The complex symplectic geometry of the deformation space of complex projective structures

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This article investigates the complex symplectic geometry of the deformation space of complex projective structures on a closed oriented surface of genus at least 2. The cotangent symplectic structure given by the Schwarzian parametrization is studied carefully and compared to the Goldman symplectic structure on the character variety, clarifying and generalizing a theorem of S Kawai. Generalizations of results of C McMullen are derived, notably quasifuchsian reciprocity. The symplectic geometry is also described in a Hamiltonian setting with the complex Fenchel–Nielsen coordinates on quasifuchsian space, recovering results of I Platis.

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1 Introduction

Complex projective structures on surfaces are rich examples of geometric structures. They include in particular the three classical homogeneous Riemannian geometries on surfaces (Euclidean, spherical, hyperbolic) and they extend the theory of complex structures on surfaces, ie Teichmüller theory. They also have a strong connection to hyperbolic structures on 3–manifolds. Another feature is their analytic description using the Schwarzian derivative, which turns the deformation space of complex projective structures into a holomorphic affine bundle modeled on the cotangent bundle of the Teichmüller space. A natural complex symplectic geometry shows through these different perspectives, which has been discussed by various authors, eg Kawai [19], Platis [32] and Goldman [15]. This article attempts a unifying picture of the complex symplectic geometry of the deformation space of complex projective structures on surfaces, one that carefully relates the different approaches.

Let S be a closed oriented surface of genus $g \geqslant 2$. A complex projective structure on S is given by an atlas of charts mapping open sets of S into the projective line $\mathbb{C}\mathbf{P}^1$ such that the transition maps are restrictions of projective linear transformations. The deformation space of projective structures $\mathcal{CP}(S)$ is the space of equivalence classes of projective structures on S, where two projective structures are considered equivalent

if they are diffeomorphic.[1] Any projective atlas is in particular a holomorphic atlas, therefore a projective structure defines an underlying complex structure. This gives a forgetful projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$, where $\mathcal{T}(S)$ is the Teichmüller space of S, defined as the deformation space of complex structures on S.

The Schwarzian derivative is a differential operator that turns the fibers of p into complex affine spaces. Globally, $\mathcal{CP}(S)$ is a holomorphic affine bundle modeled on the holomorphic cotangent bundle $T^*\mathcal{T}(S)$. This yields a noncanonical identification $\mathcal{CP}(S) \cong T^*\mathcal{T}(S)$; it depends on the choice of the "zero section" $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$. There are at least two natural choices of sections to be considered. The Fuchsian section $\sigma_{\mathcal{F}}$ assigns to a Riemann surface X its Fuchsian projective structure given by the uniformization theorem. However, $\sigma_{\mathcal{F}}$ is not holomorphic. The other natural choice is that of a Bers section, given by Bers' simultaneous uniformization theorem. Bers sections are a family of holomorphic sections parametrized by Teichmüller space. Like any holomorphic cotangent bundle, $T^*\mathcal{T}(S)$ is equipped with a canonical complex symplectic form ω_{can}. Each choice of a zero section σ thus yields a symplectic structure ω^σ on $\mathcal{CP}(S)$, simply by pulling back the canonical symplectic form of $T^*\mathcal{T}(S)$. A first natural question is: How is ω^σ affected by σ? A small computation shows:

Proposition 3.3 *For any two sections σ_1 and σ_2 to $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$,*

$$\omega^{\sigma_2} - \omega^{\sigma_1} = -p^*d(\sigma_2 - \sigma_1).$$

A significantly different description of $\mathcal{CP}(S)$ is given by the holonomy of complex projective structures. Holonomy is a concept defined for any geometric structures and in this situation it gives a local identification *hol*$\colon \mathcal{CP}(S) \to \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$, where the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ is defined as a quotient of the set of representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. By a general construction of Goldman, $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C}))$ enjoys a natural complex symplectic structure ω_G. How does this symplectic structure compare to the cotangent symplectic structures ω^σ introduced above? A theorem of Kawai [19] gives a pleasant answer to that question: If σ is any Bers section, then ω^σ and ω_G agree up to some constant. Kawai's proof is highly technical and not very insightful though. Also, the conventions chosen in his paper can be misleading.[2] Relying on theorems of other authors, we give a simple alternative proof of Kawai's result. In fact,

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[1]More precisely, diffeomorphic by a homotopically trivial diffeomorphism; see Section 2.1.

[2]With the conventions chosen in his paper, Kawai finds $\omega^\sigma = \pi\omega_G$ while we get $\omega^\sigma = -i\omega_G$. Kawai's choices imply that ω_G takes imaginary values when restricted to the Fuchsian slice, but Goldman showed in [14] that (with appropriate conventions) ω_G is just the Weil–Petersson Kähler form on the Fuchsian slice. For the interested reader, we believe that, even after rectifying the conventions, there is a factor of 2 missing in Kawai's result.

we are able to do a little better and completely answer the question raised above. Our argument is based on the observation that there is an intricate circle of related ideas:

(i) $p: \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration (with respect to ω_G).

(ii) Bers sections $\mathcal{T}(S) \to \mathcal{CP}(S)$ are Lagrangian (with respect to ω_G).

(iii) If M is a 3–manifold diffeomorphic to $S \times \mathbb{R}$, then the Bers simultaneous uniformization map $\beta: \mathcal{T}(\partial_\infty M) \to \mathcal{CP}(\partial_\infty M)$ is Lagrangian (with respect to ω_G).

(iv) ω_G restricts to the Weil–Petersson Kähler form ω_{WP} on the Fuchsian slice.

(v) If σ is any Bers section, then $d(\sigma_\mathcal{F} - \sigma) = -i\,\omega_{\mathrm{WP}}$.

(vi) McMullen's quasifuchsian reciprocity (see [31] and Theorem 6.18).

(vii) For any Bers section σ, $\omega^\sigma = -i\,\omega_G$.

Let us briefly comment on these. Item (iv) is a result of Goldman; see [14]. Items (v) and (vi) are closely related and due to McMullen; see [31]. Steven Kerckhoff discovered that (iii) easily follows from a standard argument; we include this argument in our presentation, Theorem 4.3, for completeness. Item (vii) appears to be the strongest result, as it is not too hard to see that it implies all other results.[3] However, using Proposition 3.3 written above and a simple analytic continuation argument (Theorem 6.7) we show that (iv) and (v) imply (vii). In fact, we give a characterization of sections σ such that ω^σ agrees with ω:

Theorem 6.8 *Let* $\sigma: \mathcal{T}(S) \to \mathcal{CP}(S)$ *be a section of* p. *Then* ω^σ *agrees with the standard complex symplectic structure* ω_G *on* $\mathcal{CP}(S)$ *if and only if* $\sigma_\mathcal{F} - \sigma$ *is a primitive for the Weil–Petersson metric on* $\mathcal{T}(S)$:

$$\omega^\sigma = \omega_G \iff d(\sigma_\mathcal{F} - \sigma) = \omega_{\mathrm{WP}}.$$

(vii) then follows from McMullen's theorem (v):

Theorem 6.10 *If* $\sigma: \mathcal{T}(S) \to \mathcal{CP}(S)$ *is any Bers section, then*

$$\omega^\sigma = -i\,\omega_G.$$

We also get the expression of the symplectic structure pulled back by the Fuchsian identification:

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[3] This is not entirely true per se, but we do not want to go into too much detail here.

Corollary 6.13 *Let $\sigma_{\mathcal{F}}\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section. Then*

$$\omega^{\sigma_{\mathcal{F}}} = -i\,(\omega_G - p^*\omega_{\mathrm{WP}}).$$

Generalizing these ideas in the setting of convex cocompact 3–manifolds, we prove a generalized version of Theorem 6.10, by indirectly bootstrapping from Theorem 6.10 and relying on a result of Takhtajan–Teo [35]:

Theorem 6.15 *Let $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section. Then*

$$\omega^{\sigma} = -i\,\omega_G.$$

We derive a generalization of McMullen's result (v):

Corollary 6.17 *Let $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section. Then*

$$d(\sigma_{\mathcal{F}} - \sigma) = -i\,\omega_{\mathrm{WP}}.$$

And a generalized version of McMullen's quasifuchsian reciprocity:

Theorem 6.18 *Let $f\colon \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g\colon \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ be reciprocal generalized Bers embeddings. Then $D_{X_j} f$ and $D_{X_k} g$ are dual maps. In other words, for any $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$,*

$$\langle D_{X_j} f(\mu), \nu \rangle = \langle \mu, D_{X_k} g(\nu) \rangle.$$

Finally, we discuss the symplectic geometry of $\mathcal{CP}(S)$ in relation to complex Fenchel–Nielsen coordinates on the quasifuchsian space $\mathcal{QF}(S)$. These are global holomorphic coordinates on $\mathcal{QF}(S)$ introduced by Kourouniotis [24] and Tan [36] that are the complexification of the classical Fenchel–Nielsen coordinates on Teichmüller space $\mathcal{T}(S)$, or rather the Fuchsian space $\mathcal{F}(S)$. In [39; 40; 41], Wolpert showed that the Fenchel–Nielsen coordinates on $\mathcal{F}(S)$ encode the symplectic structure. For any simple closed curve γ on the surface S, there is a hyperbolic length function $l_\gamma\colon \mathcal{F}(S) \to \mathbb{R}$ and a twist flow $\mathrm{tw}_\gamma\colon \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S)$. Given a pants decomposition $\alpha = (\alpha_1, \dots, \alpha_N)$ on S (ie a maximal collection of nontrivial distinct free homotopy classes of simple closed curves; see Section 5.1), choosing a section of $l_\alpha = (l_{\alpha_1}, \dots, l_{\alpha_N})$ yields the classical Fenchel–Nielsen coordinates on Teichmüller space, $(l_\alpha, \tau_\alpha)\colon \mathcal{F}(S) \to (\mathbb{R}_{>0})^N \times \mathbb{R}^N$. Wolpert showed that the twist flow associated to a curve γ is the Hamiltonian flow of the length function l_γ. He also gave formulas for the Poisson bracket of two length functions, which show in particular that the length functions l_{α_i} associated to a pants decomposition α define an integrable Hamiltonian system, for which the functions l_{α_i}

are the action variables and the twist functions τ_{α_i} are the angle variables. In [32], Platis shows that this very nice "Hamiltonian picture" remains true in its complexified version on the quasifuchsian space for some complex symplectic structure ω_P, giving complex versions of Wolpert's results. This Hamiltonian picture is also extensively explored on the $\mathrm{SL}_2(\mathbb{C})$–character variety by Goldman in [15]. Independently from Platis' work, our analytic continuation argument shows that complex Fenchel–Nielsen coordinates are Darboux coordinates for the symplectic structure on $\mathcal{QF}(S)$:

Theorem 6.19 *Let α be any pants decomposition of S. Complex Fenchel–Nielsen coordinates $(l_\alpha^\mathbb{C}, \beta_\alpha^\mathbb{C})$ on the quasifuchsian space $\mathcal{QF}(S)$ are Darboux coordinates for the standard complex symplectic structure*

$$\omega_G = \sum_{i=1}^{N} dl_{\alpha_i}^\mathbb{C} \wedge d\tau_{\alpha_i}^\mathbb{C}.$$

This shows in particular the following, which was mentioned as "apparent" in [32] and is implied in [15], but does not seem to have been formally proved:

Corollary 6.20 *Platis' symplectic structure ω_P is equal to the standard complex symplectic structure ω_G on the quasifuchsian space $\mathcal{QF}(S)$.*

We thus recover Platis' and some of Goldman's results, in particular that the complex twist flow is the Hamiltonian flow of the associated complex length function. Although in the Fuchsian case it would seem unnecessarily sophisticated to use this as a definition of the twist flow, this approach might be fruitful in the space of projective structures. This transformation relates to what other authors have called *quakebends* or *complex earthquakes* discussed by Epstein–Marden [11], McMullen [30] and Series [33] among others.

Note The study of Taubes' symplectic structure [37] on the deformation space of *minimal hyperbolic germs* (in restriction to *almost-Fuchsian space*, which embeds as an open subspace of quasifuchsian space $\mathcal{QF}(S)$) is addressed in a forthcoming paper [26]. Both that article and the current one are based on the author's PhD thesis [27].

Structure of the paper Section 2 reviews complex projective structures, Fuchsian and quasifuchsian projective structures, the relation between complex projective structures and hyperbolic 3–manifolds, (generalized) Bers sections and embeddings. Section 3 introduces the affine cotangent symplectic structures given by the Schwarzian parametrization of $\mathcal{CP}(S)$. Section 4 reviews the character variety, holonomy of projective structures, Goldman's symplectic structure and some of its properties. In Section 5, we

briefly review Wolpert's "Hamiltonian picture" of Teichmüller space, then describe the complex Fenchel–Nielsen coordinates in quasifuchsian space and Platis' symplectic structure. In Section 6, we describe an analytic continuation argument then discuss and compare the different symplectic structures previously introduced. Our results are essentially contained in that last section.

2 Teichmüller space and the deformation space of complex projective structures

2.1 $\mathcal{T}(S)$ and $\mathcal{CP}(S)$

Let S be a surface. Unless otherwise stated, we will assume that S is connected[4], oriented, smooth, closed and with genus $g \geqslant 2$.

A complex structure on S is a maximal atlas of charts mapping open sets of S into the complex line $\mathbb{C}$ such that the transition maps are holomorphic. The atlas is required to be compatible with the orientation and smooth structure on S. A Riemann surface X is a surface S equipped with a complex structure.

