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Fuchsian groups, circularly ordered groups and dense invariant laminations on the circle

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We propose a program to study groups acting faithfully on S^1 in terms of numbers of pairwise transverse dense invariant laminations. We give some examples of groups that admit a small number of invariant laminations as an introduction to such groups. The main focus of the present paper is to characterize Fuchsian groups in this scheme. We prove a group acting on S^1 is conjugate to a Fuchsian group if and only if it admits three very full laminations with a variation on the transversality condition. Some partial results toward a similar characterization of hyperbolic 3–manifold groups that fiber over the circle have been obtained. This work was motivated by the universal circle theory for tautly foliated 3–manifolds developed by Thurston, Calegari and Dunfield.

20H10, 37C85; 37E30, 57M60

This paper is dedicated to the memory of William Thurston (1946–2012)

1 Introduction

We say a group is CO if it is circularly orderable. See Calegari [2] for general background for circular ordering of groups. It is well known that a group is CO if and only if it acts faithfully on S^1. In this paper, we only talk about circularly ordered groups. More precisely, a group G comes with an injective homomorphism from G to $\mathrm{Homeo}_+(S^1)$, where $\mathrm{Homeo}_+(S^1)$ is the group of all orientation-preserving homeomorphisms of S^1. Abusing notation, we identify G with its image under this representation and regard it as a subgroup of $\mathrm{Homeo}_+(S^1)$, ie we consider the subgroups of $\mathrm{Homeo}_+(S^1)$. There is a reason why we emphasize this: the properties we will define may depend on the circular order on a group. So, if we just talk about an abstract group which is circularly orderable without specifying the actual circular order, there is a possible ambiguity. Since we care only about topological dynamics, the groups are considered up to topological conjugacy, ie conjugacy by elements of $\mathrm{Homeo}_+(S^1)$.

Circularly orderable groups arise naturally in low-dimensional topology. Thurston showed that for a 3–manifold M admitting a taut foliation, $\pi_1(M)$ admits a faithful

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action on the circle (which is now called a universal circle) in his unfinished manuscript [14]. In [4], Calegari and Dunfield completed the construction and generalized this to 3–manifolds admitting essential laminations with solid torus guts. Universal circles from taut foliations come with a pair of transverse dense invariant laminations. This provides a motivation to study those groups acting on S^1 with some invariant laminations. We suggest a new classification of the subgroups of $\mathrm{Homeo}_+(S^1)$ in terms of the number of dense invariant laminations they admit. In this paper, we mainly focus on the case of groups acting faithfully on S^1 with two or three different very full invariant laminations. We also give motivation for this classification by demonstrating interesting examples and questions.

By a Fuchsian group, we mean a torsion-free discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (up to conjugacy by an element of $\mathrm{Homeo}_+(S^1)$). Recall that $\mathrm{PSL}_2(\mathbb{R})$ is naturally identified with the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$. For a collection $\mathcal{C}$ of G–invariant laminations, being *pants-like* means that a pair of leaves from two different laminations in $\mathcal{C}$ share a common endpoint if and only if the shared endpoint is the fixed point of a parabolic element of G. For other terminologies, see Section 2.

Main theorem *Let G be a torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$*. Then G is a Fuchsian group such that $\mathbb{H}^2/G$ is not the thrice-punctured sphere if and only if G admits a pants-like collection $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of three very full G–invariant laminations.*

As we pointed out earlier, saying a group G is Fuchsian means G is conjugate to a group $G' \subset \mathrm{PSL}_2(\mathbb{R})$ by an element of $\mathrm{Homeo}_+(S^1)$, and $\mathbb{H}^2/G$ in the statement of the theorem should be understood as $\mathbb{H}^2/G'$. The theorem provides an alternative characterization of Fuchsian groups in terms of invariant laminations. Note that we do not assume that G is finitely generated. The following is an immediate corollary of the Main theorem.

Corollary *Let G be a torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$*. Then G is a Fuchsian group such that $\mathbb{H}^2/G$ has no cusps if and only if G admits a collection $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of three very full G–invariant laminations such that no leaf of Λ_i has a common endpoint of a leaf of Λ_j for $i \neq j$.*

In Section 3, we present some explicit examples of groups acting on the circle with a specified number of dense invariant laminations. The most interesting case is when a group has exactly two dense invariant laminations. A class of examples will be constructed by considering pseudo-Anosov homeomorphisms of hyperbolic surfaces. One should note that those examples are not Fuchsian groups. In some sense, the

Main theorem shows that there are clear differences between having two invariant laminations and having three invariant laminations as long as the structure of the invariant laminations is restricted enough.

Nevertheless, groups admitting a pant-like collection of two very full laminations are already interesting. We study those groups in Section 8 and the following theorem is a summary of the results.

Theorem *Let G be a torsion-free discrete subgroup of* $\mathrm{Homeo}_+(S^1)$. *Suppose G admits a pants-like collection of two very full laminations $\{\Lambda_1, \Lambda_2\}$. Then an element of G either behaves like a parabolic or hyperbolic isometry of $\mathbb{H}^2$ or has even number of fixed points alternating between attracting and repelling. In the latter case, one lamination contains the boundary leaves of the convex hull of the attracting fixed points and the other lamination contains the boundary leaves of the convex hull of the repelling fixed points. If we further assume that G has no element with a single fixed point, then G acts faithfully on S^2 by orientation-preserving homeomorphisms and each element of G has two fixed points on S^2.*

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2 Definitions and set-up

In the present paper, a group is always assumed to be countable. A faithful (orientation-preserving) action of a group G on the circle is an injective homomorphism

$$\rho\colon G \to \mathrm{Homeo}_+(S^1).$$

Once we fix the action, we often identify G with its image under ρ. For general background on group actions on the circle, we suggest reading Ghys [9].

The ideal boundary of the hyperbolic plane $\mathbb{H}^2$ is topologically a circle. A geodesic lamination of $\mathbb{H}^2$ is a disjoint union of geodesics which is a closed subset of $\mathbb{H}^2$. If

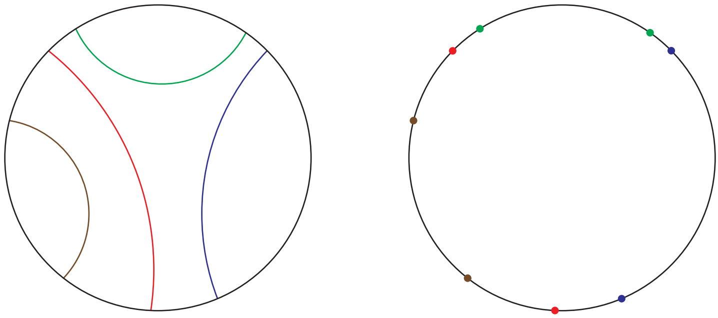


Figure 1: On the left, a geodesic lamination on $\mathbb{H}^2$ with four leaves. On the right, one can see the corresponding lamination of the circle after removing the geodesics and leaving only the endpoints.

one forgets about actual geodesics of a geodesic lamination of $\mathbb{H}^2$ and considers only the endpoints of geodesics on the ideal boundary, one gets a set of pairs of points of the circle. A lamination on the circle is defined as a set of pairs of points of the circle to capture this endpoint data of a geodesic lamination of $\mathbb{H}^2$ (see Figure 1).

Two pairs (p_1, p_2), (q_1, q_2) of unordered two distinct points of S^1 are called *linked* if the chord joining p_1 to p_2 crosses the chord joining q_1 to q_2 in the interior of the disk bounded by S^1. The space of all unordered pairs of two distinct points of S^1 is $((S^1 \times S^1) \setminus \Delta)/(x, y) \sim (y, x)$, where $\Delta = \{(x, x) \in S^1 \times S^1\}$. This is homeomorphic to an open Möbius band and we will denote this space by $\mathcal{M}$. A group action on S^1 induces an action on $\mathcal{M}$ in the obvious way; this action is not minimal in our examples, since otherwise there could not be any invariant laminations.

A *lamination* on S^1 is a set of unordered and unlinked pairs of two distinct points of S^1 which is a closed subset of $\mathcal{M}$. The elements of a lamination (which are pairs of points of S^1) are called *leaves* of the lamination. If a leaf is the pair of points (p, q), then the points p, q are called *ends* or *endpoints* of the leaf. For a lamination Λ, let E_Λ or $E(\Lambda)$ denote the set of all ends of leaves of Λ. We also use $\overline{\mathcal{M}}$ to denote the closed Möbius band and the points on $\partial\mathcal{M} := \overline{\mathcal{M}} \setminus \mathcal{M}$ are called *degenerate* leaves, which are single points of S^1.

Alternatively, one can identify the circle with the ideal boundary of $\mathbb{H}^2$ and consider only the ends of leaves of some geodesic lamination of $\mathbb{H}^2$. Every lamination on S^1 is of this form. Even though the group action on S^1 does not extend to the interior of the disk, it is usually better to picture a lamination of S^1 as a geodesic lamination of $\mathbb{H}^2$. Consider a connected component of the complement of the lamination in the open disk. Its closure in the closed disk is called a *gap* or a *complementary region* of the lamination. In other words, a gap of a lamination Λ of S^1 is the metric completion of a connected component of the complement of the corresponding lamination in $\mathbb{D}$ with respect to the path metric. We will use $\mathbb{D}$ to denote the open disk bounded by S^1 where the groups we consider act. The disk $\mathbb{D}$ will be freely identified with the Poincaré disk model of $\mathbb{H}^2$, often without mention if there is no confusion.

Once a group G acts on S^1 by homeomorphisms, there is a diagonal action on $\mathcal{M}$. A lamination Λ of S^1 is said to be G–invariant if it is an invariant subset of $\mathcal{M}$ under this induced action of G.

We give names to some properties of laminations.

Definition 2.1 Let G be a group acting on S^1 faithfully. A G–invariant lamination Λ is called

- *dense* if the endpoints of the leaves of Λ form a dense subset of S^1,
- *very full* if all the gaps of Λ are finite-sided ideal polygons,
- *minimal* if the orbit closure of any leaf of Λ is the whole Λ,
- *totally disconnected* if no open subset of $\mathbb{D}$ is foliated by Λ,
- *solenoidal* if it is totally disconnected and has no isolated leaves,
- *boundary-full* if the closure of the lamination in $\overline{\mathcal{M}}$ contains the entire $\partial\mathcal{M}$.

In fact, all properties above except the minimality are independent on the group action. Hence we use those notions for laminations on S^1 even when we do not have a group action in consideration. In this paper, very full laminations are of a particular interest. See Figure 2 for an example.[1]

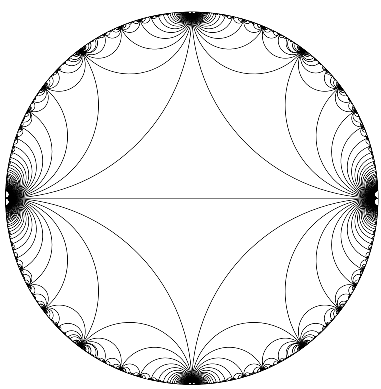


Figure 2: The Farey diagram is a famous example of very full laminations.

