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Generating the Johnson filtration

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For $k \geq 1$, let $\mathcal{I}_g^1(k)$ be the k^{th} term in the Johnson filtration of the mapping class group of a genus g surface with one boundary component. We prove that for all $k \geq 1$, there exists some $G_k \geq 0$ such that $\mathcal{I}_g^1(k)$ is generated by elements which are supported on subsurfaces whose genus is at most G_k. We also prove similar theorems for the Johnson filtration of $\mathrm{Aut}(F_n)$ and for certain mod-p analogues of the Johnson filtrations of both the mapping class group and of $\mathrm{Aut}(F_n)$. The main tools used in the proofs are the related theories of FI–modules (due to the first author with Ellenberg and Farb) and central stability (due to the second author), both of which concern the representation theory of the symmetric groups over $\mathbb{Z}$.

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1 Introduction

In this paper, we use techniques from representation theory to prove that the terms of the Johnson filtrations of both the mapping class group and the automorphism group of a free group are generated by elements whose complexity is bounded in a sense to be made precise below.

Mapping class group Let Σ_g^k denote a compact oriented genus-g surface with k boundary components. Let Mod_g^1 be the *mapping class group* of Σ_g^1, ie the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g^1 that restrict to the identity on $\partial\Sigma_g^1$.

Choosing a basepoint $* \in \partial\Sigma_g^1$, the group Mod_g^1 acts on $\pi := \pi_1(\Sigma_g^1, *)$. For a group G, let $\gamma_k(G)$ be the k^{th} term in the lower central series of G, so $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [\gamma_k(G), G]$ for $k \geq 1$. The action of Mod_g^1 on π preserves $\gamma_k(\pi)$, so there is an induced action of Mod_g^1 on $\pi/\gamma_k(\pi)$. The k^{th} term of the *Johnson filtration* of Mod_g^1, denoted $\mathcal{I}_g^1(k)$, is the kernel of the action of Mod_g^1 on $\pi/\gamma_{k+1}(\pi)$. The Johnson filtration was defined by Johnson in [18] and has connections to number theory (see Matsumoto [23]) and 3–manifolds (see Garoufalidis and Levine [12]); however, many basic questions about it remain open.

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Generators in low degree Let $T_x \in \mathrm{Mod}_g^1$ denote the Dehn twist about a simple closed curve x on Σ_g^1. It was proved independently by Lickorish [21] and Mumford [24], building on the work of Dehn, that Mod_g^1 is generated by Dehn twists about nonseparating simple closed curves.

Let $\mathcal{I}_g^1 := \mathcal{I}_g^1(1)$. The group $\mathcal{I}_g^1$ is known as the *Torelli group*; it is the kernel of the action of Mod_g^1 on $\pi/\gamma_2(\pi) \cong \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$. A *genus-$\ell$ bounding pair map* is a product $T_y T_z^{-1}$, where y and z are disjoint nonseparating simple closed curves on Σ_g^1 whose union $y \cup z$ separates Σ_g^1 into two subsurfaces, one homeomorphic to Σ_ℓ^2 and the other to $\Sigma_{g-\ell-1}^3$ (see Figure 1). Making essential use of work of Powell [27], Johnson [15] proved that $\mathcal{I}_g^1$ is generated by genus-1 bounding pair maps for $g \geq 3$. See [28] and Hatcher and Margalit [13] for modern proofs of the necessary results of Powell.

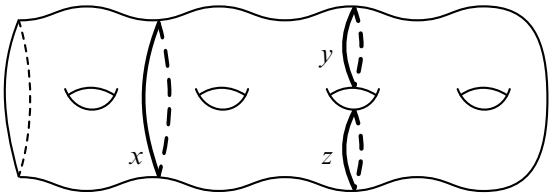

Figure 1: A genus-3 separating twist T_x and a genus-1 bounding pair map $T_y T_z^{-1}$

The group $\mathcal{I}_g^1(2)$ is known as the *Johnson kernel*. A *genus-ℓ separating twist* is a mapping class T_x, where x is a simple closed curve that separates Σ_g^1 into two subsurfaces, one homeomorphic to Σ_ℓ^1 and the other to $\Sigma_{g-\ell}^2$ (see Figure 1). Johnson [19] proved that $\mathcal{I}_g^1(2)$ is generated by genus 1 and 2 separating twists.

Higher degree For $k \geq 3$, no interesting generating set for $\mathcal{I}_g^1(k)$ is known (of course, one could do uninteresting things like taking the entire group as a generating set). An appealing feature of the generating sets above is that the generators are "simple", in the sense that they are supported on small subsurfaces (ie subsurfaces with 1 boundary component and bounded genus). Our first main theorem says that for every $k \geq 1$ the group $\mathcal{I}_g^1(k)$ can be generated by elements supported on subsurfaces of uniformly bounded size.

In fact, we can be somewhat more precise. Fix a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, ie a free basis such that

$$\widehat{i}(a_i, a_j) = \widehat{i}(b_i, b_j) = 0 \quad \text{and} \quad \widehat{i}(a_i, b_j) = \delta_{ij},$$

where $\widehat{i}(\cdot,\cdot)$ is the algebraic intersection pairing. Say that a subsurface S of Σ_g^1 is *homologically standard* if S has one boundary component and the image of $\mathrm{H}_1(S;\mathbb{Z})$

in $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$ is $\langle a_i, b_i \mid i \in I\rangle$ for some $I \subset \{1,\dots,g\}$. Our theorem is then as follows.

Theorem A (Generators for Johnson filtration) *For every $k \geq 1$, there exists some $G_k \geq 0$ such that for all $g \geq 1$, the group $\mathcal{I}_g^1(k)$ is generated by elements which are supported on homologically standard subsurfaces of Σ_g^1 whose genus is at most G_k.*

Remark 1.1 We emphasize that the constant G_k in Theorem A depends only on k, not on g. Otherwise, the theorem would be rather trivial!