The group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms of S acts on the set of all complex structures on S in a natural way: a compatible complex atlas on S is pulled back to another one by such diffeomorphisms. Denote by $\mathrm{Diff}_0^+(S)$ the identity component of $\mathrm{Diff}^+(S)$; its elements are the orientation-preserving diffeomorphisms of S that are homotopic to the identity. The quotient $\mathcal{T}(S)$ of the set of all complex structures on S by $\mathrm{Diff}_0^+(S)$ is called the *Teichmüller space* of S; its elements are called *marked Riemann surfaces*.

In a similar fashion, define a *complex projective structure* on S as a maximal atlas of charts mapping open sets of S into the complex projective line $\mathbb{C}\mathbf{P}^1$ such that

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[4]In some sections (eg Section 2.3), we will allow S to be disconnected to be able to consider the case where S is the boundary of a compact 3–manifold. This does not cause any issue in the exposition above.

the transition maps are (restrictions of) projective linear transformations (*ie* Möbius transformations of the Riemann sphere). The atlas is also required to be compatible with the orientation and smooth structure on S. A complex projective surface Z is a surface S equipped with a complex projective structure. In terms of geometric structures (see eg [38]), a complex projective structure is a $(\mathbb{C}\mathbf{P}^1, \mathrm{PSL}_2(\mathbb{C}))$–structure.

Again, $\mathrm{Diff}^+(S)$ naturally acts on the set of all complex projective structures on S. The quotient $\mathcal{CP}(S)$ by the subgroup $\mathrm{Diff}_0^+(S)$ is called the *deformation space of complex projective structures* on S; its elements are *marked complex projective surfaces*.

$\mathcal{T}(S)$ and $\mathcal{CP}(S)$ are complex manifolds Kodaira–Spencer deformation theory (see [10; 21]) applies and shows that $\mathcal{T}(S)$ is naturally a complex manifold whose holomorphic tangent space is $T_X^{1,0}\mathcal{T}(S) = \check{H}^1(X, \Theta_X)$, where Θ_X is the sheaf of holomorphic vector fields on X. Denote by K the canonical bundle over X (the holomorphic cotangent bundle of X). By Dolbeault's theorem, $\check{H}^1(X, \Theta_X)$ is isomorphic to the Dolbeault cohomology space $H^{-1,1}(X)$. Elements of $H^{-1,1}(X)$ are Dolbeault classes of smooth sections of $K^{-1} \otimes \bar{K}$ and are called *Beltrami differentials*. In a complex chart $z\colon U \subset S \to \mathbb{C}$, a Beltrami differential μ has an expression of the form $\mu = u(z)\frac{d\bar{z}}{dz}$, where u is a smooth function. The fact that we only consider (Dolbeault) classes of Beltrami differentials can be expressed as follows: if V is a vector field on X of type $(1,0)$, then the Beltrami differential $\bar{\partial}V$ induces a trivial (infinitesimal) deformation of the complex structure X. Recall that X carries a unique hyperbolic metric within its conformal class (called the Poincaré metric) by the uniformization theorem. By Hodge theory, every Dolbeault cohomology class has a unique harmonic representative μ. The tangent space $T_X\mathcal{T}(S)$ is thus also identified with the space $HB(X)$ of harmonic Beltrami differentials.

We can also derive a nice description of the Teichmüller cotangent space using cohomological machinery. Because $\dim_{\mathbb{C}} X = 1$ we have $\check{H}^1(X, \Theta_X) = H^1(X, K^{-1})$ and this space is dual to $H^0(X, K^2)$ by Serre duality. An element $\varphi \in Q(X) := H^0(X, K^2)$ is called a *holomorphic quadratic differential*. In a complex chart $z\colon U \subset S \to \mathbb{C}$, φ has an expression of the form $\varphi = \phi(z)dz^2$, where ϕ is a holomorphic function. The holomorphic cotangent space $T_X^*\mathcal{T}(S)$ is thus identified with the space $Q(X)$ of holomorphic quadratic differentials. The duality pairing $Q(X) \times H^{-1,1}(X) \to \mathbb{C}$ is just given by $(\varphi, \mu) \mapsto \int_S \varphi \cdot \mu$. Note that we systematically use tensor contraction (when dealing with line bundles over X): $\varphi \cdot \mu$ is a section of $K \otimes \bar{K} \cong |K|^2$, so it defines a conformal density and can be integrated over S. With the notations above, $\varphi \cdot \mu$ has local expression $\phi(z)u(z)|dz|^2$.

An easy consequence of the Riemann–Roch theorem is that $\dim_{\mathbb{C}} Q(X) = 3g - 3$, so that $\mathcal{T}(S)$ is a complex manifold of dimension $3g - 3$.

Similarly, Kodaira–Spencer deformation theory applies to show that $\mathcal{CP}(S)$ is naturally a complex manifold with tangent space $T_Z\mathcal{CP}(S) = \check{H}^1(Z, \Xi_Z)$, where Ξ_Z is the sheaf of projective vector fields on Z (see also [18]). It follows that $\mathcal{CP}(S)$ is a complex manifold of dimension $6g - 6$.

Unlike Teichmüller tangent vectors, there is no immediate way to describe tangent vectors to $\mathcal{CP}(S)$ in a more tangible fashion. However, note that a complex projective atlas is in particular a holomorphic atlas, so that a complex projective surface Z has an underlying structure of a Riemann surface X. This yields a forgetful map

$$p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$$

which is easily seen to be holomorphic. We will see in Section 3.1 that the fiber $p^{-1}(X)$ is naturally a complex affine space whose underlying vector space is $Q(X)$. In particular $\dim_{\mathbb{C}} \mathcal{CP}(S) = \dim_{\mathbb{C}} \mathcal{T}(S) \times \dim_{\mathbb{C}} Q(X) = 6g - 6$ as expected.

The Weil–Petersson Kähler metric on $\mathcal{T}(S)$ The *Weil–Petersson product* of two holomorphic quadratic differentials ϕ and ψ is given by

$$\langle \varphi, \psi \rangle_{\text{WP}} = -\frac{1}{4} \int_X \varphi \cdot \sigma^{-1} \cdot \overline{\psi}$$

where σ^{-1} is the dual current of the area form σ for the Poincaré metric.[5] It is a Hermitian inner product on the complex vector space $Q(X)$.

By duality, this gives a Hermitian product also denoted by $\langle \cdot, \cdot \rangle_{\text{WP}}$ on $H^{-1,1}(X)$ and globally a Hermitian metric $\langle \cdot, \cdot \rangle_{\text{WP}}$ on the manifold $\mathcal{T}(S)$. It was first shown to be Kähler by Ahlfors [1] and Weil. The Kähler form of the Weil–Petersson metric on $\mathcal{T}(S)$ is the real symplectic form

$$\omega_{\text{WP}} = -\text{Im}\langle \cdot, \cdot \rangle_{\text{WP}}.$$

2.2 Fuchsian and quasifuchsian projective structures

Note that whenever a Kleinian group Γ (ie a discrete subgroup of $\text{PSL}_2(\mathbb{C})$) acts freely and properly on some open subset U of the complex projective line $\mathbb{CP}^1$, the quotient surface U/Γ inherits a complex projective structure. This gives a variety (but not all) of complex projective surfaces, called embedded projective structures.

Fuchsian projective structures are a fundamental example of embedded projective structures. Given a marked complex structure X, the uniformization theorem provides

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[5]In a complex chart with values in the upper half-plane $z = x + iy : U \subset X \to \mathbb{H}^2$, the tensor product $-\frac{1}{4}\varphi \cdot \sigma^{-1} \cdot \overline{\psi}$ reduces to the classical expression $y^2 \varphi(z)\overline{\psi(z)}dx \wedge dy$.

a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ such that X is isomorphic to $\mathbb{H}^2/\rho(\pi_1(S))$ as a Riemann surface (where $\mathbb{H}^2$ is the upper half-plane). $\mathbb{H}^2$ can be seen as an open set (a disk) in $\mathbb{CP}^1$ and the Fuchsian group $\rho(\pi_1(S)) \subset \mathrm{PSL}_2(\mathbb{R})$ is in particular a Kleinian group, so the quotient $X \cong \mathbb{H}^2/\rho(\pi_1(S))$ inherits a complex projective structure Z. This defines a section

$$\tag{1} \sigma_{\mathcal{F}}\colon \mathcal{T}(S) \to \mathcal{CP}(S)$$

to p, called the *Fuchsian section*. It shows in particular that the projection p is surjective. We call $\mathcal{F}(S) := \sigma_{\mathcal{F}}(\mathcal{T}(S))$ the (deformation) space of (standard) Fuchsian (projective) structures on S, it is an embedded copy of $\mathcal{T}(S)$ in $\mathcal{CP}(S)$.

Quasifuchsian structures are another important class of embedded projective structures. Given two marked complex structures $(X^+, X^-) \in \mathcal{T}(S) \times \mathcal{T}(\overline{S})$[6] (where $\overline{S}$ is the surface S with reversed orientation), Bers' simultaneous uniformization theorem states that there exists a unique representation $\rho\colon \pi_1(S) \xrightarrow{\sim} \Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ up to conjugation such that:

- The limit set[7] Λ is a Jordan curve. The domain of discontinuity Ω is then the disjoint union of two simply connected domains Ω^+ and Ω^-. Such a Γ is called a quasifuchsian group.

- As marked Riemann surfaces, $X^+ \cong \Omega^+/\Gamma$ and $X^- \cong \Omega^-/\Gamma$.

Again, both Riemann surfaces X^+ and X^- inherit embedded complex projective structures Z^+ and Z^- by this construction. This defines a map

$$\beta = (\beta^+, \beta^-)\colon \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \to \mathcal{CP}(S) \times \mathcal{CP}(\overline{S})$$

which is a holomorphic section of $p \times p\colon \mathcal{CP}(S) \times \mathcal{CP}(\overline{S}) \to \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ by Bers' theorem. The map β has the obvious symmetry property:

$$\overline{\beta^-(X^+, X^-)} = \beta^+(\overline{X^-}, \overline{X^+}).$$

In particular, when $X^- \in \mathcal{T}(\overline{S})$ is fixed, the map $\sigma_{X^-} := \beta^+(\cdot, X^-)\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ is a holomorphic section of p, called a *Bers section*, and its image $\sigma_{X^-}(\mathcal{T}(S))$ in $\mathcal{CP}(S)$ is called a *Bers slice*. On the other hand, when $X^+ \in \mathcal{T}(S)$ is fixed, the map $f_{X^+} = \beta^+(X^+, \cdot)$ is an embedding of $\mathcal{T}(\overline{S})$ in the fiber $P(X^+) := p^{-1}(X^+) \subset$

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[6] Note that $\mathcal{T}(\overline{S})$ is canonically identified with $\overline{\mathcal{T}(S)}$, which denotes the manifold $\mathcal{T}(S)$ equipped with the opposite complex structure. The same remark holds for $\mathcal{CP}(\overline{S})$ and $\overline{\mathcal{CP}(S)}$.

[7] The *limit set* $\Lambda = \Lambda(\Gamma)$ is defined as the complement in $\mathbb{CP}^1$ of the domain of discontinuity Ω, which is the maximal open set on which Γ acts freely and properly. Alternatively, Λ is described as the closure in $\mathbb{CP}^1$ of the set of fixed points of elements of Γ.

$\mathcal{CP}(S)$ (which has the structure of a complex affine space as we will see in Section 3.1), f_{X^+} is called a *Bers embedding*. Also note that

$$\sigma_{\mathcal{F}}(X) = \beta^+(X, \bar{X}) = \overline{\beta^-(X, \bar{X})} = \sigma_{\bar{X}}(X).$$

This shows that the Fuchsian section $\sigma_{\mathcal{F}}$ is real analytic but not holomorphic, in fact it is a maximal totally real analytic embedding; see Section 6.1.

$\mathcal{QF}(S) := \beta^+(\mathcal{T}(S) \times \mathcal{T}(\bar{S})) \subset \mathcal{CP}(S)$ is called the (deformation) space of (standard) quasifuchsian (projective) structures on S. It is an open neighborhood of $\mathcal{F}(S)$ in $\mathcal{CP}(S)$ (this is a consequence of general arguments mentioned in the next paragraph), and it follows from the discussion above that Bers slices and Bers embeddings define two transverse foliations of $\mathcal{QF}(S)$ by holomorphic copies of $\mathcal{T}(S)$.

2.3 Complex projective structures and hyperbolic 3–manifolds

In this paragraph, we briefly review the relation between complex projective structures on the boundary of a compact 3–manifold $\hat{M}$ and hyperbolic structures on its interior. The quasifuchsian projective structures presented in the previous section occur as a particular case of this discussion. We then define *generalized Bers sections* and *generalized Bers embeddings*, and fix a few notations for later sections.

Let M be a connected complete hyperbolic 3–manifold. The universal cover of M is isometric to hyperbolic 3–space $\mathbb{H}^3$, this defines a unique faithful representation $\rho\colon \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ up to conjugation such that $\Gamma := \rho(\pi_1(M))$ acts freely and properly on $\mathbb{H}^3$ and $M \cong \mathbb{H}^3/\Gamma$. Let $\Omega \subset \mathbb{CP}^1$ be the domain of discontinuity of the Kleinian group Γ, it is the maximal open set on which Γ acts freely and properly. Here $\mathbb{CP}^1$ is seen as the "ideal boundary" of $\mathbb{H}^3$, also denoted $\partial_\infty \mathbb{H}^3$. The possibly disconnected surface $\partial_\infty M := \Omega/\Gamma$ is called the *ideal boundary* of M and it inherits an embedded complex projective structure as the quotient of $\Omega \subset \mathbb{CP}^1$ by the Kleinian group Γ. Conversely, any torsion-free Kleinian group Γ acts freely and properly on $\mathbb{H}^3 \sqcup \Omega$ (where Ω is the domain of discontinuity of Γ), and the quotient consists of a 3–manifold $\hat{M} = M \sqcup \partial_\infty M$, where $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3–manifold and $\partial_\infty M = \Omega/\Gamma$ is its ideal boundary. In general the manifold $\hat{M}$ is not compact, if it is then $\hat{M}$ is topologically the end compactification of M. In that case we say that the hyperbolic structure on M is *convex cocompact*. The *convex core* of M is the quotient of the convex hull of the limit set Λ in $\mathbb{H}^3$ by Γ. It is well-known that M is convex cocompact if and only if its convex core is a compact deformation retract of M.

