 2.1 Conformal change around a compact submanifold

**Theorem 2.1** *Let $(M, g)$ be a compact Riemannian $n$–manifold. Let $S$ be a compact, smooth submanifold of $(M, g)$. For any $\varepsilon$, $K > 0$ there are neighborhoods $\Omega_1 \subset \Omega_3$ of $S$ and a metric $\tilde{g} = e^{2f} g$ with the following properties.*

(1) *The metrics $\tilde{g}$ and $g$ coincide on $M \setminus \Omega_3$.*

(2) *For all $V \in \mathrm{span}\{X\} \oplus \mathcal{V}$ and for all $Z \in T\Omega_1$,*

$$\widetilde{\sec}(V, Z)|_{\Omega_1} > K. \tag{2.1.1}$$

(3) *If $\{E_1, \dots, E_n\}$ is a local orthonormal frame for $\Omega_3$ with $X = E_1$ and $\mathrm{span}\{E_2, \dots, E_r\} = \mathcal{V}$ for $2 \le r \le n$, then*

$$|\tilde{R}(E_i, E_j, E_k, E_l) - R(E_i, E_j, E_k, E_l)| < \varepsilon,$$

*except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \mathrm{span}\{X\} \cup \mathcal{V}$.*

(4) *For all $U, W \in TM$,*

$$\widetilde{\sec}(U, W) > \sec(U, W) - \varepsilon.$$

(5) *If $G$ acts isometrically on $(M, g)$ and $S$ is $G$–invariant, then we may choose $\tilde{g}$ to be $G$–invariant.*

**Remark 2.2** While this theorem does not imply that $\Omega_1$ is almost nonnegatively curved, we can conclude, with appropriate choices of $\varepsilon$ and $K$, that $\mathrm{Ric}_{\widetilde{g}}|_{\Omega_1} > 1$.

We get Theorem E by applying Theorem 2.1 in the special case when $S$ is a point.

In our proof of Theorem 2.1, our conformal factor will have the form $e^{2f}$, where $f = \rho \circ \mathrm{dist}(S, \cdot)$ and $\rho$: $[0,\infty) \to \mathbb{R}$ is $C^\infty$, satisfies $\rho|_{(\mathrm{inj}(S)/2,\infty)} \equiv 0$ and will be further specified later. We set $\widetilde{g} = e^{2f} g$.

For ease of notation we set

$$f' \equiv \rho' \circ \mathrm{dist}(S,\cdot), \quad \mathrm{grad}\, f = f'X \quad \text{and} \quad f'' \equiv \rho'' \circ \mathrm{dist}(S,\cdot).$$

The main step to prove Theorem 2.1 is the following.

**Key lemma 2.3** *For every $\varepsilon$, $K > 0$, there is a $\delta > 0$ and a $\sigma_1 \in (0, \mathrm{inj}(S)/2)$ so that the following holds.*

*Suppose that for all $Z \in T\Omega$, for all $V \in \mathrm{span}\{X, \mathcal{V}\}$ and for some $\sigma_3 \in (\sigma_1, \mathrm{inj}(S)/2)$,*

(2.3.1)
$$R(Z,V,V,Z)|_{B(S,\sigma_1)} - f''|_{B(S,\sigma_1)} |Z|^2 |V^{\mathrm{span}\{X\}}|^2 - \frac{f'}{\mathrm{dist}(S,\cdot)}\Big|_{B(S,\sigma_1)} |V^{\mathcal{V}}|^2 |Z|^2 \geq (K+1)|V|^2|Z|^2,$$

$$f' \leq 0, \quad f''|_{B(S,\sigma_1)} \leq 0, \quad |f| + |f'| < \delta, \quad f'' < \delta \quad \textit{and} \quad f|_{M \setminus B(S,\sigma_3)} \equiv 0.$$

*Then:*

(1) *We have*

(2.3.2)
$$\widetilde{\sec}(V,Z)|_{B(S,\sigma_1)} > K.$$

(2) *If $\{E_1, \ldots, E_n\}$ is a local orthonormal frame for $B(S,\sigma_3)$ with $X = E_1$ and $\mathrm{span}\{E_2, \ldots, E_r\} = \mathcal{V}$ for $2 \leq r \leq n$, then*

(2.3.3)
$$|\widetilde{R}(E_i,E_j,E_k,E_l) - R(E_i,E_j,E_k,E_l)| < \varepsilon,$$

*except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \mathrm{span}\{X\} \cup \mathcal{V}$.*

(3) *For all $Z, W \in TM$,*

(2.3.4)
$$\widetilde{\sec}(Z,W) > \sec(Z,W) - \varepsilon.$$

Recall from Walschap [43, page 144] that

$$\begin{aligned} e^{-2f}\,\widetilde{R}(V,Y,Z,U) = R(V,Y,Z,U) &- g(V,U)\operatorname{Hess}_f(Y,Z) - g(Y,Z)\operatorname{Hess}_f(V,U) \\ &+ g(V,Z)\operatorname{Hess}_f(Y,U) + g(Y,U)\operatorname{Hess}_f(V,Z) \\ &+ g(V,U)D_Y f D_Z f + g(Y,Z) D_V f D_U f \\ &- g(Y,U) D_V f D_Z f - g(V,Z) D_Y f D_U f \\ &- g(Y,Z) g(V,U)|\operatorname{grad} f|^2 + g(V,Z) g(Y,U)|\operatorname{grad} f|^2. \end{aligned}$$

Since we assume $|f'| < \delta$ this becomes

$$\begin{aligned} e^{-2f}\,\widetilde{R}(V,Y,Z,U) = R(V,Y,Z,U) &- g(V,U)\operatorname{Hess}_f(Y,Z) \\ &- g(Y,Z)\operatorname{Hess}_f(V,U) + g(V,Z)\operatorname{Hess}_f(Y,U) \\ &+ g(Y,U)\operatorname{Hess}_f(V,Z) \pm O(\delta^2)|V||Y||Z||U|. \end{aligned} \tag{2.3.5}$$

So, to prove the Key lemma we need to understand $\operatorname{Hess}_f$, which we work on in the next subsection.

## 2.2 Universal infinitesimal geometry of tubular neighborhoods

**Proposition 2.4** *Let $X, \mathcal{V}$, and $\mathcal{H}$ be as in (1.0.1). Along a unit speed geodesic, $\gamma$ in $\Omega$, that leaves $S$ orthogonally at $\gamma(0)$ we have the following.*

(1) *At $\gamma(t)$, any vector in $\mathcal{V}$ has the form $J(t)$, where $J$ is a Jacobi field along $\gamma$ that satisfies*

$$J(0) = 0, \quad J'(0) \in \nu_{\gamma(0)}(S) \cap \gamma'(0)^{\perp}. \tag{2.4.1}$$

(2) *At $\gamma(t)$, any vector in $\mathcal{H}$ has the form $J(t)$, where $J$ is a Jacobi field along $\gamma$ that satisfies*

$$J(0), J'(0) \in T_{\gamma(0)} S. \tag{2.4.2}$$

(3) *Let $\operatorname{Sh}_{\gamma'(0)}$ be the shape operator of $S$ at $\gamma(0)$ in the direction of $\gamma'(0)$. That is,*

$$\operatorname{Sh}_{\gamma'(0)}\colon T_{\gamma(0)} S \to T_{\gamma(0)} S$$

*is $\operatorname{Sh}_{\gamma'(0)}(v) \equiv (\nabla_v Z)^{T_{\gamma(0)}S}$, where $Z$ is any extension of $\gamma'(0)$ to a field in $\nu(S)$. Then the Jacobi fields in part (2) also satisfy*

$$J'(0) = \operatorname{Sh}_{\gamma'(0)}(J(0)).$$

(4) *The distribution*

$$\overline{\mathcal{V}}_{\gamma(t)} \equiv \{\mathcal{V}_{\gamma(t)} \mid t > 0\} \cup \{\nu_{\gamma(0)}(S) \cap \gamma'(0)^{\perp}\} \tag{2.4.3}$$

*is smooth along $\gamma$.*

**Proof** A vector in $\widetilde{\mathcal{V}}$ is a value of a variation field of a variation of lines leaving the origin in a normal fiber. Since $\mathcal{V} \equiv d \exp_S^{\perp}(\widetilde{\mathcal{V}})$, a vector in $\mathcal{V}$ is tangent to a variation of geodesics that leave $S$ orthogonally from a single point, and part (1) follows.

With respect to the Sasaki metric, $\nu(S) \to S$ is a Riemannian submersion whose horizontal spaces $\widetilde{\mathcal{H}}$ are given by normal parallel transport of vectors in $\nu(S)$ along curves in $S$. That is, if $Z\colon [a,b] \to \nu(S)$ is a horizontal lift of a curve $c\colon [a,b] \to S$, then

$$(\nabla_{c'} Z)^{\nu(S)} = 0.$$

Exponentiating all real multiples of such a field $Z$ gives a variation of geodesics whose tangent field is $X$. Along the geodesic $t \mapsto \exp_{c(0)} tZ(0)$ the variation field $J$ satisfies

$$J(0) = c'(0) \in T_{\gamma(0)} S.$$

Since $J'(0) = \nabla_{J(0)} Z = \nabla_{c'(0)} Z$ and $(\nabla_{c'} Z)^{\nu(S)} = 0$, it follows that

$$J'(0) \in T_{\gamma(0)} S,$$

proving part (2), and also part (3) since

$$\mathrm{Sh}_{\gamma'(0)}(J(0)) = \nabla_{c'(0)} Z = J'(0).$$

Combining the proofs of parts (1) and (2), we see that together the families of Jacobi fields that span $\mathcal{H} \oplus \mathcal{V}$ come from variations of geodesics that leave $S$ orthogonally. In particular, they form an $(n-1)$–dimensional family of Jacobi fields on which the Riccati operator is self-adjoint; see Wilking [46].

Let $\mathcal{J}^V$ be the family of Jacobi fields along $\gamma$ from part (1). That is,

$$\mathcal{J}^V \equiv \{J \mid J(0) = 0,\, J'(0) \in \nu_{\gamma(0)}(S) \cap \gamma'(0)^{\perp}\}.$$

It follows from part (1) that for $t \in (0, \mathrm{inj}(S))$,

$$\{\mathcal{V}_{\gamma(t)} \mid t > 0\} = \mathrm{span}\{J(t) \mid J \in \mathcal{J}^V\}.$$

For $t = 0$, we have that

$$\nu_{\gamma(0)}(S) \cap \gamma'(0)^{\perp} = \mathrm{span}\{J'(0) \mid J \in \mathcal{J}^V\}.$$

On the other hand, given a nonzero $J \in \mathcal{J}^V$, then for all $t \in (0, \mathrm{inj}(S))$, $J(t) \neq 0$; so

$$\mathrm{span}\{J'(t) \mid J \in \mathcal{J}^V, J(t) = 0\} = \mathrm{span}\{J'(0) \mid J \in \mathcal{J}^V\}.$$

Therefore for $t \in (-\mathrm{inj}(S), \mathrm{inj}(S))$

$$\overline{\mathcal{V}}_{\gamma(t)} = \mathrm{span}\{J(t) \mid J \in \mathcal{J}^V\} \oplus \mathrm{span}\{J'(t) \mid J \in \mathcal{J}^V, J(t) = 0\}. \tag{2.4.4}$$

As asserted on [46, page 1300], $\overline{\mathcal{V}}_{\gamma(t)}$ depends smoothly on $t$; cf Gromoll and Walschap [16, Lemma 1.7.1]. This proves part (4). □

**Remark 2.5** Note that the first summand in (2.4.4) vanishes only at $t=0$ and the second summand is only nonzero at $t=0$.

**Lemma 2.6** *Let $S$ be a compact submanifold of a Riemannian $n$–manifold $M$. There are constants $C_1, C_2$ so that if $\gamma\colon [0,l]\to\Omega\subset B(S,\mathrm{inj}(S)/2)$ is any unit speed geodesic that leaves $S$ orthogonally and $J$ is any Jacobi field along $\gamma$ as in (2.4.2), then*

$$|J(t)|\geq C_1|J(0)| \quad \textit{and} \quad |J'(t)|\leq C_2|J(0)|.$$

**Proof** All constants that we discuss in this proof are independent of $\gamma$. For simplicity we also suppose that $|J(0)|=1$.

Let $\widetilde{V}$ be the variation of lines in $\nu(S)$ that corresponds to $J$. Then $\widetilde{V}(t,0)=t\gamma'(0)$ and the variation field $\frac{\partial}{\partial s}\widetilde{V}|_{s=0}$ consists of lifts of $J(0)$ to the normal bundle, $\nu(S)$, along $t\gamma'(0)$. In particular, $|\frac{\partial}{\partial s}\widetilde{V}|_{s=0}|\equiv|J(0)|=1$. Since $J$ is the image of $\frac{\partial}{\partial s}\widetilde{V}|_{s=0}$ under $d\exp$, it follows from compactness that there is a constant $C_3>0$ so that

$$|J(t)|\leq C_3. \tag{2.6.1}$$

Also since $\gamma(t)\subset B(S,\mathrm{inj}(S)/2)$, $J(t)\neq 0$, so there is a constant $C_1>0$ so that

$$|J(t)|\geq C_1=C_1|J(0)|.$$

Since $J'(0)=\mathrm{Sh}_{\gamma'(0)}(J(0))$, by continuity of the shape operator and compactness of the unit normal bundle of $S$ there is a constant $C_4>0$ so that

$$|J'(0)|\leq C_4=C_4|J(0)|. \tag{2.6.2}$$

Let $\{E_i\}_{i=1}^n$ be an orthonormal parallel frame along $\gamma$ with $E_1(t)=\gamma'(t)$ and write $J(t)=\sum_{i=2}^n e^i(t)E_i$ . Then

$$J'(t)=\sum_{i=2}^n (e^i)'(t)E_i \quad \text{and} \quad -R(J,\gamma')\gamma'=J''(t)=\sum_{i=2}^n (e^i)''(t)E_i,$$

so

$$\begin{aligned}\left|\sum_{i=2}^n (e^i)''(t)E_i\right| &\leq |R||J|\leq C_3|R||J(0)| && \text{(by (2.6.1))}\\ &\leq C_5|J(0)| && \text{(by compactness of } M)\end{aligned}$$

for some constant $C_5>0$. Combining this with inequality (2.6.2) and the fundamental theorem of calculus completes the proof. □

The following is from a revised version of [32].

**Lemma 2.7** *There is a constant $C > 0$ so that on $\Omega \setminus S$ we have:*

(1) $(\mathrm{Hess}_{\mathrm{dist}(S,\cdot)})(X, \cdot) = 0$.

(2) *For $Z \in \mathcal{H}$ and $Y \in \mathcal{V} \oplus \mathcal{H}$,*

$$|(\mathrm{Hess}_{\mathrm{dist}(S,\cdot)})(Y, Z)| < C|Y||Z|.$$

(3) *For $V \in \mathcal{V}$ and $W \in \mathcal{V} \oplus \mathcal{H}$,*

$$\left|\left(\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}(V, W) - \frac{1}{\mathrm{dist}(S,\cdot)} g(V, W)\right)\right| < O(\mathrm{dist}(S,\cdot))|V||W|. \tag{2.7.1}$$

**Proof** Recall that $X = \mathrm{grad}(\mathrm{dist}(S,\cdot))$. So $\nabla_X X = 0$, thus $(\mathrm{Hess}_{\mathrm{dist}(S,\cdot)})(X, \cdot) = 0$.

To prove the estimates in parts (2) and (3), we first focus on a fixed geodesic $\gamma\colon [0, l] \to \Omega$ that leaves $S$ orthogonally at time $0$.

For the second estimate, we let $J$ be a Jacobi field along $\gamma$ as in (2.4.2). Then for $Y \in \mathcal{V}_{\gamma(t)} \oplus \mathcal{H}_{\gamma(t)}$,

$$\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}(J(t), Y) = g(\nabla_{J(t)} X, Y) = g(J'(t), Y). \tag{2.7.2}$$

By Lemma 2.6, there are constants $C_1, C_2$ so that $|J(t)| \geq C_1|J(0)|$ and $|J'(t)| \leq C_2|J(0)|$. So for $Y \in \mathcal{V}_{\gamma(t)} \oplus \mathcal{H}_{\gamma(t)}$ and $Z \in \mathcal{H}_{\gamma(t)}$ with $Z = J(t)$ as in (2.4.2) we get from (2.7.2) that

$$\left|(\mathrm{Hess}_{\mathrm{dist}(S,\cdot)})(Z, Y)\right| \leq |J'(t)||Y| \leq C_2|J(0)||Y| \leq \frac{C_2}{C_1}|J(t)||Y| \leq C|Y||Z|$$

for $C = C_2/C_1$, proving part (2), along $\gamma$.

Similarly, for $J$ as in (2.4.1) and $Y \in \mathcal{V} \oplus \mathcal{H}$, we have $\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}(J(t), Y) = g(J'(t), Y)$, so

$$\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}\left(\frac{J(t)}{|J(t)|}, Y\right) = \frac{1}{|J(t)|} g(J'(t), Y)$$

and

$$\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}\left(\frac{J(t)}{|J(t)|}, \frac{J(t)}{|J(t)|}\right) = \frac{1}{|J(t)|^2} g(J'(t), J(t)).$$

Write $J(t) = \sum_{i=2}^{n} e^i E_i$, where $\{E_i\}_{i=1}^{n}$ is an orthonormal parallel frame along $\gamma$ with $E_1(t) = \gamma'(t)$ and $E_2(0) = J'(0)$. Then

$$\begin{aligned} &e^i(0) = 0 && \text{for } i \geq 2, \\ &(e^2)'(0) = 1, \quad (e^i)'(0) = 0 && \text{for } i \geq 3, \end{aligned}$$

and since $0 = -R(J, \dot{\gamma})\dot{\gamma}|_0 = J''(0) = \sum_{i=2}^{n} (e^i)''(0) E_i(0)$,

$$(e^i)''(0) = 0 \quad \text{for } i \geq 2.$$

Thus we get

$$\begin{aligned} J(t) &= (t + O(t^3))E_2(t) + \sum_{i=3}^{n} O(t^3) E_i, \\ J'(t) &= (1 + O(t^2))E_2(t) + \sum_{i=3}^{n} O(t^2) E_i, \\ |J(t)| &= t + O(t^3), \qquad |J(t)|^2 = t^2 + O(t^4), \end{aligned} \tag{2.7.3}$$

so

$$\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}\left(\frac{J(t)}{|J(t)|}, \frac{J(t)}{|J(t)|}\right) = \frac{1}{|J(t)|^2} g(J'(t), J(t)) = \frac{t + O(t^3)}{t^2 + O(t^4)} = \frac{1}{t} + O(t),$$

and part (3) holds along $\gamma$ when $V = W$.

For $Y \in \mathcal{V} \oplus \mathcal{H}$ with $Y \perp J(t)$, $|Y| = 1$, we write $Y(t) = \sum_{i=2}^{n} \alpha_i E_i$ . Since $|Y| = 1$, $|\alpha_i| \leq 1$. Combining this with $Y \perp J(t)$, we get $\alpha_2 = O(t^2)$, so

$$\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}\left(\frac{J(t)}{|J(t)|}, Y\right) = \frac{1}{|J(t)|} g(J'(t), Y) = \frac{O(t^2)}{t + O(t^3)} = O(t),$$

and part (3) holds along $\gamma$.

The result follows in general from continuity and the compactness of the unit normal bundle of $S$. $\square$

The distributions $\mathcal{H}$ and $\mathcal{V}$ are not orthogonal, but they are asymptotically orthogonal to a high order as $\mathrm{dist}(S, \cdot) \to 0$, as we show in the following proposition.

**Proposition 2.8** *There is a constant $C > 0$ with the following property. Let $\gamma$ be a unit speed geodesic in $\Omega$, that leaves $S$ orthogonally at $\gamma(0)$. Let $J_1$ and $J_2$ be Jacobi fields along $\gamma$ with*

$$\begin{aligned} J_1(0) = 0,\ J_1'(0) \in \nu_{\gamma(0)}(S), &\quad |J_1'(0)| = 1, \\ J_2(0),\ J_2'(0) \in T_{\gamma(0)}S, &\quad |J_2(0)| = 1. \end{aligned}$$

*Then*

$$|g(J_1(\gamma(t)), J_2(\gamma(t)))| \leq C t^3.$$

**Remark 2.9** Since $J_1$ satisfies (2.4.1) and $J_2$ satisfies (2.4.2) this tells us that near $S$ the distributions $\mathcal{H}$ and $\mathcal{V}$ are almost orthogonal.

**Proof** Just notice that

$$\begin{aligned}
g(J_1,J_2)(0) &= 0 && \text{since } J_1(0)=0,\\
g(J_1,J_2)'(0) &= g(J_1',J_2)(0)+g(J_1,J_2')(0)\\
&= 0 && \text{since } J_1(0)=0,\, J_2(0)\in T_{\gamma(0)}S,\, J_1'(0)\in \nu_{\gamma(0)}(S),\\
g(J_1,J_2)''(0) &= g(J_1'',J_2)(0)+2g(J_1',J_2')+g(J_1,J_2'')(0)\\
&= 0 && \text{since } J_1'(0)\in \nu_{\gamma(0)}(S),\ J_2'(0)\in T_{\gamma(0)}S,\\
& && \quad J_1(0)=0,\ J_1''(0)=-R(J_1,\dot\gamma)\dot\gamma|_0=0.
\end{aligned}$$

This gives us the desired estimate for any particular choice of $\gamma$, $J_1$ and $J_2$. We then get the result with a uniform constant $C$ from compactness of the unit tangent bundle of $M$ along $S$. □

Proposition 2.8 allows us to estimate the entire Hessian of $f$ near $S$ by estimating its values on vectors in $X\cup\mathcal{V}\cup\mathcal{H}$.