Somewhat surprisingly, our proof of Theorem A is purely an existence proof; it gives no information about how large the constants G_k must be. The following theorem, however, implies that the bounds G_k must tend to infinity.

Theorem B (Lower bound on genus) *For all $k \geq 1$ and $g > k$, the group $\mathcal{I}_g^1(k)$ is not generated by elements supported on subsurfaces with one boundary component and genus less than $k/2$.*

Automorphism groups of free groups The Johnson filtration can also be defined on the automorphism group $\mathrm{Aut}(F_n)$ of the free group $F_n = \langle x_1,\dots,x_n\rangle$. Let $\mathrm{IA}_n(k)$ denote the kernel of the action of $\mathrm{Aut}(F_n)$ on $F_n/\gamma_{k+1}(F_n)$. The group $\mathrm{IA}_n := \mathrm{IA}_n(1)$ consists of automorphisms in $\mathrm{Aut}(F_n)$ acting trivially on $F_n/\gamma_2(F_n) \cong \mathbb{Z}^n$, and is often known as the Torelli subgroup of $\mathrm{Aut}(F_n)$. Magnus found a finite generating set for IA_n consisting of the following two types of elements.

- For distinct $1 \leq i, j \leq n$, let $c_{ij} \in \mathrm{IA}_n$ be the automorphism defined by
$$c_{ij}(x_\ell) = \begin{cases} x_j^{-1} x_\ell x_j & \text{if } \ell = i, \\ x_\ell & \text{otherwise.} \end{cases}$$
- For distinct $1 \leq i, j, k \leq n$, let $m_{ijk} \in \mathrm{IA}_n$ be the automorphism defined by
$$m_{ijk}(x_\ell) = \begin{cases} x_\ell [x_j, x_k] & \text{if } \ell = i, \\ x_\ell & \text{otherwise.} \end{cases}$$

Magnus [22] proved that IA_n is generated by the automorphisms c_{ij} and m_{ijk}; see Bestvina, Bux and Margalit [2] and Day and Putman [8] for modern proofs of Magnus's theorem. For $k \geq 2$, a generating set for $\mathrm{IA}_n(k)$ is not known.

Subsurfaces for free groups To state a version of Theorem A for $\mathrm{IA}_n(k)$, we need an appropriate analogue of "supported on a subsurface" for $\mathrm{Aut}(F_n)$. A *splitting* of F_n consists of subgroups $A, B < F_n$ such that F_n splits as the free product $F_n = A * B$. The *rank* of a splitting $A * B$ is the rank of the free group A (notice that this is different

from the rank of the splitting $B * A$). We will say that an element $\varphi \in \mathrm{Aut}(F_n)$ is *supported* on a splitting $A * B$ if $\varphi(A) = A$ and $\varphi|_B = \mathrm{id}$. For example, Magnus's generator c_{ij} is supported on a splitting of rank 2, and m_{ijk} is supported on a splitting of rank 3. We will prove that for all $k \geq 1$, the group $\mathrm{IA}_n(k)$ is generated by elements supported on splittings whose rank is uniformly bounded.

Just as for the mapping class group, we will actually prove something a bit more precise. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$. Say that a splitting $A * B$ of F_n is *homologically standard* if there is some $I \subset \{1, \ldots, n\}$ such that the images of A and B in F_n^{ab} are $A^{\mathrm{ab}} = \langle e_i \mid i \in I \rangle$ and $B^{\mathrm{ab}} = \langle e_i \mid i \notin I \rangle$. We then have the following theorem.

Theorem C (Generators for Johnson filtration of $\mathrm{Aut}(F_n)$) *For every $k \geq 1$, there exists some $N_k \geq 0$ such that for all $n \geq 1$, the group $\mathrm{IA}_n(k)$ is generated by elements which are supported on homologically standard splittings whose rank is at most N_k.*

We will also prove the following analogue of Theorem B.

Theorem D (Lower bound on rank) *For all $k \geq 1$ and $n > k$, the group $\mathrm{IA}_n(k)$ is not generated by elements supported on splittings of rank less than k.*

Mod-p lower central series Fix a prime p. In recent work [7], Cooper has introduced two mod-p analogues of the Johnson filtration. The starting points are two different mod-p analogues of the lower central series of a group G. If G' is a subgroup of G and $\ell \geq 1$, then denote by $(G')^{\ell}$ the subgroup of G generated by $\{x^{\ell} \mid x \in G'\}$.

- The *mod-p Stallings filtration* of G is the inductively defined filtration

$$\gamma_1^S(G) = G \quad \text{and} \quad \gamma_{k+1}^S(G) = [\gamma_k^S(G), G] \cdot (\gamma_k^S(G))^p \quad \text{for } k \geq 1.$$

  This filtration first appeared in Stallings [33].

- The *mod-p Zassenhaus filtration* of G is defined in terms of the usual lower central series via the formula

$$\gamma_k^Z(G) = \prod_{ip^j \geq k} (\gamma_i(G))^{p^j}.$$

  This filtration first appeared in Zassenhaus [37].

If G is finitely generated, the quotients $G/\gamma_k^S(G)$ and $G/\gamma_k^Z(G)$ are both finite p–groups. We have

$$G/\gamma_2^Z(G) \cong G/\gamma_2^Z(G) \cong \mathrm{H}_1(G; \mathbb{Z}/p);$$

however, for $k \geq 3$ these two filtrations differ.