Consider now a smooth 3–manifold with boundary $\widehat{M}$ with the following topological restrictions: $\widehat{M}$ is connected, oriented, compact, irreducible (every embedded 2–sphere bounds a ball) atoroidal (it contains no embedded, non-boundary parallel, incompressible tori) and with infinite fundamental group. Let $M = \widehat{M} \setminus \partial \widehat{M}$ denote the interior of $\widehat{M}$. For simplicity, we also assume that the boundary $\partial \widehat{M}$ is incompressible (the map $\iota_*\colon \pi_1(\partial \widehat{M}) \to \pi_1(\widehat{M})$ induced by the inclusion map ι is injective) and contains no tori, so that it consists of a finite number of surfaces $S_1, \ldots, S_N$ of genera at least 2. The Teichmüller space $\mathcal{T}(\partial \widehat{M})$ is described as the direct product $\mathcal{T}(\partial \widehat{M}) = \mathcal{T}(S_1) \times \cdots \times \mathcal{T}(S_N)$, similarly $\mathcal{CP}(\partial \widehat{M}) = \mathcal{CP}(S_1) \times \cdots \times \mathcal{CP}(S_N)$ and there is a holomorphic "forgetful" projection $p = p_1 \times \cdots \times p_N \colon \mathcal{CP}(\partial \widehat{M}) \to \mathcal{T}(\partial \widehat{M})$. Let $\mathrm{pr}_k \colon \mathcal{CP}(\partial \widehat{M}) \to \mathcal{CP}(S_k)$ denote the kth projection map. Let us consider the space $\mathcal{HC}(M)$ of convex cocompact hyperbolic structures on M up to homotopy. In other words, we define $\mathcal{HC}(M)$ as the quotient of the set of convex cocompact hyperbolic metrics on M by the group of orientation-preserving diffeomorphisms of M that are homotopic to the identity. Let us mention that Marden [28] and Sullivan [34] showed that $\mathcal{HC}(M)$ is a connected component of the interior of the subset of discrete and faithful representations in the character variety $\mathcal{X}(M, \mathrm{PSL}_2(\mathbb{C}))$. By the discussion above, any element of $\mathcal{HC}(M)$ determines a marked complex projective structure $Z \in \mathcal{CP}(\partial \widehat{M})$. We thus have a map $\varphi \colon \mathcal{HC}(M) \to \mathcal{CP}(\partial \widehat{M})$ that is holomorphic; that is a straightforward consequence of the fact that the holonomy map is holomorphic (see Section 4.3). Considering the induced conformal structure on $\partial \widehat{M}$, define the map $\psi = p \circ \varphi$ as in the following diagram:

$$
\begin{array}{ccc}
\mathcal{HC}(M) & \xrightarrow{\ \varphi\ } & \mathcal{CP}(\partial \widehat{M}) \\
& \searrow{\scriptstyle \psi} & \downarrow{\scriptstyle p} \\
& & \mathcal{T}(\partial \widehat{M})
\end{array}
$$

We now recall a version of a powerful theorem mainly due to Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurson (see [7, Chapter 7] for a detailed exposition of this theorem; also [2; 3; 6; 25; 28; 29; 34]):

Theorem 2.1 *The map $\psi \colon \mathcal{HC}(M) \to \mathcal{T}(\partial \widehat{M})$ is bijective.*

Let us mention that this statement has to be slightly modified if $\widehat{M}$ has compressible boundary. A consequence of this theorem is the following.

Proposition 2.2 *The map*

$$
\beta = \varphi \circ \psi^{-1} \colon \mathcal{T}(\partial \widehat{M}) \to \mathcal{CP}(\partial \widehat{M})
$$

is a holomorphic section of $p \colon \mathcal{CP}(\partial \widehat{M}) \to \mathcal{T}(\partial \widehat{M})$.

We call β the *(generalized) simultaneous uniformization section*. This map allows us to define "generalized Bers sections" and "generalized Bers embeddings" by letting only one of the boundary components' conformal structure vary and by looking at the resulting complex projective structure on some other (or the same) boundary component. This idea is made precise as follows. If an index $j \in \{1, \ldots, N\}$ and marked complex structures $X_i \in \mathcal{T}(S_i)$ are fixed for all $i \neq j$, we denote by $\iota_{(X_i)}$ the injection

$$\iota_{(X_i)} \colon \mathcal{T}(S_j) \to \mathcal{T}(\partial \hat{M}),$$

$$X \mapsto (X_1, \ldots, X_{j-1}, X, X_{j+1}, \ldots, X_N).$$

Let $f_{(X_i),k} = \mathrm{pr}_k \circ \beta \circ \iota_{(X_i)}$ as in the following diagram:

$$
\begin{array}{ccc}
\mathcal{T}(\partial \hat{M}) & \xrightarrow{\quad\beta\quad} & \mathcal{CP}(\partial \hat{M}) \\
{\scriptstyle \iota_{(X_i)}}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{pr}_k} \\
\mathcal{T}(S_j) & \xrightarrow{\quad f_{(X_i),k}\quad} & \mathcal{CP}(S_k)
\end{array}
$$

If $j = k$, then $\sigma_{(X_i)} := f_{(X_i),j}$ is a holomorphic section of $p_j \colon \mathcal{CP}(S_j) \to \mathcal{T}(S_j)$ that we call a *generalized Bers section*. On the other hand, if $j \neq k$, then $f_{(X_i),k}$ maps $\mathcal{T}(S_j)$ into the affine (see Section 3.1) fiber $P(X_k) \subset \mathcal{CP}(S_k)$ and we call a $f_{(X_i),k}$ a *generalized Bers embedding*. We apologize for this misleading terminology: a "generalized Bers embedding" is *not* an embedding in general.

Note that quasifuchsian structures discussed in the previous paragraph just correspond to the case where $M = S \times \mathbb{R}$. Let us also mention that this discussion is easily adapted when ∂M contains tori or is no longer assumed incompressible, with a few precautions. When $\hat{M}$ only has one boundary component, this gives the notion of a *Schottky section*.

3   The cotangent symplectic structures

3.1 $\mathcal{CP}(S)$ as an affine holomorphic bundle over $\mathcal{T}(S)$

The Schwarzian derivative   Given a locally injective holomorphic function $f \colon Z_1 \to Z_2$ where Z_1 and Z_2 are complex projective surfaces, define the *osculating map* $\tilde{f}$ to f at a point $m \in Z_1$ as the germ of a (locally defined) projective map that has the best possible contact with f at m. In some sense, one can take a flat covariant derivative $\nabla \tilde{f}$ and identify it as holomorphic quadratic differential $\mathcal{S}f \in Q(X)$, called the *Schwarzian derivative* of f. We refer to [4] and [9] for details.

In local projective charts, the Schwarzian derivative of f has the classical expression $\mathcal{S}f = Sf(z)dz^2$, where

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian operator enjoys the following properties:

Proposition 3.1 • If f is a projective map, then $\mathcal{S}f = 0$ (and conversely).

• If $f\colon Z_1 \to Z_2$ and $g\colon Z_2 \to Z_3$ are locally injective holomorphic functions between complex projective surfaces, then

$$\mathcal{S}(g \circ f) = \mathcal{S}(f) + f^*\mathcal{S}(g).$$

The Schwarzian derivative also satisfies an existence theorem:

Proposition 3.2 If $U \subset \mathbb{C}$ is simply connected and $\varphi \in Q(U)$, then $\mathcal{S}f = \varphi$ can be solved for $f\colon U \to \mathbb{C}\mathbf{P}^1$.

An elementary and constructive proof of this fact is given in [9], see also [4] for a more abstract argument.

Schwarzian parametrization of a fiber Recall that there is a holomorphic "forgetful" map $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$. Let X be a fixed point in $\mathcal{T}(S)$ and $P(X) := p^{-1}(\{X\})$ the set of marked projective structures on S whose underlying complex structure is X.

Given $Z_1, Z_2 \in P(X)$, the identity map $\mathrm{id}_S\colon Z_1 \to Z_2$ is holomorphic but not projective if $Z_1 \neq Z_2$. Taking its Schwarzian derivative accurately measures the "difference" of the two projective structures Z_1 and Z_2. Let us make this observation more precise. A consequence of Proposition 3.2 is that given $Z_1 \in P(X)$ and $\varphi \in Q(X)$, there exists $Z_2 \in P(X)$ such that $\mathcal{S}(\mathrm{id}_S\colon Z_1 \to Z_2) = \varphi$. This defines a map $Q(X) \times P(X) \to P(X)$, which is now easily seen to be a freely transitive action of $Q(X)$ on $P(X)$ as a consequence of Proposition 3.1. In other words, $P(X)$ is equipped with a complex affine structure, modeled on the vector space $Q(X)$.

Recall that $Q(X)$ is also identified with the complex dual space $T_X^*\mathcal{T}(S)$. As a result of this discussion, $\mathcal{CP}(S)$ is an affine holomorphic bundle modeled on the holomorphic cotangent vector bundle $T^*\mathcal{T}(S)$.

Consequently, $\mathcal{CP}(S)$ can be identified with $T^*\mathcal{T}(S)$ by choosing a "zero section" $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$. Explicitly, we get an isomorphism of complex affine bundles

$\tau^\sigma\colon Z \mapsto Z - \sigma(p(Z))$ as in the following diagram:

τ^σ is characterized by the fact that $\tau^\sigma \circ \sigma$ is the zero section of $\pi\colon T^*\mathcal{T}(S) \to \mathcal{T}(S)$. It is an isomorphism of holomorphic bundles whenever σ is a holomorphic section of p, such as a (generalized) Bers section (see Sections 2.2 and 2.3).

3.2 Complex symplectic structure on $T^*\mathcal{T}(S)$

A basic fact is that if M is any complex manifold (in particular when $M = \mathcal{T}(S)$), the total space of its holomorphic cotangent bundle T^*M is equipped with a canonical complex symplectic structure. We briefly recall this and a few useful properties.

The *canonical 1–form* ξ is the holomorphic $(1,0)$–form on T^*M defined at a point $\varphi \in T^*M$ by $\xi_\varphi := \pi^*\varphi$, where $\pi\colon T^*M \to M$ is the canonical projection and φ is seen as a complex covector on M in the right-hand side of the equality. The *canonical complex symplectic form* on T^*M is then simply defined by $\omega_{\mathrm{can}} = d\xi$.[8] If (z_k) is a system of holomorphic coordinates on M so that an arbitrary $(1,0)$–form has an expression of the form $\alpha = \sum w_k dz_k$, then (z_k, w_k) is a system of holomorphic coordinates on T^*M for which $\xi = \sum w_k dz_k$ and $\omega_{\mathrm{can}} = \sum dw_k \wedge dz_k$.

The canonical 1–form satisfies the following reproducing property. If α is any $(1,0)$–form on M, it is in particular a map $M \to T^*M$ and as such it can be used to pull back differential forms from T^*M to M. It is then not hard to show that $\alpha^*\xi = \alpha$ and as a consequence $\alpha^*\omega_{\mathrm{can}} = d\alpha$.

Note that if u is a vertical tangent vector to T^*M, ie $\pi_* u = 0$, then u can be identified with an element of the fiber containing its base point α (since the fibers of the projection are vector spaces). Under that identification, the symplectic pairing between u and any other tangent vector $v \in T_\alpha T^*M$ is just given by $\omega_{\mathrm{can}}(u, v) = \langle u, \pi_* v \rangle$, where $\langle \cdot, \cdot \rangle$ is the duality pairing on $T_{\pi(\alpha)}M$.

Note that the fibers of the projection $\pi\colon T^*M \to M$ are Lagrangian submanifolds of T^*M, ie π is a Lagrangian fibration. The zero section $s_0\colon M \hookrightarrow T^*M$ is also a Lagrangian embedding. These are direct consequences of the previous observation.

---

[8]Some authors take the opposite sign convention for ω_{can}.

3.3 Affine cotangent symplectic structures

As we have seen in Section 3.1, any choice of a "zero section" $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ yields an affine isomorphism $\tau^\sigma\colon \mathcal{CP}(S) \xrightarrow{\sim} T^*\mathcal{T}(S)$. We can use this to pull back the canonical symplectic structure of $T^*\mathcal{T}(S)$ on $\mathcal{CP}(S)$; define

$$\omega^\sigma := (\tau^\sigma)^* \omega_{\mathrm{can}}.$$

It is clear that ω^σ is a complex symplectic form on $\mathcal{CP}(S)$ whenever σ is a holomorphic section of p. Otherwise, it is just a complex-valued non-degenerate 2–form on $\mathcal{CP}(S)$, whose real and imaginary parts are both real symplectic forms.

How is ω^σ affected by the choice of the "zero section" σ? The following statement is both straightforward and key to our arguments:

Proposition 3.3 *For any two sections σ_1 and σ_2 to $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$,*

$$\omega^{\sigma_2} - \omega^{\sigma_1} = -p^* d(\sigma_2 - \sigma_1).$$

Here $\sigma_2 - \sigma_1$ is the "affine difference" between σ_2 and σ_1; it is a 1–form on $\mathcal{T}(S)$. In particular, the symplectic structures induced by the respective choices of two sections agree if and only if their affine difference is a closed 1–form.