[1]This figure is borrowed from Lars Madsen at Aarhus University.

A continuous map f from S^1 to itself of degree 1 is called a *monotone* map if the pre-image of each point in the range under f is connected. Let $\rho_1\colon G \to \mathrm{Homeo}_+(S^1)$ and $\rho_2\colon G \to \mathrm{Homeo}_+(S^1)$ be faithful group actions on S^1. We say that ρ_1 is *semi-conjugate* to ρ_2 if there exists a monotone map $f\colon S^1 \to S^1$ such that $f \circ \rho_1(g) = \rho_2(g) \circ f$ for all $g \in G$. If f could be taken to be a homeomorphism, then ρ_1 is said to be *conjugate* to ρ_2. Note that a semi-conjugacy (or rather a monotone map) gives a map from $\mathcal{M}$ to $\overline{\mathcal{M}}$. For general background on the laminations on S^1 and monotone maps, we highly recommend Calegari [3, Chapter 2].

A group G is said to act minimally on S^1 if all orbits are dense. One immediate consequence of an action being minimal is that the only non-empty closed G–invariant subset of S^1 is the entire S^1. Note that the minimality of an action of a group G is not equivalent to the minimality of a G–invariant lamination.

Some elements of $\mathrm{Homeo}_+(S^1)$ are particularly interesting to us.

Definition 2.2 For $g \in \mathrm{Homeo}_+(S^1)$, let Fix_g be the fixed-point set $\{x \in S^1 : g(x) = x\}$. An element g of $\mathrm{Homeo}_+(S^1)$ is said to be:

- *Elliptic* if $|\mathrm{Fix}_g| = 0$.
- *Parabolic* if $|\mathrm{Fix}_g| = 1$.
- *Hyperbolic* if $|\mathrm{Fix}_g| = 2$ and one fixed point is attracting and the other is repelling.
- *Pseudo-Anosov-like* if there exists $m > 0$ such that $|\mathrm{Fix}_{g^m}| = 2n$ for some $n > 1$ and the elements of Fix_{g^m} alternate between attracting and repelling fixed points along S^1.

Once a group $G \subset \mathrm{Homeo}_+(S^1)$ is given, a point p on S^1 is called a *cusp point* if p is the fixed point of a parabolic element of G.

Once we consider more than one lamination at the same time, we need some more definitions.

Definition 2.3 Two laminations Λ_1, Λ_2 of S^1 are *transverse* if they have no leaf in common, ie $\Lambda_1 \cap \Lambda_2 = \varnothing$ as subsets of $\mathcal{M}$. They are said to be *strongly transverse* if no leaf of Λ_1 shares any endpoints with a leaf of Λ_2, ie $E(\Lambda_1) \cap E(\Lambda_2) = \varnothing$.

For a collection of very full laminations, each of which is invariant under some group G, one can define a notion that lies between pairwise transversality and pairwise strong-transversality. The motivation of the following definition will be explained later.

Definition 2.4 Let G be a group acting on S^1 faithfully and let $\mathcal{C} = \{\Lambda_\alpha\}_{\alpha \in J}$ be a collection of G–invariant very full laminations, where J is an index set. Then $\mathcal{C}$ is called *pants-like* if the laminations in $\mathcal{C}$ are pairwise transverse, and each point $p \in S^1$ is either fixed by a parabolic element of G or an endpoint of a leaf of at most one lamination Λ_α. In other words, for $\alpha \neq \beta \in J$, $E(\Lambda_\alpha) \cap E(\Lambda_\beta) = \{\text{cusp points of } G\}$.

For a group $G \subset \mathrm{Homeo}_+(S^1)$, we say G is COL_n for some $n \in \mathbb{N}$ if it admits n pairwise transverse dense invariant laminations. We use COL_∞ to denote the groups which admit an infinite collection of transverse dense invariant laminations.

By definition, we have the inclusions

$$\mathrm{COL}_1 \supset \mathrm{COL}_2 \supset \mathrm{COL}_3 \supset \cdots .$$

We say a group is strictly COL_n if it is COL_n but not COL_{n+1}. A COL_n group G is said to be pants-like COL_n if the collection of n pairwise transverse dense G–invariant laminations could be chosen to be pants-like.

For an abstract group G and an injective homomorphism $\rho\colon G \to \mathrm{Homeo}_+(S^1)$, ρ is called a COL_n–representation if $\rho(G)$ is a COL_n group. One of our aims is to deduce interesting properties of a COL_n group from the dynamical and geometric data of its invariant laminations.

We will consider the following natural questions.

Question 2.5 Is the set of COL_i groups strictly bigger than the set of COL_{i+1} groups for any i? Can one characterize those groups in an interesting way?

Question 2.6 Is COL_n nonempty for all n?

We will get partial answers to Question 2.5 and provide an affirmative answer to Question 2.6. Our main result of the present paper is to show that pants-like COL_3 groups are Fuchsian.

3 Groups with specified number of invariant laminations

3.1 Strictly COL_1 groups

In this section, we construct an example of a strictly COL_1 group. Let R be a rigid rotation by an irrational angle and pick a point $p \in S^1$. Let $\mathcal{O}_p$ be the orbit of p under the forward and backward iterates of R. Then it is a countable dense subset of S^1. Let $p_i = R^i(p)$, where R^i is the i^{th} iterate of R. We blow up all points in $\mathcal{O}_p$ and replace them by intervals. More precisely, replace p_j by an interval of length $1/2^{|j|}$, and call this interval I_j. Since the sum of the lengths of the I_j is finite, we get, again, a circle. The action of R on the new circle is the same as in the original circle in the complements of the I_j and $R(I_j) = I_{j+1}$ is defined as a unique affine homeomorphism between closed intervals for all j. This type of process is called a Denjoy blow-up (for instance, see [3, Construction 2.45]). We use $\widetilde{R}$ to denote the new action obtained from R as above.

Now consider this circle as $\partial\mathbb{H}^2$. For each j, connect the endpoints of I_j by a geodesic of $\mathbb{H}^2$. Then we get a lamination, and call it Λ_R, which is invariant under the cyclic group G_R generated by $\widetilde{R}$. Let P_R be the unique complementary region of Λ_R which does not contain any open arc of S^1. Then the following lemma holds.

Lemma 3.1 *No G_R–invariant lamination meets the interior of P_R.*

Proof Let l be a leaf intersecting the interior of P_R. The G_R–action is semi-conjugate to R via the monotone map $f\colon S^1 \to S^1$ that collapses each I_j, reversing the process of a Denjoy blow-up. If the orbit closure of l under the G_R–action gives a G_R–invariant lamination, then so does the orbit closure of $f(l)$ under R–action. Since l intersects the interior of P_R, $f(l)$ is not degenerate. But R cannot have any invariant lamination, since an irrational rotation maps any pair to a linked pair under some power of R, a contradiction. Hence the orbit closure of l under the G_R–action cannot be a lamination. This implies that no invariant lamination of G_R has a leaf intersecting the interior of P_R. □

Λ_R is not a dense lamination. We can fix this by putting infinitely many copies of Λ_R together in a nice way.

Pick a leaf l of Λ_R and consider a larger group: the maximal orientation-preserving subgroup G of the group $G' = \langle \widetilde{R}, r(l)\rangle$ generated by $\widetilde{R}$ and the reflection $r(l)$ along the leaf l. Note that G is simply $G' \cap \mathrm{Homeo}_+(S^1)$. We claim that G is strictly COL_1. The orbit closure of l under the G'–action is a dense lamination, call it $\Lambda(R)$. The images of P_R under the elements of G' tessellate the open disk.

Suppose there exists another G–invariant lamination $\widehat{\Lambda}$ and let L be a leaf of $\widehat{\Lambda}$ that is not contained in $\Lambda(R)$. Then L must intersect the interior of some gap P. But the action of $\mathrm{Stab}(P)$ is like the one of G_R by construction where $\mathrm{Stab}(P) = \{\gamma \in G : \gamma(P) = P\}$. By Lemma 3.1, the $\mathrm{Stab}(P)$–orbit of L has linked elements so $\widehat{\Lambda}$ cannot be a lamination. Hence $\Lambda(R)$ is the only invariant lamination of G'.

There are some questions we can ask. If we take a rotation R' by a different irrational angle, are the $\Lambda(R)$ topologically conjugate to $\Lambda(R')$? What can we say about the structure of the group G? It would be very interesting to know what makes the difference between strictly COL_1 and COL_2.

3.2 Strictly COL_2 groups

We shall now construct an example of a strictly COL_2 group. Let S be a closed orientable surface with genus $g \geq 2$. Thus it admits $\mathbb{H}^2$ as its universal cover. Let $\phi\colon S \to S$ be a pseudo-Anosov homeomorphism from S to itself. Since $\mathbb{H}^2$ is simply connected, ϕ lifts to $\widetilde{\phi}\colon \mathbb{H}^2 \to \mathbb{H}^2$. Since $\widetilde{\phi}$ is a quasi-isometry, it extends continuously to $\partial\mathbb{H}^2$. The restriction of this extension to the boundary circle gives a homeomorphism $\overline{\phi}\colon S^1 \to S^1$, where $S^1 = \partial\mathbb{H}^2$. Let G_ϕ be the infinite cyclic subgroup of $\mathrm{Homeo}_+(S^1)$ generated by $\overline{\phi}$.

It is well known that any pseudo-Anosov homeomorphism of a hyperbolic surface has a pair of transverse invariant laminations, the stable and unstable laminations. One can obtain them as limits of images of a simple closed curve under the forward and backward iterates of the pseudo-Anosov map. Let $\Lambda^\pm$ denote those two laminations on S invariant under ϕ. Then these laminations lift to laminations $\widetilde{\Lambda}^\pm$ in $\mathbb{H}^2$ invariant under $\widetilde{\phi}$. Then the endpoints of the leaves of $\widetilde{\Lambda}^\pm$ form laminations $\overline{\Lambda}^\pm$ in S^1 invariant under $\overline{\phi}$.

Lemma 3.2 *$\overline{\Lambda}^\pm$ are dense in S^1.*

Proof It suffices to show that the endpoints of the lifts of any leaf of $\Lambda^\pm$ are dense in S^1. This is obvious from the following easy observation. Let γ be any leaf of $\overline{\Lambda}^\pm$. For arbitrary geodesic l of $\mathbb{H}^2$ and for a half-space H bounded by l, some fundamental domain of S should intersect H. Hence one of the lifts of γ intersects H and then one end hits the arc of $\partial_\infty\mathbb{H}^2$ bounded by the endpoints of l, on the same side as H. This shows that for arbitrary open interval of S^1, some leaf has one endpoint in there. □

Proposition 3.3 *The $G := \pi_1(S) \rtimes \langle\overline{\phi}\rangle$–action on $\partial_\infty\mathbb{H}^2$ is strictly* COL_2.