Mod-p Johnson filtrations We define $\mathcal{I}_g^{1,S}(k)$ and $\mathcal{I}_g^{1,Z}(k)$ to be the kernels of the actions of Mod_g^1 on $\pi/\gamma_{k+1}^S(\pi)$ and $\pi/\gamma_{k+1}^Z(\pi)$, respectively. Observe that both $\mathcal{I}_g^{1,Z}(1)$ and $\mathcal{I}_g^{1,S}(1)$ coincide with the *level-p congruence subgroup* $\mathrm{Mod}_g^1(p)$, that is, the kernel of the action of Mod_g^1 on $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z}/p)$. All the groups $\mathcal{I}_g^{1,S}(k)$ and $\mathcal{I}_g^{1,Z}(k)$ in these filtrations are finite-index subgroups of Mod_g^1.

Similarly, we define $\mathrm{IA}_n^S(k)$ and $\mathrm{IA}_n^Z(k)$ to be the kernels of the actions of $\mathrm{Aut}(F_n)$ on $F_n/\gamma_{k+1}^S(F_n)$ and $F_n/\gamma_{k+1}^Z(F_n)$, respectively. Both $\mathrm{IA}_n^S(1)$ and $\mathrm{IA}_n^Z(1)$ coincide with the *level-p congruence subgroup* $\mathrm{Aut}(F_n, p)$, that is, the kernel of the action of $\mathrm{Aut}(F_n)$ on $\mathrm{H}_1(F_n;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^n$. Again, all of the terms in these filtrations are finite-index subgroups of $\mathrm{Aut}(F_n)$.

Remark 1.2 Yet another mod-p Johnson filtration was defined by Perron in [26] using the Fox calculus, but Cooper [7] proved that Perron's filtration equals the Zassenhaus filtration.

Generators for mod-p Johnson filtrations Cooper [7] proved many interesting results about these filtrations. In particular, he found simple generating sets for $\mathcal{I}_g^{1,S}(k)$ and $\mathcal{I}_g^{1,Z}(k)$ for $k = 1$ and $k = 2$. We are able to prove analogues of Theorems A and C for these filtrations. Let $\{a_1, b_1, \dots, a_g, b_g\}$ the standard symplectic basis for $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z}/p)$. Say that a subsurface S of Σ_g^1 is *$\mathbb{Z}/p$–homologically standard* if S has one boundary component and the image of $\mathrm{H}_1(S;\mathbb{Z}/p)$ in $\mathrm{H}_1(\Sigma_g^1;\mathbb{Z}/p)$ is $\langle a_i, b_i \mid i \in I\rangle$ for some $I \subset \{1,\dots,g\}$. We then have the following.

Theorem E (Generators for mod-p Johnson filtrations) *Fix a prime p. For all $k \geq 1$, there exists some $G_k \geq 0$ (depending on p) such that for all $g \geq 1$, both $\mathcal{I}_g^{1,S}(k)$ and $\mathcal{I}_g^{1,Z}(k)$ are generated by elements which are supported on a $\mathbb{Z}/p$–homologically standard subsurface of Σ_g^1 of genus at most G_k.*

Similarly, let $\{e_1,\dots,e_n\}$ be the standard basis for $\mathrm{H}_1(F_n;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^n$. Say that a splitting $A * B$ of F_n is *$\mathbb{Z}/p$–homologically standard* if for some $I \subset \{1,\dots,n\}$, the images of A and B in $\mathrm{H}_1(F_n;\mathbb{Z}/p)$ are $\mathrm{H}_1(A;\mathbb{Z}/p) = \langle e_i \mid i \in I\rangle$ and $\mathrm{H}_1(B;\mathbb{Z}/p) = \langle e_i \mid i \notin I\rangle$. We then have the following.

Theorem F (Generators for mod-p Johnson filtrations of $\mathrm{Aut}(F_n)$) *Fix a prime p. For all $k \geq 1$, there exists some $N_k \geq 0$ (depending on p) such that for all $n \geq 1$, both $\mathrm{IA}_n^S(k)$ and $\mathrm{IA}_n^Z(k)$ are generated by elements which are supported on a $\mathbb{Z}/p$–homologically standard splitting of F_n of rank at most N_k.*

Central stability Though our theorems concern topology and infinite group theory, the main tools used in their proofs concern the representation theory of the symmetric group. In particular, we use the notion of *central stability* for representations of the symmetric group, which was introduced by the second author in [30] to study the homology groups of congruence subgroups of $\mathrm{GL}_n(\mathbb{Z})$. Roughly speaking, this allows us to give an inductive description of the images of the *higher Johnson homomorphisms*, which are an important sequence of abelian quotients of the terms of the Johnson filtrations. The key advance that makes this possible is the recent theorem of the first author with Ellenberg, Farb and Nagpal [6], which establishes a Noetherian property for FI–modules over $\mathbb{Z}$. This theorem allows one to prove that certain sequences of representations are centrally stable almost for free (in particular, with no detailed understanding of their structure, which seems quite hard to achieve for the images of the higher Johnson homomorphisms).

FI–groups To formulate the technical framework for our arguments, we introduce FI–groups and weak FI–groups. An FI–group G consists of a group G_I for each finite subset $I \subset \mathbb{N}$, together with homomorphisms $G_I \to G_J$ for each injection $I \hookrightarrow J$ satisfying some natural compatibility conditions (see Definition 2.1 below). A weak FI–group consists of similar data, except that for some of these homomorphisms, we require only that they be compatible up to conjugacy. The main technical result of the paper is the following theorem. The terms involved have not yet been defined; see Section 2.1 below for their definitions.