**Lemma 2.10** *On $\Omega\setminus S$:*

(1) $\mathrm{Hess}_f(X,X)=f''$.

(2) *For $Y\in\mathrm{span}\{X,\mathcal{V},\mathcal{H}\}$ and $Z\in\mathcal{H}$ and $\delta>0$ as in the Key lemma,*
$$|\mathrm{Hess}_f(Y,Z)|<O(\delta)|Y||Z|.$$

(3) *For $Y\in\mathcal{V}$ and $Z\in\mathrm{span}\{X,\mathcal{V}\}$,*
$$\left|\mathrm{Hess}_f(Y,Z)-\frac{f'}{\mathrm{dist}(S,\cdot)}g(Y,Z)\right|\le \delta O(\mathrm{dist}(S,\cdot))|Y||Z|.$$

**Proof** Recall our notational shorthand $f=\rho\circ\mathrm{dist}(S,\cdot)$, $f'\equiv\rho'\circ\mathrm{dist}(S,\cdot)$ and $f''\equiv\rho''\circ\mathrm{dist}(S,\cdot)$. So $\mathrm{grad}\, f=f'X$ and $\mathrm{grad}\, f'=f''X$. Thus

$$\begin{aligned}
\text{(2.10.1)}\qquad \mathrm{Hess}_f(Y,Z) &= g(\nabla_Y f'X,Z)\\
&= (D_Y f')g(X,Z)+f'g(\nabla_Y X,Z)\\
&= g(Y,\mathrm{grad}\, f')g(X,Z)+f'\,\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}(Y,Z)\\
&= f''g(Y,X)g(X,Z)+f'\,\mathrm{Hess}_{\mathrm{dist}(S,\cdot)}(Y,Z).
\end{aligned}$$

The lemma follows from Lemma 2.7, (2.10.1) and our hypothesis that $|f'|<\delta$. □

Combining the previous two results gives us the following.

**Corollary 2.11** *For $\bar{Y} \in \operatorname{span}\{X, \mathcal{V}, \bar{\mathcal{H}}\}$, $\bar{Z} \in \bar{\mathcal{H}}$, with foot point in $\Omega \setminus S$ sufficiently close to $S$, and for $\delta$ as in the Key lemma*

$$|\mathrm{Hess}_f(\bar{Y}, \bar{Z})| < O(\delta)|\bar{Y}||\bar{Z}|.$$

**Proof** Write $\bar{Y} = Y + V$ with $Y \in \mathcal{H}$ and $V \in \operatorname{span}\{X, \mathcal{V}\}$, and write $\bar{Z} = Z + W$ with $Z \in \mathcal{H}$ and $W \in \operatorname{span}\{\mathcal{V}\}$. So

$$|\mathrm{Hess}_f(\bar{Y}, \bar{Z})| \le |\mathrm{Hess}_f(Y, Z)| + |\mathrm{Hess}_f(Y, W)| + |\mathrm{Hess}_f(V, Z)| + |\mathrm{Hess}_f(V, W)|.$$

By Proposition 2.8, $|W| = |\bar{Z}|O(\operatorname{dist}(S, \cdot)^2)$. Combining this with Lemma 2.10 and our hypothesis that $\Omega \subset B(S, \operatorname{inj}(S)/2)$ gives us

$$|\mathrm{Hess}_f(\bar{Y}, \bar{Z})| \le O(\delta)|\bar{Y}||\bar{Z}|. \qquad \square$$

We are now in a position to prove the Key lemma.

**Proof of the Key lemma** From (2.3.5), we have for $Y \perp U$ that

$$\text{(2.11.1)} \quad \begin{aligned} e^{-2f}\tilde{R}(U, Y, Y, U) \ge R(U, Y, Y, U) &- g(U, U)\,\mathrm{Hess}_f(Y, Y) \\ &- g(Y, Y)\,\mathrm{Hess}_f(U, U) - |O(\delta^2)||Y|^2|U|^2. \end{aligned}$$

Combining this with Lemma 2.10 we have for $Z \in T\Omega$ and all $V \in \operatorname{span}\{X, \mathcal{V}\}$ with $Z \perp V$

$$\begin{aligned} e^{-2f}\tilde{R}(Z, V, V, Z)|_{B(S,\sigma_1)} \ge R(Z, V, V, Z)|_{B(S,\sigma_1)} &- f''|_{B(S,\sigma_1)}(|Z|^2|V^{\operatorname{span}\{X\}}|^2 + |Z^{\operatorname{span}\{X\}}|^2|V|^2) \\ &- \frac{f'}{\operatorname{dist}(S, \cdot)}\bigg|_{B(S,\sigma_1)}(|V^{\mathcal{V}}|^2|Z|^2 + |Z^{\mathcal{V}}|^2|V|^2) \\ &- |O(\delta)||Z|^2|V|^2 \end{aligned}$$

provided $\sigma_1$ is sufficiently small. Since we assumed that

$$\begin{aligned} R(Z, V, V, Z)|_{B(S,\sigma_1)} - f''|_{B(S,\sigma_1)}|Z|^2|V^{\operatorname{span}\{X\}}|^2 - \frac{f'}{\operatorname{dist}(S, \cdot)}\bigg|_{B(S,\sigma_1)}|V^{\mathcal{V}}|^2|Z|^2 \\ \ge (K+1)|V|^2|Z|^2, \end{aligned}$$

as well as $f'|_{B(S,\sigma_1)} \le 0$, $f''|_{B(S,\sigma_1)} \le 0$ and $|f| < \delta$, we obtain

$$\widetilde{\sec}(V, Z)|_{B(S,\sigma_1)} > K,$$

provided $\delta$ is sufficiently small.

Now consider (not necessarily distinct) orthonormal vectors $E, Y, Z, U \in \operatorname{span}\{X\} \cup \mathcal{V} \cup \overline{\mathcal{H}}$. Then

$$\begin{aligned}(2.11.2)\quad e^{-2f}\widetilde{R}(E,Y,Z,U) = R(E,Y,Z,U)& \\ &- g(E,U)\operatorname{Hess}_f(Y,Z) - g(Y,Z)\operatorname{Hess}_f(E,U) \\ &+ g(E,Z)\operatorname{Hess}_f(Y,U) + g(Y,U)\operatorname{Hess}_f(E,Z) \\ &\pm O(\delta^2)|E||Y||Z||U|.\end{aligned}$$

If we further assume that $R(E,Y,Z,U)$ does not correspond, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \operatorname{span}\{X\} \cup \mathcal{V}$, it then follows from Lemma 2.10 and Corollary 2.11 that all four Hessian terms are bounded from above by $O(\delta)$. So

$$e^{-2f}\widetilde{R}(E,Y,Z,U) = R(E,Y,Z,U) \pm O(\delta)|E||Y||Z||U|.$$

We then get inequality (2.3.3) by choosing $\delta$ to be sufficiently small.

On $M \setminus B(S,\sigma_3)$ inequality (2.3.4) follows from the hypothesis that $f|_{M\setminus B(S,\sigma_3)} \equiv 0$. We get inequality (2.3.4) on $B(S,\sigma_3)$ by combining inequalities (2.3.3) and (2.11.1) with Lemma 2.10, Corollary 2.11 and the hypothesis that $|f'| + f'' < 2\delta$. □

Now we prove Theorem 2.1.

**Proof of Theorem 2.1** Given $\varepsilon, K > 0$, choose $\delta$ and $\sigma_1$ as in the Key lemma. Let $\sigma_2, \sigma_3$, and $\sigma_4$ be such that $\sigma_1 < \sigma_2 \ll \sigma_3 < \sigma_4 < \min\{\operatorname{inj}(S)/2, \frac{1}{4}\}$, and let $\rho\colon [0,\infty) \to \mathbb{R}$ satisfy the following conditions.

(1) All derivatives of $\rho$ of odd order at $0$ are equal to $0$.

(2) $K + 2 > -\rho''(t)|_{[0,\sigma_1]} + \min\sec_g > K + 1$.

(3) $\rho''(t)|_{[0,\sigma_2]} \leq 0$, $\rho'(t) \leq 0$.

(4) $0 \leq \rho''|_{(\sigma_2,\infty)} < \delta$.

(5) $|\rho'| + |\rho| < \delta$.

(6) $\rho|_{[\sigma_3,\infty)} \equiv 0$.

Since $f = \rho \circ \operatorname{dist}(S,\cdot)$, condition (1) gives us that our conformal factor $e^{2f}$ is a smooth function on $M$.

The fundamental theorem of calculus and condition (2) give

$$-\rho'(t)|_{[0,\sigma_1]} > (K + 1 - \min\sec_g)t,$$

so

$$-\frac{\rho'(t)|_{[0,\sigma_1]}}{t} + \min \sec_g > (K+1). \tag{2.11.3}$$

For $V \in \operatorname{span}\{X, \mathcal{V}\}$, write $V = V^{\operatorname{span}\{X\}} + V^{\mathcal{V}}$. Then condition (2) gives

$$-\rho''(t)|_{[0,\sigma_1]} |V^{\operatorname{span}\{X\}}|^2 + \min \sec_g |V^{\operatorname{span}\{X\}}|^2 > (K+1)|V^{\operatorname{span}\{X\}}|^2,$$

and inequality (2.11.3) gives

$$-\frac{\rho'(t)|_{[0,\sigma_1]}}{t} |V^{\mathcal{V}}|^2 + \min \sec_g |V^{\mathcal{V}}|^2 > (K+1)|V^{\mathcal{V}}|^2.$$

Adding the previous two inequalities we get

$$-\rho''(t)|_{[0,\sigma_1]} |V^{\operatorname{span}\{X\}}|^2 - \frac{\rho'(t)|_{[0,\sigma_1]}}{t} |V^{\mathcal{V}}|^2 + \min \sec_g |V|^2 > (K+1)|V|^2.$$

Let $t = \operatorname{dist}(S, \cdot)$; then $f' \equiv \rho'(t)$ and $f'' \equiv \rho''(t)$. Making these substitutions, multiplying both sides by $|Z|^2$, and using $R(Z, V, V, Z)|_{B(S,\sigma_1)} \geq \min \sec_g |V|^2|Z|^2$ gives

$$R(Z,V,V,Z)|_{B(S,\sigma_1)} - f''|_{B(S,\sigma_1)} |Z|^2 |V^{\operatorname{span}\{X\}}|^2 - \frac{f'}{\operatorname{dist}(S,\cdot)}\bigg|_{B(S,\sigma_1)} |V^{\mathcal{V}}|^2|Z|^2 \geq (K+1)|V|^2|Z|^2.$$

This establishes (2.3.1) of the Key lemma. The other hypotheses of the Key lemma follow from the properties of $\rho$ (numbered (3)–(6) above). We then apply the Key lemma to obtain the curvature bounds of Theorem 2.1. Finally, if $G$ acts isometrically on $M$ and $S$ is $G$–invariant, then $\tilde{g}$ is as well, since $f = \rho \circ \operatorname{dist}(S, \cdot)$. □

**Remark 2.12** Given $\varepsilon$ and $K$, if the Key lemma holds for $\delta = \delta_0$, then it also holds for all $\delta \in (0, \delta_0)$. Since $|\rho'| + |\rho| < \delta$, and $f = \rho \circ \operatorname{dist}(S, \cdot)$, our conformal factor $e^{2f}$ can be as close as we please in the $C^1$–topology to $1$.

## 2.3 Conformal change near a compact subset of a noncompact submanifold

Since the strata can be noncompact manifolds, we will need to generalize Theorem 2.1. Let $(M, g)$ be a compact Riemannian $n$–manifold. Let $S$ be a smooth submanifold of $(M, g)$. Let $\mathcal{C}_1$ be a compact subset of $S$. Let $\operatorname{inj}(\mathcal{C}_1)$ be the injectivity radius of the normal bundle $\nu(S)|_{\mathcal{C}_1}$. Let $\nu_0(S)|_{\mathcal{C}_1}$ be the image of the zero section of $\nu(S)|_{\mathcal{C}_1} \to \mathcal{C}_1$. Let

$$\Omega \equiv \exp_S^{\perp}\Big(B\Big(\nu_0(S)|_{\mathcal{C}_1}, \frac{\operatorname{inj}(\mathcal{C}_1)}{2}\Big)\Big),$$

and let

$$X \oplus \mathcal{V} \oplus \mathcal{H}$$

be the splitting of $T\Omega$ given in (1.0.1).

**Theorem 2.13** *Let $(M,g)$, $S$, $\mathcal{C}_1$, $X$ and $\mathcal{V}$ be as above, and let $\mathcal{C}_3$ be any compact subset of $S$ with $\mathcal{C}_1 \subset \operatorname{int}(\mathcal{C}_3)$. For any $\varepsilon, K > 0$ there are numbers $\sigma_1, \sigma_3$ with $0 < \sigma_1 < \sigma_3 < \operatorname{inj}(\mathcal{C}_3)/2$ and a metric $\tilde{g} = e^{2f} g$ with the following properties.*

1. *Setting $\Omega_1 \equiv \exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_1}, \sigma_1)$ and $\Omega_3 \equiv \exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_3}, \sigma_3)$, the metrics $\tilde{g}$ and $g$ coincide on $M \setminus \Omega_3$.*
2. *For all $Z \in T\Omega_1$ and all $V \in \operatorname{span}\{X, \mathcal{V}\}$*

$$\widetilde{\sec}(V,Z)|_{\Omega_1} > K. \tag{2.13.1}$$

3. *If $\{E_1, \dots, E_n\}$ is a local orthonormal frame for $\Omega_3$ with $X = E_1$ and $\operatorname{span}\{E_2, \dots, E_r\} = \mathcal{V}$ for $2 \le r \le n$, then*

$$|\tilde{R}(E_i, E_j, E_k, E_l) - R(E_i, E_j, E_k, E_l)| < \varepsilon, \tag{2.13.2}$$

*except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \operatorname{span}\{X\} \cup \mathcal{V}$.*

4. *We have*

$$\widetilde{\sec}(V,W) > \sec(V,W) - \varepsilon \tag{2.13.3}$$

*for all $V, W \in TM$.*

*Moreover, if $G$ acts isometrically on $(M,g)$ and both $S$ and $\mathcal{C}_1$ are $G$–invariant, then we may choose $\tilde{g}$ to be $G$–invariant.*

**Remark 2.14** As was the case for Theorem 2.1, with appropriate choices of $\varepsilon$ and $K$, $\operatorname{Ric}_{\tilde{g}}|_{\Omega_1} > 1$.

**Proof of Theorem 2.13** Let $\mathcal{C}_2$ and $\mathcal{C}_4$ be compact subsets of $S$ with $\mathcal{C}_1 \subset \operatorname{Int}(\mathcal{C}_2)$, $\mathcal{C}_2 \subset \operatorname{Int}(\mathcal{C}_3)$ and $\mathcal{C}_3 \subset \operatorname{Int}(\mathcal{C}_4)$. Let $\operatorname{inj}(\mathcal{C}_4)$ be the injectivity radius of the normal bundle $\nu(S)|_{\mathcal{C}_4}$. Let $\bar{\varphi}\colon S \to [0,1]$ be $C^\infty$ and satisfy

$$\bar{\varphi} = \begin{cases} 1 & \text{on } \mathcal{C}_2, \\ 0 & \text{on } S \setminus \mathcal{C}_3. \end{cases}$$

Given $\sigma_4 \in (0, \operatorname{inj}(\mathcal{C}_4)/2)$, extend $\bar{\varphi}$, by exponentiation, to a function $\varphi$ defined on $\exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_4}, \sigma_4)$ by setting

$$\varphi(x) = \bar{\varphi}(\text{footpoint}((\exp_S^{\perp})^{-1}(x))).$$

Our conformal factor is $e^{2f}$, where

$$f(x) \equiv \begin{cases} (\rho \circ \operatorname{dist}(S,x)) \cdot \varphi(x) & \text{for } x \in \exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_4}, \sigma_4), \\ 0 & \text{for } x \in M \setminus \exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_3}, \sigma_3), \end{cases}$$

and $\rho$ is as in the proof of Theorem 2.1. Since $(\rho \circ \operatorname{dist}(S,x)) \cdot \varphi(x)$ is $0$ on $(\exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_4}, \sigma_4)) \setminus (\exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_3}, \sigma_3))$, $f$ is a well-defined $C^\infty$ function.

Setting $\tilde{f} \equiv \rho \circ \operatorname{dist}(S, \cdot)$, we have that on $\exp_S^{\perp} B(\nu_0(S)|_{\mathcal{C}_4}, \sigma_4)$,

$$f = \varphi \cdot \tilde{f},$$

$$\operatorname{grad}(f) = \varphi \operatorname{grad}(\tilde{f}) + \tilde{f} \operatorname{grad}(\varphi). \tag{2.14.1}$$

Since $|\tilde{f}|, |\operatorname{grad}(\tilde{f})| < \delta$ and $|\varphi| \le 1$, we see from (2.14.1) that if $\delta$ is sufficiently small compared to $|\operatorname{grad} \varphi|$, then

$$|\operatorname{grad}(f)| < O(\delta). \tag{2.14.2}$$

Since $\operatorname{grad}(f) = \varphi \operatorname{grad}(\tilde{f}) + \tilde{f} \operatorname{grad}(\varphi)$,

$$\begin{aligned} \operatorname{Hess}_f(V,W) &= g(\nabla_V(\varphi \operatorname{grad}(\tilde{f}) + \tilde{f} \operatorname{grad}(\varphi)), W) \\ &= (D_V \varphi) g(\operatorname{grad}(\tilde{f}), W) + \varphi g(\nabla_V(\operatorname{grad}(\tilde{f})), W) \\ &\qquad + (D_V \tilde{f}) g(\operatorname{grad}(\varphi), W) + \tilde{f} g(\nabla_V \operatorname{grad}(\varphi), W) \\ &= (D_V \varphi) D_W \tilde{f} + \varphi \operatorname{Hess}_{\tilde{f}}(V,W) + (D_V \tilde{f}) D_W \varphi + \tilde{f} \operatorname{Hess}_\varphi(V,W). \end{aligned}$$

Using $|\tilde{f}|, |\operatorname{grad}(\tilde{f})| < \delta$ and choosing $\delta$ small compared to both $|\operatorname{grad} \varphi|$ and $|\operatorname{Hess}_\varphi|$ gives us

$$\operatorname{Hess}_f(V,W) = \varphi \operatorname{Hess}_{\tilde{f}}(V,W) + O(\delta)|V||W|. \tag{2.14.3}$$

Inequality (2.14.2) and (2.14.3) allow us to argue as in the proof of Theorem 2.1 to obtain the curvature estimates in (2.13.1), (2.13.2) and (2.13.3).

If $S$ and $\mathcal{C}_1$ are $G$–invariant, we take $\mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ to be metric neighborhoods of $\mathcal{C}_1$ within $S$. Let $\overline{\varphi}$ have the form $\overline{\varphi} = \psi \circ \operatorname{dist}(\mathcal{C}_1, \cdot)$, where $\psi \colon \mathbb{R} \to \mathbb{R}$. Such functions $\overline{\varphi}$ are $G$–invariant and have $G$–invariant smoothings, using the Riemannian convolution technique of, for example, Greene and Wu [13; 14] or Grove and Shiohama [17]. Extending $\overline{\varphi}$ by exponentiation as above then gives a smooth $G$–invariant $\varphi$ and hence a $G$–invariant $\tilde{g}$. □

**Remark 2.15** As was the case for Theorem 2.1, the conformal factor $e^{2f}$ can be as close to $1$ as we please in the $C^1$–topology.

## 2.4 Conformal change in a neighborhood of the entire singular strata

The conformal change that we actually use to prove Theorems A and B is the one obtained from the following theorem.

**Theorem 2.16** *Let $G$ be a compact, connected Lie group acting isometrically and effectively on a compact Riemannian $n$–manifold $(M, g)$ with singular strata $S_1, S_2, \dots, S_p$.*

*For any $\varepsilon, K > 0$ there are neighborhoods $\Omega_1 \subset \Omega_3$ of $S_1 \cup S_2 \cup \cdots \cup S_p$, and a $G$–invariant metric $\tilde{g} = e^{2f} g$ that have the following properties.*

(1) *For each $S_i$ there is a compact subset $\mathcal{C}_i \subset S_i$ and tubular neighborhoods $\Omega_1^i \subset \Omega_3^i$ as in Theorem 2.13 with $\Omega_1 = \bigcup \Omega_1^i$ and $\Omega_3 = \bigcup \Omega_3^i$. Let $T\Omega_3^i = \mathrm{span}\{X_i\} \oplus \mathcal{V}_i \oplus \mathcal{H}_i$ be the splitting as in (1.0.1).*

(2) *The metrics $\tilde{g}$ and $g$ coincide on $M \setminus \Omega_3$.*

(3) *For all $i \in \{1, \dots, p\}$, all $Z \in T\Omega_1^i$ and all $V \in \mathrm{span}\{X_i\} \oplus \mathcal{V}_i$*

$$\widetilde{\sec}(V, Z)|_{\Omega_1^i} > K. \tag{2.16.1}$$

(4) *If $\{E_1, \dots, E_n\}$ is a local orthonormal frame for $\Omega_3^i$ with $X = E_1$ and $\mathrm{span}\{E_2, \dots, E_r\} = \mathcal{V}_i$ for $2 \le r \le n$, then*

$$|\tilde{R}(E_i, E_j, E_k, E_l) - R(E_i, E_j, E_k, E_l)| < \varepsilon, \tag{2.16.2}$$

*except if the quadruple corresponds, up to a symmetry of the curvature tensor, to the sectional curvature of a plane containing a vector $V \in \mathrm{span}\{X_i\} \cup \mathcal{V}_i$.*

(5) *We have*

$$\widetilde{\sec}(V, W) > \sec(V, W) - \varepsilon \tag{2.16.3}$$

*for all $V, W \in TM$.*

**Proof** Our proof is by induction on descendant number, which was defined in the proof of Proposition 1.1. A stratum with descendant number 1 contains no stratum other than itself and hence is a compact submanifold. We apply Theorem 2.1 to obtain a $G$–invariant conformal change and neighborhoods $\Omega_1^1$ and $\Omega_3^1$ of all the strata with descendant number 1 on which the inequalities (2.16.1), (2.16.2) and (2.16.3) hold.