Proof Let Λ be an invariant lamination under G. Then, it projects down to a lamination on S, which is invariant under ϕ. However, Λ^+ or Λ^- are minimal and filling (meaning that every simple closed curve on S intersects the lamination). Hence, the projected lamination on S contains either Λ^+ or Λ^- as a sub-lamination. In particular, Λ cannot be transverse to both $\overline{\Lambda}^+$ and $\overline{\Lambda}^-$. □

We just saw that one can produce a large family of examples of strictly COL_2 groups via pseudo-Anosov surface homeomorphisms.

Note that we also saw that any group containing irrational rotations is an example of a strictly COL_0 group. The results of this section prove the following proposition.

Proposition 3.4 $\mathrm{CO} \supsetneq \mathrm{COL}_1 \supsetneq \mathrm{COL}_2 \supsetneq \mathrm{COL}_3$

3.3 COL_∞ groups

We have seen some examples of groups that have a very small number of invariant laminations. In the other extreme, there are groups that admit infinitely many invariant laminations.

Proposition 3.5 COL_∞ *is nonempty.*

Proof Let S be a closed hyperbolic surface. One can find infinitely many nonhomotopic simple closed curves. In the homotopy class of each simple closed curve, there exists a unique simple closed geodesic. Identify the universal cover of S with $\mathbb{H}^2$. The lift of a simple closed geodesic becomes a geodesic lamination, that is invariant under the action of $\pi_1(S)$. By the same argument as in the proof of Lemma 3.2, its endpoints form a dense subset of S^1. Now we found an infinite family of dense invariant laminations of S^1 where the action of $\pi_1(S)$ is the restriction of the natural extension of the deck transformation action on $\partial_\infty \mathbb{H}^2$. By construction, they are obviously transverse to each other. □

As we saw in the proof of above proposition, a surface group admits an infinite collection of transverse dense invariant laminations. Hence they lie in COL_∞. There are two natural questions to ask.

Question 3.6 Is COL_∞ the same as $\bigcap_n \mathrm{COL}_n$?

Question 3.7 Are there examples of COL_∞ other than surface groups?

For Question 3.6, if we add a condition to the definition of COL_n that the n transverse dense invariant laminations are minimal, then the answer is *yes*.

Proposition 3.8 *Let G be COL_n with minimal laminations for all n. Then G is COL_∞.*

Proof Pick an arbitrary n. Then G admits n transverse dense minimal invariant laminations $\Lambda_1, \dots, \Lambda_n$. We will show that there exists a minimal dense G–invariant lamination Λ_{n+1} that is transverse to each Λ_i for $i = 1, \dots, n$.

Note that a minimal G–invariant lamination Λ is simply the orbit closure of an arbitrary leaf of Λ under the G–action. Hence, any two different minimal G–invariant laminations are transverse to each other. Since G is COL_{n+1}, there must be a minimal dense invariant lamination that is different from $\Lambda_1, \dots, \Lambda_n$. Thus, there exists a pair $l = (a, b)$ of points of S^1 that is not a leaf of any Λ_i for $i = 1, 2, \dots, n$, and the orbit closure of l forms a minimal dense invariant lamination transverse to $\Lambda_1, \dots, \Lambda_n$. This new lamination can be taken as Λ_{n+1}.

What we proved is that for any existing collection of pairwise minimal dense invariant laminations of G, we can add an extra minimal dense G–invariant lamination so that the new collection is still pairwise transverse. One obtains an infinite collection of pairwise transverse minimal dense G–invariant laminations by performing this process infinitely many times. □

In the proof, we need the minimality of the laminations in order to add a lamination to an existing collection. We suspect that Question 3.6 has an affirmative answer in general, but could not prove it without the minimality assumption.

For Question 3.7, the answer is still *yes*. One can construct an example using Denjoy blow-up. In the subsequent sections, however, we will see that the situation is very different as long as one requires the invariant laminations to be very full.

4 Laminations on the hyperbolic surfaces

In this section, we will study the laminations on hyperbolic surfaces.

Definition 4.1 A surface S admitting a complete hyperbolic metric is called *pants-decomposable* if there exists a non-empty multi-curve X on S consisting of simple closed geodesics so that the closure of each connected component of the complement of X is a pair of pants. The fundamental group of some pants-decomposable surface is called a pants-decomposable surface group. The multi-curve X used in the pants-decomposition of S will be called a *pants-curve*.

Note that all hyperbolic surfaces of finite area except the thrice-puncture sphere are pants-decomposable. The thrice-puncture sphere is excluded by the definition, since we required the existence of a "non-empty" pants-curve. For a hyperbolic surface with infinite area, we still have a similar decomposition but some component of complement of the closure of a multi-curve could be a half-annulus or a half-plane. For the precise statement, we refer to [10, Theorem 3.6.2].

Lemma 4.2 *Let S be a hyperbolic surface of finite area which is not the thrice punctured sphere. For any pseudo-Anosov homeomorphism f of S and two arbitrary finite sets of simple closed curves F_1, F_2, there exists a large enough n such that no curve in F_1 is homotopic to a curve in $f^n(F_2)$, where f^n is the n^{th} iterate of f.*

Proof This is an immediate consequence of the fact that a pseudo-Anosov map has no reducible power. □

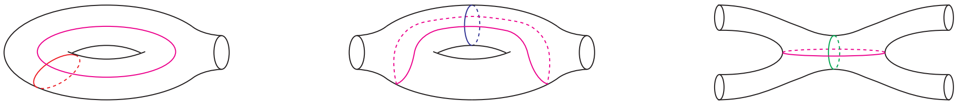
Proposition 4.3 *Let S be a pants-decomposable surface. Then there exist pants-curves X_0, X_1, X_2 so that no curve in X_i is homotopic to a curve in X_j for all $i \neq j$.*

Proof Note that this claim is clear if S is of finite area. We take an arbitrary pants-curve X_0 and a pseudo-Anosov map $f\colon S \to S$. Then by Lemma 4.2, there exist large enough positive integers n_1, n_2 such that $X_0, X_1 = f^{n_1}(X_0), X_2 = f^{n_2}(X_0)$ are such pants-curves. But we have no well-understood notion of pseudo-Anosov map for an arbitrary surface of infinite area.

Let S be a pants-decomposable surface of infinite area.

First we take an arbitrary pants-curve X_0. Seeing X_0 as some set of simple closed geodesics, choose a subset B of X_0 such that no two curves in B are boundary components of a single pair of pants, and each connected component of $S \setminus B$ is a finite union of pairs of pants, ie of finite area. Let $(S_i)_{i\in\mathbb{N}}$ be the enumeration of the connected components of $S \setminus B$. For each i, choose a pseudo-Anosov map f_i on S_i rel ∂S_i. By Lemma 4.2, there exists $n_i, m_i \in \mathbb{N}$ so that $X_0 \cap S_i$, $f^{n_i}(X_0 \cap S_i)$, $f^{m_i}(X_0 \cap S_i)$ are desired pants-curves on S_i.

Let $X_1 := B \cup (\bigcup_{i\in\mathbb{N}} f_i^{n_i}(X_0 \cap S_i))$, $X_2 := B \cup (\bigcup_{i\in\mathbb{N}} f_i^{m_i}(X_0 \cap S_i))$. We are not quite done yet, since all X_0, X_1, X_2 contain B. For each curve γ in B, we choose a simple closed curve $\delta(\gamma)$ as in Figure 3. They show three different possibilities for γ as red, blue and green curves, and in each case, $\delta(\gamma)$ is drawn as the curve colored in magenta. By definition of B, $\delta(\gamma)$ is disjoint from $\delta(\gamma')$ for $\gamma \neq \gamma' \in B$. Let D be the positive multi-twist along the multi-curve $Y = \cup_{\gamma\in B}\delta(\gamma)$. Let $X_1' = D(X_1)$. A curve in X_1 that had zero geometric intersection number with Y remains unchanged, and clearly it has no homotopic curves in X_0. A curve in X_1 that had non-zero intersection number with Y now has positive geometric intersection number with B. Since no



(a) Boundary component of a single pair of pants (b) 1 of 2 shared boundary components of two pants pairs (c) Unique shared boundary component of two pants pairs

Figure 3: This shows how to choose the multi-curve along which we will perform the positive multi-twist to produce a new pants-curve.

curve in X_0 has positive geometric intersection number with B, we are done for this case too.

Changing X_2 is a bit trickier. Let $(b_i)_{i\in\mathbb{N}}$ be an enumeration of the curves in B. Let D_i be the positive Dehn twist along $\delta(b_i)$. One can take k_i for each i so that each curve in $D_i^{k_i}(X_2)$ either is the image of a curve in X_2, which has zero geometric intersection number with $\delta(b_i)$ (so remains unchanged), or has positive geometric number with b_i, which is strictly larger than 2. Since $\delta(b_i)$ are disjoint, the infinite product $D' := \prod_{i\in\mathbb{N}} D_i^{k_i}$ is well-defined. Define X_2' as $X_2' = D'(X_2)$. Note that the geometric intersection number between a curve γ in X_1' and b_i for some i is at most 2. Now it is clear that X_0, X_1', X_2' are desired pants-curves. □

The next two lemmas are preparation to produce a pants-like collection of laminations out of the pants-curve we produced above.