Theorem G (Bounded generation for central filtrations) *Let G be a weak FI–group with a central filtration $\{G(k)\}_{k=1}^{\infty}$ of finite rank. If G is boundedly generated, then $G(k) \lhd G$ is boundedly normally generated for every $k \geq 1$.*

Generating sets for Torelli To apply Theorem G to the Torelli groups $\mathcal{I}_g^1$, we need a strengthening of a recent theorem of the second author [29] concerning generating sets for $\mathcal{I}_g^1$. Johnson [17] proved that the Torelli groups $\mathcal{I}_g^1$ are finitely generated for $g \geq 3$ with a generating set whose size is exponential in g. Johnson [20] also proved that the rank of the abelianization of $\mathcal{I}_g^1$ is cubic in g, which gives a lower bound on the size of any generating set for $\mathcal{I}_g^1$.

The second author's theorem [29] says that $\mathcal{I}_g^1$ is generated by $57\binom{g}{3} + 2g + 1$ elements for $g \geq 3$. What is important to us is not the size of his generating set per se, but rather the fact that his generators are supported on fairly simple subsurfaces of Σ_g^1: each element is supported on a genus-3 subsurface with multiple boundary components. To prove Theorem A, we need to improve this generating set slightly, so that the generators are supported on $\binom{g}{3}$ different genus-3 subsurfaces with only *one* boundary component.

We refer to Proposition 4.5 below for a precise description of our new generating set, but we point out the following corollary. Denote by $\mathcal{I}_g$ the Torelli group on a closed genus-g surface.

Theorem H *For all $g \geq 3$, the groups $\mathcal{I}_g$ and $\mathcal{I}_g^1$ are each generated by $42\binom{g}{3}$ elements.*

Outline In Section 2, we introduce FI–groups, weak FI–groups, and their central filtrations; the main result of this section is Theorem G. In Section 3 we show how to apply this to the automorphism group of a free group and prove Theorem C. In Section 4 we show how to apply this to the mapping class group and prove Theorem A; to do this, we first prove Theorem H. Next, in Section 5 we discuss how to modify our proof of Theorems C and A to prove Theorems E and F. Finally, in Section 6 we prove Theorems B and D.

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2 FI–groups and their central filtrations

This section contains all our general results on FI–groups and weak FI–groups. The key result is Theorem G, which we will later apply to prove Theorems A, C, E and F.

We begin in Section 2.1 with general definitions, including all the definitions that are used in the statement of Theorem G. We then discuss some technical results in Section 2.2. In Section 2.3, we describe the related theories of central stability and FI–modules as they will be used in this paper. Finally, we prove Theorem G in Section 2.4.

2.1 FI–groups and weak FI–groups

In this section, we introduce FI–groups, weak FI–groups, and central filtrations of weak FI–groups, leading up to the statement of the key Theorem G.

FI–groups Let $\mathbb{N}$ be the set of natural numbers, and let $\mathtt{FI}$ be the category whose objects are finite subsets of $\mathbb{N}$ and whose morphisms are injections. Let $\mathtt{Grp}$ be the category of groups and homomorphisms.

Definition 2.1 An *FI–group* is a functor from `FI` to `Grp`. In other words, an FI–group G consists of the following data:

(i) For each finite set $I \subset \mathbb{N}$, a group G_I.

(ii) For each injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, a homomorphism $G_f\colon G_I \to G_J$. These homomorphisms must satisfy the following compatibility conditions:
   (a) For all finite sets $I \subset \mathbb{N}$, we have $G_{\mathrm{id}_I} = \mathrm{id}$, where $\mathrm{id}_I\colon I \to I$ is the identity.
   (b) For all finite sets $I, J, K \subset \mathbb{N}$ and all injections $f\colon I \hookrightarrow J$ and $g\colon J \hookrightarrow K$, we have $G_{g\circ f} = G_g \circ G_f$.

A morphism $\Psi\colon G \to H$ of FI–groups is a natural transformation of functors. In other words, Ψ consists of a homomorphism $\Psi_I\colon G_I \to H_I$ for each finite set $I \subset \mathbb{N}$, so that for every injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} G_I & \xrightarrow{G_f} & G_J \\ {\scriptstyle \Psi_I}\downarrow & & \downarrow{\scriptstyle \Psi_J} \\ H_I & \xrightarrow{H_f} & H_J \end{array}$$

The morphism Ψ is an isomorphism (resp. an injection, resp. a surjection) if Ψ_I is an isomorphism (resp. an injection, resp. a surjection) for all finite sets $I \subset \mathbb{N}$.

Remark 2.2 FI–groups (and the related notion of FI–modules; see Section 2.2 below) were originally defined by the first author with Ellenberg and Farb in [5]. The definitions in that paper were slightly different from ours, in that in [5] the category `FI` had *all* finite sets as its objects; however, this larger category is equivalent to our category.

Remark 2.3 Let $[n] = \{1, \dots, n\}$. For each bijection $\sigma\colon [n] \to [n]$, we have a homomorphism $G_\sigma\colon G_{[n]} \to G_{[n]}$. Together these give an action of the symmetric group $\mathfrak{S}_n$ on $G_{[n]}$.

Weak FI–groups In Section 3, we will see that the automorphism groups of free groups can be naturally viewed as an FI–group. Unfortunately, the mapping class groups of surfaces do not form an FI–group. However, they do satisfy a weaker form of functoriality that is sufficient for our purposes.

If A and B are groups, then B acts by conjugation on the set of homomorphisms from A to B. A *homomorphism-modulo-conjugacy* is an equivalence class of homomorphisms

under this action. Homomorphisms-modulo-conjugacy can be composed (by composing representatives), so there is a category `CGrp` of groups and homomorphisms-modulo-conjugacy. Given a pair of finite sets $I \subset J \subset \mathbb{N}$, let $i_I^J\colon I \hookrightarrow J$ denote the inclusion.

Definition 2.4 A *weak FI–group* G consists of the following data:

(i) For each finite set $I \subset \mathbb{N}$, a group G_I.