Now suppose we have such a $G$–invariant conformal change and neighborhoods $\Omega_1^l$ and $\Omega_3^l$ for all strata whose descendant number is $\le l$. For each stratum $S$ with descendant number $l+1$, we choose a compact subset $\mathcal{C}$ of $S$ so that $\overline{S} \subset \mathcal{C} \cup \Omega_1^l$. Applying Theorem 2.13 to each such $S$ yields a $G$–invariant conformal change and neighborhoods $\Omega_1^{l+1}$ and $\Omega_3^{l+1}$ of all the strata with descendant number $l+1$ that satisfy inequalities (2.16.1), (2.16.2) and (2.16.3). □

**Remark 2.17** Our conformal factor can be as close as we please in the $C^1$–topology to 1 since it comes from repeated applications of Theorem 2.13.

# 3 Cheeger deformations

In the presence of a group of isometries $G$, a method for deforming the metric on a manifold $M$ of nonnegative sectional curvature is given in Cheeger [6]. It is based on the Gray–O'Neill principle that Riemannian submersions do not decrease the curvatures of horizontal planes. We briefly review the basics of this construction here, largely following the exposition from Petersen and the second author [31].

Let $G$ be a compact group of isometries of $(M, g_M)$, $g_{\mathrm{bi}}$ a bi-invariant metric on $G$, and consider the one parameter family $l^2 g_{\mathrm{bi}} + g_M$ of metrics on $G \times M$. Then $G$ acts on $(G \times M, l^2 g_{\mathrm{bi}} + g_M)$ via

(3.0.1) $$g(p, m) = (pg^{-1}, gm).$$

Modding out by the action (3.0.1) we obtain a one-parameter family $g_l$ of metrics on $M \cong (G \times M)/G$. As $l \to \infty$, $(M, g_l)$ converges to $g_M$ [31].

The quotient map for the action (3.0.1) is

$$q_{G\times M}\colon (g, m) \longmapsto gm.$$

The vertical space for $q_{G\times M}$ at $(g, m) \in G \times M$ is

(3.0.2) $$\mathcal{V}_{q_{G\times M}} = \{(-k_G(g), k_M(m)) \mid k \in \mathfrak{g}\},$$

where we are employing the convention that for $k \in \mathfrak{g}$, $k_G$ is the Killing field on $G$ generated by $k$ and $k_M$ is the Killing field on $M$ generated by $k$.

We recall from [6; 31] that there is a reparametrization of the tangent space that we call the Cheeger reparametrization. We denote it by

$$\mathrm{Ch}_l\colon TM \longrightarrow TM.$$

It is defined by

$$\mathrm{Ch}_l(v) = d(q_{G\times M})(\hat{v}_l),$$

where $\hat{v}_l \in TG \times TM$ is the horizontal vector for $q_{G\times M}\colon (G \times M, l^2 g_{\mathrm{bi}} + g_M) \to (M, g_l)$ that maps to $v$ under the projection $d\pi_2\colon T(G \times M) \to TM$.

Note that every $G$–orbit in $G \times M$ has a point of the form $(e, x)$. At such a point, we let $\kappa_v$ be the element of $TG$ so that when $l = 1$

$$\hat{v}_1 = (\kappa_v, v).$$

Because $\hat{v}_l$ is $q_{G\times M}$–horizontal, $\kappa_v$ is orthogonal to the Lie algebra of the isotropy at $x$, that is, $\kappa_v \in \mathfrak{m}_x$. For any $l$ we then have

$$\hat{v}_l = \left(\frac{\kappa_v}{l^2}, v\right).$$

For simplicity we will write $\hat{v}$ for $\hat{v}_l$.

Although $\kappa_v$ is completely determined by $v$, $g_{\mathrm{bi}}$, $g_M$ and the $G$–action, we will not give its explicit formula since it is somewhat unpleasant. Instead, we develop some key abstract properties in the following proposition.

**Proposition 3.1** (1) *There is a constant $C_1 > 0$ so that for all unit vectors $V \in TM$,*

$$|\kappa_V|_{g_{\mathrm{bi}}} \leq C_1.$$

(2) *For any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$ there is a constant $C_2 > 0$ so that for all $x \in \mathcal{K}$ and all units $V \in TG(x)$,*

$$|\kappa_V|_{g_{\mathrm{bi}}} \geq C_2.$$

(3) *The maps $T_pM \to T_{(e,p)}(G \times M)$, $v \longmapsto \hat{v}_l$ and $T_pM \to T_pM$, $v \longmapsto \mathrm{Ch}_l(v)$ are linear.*

**Proof** By definition, $(\kappa_V, V)$ is $q_{G\times M}$–horizontal with respect to $(g_{\mathrm{bi}} + g_M)$, so for all $k \in \mathfrak{g}$ with $|k_G|_{g_{\mathrm{bi}}} = 1$,

$$0 = (g_{\mathrm{bi}} + g_M)((\kappa_V, V), (-k_G, k_M)),$$

so

$$\begin{aligned}
|\kappa_V|_{g_{\mathrm{bi}}} &= \max\{|g_{\mathrm{bi}}(\kappa_V, -k_G)| \text{ such that } k \in \mathfrak{g} \text{ with } |k_G|_{g_{\mathrm{bi}}} = 1\} \\
&= |g_{\mathrm{bi}}(\kappa_V, -(k_{\max})_G)| \quad \text{for some } k_{\max} \in \mathfrak{g} \text{ with } |(k_{\max})_G|_{g_{\mathrm{bi}}} = 1 \\
&= |g_M(V, (k_{\max})_M)| \\
&\leq \max\{|k_M|_{g_M} \text{ such that } k \in \mathfrak{g} \text{ with } |k_G|_{g_{\mathrm{bi}}} = 1\} \quad \text{since } |V|_{g_M} = 1.
\end{aligned}$$

By compactness, the right-hand side is bounded from above by some constant $C_1 > 0$, proving part (1).

For (2), take $x \in \mathcal{K} \subset M^{\mathrm{reg}}$ and choose $k \in \mathfrak{m}_x$ with $|k_G|_{g_{\mathrm{bi}}} = 1$ and $k_M/|k_M|_{g_M} = V$; then

$$0 = (g_{\mathrm{bi}} + g_M)((\kappa_V, V), (-k_G, k_M)).$$

So

$$|\kappa_V|_{g_{\mathrm{bi}}} \geq |g_{\mathrm{bi}}(\kappa_V, -k_G)| = |g_M(V, k_M)| = |k_M|_{g_M}.$$

Since $k \in \mathfrak{m}_x$ and $|k_G|_{g_{\mathrm{bi}}} = 1$, $|k_M(x)|_{g_M} > 0$. By compactness, there is a positive constant $C_2$ so that

$$\min_{x \in \mathcal{K}} \left\{ \min_{k \in \mathfrak{m}_x,\ |k_G|_{g_{\mathrm{bi}}}=1} |k_M(x)|_{g_M} \right\} > C_2 > 0,$$

and part (2) follows.

Part (3) is an immediate consequence of the definitions of $\hat{v}_l$ and $\mathrm{Ch}_l(v)$. □

Next we bound the sectional curvatures of $g_l$ from below.

**Proposition 3.2** *If $\{V, W\}$ is $g_M$–orthonormal, then*

$$\begin{aligned}\sec_{g_l}(\mathrm{Ch}_l(W), \mathrm{Ch}_l(V)) &\geq \sec_{l^2 g_{\mathrm{bi}} + g_M}(\widehat{W}, \widehat{V}) \\ &\geq \max\left\{-1, -\frac{l^2}{|\kappa_V|^2_{g_{\mathrm{bi}}}}, -\frac{l^2}{|\kappa_W|^2_{g_{\mathrm{bi}}}}\right\} |\sec_{g_M}(V, W)|.\end{aligned}$$

*Moreover, if $V$ and $W$ are perpendicular to the orbits of $G$, that is, if $V, W \in (TG(x))^{\perp}$, then*

$$\sec_{g_l}(\mathrm{Ch}_l(W), \mathrm{Ch}_l(V)) \geq \sec_{g_M}(V, W), \tag{3.2.1}$$

*and if $\sec_{g_M}(V, W) > 0$, then $\sec_{g_l}(\mathrm{Ch}_l(W), \mathrm{Ch}_l(V)) > 0$.*

**Proof** From the Gray–O'Neill horizontal curvature equation we have

$$\begin{aligned}\mathrm{curv}_{g_l}(\mathrm{Ch}_l(V), \mathrm{Ch}_l(W)) &\geq \mathrm{curv}_{l^2 g_{\mathrm{bi}} + g_M}(\widehat{W}, \widehat{V}) \\ &= \mathrm{curv}_{l^2 g_{\mathrm{bi}}}\left(\frac{\kappa_V}{l^2}, \frac{\kappa_W}{l^2}\right) + \mathrm{curv}_{g_M}(V, W) \\ &\geq \mathrm{curv}_{g_M}(V, W) \\ &= \sec_{g_M}(V, W).\end{aligned} \tag{3.2.2}$$

For $U \in (TG(x))^{\perp}$, we have $\widehat{U} = (0, U)$. So for $U, V \in (TG(x))^{\perp}$, we have $|\mathrm{Ch}_l(U \wedge V)|_{g_l} = |U \wedge V|_{g_M}$, and inequality (3.2.1) follows from (3.2.2). It also follows that $\sec_{g_l}(\mathrm{Ch}_l(W), \mathrm{Ch}_l(V)) > 0$, if $\sec_{g_M}(V, W) > 0$.

In general we have

$$\begin{aligned}
|\operatorname{Ch}_l(V)|^2_{g_l} &= \left|\frac{\kappa_V}{l^2}\right|^2_{l^2 g_{\mathrm{bi}}} + |V|^2_{g_M} = \frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}} + 1, \\
|\operatorname{Ch}_l(W)|^2_{g_l} &= \frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}} + 1, \\
g_l(\operatorname{Ch}_l(V), \operatorname{Ch}_l(W))^2 &= l^4 g_{\mathrm{bi}}\Big(\frac{\kappa_V}{l^2}, \frac{\kappa_W}{l^2}\Big)^2 = \frac{1}{l^4} g_{\mathrm{bi}}(\kappa_V, \kappa_W)^2,
\end{aligned}$$

so

$$\begin{aligned}
|\operatorname{Ch}_l(V) \wedge \operatorname{Ch}_l(W)|^2_{g_l} &= \Big(\frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}} + 1\Big)\Big(\frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}} + 1\Big) - \frac{1}{l^4} g_{\mathrm{bi}}(\kappa_V, \kappa_W)^2 \\
&= \frac{1}{l^4}|\kappa_V \wedge \kappa_W|^2_{g_{\mathrm{bi}}} + \frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}} + \frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}} + 1 \\
&\geq \frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}} + \frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}} + 1 \\
&\geq \max\Big\{\frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}}, \frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}}, 1\Big\}, \\
-|\operatorname{Ch}_l(V) \wedge \operatorname{Ch}_l(W)|^2_{g_l} &\leq -\max\Big\{\frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}}, \frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}}, 1\Big\} \\
&= \min\Big\{-\frac{1}{l^2}|\kappa_V|^2_{g_{\mathrm{bi}}}, -\frac{1}{l^2}|\kappa_W|^2_{g_{\mathrm{bi}}}, -1\Big\}, \\
-\frac{1}{|\operatorname{Ch}_l(V) \wedge \operatorname{Ch}_l(W)|^2_{g_l}} &\geq \frac{1}{\min\{-(1/l^2)|\kappa_V|^2_{g_{\mathrm{bi}}}, -(1/l^2)|\kappa_W|^2_{g_{\mathrm{bi}}}, -1\}} \\
&= \max\Big\{-\frac{l^2}{|\kappa_V|^2_{g_{\mathrm{bi}}}}, -\frac{l^2}{|\kappa_W|^2_{g_{\mathrm{bi}}}}, -1\Big\}.
\end{aligned}$$

Combining this with (3.2.2),

$$\begin{aligned}
\sec_{g_l}(\operatorname{Ch}_l(V), \operatorname{Ch}_l(W)) &\geq -\frac{1}{|\operatorname{Ch}_l(V) \wedge \operatorname{Ch}_l(W)|^2_{g_l}} |\sec_{g_M}(V, W)| \\
&\geq \max\Big\{-\frac{l^2}{|\kappa_V|^2_{g_{\mathrm{bi}}}}, -\frac{l^2}{|\kappa_W|^2_{g_{\mathrm{bi}}}}, -1\Big\} |\sec_{g_M}(V, W)|,
\end{aligned}$$

as desired. □

**Remark 3.3** In particular, Proposition 3.2 shows that the family $\{(M, g_l)\}_{l>0}$ has a uniform lower curvature bound. Since $\{(M, g_l)\}_{l>0}$ converges to $M/G$ in the Gromov–Hausdorff topology, this provides a simple proof that any $G$–manifold collapses with a lower curvature bound to $M/G$, as remarked by Yamaguchi in [49, Example 1.2(c)].

## 3.1 The $A$–tensor for the Cheeger deformation on the regular set

In this subsection, we show the curvature of a horizontal plane for $\pi^{\mathrm{reg}}\colon (M^{\mathrm{reg}}, g_l) \to M^{\mathrm{reg}}/G$ converges to the curvature of its projection in $M^{\mathrm{reg}}/G$ as $l \to 0$.

**Proposition 3.4** *Let $A^{\mathrm{Ch}}$ denote the $A$–tensor of the Cheeger submersion $q_{G\times M}\colon G\times M \to M$, and let $A^{\mathrm{reg}}$ denote the $A$–tensor of the Riemannian submersion*

$$\pi^{\mathrm{reg}}\colon M^{\mathrm{reg}} \longrightarrow M^{\mathrm{reg}}/G.$$

*Given any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$, $x \in \mathcal{K}$ and any unit vectors $Z_1, Z_2 \in T_x G(x)^{\perp}$,*

$$\left| A^{\mathrm{Ch}}_{\widehat{Z}_1} \widehat{Z}_2 \right|_{l^2 g_{\mathrm{bi}} + g_M} \longrightarrow \left| A^{\mathrm{reg}}_{Z_1} Z_2 \right|_{g_M} \quad \textit{uniformly on } \mathcal{K} \textit{ as } l \to 0.$$

**Remark 3.5** Let $(-k_G, k_M) \in T(G\times M)$, as in (3.0.2), be vertical for $q_{G\times M}$. Notice that

$$(-k_G, k_M) = (-k_G, 0) + (0, k_M)$$

is the sum of a vector field $(-k_G, 0)$ that only depends on the $G$–coordinate of the foot point and a vector field $(0, k_M)$ that only depends on the $M$–coordinate of the foot point. In the proof below, we exploit the fact that $k_G$ does not depend on the $M$–coordinate.

**Proof** Notice that $Z_1, Z_2 \in T_x G(x)^{\perp}$ implies that $\widehat{Z}_1 = (0, Z_1)$ and $\widehat{Z}_2 = (0, Z_2)$ for all $l$. So

$$\begin{aligned}
&(l^2 g_{\mathrm{bi}} + g_M)\big(A^{\mathrm{Ch}}_{\widehat{Z}_1} \widehat{Z}_2, (-k_G, k_M)\big) \\
&\qquad = -(l^2 g_{\mathrm{bi}} + g_M)((0, Z_2), A^{\mathrm{Ch}}_{(0,Z_1)}(-k_G, k_M)) \\
&\qquad = -(l^2 g_{\mathrm{bi}} + g_M)((0, Z_2), (0, \nabla^{g_M}_{Z_1} k_M)) \quad \text{(by Remark 3.5)} \\
&\qquad = -g_M(Z_2, A^{\mathrm{reg}}_{Z_1} k_M) \\
&\qquad = g_M(A^{\mathrm{reg}}_{Z_1} Z_2, k_M).
\end{aligned}$$

For $k \in \mathfrak{g}_x$, we have $k_M(x) = 0$, and therefore $(l^2 g_{\mathrm{bi}} + g_M)(A^{\mathrm{Ch}}_{\widehat{Z}_1} \widehat{Z}_2, (-k_G, k_M)) = g_M(A^{\mathrm{reg}}_{Z_1} Z_2, k_M) = 0$.

So we may assume that $k \in \mathfrak{m}_x$. In this case, divide both sides of the previous display by $|(-k_G, k_M)|_{l^2 g_{\mathrm{bi}} + g_M}$ and observe that for $k \in \mathfrak{m}_x$ with $|k_G|_{g_{bi}} = 1$, $|k_M|_{g_M}$ is uniformly bounded from below on $\mathcal{K}$, so

$$\frac{1}{|(-k_G, k_M)|_{l^2 g_{\mathrm{bi}} + g_M}} = \frac{1}{\sqrt{l^2 + |k_M|^2_{g_M}}} \longrightarrow \frac{1}{|k_M|_{g_M}}$$

uniformly on $\mathcal{K}$ as $l \to 0$, and the result follows. $\square$

# 4 Infinitesimal geometry near the singular orbits

This section culminates with the proof of Lemma 4.7, which shows that for planes that are orthogonal to both the orbits of $G$ and to the fibers of the metric projection to the singular strata, the sectional curvatures of $g_l$ converge to the curvatures of their images under $d\pi^{\mathrm{reg}}$ as $l$ approaches $0$. The other crucial result in this section is Corollary 4.4. It gives a lower bound for $|\kappa_v|$ for certain vectors $v$ whose foot points are near the singular strata.

Recall from Proposition 1.1 that the singular strata have a neighborhood $\Omega = \cup\Omega^i$ so that for each $i$, we have a splitting of $T(\Omega^i \setminus S_i)$, $\mathcal{H}^i \oplus X^i \oplus \mathcal{V}^i$. Recall that $\mathcal{H}^i$ need not be orthogonal to $\operatorname{span}\{X^i, \mathcal{V}^i\}$. So we define $\overline{\mathcal{H}}^i$ to be the distribution that is orthogonal to $\operatorname{span}\{X^i, \mathcal{V}^i\}$. We let $\mathrm{Pr}_i\colon B(\mathcal{C}_i, \operatorname{inj}(\mathcal{C}_i)/2) \to \mathcal{C}_i$ be

$$\mathrm{Pr}_i(x) = \text{foot point}((\exp^{\perp}_{S_i})^{-1}(x)).$$

That is, $\mathrm{Pr}_i$ is the metric projection map onto $\mathcal{C}_i \subset S_i$.

**Proposition 4.1** *For $x \in \exp^{\perp}_{S_i} B(\nu_0(S_i)|_{\mathcal{C}_i}, \operatorname{inj}(\mathcal{C}_i)/2)$,*

$$\mathcal{V}^i_x \supset T_x G_{\mathrm{Pr}_i(x)}(x). \tag{4.1.1}$$

*In fact,*

$$\mathcal{V}^i_x \cap T_x G(x) = T_x G_{\mathrm{Pr}_i(x)}(x).$$

**Proof** Let $\gamma$ be the minimal geodesic from $\mathrm{Pr}_i(x)$ to $x$. Then $G_{\mathrm{Pr}_i(x)}(\gamma)$ is a family of minimal geodesics emanating from $\mathrm{Pr}_i(x)$, normal to $S_i$. In particular,

$$G_{\mathrm{Pr}_i(x)}(\dot\gamma(0)) \subset \nu_{\mathrm{Pr}_i(x)}(S_i). \tag{4.1.2}$$

Since $\{\widetilde{X}^i, \widetilde{\mathcal{V}}^i\}$ spans the vertical distribution of $\nu(S) \to S$, and

$$\mathcal{V}^i \equiv d\exp^{\perp}_S(\widetilde{\mathcal{V}}^i),$$
$$X^i = d\exp^{\perp}_S(\widetilde{X}^i),$$

exponentiating (4.1.2) then gives

$$TG_{\mathrm{Pr}_i(x)}(\gamma) \subset \operatorname{span}\{X^i, \mathcal{V}^i\}.$$

Since $X^i(x) \perp T_x G_{\mathrm{Pr}_i(x)}(x)$ and $X^i \perp \mathcal{V}^i$, we get that for any small fixed number $t_0 > 0$, $T_{\gamma(t_0)} G_{\mathrm{Pr}_i(x)}(\gamma(t_0)) \subset \operatorname{span}\{\mathcal{V}^i_{\gamma(t_0)}\}$, proving (4.1.1).