Lemma 4.4 *Let G be a* COL *group with an invariant lamination Λ and $g \in G$ be a hyperbolic element. If Λ has leaf l, one end of which is fixed by g, then Λ has a leaf joining two fixed points of g.*

Proof Either $g^n(l)$ or $g^{-n}(l)$ converges to the axis of g as n goes to ∞. □

Lemma 4.5 *Let G be a* COL_n *group for some $n \geq 1$ and let $\{\Lambda_\alpha\}$ be a collection of n pairwise transverse dense invariant laminations of G. If $x \in S^1$ is a fixed point of a hyperbolic element g of G, then there exists at most one lamination Λ_α that has a leaf with x as an endpoint.*

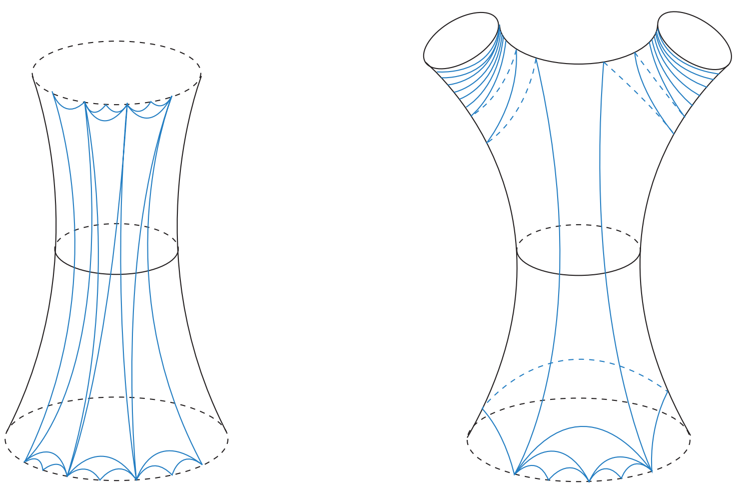
Proof This is a consequence of Lemma 4.4 and the transversality of the laminations. □

Theorem 4.6 *Any Fuchsian group G such that $\mathbb{H}^2/G$ is not the thrice-punctured sphere is a pants-like* COL_3 *group.*

Proof We start with the case when G is the fundamental group of a pants-decomposable surface S. Let $(X_i)_{i=0,1,2}$ be the pants-curves as in Proposition 4.3. For each i, let L_i be the lamination on S obtained from X_i by decomposing the interior of each pair of pants into two ideal triangles. It is possible to put a hyperbolic metric on S so that L_i is a geodesic lamination. Identify the universal cover of S with $\mathbb{H}^2$. G acts on the circle at infinity. Let Λ_i be the lamination of the circle at infinity obtained by lifting L_i to $\mathbb{H}^2$ and taking the end-points data. Since all the complementary regions are ideal triangles, it is very full.

Also any leaf of L_i is either a simple closed geodesic, or it is an infinite geodesic, each of whose ends either accumulates to a simple closed geodesic or escape to a cusp. Hence each end is a fixed point of some parabolic or hyperbolic element of G. Now the pants-like property follows from Lemma 4.4 and the transversality of the laminations.

We would like to get the same conclusion as before in the general case if G is a Fuchsian group but its quotient surface $\mathbb{H}^2/G$ is neither a thrice-punctured sphere nor pants-decomposable.



(a) The case when the quotient surface is an infinite annulus

(b) The case when there exists an annulus component glued to a pair of pants

Figure 4: Indeed one can put arbitrarily many pairwise transverse very full laminations on the annulus components in this way.

Let us first deal with the half-annulus components. Suppose that X is a multi-curve on the quotient surface S such that $S \setminus X$ consists of pairs of pants and half-annuli. If two half-annuli are glued along a simple closed geodesic, our surface is actually an annulus and the lamination could be taken as in Figure 4(a). Since we can take the ends of such a lamination arbitrarily, it is obvious that there are arbitrarily many such invariant laminations that are pairwise transverse. If the surface is not an annulus, a half-annulus component needs to be attached to a pair of pants. Let X_0 be the collection of simple closed geodesics obtained from the X by removing those boundaries of half-annulus components. Let S' be the complement of the half-annuli. As in the proof of Proposition 4.3, we can find other pants-curves X_1, X_2 on S' so that X_0, X_1, X_2 are disjoint in the curve complex of S'. Now we decompose the interior of each pair of pants into two ideal triangles as before.

We need to put more leaves on each component of $S \setminus X_i$ for any i that is the union of one half-annulus and one pair of pants glued along a cuff. We construct a lamination inside such a component as in Figure 4(b). Again, we can put an arbitrary lamination on the ideal boundary part of the half-annulus. Note that we construct each lamination so that all gaps are finite-sided, thus we are done.

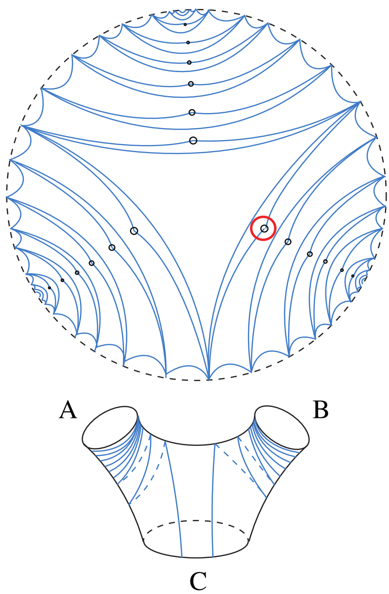


Figure 5: An example of the subsurface that we consider in the proof of Theorem 4.6. One can choose the endpoints of the leaves on the ideal boundary arbitrarily so that we can put as many pairwise transverse very full laminations on such a subsurface as we want. Look at the part where the red circle is. Here a pair of pants is attached as in the figure below. The boundary component labeled by "C" is not included in X_0 but those labeled by "A" and "B" are. The boundary curves A and B could be cusps or glued along each other.

Now we consider the case where S has even half-plane components. In [10, Theorem 3.6.2], it is also shown that if Z denotes the set of points of a pants-curve X, then components of $\overline{Z} \setminus Z$ are simple infinite geodesics bounding half-planes, ie we know exactly how the half-plane components arise in the decomposition of a complete hyperbolic surface. Let X be a multi-curve and let Z be the set of points of simple closed curves in X such that $S \setminus \overline{Z}$ consists of pairs of pants, half-annuli, and half-planes, and the boundaries of half-plane components form the set $\overline{Z} \setminus Z$. We will define X_0 by removing some geodesics from X. As before, we remove all the boundary curves of half-annulus components. Observe that there is a part of a surface that is homeomorphic to a half-plane with families of cusps and geodesic boundaries that converge to the ideal boundary (see [10, Figure 3.6.3 on page 86] for example). On this subsurface, there are infinitely many components of $\overline{Z}$ so that this subsurface is decomposed into pairs of pants and some half-planes. We remove all the components of X appear on this type of subsurface. Again, X_0 is a pants-curve of a pants-decomposable subsurface S' of S with geodesic boundaries. On S', we construct X_1, X_2 as before. Among the connected components of $S \setminus S'$, the one containing a half-annulus can be laminated as we explained in the previous paragraph. In the connected component that is homeomorphic to an open disk with punctures, we can do this as in Figure 5. Once again, since the ideal boundary part is invariant, we can put an arbitrary lamination there. It is also obvious that the way we construct a lamination gives a very full lamination. □

Remark 4.7 We constructed a pants-like collection of laminations for Fuchsian groups using pants-decompositions in the proof of Theorem 4.6. This is where the name "pants-like" comes from.

We would like to see if the converse of Theorem 4.6 is also true. In order to answer that question, one needs to analyze the properties of pants-like COL_3 groups.

5 Rainbows in very full laminations

Before we move on, we would like to understand better the structure of very full laminations. Recall that $\mathcal{M}$ is the set of all pairs of two distinct points of S^1, which is homeomorphic to an open Möbius band.

Let $p \in S^1$ and Λ be a dense lamination on S^1. Suppose that there is a sequence of leaves of Λ, both of whose ends converge to p but from opposite sides. We call such a sequence a *rainbow* at p. Imagine the upper half-plane model of $\mathbb{H}^2$ and that we stand

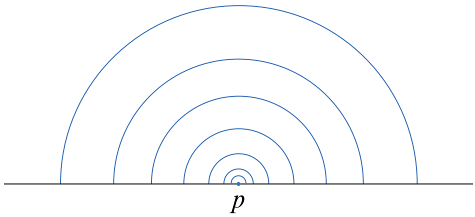


Figure 6: This is a schematic picture of a rainbow at p

at x in the real line, which is not an endpoint of a lamination. The name "rainbow" would make sense in this picture. See Figure 6.

The following lemma is more or less an observation.

Lemma 5.1 *Let Λ be a very full lamination of S^1. Then Λ is dense. Further, for any gap P of Λ, if $x \in S^1 \cap P$, then x is an endpoint of some leaf of Λ.*

Proof Suppose Λ is not dense. Then we can take an open connected arc I of S^1 where the leaves of Λ have no endpoints. Let l be a geodesic connecting the endpoints of I. Λ has no leaf intersecting l. Take a point p on l. Clearly the gap containing p cannot be a finite-sided ideal polygon. Suppose P is a gap of Λ and $x \in S^1 \cap P$. Since P is a finite-sided ideal polygon, it intersects S^1 only at points to which two sides of P converges. Hence x is an endpoint of some leaf. □

The proof of the following lemma is easily provided from basic facts of hyperbolic geometry.

Lemma 5.2 *Consider a very full lamination Λ of S^1. Let $x \in \mathbb{D}$. For $p \in S^1$, a gap of Λ containing x contains p if and only if there is no leaf of Λ crossing the geodesic ray from x to p.*

Recall that for a lamination Λ on S^1, E_Λ denotes the set of endpoints of the leaves of Λ. There is a nice dichotomy.

Theorem 5.3 (There are enough rainbows) *Let Λ be a very full lamination of S^1. For $p \in S^1$, either p is in E_Λ or p has a rainbow. These two possibilities are mutually exclusive.*

Proof It is clear that if $p \in E_\Lambda$, there is no rainbow. Suppose there is no rainbow for p. Then p has a neighborhood U so that if a leaf of Λ has both endpoints in U, then both endpoints are contained in the same connected component of $U \setminus \{p\}$. Replacing U by a smaller neighborhood, we may assume that no leaf connects the endpoints of U.

Identify S^1 with the boundary of the hyperbolic plane $\mathbb{D}$ and realize Λ as a geodesic lamination on $\mathbb{D}$. Let q_1 be a point on the geodesic connecting the endpoints of U. We may assume that there is no leaf passing through q_1. The only way of not having such a point is that the entire $\mathbb{D}$ is foliated and p is an endpoint of one of the leaves. Let L be the geodesic passing through q_1 and ending at p (see Figure 7). We denote the part of L between a point x on L and p by L_x. Note that any leaf of Λ crossing L_{q_1} has one end in U so that the other end must be outside U by the assumption on U.

If there is no leaf of Λ intersecting L_{q_1}, then the gap containing q_1 contains p by Lemma 5.2. Hence p must be an endpoint of a leaf by Lemma 5.1 and we are done. Suppose there is a leaf l_1 which crosses L_{q_1} at x. Then let q_2 be a point in L_x so that there is no leaf of Λ passing through it. If there were no such q_2, there is a leaf passing through each point of L_x, so there must be a leaf ending at p whose other end is necessarily outside U. So, we may assume that such a q_2 exists.

If there is no leaf of Λ crossing L_{q_2}, then the gap containing q_2 contains p, and we are done. Otherwise, a leaf, say l_2, crosses L_{q_2} at x_2. Repeat the process until we obtain an infinite sequence (q_i) on L which converges to p. This is possible, since otherwise we must have some x on L such that no leaf of Λ crosses L_x. Since (q_i) converges to p, the endpoints of the sequence (l_i) of leaves in U form a sequence converging to p, and the other endpoints are all outside U. By the compactness of S^1, we can take a convergent subsequence so that p is an endpoint of the limiting leaf. In any case, p must be an endpoint of a leaf.