(ii) For each injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, a homomorphism-modulo-conjugacy $G_f\colon G_I \to G_J$. These homomorphisms-modulo-conjugacy must satisfy the following compatibility conditions:
  (a) For all finite sets $I \subset \mathbb{N}$, we have $G_{\mathrm{id}_I} = \mathrm{id}$.
  (b) For all finite sets $I, J, K \subset \mathbb{N}$ and all injections $f\colon I \hookrightarrow J$ and $g\colon J \hookrightarrow K$, we have $G_{g\circ f}$ equal to $G_g \circ G_f$ in `CGrp`.

(iii) For each pair of finite sets $I \subset J \subset \mathbb{N}$, a homomorphism $G_I^J\colon G_I \to G_J$; these homomorphisms must satisfy the following compatibility conditions:
  (a) For all pairs of finite sets $I \subset J \subset \mathbb{N}$, the homomorphism-modulo-conjugacy $G_{i_I^J}$ is represented by the homomorphism G_I^J.
  (b) For all triples of finite sets $I \subset J \subset K \subset \mathbb{N}$, we have $G_J^K \circ G_I^J = G_I^K$.

In Section 4, we will see that the mapping class groups of surfaces with one boundary component can be naturally viewed as a weak FI–group.

Remark 2.5 Every FI–group G can be canonically considered as a weak FI–group, by considering the homomorphisms $G_f\colon G_I \to G_J$ only as homomorphisms-modulo-conjugacy (and setting $G_I^J := G_{i_I^J}$). The conditions of Definition 2.1 imply that all the conditions of Definition 2.4 are satisfied. Throughout the paper, wherever necessary we consider FI–groups as weak FI–groups via this "forgetful" process. As a result, many of our technical results and definitions will be stated for weak FI–groups, but they apply equally well to FI–groups.

Remark 2.6 The conditions on the G_f in Definition 2.4(ii) are equivalent to the assertion that they piece together to give a functor from `FI` to `CGrp`. In [5], such functors were called "FI–groups up to conjugacy". Similarly, the conditions on the homomorphisms G_I^J in Definition 2.4(iii) are equivalent to the assertion that they piece together to give a functor from the category of finite subsets of $\mathbb{N}$ and *inclusions* to `Grp`.

FI–modules An *FI–module* is an FI–group W such that W_I is an abelian group for all finite sets $I \subset \mathbb{N}$. We say that an FI–module W has *finite rank* if for all finite sets $I \subset \mathbb{N}$, the abelian group W_I is finitely generated.

Remark 2.7 FI–modules were originally defined by the first author with Ellenberg and Farb in [5], and we refer the reader to [5] for many examples of them. The paper [5] considers FI–modules over an arbitrary ring R; in that language, our FI–modules are FI–modules over the ring $\mathbb{Z}$. Observe that there would be no point in defining "weak FI–modules"; indeed, since homomorphisms-modulo-conjugacy coincide with homomorphisms when the groups involved are abelian, the conditions of Definition 2.4 reduce to the conditions of Definition 2.1 in this case.

Normal weak FI–subgroups Let A and B be groups, and let $f\colon A \to B$ be some homomorphism-modulo-conjugacy. Observe that if $N \lhd A$ is a normal subgroup, the subgroup $f(N) \subset B$ is well defined, even though f is not a well-defined homomorphism.

Definition 2.8 Let G be either an FI–group or a weak FI–group. A *normal weak FI–subgroup* H of G, denoted $H \lhd G$, consists of a normal subgroup $H_I \lhd G_I$ for each finite set $I \subset \mathbb{N}$ satisfying the following property.

- For all injections $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, $G_f(H_I) \subset H_J$.

Given $H \lhd G$ and $H' \lhd G$, we write $H \subset H'$ if $H_I \subset H'_I$ for all finite sets $I \subset \mathbb{N}$.

Remark 2.9 By the remark preceding Definition 2.8, the fact that $H_I \lhd G_I$ guarantees that the condition $G_f(H_I) \subset H_J$ is well defined, even when G is only a weak FI–group. This issue is the reason we do not define nonnormal weak FI–subgroups.

Remark 2.10 If G is an FI–group and $H \lhd G$, then H is itself an FI–group. However, we warn the reader that if G is only a weak FI–group and $H \lhd G$, then H is *not* necessarily a weak FI–group. The reason is that a homomorphism-modulo-conjugacy $G_I \to G_J$ cannot be restricted to a homomorphism-modulo-conjugacy $H_I \to H_J$, since homomorphisms conjugate by an element of G_J need not be conjugate by an element of its subgroup H_J.

Bounded generation The notion of bounded generation, which we define in this subsection, captures the idea that an FI–group (or weak FI–group) is generated by elements "supported on subsets of bounded size".

Definition 2.11 Let G be a weak FI–group, and $H \lhd G$. Given a pair of finite sets $I \subset J \subset \mathbb{N}$, we denote by $H_J(I)$ the image $H_J(I) := G_I^J(H_I) \subset H_J$.

One should regard $H_J(I)$ as the subgroup of H_J which is "supported on the subset I". Given $I \subset K \subset J$, the identity $G_I^J = G_K^J \circ G_I^K$ implies that $H_J(I) \subset H_J(K)$. Taking $H = G$, we have $G_J(I) := G_I^J(G_I) \subset G_J$.

Definition 2.12 Let G be a weak FI–group. Given $A \geq 0$, we say that G is *boundedly generated in degree* A if for all finite sets $J \subset \mathbb{N}$,

(2-1) G_J is generated by its subgroups $G_J(I)$ for those $I \subset J$ satisfying $|I| \leq A$.

We say that G is *boundedly generated* if (2-1) holds for some $A \geq 0$.