On the other hand, if $k \in \mathfrak{g} \setminus \mathfrak{g}_{\mathrm{Pr}_i(x)}$, then

$$\frac{\partial}{\partial s} \exp_G(sk) \cdot \gamma(0)\Big|_{s=0} \neq 0,$$

so from part (1) of Proposition 2.4 we have for any small fixed number $t_0 > 0$

$$\frac{\partial}{\partial s} \exp_G(sk) \cdot \gamma(t_0)\Big|_{s=0} \notin \mathcal{V}^i_x.$$

Thus

$$\mathcal{V}^i_x \cap T_x G(x) = T_x G_{\mathrm{Pr}_i(x)}(x). \qquad \square$$

**Proposition 4.2** *For $i \in \{1, 2, \dots, p\}$, let $U$ be a neighborhood of the vectors*

$$\bigcup_{x \in \Omega^i \setminus \mathcal{C}_i} \operatorname{span}\{\overline{\mathcal{H}}^i_x \cap T_x G(x)^\perp, X^i_x, \mathcal{V}^i_x\} \cup \bigcup_{x \in \mathcal{C}_i} \operatorname{span}\{T_x(S_i) \cap T_x G(x)^\perp, \nu_x(S_i)\}.$$

*There is a constant $C > 0$ so that for all $x \in \Omega^i$ that are close enough to $\mathcal{C}_i$ and all $v \in (T\Omega^i) \setminus U$, there is a $k \in \mathfrak{g}$ with $|k|_{g_{\mathrm{bi}}} = 1$ so that*

$$g_M(v, k_M) \geq C|v|_{g_M}.$$

**Remark 4.3** It follows from part (4) of Proposition 2.4 that along any geodesic $\gamma$ that leaves $S_i$ orthogonally at $\gamma(0)$,

$$\text{(4.3.1)} \quad \begin{aligned} &\lim_{t \to 0} \operatorname{span}\{\overline{\mathcal{H}}^i_{\gamma(t)} \cap T_{\gamma(t)} G(\gamma(t))^\perp, X^i_{\gamma(t)}, \mathcal{V}^i_{\gamma(t)}\} \\ &\qquad = \operatorname{span}\{T_{\gamma(0)}(S_i) \cap T_{\gamma(0)} G(\gamma(0))^\perp, \nu_{\gamma(0)}(S_i)\}. \end{aligned}$$

**Proof** Since $S_i$ is $G$–invariant, $T_x(S_i) = \{T_x(S_i) \cap T_x G(x)^\perp\} \oplus T_x G(x)$. So for $x \in \mathcal{C}_i$ we have the orthogonal splitting

$$T_x M = \{T_x(S_i) \cap T_x G(x)^\perp\} \oplus \nu_x(S_i) \oplus T_x G(x).$$

So if $v \in T_x M$ is not in the span of the first two summands, its projection to $TG(x)$ is nonzero.

Combining this with (4.3.1) and continuity, we have that for $x \in \Omega^i$ close enough to $\mathcal{C}_i$, if

$$v \notin [\overline{\mathcal{H}}^i_x \cap T_x G(x)^\perp] \oplus X^i_x \oplus \mathcal{V}^i_x,$$

then its projection to $TG(x)$ is nonzero.

The result then follows from compactness of the unit vectors in $TM \setminus U$. $\square$

**Corollary 4.4** *For $i \in \{1, 2, \dots, p\}$, let $U$ be a neighborhood of the vectors*

$$\bigcup_{x \in \Omega^i} \operatorname{span}\{\overline{\mathcal{H}}^i_x \cap T_x G(x)^\perp, X^i_x, \mathcal{V}^i_x\} \cup \bigcup_{x \in \mathcal{C}_i} \operatorname{span}\{T_x(S_i) \cap T_x G(x)^\perp, \nu_x(S_i)\}.$$

*There is a constant $C > 0$ so that for all $x \in \Omega^i$, and all $v \in (T_x\Omega^i) \setminus U$,*

$$|\kappa_v|_{g_{\mathrm{bi}}} \geq C|v|_{g_M}.$$

**Proof** By definition of $\kappa_v$ we have

$$0=(g_{\mathrm{bi}}+g_M)((\kappa_v,v),(-k_G,k_M))=-g_{\mathrm{bi}}(\kappa_v,k_G)+g_M(v,k_M)$$

for all $k\in\mathfrak{g}$. From the previous proposition we have a constant $C>0$ and a $k\in\mathfrak{g}$ with $|k|_{g_{\mathrm{bi}}}=1$ so that

$$g_M(v,k_M)\geq C|v|_{g_M}.$$

The result follows by combining the previous two displays. □

## 4.1 The $A$–tensor near the singular orbits

In this subsection we prove Lemma 4.7, which shows that for planes that are orthogonal to both the orbits of $G$ and to the fibers of the metric projection to the singular strata, the sectional curvatures of $g_l$ converge to the curvatures of their images under $d\pi^{\mathrm{reg}}$ as $l$ approaches $0$.

**Lemma 4.5** *For each $i\in\{1,2,\dots,p\}$, there is a constant $C_i>0$ so that the following hold.*

(1) *If our foot point $x$ is in $\Omega^i$, $k\in\mathfrak{m}_{\mathrm{Pr}_i(x)}$ and $Y,Z\in\mathrm{span}\{TG(x)^{\perp}\cap\overline{\mathcal{H}}^i_x\}$,*

$$|g(A^{\mathrm{reg}}_Z Y,k_M)|\leq C_i|k_M||Y||Z|. \tag{4.5.1}$$

(2) *If our foot point $x$ is in $\Omega^i$, $k\in\mathfrak{g}_{\mathrm{Pr}_i(x)}$, $Y\in\mathrm{span}\{TG(x)^{\perp}\cap\overline{\mathcal{H}}^i_x\}$ and $Z\in\mathrm{span}\{X^i,TG(x)^{\perp}\cap\overline{\mathcal{H}}^i_x\}$,*

$$|g(A^{\mathrm{reg}}_Z Y,k_M)|\leq C_i|Z||Y||k_M|\,\mathrm{dist}(\mathcal{C}_i,x).$$

**Proof** First we prove both inequalities for the special case of a fixed $k\in\mathfrak{g}$ and a fixed geodesic $\gamma\colon[-l,l]\to\Omega^i$ with $\gamma(0)\in\mathcal{C}_i$, $\gamma'(0)\in\nu_{\gamma(0)}S_i$.

If $k\in\mathfrak{m}_{\gamma(0)}\setminus\{0\}$, then

$$t\longmapsto\left.\frac{|\nabla_{\bullet}k_M|}{|k_M|}\right|_{\gamma(t)}$$

is continuous on $[-l,l]$ and hence has a maximum, proving part (1) for a fixed $k\in\mathfrak{g}$ and a fixed geodesic $\gamma$.

For part (2), we first consider the case when both $Y,Z$ are in $T_{\gamma(t)}G(\gamma(t))^{\perp}\cap\overline{\mathcal{H}}^i_{\gamma(t)}$. Since $\overline{\mathcal{H}}^i$ is the orthogonal complement of $\mathrm{span}\{X^i,\mathcal{V}^i\}$, it follows from part (4) of Proposition 2.4 that $\bigcup_{t\in[-l,l]\setminus\{0\}}\overline{\mathcal{H}}^i_{\gamma(t)}$ has an extension to a smooth distribution along $\gamma$. Let $\bar{\mathcal{H}}^i_{\gamma(0)}$ be the vectors at $\gamma(0)$ that are in this distribution. From part (4) of Proposition 2.4, it follows that

$$\bar{\mathcal{H}}^i_{\gamma(0)} = T_{\gamma(0)} S_i.$$

By the slice theorem we have

$$T_{\gamma(0)} G(\gamma(0))^{\perp} \cap T_{\gamma(0)} S_i \subset T_{\gamma(0)} \operatorname{Fix}(M; G_{\gamma(0)}),$$

where $\operatorname{Fix}(M; G_{\gamma(0)})$ is the fixed point set of the $G_{\gamma(0)}$–action on $M$. On the other hand, $G_{\gamma(0)}$ acts on $\nu_{\gamma(0)} S_i$ without fixed points, so

$$T_{\gamma(0)} \operatorname{Fix}(M; G_{\gamma(0)}) \subset T_{\gamma(0)} S_i.$$

Combining the previous three displays gives

$$T_{\gamma(0)} G(\gamma(0))^{\perp} \cap \bar{\mathcal{H}}^i_{\gamma(0)} \subset T \operatorname{Fix}(M; G_{\gamma(0)}) \subset T S_i.$$

As components of fixed point sets are totally geodesic, it follows that for $Y, Z \in T_{\gamma(0)} G(\gamma(0))^{\perp} \cap \bar{\mathcal{H}}^i_{\gamma(0)}$, $\nabla_Z Y \in T \operatorname{Fix}(M; G_{\gamma(0)}) \subset T S_i$. In particular, for any vector $W$ normal to $S_i$,

$$g(\nabla_Z Y, W) = 0. \tag{4.5.2}$$

Combining part (4) of Proposition 2.4, Proposition 4.1 and (4.5.2) yields

$$\left| g\left( A^{\mathrm{reg}}_Z Y, \frac{k_M}{|k_M|} \right) \right| = \left| g\left( \nabla_Z Y, \frac{k_M}{|k_M|} \right) \right| \le C \cdot \operatorname{dist}(\mathcal{C}_i, \gamma(t))$$

for unit $Y, Z \in \operatorname{span}\{T G(\gamma(t))^{\perp} \cap \bar{\mathcal{H}}^i_{\gamma(t)}\}$, $k \in \mathfrak{g}_{\mathrm{Pr}_i(\gamma(t))}$ and some $C > 0$.

For the case when $Z = X$ and $Y \in \operatorname{span}\{T G(\gamma(t))^{\perp} \cap \bar{\mathcal{H}}^i_{\gamma(t)}\}$ is unit we write $Y = Y^{\mathcal{H}} + Y^{\mathcal{V}}$. For $k \in \mathfrak{g}_{\mathrm{Pr}_i(\gamma(t))}$ by Proposition 4.1 we then have

$$0 = g\left( Y, \frac{k_M}{|k_M|} \right) = g\left( Y^{\mathcal{H}}, \frac{k_M}{|k_M|} \right) + g\left( Y^{\mathcal{V}}, m \frac{k_M}{|k_M|} \right).$$

By Propositions 2.8 and 4.1, we have that $|g(Y^{\mathcal{H}}, k_M/|k_M|)| \le C(t^2)$. It follows that $|g(Y^{\mathcal{V}}, k_M/|k_M|)| \le C(t^2)$.

From part (3) of Lemma 2.7 we then conclude that

$$\begin{aligned} \left| g\left( A^{\mathrm{reg}}_Y X, \frac{k_M}{|k_M|} \right) \right| &\le \left| g\left( \nabla_{Y^{\mathcal{H}}} X, \frac{k_M}{|k_M|} \right) \right| + \left| g\left( \nabla_{Y^{\mathcal{V}}} X, \frac{k_M}{|k_M|} \right) \right| \\ &\le C \operatorname{dist}(\mathcal{C}_i, \gamma(t)), \end{aligned}$$

for some $C > 0$. So part (2) follows for a fixed $k \in \mathfrak{g}$ and a fixed geodesic $\gamma$.

Now observe that in both cases, the left-hand side depends continuously on the choice of $k \in \mathfrak{g}$ and the choice of normal geodesic $\gamma$. Thus the theorem follows in general from the compactness of the unit sphere in $\mathfrak{g}$ and the unit normal bundle of $\mathcal{C}_i$. □

**Proposition 4.6** *Let $\gamma$ be a geodesic that leaves $S_i$ orthogonally from a point of $\mathcal{C}_i$. For $k^1 \in \mathfrak{g}_{\gamma(0)}$ and $k^2 \in \mathfrak{m}_{\gamma(0)}$*

$$g_M(k^1_M(\gamma(t)), k^2_M(\gamma(t))) = O(t)|k^1_M(\gamma(t))||k^2_M(\gamma(t))|.$$

*In particular, the angles between the subspaces*

$$\{(-k_G, k_M) \mid k \in \mathfrak{g}_{\gamma(0)}\} \quad \textit{and} \quad \{(-k_G, k_M) \mid k \in \mathfrak{m}_{\gamma(0)}\}$$

*and the subspaces*

$$\{k_M \mid k \in \mathfrak{g}_{\gamma(0)}\} \quad \textit{and} \quad \{k_M \mid k \in \mathfrak{m}_{\gamma(0)}\}$$

*are both $\frac{\pi}{2} \pm O(t)$.*

**Proof** The action of the circle generated by $k^1$ on $\gamma$ produces a variation of $\gamma$ by geodesics that leave $S_i$ orthogonally from $\gamma(0)$, and hence shows that $\nabla_{\gamma'(0)} k^1_M \in \nu_{\gamma(0)} S_i$.

Since $k^1_M(\gamma(0)) = 0$, we have

$$g_M(k^1_M(\gamma(0)), k^2_M(\gamma(0))) = 0$$

and

$$\frac{d}{dt} g_M(k^1_M(\gamma(t)), k^2_M(\gamma(t)))\Big|_{t=0} = g_M(\nabla_{\gamma'(0)} k^1_M, k^2_M) + g_M(k^1_M, \nabla_{\gamma'(0)} k^2_M) = 0$$

since $\nabla_{\gamma'(0)} k^1_M \in \nu_{\gamma(0)} S_i$ and $k^2_M \in TS_i$ and $k^1_M(\gamma(0)) = 0$. On the other hand,

$$\begin{aligned}
\frac{d^2}{dt^2} g_M(k^1_M(\gamma(t)), k^2_M(\gamma(t)))\Big|_{t=0}
&= g_M(\nabla_{\gamma'(0)} \nabla_{\gamma'(0)} k^1_M, k^2_M) \\
&\qquad + 2 g_M(\nabla_{\gamma'(0)} k^1_M, \nabla_{\gamma'(0)} k^2_M) + g_M(k^1_M, \nabla_{\gamma'(0)} \nabla_{\gamma'(0)} k^2_M) \\
&= -R(k^1_M, \gamma', \gamma', k^2_M)\Big|_{t=0} + 2 g_M(\nabla_{\gamma'(0)} k^1_M, \nabla_{\gamma'(0)} k^2_M) \\
&= 2 g_M(\nabla_{\gamma'(0)} k^1_M, \nabla_{\gamma'(0)} k^2_M)
\end{aligned}$$

since $k^1_M(\gamma(0)) = 0$. Thus

$$\begin{aligned}
|g_M(k^1_M(\gamma(t)), k^2_M(\gamma(t)))| &= |g_M(\nabla_{\gamma'(0)} k^1_M, \nabla_{\gamma'(0)} k^2_M)| t^2 + O(t^3) \\
&\le 2 |\nabla_{\gamma'(0)} k^1_M|_{g_M} |\nabla_{\gamma'(0)} k^2_M|_{g_M} t^2.
\end{aligned} \tag{4.6.1}$$

Since $k_M^1(\gamma(0)) = 0$,

$$|k_M^1(\gamma(t))|_{g_M} = |\nabla_{\gamma'(0)} k_M^1|_{g_M} t + O(t^2). \tag{4.6.2}$$

By compactness of $\mathcal{C}_i$ and the unit sphere in $\mathfrak{m}_{\gamma(0)}$,

$$|\nabla_{\gamma'(0)} k_M^2|_{g_M} \leq C_1 |k_G^2|_{g_{\mathrm{bi}}}$$

for some constant $C_1 > 0$. By compactness of the unit sphere in $\mathfrak{m}_{\gamma(0)}$,

$$|k_G^2|_{g_{\mathrm{bi}}} \leq C_2 |k_M^2(\gamma(0))|_{g_M}$$

for some constant $C_2 > 0$, and for all sufficiently small $t$,

$$|k_M^2(\gamma(0))|_{g_M} \leq 2|k_M^2(\gamma(t))|_{g_M}.$$

Combining the previous three inequalities gives

$$|\nabla_{\gamma'(0)} k_M^2|_{g_M} \leq C|k_M^2(\gamma(t))|_{g_M}$$

for all sufficiently small $t > 0$ and some $C > 0$. Together with inequalities (4.6.1) and (4.6.2) this gives

$$|g_M(k_M^1(\gamma(t)), k_M^2(\gamma(t)))| = O(t)|k_M^1(\gamma(t))|\,|k_M^2(\gamma(t))|$$

for all sufficiently small $t$.

So the angle between

$$\{k_M \mid k \in \mathfrak{g}_{\gamma(0)}\} \quad \text{and} \quad \{k_M \mid k \in \mathfrak{m}_{\gamma(0)}\}$$

is $\frac{\pi}{2} \pm O(t)$, and the angle between

$$\{(-k_G, k_M) \mid k \in \mathfrak{g}_{\gamma(0)}\} \quad \text{and} \quad \{(-k_G, k_M) \mid k \in \mathfrak{m}_{\gamma(0)}\}$$

is only closer to $\frac{\pi}{2}$. □

**Lemma 4.7** *For all points $x \in \bigcup_i \Omega^i$, for all $Y \in \mathrm{span}\{TG(x)^\perp \cap \overline{\mathcal{H}}^i_x\}$ and for all $Z \in \mathrm{span}\{X^i, TG(x)^\perp \cap \overline{\mathcal{H}}^i_x\}$, we have*

$$\Big| |A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}| - |A^{\mathrm{reg}}_Y Z| \Big| \leq \big[O(l) + O(\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))\big] |Z|\,|Y|,$$

*where $A^{\mathrm{Ch}}$ is the $A$–tensor of the Riemannian submersion $(G \times M, l^2 g_{\mathrm{bi}} + g_M) \to (M, g_l)$.*

**Proof** Since the splittings

$$\{(-k_G, k_M) \mid k \in \mathfrak{g}_{\mathrm{Pr}_i(x)}\} \oplus \{(-k_G, k_M) \mid k \in \mathfrak{m}_{\mathrm{Pr}_i(x)}\},$$
$$\{k_M \mid k \in \mathfrak{g}_{\mathrm{Pr}_i(x)}\} \oplus \{k_M \mid k \in \mathfrak{m}_{\mathrm{Pr}_i(x)}\}$$

are nearly orthogonal, it is enough to compare the projections of $A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}$ and $A^{\mathrm{reg}}_{Y} Z$ onto the corresponding subspaces.

For all $Y \in \mathrm{span}\{TG(x)^\perp \cap \overline{\mathcal{H}}^i_x\}$ and $Z \in \mathrm{span}\{X^i, TG(x)^\perp \cap \overline{\mathcal{H}}^i_x\}$, as in the proof of Proposition 3.4, we have

$$(l^2 g_{\mathrm{bi}} + g_M)\big(A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}, (-k_G, k_M)\big) = -g_M(Z, \nabla_Y k_M) \tag{4.7.1}$$
$$= g_M(A^{\mathrm{reg}}_{Y} Z, k_M).$$

The set of real numbers

$$\{|k_M(x)|_{g_M} \mid x \in \Omega^i, k \in \mathfrak{m}_{\mathrm{Pr}_i(x)}, |k_G|_{g_{bi}} = 1\}$$

has a positive lower bound. So for $x \in \Omega^i$ and $k \in \mathfrak{m}_{\mathrm{Pr}_i(x)}$ with $|k_G|_{g_{bi}} = 1$ we have

$$\frac{1}{|(-k_G, k_M)|^2_{l^2 g_{\mathrm{bi}} + g_M}} = \frac{1}{l^2 |k_G|^2_{g_{\mathrm{bi}}} + |k_M|^2_{g_M}} \longrightarrow \frac{1}{|k_M|^2_{g_M}} \quad \text{as } l \to 0$$

uniformly on $\Omega^i$. So for $k \in \mathfrak{m}_{\mathrm{Pr}_i(x)}$ and unit $Y$ and $Z$ we have

$$\frac{(l^2 g_{\mathrm{bi}} + g_M)\big(A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}, (-k_G, k_M)\big)}{|(-k_G, k_M)|_{l^2 g_{\mathrm{bi}} + g_M}} \longrightarrow g_M\left(A^{\mathrm{reg}}_{Y} Z, \frac{k_M}{|k_M|_{g_M}}\right) \quad \text{as } l \to 0$$

uniformly on $\Omega^i$.

On the other hand, if $k \in \mathfrak{g}_{\mathrm{Pr}_i(x)}$, we combine (4.7.1) with part (2) of Lemma 4.5 to get

$$(l^2 g_{\mathrm{bi}} + g_M)\big(A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}, (-k_G, k_M)\big)^2 \le C |Z|^2_{g_M} |Y|^2_{g_M} |k_M|^2_{g_M} (\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))^2,$$

for $C = \max\{C_i\}$. Dividing both sides by $|(-k_G, k_M)|^2_{l^2 g_{\mathrm{bi}} + g_M}$, we have

$$\begin{aligned} &\frac{(l^2 g_{\mathrm{bi}} + g_M)\big(A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z}, (-k_G, k_M)\big)^2}{|(-k_G, k_M)|^2_{l^2 g_{\mathrm{bi}} + g_M}} \\ &\qquad \le C |Z|^2_{g_M} |Y|^2_{g_M} (\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))^2 \frac{|k_M|^2_{g_M}}{l^2 |k_G|^2_{g_{\mathrm{bi}}} + |k_M|^2_{g_M}} \\ &\qquad \le C |Z|^2_{g_M} |Y|^2_{g_M} (\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))^2. \end{aligned}$$

Finally, by part (2) of Lemma 4.5

$$\frac{g_M(A_Y^{\mathrm{reg}} Z, k_M)^2}{|k_M|^2_{g_M}} \le C|Z|^2_{g_M}|Y|^2_{g_M}(\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))^2$$

for $C = \max\{C_i\}$, and the result follows. □

## 5 Two steps to better curvature

In this section, we prove two results that are the first two steps in the proofs of Theorems A, B and C. They track the effects of first Cheeger deforming $g_M$ and then performing the conformal change of Theorem 2.16, allowing us to improve the curvature of $M$.