Therefore, $p \in E_\Lambda$ if and only if p has no rainbow. □

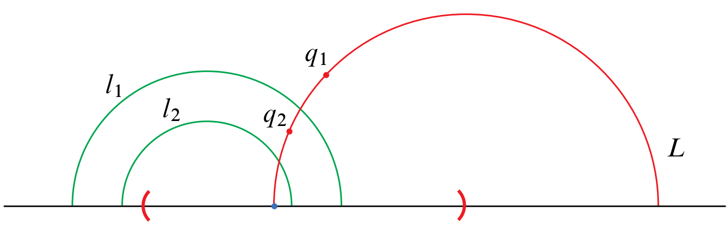


Figure 7: A situation where we have no rainbow for p

Corollary 5.4 *Let G be a group acting on S^1 and Λ be a G–invariant very full lamination. For $x \in S^1$, which is the fixed point of a parabolic element g of G, there exist infinitely many leaves which have x as an endpoint.*

Proof Let $x \in S^1$ be the fixed point of a parabolic element g of G and pick Λ_α. By Theorem 5.3, if x is not an endpoint of a leaf of Λ_α, then x has a rainbow. But any leaf whose ends are all not x must be contained in a single fundamental domain of g to stay unlinked under the iterates of g (here, a fundamental domain is the arc connecting y and $g(y)$ in S^1 for some y different from x). Then the existence of a rainbow would imply that g is a constant map whose image is x but it is impossible since g is a homeomorphism. Hence x is an endpoint of some leaf l of Λ_α. Then the $(g^n(l))_{n\in\mathbb{Z}}$ are infinitely many distinct leaves of Λ_α, all of which have x as an endpoint. □

Corollary 5.5 (Boundary-full laminations) *Suppose a group G acts on S^1 faithfully and minimally. Let Λ be a lamination of S^1 invariant under the G–action. If Λ is very full and totally disconnected, then Λ is boundary-full.*

Proof The minimality of the action implies that once the closure of the lamination in $\overline{\mathcal{M}}$ contains at least one point in $\partial\mathcal{M}$, then it contains $\partial\mathcal{M}$ and thus the lamination is very full (this is a simple diagonalization argument).

Let l_1 be any leaf of Λ. Due to the minimality, some element of G maps one of the ends of l_1 somewhere in the middle of the shortest two arcs joining the endpoints of l. Let l_2 be the image of l_1 under the action of this element. Again due to the minimality, one can find an element of G which maps one of the ends of l_2 somewhere in the middle of the shortest arcs in the complement of the endpoints of l_1 and l_2 in S^1. Let l_3 be the image of l_2 under that element. Repeating this procedure, one gets a sequence (l_n) of leaves for which the distance between their endpoints tends to zero, hence giving a desired point in ∂M. □

In fact, the laminations we constructed for pants-decomposable surface groups satisfy the hypotheses of Corollary 5.5. Hence all of them are boundary-full laminations.

6 Classification of elements of pants-like COL_3 groups

Any element of a Fuchsian group has at most two fixed points on $\partial_\infty\mathbb{H}^2$. Hence, it might be useful to check how many fixed points an element of a pants-like COL_3 group can have.

Lemma 6.1 *Let f be a non-identity orientation-preserving homeomorphism of S^1 with $3 \le |\mathrm{Fix}_f|$. Then any very full lamination Λ invariant under f has a leaf connecting two fixed point of f. Moreover, for any connected component I of $S^1 \setminus \mathrm{Fix}_f$ with endpoints a and b, at least one of a and b is an endpoint of a leaf of Λ.*

Proof Let I be a connected component of $S^1 \setminus \mathrm{Fix}_f$ with endpoints a and b. Since Fix_f has at least three points, one can take $c \in \mathrm{Fix}_f \setminus \{a, b\}$. Relabeling a and b if necessary, we may assume that the triple a, b, c are counterclockwise oriented.

Suppose a is not an endpoint of a leaf of Λ. Then there exists a rainbow in Λ at a by Theorem 5.3. In particular, there exists a leaf l such that one end of l lies in I and the other end lies outside I; call the second one d. If d is a fixed point of f, then replace c by d. Otherwise, we may assume that a, c, d are counterclockwise-oriented and there is no fixed point of f between c and d after replacing c by another fixed point if necessary. Clearly, either $f^n(l)$ or $f^{-n}(l)$ converges to the leaf connecting b and c (this may not be the same c as the c at the beginning).

This proves the lemma. □

Corollary 6.2 *Let G be a pants-like COL_3 group. Then for any $g \in G$, one must have $|\mathrm{Fix}_g| \leq 2$.*

Proof Let $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ be a pants-like collection of G–invariant laminations. Suppose that there exists an element g of G which has at least three fixed points on S^1. Let I be a connected component of $S^1 \setminus \mathrm{Fix}_g$ with endpoints a and b. Then by Lemma 6.1, each of a and b is an endpoint of a leaf of some Λ_i. Hence, if none of a, b is the fixed point of a parabolic element of G, we get a contradiction to the pants-like property.

Suppose a is the fixed point of a parabolic element $h \in G$. By Corollary 5.4 (or rather the proof of it), there must be a leaf l of Λ_i for any choice of i such that one end of l is a and the other end lies in I. Then either $g^n(l)$ or $g^{-n}(l)$ converges to the leaf connecting a and b as n increases. Hence each Λ_i must have the leaf connecting a and b, contradicting to the transversality. Similarly b cannot be a cusp point either. This completes the proof. □

Lemma 6.3 *Let G be a group acting on S^1 and Λ be a very full G–invariant lamination. For each hyperbolic element $g \in G$ with fixed points a and b, if Λ does not have (a, b) as a leaf, there must be a leaf l of Λ_α that separates a, b (ie not both endpoints of l lies in the same connected component of $S^1 \setminus \{a, b\}$).*

Proof This is just an observation using the existence of a rainbow. □

Lemma 6.4 *Let G be a pants-like COL_3 group. If $f \in G$ is parabolic, then its fixed point is a parabolic fixed point, ie the fixed point behaves as a sink on one side and as a source on the other side. If f is hyperbolic, it has one attracting and one repelling fixed point, ie it has North–South pole dynamics.*

Proof For the parabolic case, it is an obvious observation. Suppose f is hyperbolic. If there are no North–South pole dynamics, then both fixed points are parabolic fixed points. But we have G–invariant laminations with no leaves connecting the fixed points of g (by the transversality, all but at most one lamination are like that; see Lemma 4.5). For each of those laminations, there must be a leaf connecting two components of the complement of the fixed points by Lemma 6.3. They cannot stay unlinked under f if both fixed points are parabolic. □

Lemma 6.5 *Let G be a pants-like COL_3 group. Any elliptic element of G is of finite order.*

Proof Let $f \in G$ be an elliptic element. If its rotation number is rational, then some power f^n of f must have fixed points. By Corollary 6.2, Fix_{f^n} has either one or two points unless f^n is identity. Suppose first f^n has only one fixed point. Then f must have one fixed point too, contradicting to the assumption. Thus f^n has two fixed points. But since f has no fixed points, both fixed points must be parabolic fixed points, which contradicts Lemma 6.4. Hence the only possibility is $f^n = \mathrm{Id}$, so f is of finite order.

Suppose f has irrational rotation number. It cannot be conjugate to a rigid rotation with irrational angle, since any irrational rotation has no invariant lamination at all as we observed before. So f must be semiconjugate to a irrational rotation, say R. We may assume that the action of f on S^1 is obtained by Denjoy blow-up for one or several orbits under R. Any invariant lamination should be supported by the blown-up orbit. But such a lamination cannot be very full. Hence f cannot have an irrational rotation number. □

Theorem 6.6 *Suppose $G \subset \mathrm{Homeo}_+(S^1)$ is a pants-like COL_3 group. Then each element of G is either a torsion, parabolic, or hyperbolic element.*

Proof This follows from Corollary 6.2, Lemma 6.4 and Lemma 6.5. □

Theorem 6.6 provides a classification of elements of pants-like COL_3 groups just like the one for Fuchsian groups. In fact, this is not a coincidence. It is hard in general to extend the action to the interior of $\mathbb{D}$. Instead, we will try to show that pants-like COL_3 groups are convergence groups. The convergence group theorem says that a group acting faithfully on the circle is a convergence group if and only if it is a Fuchsian group (this theorem was proved for a large class of groups in [15], and in full generality in [8] and [7]). For the general background for convergence groups, see [15].

7 Pants-like COL_3 groups are Fuchsian groups

For general reference, we state the following well-known lemma without proof.

Lemma 7.1 *Let G be a group acting on a space X. Let K be a compact subset of X such that $g(K) \cap K \neq \varnothing$ for infinitely many g_i. Then there exists a sequence (x_i) in K converging to x and a sequence of the set $\{(g_i)\}$, also called (g_i) (abusing notation), such that $g_i(x_i)$ converges to a point x' in K.*

Let G be a discrete subgroup of $\mathrm{Homeo}_+(S^1)$. Then a sequence (g_i) of elements of G is said to have the *convergence property* if there exist two points $a, b \in S^1$ (not necessarily distinct) and a subsequence (g_{i_j}) of (g_i) so that g_{i_j} converges to a uniformly on compacts subsets of $S^1 \setminus \{b\}$. If every sequence of elements of G has the convergence property, then we say G is a *convergence group*.

Let T be the space of ordered triples of three distinct elements of S^1. By [15, Theorem 4.A], a group $G \subset \mathrm{Homeo}(S^1)$ is a convergence group if and only if it acts on T properly discontinuously. If one looks at the proof of this theorem, we do not really use the group operation. Hence we get the following statement from the exact same proof.

Proposition 7.2 *Let C be a set of homeomorphisms of S^1. C has the convergence property if and only if C acts on T properly discontinuously.*

We can define the limit set of a pants-like COL_3 group G in a way similar to that for the case of Fuchsian groups. Let $\Omega(G)$ be the set of points of S^1 where G acts discontinuously, ie $\Omega(G) = \{x \in S^1 :$ there exists a neighborhood U of x such that $g(U) \cap U = \varnothing$ for all but finitely many $g \in G\}$ and call it *domain of discontinuity* of G. Let $L(G) = S^1 \setminus \Omega(G)$ and call it the *limit set* of G. For our conjecture to have a chance to be true, $\Omega(G)$ and $L(G)$ have the same properties as those for Fuchsian groups.

For the rest of this section, we fix a torsion-free pants-like COL_3 group G with a pants-like collection $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of G–invariant laminations. Let $F(G)$ be the set of all fixed points of elements of G, ie $F(G) = \cup_{g \in G} \mathrm{Fix}_g$. We do not need the following lemma but it shows another similarity of pants-like COL_3 groups with Fuchsian groups.