Lemma 2.13 *Let G be an FI–group. Fix $A \geq 0$, and assume that for all $n \in \mathbb{N}$, the condition* (2-1) *holds for the set $J = [n]$. Then G is boundedly generated in degree A.*

Proof Given any set $J \subset \mathbb{N}$, let $n = |J|$, and choose a bijection $f\colon [n] \to J$. Given any $I \subset J$, set $I' := f^{-1}(I) \subset [n]$. By Definition 2.11 we have

$$G_f(G_{[n]}(I)) = G_f(G^{[n]}_{I'}(G_{I'})) = G^J_I(G_f(G_{I'})) = G^J_I(G_I) = G_J(I),$$

where the equality $G_f \circ G^{[n]}_{I'} = G^J_I \circ G_f$ holds because G is an FI–group. Therefore the condition (2-1) for J follows from condition (2-1) for $[n]$. □

Remark 2.14 When G is a weak FI–group it is *not enough* to check (2-1) for $J = [n]$. The proof of Lemma 2.13 breaks down not just because $G_f \circ G^{[n]}_{I'} = G^J_I \circ G_f$ need not hold, but because $G_f(G_{[n]}(I))$ is not even a well-defined subgroup. The best we could conclude is that G_J is *normally* generated by the subgroups $G_J(I)$ with $|I| \leq A$, a far weaker condition. Indeed, choosing the homomorphisms G^J_I so that a given weak FI–group is boundedly generated can be quite delicate. This issue is the main reason that we must be so careful in Section 4 when making the Torelli group into a weak FI–group.

Definition 2.15 Let G be a weak FI–group, and let $H \lhd G$ be a normal weak FI–subgroup. Given $B \geq 0$, we say that $H \lhd G$ is *boundedly normally generated in degree* B if for all finite sets $J \subset \mathbb{N}$,

(2-2) H_J is generated by the G_J–conjugates of its subgroups $H_J(I)$ for those $I \subset J$ satisfying $|I| \leq B$.

We say that $H \lhd G$ is *boundedly normally generated* if this holds for some $B \geq 0$.

Remark 2.16 The condition (2-2) is vacuous for $|J| \leq B$, since $H_J = H_J(J)$; similarly the condition (2-1) is vacuous for $|J| \leq A$.

Central filtrations Let G be a weak FI–group. Given $H \lhd G$, we can define $[G, H]$ via the formula $[G, H]_I = [G_I, H_I]$ for finite sets $I \subset \mathbb{N}$; it is easy to check that $[G, H] \lhd G$ and $[G, H] \subset H$.

Definition 2.17 Let G be a weak FI–group. A *central filtration* of G consists of normal weak FI–subgroups $G(k) \lhd G$ for each $k \geq 1$ satisfying

$$G = G(1) \supset G(2) \supset \cdots \supset G(k) \supset G(k+1) \supset \cdots$$

and $[G, G(k)] \subset G(k+1)$ for all $k \geq 1$. This latter condition implies $G(k)_I / G(k+1)_I$ is an abelian group for all finite sets $I \subset \mathbb{N}$, and we say that our central filtration is of *finite rank* if the abelian group $G(k)_I / G(k+1)_I$ is finitely generated for all $k \geq 1$ and all finite sets $I \subset \mathbb{N}$.

We can now state our main technical theorem, which we will prove in Section 2.4 below.

Theorem G *Let G be a weak FI–group with a central filtration $\{G(k)\}_{k=1}^{\infty}$ of finite rank. If G is boundedly generated, then $G(k) \lhd G$ is boundedly normally generated for every $k \geq 1$.*

2.2 Technical results about FI–groups

This section collects a number of technical results about FI–groups that we will need in the proof of Theorem G.

Controlling the support We begin with the following lemma, which allows us to control the support of certain commutators.

Lemma 2.18 *Let G be a weak FI–group with a central filtration $\{G(k)\}_{k=1}^{\infty}$. Fix some $k \geq 1$ and let $I, I', J \subset \mathbb{N}$ be finite sets satisfying $I, I' \subset J$. Consider $w \in G_J(I)$ and $z \in G(k)_J(I')$. Then $[w, z] \in G(k+1)_J(I \cup I')$.*

Lemma 2.18 follows immediately from the inclusions $G_J(I) \subset G_J(I \cup I')$ and $G(k)_J(I') \subset G(k)_J(I \cup I')$ together with the definition of a central filtration.

Subgroups normally generated on sets of a fixed size Let G be a weak FI–group with $H \lhd G$, and fix $N \geq 0$. For each finite set $J \subset \mathbb{N}$, define $H_J^{\leq N}$ to be the subgroup generated by the G_J–conjugates of the subgroups $H_J(I)$ for those $I \subset J$ satisfying $|I| \leq N$. Since $H_J \lhd G_J$ is a normal subgroup, we have $H_J^{\leq N} \subset H_J$.

Lemma 2.19 *Let G be a weak FI–group, let $H \lhd G$ be a normal weak FI–subgroup, and fix $N \geq 0$. Then $H^{\leq N} \lhd G$ is a normal weak FI–subgroup of G.*

Comparing the definition of $H^{\leq N}$ with (2-2), we see that by definition

(2-3) $H \lhd G$ is boundedly normally generated in degree $N \iff H = H^{\leq N}$.

Proof of Lemma 2.19 We must prove that for any injection $f\colon J \hookrightarrow K$ between finite sets $J, K \subset \mathbb{N}$, we have $G_f(H_J^{\leq N}) \subset H_K^{\leq N}$. Choose a homomorphism representing the homomorphism-modulo-conjugacy G_f, which by abuse of notation we also denote G_f. Since $H_K^{\leq N}$ is a normal subgroup of G_K, it is enough to show $G_f(H_J(I)) \subset H_K^{\leq N}$ for all $I \subset J$ with $|I| \leq N$.