Proof This is an easy computation as

$$
\begin{aligned}
-p^* d(\sigma_2 - \sigma_1) &= -p^*((\sigma_2 - \sigma_1)^* \omega_{\mathrm{can}}) \\
&= (-(\sigma_2 - \sigma_1) \circ p)^* \omega_{\mathrm{can}} \\
&= (\tau_2^\sigma - \tau_1^\sigma)^* \omega_{\mathrm{can}} = (\tau^{\sigma_2})^* \omega_{\mathrm{can}} - (\tau^{\sigma_1})^* \omega_{\mathrm{can}},
\end{aligned}
$$

where the first equality follows from Section 3.2. The last step is not as trivial as it would seem because one has to be careful about base points. Also, note that in

$$\tau^{\sigma_2}(Z) - \tau^{\sigma_1}(Z) = (Z - \sigma_2 \circ p(Z)) - (Z - \sigma_1 \circ p(Z)) = -(\sigma_2 - \sigma_1) \circ p(Z)$$

some minus signs are "affine" ones (hiding the Schwarzian derivative) and others are "genuine" minus signs, but this can be ignored in computation. □

A straightforward calculation also gives an explicit expression of $\omega^\sigma(u, v)$ whenever u is a vertical tangent vector to $\mathcal{CP}(S)$, it is exactly the same as the one obtained for the symplectic structure on $T^*\mathcal{T}(S)$:

Proposition 3.4 *Let $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a section of p. Let Z be a point in $\mathcal{CP}(S)$, and u, v be tangent vectors at Z such that u is vertical, ie $p_* u = 0$. Then*

$$\omega^\sigma(u, v) = \langle u, p_* v \rangle.$$

In this expression, u is seen as an element of $T_X{}^*T(S)$ (where $X = p(Z)$) under the identification $T_Z P(X) = Q(X) = T_X{}^*T(S)$. Note that this expression not involving σ is compatible with the previous proposition, which implies that $\omega^{\sigma_2} - \omega^{\sigma_1}$ is a horizontal 2–form.

As a consequence, just like for the cotangent space, we have the following.

Proposition 3.5 *Let* $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ *be any section. The projection* $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ *is a Lagrangian fibration for* ω^σ. *Also,* σ *is a Lagrangian embedding.*

4 The character variety and Goldman's symplectic structure

4.1 The character variety

References for this section include [9; 14; 15; 17].

Let $G = \mathrm{PSL}_2(\mathbb{C})$ and $\mathcal{R}(S)$ be the set of group homomorphisms from $\pi := \pi_1(S)$ to G. It has a natural structure of a complex affine algebraic set as follows. Choose a finite presentation $\pi = \langle \gamma_1, \dots, \gamma_N \mid (r_i)_{i \in I} \rangle$ of π. Evaluating a representation $\rho \in \mathcal{R}(S)$ on the generators γ_k embeds $\mathcal{R}(S)$ as an algebraic subset of G^N. This gives $\mathcal{R}(S)$ an affine structure indeed because of the identification $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$ (given by the adjoint representation of $\mathrm{PSL}_2(\mathbb{C})$ on its Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$). One can check that this structure is independent of the presentation.

G acts algebraically on $\mathcal{R}(S)$ by conjugation. The character variety $\mathcal{X}(S)$ is defined as the quotient in the sense of invariant theory. Specifically, the action of G on $\mathcal{R}(S)$ induces an action on the ring of regular functions $\mathbb{C}[\mathcal{R}(S)]$. Denote by $\mathbb{C}[\mathcal{R}(S)]^G$ the ring of invariant functions, it is finitely generated because $\mathcal{R}(S)$ is affine and G is reductive.

Lemma 4.1 [17] *The ring* $\mathbb{C}[\mathcal{R}(S)]^G$ *is generated in this case* ($G = \mathrm{PSL}_2(\mathbb{C})$) *by a finite number of the complex valued functions on* $\mathcal{R}(S)$ *of the form* $\rho \mapsto \mathrm{tr}^2(\rho(\gamma))$, *where* $\gamma \in \pi$.

$\mathcal{X}(S)$ is the affine set such that $\mathbb{C}[\mathcal{X}(S)] = \mathbb{C}[\mathcal{R}(S)]^G$, it is called the *character variety* of S. A consequence of the lemma is that the points of $\mathcal{X}(S)$ are in one-to-one correspondence with the set of *characters*, ie complex-valued functions of the form $\gamma \in \pi \mapsto \mathrm{tr}^2(\rho(\gamma))$.

The affine set $\mathcal{X}(S)$ splits into two irreducible components $\mathcal{X}(S)_l \cup \mathcal{X}(S)_r$, where elements of $\mathcal{X}(S)_l$ are characters of representations that lift to $\mathrm{SL}(2, \mathbb{C})$.

The set-theoretic quotient $\mathcal{R}(S)/G$ is rather complicated, but G acts freely and properly on the subset $\mathcal{R}(S)^s$ of irreducible[9] ("stable") representations, so that the quotient $\mathcal{R}(S)^s/G$ is a complex manifold. Furthermore, an irreducible representation is determined by its character, so that $\mathcal{X}(S)^s := \mathcal{R}(S)^s/G$ embeds (as a Zariski-dense open subset) in the smooth locus of $\mathcal{X}(S)$. Its dimension is $6g-6$. Let us mention that more generally, $\mathcal{X}(S)$ is in bijection with the set of orbits of "semistable" (ie reductive[10]) representations.

It is relatively easy to see that the Zariski tangent space at a point $\rho \in \mathcal{R}(S)$ is described as the space of crossed homomorphisms $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$ (ie 1–cocycles in the sense of group cohomology), specifically maps $u\colon \pi \to \mathfrak{sl}_2(\mathbb{C})$ such that $u(\gamma_1\gamma_2) = u(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)}u(\gamma_2)$ (where of course $\mathrm{Ad}\colon G \to \mathrm{Aut}\mathfrak{g}$ is the adjoint representation). The subspace corresponding to the tangent space of the G–orbit of ρ is the space of principal crossed homomorphisms $B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$ (ie 1–coboundaries in the sense of group cohomology), specifically maps $u\colon \pi \to \mathfrak{sl}_2(\mathbb{C})$ such that $u(\gamma) = \mathrm{Ad}_{\rho(\gamma)}u_0 - u_0$ for some $u_0 \in \mathfrak{sl}_2(\mathbb{C})$. Hence for (at least) smooth points $[\rho] \in \mathcal{X}(S)$, the tangent space is given by $T_{[\rho]}\mathcal{X}(S) = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho})$.

4.2 The complex symplectic structure on the character variety

By the general construction of [14], the character variety enjoys a complex symplectic structure defined in this situation as follows.

Recall that the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is equipped with its complex Killing form B. It is a non-degenerate complex bilinear symmetric form preserved by G under the adjoint action. Let $\widetilde{B} = \frac{1}{4}B$, it is explicitly given by $\widetilde{B}(u, v) = \mathrm{tr}(uv)$, where $u, v \in \mathfrak{sl}_2(\mathbb{C})$ are represented by trace-free 2×2 matrices.

One can compose the standard cup-product in group cohomology with $\widetilde{B}$[11] as "coefficient pairing" to get a dual pairing

$$H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \times H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \overset{\cup}{\to} H^2(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \overset{\widetilde{B}}{\to} H^2(\pi, \mathbb{C}) \cong \mathbb{C}.$$

This pairing defines a non-degenerate complex bilinear alternate 2–form on the complex vector space $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \cong T_{[\rho]}\mathcal{X}(S)$. It globalizes into a non-degenerate 2–form

---

[9] A representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ is called *irreducible* if it fixes no point in $\mathbb{C}\mathbf{P}^1$.

[10] A nontrivial representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ is called *reductive* if it is either irreducible of it fixes a pair of distinct points in $\mathbb{C}\mathbf{P}^1$.

[11] It would look somewhat more natural to use the actual Killing form B instead of $\widetilde{B} = \frac{1}{4}B$, but we choose to go with $\widetilde{B}$ because it is the convention used by most authors. Moreover, it gives a slightly simpler expression of our Theorems 6.10, 6.15 and 6.19.

ω_G on $\mathcal{X}(S)^s$. By arguments of Goldman [14] following Atiyah–Bott [5] this form is closed, in other words it is a complex symplectic form on the smooth quasi-affine variety $\mathcal{X}(S)^s$. (In fact, it defines an algebraic tensor on the whole character variety; see [14].)

4.3 Holonomy of projective structures

Just like any geometric structure, a complex projective structure Z defines a *developing map* and a *holonomy representation* (see eg [38]). The developing map is a locally injective projective map $f: \widetilde{Z} \to \mathbb{C}\mathbf{P}^1$ and it is equivariant with respect to the holonomy representation $\rho: \pi \to \mathrm{PSL}_2(\mathbb{C})$ in the sense that $f \circ \gamma = \rho(\gamma) \circ f$ for any $\gamma \in \pi$.

Holonomy of complex projective structures defines a map

$$hol: \mathcal{CP}(S) \to \mathcal{X}(S).$$

It is differentiable and its differential is "the identity map" in the sense that it is the canonical identification

$$d\,hol: T_Z\mathcal{CP}(S) = \check{H}^1(Z, \Xi_Z) \xrightarrow{\sim} H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) = T_{[hol(Z)]}\mathcal{X}(S).$$

A consequence of this observation is that *hol* is a local biholomorphism.

The holonomy representation ρ of a complex projective structure satisfies the following properties:

- ρ is liftable to $\mathrm{SL}_2(\mathbb{C})$ (a lift is provided by the monodromy of the Schwarzian equation). The image of the holonomy map thus lies in the irreducible component $\mathcal{X}(S)_l$ of $\mathcal{X}(S)$.

- The action of $\Gamma := \rho(\pi)$ on hyperbolic 3–space $\mathbb{H}^3$ does not fix any point or ideal point, nor does it preserve any geodesic. Representations having this property are called non-elementary. They are in particular irreducible representations, hence smooth points of the character variety as expected.

Conversely, Gallo, Kapovich and Marden [13] showed that any non-elementary liftable representation is the holonomy of a complex projective structure.

Although the holonomy map *hol*: $\mathcal{CP}(S) \to \mathcal{X}(S)$ is a local biholomorphism, it is neither injective nor a covering onto its image [16]. Nonetheless, we get a complex symplectic structure on $\mathcal{CP}(S)$ simply by pulling back that of $\mathcal{X}(S)^s$ by the holonomy map. Abusing notations, we will still call this symplectic structure ω_G. Alternatively, one could directly define ω_G on $\mathcal{CP}(S)$ in terms of the exterior product of 1–forms with values in some flat bundle (recall that $T_Z\mathcal{CP}(S) = \check{H}^1(Z, \Xi_Z)$, where Ξ_Z is

the sheaf of projective vector fields on Z; see Section 2.1). We will consider ω_G as the *standard complex symplectic structure on $\mathcal{CP}(S)$* (notably because it does not depend on any choice).

4.4 Fuchsian structures and a theorem of Goldman

Let $\mathcal{F}(S)$ be the space of marked hyperbolic structures on S (we abusively use the same notation as for the Fuchsian space). More precisely, $\mathcal{F}(S)$ is the space of complete hyperbolic metrics on S quotiented by $\mathrm{Diff}_0^+(S)$. In terms of geometric structures, $\mathcal{F}(S)$ is the deformation space of $(\mathbb{H}^2, \mathrm{PSL}_2(\mathbb{R}))$–structures on S (this is a consequence of Cartan–Hadamard's theorem). Holonomy identifies $\mathcal{F}(S)$ as the connected component of the character variety $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{R}))$ corresponding to faithful and discrete representations. $\mathcal{F}(S)$ is sometimes called the Fricke space of S.

The uniformization theorem states that there is a unique hyperbolic metric in each conformal class of Riemannian metrics on S. Since S is oriented, the choice of a conformal structure on S is equivalent to that of a complex structure on S. The uniformization theorem thus provides a bijective map

$$u \colon \mathcal{T}(S) \to \mathcal{F}(S).$$

By definition of the Fuchsian section $\sigma_{\mathcal{F}}$, the map u is precisely identified as $\sigma_{\mathcal{F}}$ if hyperbolic structures are considered as special examples of complex projective structures. Putting it differently, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{CP}(S) & \xrightarrow{\ hol\ } & \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{C})) \\
\sigma_{\mathcal{F}} \uparrow & & \uparrow \iota \\
\mathcal{T}(S) & \xrightarrow{\ u\ } & \mathcal{F}(S) \hookrightarrow \mathcal{X}(S, \mathrm{PSL}_2(\mathbb{R}))
\end{array}
$$

It is derived from this diagram that $\sigma_{\mathcal{F}}$ is a maximal totally real (see Section 6.1 for a definition of this notion and a different argument for this fact) analytic embedding of $\mathcal{T}(S)$ in $\mathcal{CP}(S)$.

By Goldman's general construction in [14] (described above in the case of $G = \mathrm{PSL}_2(\mathbb{C})$), $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{R}))$ is equipped with a real symplectic structure $\omega_{G,\mathrm{PSL}_2(\mathbb{R})}$. Of course it is just the restriction of the symplectic structure $\omega_G = \omega_{G,\mathrm{PSL}_2(\mathbb{C})}$ on $\mathcal{X}(S, \mathrm{PSL}_2(\mathbb{R}))$. Recall that $\mathcal{T}(S)$ is also equipped with a symplectic structure, the Weil–Petersson Kähler form ω_{WP}. In the same paper, Goldman shows that they are the same. More precisely, this is expressed in our setting as follows:

Theorem 4.2 (Goldman [14])

(2) $(\sigma_{\mathcal{F}})^* \omega_G = \omega_{\mathrm{WP}}.$

4.5   A Lagrangian embedding

Let $\hat{M}$ be a compact 3–manifold as in Section 2.3. We will use here the same notations as in Section 2.3, let us briefly recall these. The boundary $\partial\hat{M}$ is the disjoint union of N surfaces S_k of genera at least 2. The Teichmüller space of the boundary is given by $\mathcal{T}(\partial\hat{M}) = \mathcal{T}(S_1) \times \cdots \times \mathcal{T}(S_N)$, and similarly $\mathcal{CP}(\partial\hat{M}) = \mathcal{CP}(S_1) \times \cdots \times \mathcal{CP}(S_N)$. The forgetful projection is the holomorphic map $p = p_1 \times \cdots \times p_N \colon \mathcal{CP}(\partial\hat{M}) \to \mathcal{T}(\partial\hat{M})$, and $\beta \colon \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ is the "simultaneous uniformization section".