One can also omit the first step and still prove Theorem A. So the reader who is only interested in the proof of Theorem A can skip this section.

Recall that $g_l$ is the metric on $M$ induced by the Riemannian submersion

$$q_{G\times M}\colon (G \times M, l^2 g_{\mathrm{bi}} + g) \longrightarrow M,$$

and $d\pi^{\mathrm{reg}}(g_l)$ is the Riemannian metric on $(M^{\mathrm{reg}}/G)$ induced by the Riemannian submersion $\pi^{\mathrm{reg}}\colon (M^{\mathrm{reg}}, g) \to M^{\mathrm{reg}}/G$.

**Theorem 5.1** **(Step 1)** *Let $G$ be a compact Lie group acting isometrically on a Riemannian $n$–manifold $(M, g)$. For any $\varepsilon > 0$ there is a neighborhood $\Omega'$ of $S_1 \cup S_2 \cup \cdots \cup S_p$ as in Proposition 1.1 and a Cheeger parameter $l_1$ such that for all $l \in (0, l_1)$*

$$|\sec_{g_l}(Y, Z) - \sec_{d\pi(g_l)}(d\pi^{\mathrm{reg}}(Y), d\pi^{\mathrm{reg}}(Z))| < \frac{\varepsilon}{2} \tag{5.1.1}$$

*if either $Y, Z \in TG(x)^{\perp}|_{M\setminus\Omega'}$ or $Y, Z \in \{TG(x)^{\perp} \cap \overline{\mathcal{H}}^i\}|_{\Omega'\setminus S_1\cup S_2\cup\cdots\cup S_p}$ for some $i \in \{1, 2, \ldots, p\}$.*

*Moreover, $d\pi^{\mathrm{reg}}(g_l)$ is independent of $l$ and is equal to $d\pi^{\mathrm{reg}}(g)$.*

**Proof** For orthonormal $Y, Z \in TG(x)^{\perp}$ we have $\widehat{Y} = (0, Y)$ and $\widehat{Z} = (0, Z)$, so using the horizontal curvature equation we obtain

$$\begin{aligned}\sec_{g_l}(Y, Z) &= \sec_{l^2 g_{\mathrm{bi}}+g_M}((0, Y), (0, Z)) + 3\big|A^{\mathrm{Ch}}_{\widehat{Y}}\widehat{Z}\big|^2_{l^2 g_{\mathrm{bi}}+g_M} \\ &= \sec_{g_M}(Y, Z) + 3\big|A^{\mathrm{Ch}}_{\widehat{Y}}\widehat{Z}\big|^2_{l^2 g_{\mathrm{bi}}+g_M}.\end{aligned}$$

On the other hand, for $\pi^{\mathrm{reg}}\colon M \to M/G$, the horizontal curvature equation becomes

$$\sec(d\pi^{\mathrm{reg}} Y, d\pi^{\mathrm{reg}} Z) = \sec_{g_M}(Y, Z) + 3\big|A^{\mathrm{reg}}_Y Z\big|^2_{g_M}.$$

From Lemma 4.7 we have

$$\left| \left| A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z} \right|^2_{l^2 g_{\mathrm{bi}} + g_M} - \left| A^{\mathrm{reg}}_{Y} Z \right|^2_{g_M} \right| < O(l) + O(\mathrm{dist}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_p, x))$$

for $Y, Z \in \{TG(x)^{\perp} \cap \overline{\mathcal{H}}^i\}|_{\Omega' \setminus S_1 \cup S_2 \cup \cdots \cup S_p}$, and from Proposition 3.4 we have

$$\left| \left| A^{\mathrm{Ch}}_{\widehat{Y}} \widehat{Z} \right|^2_{l^2 g_{\mathrm{bi}} + g_M} - \left| A^{\mathrm{reg}}_{Y} Z \right|^2_{g_M} \right| < \frac{\varepsilon}{6}$$

for all $Y, Z \in TG(x)^{\perp}|_{M \setminus \Omega'}$, provided $l$ is sufficiently small.

Inequality (5.1.1) follows by combining the previous four displays.

Finally, since a Cheeger deformation does not change the metric on the distribution that is orthogonal to the orbits, $d\pi^{\mathrm{reg}}(g_l)$ is independent of $l$ and equal to $d\pi^{\mathrm{reg}}(g)$. $\square$

Next we apply Theorem 2.16 to the metrics $g_l$ and obtain the following.

**Theorem 5.2** **(Step 2)** *Let $M$ and $G$ be as in Theorem 5.1. For any $\varepsilon > 0$, let $g_l$ be a metric that satisfies the conclusion of Theorem 5.1. For any $K > 0$, there is a neighborhood $\Omega_1$ of $S_1 \cup S_2 \cup \cdots \cup S_p$ and a $G$–invariant metric $\widetilde{g_l} = e^{2f} g_l$ so that if $V \in \mathrm{span}\{\mathcal{V}^i, X^i\}|_{\Omega_1}$ for some $i \in \{1, \ldots, p\}$, then*

$$\sec_{\widetilde{g_l}}(V, W) \geq K$$

*for all $W \in T\Omega_1$. Moreover,*

$$\sec_{\widetilde{g_l}}(V, W) \geq \sec_{g_l}(V, W) - \frac{\varepsilon}{2}$$

*for all $V, W \in TM$.*

## 6 Lifting positive Ricci curvature

In this section we prove Theorem A. For convenience, rescale so that $\mathrm{Ric}_{M^{\mathrm{reg}}/G} \geq 2$.

To the best of our knowledge, the most efficient metric construction to prove Theorem A is to perform the conformal change of Theorem 2.16, and then to Cheeger deform the resulting metric. In contrast, to prove Theorem B we first Cheeger deform, then perform a conformal change, and then further Cheeger deform. Consequently we also use this 3–step deformation to prove Theorem C.

If $(M, g)$ satisfies the hypotheses of Theorem A and $g_l$ is a Cheeger deformation of $g$, then $(M, g_l)$ also satisfies the hypotheses of Theorem A. So for simplicity of notation we will write $g$ for $g_l$ in this section. On the one hand, this points to the most efficient path to proving Theorem A; on the other hand, since $g_l$ satisfies the hypotheses of Theorem A we will simultaneously verify the positive Ricci curvature portion of the conclusion of Theorem C.

We obtain Theorem A by combining the conformal change of Theorem 2.16 and the following two results, both of which are proven in this section. The first result shows that all Cheeger deformations of $\widetilde{g}$ have positive Ricci curvature on $\Omega_1$, where $\widetilde{g}$ and $\Omega_1$ are as in Theorem 2.16. The second result shows that there are Cheeger deformations of $\widetilde{g}$ that have positive Ricci curvature on $M \setminus \Omega_1$.

**Theorem 6.1** *Let $(M, g)$ be as in Theorem A. Given $\varepsilon, K > 0$, let $\widetilde{g}$ be the $G$–invariant metric on $M$ from Theorem 2.16. If $\varepsilon$ is sufficiently small and $K$ is sufficiently large, then for all $\lambda \in (0, \infty)$*

$$\mathrm{Ric}_{\widetilde{g}_\lambda} |_{\Omega_1} > 0,$$

*where $\widetilde{g}_\lambda$ is the metric on $M$ induced by the Riemannian submersion*

$$q_{G\times M}\colon (G \times M, \lambda^2 g_{\mathrm{bi}} + \widetilde{g}) \longrightarrow M,$$

*and $\Omega_1$ is as in Theorem 2.16.*

**Theorem 6.2** *Given $(M, g)$ as in Theorem A, let $\widetilde{g}_\lambda$ be the metric on $M$ from Theorem 6.1, and let $\Omega_1$ be as in Theorem 2.16. Then*

$$\mathrm{Ric}_{\widetilde{g}_\lambda} |_{M\setminus\Omega_1} > 0,$$

*provided $\lambda$ is sufficiently small.*

Before proceeding with the proofs, we record the following result, which is obtained by taking the trace of the horizontal curvature equation.

**Proposition 6.3** *Let $\pi\colon (E, g) \to B$ be a Riemannian submersion with horizontal distribution $H$. Using the superscript $(\cdot)^{\mathrm{Horiz}}$ to denote the $H$–component of a vector, for $x, y, z \in H$ we define*

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{Horiz}}(x, y) &\equiv \mathrm{Trace}\big(z \longmapsto \{R(z, x)y\}^{\mathrm{Horiz}}\big),\\
R^A(z, x)y &\equiv 2A_y A_z x - A_z A_x y - A_x A_y z,\\
\mathrm{Ric}^A(x, y) &\equiv \mathrm{Trace}\big(z \longmapsto R^A(z, x)y\big).
\end{aligned}$$

*Extend $\mathrm{Ric}^{\mathrm{Horiz}}$ and $\mathrm{Ric}^A$ to be $(0,2)$–tensors on $M$ by setting $\mathrm{Ric}^{\mathrm{Horiz}}(v, \cdot) = \mathrm{Ric}^A(v, \cdot) = 0$, if $v$ is vertical. Then*

$$\pi^*(\mathrm{Ric}_B) = \mathrm{Ric}^{\mathrm{Horiz}} + 3\,\mathrm{Ric}^A. \tag{6.3.1}$$

**Remark 6.4** Let $\{e_i\}_{i=2}^{\dim B}$ be an extension of $x$ to an orthonormal basis for the horizontal distribution. Then we have

$$\begin{aligned}\mathrm{Ric}^A(x,x) = \sum_{i=2}^{\dim B} g(R^A(e_i,x)x,e_i) &= \sum_{i=2}^{\dim B} (2g(A_x A_{e_i} x, e_i) - g(A_x A_x e_i, e_i)) \\ &= \sum_{i=2}^{\dim B} 3g(A_x e_i, A_x e_i) \geq 0.\end{aligned}$$

Combined with (6.3.1), this yields $\pi^*(\mathrm{Ric}_B) \geq \mathrm{Ric}^{\mathrm{Horiz}}$. In contrast, the inequality $\pi^*(\mathrm{Ric}_B) \geq \mathrm{Ric}_{(E,g)}$ does not hold for all Riemannian submersions; see Pro and the second author [35].

**Proof of Theorem 6.1** Recall that $\Omega_1 = \bigcup \Omega_1^i$, where $\Omega_1^i$ is as in Proposition 1.1, and for each $\Omega_1^i$ we have a splitting

$$T(\Omega_1^i) = \mathcal{H}^i \oplus \mathcal{V}^i \oplus \mathrm{span}\{X^i\},$$

as in (1.1.1). For simplicity, throughout this proof, we will write $X$ for any of the $X^i$ s.

Let $W \in T\Omega_1$ be any vector with $|\mathrm{Ch}_\lambda(W)|_{\tilde{g}_\lambda} = 1$. Since $X \in TG(x)^\perp$, we have $\tilde{g}(X,V) = 0$ if and only if $\tilde{g}_\lambda(\mathrm{Ch}_\lambda(X), \mathrm{Ch}_\lambda(V)) = 0$. So we may write

$$\mathrm{Ch}_\lambda(W) = \mathrm{Ch}_\lambda(X)\cos\sigma + \mathrm{Ch}_\lambda(V)\sin\sigma$$

with $X \perp V$ and $|\mathrm{Ch}_\lambda(V)| = 1$.

Choose $\{E_i\}_{i=2}^n \subset T\Omega_1$ so that $\{\mathrm{Ch}_\lambda(W), \mathrm{Ch}_\lambda(E_2), \{\mathrm{Ch}_\lambda(E_i)\}_{i=3}^n\}$ is an orthonormal basis with $E_2 \in \mathrm{span}\{X,V\}$. By the horizontal curvature equation and Theorem 2.16,

$$\begin{aligned}\sec_{\tilde{g}_\lambda}(\mathrm{Ch}_\lambda(W), \mathrm{Ch}_\lambda(E_2)) &= \sec_{\tilde{g}_\lambda}(\mathrm{Ch}_\lambda(X), \mathrm{Ch}_\lambda(V)) \geq \sec_{\lambda^2 g_{\mathrm{bi}}+\tilde{g}}(\widehat{X}, \widehat{V}) \\ &= \mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+\tilde{g}}\Big((0,X), \Big(\frac{\kappa_V}{\lambda^2}, V\Big)\Big) \\ &= \mathrm{curv}_{\tilde{g}}(X,V) \geq K|V|^2_{\tilde{g}}.\end{aligned} \tag{6.4.1}$$

For $i \geq 3$ we have

$$\begin{aligned}\sec_{\tilde{g}_\lambda}(\mathrm{Ch}_\lambda(W), \mathrm{Ch}_\lambda(E_i)) &\geq \mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+\tilde{g}}(\widehat{W}, \widehat{E}_i) \\ &\geq \mathrm{curv}_{\tilde{g}}(W, E_i) \\ &= \mathrm{curv}_{\tilde{g}}(X\cos\sigma + V\sin\sigma, E_i) \\ &\geq \cos^2\sigma K|E_i|^2 + 2\sin\sigma\cos\sigma R^{\tilde{g}}(X,E_i,E_i,V) \\ &\qquad - 2\sin^2\sigma|\min\sec_g||V|^2|E_i|^2,\end{aligned} \tag{6.4.2}$$

where we applied Theorem 2.16(5) to replace $-|\min\sec_{\tilde{g}}|$ by $-2|\min\sec_g|$.

By the antisymmetry of $R^{\widetilde{g}}$, $R^{\widetilde{g}}(X, E_i, E_i, V) = R^{\widetilde{g}}(X, E_i, E_i, V^{\perp, E_i})$, where $V^{\perp, E_i}$ is the component of $V$ that is perpendicular to $E_i$. For $i \geq 3$, $X \perp E_i$, so $X \perp V^{\perp, E_i}$. Combining this with Lemma 2.10 and (2.3.5) we conclude

$$\begin{aligned}
-|R^{\widetilde{g}}(X, E_i, E_i, V)| &= -|R^{\widetilde{g}}(X, E_i, E_i, V^{\perp, E_i})| \\
&\geq -|R^{g}(X, E_i, E_i, V^{\perp, E_i})| - |E_i|_g^2 |V|_g \\
&\geq -(|R^g| + 1)|E_i|_g^2 |V|_g.
\end{aligned}$$

Thus

$$\begin{aligned}
\sec_{\widetilde{g}_\lambda}(\mathrm{Ch}_\lambda(W), \mathrm{Ch}_\lambda(E_i)) \geq \cos^2\sigma K |E_i|^2 - 2\sin\sigma\cos\sigma(|R^g| + 1)|E_i|_g^2 |V|_g \\
- 2\sin^2\sigma |\min \sec_g| |V|_g^2 |E_i|_g^2.
\end{aligned}$$

Combining this with (6.4.1) we have

$$\begin{aligned}
\mathrm{Ric}(W, W) \geq K|V|_g^2 + \sum_{i=3}^{n} (\cos^2\sigma K - 2\sin\sigma\cos\sigma(|R^g| + 1)|V|_g)|E_i|_g^2 \\
- \sum_{i=3}^{n} (2\sin^2\sigma |\min \sec_g| |V|_g^2)|E_i|_g^2.
\end{aligned}$$

Since $|E_i|_g \leq 1$ and we can choose $K$ to be as large as we please compared to $|\min \sec_g|$, we have

(6.4.3) $\mathrm{Ric}(W, W)$

$$\begin{aligned}
&\geq \frac{K}{2}|V|_g^2 + \sum_{i=3}^{n} \big(\cos^2\sigma K - 2\sin\sigma\cos\sigma(|R^g| + 1)|V|_g\big)|E_i|_g^2 \\
&\geq \sum_{i=3}^{n} \left(\cos^2\sigma K - 2\cos\sigma(|R^g| + 1)|V|_g + \frac{K}{2(n-2)}|V|_g^2\right)|E_i|_g^2 \\
&= \sum_{i=3}^{n} \big((\cos\sigma - (|R^g| + 1)|V|_g)^2 - \cos^2\sigma - (|R^g| + 1)^2|V|_g^2\big)|E_i|_g^2 \\
&\qquad + \sum_{i=3}^{n} \left(\cos^2\sigma K + \frac{K}{2(n-2)}|V|_g^2\right)|E_i|_g^2 \\
&\geq \sum_{i=3}^{n} \left(\cos^2\sigma \frac{K}{2} + \frac{K}{3(n-2)}|V|_g^2\right)|E_i|_g^2,
\end{aligned}$$

since $K$ can be arbitrarily large. This gives us a positive lower bound for the Ricci curvature on the regular part of $\Omega_1$. Since the regular part of $\Omega_1$ is not compact we also need to see that this bound is uniformly positive. For this we analyze the norms $|V|_g$ and $|E_i|_g$.

Writing $Y$ for either $V/|V|_{\widetilde{g}}$ or $E_i/|E_i|_{\widetilde{g}}$, we have

$$|\widehat{Y}|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}} = \frac{|\kappa_Y|^2_{g_{\mathrm{bi}}}}{\lambda^2} + |Y|^2_{\widetilde{g}}.$$

Since $|Y|_{\widetilde{g}} = 1$, by part (1) of Proposition 3.1 we have

$$|\widehat{Y}|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}} \leq \frac{C}{\lambda^2} + 1$$

for some $C > 0$. So when $Y = E_i/|E_i|_{\widetilde{g}}$ we have

$$\frac{|\widehat{E}_i|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}}}{|E_i|^2_{\widetilde{g}}} = |\widehat{Y}|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}} \leq \frac{C}{\lambda^2} + 1.$$

Since $|\widehat{E}_i|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}} = 1$, we conclude

$$\frac{\lambda^2}{C+\lambda^2} = \frac{1}{\frac{C}{\lambda^2}+1} \leq |E_i|^2_{g}.$$

The same argument gives us

$$\frac{\lambda^2}{C+\lambda^2} \leq |V|^2_{g}.$$

Combining the previous two displays with inequality (6.4.3) gives a uniform positive lower bound for $\mathrm{Ric}_{\Omega_1}$. $\square$

**Remark 6.5** The positive lower bound on $\mathrm{Ric}_{\widetilde{g}_\lambda}|_{\Omega_1}$ above is far from optimal. In fact, the lower bound on $|E_i|^2_g$ is minimal on the vectors $E_i$ for which $|\kappa_{E_i}|^2_{g_{\mathrm{bi}}}$ is maximal, and one of the nonnegative terms that we dropped in inequality (6.4.2) is very large for these vectors if $\lambda$ is small; cf the statement and proof of Proposition 6.7.

We will make use of [3, Theorem 1], which is a consequence of the proof of [27, Proposition 3.4].

**Proposition 6.6** [3, Theorem 1] *Let $M = G/H$ be an effective homogeneous space with $G$ a connected Lie group and $H$ a compact subgroup. Let $\mathfrak{h}$ be the Lie algebra of $H$ and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ with respect to $g_{\mathrm{bi}}$. Let*

$$C(\mathfrak{m}) \equiv \{v \in \mathfrak{m} \mid [v, w] = 0 \text{ for all } w \in \mathfrak{m}\}.$$

*If $\pi_1(M)$ is finite, then $C(\mathfrak{m}) = 0$.*

Before proving Theorem 6.2 we establish three preliminary results.