Lemma 7.3 *$\overline{F(G)}$ is either a finite subset of at most 2 points or an infinite set. When $\overline{F(G)}$ is infinite, it is either the entire S^1 or a perfect nowhere-dense subset of S^1.*

Proof When all the elements of G share fixed points, then $|\overline{F(G)}| \leq 2$ as $|\mathrm{Fix}_g| \leq 2$ for all $g \in G$. If that is not the case, then say we have $g, h \in G$ whose fixed point sets are distinct. Then for $x \in \mathrm{Fix}_h \setminus \mathrm{Fix}_g$, the $g^n(x)$ are all distinct for $n \in \mathbb{Z}$ and $g^n(x)$ is a fixed point of $g^n h g^{-n}$, hence $\overline{F(G)}$ is infinite. Note that $\overline{F(G)}$ is a closed minimal G–invariant subset of S^1. An infinite minimal set under the group action on the circle has no isolated points. Thus $\overline{F(G)}$ is a perfect set. □

The next lemma will itself not be used to prove our main theorem, but the proof is important.

Lemma 7.4 $$L(G) = \overline{F(G)}$$

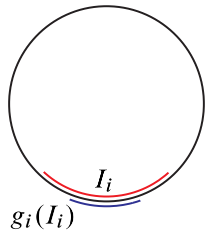
Proof Since $\overline{F(G)}$ is a minimal closed G–invariant subset of S^1, it is obvious that $\overline{F(G)} \subset L(G)$.

For the converse, we use laminations. Let $x \in L(G)$. If x is the fixed point of a parabolic element, then we are done. Hence we may assume that it is not the case and take Λ_α such that $x \notin E_{\Lambda_\alpha}$. Let (l_i) be a sequence of leaves which forms a rainbow for x (such a sequence exists by Theorem 5.3) and let (I_i) be a sequence of open arcs in S^1 such that each I_i is the component of the complement of the endpoints of l_i containing x. Since G does not act discontinuously at x, we can choose $g_i \in G$ such that $g_i(I_i)$ intersects I_i nontrivially. This cannot happen arbitrarily, but one must have either $g_i(\overline{I_i}) \subset \overline{I_i}$, $\overline{I_i} \subset g_i(\overline{I_i})$ or $I_i \cup g_i(I_i) = S^1$, since the endpoints of I_i form a leaf.

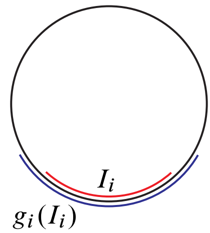
In the former two cases, an application of Brouwer's fixed point theorem implies the existence of a fixed point in $\overline{I_i}$. In the latter one, take any point in $F(G)$: either it belongs to $\overline{I_i}$, or its image under g_i^{-1} does (see Figure 8 for possible configurations, and the reason why our situation is restricted to these cases is described in Figure 9). Since I_i shrinks to x, this implies that x is a limit point of fixed points of (g_i). □

Lemma 7.5 *Suppose (g_i) is a sequence of elements of G and $x \in S^1$ such that for any neighborhood U of x, $g_i(U)$ intersects U nontrivially for all i large enough. Then x is a limit point of the fixed points of g_i in the sense that there exists a choice of a fixed point a_i of g_i for all i such that the sequence (a_i) converges to x.*

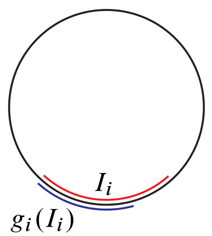
Proof If we can find Λ_α such that $x \notin E_{\Lambda_\alpha}$, then we are done by the proof of Lemma 7.4. Suppose that is not the case, ie x is the fixed point of a parabolic element h. The key point here is to figure out how to construct I_i for x in order to mimic the proof of Lemma 7.4. Take arbitrary invariant lamination Λ. There exists a



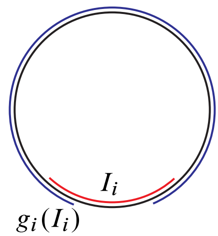
(a) $g_i(I_i)$ is completely contained in I_i



(b) $g_i(I_i)$ completely contains I_i



(c) $g_i(I_i)$ is completely contained in I_i but one endpoint of I_i is fixed



(d) The union of I_i and $g_i(I_i)$ is the whole circle

Figure 8: Possibilities of the image of I_i under g_i. I_i is the red arc (inside the disc) and $g_i(I_i)$ is the blue arc (outside the disc) in each figure.

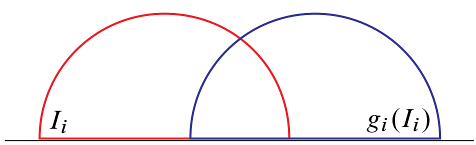
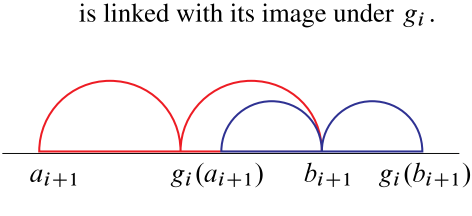
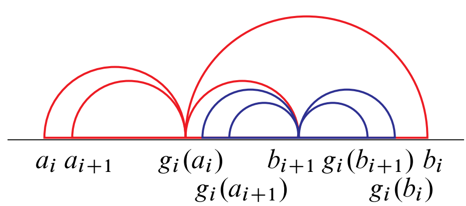


Figure 9: This case is excluded, since the leaf connecting the endpoints of I_i is linked with its image under g_i.



(a) Both $g_i(I_{i+1})\setminus I_{i+1}$ and $I_{i+1}\setminus g_i(I_{i+1})$ could be non-empty in this case.



(b) But even in case (a), there is no such problem for I_i and $g_i(I_i)$.

Figure 10: The nested intervals for a cusp point need more care. (a) shows that it might have a problematic intersection, but (b) shows we can take a one-step bigger intervals to avoid that. The endpoints of I_i are marked as a_i, b_i and the endpoints of I_{i+1} are marked as a_{i+1}, b_{i+1}.

leaf l that has x as an endpoint. Then the $h^n(l)$ form an infinite family of such leaves. For all $i \in \mathbb{N}$, let a_i be the other endpoint of $h^i(l)$ and b_i be the other endpoint of $h^{-i}(l)$. Let I_n be an open interval containing x with endpoints a_n, b_n for each $n \in \mathbb{N}$. Now we have a sequence of intervals shrinking to x. Take a subsequence of (g_i), with the sequence of I_j already given (including I_{i+1}) so that $g_i(I_{i+1}) \cap I_{i+1} \neq \varnothing$ for all i.

Note that it is possible that neither $g_i(I_{i+1}) \subset I_{i+1}$ nor $I_{i+1} \subset g_i(I_{i+1})$ holds (see Figure 10(a)), but one must have either $g_i(I_i) \subset I_i$ or $I_i \subset g_i(I_i)$ to avoid having any linked leaves (see Figure 10(b)). Now the same argument shows that x is a limit point of fixed points of g_i. □

Proposition 7.6 *Suppose we have a sequence (x_i) of points in S^1 that converges to $x \in S^1$ and a sequence (g_i) of elements of G such that $g_i(x_i)$ converges to $x' \in S^1$.*

Then either x or x' lies in the limit set $L(G)$. In fact, by passing to a subsequence if necessary, either x is an accumulation point of the fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or x' is an accumulation point of the fixed points of the sequence $(g_i \circ g_{i+1}^{-1})$.

Moreover, one can apply this to multiple sequences. More precisely, suppose we also have a sequence (y_i) of points in S^1 such that (y_i) converges to $y \in S^1$ and $g_i(y_i)$ converges to $y' \in S^1$. Then one can pass to a subsequence such that the conclusion for x, x' is also true for y, y', ie either y is an accumulation point of the fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or y' is an accumulation point of the fixed points of the sequence $(g_i \circ g_{i+1}^{-1})$.

Proof Taking a subsequence of x_i, we may assume that (x_i) converges to x monotonically. Take any neighborhood U of x'. Then for large enough N, we have $g_i(x_i) \in U$ for all $i \geq N$. We show there is a dichotomy here: either the preimages of U shrink to a point, and we are done quickly, or by passing to a subsequence we can assume all of them to be large.

Suppose $g_i^{-1}(U)$ does not contain any x_j with $j \neq i$ for each $i \geq N$. But $g_i^{-1}(U)$ contains x_i and the sequence (x_i) converges to x. Hence $g_i^{-1}(U)$ for $i \geq N$ form a sequence of disjoint open intervals shrinking to x, implying that $g_i^{-1}(x')$ converges to x. Now let V be a neighborhood of x. Replacing N by a larger number if necessary, $g_i^{-1}(x')$ lies in V for all $i \geq N$. Then $(g_{i+1}^{-1} \circ g_i)(V)$ intersects nontrivially V for all $i \geq N$.

Now suppose such a U does not exist. Take an arbitrary neighborhood U of x'. As $g_i(x_i) \to x'$ for all large enough i, one has $g_i(x_i) \in U$. By the assumption, for each i, there exists $n_i \neq i$ such that $x_{n_i} \in g_i^{-1}(U)$. Thus we are allowed to assume that either

$n_i = i+1$ or $n_i = i-1$ for all i. In the former case, $g_{i+1} \circ g_i^{-1}(U)$ intersects U nontrivially for each i, and in the latter case, $g_i \circ g_{i+1}^{-1}$ does the same thing (note that $g_i \circ g_{i+1}^{-1}$ and $g_{i+1} \circ g_i^{-1}$ have the same fixed points).

Now we are in the assumptions of Lemma 7.5. Hence, either x is an accumulation point of the fixed points of the sequence $(g_{i+1}^{-1} \circ g_i)$ or x' is an accumulation point of the fixed points of the sequence $(g_i \circ g_{i+1}^{-1})$.

To see the last paragraph of the statement, one can first construct a subsequence for x, x', and then apply the same argument to this sequence to obtain a further subsequence for y, y'. This can be done because the argument above only depends on the existence of a neighborhood U with monotonically shrinking preimages $g_i^{-1}(U)$ that is preserved under taking a subsequence. □

Lemma 7.7 *Let G be a torsion-free discrete Möbius-like subgroup of* $\mathrm{Homeo}_+(S^1)$. *Suppose x, x', z, z' are four points of S^1 such that $x \neq z$, $x' \neq z'$, (h_i) is a sequence of elements of G, $(a_i), (z_i), (x_i)$ are sequences of points in S^1 and they satisfy all of the following:*

(1) *(x_i) converges to x, and both (z_i) and (a_i) converge to z.*

(2) *$(h_i(a_i))$ converges to x'.*

(3) *$(h_i(z_i))$ converges to z'.*

(4) *$(h_i(x_i))$ converges to x'.*

Further assume that G has a very full invariant lamination Λ such that each of x' and z either has a rainbow in Λ or is a cusp point of G. Then the sequence (h_i) has the convergence property.