By Goldman's construction discussed above, $\mathcal{CP}(\partial\hat{M})$ is equipped with a complex symplectic structure ω_G, which is obtained here as

$$\omega_G = \mathrm{pr}_1{}^* \omega_G{}^{(1)} + \cdots + \mathrm{pr}_N{}^* \omega_G{}^{(N)},$$

where $\omega_G{}^{(k)}$ is the complex symplectic structure on $\mathcal{CP}(S_k)$ and pr_k is the kth projection map $\mathcal{CP}(\partial\hat{M}) \to \mathcal{CP}(S_k)$.

There is a general argument, discovered in this setting by Steven Kerckhoff, which shows the following.

Theorem 4.3 $\beta \colon \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ *is a Lagrangian embedding.*

Although this is a consequence of our Theorem 6.14, we briefly explain this nice argument, based on Poincaré duality in cohomology. This could be done directly on the manifolds $\mathcal{HC}(M)$ and $\mathcal{CP}(\partial\hat{M})$, but we prefer to transport the situation to character varieties, where it is simpler.

Recall that the simultaneous uniformization section β was defined as the composition $\beta = \psi \circ \varphi^{-1}$, where $\psi \colon \mathcal{HC}(M) \to \mathcal{CP}(\partial\hat{M})$ is the map which assigns the induced projective structure on $\partial\hat{M}$ to each cocompact hyperbolic structure on the interior M of $\hat{M}$, and $\varphi = p \circ \psi \colon \mathcal{HC}(M) \to \mathcal{T}(\partial\hat{M})$ is a biholomorphism. By definition, the embedding β is Lagrangian if it is isotropic ($\beta^* \omega_G = 0$) and $\dim \mathcal{CP}(\partial\hat{M}) = 2 \dim \mathcal{T}(\partial\hat{M})$. We already know the second statement to be true (see Section 2.1). It remains to show that β is isotropic, but since ϕ is a diffeomorphism, this amounts to showing that $\psi \colon \mathcal{HC}(M) \to \mathcal{CP}(\partial\hat{M})$ is isotropic.

Let us have a look at the equivalent statement on holonomy: there is a commutative diagram

$$\begin{array}{ccc} \mathcal{HC}(M) & \xrightarrow{\psi} & \mathcal{CP}(\partial\widehat{M}) \\ \downarrow{\scriptstyle hol} & & \downarrow{\scriptstyle hol} \\ \widehat{\mathcal{X}}(M,\mathrm{PSL}_2(\mathbb{C})) & \xrightarrow{f} & \mathcal{X}(\partial\widehat{M},\mathrm{PSL}_2(\mathbb{C})), \end{array}$$

where $f\colon \mathcal{X}(M,\mathrm{PSL}_2(\mathbb{C})) \to \mathcal{X}(\partial\widehat{M},\mathrm{PSL}_2(\mathbb{C}))$ is the map between character varieties induced by the "restriction" map $\iota_*\colon \pi_1(\partial\widehat{M}) \to \pi_1(\widehat{M})$.[12] Since the property of being isotropic is local, it is enough to show the following proposition:

Proposition 4.4 *The map* $f\colon \mathcal{X}(\widehat{M},\mathrm{PSL}_2(\mathbb{C})) \to \mathcal{X}(\partial\widehat{M},\mathrm{PSL}_2(\mathbb{C}))$ *is isotropic.*

Proof Let $[\rho] \in \mathcal{X}(\widehat{M},\mathrm{PSL}_2(\mathbb{C}))$. The map

$$df\colon T_{[\rho]}\mathcal{X}(\widehat{M},\mathrm{PSL}_2(\mathbb{C}) \to T_{[\rho\circ\iota_*]}\mathcal{X}(\partial\widehat{M},\mathrm{PSL}_2(\mathbb{C}))$$

is the map α that appears in long exact sequence in cohomology of the pair $(M,\partial M)$ as follows. This exact diagram shows a piece of this sequence written in terms of group cohomology, where vertical arrows are given by Poincaré duality:

$$\begin{array}{ccccc} H^1(\pi_1(\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho}) & \xrightarrow{\alpha} & H^1(\pi_1(\partial\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho}) & \xrightarrow{\beta} & H^2(\pi_1(\widehat{M}),\pi_1(\partial\widehat{M});\mathfrak{g}_{\mathrm{Ado}\rho}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(\pi_1(\widehat{M}),\pi_1(\partial\widehat{M});\mathfrak{g}_{\mathrm{Ado}\rho})^* & \xrightarrow{\beta^*} & H^1(\pi_1(\partial\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho})^* & \xrightarrow{\alpha^*} & H^1(\pi_1(\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho})^* \end{array}$$

Note that if $u \in H^1(\pi_1(\partial\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho})$, the Poincaré dual of u is defined by the relation $\langle u^*,v\rangle = \widetilde{B}(u\cup v)\cap[\partial\widehat{M}]$ for all $v \in H^1(\pi_1(\partial\widehat{M}),\mathfrak{g}_{\mathrm{Ado}\rho})$, where $[\partial\widehat{M}]$ is the fundamental class of $\partial\widehat{M}$. This is precisely saying that $\langle u^*,v\rangle = \omega_G(u,v)$. It follows that α is isotropic: using the commutativity and exactness of the diagram, we can write

$$\omega_G(\alpha(u),\alpha(v)) = \langle\alpha(u)^*,\alpha(v)\rangle = \langle\beta^*(u^*),\alpha(v)\rangle = \langle u^*,\beta\circ\alpha(v)\rangle = 0. \qquad\square$$

Remark 4.5 Note that in the quasifuchsian situation $M = S \times \mathbb{R}$, Theorem 4.3 is trivial, or rather its formulation in terms of holonomy (cf Proposition 4.4). Indeed, the map $f\colon \mathcal{X}(\widehat{M},\mathrm{PSL}_2(\mathbb{C})) \to \mathcal{X}(\partial\widehat{M},\mathrm{PSL}_2(\mathbb{C}))$ in that case is just the diagonal embedding of $\mathcal{X}(\pi,\mathrm{PSL}_2(\mathbb{C}))$ (where $\pi = \pi_1(\widehat{M}) = \pi_1(S)$) into $\mathcal{X}(\pi,\mathrm{PSL}_2(\mathbb{C})) \times$

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[12] Note that if $\partial\widehat{M}$ is disconnected, we define its fundamental group $\pi_1(\partial\widehat{M})$ as the free product of the fundamental groups of its components, so that a representation $\rho\colon \pi_1(\partial\widehat{M}) \to \mathrm{PSL}_2(\mathbb{C})$ is just a N–tuple of representations $\rho_k\colon \pi_1(S_k) \to \mathrm{PSL}_2(\mathbb{C})$.

$\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$. (Here $\mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C})) \times \mathcal{X}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ is equipped with the complex symplectic structure $\mathrm{pr}_1{}^* \omega_G - \mathrm{pr}_2{}^* \omega_G$ (the minus sign is due to the opposite orientation of $\partial^+ \widehat{M}$ and $\partial^- \widehat{M}$). The fact that the diagonal is Lagrangian is a particular case of the following general fact: if (X, ω) is a symplectic manifold and $X \times X$ is equipped with the symplectic structure $\mathrm{pr}_1{}^* \omega - \mathrm{pr}_2{}^* \omega$, then the graph of a function $h\colon X \to X$ is a Lagrangian submanifold of $X \times X$ if and only if h is a symplectomorphim.)

5 Complex Fenchel–Nielsen coordinates and Platis' symplectic structure

5.1 Fenchel–Nielsen coordinates and Wolpert theory

Pants decomposition and Fenchel–Nielsen coordinates In this paragraph, we consider the Fuchsian (or Fricke) space $\mathcal{F}(S)$ of marked hyperbolic structures on S (or marked Fuchsian projective structures) rather than Teichmüller space of $\mathcal{T}(S)$ (see Section 4.4). Let us first briefly recall the construction of the classical Fenchel–Nielsen coordinates on $\mathcal{F}(S)$, as it will be useful for the subsequent paragraphs. These depend on the choice of a *pants decomposition* of S, ie an ordered maximal collection of distinct, disjoint (in the sense that for $j \neq k$, there exists disjoint representatives of α_j and α_k) nontrivial free homotopy classes of simple (meaning that there exist simple representatives) closed curves $\alpha = (\alpha_1, \ldots, \alpha_N)$.

The following are classical facts:

- $N = 3g - 3$.

- If $c_1, \ldots, c_N$ are disjoint representatives of $\alpha_1, \ldots, \alpha_N$ (respectively), then $S \setminus \bigcup_{i=1}^{N} c_i$ is a disjoint union of $M = 2g - 2$ topological pair of pants P_k (thrice-punctured spheres).

- If X is a hyperbolic structure on S, every nontrivial free homotopy class of simple closed curves γ is uniquely represented by a simple closed geodesic γ^X.

Given a hyperbolic structure X on S, denote by $l_\gamma(X)$ the hyperbolic length of γ^X. This defines a *length function* $l_\gamma\colon \mathcal{F}(S) \to \mathbb{R}_{>0}$. In particular, given a pants decomposition α, one gets a function $l_\alpha\colon \mathcal{F}(S) \to (\mathbb{R}_{>0})^N$. The components l_{α_i} of l_α are called the *Fenchel–Nielsen length parameters*.

Any hyperbolic structure X on S induces a hyperbolic structure (with geodesic boundary) on each one of the closed pair of pants $\overline{P_k}$ in the decomposition

$$S \setminus \bigcup_{i=1}^{N} \alpha_i^X = \bigsqcup_{k=1}^{M} P_k.$$

It is well-known that a hyperbolic structure on a closed pair of pants is uniquely determined by the lengths of its three boundary components. This follows from the observation that a hyperbolic pair of pants is obtained by gluing two isometric oppositely oriented right-angled hexagons in $\mathbb{H}^2$ and the following elementary theorem in plane hyperbolic geometry.

Proposition 5.1 *Up to isometry, there exists a unique right-angled hexagon in $\mathbb{H}^2$ with prescribed lengths on every other side.*

As a consequence, a hyperbolic structure on S is completely determined by the lengths of the curves α_i, and the parameters τ_i that prescribe how the gluing occurs along these curves, ie by which amount of "twisting". However, these parameters τ_i are not very well defined: there is no obvious choice of the hyperbolic structure obtained by "not twisting at all before gluing". Also, note that assuming that such a choice is made, each of these parameters should live in $\mathbb{R}$ indeed and not $\mathbb{R}/2\pi\mathbb{Z}$: although there is a natural isometry $f: X \to Y$, where Y is obtained by 2π–twisting X along some curve α_i, f is not homotopic to the identity.

Let us make this more precise. For any nontrivial free homotopy class of simple closed curves γ, there is a flow (an $\mathbb{R}$–action) called *twisting along γ*

$$\mathrm{tw}_\gamma \colon \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S).$$

The flow is freely transitive in the fibers of l_γ. Let us mention that twist deformations along simple closed curves are naturally generalized first to weighted multicurves, then to the completion $\mathcal{ML}(S)$ of measured laminations. This generalization is the notion of *earthquake* introduced by Thurston (see eg [20]).

Denote by tw_α the $\mathbb{R}^N$–action $\mathrm{tw}_\alpha = (\mathrm{tw}_{\alpha_1}, \ldots, \mathrm{tw}_{\alpha_N}) \colon \mathbb{R}^N \times \mathcal{F}(S) \to \mathcal{F}(S)$. The fact that a hyperbolic structure on S is uniquely determined by the lengths parameters l_{α_i} and the amount of twisting along each α_i is precisely stated as: The $\mathbb{R}^N$–action tw_α is freely transitive in the fibers of l_α, and the reunion of these fibers is the whole Fricke space $\mathcal{F}(S)$. In particular, we get the following.

Theorem 5.2 *Choosing a smooth section of l_α determines a diffeomorphism*

$$(l_\alpha, \tau_\alpha) \colon \mathcal{F}(S) \to (\mathbb{R}_{>0})^N \times \mathbb{R}^N.$$

The function τ above is naturally defined by $\mathrm{tw}_\alpha(\tau_\alpha, \sigma \circ l_\alpha) = \mathrm{id}_{\mathcal{F}(S)}$, where σ is the chosen section. The components $\tau_{\alpha_1}, \ldots, \tau_{\alpha_N}$ of τ are called the *Fenchel–Nielsen twist parameters*. The theorem above thus says that Fenchel–Nielsen length and twist parameters are global coordinates on $\mathcal{F}(S)$. In particular, one recovers $\dim_{\mathbb{R}} \mathcal{T}(S) = \dim_{\mathbb{R}} \mathcal{F}(S) = 2N = 6g - 6$. It also appears that $\mathcal{T}(S) \cong \mathcal{F}(S)$ is topologically a cell, and it follows that $\mathcal{CP}(S)$ is also a cell.