Set $I' := f(I) \subset K$ and $f' := f|_I\colon I \to I'$, and choose a representative homomorphism $G_{f'}\colon G_I \to G_{I'}$. Since $f'\colon I \to I'$ is invertible, $G_{f'}$ must be an isomorphism, and restricts to an isomorphism $H_I \to H_{I'}$.

Definition 2.4(ii) implies that $G_f \circ G_I^J$ is G_K–conjugate to $G_{I'}^K \circ G_{f'}$. Therefore

$$\text{(2-4)}\quad G_f(H_J(I)) = G_f(G_I^J(H_I)) \text{ is } G_K\text{–conjugate to } G_{I'}^K(G_{f'}(H_I)) = G_{I'}^K(H_{I'}) = H_K(I').$$

Since $|I'| = |I| \leq N$, certainly $H_K(I')$ is contained in $H_K^{\leq N}$ (being among its normal generators). Since $H_K^{\leq N}$ is normal in G_K, any G_K–conjugate of this subgroup is also contained in $H_K^{\leq N}$. We conclude that $G_f(H_J(I)) \subset H_K^{\leq N}$, as desired. □

The graded quotients of a central filtration In this paper, the key examples of FI–modules are the graded quotients of a central filtration of a weak FI–group. The following lemma asserts that these do indeed form FI–modules.

Lemma 2.20 *Let G be a weak FI–group and let $\{G(k)\}_{k=1}^{\infty}$ be a central filtration of G. Fix some $k \geq 1$. For each finite set $I \subset \mathbb{N}$, define $Q(k)_I := G(k)_I / G(k+1)_I$. Then the weak FI–group structure on G induces an FI–module structure on $Q(k)$.*

Lemma 2.20 is a special case of the following more general lemma.

Lemma 2.21 *Let G be a weak FI–group and assume that $K \lhd G$ and $H \lhd G$ satisfy $[G, H] \subset K \subset H$. Then there exists an FI–module Q defined as follows: for each finite set $I \subset \mathbb{N}$ define $Q_I := H_I / K_I$, and for each injection $f\colon I \hookrightarrow J$ let $Q_f\colon Q_I \to Q_J$ be the map induced by $G_f\colon H_I \to H_J$.*

Proof For all finite sets $I \subset \mathbb{N}$, we have $[H_I, H_I] \subset [G_I, H_I] \subset K_I$, so Q_I is an abelian group. It remains to prove that the maps Q_f are well defined, and that they satisfy the conditions of Definition 2.1(ii).

Consider an injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$. The key to the lemma is that, since $[G_J, H_J] \subset K_J$, the conjugation action of G_J on H_J descends to the trivial action on Q_J. Therefore even though $G_f\colon H_I \to H_J$ is only defined

up to G_J–conjugacy, it descends to a well-defined homomorphism $Q_f\colon Q_I \to Q_J$. Given another injection $g\colon J \hookrightarrow K$, Definition 2.4(ii) guarantees that $G_g \circ G_f$ is G_K–conjugate to $G_{g\circ f}\colon H_I \to H_K$. It follows that the induced maps $Q_g \circ Q_f$ and $Q_{g\circ f}\colon Q_I \to Q_K$ coincide, so Q is an FI–group. □

2.3 Central stability and FI–modules

To prove Theorem G, we will need the notion of *central stability*, which was introduced by the second author in [30]. The definitions in [30] were in terms of the representation theory of the symmetric group. Here we give an equivalent definition in the language of FI–modules.

Bounded generation Let W be an FI–module, so all the groups W_I are abelian. In this case, for any finite set $J \subset \mathbb{N}$ we have a map

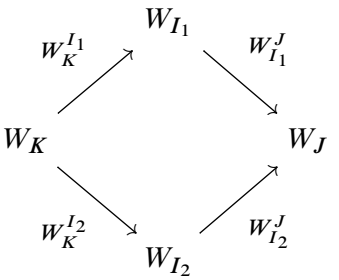
$$\bigoplus_{\substack{I\subset J,\\ |I|\le A}} W_I \longrightarrow W_J \tag{2-5}$$

induced by the homomorphisms $W_I^J\colon W_I \to W_J$. Definition 2.12 says that W is boundedly generated in degree A if (2-5) is surjective for every finite set $J \subset \mathbb{N}$. (In [5, Definition 2.1], the term "generated in degree $\le A$" was used instead.)

Central stabilization Let W be an FI–module, and consider some finite set $J \subset \mathbb{N}$. We have a homomorphism

$$\psi\colon \bigoplus_{\substack{I\subset J,\\ |I|=|J|-1}} W_I \longrightarrow W_J.$$

If $|J| > A$, then the map (2-5) factors through ψ, so ψ is surjective if W is boundedly generated in some degree less than $|J|$. We wish to understand the kernel of ψ. One source of elements in $\ker(\psi)$ is as follows. Consider a finite set $K \subset \mathbb{N}$ such that $K \subset J$ and $|K| = |J|-2$. Let $I_1, I_2 \subset \mathbb{N}$ be the two distinct sets satisfying $K \subset I_i \subset J$ and $|I_i| = |J| - 1$. We then have a commutative diagram



There is thus a map $W_K \to \ker(\psi)$ that takes $x \in W_K$ to

$$(W_K^{I_1}(x), -W_K^{I_2}(x)) \in W_{I_1} \oplus W_{I_2} \subset \bigoplus_{|I|=|J|-1} W_I.$$

Collecting all of these maps, we obtain a map

$$\eta\colon \bigoplus_{\substack{K\subset J,\\ |K|=|J|-2}} W_K \longrightarrow \bigoplus_{\substack{I\subset J,\\ |I|=|J|-1}} W_I$$

whose image lies in $\ker(\psi)$. The *J–central stabilization* of W, denoted $\mathcal{C}(W, J)$, is the cokernel of η.