**Proposition 6.7** *Let $g$ be any $G$–invariant metric on $M$, and let $\mathcal{K}$ be any compact subset of $M^{\mathrm{reg}}$. Given any $C > 0$, there is an $l(C) > 0$ so that for all $V \in TG(x)$ with $x \in \mathcal{K}$*

$$\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{V}) > C|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g} > 0$$

*for all $\lambda \in (0, l(C))$.*

**Proof** Let $\{\widehat{V}, \widehat{W}_1, \dots, \widehat{W}_p, \widehat{Y}_1, \dots, \widehat{Y}_m\}$ be a $(\lambda^2 g_{\mathrm{bi}} + g)$–orthogonal basis for the horizontal space of our Riemannian submersion $q_{G\times M}\colon (G \times M, \lambda^2 g_{\mathrm{bi}} + g) \to M$ with $W_1, \dots, W_p \in TG(x)$ and $Y_1, \dots, Y_m \in TG(x)^{\perp}$, $|V|_g = |W_i|_g = |Y_i|_g = 1$. Since each $Y_i$ is in $TG(x)^{\perp}$, $\widehat{Y}_i = (0, Y_i)$. Therefore

$$\sum_{i=1}^{m} \mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{Y}_i) = \sum_{i=1}^{m} \mathrm{curv}_g(V, Y_i) = \sum_{i=1}^{m} \sec_g(V, Y_i), \tag{6.7.1}$$

whereas

$$\begin{aligned}\sum_{i=1}^{p} \mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{W}_i) &= \sum_{i=1}^{p}\Big[\mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}}\Big(\frac{\kappa_V}{\lambda^2}, \frac{\kappa_{W_i}}{\lambda^2}\Big) + \mathrm{curv}_g(V, W_i)\Big] \\ &= \sum_{i=1}^{p}\Big[\frac{1}{\lambda^6}\,\mathrm{curv}_{g_{\mathrm{bi}}}(\kappa_V, \kappa_{W_i}) + \sec_g(V, W_i)\Big].\end{aligned} \tag{6.7.2}$$

Combining our hypothesis that $|\pi_1(\text{principal orbit})| < \infty$ with Proposition 6.6, and the fact that for all $i$, $\kappa_V, \kappa_{W_i} \in \mathfrak{m}_x$, we conclude that for at least one $i$, $\mathrm{curv}_{g_{\mathrm{bi}}}(\kappa_V, \kappa_{W_i}) > 0$. Moreover, our normalization $|V|_g = |W_i|_g = 1$ gives us a constant $C_1 > 0$ so that throughout $\mathcal{K}$

$$\max_i \mathrm{curv}_{g_{\mathrm{bi}}}(\kappa_V, \kappa_{W_i}) \geq C_1. \tag{6.7.3}$$

By part (1) of Proposition 3.1, there is another constant $C_3 > 0$ so that throughout $\mathcal{K}$ for $\lambda$ sufficiently small

$$|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g} = \frac{g_{\mathrm{bi}}(\kappa_V, \kappa_V)}{\lambda^2} + 1 \leq \frac{C_3}{\lambda^2} \tag{6.7.4}$$

and similarly

$$|\widehat{W}_i|^2_{\lambda^2 g_{\mathrm{bi}}+g} \leq \frac{C_3}{\lambda^2}.$$

Combining (6.7.1) with $|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g} \geq 1$ and $|\widehat{W}_i|^2_{\lambda^2 g_{\mathrm{bi}}+g} \geq 1$ we have

$$\begin{aligned}\sum_{i=1}^{m} \sec_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{Y}_i) &= \frac{1}{|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \sum_{i=1}^{m} \mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{Y}_i) \\ &= \frac{1}{|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \sum_{i=1}^{m} \sec_g(V, Y_i) \\ &\geq -m|\min \sec_g|.\end{aligned}$$

Equation (6.7.2) and inequality (6.7.3) give

$$\begin{aligned}&\sum_{i=1}^{p} \sec_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{W}_i) \\ &\qquad = \frac{1}{|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \sum_{i=1}^{p} \frac{\mathrm{curv}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{W}_i)}{|\widehat{W}_i|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \\ &\qquad \geq \frac{1}{|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \left( \frac{1}{\lambda^6} C_1 \frac{1}{|\widehat{W}_{j_0}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} + \sum_{i=1}^{p} \frac{1}{|\widehat{W}_i|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \sec_g(V, W_i) \right)\end{aligned}$$

for some $j_0$. So using inequality (6.7.4) and $|\widehat{W}_i|^2_{\lambda^2 g_{\mathrm{bi}}+g} \geq 1$,

$$\sum_{i=1}^{p} \sec_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{W}_i) \geq \frac{\lambda^4}{C_3^2} \frac{1}{\lambda^6} C_1 - \sum_{i=1}^{p} |\min \sec_g| \geq \frac{C_4}{\lambda^2} - p|\min \sec_g|,$$

for $C_4 = C_1/C_3^2$. Since $m + p = n - 1$, combining gives

$$\frac{1}{|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}} \mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{V}, \widehat{V}) \geq \frac{C_4}{\lambda^2} - (n-1)|\min \sec_{\widetilde{g}}|.$$

Since $C_4$ and $|\min \sec_g|$ are independent of $\lambda$, the right-hand side becomes arbitrarily large as $\lambda \to 0$. □

**Proposition 6.8** *Let $g$ be any $G$–invariant metric on $M$. For any compact subset $\mathcal{K} \subset M^{\mathrm{reg}}$, all $x \in \mathcal{K}$, any unit vector $Z \in T_x G(x)^{\perp}$, and any $V \in T_x G(x)$*

$$\frac{1}{|\widehat{V}|_{\lambda^2 g_{\mathrm{bi}}+g}} \left| \mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{Z}, \widehat{V}) \right| \longrightarrow 0$$

*uniformly on $\mathcal{K}$ as $\lambda \to 0$.*

**Proof** Since $\widehat{Z} = (0, Z)$ and $\lambda^2 g_{\mathrm{bi}} + g$ is a product metric,

$$R_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{U}, \widehat{V}, \widehat{Z}, \widehat{U}) = R_g(U, V, Z, U).$$

If we assume that $|\widehat{U}|^2_{\lambda^2 g_{\mathrm{bi}}+g}=1$, it follows that $|U|^2_g \le |\widehat{U}|^2_{\lambda^2 g_{\mathrm{bi}}+g}=1$, so dividing we get

$$\text{(6.8.1)}\qquad \begin{aligned}\frac{1}{|\widehat{V}|_{\lambda^2 g_{\mathrm{bi}}+g}}|R_{\lambda^2 g_{\mathrm{bi}}+g}(\widehat{U},\widehat{V},\widehat{Z},\widehat{U})| &\le |R_g||U|^2_g\frac{|V|_g}{|\widehat{V}|_{\lambda^2 g_{\mathrm{bi}}+g}}\\ &\le |R_g|\frac{|V|_g}{|\widehat{V}|_{\lambda^2 g_{\mathrm{bi}}+g}}.\end{aligned}$$

Normalize so that $|V|_g=1$, and combine part (2) of Proposition 3.1 with

$$|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+g}=\frac{g_{\mathrm{bi}}(\kappa_V,\kappa_V)}{\lambda^2}+1$$

to conclude that $|\widehat{V}|^2_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}_l}\to\infty$ uniformly on $\mathcal{K}$ as $\lambda\to 0$.

So $|V|_g/|\widehat{V}|_{\lambda^2 g_{\mathrm{bi}}+g}\to 0$ uniformly on $\mathcal{K}$ as $\lambda\to 0$, and the result follows. □

In the previous two propositions we estimated $\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+g}$ for an *abstract* $G$–invariant metric $g$. In part we did this because it is simpler to drop the terms involving the Cheeger $A$–tensor, $A^{\mathrm{Ch}}$. The metrics were abstract because we will still need the $A^{\mathrm{Ch}}$–terms to control the Ricci curvature on vectors in $T_xG(x)^{\perp}$. This will be achieved with the next result, Proposition 6.9, where we study the $\mathrm{Ric}^{\mathrm{Horiz}}$ tensor on vectors in $T_xG(x)^{\perp}$ for an iterated Cheeger deformation $q_{G\times M}\colon (G\times M,\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0})\to M$. It is of course true that $(\widetilde{g}_{\lambda_0})_{\lambda_1}=\widetilde{g}_{\lambda}$ for some $\lambda$, but the tensors $\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1{}^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}$ and $\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+\widetilde{g}}$ can differ significantly since the corresponding Cheeger $A$–tensors that are dropped can be quite different.

Since the previous two propositions contain estimates for $\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda^2 g_{\mathrm{bi}}+g}$ with respect to an abstract $G$–invariant metric $g$, we will then be able to combine Propositions 6.7, 6.8 and 6.9 to prove Theorem 6.2.

**Proposition 6.9** *Let $(\widetilde{g}_{\lambda_0})_{\lambda_1}$ be the metric on $M$ induced by the Riemannian submersion $q_{G\times M}\colon (G\times M,\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0})\to M$. Write $\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1{}^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}$ for the tensor $\mathrm{Ric}^{\mathrm{Horiz}}_g$ of Proposition 6.3 when the submersion is $q_{G\times M}$. Let $\mathcal{K}$ be any compact subset of $M^{\mathrm{reg}}$. If $\lambda_0$ is sufficiently small, then for all $x\in\mathcal{K}$ and for all $Z\in T_xG(x)^{\perp}$,*

$$\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Z},\widehat{Z})>|Z|^2_{\widetilde{g}_{\lambda_0}}$$

*for all $\lambda_1\in(0,\infty)$.*

**Proof** First we study the sectional curvature of $\widetilde{g}_{\lambda_0}$. For $\varepsilon>0$ as in Theorem 2.16 and $Y,Z\in TM$ and we have

$$\sec_{\widetilde{g}}(Y,Z)\ge\sec_g(Y,Z)-\varepsilon.$$

Combining this with Proposition 3.4 we have that for $x \in \mathcal{K}$ and $Y, Z \in T_x G(x)^{\perp}$,

$$\sec_{\tilde{g}_{\lambda_0}}(Y, Z) \geq \sec_{d\pi(g)}(d\pi^{\mathrm{reg}}(Y), d\pi^{\mathrm{reg}}(Z)) - 2\varepsilon, \tag{6.9.1}$$

provided $\lambda_0$ is sufficiently small.

For $V \in T_x G(x)$ with $|V|_{\tilde{g}} = 1$ and $Z \in T_x G(x)^{\perp}$ with $|Z|_{\tilde{g}_{\lambda_0}} = |Z|_{\tilde{g}} = 1$,

$$\begin{aligned}
\sec_{\tilde{g}_{\lambda_0}}(\mathrm{Ch}_{\lambda_0}(Z), \mathrm{Ch}_{\lambda_0}(V)) &\geq \sec_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}}(\hat{Z}, \hat{V}) \\
&= \frac{1}{|\hat{V}|^2_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}}} \operatorname{curv}_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}}\Big((0, Z), \Big(\frac{k_V}{\lambda_0^2}, V\Big)\Big) \\
&= \frac{\sec_{\tilde{g}}(Z, V)}{|\hat{V}|^2_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}}}.
\end{aligned}$$

Combining part (2) of Proposition 3.1 with

$$|\hat{V}|^2_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}} = \frac{g_{\mathrm{bi}}(\kappa_V, \kappa_V)}{\lambda_0^2} + 1,$$

we have

$$|\hat{V}|^2_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}} \to \infty$$

as $\lambda_0 \to 0$. So $\sec_{\tilde{g}}(Z, V)/|\hat{V}|^2_{\lambda_0^2 g_{\mathrm{bi}} + \tilde{g}}$ goes to $0$ uniformly on $M \setminus \Omega_1$ as $\lambda_0 \to 0$, since $|\sec_{\tilde{g}}(Z, V)|$ is bounded from above by a bound that is independent of $\lambda_0$.

It follows that

$$\sec_{\tilde{g}_{\lambda_0}}(\mathrm{Ch}_{\lambda_0}(Z), \mathrm{Ch}_{\lambda_0}(V)) \geq -\tau(\lambda_0),$$

where $\tau$ is as in (1.0.2). Since $\mathrm{Ch}_{\lambda_0}\colon T_x M \to T_x M$ is an isomorphism that preserves the splitting $T_x M = T_x G(x) \oplus T_x G(x)^{\perp}$, we conclude that for any $Z \in T_x G(x)^{\perp}$ and any $W \in T_x G(x)$,

$$\sec_{\tilde{g}_{\lambda_0}}(Z, W) \geq -\tau(\lambda_0). \tag{6.9.2}$$

Now let $\{\hat{Z}, \hat{W}_1, \ldots, \hat{W}_p, \hat{Y}_1, \ldots, \hat{Y}_m\}$ be a $(\lambda_1^2 g_{\mathrm{bi}} + \tilde{g}_{\lambda_0})$–orthogonal basis for the horizontal space of our Riemannian submersion

$$q_{G \times M}\colon (G \times M, \lambda_1^2 g_{\mathrm{bi}} + \tilde{g}_{\lambda_0}) \to M$$

with $W_1,\dots,W_p\in TG(x)$ and $Y_1,\dots,Y_m\in TG(x)^\perp$, $|Z|_{\widetilde{g}_{\lambda_0}}=|W_i|_{\widetilde{g}_{\lambda_0}}=|Y_i|_{\widetilde{g}_{\lambda_0}}=1$. Then

$$\begin{aligned}
&\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Z},\widehat{Z})\\
&\quad=\sum_{i=1}^{p}\sec_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{W}_i,\widehat{Z})+\sum_{j=1}^{m}\sec_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Y}_i,\widehat{Z})\\
&\quad=\sum_{i=1}^{p}\frac{1}{|\widehat{W}_i|^2_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}}\,\mathrm{curv}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{W}_i,\widehat{Z})+\sum_{j=1}^{m}\sec_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Y}_i,\widehat{Z})\\
&\quad=\sum_{i=1}^{p}\frac{1}{|\widehat{W}_i|^2_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}}\,\sec_{\widetilde{g}_{\lambda_0}}(W_i,Z)+\sum_{j=1}^{m}\sec_{\widetilde{g}_{\lambda_0}}(Y_i,Z)\\
&\quad\geq-\sum_{i=1}^{p}\frac{\tau(\lambda_0)}{|\widehat{W}_i|^2_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}}+\sum_{j=1}^{m}\sec_{\widetilde{g}_{\lambda_0}}(Y_i,Z)\quad\text{by (6.9.2)}\\
&\quad=-\tau(\lambda_0)+\sum_{j=1}^{m}\sec_{\widetilde{g}_{\lambda_0}}(Y_i,Z).
\end{aligned}$$

Combining this with our hypothesis that $\mathrm{Ric}_{(M^{\mathrm{reg}}/G)}\geq 2$ and with inequality (6.9.1) gives

$$\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Z},\widehat{Z})\geq\left(-\tau(\lambda_0)+\tfrac{3}{2}\right)>1=|Z|^2_{\widetilde{g}_{\lambda_0}}$$

as claimed. □

**Proof of Theorem 6.2** Let $\lambda_0$ be small enough so the conclusion of Proposition 6.9 holds. An arbitrary unit vector that is horizontal for $q_{G\times M}\colon (G\times M,\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0})\to M$ has the form $\cos\sigma\widehat{V}+\sin\sigma\widehat{Z}$, where $V\in TG(x)$ and $Z\in TG(x)^\perp$, $|\widehat{V}|_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}=|\widehat{Z}|_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}=1$. So

$$\begin{aligned}
\mathrm{Ric}_{(\widetilde{g}_{\lambda_0})_{\lambda_1}}|_{M\setminus\Omega_1}&\geq\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\cos\sigma\widehat{V}+\sin\sigma\widehat{Z},\cos\sigma\widehat{V}+\sin\sigma\widehat{Z})\\
&=\cos^2\sigma\,\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{V},\widehat{V})+\sin 2\sigma\,\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{V},\widehat{Z})\\
&\qquad+\sin^2\sigma\,\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{Z},\widehat{Z}).
\end{aligned}$$

By Proposition 6.8 we have

$$\left|\mathrm{Ric}^{\mathrm{Horiz}}_{\lambda_1^2 g_{\mathrm{bi}}+\widetilde{g}_{\lambda_0}}(\widehat{V},\widehat{Z})\right|<\tfrac{1}{100},$$

provided $\lambda_1$ is sufficiently small. If we choose the constant $C$ in Proposition 6.7 to be $100$, and apply Proposition 6.9 we then get

$$\mathrm{Ric}_{(\widetilde{g}_{\lambda_0})_{\lambda_1}}|_{M\setminus\Omega_1} \geq 100\cos^2\sigma + \tfrac{1}{100}\sin 2\sigma + \sin^2\sigma > \tfrac{99}{100},$$

proving Theorem 6.2 and Theorem A. □

## 7 Lifting almost nonnegative curvature

Throughout this section we assume that $G$ is a compact, connected Lie group acting isometrically and effectively on a family of compact Riemannian manifolds $(M, g_\alpha)$. We further assume that the quotient $(M/G, \mathrm{dist}_\alpha)$ is an almost nonnegatively curved family of metric spaces.

We will obtain the almost nonnegatively curved family of metrics on $M$ via a sequence of three deformations, as follows.

**Step 1** Apply Theorem 5.1 to $(M, g_\alpha)$, yielding the Cheeger deformed metrics $(M, (g_\alpha)_l)$.

**Step 2** Apply Theorem 5.2 to get a family of $G$–invariant metrics $\widetilde{(g_\alpha)_l}$ of the form $\widetilde{(g_\alpha)_l} = e^{2f_\alpha}(g_\alpha)_l$, for appropriate smooth functions $f_\alpha\colon M \to \mathbb{R}$.

**Step 3** Apply a further Cheeger deformation to $(M, \widetilde{(g_\alpha)_l})$ to obtain $(M, (\widetilde{(g_\alpha)_l})_\lambda)$, which we will show is an almost nonnegatively curved family in Theorem 7.4 below.

**Remark 7.1** It seems possible that Theorem B could be proven performing these deformations in another order. The order we have chosen allows the argument to be broken into several smaller, separately verifiable pieces.

As the diameter bound is much easier to establish we will discuss it first. Since $\{(M/G, \mathrm{dist}_\alpha)\}_{\alpha=1}^\infty$ is an almost nonnegatively curved family,

$$\mathrm{Diam}(M/G, \mathrm{dist}_\alpha) \leq D$$

for some $D > 0$.

Let $(\mathrm{dist}_\alpha)_l$ be the orbital metric on $M/G$ induced by $(g_\alpha)_l$. Since a Cheeger deformation does not change the metric on the distribution that is orthogonal to the orbits,

$$\mathrm{Diam}(M/G, (\mathrm{dist}_\alpha)_l) \leq D.$$

Let $\widetilde{(\mathrm{dist}_\alpha)_l}$ be the orbital metric on $M/G$ induced by $\widetilde{(g_\alpha)_l}$. By Remark 2.17, our conformal factor $e^{2f}$ is as close as we please in the $C^0$–topology to $1$. In particular, we can easily arrange that

$$\mathrm{Diam}(M/G, \widetilde{(\mathrm{dist}_\alpha)_l}) \le 2D.$$

Finally, $(M, (\widetilde{(g_\alpha)_l})_\lambda)$ converges to $(M/G, \widetilde{(\mathrm{dist}_\alpha)_l})$ in the Gromov–Hausdorff topology as $\lambda \to 0$, so

$$\mathrm{Diam}(M, (\widetilde{(g_\alpha)_l})_\lambda) \le 3D,$$

provided $\lambda$ is sufficiently small.

Thus to prove Theorem B it suffices to show that there is a sequence of positive numbers, $\{\varepsilon_\alpha\}_{\alpha=1}^{\infty}$, Cheeger parameters $l, \lambda$ and $G$–invariant conformal factors $e^{2f_\alpha}$ so that

$$\varepsilon_\alpha \to 0 \quad \text{as } \alpha \to \infty, \quad \sec(M, (\widetilde{(g_\alpha)_l})_\lambda) \ge -\varepsilon_\alpha,$$

and $e^{2f_\alpha}$ is $C^0$–close to $1$. This in turn follows from the next three results.

Applying Theorem 5.1 gives us:

**Corollary 7.2** *Let $M$ and $g_\alpha$ be as in Theorem B. For any $\varepsilon > 0$ there is an $\alpha_0 \in \mathbb{N}$ so that for all $\alpha \ge \alpha_0$, there is a neighborhood $\Omega'(\alpha)$ of $S_1 \cup S_2 \cup \cdots \cup S_p$, and a Cheeger parameter $l_1(\alpha)$ such that for all $l \in (0, l_1(\alpha))$,*

$$\sec_{(g_\alpha)_l}(Y, Z) \ge -\frac{\varepsilon}{4} \tag{7.2.1}$$

*if either $Y, Z \in TG(x)^{\perp}|_{M \setminus \Omega'(\alpha)}$ or $Y, Z \in \{TG(x)^{\perp} \cap \overline{\mathcal{H}}^i\}|_{\Omega'(\alpha) \setminus S_1 \cup S_2 \cup \cdots \cup S_p}$ for some $i \in \{1, \dots, p\}$.*

*Here $(g_\alpha)_l$ is the metric on $M$ induced by the Riemannian submersion*

$$q_{G \times M} \colon (G \times M, l^2 g_{\mathrm{bi}} + g_\alpha) \longrightarrow M.$$

Applying Theorem 5.2 gives us the following.

**Corollary 7.3** *Let $(g_\alpha)_l$ be a metric that satisfies the conclusion of Corollary 7.2. For any $K, \varepsilon > 0$ there is a neighborhood $\Omega_1(\alpha)$ of $S_1 \cup S_2 \cup \cdots \cup S_p$ and a metric $\widetilde{(g_\alpha)_l} = e^{2f_\alpha}(g_\alpha)_l$ so that if $V \in \mathrm{span}\{\mathcal{V}^i, X^i\}|_{\Omega_1(\alpha)}$ for some $i \in \{1, \dots, p\}$, then*

$$\sec_{\widetilde{(g_\alpha)_l}}(V, W) \ge K$$

*for all $W \in T\Omega_1(\alpha)$, and*

$$\sec_{\widetilde{(g_\alpha)_l}}(V, W) \ge \sec_{(g_\alpha)_l}(V, W) - \frac{\varepsilon}{4},$$

*for all $V, W \in TM$.*

Theorem B follows from the next result.

**Theorem 7.4** *For any $\varepsilon > 0$, let $\widetilde{(g_\alpha)_l}$ be a metric that satisfies the conclusion of Corollary 7.3. There is an $l_2 > 0$ so that for all $\lambda \in (0, l_2)$,*

$$\sec_{(\widetilde{(g_\alpha)_l})_\lambda} \geq -\varepsilon,$$

*where $(\widetilde{(g_\alpha)_l})_\lambda$ is the metric on $M$ induced by the Riemannian submersion*

$$q_{G\times M}\colon (G\times M, \lambda^2 g_{\mathrm{bi}} + \widetilde{(g_\alpha)_l}) \longrightarrow M.$$

**Proof** Given $\varepsilon > 0$, from Corollaries 7.2 and 7.3 we have that there is a metric $\widetilde{(g_\alpha)_l}$ so that

$$\sec_{\widetilde{(g_\alpha)_l}}(V, W) > -\frac{\varepsilon}{2} \tag{7.4.1}$$

if $V, W \in \operatorname{span}\{TG(x)^\perp \cap \overline{\mathcal{H}}^i, \mathcal{V}^i, X^i\}|_{\Omega_1(\alpha)}$ for some $i \in \{1, \dots, p\}$ or $V, W \in TG(x)^\perp$ and $x \in M \setminus \Omega_1(\alpha)$.