Note that although the coordinates τ_{α_i} depend on the choice on a section of l_α, the 1–forms $d\tau_{\alpha_i}$ and the vector fields $\partial/\partial\tau_{\alpha_i}$ do not. In fact, $\partial/\partial\tau_\gamma$ is well-defined for any nontrivial free homotopy class of simple closed curve γ, and its flow is of course the twist flow $\mathrm{tw}_\gamma \colon \mathbb{R} \times \mathcal{F}(S) \to \mathcal{F}(S)$.

Wolpert theory We recall a few notions of symplectic geometry and the language of Hamiltonian mechanics. If (M^{2N}, ω) is a symplectic manifold, ω determines an bundle map $\omega^\flat \colon TM \to T^*M$ defined by $\omega^\flat(u) = \omega(u, \cdot)$. Since ω is nondegenerate, $\omega^\flat$ is an isomorphism, its inverse is denoted by $\omega^\sharp \colon T^*M \to TM$. If α is a one-form on M, $\omega^\sharp(\alpha)$ is thus the unique vector field X such that $i_X\omega = \alpha$. If f is a function on M, the vector field $X_f := \omega^\sharp(df)$ is called the *Hamiltonian* (or symplectic gradient) of f. Note that a vector field X is Hamiltonian is and only if the 1–form $i_X\omega$ is exact, it follows that X satisfies $\mathcal{L}_X\omega = 0$ by Cartan's magic formula. Vector fields X such that $\mathcal{L}_X\omega = 0$ are the vector fields whose flows preserve ω, they are called symplectic vector fields. The *Poisson bracket* of two functions f and g is defined by $\{f, g\} = \omega(X_f, X_g)$. f and g are said to Poisson-commute (or to be in involution) if $\{f, g\} = 0$. It is easy to see that f and g Poisson-commute if and only if f is constant along the integral curves of X_g (and vice-versa). If $f = (f_1, \ldots, f_N) \colon M \to \mathbb{R}^N$ is a regular map such that the f_i Poisson-commute, then f is a Lagrangian fibration. Moreover, the flows of the $-X_{f_i}$ (if they are complete) define a transitive $\mathbb{R}^N$–action that is transverse to the fibers of f (the reason for the choice of this minus sign will be apparent shortly). Notice already the analogy with the lengths functions and twist flows above. Such functions f_i are said to define a *(completely) integrable Hamiltonian system* on (M, ω). As in Theorem 5.2, choosing a section of f yields coordinates $g = (g_1, \ldots, g_N) \colon M \to \mathbb{R}^N$ (to be accurate, g takes values in $\mathbb{R}^{N-k} \times \mathbb{T}^k$ in general, where k is some integer and $\mathbb{T}^k$ is the k–dimensional torus) such that the $\mathbb{R}^N$–action is given by the flows of the $\partial/\partial g_i$, in other words $\partial/\partial g_i = -X_{f_i}$. In general though, (f_i, g_i) is not a system of Darboux coordinates for ω, but the classical Arnold–Liouville theorem states that such a choice of coordinates is possible in a way that is compatible with the Lagrangian fibration and the $\mathbb{R}^N$–action (see [8] for a precise statement and proof of this theorem). The Darboux coordinates obtained by Arnold–Liouville's theorem are called *action-angle coordinates*.

In [39; 40; 41], Wolpert developed a very nice theory describing the symplectic geometry of $\mathcal{F}(S)$ in relation to Fenchel–Nielsen coordinates. Let us present some of his results. In the following, $\mathcal{F}(S)$ is equipped with its standard symplectic structure ω_G ($= \omega_{\mathrm{WP}}$ under the identification $\mathcal{T}(S) \cong \mathcal{F}(S)$; see Section 4.4).

Theorem 5.3 (Wolpert) *Let γ be any nontrivial free homotopy class of simple closed curves on S. The flow of the Hamiltonian vector field $-X_{l_\gamma}$ is precisely the twist flow* tw_γ.

In other words, $\frac{\partial}{\partial \tau_\gamma} = -X_{l_\gamma}$.

Theorem 5.4 (Wolpert) *Let γ and γ' be distinct nontrivial free homotopy classes of simple closed curves on S. Then at any point $X \in \mathcal{F}(S)$,*

$$\omega_G\left(\frac{\partial}{\partial \tau_\gamma}, \frac{\partial}{\partial \tau_{\gamma'}}\right) = \sum_{p \in (\gamma^X \cap \gamma'^X)} \cos \theta_p,$$

where θ_p is the angle between the geodesics γ^X and γ'^X at p.

A direct consequence of these two theorems is the following.

Theorem 5.5 *If α is a pants decomposition of S, then Fenchel–Nielsen length functions l_{α_i} define an integrable Hamiltonian system. The Hamiltonian $\mathbb{R}^N$–action associated to this system is the twist flow* tw_γ.

Wolpert also shows the next result.

Proposition 5.6 (Wolpert) *If α is a pants decomposition of S, then for any $i, j \in \{1, \ldots, N\}$,*

$$\omega_G\left(\frac{\partial}{\partial l_{\alpha_i}}, \frac{\partial}{\partial l_{\alpha_j}}\right) = 0.$$

It follows that we are in the best possible situation:

Theorem 5.7 (Wolpert) *Let α be a pants decomposition of S. Fenchel–Nielsen length and twist parameters associated to α are respectively action and angle variables for the integrable Hamiltonian system defined by the functions l_{α_i}. In particular, Fenchel–Nielsen coordinates are Darboux coordinates for the symplectic structure:*

$$\omega_G = \sum_{i=1}^{N} dl_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

It is remarkable in particular that this does not depend on the choice of the pants decomposition α.

5.2 Complex Fenchel–Nielsen coordinates

Kourouniotis [24] (see also [22; 23]) and Tan [36] introduced a system of global holomorphic coordinates $(l^{\mathbb{C}}, \tau^{\mathbb{C}}) \colon \mathcal{QF}(S) \to \mathbb{C}^N \times \mathbb{C}^N$ that can be thought of as a complexification of Fenchel–Nielsen coordinates on the Fuchsian slice $\mathcal{F}(S)$. We outline this construction and refer to [24; 36; 33] for details.

Complex distance and displacement in hyperbolic space Let α and β be two geodesics in the hyperbolic space $\mathbb{H}^3$. The *complex distance* between α and β is the complex number $\sigma = \sigma(\alpha, \beta)$ (defined modulo $2i\pi\mathbb{Z}$) such that $\mathrm{Re}(\sigma)$ is the hyperbolic distance between α and β and $\mathrm{Im}(\sigma)$ is the angle between them (meaning the angle between the two planes containing their common perpendicular and either α or β). In the upper half-space model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}^*_+$, after applying an isometry so that α has endpoints $(u, -u)$ and β has endpoints $(p, -p)$ (where $u, p \in \mathbb{CP}^1$), σ is determined by $e^{\sigma} u = p$. Note that one has to be careful about orientations and sign to define σ unambiguously; see [24] and [33] for details.

Let f be a non-parabolic isometry of $\mathbb{H}^3$ different from the identity, and β a geodesic perpendicular to the axis of f. The *complex displacement* of f is the complex distance φ between β and $f(\beta)$. If f is represented by a matrix $A \in \mathrm{SL}_2(\mathbb{C})$, the complex displacement of f is given by

$$2\cosh\left(\frac{\varphi}{2}\right) = \mathrm{tr}(A).$$

The complex displacement and oriented axis of a non-parabolic isometry determine it uniquely.

Right-angled hexagons and pair of pants in hyperbolic space An (oriented skew) *right-angled* hexagon in $\mathbb{H}^3$ is a cyclically ordered set of six oriented geodesics α_k indexed by $k \in \mathbb{Z}/6\mathbb{Z}$, such that α_k intersects α_{k+1} orthogonally. Define the complex length of the "side" α_k by $\sigma_k = \sigma(\alpha_{k-1}, \alpha_{k+1})$.

Proposition 5.8 [12] *(Sine rule)*

$$\frac{\sinh\sigma_1}{\sinh\sigma_4} = \frac{\sinh\sigma_3}{\sinh\sigma_6} = \frac{\sinh\sigma_5}{\sinh\sigma_2}.$$

(Cosine rule)

$$\cosh\sigma_n = \frac{\cosh\sigma_{n+3} - \cosh\sigma_{n+1}\cosh\sigma_{n-1}}{\sinh\sigma_{n+1}\sinh\sigma_{n-1}}.$$

Using these formulas, one shows that assigning complex lengths on every other side determines a unique right-angled hexagon in $\mathbb{H}^3$ up to (possibly orientation-reversing) isometry. In [24] and [36], it is showed that the construction of a hyperbolic pair of pants by gluing two right-angled hexagons can be extended to $\mathbb{H}^3$. Such a pair of pants is thus uniquely determined by the complex lengths of its boundary components. In terms of holonomy [24] we have:

Proposition 5.9 *Let P be a topological pair of pants and σ_1, σ_2, $\sigma_3 \in \mathbb{C}_+$ (ie with $\mathrm{Re}(\sigma_i) > 0$). There is a unique representation up to conjugation*

$$\rho \colon \pi_1(P) = \langle c_1, c_2, c_3 \mid c_1 c_2 c_3 = 1 \rangle \to \mathrm{PSL}_2(\mathbb{C})$$

such that $\mathrm{tr}(\rho(c_i)) = -2 \cosh \sigma_i$.

Complex lengths and complex twisting in the quasifuchsian space Let $Z \in \mathcal{QF}(S)$ be a quasifuchsian structure on S and $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ its holonomy representation. For any nontrivial free homotopy class of simple closed curves γ, define the *complex length* of γ as the complex displacement of the hyperbolic isometry $\rho(\gamma)$. This defines a holomorphic function

$$l_\gamma^{\mathbb{C}} \colon \mathcal{QF}(S) \to \mathbb{C}_+.$$

In the quasifuchsian 3–manifold M, $\rho(\gamma)$ corresponds to a geodesic of complex length $l_\gamma^{\mathbb{C}}$, ie of hyperbolic length $\mathrm{Re}(l_\gamma^{\mathbb{C}})$ and torsion $\mathrm{Im}(l_\gamma^{\mathbb{C}})$. It is easy to see that if Z is a Fuchsian structure, then $l_\gamma^{\mathbb{C}}(Z) = l_\gamma(Z)$. If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a pants decomposition of S, we call

$$l_\alpha^{\mathbb{C}} = (l_{\alpha_1}^{\mathbb{C}}, \ldots, l_{\alpha_N}^{\mathbb{C}}) \colon \mathcal{QF}(S) \to (\mathbb{C}_+)^N$$

the *complex Fenchel–Nielsen length parameters*.

As a consequence of the previous discussion, if the complex lengths $l_{\alpha_1}^{\mathbb{C}}, \ldots, l_{\alpha_N}^{\mathbb{C}}$ are fixed, a quasifuchsian structure on S is determined by how the gluings of pair of pants occur along their common boundaries. Analogously to the Fuchsian case, this is prescribed by a complex parameter $\tau_{\alpha_i}^{\mathbb{C}}$ that we will call a complex twist parameter , that describes both the amount of twisting (by $\mathrm{Re}(\tau_{\alpha_i}^{\mathbb{C}})$) and the amount of bending (by $\mathrm{Im}(\tau_{\alpha_i}^{\mathbb{C}})$) before gluing. The parameter $\tau_{\alpha_i}^{\mathbb{C}}$ can be more or less well defined as the complex distance between two adequate geodesics in $\mathbb{H}^3$, but the definition is clearer in terms of the effect of complex-twisting by $\tau_{\alpha_i}^{\mathbb{C}}$ on the holonomy of the glued pairs of pants (see [24] and [15]).

As in the Fuchsian case, it is the complex twist flow $\mathrm{tw}_\gamma^{\mathbb{C}}$ along a simple closed curve γ that is well defined rather than the twist parameter $\tau_{\alpha_i}^{\mathbb{C}}$ itself, although the complex twist vector field $\partial/\partial\tau_\gamma^{\mathbb{C}}$ is well-defined.

Let us mention this flow is called *bending* by Kourouniotis and corresponds to (or is a generalization of) what other authors have called *quakebends* or *complex earthquakes* discussed by Epstein–Marden [11], Goldman [15], McMullen [30] and Series [33] among others. It is not hard to see that starting from a Fuchsian structure Z, complex twisting by $t = t_1 + i t_2 \in \mathbb{C}$ is described as the composition of twisting by t_1 on $\mathcal{F}(S)$ and then *projective grafting* by t_2 (see eg [9] for a presentation of projective grafting).

Choosing a holomorphic section of $l_\alpha^\mathbb{C}$ determines *complex twist coordinates*

$$\tau_\alpha^\mathbb{C} = (\tau_{\alpha_1}^\mathbb{C}, \dots, \tau_{\alpha_N}^\mathbb{C}) \colon \mathcal{QF}(S) \to \mathbb{C}^N.$$

We will call $(l_\alpha^\mathbb{C}, \tau_\alpha^\mathbb{C})$ complex Fenchel–Nielsen coordinates. The conclusion of our discussion is this theorem:

Theorem 5.10 (Kourouniotis, Tan) *Complex Fenchel–Nielsen coordinates* $(l_\alpha^\mathbb{C}, \tau_\alpha^\mathbb{C})$ *are global holomorphic coordinates on* $\mathcal{QF}(S)$. *They restrict to the classical Fenchel–Nielsen coordinates* (l_α, τ_α) *on the Fuchsian slice* $\mathcal{F}(S)$.