Proof Figure 11 illustrates the sequences of points concerned here. We want to have a strictly decreasing sequence of nested intervals for each of x', z. Suppose for now that none of x', z is a cusp point. In this case, we have a rainbow for each of x', z and take intervals as in the proof of Lemma 7.4. For $p \in \{x', z\}$, let (I_i^p) be the sequence of nested decreasing intervals containing p. In Figure 11, two leaves of the lamination are drawn: one connecting endpoints of $I_i^{x'}$ and one connecting endpoints of I_i^z.

By taking subsequences, we may assume that for each i, we have $a_i, z_i \in I_i^z$ but $x_i \notin I_i^z$ and $h_i(x_i), h_i(a_i) \in I_i^{x'}$. Then, in particular, $h_i(I_i^z)$ intersects $I_i^{x'}$ non-trivially. But since h_i is a homeomorphism and $h_i(x_i) \in I_i^{x'}$, it is impossible to have $h_i(I_i^z) \supset I_i^{x'}$. Hence there are two possibilities: either $h_i(I_i^z) \subset I_i^{x'}$ or I_i^z is expanded by h_i so that $h_i(I_i^z) \cup I_i^{x'} = S^1$.

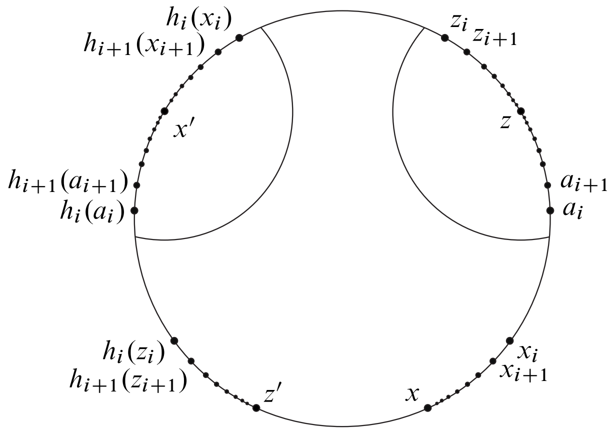


Figure 11: The sequences considered in Lemma 7.7

The former cannot happen for all large i, since $z_i \in I_i^z$ and $h_i(z_i) \to z' \notin I_i^{x'}$. Hence, the latter case should happen infinitely often. Then $S^1 \setminus I_i^z$ is mapped completely into $I_i^{x'}$ by h_i. This shows that the sequence h_i has the convergence property with the two points z, x'.

If some of them are cusp points, we take intervals as in the proof of Lemma 7.5. As we saw, one needs to be slightly more careful to choose $(I_i^{x'}), (I_i^z)$ so that the case where $h_i(I_i^{x'}) \not\subset I_i^z$, $h_i(I_i^{x'}) \not\supset I_i^z$ and $h_i(I_i^{x'}) \cup I_i^z \neq S^1$ does not happen; one can avoid this as we did in the proof of Lemma 7.5 (recall Figure 8). Then the same argument goes through. □

We are ready to prove the main theorem of the paper.

Theorem 7.8 *Let G be a torsion-free discrete subgroup* $\mathrm{Homeo}_+(S^1)$. *If G admits a pants-like collection of very full invariant laminations $\{\Lambda_1, \Lambda_2, \Lambda_3\}$, then G is a Fuchsian group.*

Proof By the convergence group theorem, it suffices to prove that G is a convergence group. Suppose not. Then there exists a sequence (g_i) of distinct elements of G that does not have the convergence property. This implies that this sequence, as a set, acts on T not properly discontinuously. Then we have three sequences $(x_i), (y_i), (z_i)$ converging to x, y, z and a sequence (h_i) of the set $\{g_i\}$ such that $h_i(x_i) \to x'$, $h_i(y_i) \to y'$, $h_i(z_i) \to z'$ where x, y, z are all distinct and x', y', z' are all distinct. Note that the sequence (h_i) could have been taken as a subsequence of (g_i), so let us assume that.

The strategy of the proof is as follows: we are going to find subsequences of the sequences $(h_i), (x_i), (z_i)$, and a new sequence (a_i) that all together satisfy the assumptions of Lemma 7.7. Since we assume that (g_i) does not have the convergence property, this leads us to a contradiction.

From Proposition 7.6, we can take a subsequence of (h_i) (call it again (h_i), abusing the notation) such that either two of x, y, z are accumulation points of the fixed points $h_{i+1}^{-1} \circ h_i$ or two of x', y', z' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$. Without loss of generality, suppose x', y' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$.

We would like to pass to subsequences so that the fixed points of the sequence $h_{i+1}^{-1} \circ h_i$ (or $h_i \circ h_{i+1}^{-1}$) have at most two accumulation points. But this cannot be done directly, since a subsequence of $(h_{i+1}^{-1} \circ h_i)$ is not from a subsequence of (h_i) in general. Instead, we proceed as follows.

Take a subsequence $(h_{i_j+1}^{-1} \circ h_{i_j})$ of $(h_{i+1}^{-1} \circ h_i)$ such that there are at most two points where the fixed points of $(h_{i_j+1}^{-1} \circ h_{i_j})$ accumulate (Such a subsequence exists due to Corollary 6.2 and the compactness of S^1). Similarly, let $(h_{i'_j+1}^{-1} \circ h_{i'_j})$ be a further subsequence of $(h_{i_j+1}^{-1} \circ h_{i_j})$ such that there are at most two points where the fixed points of $(h_{i'_j} \circ h_{i'_j+1}^{-1})$ accumulate.

Since x', y' are accumulation points of fixed points of $h_i \circ h_{i+1}^{-1}$, they are accumulation points of fixed points of $(h_{i'_j} \circ h_{i'_j+1}^{-1})$. But the fixed points of $(h_{i'_j} \circ h_{i'_j+1}^{-1})$ have at most two accumulation points and x', y', z' are three distinct points, so that z' cannot be an accumulation point of fixed points of $(h_{i'_j} \circ h_{i'_j+1}^{-1})$. This also implies that z' is not an accumulation point of fixed points of $(h_i \circ h_{i+1}^{-1})$. By our choice of (h_i), this implies that z must be an accumulation fixed points of $(h_{i+1}^{-1} \circ h_i)$ (so it is an accumulation point of fixed points of $(h_{i'_j+1}^{-1} \circ h_{i'_j})$).

Let (a_i) be such a sequence, ie a sequence of fixed points of $(h_{i+1}^{-1} \circ h_i)$ that converges to z. Now we consider a further subsequence such that $(h_{i''_j}(a_{i''_j}))$ converges to a point, say a. Note that for each i, we have $(h_i \circ h_{i+1}^{-1})(h_i(a_i)) = h_i(a_i)$. Hence, the $h_{i''_j}(a_{i''_j})$ are fixed points of a subsequence of $(h_{i''_j} \circ h_{i''_j}^{-1})$, so that a must be x' or y'. Without loss of generality, let us assume that $a = x'$.

Then we have the following:

(1) $(a_{i''_j})$ converges to z, since a_i converges to z.

(2) $(h_{i''_j}(a_{i''_j}))$ converges to x'.

(3) $(h_{i''_j}(z_{i''_j}))$ converges to z', since $h_i(z_i)$ converges to z'.

(4) $(h_{i''_j}(x_{i''_j}))$ converges to x', since $h_i(x_i)$ converges to x'.

It is now evident that the sequences $(h_{i''_j}), (a_{i''_j}), (z_{i''_j}), (x_{i''_j})$ satisfy the assumptions of Lemma 7.7. This implies the sequence (h_i) has the convergence property, hence so does the sequence (g_i). Now the result follows. □

Remark 7.9 In the proof of Theorem 7.8, the consequence of the pants-like property that we needed is that for arbitrary pair of points $p, q \in S^1$ that are not fixed by some parabolic elements, there exists an invariant lamination so that neither of p, q is an endpoint of the leaf of that lamination.

Corollary 7.10 (Main theorem) *Let G be a torsion-free discrete subgroup of $\mathrm{Homeo}_+(S^1)$. Then G is a pants-like COL_3 group if and only if G is a Fuchsian group whose quotient is not the thrice-punctured sphere.*

Proof This is a direct consequence of Theorem 4.6 and Theorem 7.8. □

Corollary 7.11 *Let G be a torsion-free discrete subgroup of $\mathrm{Homeo}_+(S^1)$. Then G admits three pairwise strongly transverse very full invariant laminations if and only if G is a Fuchsian group whose quotient has no cusps.*

Proof Replacing the pants-like property by pairwise strong transversality is equivalent to saying that there are no parabolic elements. Hence, this is an immediate corollary of the main theorem (Corollary 7.10). □

Corollary 7.12 *Let G be a torsion-free discrete pants-like COL_3 group. Then the G–action on S^1 is minimal if and only if G is a pants-decomposable surface group.*

Proof One direction is clear from the observation that the fundamental group of a pants-decomposable surface acts minimally on $\partial_\infty \mathbb{H}^2$. Suppose G is a pants-like COL_3 group. By Theorem 7.8, G is a Fuchsian group. Let S be the quotient surface $\mathbb{H}^2/G$. Note that S is not the thrice-punctured sphere, since it has infinitely many transverse laminations. If S is not pants-decomposable, then there still exists a multi-curve which decomposes S into pairs of pants, half-annuli and half-planes [10, Theorem 3.6.2]. Thus any fundamental domain of the G–action on $\overline{\mathbb{H}^2}$ contains some open arcs in $S^1 = \partial_\infty \mathbb{H}^2$. Let I be a proper closed sub-arc of such an open arc. Since it is taken as a subset of a fundamental domain, the orbit closure of I is a closed invariant subset of S^1 that has non-empty interior and is not the whole S^1. This contradicts the minimality of the G–action. □

Corollary 7.13 *Let M be a oriented hyperbolic 3–manifold whose fundamental group is finitely generated. If $\pi_1(M)$ admits a pants-like COL_3–representation into $\mathrm{Homeo}_+(S^1)$, then M is homeomorphic to $S \times \mathbb{R}$ for some surface S. If we further assume that M has no cusps and is geometrically finite, then M is either quasi-Fuchsian or Schottky.*

Proof The existence of a pants-like COL_3–representation into $\mathrm{Homeo}_+(S^1)$ implies that $\pi_1(M)$ is isomorphic to $\pi_1(S)$ for a hyperbolic surface S. The result is now a consequence of the tameness theorem (independently proved by Agol [1], and Calegari and Gabai [5]). □

Remark 7.14 There is an analogy between the cardinality of the set of ends of groups and the cardinality of the paths-like collection of laminations that subgroups of $\mathrm{Homeo}_+(S^1)$ can have. In Theorem 4.6, one can work harder to show that Fuchsian groups are in fact pants-like COL_∞ groups. The result of Section 3 says there are pants-like COL_2 groups that are not pants-like COL_3 groups (we will see the distinction in more detail in the next section). Hence, any torsion-free discrete subgroup of $\mathrm{Homeo}_+(S^1)$ is a pants-like COL_n group where n is either 0, 1, 2 or infinity, while the cardinality of the set of ends of a group has the same possibilities.