By continuity, inequality (7.4.1) continues to hold on some neighborhood $U_{\mathrm{Sing}}$ of the set of planes spanned by vectors in

$$\bigcup_{i=1}^{p} \bigcup_{x\in\Omega^i} \{\overline{\mathcal{H}}^i \cap TG(x)^\perp, \mathcal{V}^i, X^i\} \cup \bigcup_{x\in\mathcal{C}_i} \operatorname{span}\{T_x(S_i)\cap TG(x)^\perp, \nu_x(S_i)\}.$$

Continuity also gives inequality (7.4.1) on some neighborhood $U_{\mathrm{Gen}}$ of the set of planes

$$\{P \mid P = \operatorname{span}\{V, W\}, V, W \in TG(x)^\perp|_{M\setminus\Omega_1(\alpha)}\}.$$

For simplicity from this point forward we set

$$g \equiv \widetilde{(g_\alpha)_l}.$$

Then by Proposition 3.2,

$$\sec_{g_\lambda}(\mathrm{Ch}_\lambda(P)) > -\varepsilon$$

for all planes $P \in U_{\mathrm{Sing}} \cup U_{\mathrm{Gen}}$. So we only have to verify the same inequality for planes in the complement of $U_{\mathrm{Sing}} \cup U_{\mathrm{Gen}}$.

Let $P$ be any plane in the complement of $U_{\mathrm{Sing}}$ with foot point in $\Omega_1(\alpha)$ of the form $P = \operatorname{span}\{V, W\}$ with $V$ and $W$ orthonormal with respect to $g$. By Corollary 4.4 there is a $c > 0$ so that

$$\max\{|\kappa_W|^2_{g_{\mathrm{bi}}}, |\kappa_V|^2_{g_{\mathrm{bi}}}\} \geq c.$$

Similarly, let $P$ be any plane in the complement of $U_{\mathrm{Gen}}$ with foot point in $M \setminus \Omega_1(\alpha)$ of the form $P = \mathrm{span}\{V, W\}$ with $V$ and $W$ orthonormal with respect to $g$. By part (2) of Proposition 3.1, there is a (perhaps different) constant $c > 0$ so that

$$\max\{|\kappa_W|^2_{g_{\mathrm{bi}}}, |\kappa_V|^2_{g_{\mathrm{bi}}}\} \geq c.$$

So, for any plane in the complement of $U_{\mathrm{Sing}} \cup U_{\mathrm{Gen}}$,

$$\max\left\{-\frac{1}{|\kappa_W|^2_{g_{\mathrm{bi}}}}, -\frac{1}{|\kappa_V|^2_{g_{\mathrm{bi}}}}\right\} \geq -\frac{1}{c}. \tag{7.4.2}$$

On the other hand, since $M$ is compact,

$$|\sec_g| \leq K_1 \tag{7.4.3}$$

for some $K_1 \in \mathbb{R}$.

Now consider a plane $P = \mathrm{span}\{V, W\}$ in the complement of $U_{\mathrm{Sing}} \cup U_{\mathrm{Gen}}$. Combining inequalities (7.4.3) and (7.4.2) with the estimate

$$\sec_{g_l}(\mathrm{Ch}_l(W), \mathrm{Ch}_l(V)) \geq \max\left\{-\frac{\lambda^2}{|\kappa_V|^2_{g_{\mathrm{bi}}}}, -\frac{\lambda^2}{|\kappa_W|^2_{g_{\mathrm{bi}}}}\right\} |\sec_{g_M}(V, W)|$$

from Proposition 3.2 gives us

$$\sec_{g_\lambda}(\mathrm{Ch}_\lambda(W), \mathrm{Ch}_\lambda(V)) > -\varepsilon,$$

provided $\lambda$ is sufficiently small, and hence proves Theorem 7.4. □

Finally, as the deformations used to prove Theorems A and B are the same, Theorem C follows by combining the proofs of Theorems A and B.

# 8 Examples

The proof of Theorem D is based on Davis' SO(3)–actions on the class $\Sigma^7$, his $G_2$–actions on the class $\Sigma^{15}_{\mathrm{BP}}$ [7] and the following proposition.

**Proposition 8.1** *Let $(M_1, G)$ and $(M_2, G)$ be smooth $n$–dimensional $G$–manifolds with $G$ a compact Lie group, and $M_1/G = M_2/G = X$. In addition, suppose that the group diagrams and isotropy representations for $(M_1, G)$ and $(M_2, G)$ are the same when parameterized by $X$.*

*Let $(X, \mathrm{dist}_1)$ be the quotient of a $G$–invariant Riemannian metric $g_1$ on $M_1$. Then $(X, \mathrm{dist}_1)$ is also the quotient of a $G$–invariant Riemannian metric $g_2$ on $M_2$.*

**Proof** Let $\pi_1\colon M_1 \to X$ and $\pi_2\colon M_2 \to X$ be the quotient maps. By the slice theorem, for each $x \in X$ there is a neighborhood $N_x$ and a $G$–equivariant diffeomorphism

$$\Phi_x\colon \pi_2^{-1}(N_x) \longrightarrow \pi_1^{-1}(N_x)$$

so that

$$\pi_1 \circ \Phi_x = \pi_2.$$

Note that $\Phi_x^*(g_1)$ is a $G$–invariant metric on $\pi_2^{-1}(N_x)$ whose orbital distance metric is $\mathrm{dist}_1|_{N_x}$.

We glue the metrics $\Phi_{x_i}^*(g_1)$ together with a $G$–invariant partition of unity subordinate to $\{N_x\}_{x\in X}$, yielding a $G$–invariant metric $g_2$. The quotient $(M_2, g_2)/G$ is $(X, \mathrm{dist}_1)$ since for all $i$, $(\pi_2^{-1}(N_{x_i}), \Phi_{x_i}^*(g_1))/G$ is $(N_{x_i}, \mathrm{dist}_1)$. □

The key point for Davis' actions is that SO(3) and $G_2$ are the group of automorphisms of the quaternion and octonion division algebras, respectively. Davis starts by defining the actions on the subsets of $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$ that are $S^3$–bundles over $S^4$ and $S^7$–bundles over $S^8$, respectively.

Writing $\mathbb{F}$ for either $\mathbb{H}$ or $\mathbb{O}$ and $b$ for the real dimension of $\mathbb{F}$, recall that the $S^{b-1}$–bundles over $S^b$ with structure group SO$(b)$ are classified by $\mathbb{Z}\oplus\mathbb{Z}$ as follows.

The total space of the bundle $p_{m,n} : E_{m,n} \to S^b$ is obtained by gluing together two copies of $\mathbb{F}\times S^{b-1}$ via

$$\Phi_{m,n}\colon (u,v) \longmapsto \left(\frac{u}{|u|^2}, \frac{u^m}{|u|^m} v \frac{u^n}{|u|^n}\right) = (u', v').$$

To describe the map $p_{m,n}\colon E_{m,n} \to S^b$, we view $S^b$ as the disjoint union of two copies of $\mathbb{F}$ that are glued together along $\mathbb{F}\setminus\{0\}$ via $\phi\colon \mathbb{F}\setminus\{0\} \to \mathbb{F}\setminus\{0\}$, $\phi(u) = u/|u|^2$. The map $p_{m,n}\colon E_{m,n} \to S^b$ is then given by projecting onto the first factor of either copy of $\mathbb{F}\times S^{b-1}$.

Let $G$ stand for either SO(3) or $G_2$ and observe that $G$ acts by automorphisms of $\mathbb{F}$. So by letting $G$ act diagonally on both copies of $\mathbb{F}\times S^{b-1}$

$$g(u,v) = (g(u), g(v)) \tag{8.1.1}$$

we get a well defined $G$–action on $E_{m,n}$. In the quaternionic case, when $m+n=\pm1$, Milnor constructed a Morse function on $E_{m,n}$ with only two critical points and concluded that $E_{m,n}$ is homeomorphic to $S^{2b-1}$ [26], and Shimada carried out the analogous program in the octonionic case, also when $m+n=\pm1$ [39]. Davis observed that the Morse functions constructed by Milnor and Shimada are invariant under the

$G$–action, and concluded that $E_{m,n}$ is $G$–equivariantly homeomorphic to $S^{2b-1}$. In particular, $E_{m,n}/G$ is homeomorphic to $S^{2b-1}/G$.

It is easy to see the following.

**Proposition 8.2** *The action (8.1.1) is by symmetries of $p_{m,n}\colon E_{m,n} \to S^b$, and has three orbit types. In the quaternion case the isotropies are*

(1) *trivial when $uv - vu \neq 0$,*

(2) $\mathrm{SO}(2)$ *when $uv - vu = 0$, but either $\mathrm{Im}(v) \neq 0$ or $\mathrm{Im}(u) \neq 0$,*

(3) $\mathrm{SO}(3)$ *when $\mathrm{Im}(v) = \mathrm{Im}(u) = 0$.*

*In the octonion case the isotropies are*

(1) $\mathrm{SU}(2)$ *when $uv - vu \neq 0$,*

(2) $\mathrm{SU}(3)$ *when $uv - vu = 0$, but either $\mathrm{Im}(v) \neq 0$ or $\mathrm{Im}(u) \neq 0$,*

(3) $G_2$ *when $\mathrm{Im}(v) = \mathrm{Im}(u) = 0$.*

**Proposition 8.3** *The $G$–action on $E_{1,0} = S^{2b-1}$ is $G$–equivariantly diffeomorphic to an orthogonal action. It induces a $G$–action on $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$.*

**Proof** We prove the first statement by constructing explicit coordinate charts that identify $S^{2b-1}$ with $E_{1,0}$ and for which the corresponding action on $S^{2b-1}$ is

$$\left(g, \begin{pmatrix} a \\ b \end{pmatrix}\right) \longmapsto \begin{pmatrix} g(a) \\ g(b) \end{pmatrix}, \tag{8.3.1}$$

where we view $S^{2b-1}$ as the unit sphere in $\mathbb{F} \oplus \mathbb{F}$, and $G$ is acting by automorphisms of $\mathbb{F}$.

The coordinate charts are constructed as by Gromoll and Meyer in [15] or by the second author in [45]. Let $\phi\colon \mathbb{F} \to \mathbb{R}$ be

$$\phi(u) = \frac{1}{\sqrt{1+|u|^2}}.$$

The charts $h_1, h_2\colon \mathbb{F} \times S^{b-1} \to S^{2b-1}$ are defined by

$$h_1(u,q) = \begin{pmatrix} uq \\ q \end{pmatrix} \phi(u) \quad \text{and} \quad h_2(v,r) = \begin{pmatrix} r \\ \bar{v}r \end{pmatrix} \phi(v).$$

The charts $h_1$ and $h_2$ are embeddings onto the open dense sets

$$U_1 = \left\{ \begin{pmatrix} a \\ c \end{pmatrix} \,\middle|\, c \neq 0 \right\} \quad \text{and} \quad U_2 = \left\{ \begin{pmatrix} a \\ c \end{pmatrix} \,\middle|\, a \neq 0 \right\},$$

respectively. In fact, the formulas for the inverses are given by

$$h_1^{-1}\begin{pmatrix} a \\ c \end{pmatrix} = \left(\frac{a\overline{c}}{|c|^2}, \frac{c}{|c|}\right) \quad \text{and} \quad h_2^{-1}\begin{pmatrix} a \\ c \end{pmatrix} = \left(\frac{a\overline{c}}{|a|^2}, \frac{a}{|a|}\right).$$

It follows that

$$h_2^{-1} \circ h_1(u,q) = h_2^{-1}\left(\begin{pmatrix} uq \\ q \end{pmatrix}\phi(u)\right) = \left(\frac{uq\overline{q}\phi(u)^2}{|u|^2\phi(u)^2}, \frac{uq\phi(u)}{|u|\phi(u)}\right) = \left(\frac{u}{|u|^2}, \frac{u}{|u|}q\right).$$

In the case of $E_{1,0}$, it follows that the action in (8.1.1) is $G$–equivariantly diffeomorphic to the isometric action on $S^{2b-1}$ given by (8.3.1).

Since $G$ acts by symmetries of the Hopf fibration $h\colon S^{2b-1} \to S^b$, we get a well-defined $G$–action on the double mapping cylinder of the Hopf fibration

$$\mathbb{FP}^1 \cup_h \{(0, \tfrac{\pi}{2}) \times S^{2b-1}\} \cup_h \mathbb{FP}^1, \tag{8.3.2}$$

that is, on $\mathbb{FP}^2 \# -\mathbb{FP}^2$. □

To get $G$–actions on the other elements of $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$ we note that, as observed by Kervaire and Milnor, $\Sigma^7$ is a cyclic group of order $28$ and $\Sigma^{15}_{\mathrm{BP}}$ is a cyclic group of order $8,128$. In both cases $E_{2,-1}$ generates the cyclic group (see [7, page 69] and Eells and Kuiper [8, pages 101 and 106]). As observed by Davis, the fixed point set of the $G$–action is a circle. At a fixed point, we take the equivariant connected sum of $E_{2,-1}$ with itself. This produces a $G$–action on $2E_{2,-1} \equiv E_{2,-1} \# E_{2,-1}$, which is equivariantly homeomorphic to the standard $G$–action given in Proposition 8.3. Since $E_{2,-1}$ generates the cyclic groups $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$, we iterate this construction to obtain a $G$–action on each member of $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$ that is equivariantly homeomorphic to the standard $G$–action. In particular, each $G$–action has the same orbit space, group diagram and isotropy representation as the standard model.

We can therefore apply Proposition 8.1 with $M_1 = E_{1,0} = S^{2b-1}$ and the standard $G$–action and $M_2$ an arbitrary element of $\Sigma^7$ or $\Sigma^{15}_{\mathrm{BP}}$ with the $G$–action from above. This yields a $G$–invariant metric on $M_2$ whose quotient is positively curved.

We now apply Theorem C to obtain a family of $G$–invariant metrics with positive Ricci curvature that are also almost nonnegatively curved on each element of $\Sigma^7$ and $\Sigma^{15}_{\mathrm{BP}}$.

## 8.1 Fake $\mathbb{FP}^2\#-\mathbb{FP}^2$ spaces

Let $M^{2b}$ be the double mapping cylinder on

$$p_{m,n}\colon E_{m,n} \longrightarrow S^b,$$

where $m+n=\pm 1$. Since $G$ acts by symmetries of $p_{m,n}$ we get a smooth $G$–action on $M^{2b}$.

In the case when $(m,n)=(1,0)$ we get the connected sum of the standard projective plane with its negative, $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$, with a $G$–action.

As before, the two $G$–spaces $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$ and $M^{2b}$ have the same orbit space, group diagrams and isotropy representations. So as before, we will apply Proposition 8.1 and Theorem C to obtain a family of $G$–invariant metrics with positive Ricci curvature that are also almost nonnegatively curved. To do this we need a $G$–invariant metric on $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$ for which $(\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2)/G$ has both almost nonnegative sectional curvature and positive Ricci curvature.

Cheeger [6] constructed nonnegatively curved metrics on $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$ by gluing together two copies of the Hopf disk bundles that correspond to the Hopf fibration $h=p_{1,0}\colon E_{1,0}\to S^b$. To do the gluing, Cheeger constructed metrics that are products near the boundaries. Consequently, his metrics on $\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2$ and $(\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2)/G$ have $\mathrm{Ric}\geq 0$, but not $\mathrm{Ric}>0$. The zero Ricci curvatures occur for the field $X$ that is the gradient of the distance from the boundary of either disk bundle. Moreover, lower Ricci curvature bounds need not be preserved by Riemannian submersions [35], so the verification of positive Ricci curvature on $(\mathbb{F}\mathrm{P}^2 \# -\mathbb{F}\mathrm{P}^2)/G$ requires additional calculation and a minor modification of Cheeger's metric.

Since the case of $\mathbb{H}\mathrm{P}^2 \# -\mathbb{H}\mathrm{P}^2$ is essentially known by combining results of [19; 21], we will only discuss the case of $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ explicitly, noting that similar methods will also apply to the fake $\mathbb{H}\mathrm{P}^2 \# -\mathbb{H}\mathrm{P}^2$ spaces.

Fortunately, it is straightforward to modify Cheeger's construction to obtain $G$–invariant metrics on $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ for which the quotient metrics on $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2)/G$ are $\mathrm{Ric}>0$ with nonnegative curvature.

The biquotient approach indicated by Totaro [42] provides the means to achieve this with minimal calculations. Totaro observed that $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ is the quotient of a $\mathrm{Spin}(8)$–action on $\mathrm{Spin}(9)\times S^8$. Give $\mathrm{Spin}(9)\times S^8$ the product metric. Let $\mathrm{Spin}(8)$ act on $\mathrm{Spin}(9)$ on the right. Let $S^7\subset S^8$, where we view $S^7$ and $S^8$ as the unit spheres in $\mathbb{R}^8\subset\mathbb{R}^9$, respectively. We suspend the standard $\mathrm{Spin}(8)$–action on the $S^7$ to get a $\mathrm{Spin}(8)$–action on $S^8$, and denote the fixed points by $\pm e_9$. We set

$$t \equiv \mathrm{dist}_{S^8}(e^9,\cdot),$$
$$X \equiv \mathrm{grad}(\mathrm{dist}_{S^8}(e^9,\cdot)).$$

We write points in $S^8\setminus\{\pm e_9\}$ as $(x,t)\in S^7\times(0,\pi)$.

We then get a free Spin(8)–action on $\mathrm{Spin}(9) \times S^8$, and call the quotient map

$$q\colon \mathrm{Spin}(9) \times S^8 \longrightarrow (\mathrm{Spin}(9) \times S^8)/\,\mathrm{Spin}(8).$$

As observed in [42], $(\mathrm{Spin}(9) \times S^8)/\,\mathrm{Spin}(8)$ is diffeomorphic to $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$. To see this we first point out:

**Proposition 8.4** *The quotient space* $(\mathrm{Spin}(9) \times S^7)/\,\mathrm{Spin}(8)$ *is diffeomorphic to* $\mathrm{Spin}(9)/\,\mathrm{Spin}(7)$, *which in turn is diffeomorphic to* $S^{15}$.

**Proof** We identify $S^7$ with $\mathrm{Spin}(8)/\,\mathrm{Spin}(7)$ and write elements of $S^7$ as $\sigma\,\mathrm{Spin}(7)$ with $\sigma \in \mathrm{Spin}(8)$. This gives us a diffeomorphism $\Phi\colon (\mathrm{Spin}(9) \times S^7)/\,\mathrm{Spin}(8) \to \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$,

$$\Phi\colon (A, \sigma\,\mathrm{Spin}(7)) \cdot \mathrm{Spin}(8) \longmapsto (A\sigma)\,\mathrm{Spin}(7).$$

Finally, we identity $\mathrm{Spin}(9)/\,\mathrm{Spin}(7)$ with $S^{15}$, since, as was shown by Gluck, Warner and Ziller in [11], Spin(9) is the symmetry group of the octonionic Hopf fibration $S^{15} \to S^8$ and the isotropy is Spin(7). □

In terms of Spin(8) cosets, the octonionic Hopf fibration is

$$(\mathrm{Spin}(9) \times S^7)/\,\mathrm{Spin}(8) \longrightarrow \mathrm{Spin}(9)/\,\mathrm{Spin}(8)$$
$$(A, v) \cdot \mathrm{Spin}(8) \longmapsto A \cdot \mathrm{Spin}(8).$$

This yields the following:

**Proposition 8.5** *The quotient space* $(\mathrm{Spin}(9) \times S^8)/\,\mathrm{Spin}(8)$ *is diffeomorphic to the double mapping cylinder of the octonionic Hopf fibration, which in turn is diffeomorphic to* $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$.

We complete the proof of Theorem D by showing the following result, whose proof occupies the rest of the paper.

**Theorem 8.6** *Give* $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ *the quotient metric* $g_q$ *induced from the Riemannian submersion*

$$q\colon \mathrm{Spin}(9) \times S^8 \longrightarrow (\mathrm{Spin}(9) \times S^8)/\,\mathrm{Spin}(8) = \mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2.$$

*Then the regular part of the quotient of the* $G_2$*–action on* $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ *has uniformly positive Ricci curvature.*

To describe the horizontal space of $q$ at points of the form $(A,(x,t)) \in \mathrm{Spin}(9) \times \{S^8 \setminus \{\pm e_9\}\}$ we note that at $x \in S^7$ the isotropy $\mathrm{Spin}(8)_x$ of the Spin(8)–action on $S^7$ is isomorphic to Spin(7). For simplicity, we denote $\mathrm{Spin}(8)_x$ by Spin(7). Let $\mathfrak{spin}(7) \subset \mathfrak{spin}(8) \subset \mathfrak{spin}(9)$ be the Lie algebras of $\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$. Let $\mathfrak{m}_{\mathfrak{spin}(8)}$ and $\mathfrak{m}_{\mathfrak{spin}(9)}$ be the vector subspaces so that the splitting

$$\mathfrak{spin}(9) = \mathfrak{spin}(7) \oplus \mathfrak{m}_{\mathfrak{spin}(8)} \oplus \mathfrak{m}_{\mathfrak{spin}(9)}$$

is orthogonal and

$$\mathfrak{spin}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}_{\mathfrak{spin}(8)}.$$

**Proposition 8.7** *At any point of the form $(A,(x,t)) \in \mathrm{Spin}(9) \times \{S^8 \setminus \{\pm e_9\}\}$ the horizontal space of $q$ is spanned by vectors of the form*

$$\{(0,X), ((L_A)_* k^9, 0), (\sin^2 t (L_A)_*(k^8), k^8_{S^8})\}, \tag{8.7.1}$$

*where $X \equiv \mathrm{grad}(\mathrm{dist}_{S^8}(e^9,\cdot))$, $k^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$, $k^8 \in \mathfrak{m}_{\mathfrak{spin}(8)}$ and $t = \mathrm{dist}_{S^8}(e_9,\cdot)$.*

*At a point of the form $(A, \pm e_9) \in \mathrm{Spin}(9) \times \{\pm e_9\}$ the horizontal space of $q$ is spanned by vectors of the form*

$$\{(0,X), ((L_A)_* k^9, 0)\}, \tag{8.7.2}$$

*where $X \in T_{\pm e_9} S^8$, $k^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$.*

**Remark 8.8** Recall our convention that for an abstract $G$–manifold $M$ and an element $k$ of the Lie algebra $\mathfrak{g}$, $k_M$ denotes the Killing field on $M$ generated by $k$. Therefore $k^8_{S^8}$ is the Killing field on $S^8$ generated by $k^8 \in \mathfrak{m}_{\mathfrak{spin}(8)} \subset \mathfrak{spin}(8)$, and $(L_A)_* k^9$ would be written as $k^9_{\mathrm{Spin}(9)}$. However, we write $(L_A)_* k^9$, since the notation is standard.