5.3 Platis' symplectic structure

Platis [32] developed a complex version of Wolpert's theory on the quasifuchsian space. We present some of his results. First there is a complex version of Wolpert's Theorem 5.4:

Theorem 5.11 *There exists a complex symplectic structure* ω_P *on* $\mathcal{QF}(S)$ *such that if* γ *and* γ' *are distinct nontrivial free homotopy classes of simple closed curves on* S, *then at any point* $Z \in \mathcal{QF}(S)$ *with holonomy* ρ

$$\omega_P\left(\frac{\partial}{\partial \tau_\gamma^\mathbb{C}}, \frac{\partial}{\partial \tau_{\gamma'}^\mathbb{C}}\right) = \sum_{p \in (\gamma \cap \gamma')} \cosh \sigma_p,$$

where σ_p *is the complex distance between the geodesics* $\rho(\gamma)$ *and* $\rho(\gamma')$.

He also shows the complex version of Theorem 5.3 on the manifold $(\mathcal{QF}(S), \omega_P)$:

Theorem 5.12 *Let* γ *be any nontrivial free homotopy class of simple closed curves on* S. *The complex flow of the Hamiltonian vector field* $-X_{l_\gamma^\mathbb{C}}$ *is precisely the complex twist flow* $\mathrm{tw}_\gamma^\mathbb{C}$.

As in the Fuchsian case, it follows from these two theorems that complex Fenchel–Nielsen length functions associated to a pants decomposition define a complex Hamiltonian integrable system. Furthermore, he proves that the striking Theorem 5.7 is still true in its complex version on $(\mathcal{QF}(S), \omega_P)$:

Theorem 5.13 *If α is any pants decomposition of S, complex Fenchel–Nielsen coordinates are Darboux coordinates for the complex symplectic structure ω_P:*

$$\omega_P = \sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}.$$

6 Comparing symplectic structures

6.1 Analytic continuation

We are going to show the following proposition, which implies that two complex symplectic structures agree on $\mathcal{CP}(S)$ if and only if they agree in restriction to tangent vectors to the Fuchsian slice $\mathcal{F}(S)$:

Proposition 6.1 *Let ω be a closed $(2, 0)$–form on $\mathcal{CP}(S)$ and $\sigma_{\mathcal{F}}\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section (as in (1)). If $\sigma_{\mathcal{F}}^*\omega$ vanishes on $\mathcal{T}(S)$, then ω vanishes on $\mathcal{CP}(S)$.*

The proof of this proposition is based on analytic continuation. In order to use this argument, we recall a few definitions and show some elementary facts regarding totally real submanifolds of complex manifolds.

Definition 6.2 Let M be a complex manifold and $N \subset M$ be a real submanifold. N is called *totally real* if for all $x \in N$,

$$T_x N \cap J T_x N = \{0\},$$

where J is the almost complex structure on M.

If moreover, N has maximal dimension $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} M$, we say that N is a *maximal totally real submanifold* of M. There are several characterizations of maximal totally real analytic submanifolds, seemingly stronger than the definition, as in the following:

Proposition 6.3 *Let M be a complex manifold of dimension n and $N \subset M$ be a real submanifold. The following are equivalent:*

(1) *N is a maximal totally real analytic submanifold of M.*

(2) *$N \subset M$ locally looks like $\mathbb{R}^n \subset \mathbb{C}^n$. More precisely: for any $x \in N$, there is a holomorphic chart $z\colon U \to V$, where U is an open set in M containing x and V is an open set in $\mathbb{C}^n$, such that $z(U \cap N) = V \cap \mathbb{R}^n$.*

(3) *There is an antiholomorphic involution $\chi\colon M' \to M'$, where M' is a neighborhood of N in M, such that N is the set of fixed points of χ.*

If N satisfies one (equivalently all) of these conditions, M is said to be a *complexification* of N. Let us mention that any real-analytic manifold can be complexified.

Proof It is fairly easy to see that both (2) and (3) imply (1), and that in fact (2) and (3) are equivalent. Let us show that (1) implies (2). Using holomorphic charts, it is clearly enough to prove this in the case where N is a maximal totally real analytic submanifold of $\mathbb{C}^n$. Let $m \in N \subset \mathbb{C}^n$, there is a real-analytic parametrization $\varphi: D \to N$, where D is a small open disk centered at the origin in $\mathbb{R}^n$, such that $\varphi(0) = m$ and $d\varphi(0) \neq 0$. The map φ is given by a convergent power series $\varphi(x) = \sum_{|\alpha|=n} a_\alpha x^\alpha$ for all $x \in D$, where the sum is taken over all multi-indices α of length n, and the a_α are coefficients in $\mathbb{C}^n$. In order to extend φ to a holomorphic map $\Phi: D' \to M$ where D' is the disk in $\mathbb{C}^n$ such that $D = D' \cap \mathbb{R}^n$, we can just replace $x \in D$ by $z \in D'$ in the expression of φ: define $\Phi(z) = \sum_{|\alpha|=n} a_\alpha z^\alpha$. This power series converges in D' because it has the same radius of convergence as its real counterpart. Moreover, if D' is small enough, Φ is a biholomorphism onto its image because $d\Phi(0) = d\varphi(0) \neq 0$. This shows that (1) implies (2) (just take the chart given by Φ^{-1}). (Note that the simplicity of this proof relies on a little trick: the actual complexification of φ is a map $\varphi: D' \to \mathbb{C}^{2n}$ (and not $\mathbb{C}^n$), where a_α is seen as a real vector in $\mathbb{C}^n$.) $\qquad\square$

Keeping in mind that we want to consider the Fuchsian slice in $\mathcal{CP}(S)$, we make this last general observation on totally real submanifolds:

Proposition 6.4 *Let V be a complex manifold. The diagonal Δ in $V \times \overline{V}$ is a maximal totally real analytic submanifold.*

Proof This is a direct consequence of characterization (3) in the previous proposition: just take the antiholomorphic involution $\chi: V \times \overline{V} \to V \times \overline{V}$, $(x, y) \mapsto (y, x)$. $\qquad\square$

An immediate application of this is that the Fuchsian slice $\mathcal{F}(S)$ is a maximal totally real analytic submanifold of $\mathcal{CP}(S)$ (as was already pointed out in Section 4.4): it is the image of the diagonal of $\mathcal{T}(S) \times \mathcal{T}(\overline{S})$ (recall that $\mathcal{T}(\overline{S})$ is canonically identified with $\overline{\mathcal{T}(S)}$) by the holomorphic embedding β^+ (see Section 2.2). Another way to see this is that the quasifuchsian space $\mathcal{QF}(S) = \mathrm{Im}(\beta^+) \subset \mathcal{CP}(S)$ is equipped with a canonical antiholomorphic involution, which justs consists in "turning a quasifuchsian 3–manifold upside down", and $\mathcal{F}(S)$ is the set of fixed points of this involution.

Now, we prove this first elementary analytic continuation theorem:

Proposition 6.5 *Let M be a connected complex manifold and $N \subset M$ be a maximal totally real submanifold. If $f: M \to \mathbb{C}$ is a holomorphic function that vanishes on N, then f vanishes on M.*

Proof By the identity theorem for holomorphic functions, it is enough to show that f vanishes on a small open neighborhood U of some point $x \in N$. If N is analytic, this is an straightforward consequence of Proposition 6.3(2). Let us produce a proof that does not assume analyticity of N. Since the restriction $f_{|N}$ vanishes identically, we have $(df)_{|TN} = 0$. Using the fact that $T_x M = T_x N \oplus J T_x N$ for all $x \in N$ and the holomorphicity of f, it is easy to derive that df vanishes at all points of N. In particular, if $z = (z_k)_{1 \leqslant k \leqslant n} \colon U \to \mathbb{C}^n$ is a holomorphic chart, the partial derivatives $\partial f / \partial z_k$ vanish on N. But those are again holomorphic functions, so we can use the same argument: their partial derivatives must vanish on N. By an obvious induction, we see that all partial derivatives of f (at any order) vanish at points of N. Since f is holomorphic, this implies that $f = 0$. $\qquad\square$

Proposition 6.6 *Let M be a connected complex manifold and $\sigma \colon N \to M$ be a maximal totally real embedding. If ω is a closed $(2,0)$–form on M such that $\sigma^* \omega = 0$, then $\omega = 0$.*

Proof We can assume that $N \subset M$, so the hypothesis is that $\omega_{|TN} = 0$. Since $T_x M = T_x N \oplus J T_x N$ for any $x \in N$ and ω is of type $(2,0)$, it is easy to see that ω vanishes at points of N. Now, recall that a closed $(2,0)$–form is holomorphic. Let $z = (z_k)_{1 \leqslant k \leqslant n} \colon U \to \mathbb{C}^n$ be a holomorphic chart in a neighborhood of a point $x \in N$, ω has an expression of the form $\omega = \sum_{j,k} f_{jk} dz_j \wedge dz_k$ where f_{jk} are holomorphic functions on U. Since ω vanishes at points of N, the functions f_{jk} vanish on $U \cap N$, and we derive from the previous proposition that they actually vanish on U. We thus have $\omega_{|U} = 0$, and it follows once again from the identity theorem (taken in charts) that ω vanishes on M. $\qquad\square$

An immediate consequence of this, together with (2), is that a complex symplectic structure on $\mathcal{CP}(S)$ agrees with the standard complex symplectic structure if and only if it induces the Weil–Petersson Kähler form on the Fuchsian slice:

Theorem 6.7 *Let ω be a complex symplectic structure on $\mathcal{CP}(S)$. Then $\omega = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^* \omega = \omega_{\mathrm{WP}}$.*

Proof By the previous proposition, $\omega = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^*(\omega - \omega_G) = 0$. But by (2), $\sigma_{\mathcal{F}}^* \omega_G = \omega_{\mathrm{WP}}$, therefore $(\sigma_{\mathcal{F}})^*(\omega - \omega_G) = (\sigma_{\mathcal{F}})^* \omega - \omega_{\mathrm{WP}}$. $\qquad\square$

6.2 The affine cotangent symplectic structures

Recall (see Section 3.3) that any section $\sigma \colon \mathcal{T}(S) \to \mathcal{CP}(S)$ determines an affine identification $\tau^\sigma \colon \mathcal{CP}(S) \xrightarrow{\sim} T^* \mathcal{T}(S)$ and thus a complex-valued non-degenerate 2–form $\omega^\sigma = (\tau^\sigma)^* \omega$ on $\mathcal{CP}(S)$. ω^σ is a complex symplectic structure on $\mathcal{CP}(S)$ if and

only if σ is a holomorphic section of p. We will now answer this question: For which holomorphic sections σ does ω^σ agree with the standard symplectic structure ω_G?

As a consequence of Theorem 6.7, together with Proposition 3.3, we show the following:

Theorem 6.8 *Let $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a section of p. Then ω^σ agrees with the standard complex symplectic structure ω_G on $\mathcal{CP}(S)$ if and only if $\sigma_{\mathcal{F}}-\sigma$ is a primitive for the Weil–Petersson metric on $\mathcal{T}(S)$:*

$$\omega^\sigma = \omega_G \;\Leftrightarrow\; d(\sigma_{\mathcal{F}}-\sigma) = \omega_{\mathrm{WP}}.$$

More generally, if c is some complex constant,

$$\omega^\sigma = c\omega_G \;\Leftrightarrow\; d(\sigma_{\mathcal{F}}-\sigma) = c\omega_{\mathrm{WP}}.$$

Proof By Theorem 6.7, $\omega^\sigma = \omega_G$ if and only if $(\sigma_{\mathcal{F}})^*\omega^\sigma = \omega_{\mathrm{WP}}$. However, it follows from Proposition 3.3 that

$$\begin{aligned}
(\sigma_{\mathcal{F}})^*\omega^\sigma &= (\sigma_{\mathcal{F}})^*[\omega_{\mathcal{F}}^\sigma - p^*d(\sigma - \sigma_{\mathcal{F}})]\\
&= (\sigma_{\mathcal{F}})^*((\tau^{\sigma_{\mathcal{F}}})^*\omega_{\mathrm{can}}) - (\sigma_{\mathcal{F}})^*(p^*d(\sigma - \sigma_{\mathcal{F}}))\\
&= (\tau^{\sigma_{\mathcal{F}}} \circ \sigma_{\mathcal{F}})^*\omega_{\mathrm{can}} - (p \circ \sigma)^*d(\sigma - \sigma_{\mathcal{F}})\\
&= s_0{}^*\omega_{\mathrm{can}} - \mathrm{id}^*d(\sigma - \sigma_{\mathcal{F}}) = -d(\sigma - \sigma_{\mathcal{F}}).
\end{aligned}$$

Let us make a couple of comments on this calculation: recall that ω_{can} denotes the canonical symplectic structure on $T^*\mathcal{T}(S)$; $\tau^\sigma \circ \sigma = s_0$ is the characterization of τ^σ; and $s_0{}^*\omega = 0$ because the zero section s_0 is a Lagrangian in $T^*\mathcal{T}(S)$ (see Section 3.2). Of course, the proof of the apparently more general second statement is just the same. □

Theorem 6.9 (McMullen [31]) *If σ is any Bers section, then*

$$d(\sigma_{\mathcal{F}}-\sigma) = -i\,\omega_{\mathrm{WP}}.$$

Using this result, we eventually obtain the next theorem as a corollary of Theorem 6.8.

Theorem 6.10 *If $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ is any Bers section, then*

$$\omega^\sigma = -i\,\omega_G.$$

In particular, we can deduce from this identification the following properties.