Remark 7.15 In Theorem 7.8, it is easy to see that the torsion-free assumption is not necessary. We conjecture that the main theorem (Corollary 7.10) could be stated without the torsion-free assumption. To show that, one needs to construct a pants-like collection of three very full laminations on hyperbolic orbifolds. It is not too clear how to do so with simple geodesics.

8 Pants-like COL_2 groups and some conjectures

We saw that being torsion-free discrete pants-like COL_3 is equivalent to being Fuchsian. In this section, we will try to see what is still true if we have one less lamination. For the rest of this section, we fix a pants-like COL_2 group G with a pants-like collection $\{\Lambda_1, \Lambda_2\}$ of G–invariant laminations. For the sake of simplicity, we also assume that $|\mathrm{Fix}_g| < \infty$ for each $g \in G$.

Proposition 8.1 *Let g be a non-parabolic element of G. Then g has no parabolic fixed point. Hence either g is elliptic or g has even number of fixed points that alternate between attracting fixed points and repelling fixed points along S^1.*

Proof Suppose $\mathrm{Fix}_g \neq \varnothing$. Let I be a connected component of $S^1 \setminus \mathrm{Fix}_g$ with endpoints a and b. In the previous section, we saw that for each i, either $a \in E_{\Lambda_i}$ or $b \in E_{\Lambda_i}$. We also know that none of a and b can be the fixed point of a parabolic element (see the second half of the proof of Corollary 6.2). Hence the pants-like property implies that there is no i such that both a and b are in E_{Λ_i}. In particular, this implies that for each $p \in \mathrm{Fix}_g$, there exists $i \in \{1, 2\}$ so that p is not in E_{Λ_i}. But this implies that there is a rainbow in Λ_i at p. But a parabolic fixed point cannot have a rainbow. This proves the claim. □

Corollary 8.2 *Each elliptic element of G is either of finite order or pseudo-Anosov-like.*

Proof This is a consequence of Lemma 6.5 and Proposition 8.1. □

We have proved the following.

Theorem 8.3 (Classification of elements of pants-like COL_2 groups) *Let G be as defined at the beginning of the section. The elements of G are either torsion, parabolic, hyperbolic or pseudo-Anosov-like.*

Conjecture 8.4 Suppose G is torsion-free discrete and $|\mathrm{Fix}_g| \leq 2$ for each $g \in G$. Then G is Fuchsian.

For each pseudo-Anosov-like element g of G, let $n = n(g)$ be the smallest positive number such that g^n has fixed points. The boundary leaves of the convex hull of the attracting fixed points form an ideal polygon; we call it the *attracting polygon* of g. The *repelling polygon* of g is defined similarly.

Theorem 8.5 *Let G, Λ_1, Λ_2 be as defined at the beginning of the section. Suppose that there exists $g \in G$ that has more than two fixed points (so there are at least 4 fixed points). Then each Λ_i contains either the attracting polygon of g or the repelling polygon of g.*

Proof Say $\mathrm{Fix}_g = \{p_1, \ldots, p_n\}$ such that if we walk from p_i along S^1 counterclockwise, then the first element of Fix_g we meet is p_{i+1} (indexes are modulo n). Suppose $p_1 \in E_{\Lambda_1}$. Then by the argument in the proof of Proposition 8.1, both p_2 and p_n are not in E_{Λ_1}. If we apply this consecutively, one can easily see that $p_i \in E_{\Lambda_1}$ if and only if i is odd.

Let j be any even number. Since p_j is not in E_{Λ_1}, there exists a rainbow at j. In particular, there exists a leaf l in Λ_1 so that one end of l lies between p_j and p_{j+1}, and the other end lies between p_j and p_{j-1}. Hence either $g^n(l)$ or $g^{-n}(l)$ converges to the leaf (p_{j-1}, p_{j+1}) as n increases. So, the leaf (p_{j-1}, p_{j+1}) should be contained in Λ_1. Since j was an arbitrary even number, this shows that Λ_1 contains the boundary leaves of the convex hull of the fixed points of g with odd indices. Similarly, one can see that Λ_2 must contain the boundary leaves of the convex hull of the fixed points of g with even indices. Since the fixed points of g alternate between attracting and repelling fixed points along S^1, the results follows. □

This shows that not only the pseudo-Anosov-like elements resemble the dynamics of pseudo-Anosov homeomorphisms but also their invariant laminations are like stable and unstable laminations of pseudo-Anosov homeomorphisms.

We introduce a following useful theorem of Moore [13] and an application in our context.

Theorem 8.6 (Moore) *Let $\mathcal{G}$ be an upper semicontinuous decomposition of S^2 such that each element of $\mathcal{G}$ is compact and nonseparating. Then $S^2/\mathcal{G}$ is homeomorphic to S^2.*

A decomposition of a Hausdorff space X is *upper semicontinuous* if and only if the set of pairs (x, y) for which x and y belong to the same decomposition element is closed in $X \times X$. A lamination Λ of S^1 is called *loose* if no point on S^1 is an endpoint of two leaves of Λ that are not edges of a single gap of Λ.

Theorem 8.7 *Let G, Λ_1, Λ_2 be as defined at the beginning of the section. We further assume that G is torsion free and each Λ_i is loose. Then G acts on S^2 by homeomorphisms such that $|\mathrm{Fix}_g(S^2) := \{p \in S^2 : g(p) = p\}| \le 2$ for each $g \in G$.*

Proof Let D_1 and D_2 be disks glued along their boundaries, and consider this boundary as the circle where G acts. We then get a 2–sphere, call it S_1, such that G acts on its equatorial circle. Put Λ_i on D_i for each $i = 1, 2$. One can first define a relation on S_1 so that two points are related if they are on the same leaf or the same complementary region of Λ_i for some i. Let $\sim$ be the closed equivalence relation generated by the relation we just defined.

It is fairly straightforward to see that $\sim$ satisfies the condition of Moore's theorem from the looseness. Looseness, in particular, implies that each equivalence class of $\sim$ has at most finitely many points in S^1.

This demonstrates that $S_2 := S_1/\sim$ is homeomorphic to a 2–sphere, and let $p\colon S_1 \to S_2$ be the corresponding quotient map. Clearly, p is surjective even after being restricted to the equatorial circle, call the restriction p again. Now we have a quotient map $p\colon S^1 \to S_2 = S^2$, hence G has an induced action on S^2 by homeomorphisms. Note that $|\mathrm{Fix}_g(S^1)| \geq |\mathrm{Fix}_g(S^2)|$ for each $g \in G$. But we know that if $g \in G$ has more than two fixed points on S^1, its attracting fixed points are mapped to a single point by p by Theorem 8.5. Similarly, the repelling fixed points are mapped to a single point too. Hence, g can have at most two fixed points in any case. □

The assumption that G does not have parabolic elements seems unnecessary, but it is probably much trickier to prove that each equivalence class of $\sim$ is non-separating under the existence of parabolic elements. It is also not so clear if the action on S^2 we obtained in the above theorem is always a convergence group action.

From what we have seen, it is conceivable that G contains a subgroup of the form $H \rtimes \mathbb{Z}$ where H is a pants-like COL_3 group and $\mathbb{Z}$ is generated by a pseudo-Anosov-like element (unless G itself is a pants-like COL_3 group). Maybe one can hope the following conjecture to be true (possibly modulo Cannon's conjecture [6]).

Conjecture 8.8 Let G be a finitely generated torsion-free discrete subgroup of $\mathrm{Homeo}_+(S^1)$. Then G is virtually a pants-like COL_2 group with loose laminations if and only if G is virtually a hyperbolic 3–manifold group.

If G is a hyperbolic 3–manifold group, then Agol's virtual fibering theorem in [11] says that G has a subgroup of finite index that fibers over the circle. Hence the result of Section 3 implies that such a subgroup is COL_2. The laminations we have are stable and unstable laminations of a pseudo-Anosov map of a hyperbolic surface, hence they form a pants-like collection of two very full laminations. This proves one direction of the conjecture. Cannon's conjecture says that if a word-hyperbolic group with ideal boundary homeomorphic to S^2 acts on its boundary faithfully, then the group is a Kleinian group. To prove the converse of Conjecture 8.8 requires one to show that a pants-like COL_2 group has a subgroup of finite index that is word-hyperbolic and acts on S^2 as a convergence group.

9 Future directions

We have seen that having two or three very full laminations restricts the dynamics of the group action quite effectively. One can still study what we can conclude about the group when we have mere dense laminations (not necessarily very full). In any case, the most interesting question is about the difference between having two laminations and three laminations. Thurston conjectured that tautly foliated 3–manifold groups are strictly COL_2 (we know that they are COL_2). Hence, one can ask following questions:

Question 9.1 What algebraic properties of a group G we could deduce from the assumption that G is strictly COL_2, ie COL_2 but not COL_3?

Question 9.2 Precisely which 3–manifold groups are strictly COL_2?

More ambitiously, one may ask:

Question 9.3 Can one construct an interesting geometric object like a taut foliation or an essential lamination in a 3–manifold M if we know $\pi_1(M)$ is strictly COL_2?

It would be also interesting if one can characterize the difference between strictly COL_1 groups and COL_2 groups. The example of a strictly COL_1 group we constructed suggests that in order to be COL_2, a group should not have too many homeomorphisms with irrational rotation number. It is conceivable that the way the example is constructed is essentially the only way to get the strictly COL_1 property.

Another important direction would be to classify all possible COL_n–representations of an abstract group G. This is related to the classification of all circular orderings on G. The author is preparing a paper about the action of the automorphisms of G on the space of all circular orderings which G can admit. For example, $\mathrm{Aut}(G)$ acts faithfully on the space of circular orderings of G if G is residually torsion-free nilpotent.

We also remark that the virtual fibering theorem of Agol and the universal circle theorem for the fibering case together imply that the following conjecture holds if Cannon's conjecture holds.

Conjecture 9.4 Let G be a word-hyperbolic group whose ideal boundary is homeomorphic to a 2–sphere, and suppose that G acts faithfully on its boundary. Then G is virtually COL_2.

At the end of the last section, we formulated a conjecture about being virtually a 3–manifold group that fibers over S^1. In the view of Vlad Markovic's recent work [12], pants-like COL_3 subgroups of a pants-like COL_2 word-hyperbolic group G are good candidates for quasi-convex codimension-1 subgroups whose limit sets separate pairs of points in the boundary of G.

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