**Proof** The definitions of $(0,X)$, $((L_A)_* k^9, 0)$ and the Spin(8)–action give us that $(0,X)$ and $((L_A)_* k^9, 0)$ are $q$–horizontal at points of $\mathrm{Spin}(9) \times \{S^8 \setminus \{\pm e_9\}\}$.

For any $k \in \mathfrak{m}_{\mathfrak{spin}(8)}$ we have

$$\begin{aligned}(g_{\mathrm{bi}} + g_{S^8})\big((\sin^2 t (L_A)_*(k^8), k^8_{S^8}), (-(L_A)_* k, k_{S^8})\big) \\ = -\sin^2 t g_{\mathrm{bi}}(k^8, k) + g_{S^8}(k^8_{S^8}, k_{S^8}).\end{aligned} \tag{8.8.1}$$

Since $S^7 = \mathrm{Spin}(8)/\mathrm{Spin}(7)$, we have the Riemannian submersion

$$\mathrm{Spin}(8) \longrightarrow S^7 = S^7 \times \{\tfrac{\pi}{2}\} \subset S^8.$$

Recall that we have used $\mathfrak{m}_{\mathfrak{spin}(8)}$ to denote the horizontal space at $x$ and, in our notation, the differential is

$$\mathfrak{m}_{\mathfrak{spin}(8)} \longmapsto TS^7, \quad k \longmapsto k_{S^8}.$$

Thus

$$g_{S^8}(k^8_{S^8}, k_{S^8})|_{(x,\pi/2)} = g_{S^7}(k^8_{S^8}, k_{S^8})|_x = g_{\mathrm{bi}}(k^8, k),$$
$$g_{S^8}(k^8_{S^8}, k_{S^8})|_{(x,t)} = \sin^2 t g_{\mathrm{bi}}(k^8, k),$$

so the right-hand side of (8.8.1) is $0$.

On the other hand, for $k \in \mathfrak{spin}(7)$ we also have

$$(g_{\mathrm{bi}} + g_{S^8})\big((\sin^2 t(L_A)_*(k^8), k^8_{S^8}), (-(L_A)_*k, k_{S^8})\big)$$
$$= -\sin^2 t g_{\mathrm{bi}}(k^8, k) + g_{S^8}(k^8_{S^8}, k_{S^8})$$

The first term is $0$ since $k^8 \in \mathfrak{m}_{\mathfrak{spin}(8)}$ and $k \in \mathfrak{spin}(7)$. Further, $k_{S^8} = 0$ since $k \in \mathfrak{spin}(7)$ and $\mathrm{Spin}(7)$ is the isotropy at $(x,t)$. So the second term is $0$ and it follows that $(\sin^2 t(L_A)_*(k^8), k^8_{S^8})$ is in the horizontal space of $q$, proving the first statement.

To prove the second statement, notice that $\pm e_9$ are the fixed points of the $\mathrm{Spin}(8)$–action on $S^8$ so at a point of the form $(A, \pm e_9) \in \mathrm{Spin}(9) \times \{\pm e_9\}$, the vectors $(0, X)$, $X \in T_{\pm e_9} S^8$ are horizontal for $q$. Then observe that $(L_A)_*(\mathfrak{m}_{\mathfrak{spin}(9)})$ is the horizontal distribution for the right $\mathrm{Spin}(8)$–action on $\mathrm{Spin}(9)$. □

Combining this with the horizontal curvature equation and a linear algebra argument we will show the following.

**Proposition 8.9** *The space $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2, g_q)$ is nonnegatively curved.*

(1) *All of the zero curvature planes in $q(\mathrm{Spin}(9) \times \{S^8 \setminus \pm e_9\})$ have horizontal lifts to $\mathrm{Spin}(9) \times S^8$ of the form*

$$\mathrm{span}\{(0, X), ((L_A)_*k^9, 0)\}, \tag{8.9.1}$$

*where $k^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$.*

(2) *All of the zero curvature planes in $q(\mathrm{Spin}(9) \times \{\pm e_9\})$ have horizontal lifts to $\mathrm{Spin}(9) \times S^8$ of the form*

$$\mathrm{span}\{(0, X), ((L_A)_*k^9, 0)\}, \tag{8.9.2}$$

*where $X \in T_{\pm e_9} S^8$ and $k^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$.*

**Proof** The space $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2, g_q)$ is nonnegatively curved since

$$(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2, g_q) = (\mathrm{Spin}(9) \times S^8)/\,\mathrm{Spin}(8).$$

If a plane $P$ tangent to $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2, g_q)$ has zero curvature, then its horizontal lift to $\mathrm{Spin}(9) \times S^8$ also has zero curvature, so to prove part (1), it suffices to show that the planes of the form (8.9.1) are the only zero-curvature planes in the distribution in (8.7.1).

To prove this we set

$$\mathcal{P} \equiv \{((L_A)_* k^9, 0), (\sin^2 t (L_A)_*(k^8), k^8_{S^8})\}.$$

Notice that the structure of $\mathcal{P}$ gives us bases $\{a_i\}$ for $\mathfrak{m}_{\mathfrak{spin}(9)}$ and $\{b_i\}$ for $\mathfrak{m}_{\mathfrak{spin}(8)}$ for which

$$\mathcal{P} \equiv \mathrm{span}\{(a_i, 0), (b_i, \iota(b_i))\}, \tag{8.9.3}$$

where $\iota\colon \mathfrak{m}_{\mathfrak{spin}(8)} \to TS^7_{(x,t)} \subset TS^8_{(x,t)}$ is the isomorphism that maps $\sin^2 t (L_A)_*(k^8)$ to $k^8_{S^8}$.

Let $\pi_1\colon \mathrm{Spin}(9) \times S^8 \to \mathrm{Spin}(9)$ and $\pi_2\colon \mathrm{Spin}(9) \times S^8 \to S^8$ be the respective projections.

From the structure of $\mathcal{P}$ in (8.9.3) it follows that for $P$, a 2–plane in $\mathcal{P}$, $d\pi_1(P)$ is also 2–dimensional. Combining this with the fact that $\mathfrak{m}_{\mathfrak{spin}(9)} \oplus \mathfrak{m}_{\mathfrak{spin}(8)}$ is the horizontal space of $\mathrm{Spin}(7) \to \mathrm{Spin}(9) \to S^{15}$, it follows that

$$\sec(P) > 0 \quad \text{for all planes } P \text{ in } \mathcal{P}. \tag{8.9.4}$$

On the other hand, the horizontal distribution is

$$\mathrm{span}\{(0, X), \mathcal{P}\},$$

so, in general, we can write a horizontal plane as

$$P = \mathrm{span}\{(0, \sigma X) + V, W\},$$

where $V, W \in \mathcal{P}$, $V \perp W$ and $\sigma \in \mathbb{R}$. Using the superscripts $(\cdot)^1$ and $(\cdot)^2$ for the projections to the first and second factors of $T(\mathrm{Spin}(9) \times S^8)$ and the fact that

$\mathrm{Spin}(9)\times S^8$ has a product metric, we see that

$$\begin{aligned}
&\mathrm{curv}_{g_{\mathrm{bi}}+g_{S^8}}((0,\sigma X)+V,W)\\
&\quad=\mathrm{curv}_{g_{\mathrm{bi}}+g_{S^8}}((0,\sigma X),W)+2R_{g_{\mathrm{bi}}+g_{S^8}}((0,\sigma X),W,W,V)+\mathrm{curv}_{g_{\mathrm{bi}}+g_{S^8}}(V,W)\\
&\quad=\mathrm{curv}_{g_{S^8}}(\sigma X,W^2)+2R_{g_{S^8}}(\sigma X,W^2,W^2,V^2)+\mathrm{curv}_{g_{\mathrm{bi}}}(V^1,W^1)\\
&\qquad\qquad+\mathrm{curv}_{g_{S^8}}(V^2,W^2)\\
&\quad=\mathrm{curv}_{g_{S^8}}(\sigma X+V^2,W^2)+\mathrm{curv}_{g_{\mathrm{bi}}}(V^1,W^1).
\end{aligned}$$

Since $\mathrm{curv}_{g_{S^8}}(\sigma X+V^2,W^2)$ is a curvature of $S^8$ and $\mathrm{curv}_{g_{\mathrm{bi}}}(V^1,W^1)$ is the horizontal lift of a curvature of $S^{15}$ to $\mathrm{Spin}(9)$, both terms are nonnegative. Since $X\perp W^2$ and $X\perp V^2$, the first term is positive if both $\sigma$ and $W^2$ are not zero. If $\sigma=0$, then our plane is in $\mathcal{P}$ and has positive curvature. If $W^2=0$, then

$$\mathrm{curv}_{g_{\mathrm{bi}}+g_{S^8}}(P)=\mathrm{curv}_{g_{\mathrm{bi}}}(V^1,W^1)>0,$$

unless $V^1$ is proportional to $W^1$. Since $V\perp W$, and $W^2=0$, this would give $V^1=0$. However, from the structure of $\mathcal{P}$ in (8.9.3), we see that $V^1=0$ implies $V=0$.

So the planes $P=\mathrm{span}\{(0,\sigma X)+V,W\}$ that have zero curvature are those with $W^2=0$ and $V=0$. It follows that all horizontal zero-curvature planes tangent to $\mathrm{Spin}(9)\times S^8$ have the desired form:

$$\mathrm{span}\{(0,X),((L_A)_*k^9,0)\}.$$

Since the curvature of all these planes is zero, the proof of part (1) is complete.

Part (2) follows by combining the second statement of Proposition 8.7 and the following facts:

(1) $\mathrm{Spin}(9)\times S^8$ has the product metric.

(2) Any plane tangent plane to $\mathrm{Spin}(9)\times S^8$ with a 2–dimensional projection to $TS^8$ is positively curved.

(3) Any plane tangent plane to $\mathrm{Spin}(9)\times S^8$ with a 2–dimensional projection to $((L_A)_*\mathfrak{m}_{\mathfrak{spin}(9)},0)$ is positively curved.

This completes the proof. □

**Remark 8.10** From [34, Corollary 1] it also follows that all planes of the form (8.9.1) or (8.9.2) project to zero curvature planes in $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2, g_q)$.

View the double mapping cylinder of the octonionic Hopf fibration as

$$([0,\pi]\times S^{15})/\sim,$$

where $(0\times S^{15})/\sim$ and $(\pi\times S^{15})/\sim$ are diffeomorphic to $S^8$. We write $(0\times S^{15})/\sim$ and $(\pi\times S^{15})/\sim$ as $0\times S^8$ and $\pi\times S^8$, respectively, and we let $t\equiv \mathrm{dist}(0\times S^8,\cdot)$, where the distance is determined by $g_q$.

Under the diffeomorphism between $(\mathrm{Spin}(9)\times S^8)/\mathrm{Spin}(8)$ and the double mapping cylinder of the octonionic Hopf fibration, the equivalence classes of the sets $\mathrm{Spin}(9)\times\{\pm e_9\}$ map to $0\times S^8$ and $\pi\times S^8$, which are the distinguished $\mathbb{O}\mathrm{P}^1$s of $\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2$. The octonionic Hopf fibration $S^{15}\to S^8$ written in terms of $\mathrm{Spin}(8)$ cosets is

$$\begin{aligned}(\mathrm{Spin}(9)\times S^7)/\mathrm{Spin}(8) &\longrightarrow \mathrm{Spin}(9)/\mathrm{Spin}(8),\\ (A,v)\cdot\mathrm{Spin}(8) &\longmapsto A\cdot\mathrm{Spin}(8).\end{aligned}\tag{8.10.1}$$

The field $(0,X)$ on $\mathrm{Spin}(9)\times S^8$ is the gradient of the distance from $\mathrm{Spin}(9)\times\{e_9\}$. The vectors $((L_A)_*k^9,0)$ are horizontal for the Hopf fibration (8.10.1), so Proposition 8.9 gives us part (1) of the following.

**Corollary 8.11** *View $(\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2,g_q)$ as the double mapping cylinder of the octonionic Hopf fibration.*

(1) *The zero-curvature planes in $(\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2)\setminus\{\mathbb{O}\mathrm{P}^1\cup\mathbb{O}\mathrm{P}^1\}$ are precisely those of the form*

$$\mathrm{span}\{X,Z\},\tag{8.11.1}$$

*where $X$ is the gradient of the distance from an $\mathbb{O}\mathrm{P}^1\subset\mathbb{O}\mathrm{P}^2$ and $Z$ is tangent to the levels of the same distance function and, in addition, is horizontal for the Hopf fibration $S^{15}\to S^8$.*

(2) *The zero-curvature planes in $\{\mathbb{O}\mathrm{P}^1\cup\mathbb{O}\mathrm{P}^1\}\subset\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2$ are precisely those of the form*

$$\mathrm{span}\{X,Z\},\tag{8.11.2}$$

*where $X$ is normal to one of the $\mathbb{O}\mathrm{P}^1$ and $Z$ is tangent to the same $\mathbb{O}\mathrm{P}^1$.*

(3) *The one-parameter family of Berger metrics $\{g_q|_{\{t\}\times S^{15}}\}_{t\in(0,\pi)}$ have the following property. For any $Z\in TS^{15}$ that is horizontal for the Hopf fibration $S^{15}\to S^8$, $g_q|_{\{t\}\times S^{15}}(Z,\cdot)$ is independent of $t$.*

**Proof** For part (2), just observe that the planes in (8.9.2) are precisely the planes in (8.11.2).

For part (3), notice that by Proposition 8.7 the horizontal lift to $\mathrm{Spin}(9) \times S^8$ of a Hopf-horizontal $Z \in T(\{t\} \times S^{15})$ has the form

$$((L_A)_* k_Z^9, 0)$$

for a fixed $k_Z^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$. On the other hand, if $W \in T(\{t\} \times S^{15})$ is any vector, then its horizontal lift to $\mathrm{Spin}(9) \times S^8$ has the form

$$((L_A)_* k_W^9, 0) + (\sin^2 t (L_A)_*(k_W^8), k_{W,S^8}^8),$$

for some $k_W^9 \in \mathfrak{m}_{\mathfrak{spin}(9)}$ and some $k_W^8 \in \mathfrak{m}_{\mathfrak{spin}(8)}$. Thus

$$\begin{aligned} &g_q|_{\{t\}\times S^{15}}(Z, W) \\ &\qquad = (g_{\mathrm{bi}} + g_{S^8})\big(((L_A)_* k_Z^9, 0), ((L_A)_* k_W^9, 0) + (\sin^2 t (L_A)_*(k_W^8), k_{W,S^8}^8)\big) \\ &\qquad = g_{\mathrm{bi}}(k_Z^9, k_W^9) \end{aligned}$$

since $\mathfrak{m}_{\mathfrak{spin}(9)}$ and $\mathfrak{m}_{\mathfrak{spin}(8)}$ are orthogonal. Since the right-hand side is independent of $t$, the result follows. □

Next we relate the horizontal spaces of the $G_2$–action on $S^{15}$ and the horizontal spaces of the Hopf fibration $h : S^{15} \to S^8$.

Adopting the point of view of [45], an explicit formula for the Hopf fibration $h$: $S^{15} \to S^8$ is given as follows. View $S^{15}$ as the unit sphere in $\mathbb{O} \oplus \mathbb{O} \cong \mathbb{R}^{16}$, and view $S^8$ as the unit sphere in $\mathbb{O} \oplus \mathbb{R} \cong \mathbb{R}^9$. Then

$$h\colon \begin{pmatrix} a \\ c \end{pmatrix} \longmapsto \left(a\bar{c}, \tfrac{1}{2}(|a|^2 - |c|^2)\right).$$

The last ingredient in our proof of Theorem 8.6 is the following.

**Proposition 8.12** *For all $\binom{a}{c} \in S^{15}$, there is a vector in*

$$\left\{T_{\binom{a}{c}} G_2 \begin{pmatrix} a \\ c \end{pmatrix}\right\}^{\perp}$$

*that is not Hopf horizontal, that is, it is not in*

$$\left\{T_{\binom{a}{c}} h^{-1}\left(h \begin{pmatrix} a \\ c \end{pmatrix}\right)\right\}^{\perp}.$$

**Proof** If $\operatorname{Im}(a) \neq 0$, set $\operatorname{Im}(a)/|\operatorname{Im}(a)| = \alpha$. We claim that at $\binom{a}{c}$ the vector $\binom{a\alpha}{0}$ is in

$$\left\{T_{\binom{a}{c}} G_2 \begin{pmatrix} a \\ c \end{pmatrix}\right\}^{\perp}.$$

Indeed let $S^7(|a|)$ be the octonions with norm equal to $|a|$. The curve

$$\gamma_\alpha\colon [0, 2\pi] \longrightarrow S^7(|a|),$$
$$\gamma_\alpha\colon t \longmapsto |a| e^{\alpha t},$$

is the geodesic in $S^7(|a|)$ that passes through $\pm|a|$ and $a$. The $G_2$–action on $S^7(|a|)$ is by cohomogeneity one with singular orbits $\pm|a|$. Thus $\gamma_\alpha$ is normal to the orbits of $G_2$. On the other hand, if $\gamma_\alpha(t_0) = a$, then $\gamma'_\alpha(t_0) = a\alpha$, so at $\binom{a}{c}$,

$$\begin{pmatrix} a\alpha \\ 0 \end{pmatrix} \in \left\{T_{\binom{a}{c}} G_2 \begin{pmatrix} a \\ c \end{pmatrix}\right\}^{\perp},$$

as claimed.

To see that this vector is not Hopf horizontal, notice that since $[a, \alpha] = 0$, $a, \alpha$ and $c$ are contained in a subalgebra that is isomorphic to $\mathbb{H}$. In particular, for all $t \in \mathbb{R}$ the three octonions $a$, $c$ and $e^{\alpha t}$ associate. So

$$\begin{pmatrix} ae^{\alpha t} \\ ce^{\alpha t} \end{pmatrix} \in h^{-1}\left(h\begin{pmatrix} a \\ c \end{pmatrix}\right),$$

and it follows that

$$\begin{pmatrix} a\alpha \\ c\alpha \end{pmatrix} \in T_{\binom{a}{c}} h^{-1}\left(h\begin{pmatrix} a \\ c \end{pmatrix}\right).$$

So

$$\begin{pmatrix} a\alpha \\ 0 \end{pmatrix} \notin \left\{T_{\binom{a}{c}} h^{-1}\left(h\begin{pmatrix} a \\ c \end{pmatrix}\right)\right\}^{\perp}.$$

A similar argument covers points for which $\operatorname{Im}(c) \neq 0$.

Finally, if $\operatorname{Im}(a) = \operatorname{Im}(c) = 0$, then $\binom{a}{c}$ is a fixed point of $G_2$ and all vectors are in

$$\left\{T_{\binom{a}{c}} G_2 \begin{pmatrix} a \\ c \end{pmatrix}\right\}^{\perp}. \qquad \square$$

**Proof of Theorem 8.6** Combining Proposition 8.9 and Corollary 8.11 we see that $\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ is nonnegatively curved and every zero plane in $(\mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2) \setminus (\mathbb{O}\mathrm{P}^1 \cup \mathbb{O}\mathrm{P}^1)$ contains $X$ and a Hopf horizontal vector. Similarly, every zero plane in $(\mathbb{O}\mathrm{P}^1 \cup \mathbb{O}\mathrm{P}^1) \subset \mathbb{O}\mathrm{P}^2 \# -\mathbb{O}\mathrm{P}^2$ is spanned by a vector tangent to an $\mathbb{O}\mathrm{P}^1$ and a vector normal to the same $\mathbb{O}\mathrm{P}^1$.

So $(\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2)^{\mathrm{reg}}/G_2$ at least has nonnegative Ricci curvature, and the only possible direction with zero Ricci curvature is $X$.

From part (3) of Corollary 8.11 and Proposition 8.12 we have an $\alpha > 0$ so that at all points of $x \in (\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2)^{\mathrm{reg}}$ there is a vector $Y \in TG_2(x)^{\perp}$ with

$$\sphericalangle(Y, \{\text{Hopf horizontal vectors}\}) > \alpha > 0. \tag{8.12.1}$$

Combining this with Corollary 8.11 we see that the planes

$$\mathrm{span}\{X, Y\}$$

are in the complement of a neighborhood $U$ of the zero planes of $\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2$. Hence by compactness of the complement of $U$, there is a $\beta > 0$ so that $\sec(X, Y) > \beta > 0$. Since all other sectional curvatures are at least nonnegative, we have

$$\mathrm{Ric}_{(\mathbb{O}\mathrm{P}^2\#-\mathbb{O}\mathrm{P}^2)^{\mathrm{reg}}/G_2}(X, X) > \beta > 0. \qquad \square$$

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