Corollary 6.11 *Consider the space $\mathcal{CP}(S)$ equipped with its standard symplectic structure ω_G.*

(1) *The canonical projection $p\colon \mathcal{CP}(S) \to \mathcal{T}(S)$ is a Lagrangian fibration.*

(2) *Bers slices are leaves of a Lagrangian foliation of the quasifuchsian space $\mathcal{QF}(S)$.*

We also derive an explicit expression of $\omega_G(u, v)$ when u is a vertical tangent vector (by Proposition 3.4):

Corollary 6.12 *Let u, v be tangent vectors at $Z \in \mathcal{CP}(S)$ such that u is vertical, ie $p_* u = 0$. Then*

$$\omega_G(u, v) = i \langle u, p_* v \rangle.$$

Looking back at Proposition 3.3, we also get the expression of the 2–form $\omega^{\sigma_\mathcal{F}}$ obtained under the Fuchsian identification:

Corollary 6.13 *Let $\sigma_\mathcal{F}: \mathcal{T}(S) \to \mathcal{CP}(S)$ be the Fuchsian section. Then*

$$\omega^{\sigma_\mathcal{F}} = -i(\omega_G - p^* \omega_{\mathrm{WP}}).$$

It should not come as a surprise that we see from this equality that $\omega^{\sigma_\mathcal{F}}$ vanishes on the Fuchsian slice. Notice the equality between real symplectic structures,

$$\mathrm{Re}(\omega^{\sigma_\mathcal{F}}) = \mathrm{Im}(\omega_G),$$

and that $\mathrm{Re}(\omega^{\sigma_\mathcal{F}})$ is (half) the real canonical symplectic structure on $T^* \mathcal{T}(S)$ pulled back by the Fuchsian identification.

Finally, we note that McMullen's *quasifuchsian reciprocity* theorem showed in [31] can be seen as a consequence of Theorem 6.10. We will give a precise statement and proof of a generalized version of this theorem in the setting of convex cocompact 3–manifolds (Theorem 6.18).

Generalizations in the setting of convex cocompact hyperbolic 3–manifolds Let $\hat{M}$ be a compact 3–manifold as in Section 2.3. We will use here the same notations as in Section 2.3. Recall that we have defined there a canonical holomorphic section $\beta: \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$.

McMullen and Takhtajan–Teo gave generalized versions of quasifuchsian reciprocity, which they called *Kleinian reciprocity*, notably in [31, Appendix] and [35]. In particular, [35, Theorem 6.3] says the following:

Theorem *Let $\sigma_\mathcal{F}: \mathcal{T}(\partial\hat{M}) \to \mathcal{CP}(\partial\hat{M})$ denote the Fuchsian section. Then*

$$d(\sigma_\mathcal{F} - \beta) = -i\omega_G.$$

Since our Theorem 6.8 above does not assume that S is connected, we obtain this:

Theorem 6.14 *Let ω_G be the standard complex symplectic structure on $\mathcal{CP}(\partial\hat{M})$ and $\omega^\beta = (\tau^\beta)^*\omega$ be the complex symplectic structure obtained by the identification $\tau^\beta\colon \mathcal{CP}(\partial\hat{M}) \overset{\sim}{\longrightarrow} T^*\mathcal{T}(\partial\hat{M})$ as in Section 3.3. Then*

$$\omega^\beta = -i\,\omega_G. \tag{3}$$

A first corollary is that we recover Theorem 4.3: β is a Lagrangian embedding.

Another consequence of this theorem and of the fact that the projections $p_k\colon \mathcal{CP}(S_k) \to \mathcal{T}(S_k)$ are Lagrangian (Corollary 6.11) is a generalization of Theorem 6.10:

Theorem 6.15 *Let $\sigma\colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section (see Section 2.3). Then*

$$\omega^\sigma = -i\,\omega_G.$$

Proof By definition, σ is map defined by $\sigma = f_{(X_i),j}$ as in Section 2.3, where $S = S_j$ and X_i is a fixed point in $\mathcal{T}(S_i)$ for $i \neq j$. Recall that

$$\omega_G = (\mathrm{pr}_1)^*\omega_G{}^{(1)} + \cdots + (\mathrm{pr}_N)^*\omega_G{}^{(N)}$$

where $\omega_G{}^{(k)}$ is the standard complex symplectic structure on $\mathcal{CP}(S_k)$, and similarly

$$\omega = (\mathrm{pr}_1)^*\omega^{(1)} + \cdots + (\mathrm{pr}_N)^*\omega^{(N)}$$

where $\omega^{(k)}$ is the canonical complex symplectic structure on $T^*\mathcal{T}(S_k)$. Beware that in this equality, pr_k stands for the kth projection map $T^*\mathcal{T}(\partial\hat{M}) \to T^*\mathcal{T}(S_k)$ (whereas it stood for the kth projection map $\mathcal{CP}(\partial\hat{M}) \to \mathcal{CP}(S_k)$ in the previous equality). We apologize for these (slightly) misleading notations.

The equality (3) can thus be rewritten as

$$\left(\mathrm{pr}_1 \circ \tau^\beta\right)^*\omega^{(1)} + \cdots + \left(\mathrm{pr}_N \circ \tau^\beta\right)^*\omega^{(N)} = -i\left[(\mathrm{pr}_1)^*\omega_G{}^{(1)} + \cdots + (\mathrm{pr}_N)^*\omega_G{}^{(N)}\right].$$

Fix $Z_i \in P(X_i)$ for $i \neq j$ and let us pull back this equality on $\mathcal{CP}(S_j)$ by the map

$$\iota_{(\widetilde{Z}_i)}\colon \mathcal{CP}(S_j) \to \mathcal{CP}(\partial\hat{M}),$$
$$Z \mapsto (Z_1, \ldots, Z_{j-1}, Z, Z_{j+1}, \ldots, X_N).$$

For $k \neq j$, the map $\mathrm{pr}_k \circ \tau^\beta \circ \widetilde{\iota}_{(Z_i)}$ takes values in the fiber $T_{X_k}{}^*\mathcal{T}(S_k)$, so that

$$(\widetilde{\iota}_{(Z_i)})^*\left((\mathrm{pr}_k \circ \tau^\beta)^*\omega^{(k)}\right) = 0.$$

Similarly, the map $\mathrm{pr}_k \circ \widetilde{\iota}_{(Z_i)}$ maps into the fiber $P(X_k) \subset \mathcal{CP}(S_k)$, so that

$$(\widetilde{\iota}_{(Z_i)})^*\left((\mathrm{pr}_k)^*\omega_G{}^{(k)}\right) = 0$$

because p_k is a Lagrangian fibration. For $k = j$, $\mathrm{pr}_k \circ \tau^\beta \circ \widetilde{\iota}_{(Z_i)}$ is the map

$$\tau^\sigma \colon \mathcal{CP}(S_j) \to T^*\mathcal{T}(S_j)$$

and $\mathrm{pr}_k \circ \widetilde{\iota}_{(Z_i)}$ is the identity in $\mathcal{CP}(S_j)$. We therefore obtain the desired equality $(\tau^\sigma)^* \omega^{(j)} = -i\omega_G^{(j)}$. $\qquad\square$

Corollary 6.16 *Generalized Bers sections $\mathcal{T}(S) \to \mathcal{CP}(S)$ are Lagrangian embeddings.*

Another corollary of Theorem 6.15 and Theorem 6.8 is a generalization of McMullen's Theorem 6.9:

Corollary 6.17 *Let $\sigma \colon \mathcal{T}(S) \to \mathcal{CP}(S)$ be a generalized Bers section. Then*

$$d(\sigma_{\mathcal{F}} - \sigma) = -i\,\omega_{\mathrm{WP}}.$$

Finally, we show a generalized version of McMullen's "quasifuchsian reciprocity". To this end, we introduce the notion of "reciprocal generalized Bers embeddings": with the notations of Section 2.3, we say that $f \colon \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g \colon \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ are *reciprocal generalized Bers embeddings* if $f = f_{(X_i)_{i \neq j}, k}$ and $g = f_{(X_i)_{i \neq k}, j}$ for some fixed $X = (X_1, \ldots, X_N) \in \mathcal{T}(\partial\widehat{M})$. Since f and g take values in affine spaces, one can consider their derivatives

$$D_{X_j} f \colon T_{X_j}\mathcal{T}(S_j) \to T^*_{X_k}\mathcal{T}(S_k) \quad \text{and} \quad D_{X_k} g \colon T_{X_k}\mathcal{T}(S_k) \to T^*_{X_j}\mathcal{T}(S_j).$$

Theorem 6.18 *Let $f \colon \mathcal{T}(S_j) \to \mathcal{CP}(S_k)$ and $g \colon \mathcal{T}(S_k) \to \mathcal{CP}(S_j)$ be reciprocal generalized Bers embeddings as above. Then $D_{X_j} f$ and $D_{X_k} g$ are dual maps. In other words, for any $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$,*

$$\langle D_{X_j} f(\mu), \nu \rangle = \langle \mu, D_{X_k} g(\nu) \rangle.$$

Proof The fact that $\beta \colon \mathcal{T}(\partial\widehat{M}) \to \mathcal{CP}(\partial\widehat{M})$ is Lagrangian is written as

$$\beta^* \omega_G = (\mathrm{pr}_1 \circ \beta)^* \omega_G^{(1)} + \cdots + (\mathrm{pr}_N \circ \beta)^* \omega_G^{(N)} = 0,$$

where $\omega_G^{(i)}$ is the standard complex symplectic structure on the component $\mathcal{CP}(S_i)$. Let $\mu \in T_{X_j}\mathcal{T}(S_j)$ and $\nu \in T_{X_k}\mathcal{T}(S_k)$, and define $U, V \in T_X\mathcal{T}(\partial\widehat{M})$ by

$$U_i = \begin{cases} 0 & \text{for } i \neq j, \\ \mu & \text{for } i = j, \end{cases} \quad \text{and} \quad V_i = \begin{cases} 0 & \text{for } i \neq k, \\ \nu & \text{for } i = k. \end{cases}$$

Note that $(\mathrm{pr}_i \circ \beta)_* U$ is vertical in $\mathcal{CP}(S_i)$ whenever $i \neq j$ (resp. $(\mathrm{pr}_i \circ \beta)_* V$ is vertical in $\mathcal{CP}(S_i)$ whenever $i \neq k$). Since the fibers of $\mathcal{CP}(S_i)$ are isotropic (see

Corollary 6.11), it follows that $((\mathrm{pr}_i \circ \beta)^* \omega_G{}^{(i)})(U, V) = 0$ whenever $i \notin \{j, k\}$. For $i = k$, $(\mathrm{pr}_i \circ \beta)_* U = D_{X_j} f(\mu)$ is still vertical and $(p_k)_* ((\mathrm{pr}_i \circ \beta)_* V) = \nu$ so we can derive from Corollary 6.12 that $((\mathrm{pr}_k \circ \beta)^* \omega_G{}^{(k)})(U, V) = i \langle D_{X_j} f(\mu), \nu \rangle$. Similarly, for $i = j$ we have

$$((\mathrm{pr}_j \circ \beta)^* \omega_G{}^{(j)})(U, V) = -((\mathrm{pr}_j \circ \beta)^* \omega_G{}^{(j)})(V, U) = -i \langle D_{X_k} g(\nu), \mu \rangle.$$

In the end, $0 = (\beta^* \omega_G)(U, V) = i \langle D_{X_j} f(\mu), \nu \rangle - i \langle D_{X_k} g(\nu), \mu \rangle$ as desired. □

Note that Corollary 6.17 is not an immediate consequence of "Kleinian reciprocity": the proof requires Lagrangian information. On the other hand, Steven Kerckhoff pointed out to me that Theorem 6.18 can be derived from Kleinian reciprocity without using our previous results, rightly so (the proof is easily adapted taking ω^β instead of ω_G, avoiding the use of the symplectic structure on $\mathcal{CP}(S)$).

6.3 Darboux coordinates

It is an immediate consequence of the analytical continuation property (Theorem 6.7) and Wolpert's Theorem 5.7 that complex Fenchel–Nielsen coordinates are Darboux coordinates for the standard symplectic structure on the quasifuchsian space:

Theorem 6.19 *Let α be any pants decomposition of S. Complex Fenchel–Nielsen coordinates $(l_\alpha^{\mathbb{C}}, \beta_\alpha^{\mathbb{C}})$ on the quasifuchsian space $\mathcal{QF}(S)$ are Darboux coordinates for the standard complex symplectic structure:*

$$\omega_G = \sum_{i=1}^N dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}.$$

Proof Of course, Theorem 6.7 is still true when replacing $\mathcal{CP}(S)$ by any connected neighborhood of the Fuchsian slice $\mathcal{F}(S)$, such as the quasifuchsian space $\mathcal{QF}(S)$. Let $\omega = \sum_{i=1}^N dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}$. Since $l_{\alpha_i}^{\mathbb{C}}$ and $\tau_{\alpha_i}^{\mathbb{C}}$ are holomorphic, ω is a complex symplectic structure on $\mathcal{QF}(S)$. In restriction to the Fuchsian slice, $\iota^* \omega = \sum_{i=1}^N dl_{\alpha_i} \wedge d\tau_{\alpha_i}$, where (l_α, τ_α) are the classical Fenchel–Nielsen coordinates. By Wolpert's Theorem 5.7, it follows that $(\sigma^{\mathcal{F}})^* \omega = \omega_{\mathrm{WP}}$. This proves that $\omega = \omega_G$ according to Theorem 6.7. □

Of course, this shows in particular:

Corollary 6.20 *Platis' symplectic structure ω_P is equal to the standard complex symplectic structure ω_G in restriction to the quasifuchsian space $\mathcal{QF}(S)$.*

Notice how the analytic continuation argument provides a very simple alternative proof of Platis' result that the symplectic structure $\sum_{i=1}^{N} dl_{\alpha_i}^{\mathbb{C}} \wedge d\tau_{\alpha_i}^{\mathbb{C}}$ does not depend on a choice of a pants decomposition (relying, of course, on Wolpert's result).

In [15], Goldman gives a fairly extensive description of the complex symplectic structure ω_G on the character variety $\mathcal{X}(S, \mathrm{SL}_2(\mathbb{C}))$, discussing in particular the "Hamiltonian picture". We recover that the Hamiltonian flow of a complex length function is the associated complex twist flow.

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