There is a natural homomorphism $\mathcal{C}(W, J) \to W_J$, which is surjective if W is boundedly generated in some degree less than $|J|$. A morphism $\Psi\colon V \to W$ of FI–modules induces a map $\mathcal{C}(V, J) \to \mathcal{C}(W, J)$ consistent with the map $\Psi_J\colon V_J \to W_J$ and the maps $\mathcal{C}(V, J) \to V_J$ and $\mathcal{C}(W, J) \to W_J$.

Central stability We say that an FI–module W is *centrally stable* starting at $E \geq 0$ if for all finite sets $J \subset \mathbb{N}$ with $|J| > E$, the natural map $\mathcal{C}(W, J) \to W_J$ is an isomorphism. This implies in particular that W is boundedly generated in degree E. We say that W is *centrally stable* if it is centrally stable starting at some E. One should think of a centrally stable FI–module as being "finitely presented". The key technical result underpinning this paper is the following theorem of the first author with Ellenberg, Farb and Nagpal. It should be viewed as a "Noetherian" property of FI–modules.

Proposition 2.22 [6, Corollary 2.11] *Let W be a finite-rank FI–module. If W is boundedly generated, then W is centrally stable.*

The power of central stability If W is an FI–module which is centrally stable starting at E, then W is determined by its initial segment of size E, by which we mean the groups W_J for finite sets $J \subset \mathbb{N}$ with $|J| \leq E$ and the maps between these groups. One way of using this is as follows.

Lemma 2.23 *Let $\Psi\colon V \to W$ be a morphism between FI–modules. Assume that W is centrally stable starting at $E \geq 0$, that V is boundedly generated in degree E, and that for all finite sets $J \subset \mathbb{N}$ with $|J| \leq E$, the map $\Psi_J\colon V_J \to W_J$ is an isomorphism. Then Ψ is an isomorphism.*

Proof We will prove that $\Psi_J\colon V_J \to W_J$ is an isomorphism for all finite sets $J \subset \mathbb{N}$ by induction on $|J|$. The base cases are when $|J| \leq E$, where Ψ_J is an isomorphism

by assumption. Assume now that $|J| > E$ and that Ψ_I is an isomorphism for all sets I with $|I| < |J|$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(V,J) & \twoheadrightarrow & V_J \\ \cong\downarrow & & \downarrow\Psi_J \\ \mathcal{C}(W,J) & \xrightarrow{\cong} & W_J. \end{array}$$

The first vertical map $\mathcal{C}(V,J) \to \mathcal{C}(W,J)$ is an isomorphism because Ψ_I is an isomorphism whenever $|I| < |J|$. The first horizontal map is surjective because V is boundedly generated in degree $E < |J|$, and the second horizontal map is an isomorphism because W is centrally stable starting at $E < |J|$. We conclude that Ψ_J is an isomorphism, as desired. □

2.4 Proof of Theorem G

In this section, we prove Theorem G.

Let G be a weak FI–group with a central filtration $\{G(k)\}_{k=1}^{\infty}$ of finite rank. Assume that G is boundedly generated in degree A. Our goal is to prove for each $k \geq 1$ that $G(k) \lhd G$ is boundedly normally generated. Via the equivalence (2-3), we must prove that for each $k \geq 1$ there exists some $B_k \geq 0$ such that $G(k)^{\leq B_k} = G(k)$.

We will prove this by induction on k. In the base case $k = 1$ we have $G = G(1)$, so we may take $B_1 := A$. Now assume that for some fixed $k \geq 1$, we have constructed some $B_k \geq 0$ such that $G(k)^{\leq B_k} = G(k)$. We will find $B_{k+1} \geq 0$ such that $G(k+1)^{\leq B_{k+1}} = G(k+1)$; this will complete the inductive step.

Since the $G(k)$ form a central filtration of G, we know that $[G, G(k)] \subset G(k+1)$. Our first step will be to improve this inclusion.

Claim 1 *For all $N \geq A + B_k$, we have $[G, G(k)] \subset G(k+1)^{\leq N}$.*

Proof of claim We will use the notation $a^b = b^{-1}ab$ and $[a,b] = a^{-1}b^{-1}ab = a^{-1}a^b$. Fix some $N \geq A + B_k$, and consider a finite set $J \subset \mathbb{N}$. By definition, $[G, G(k)]_J$ is generated by the set

$$\{[x,y] \mid x \in G_J,\, y \in G(k)_J\}. \tag{2-6}$$

Our inductive hypothesis says that $G(k)_J^{\leq B_k} = G(k)_J$, so we can write $y \in G(k)_J$ as a product of elements of the set

$$\{z^g \mid g \in G_J,\, z \in G(k)_J(I) \text{ for } I \subset J \text{ with } |I| \leq B_k\}.$$

Repeatedly applying the Witt–Hall commutator identity $[a,bc]=[a,c]\cdot[a,b]^c$, we can therefore express every element of (2-6) as a product of G_J–conjugates of elements of the set

$$\{[x,z^g] \mid x,g\in G_J, z\in G(k)_J(I) \text{ for } I\subset J \text{ with } |I|\le B_k\}. \tag{2-7}$$

Consider some $[x,z^g]$ as in (2-7). We have $[x,z^g]=[x^{g^{-1}},z]^g$. Since G is boundedly generated in degree A, we can write $x^{g^{-1}}\in G_J$ as a product of elements in the set

$$\{w \mid w\in G_J(I') \text{ for some } I'\subset J \text{ with } |I'|\le A\}.$$

Repeatedly applying the Witt–Hall commutator identity $[ab,c]=[a,c]^b\cdot[b,c]$, we can therefore express $[x,z^g]$ as a product of G_J–conjugates of elements of the set

$$\{[w,z] \mid w\in G_J(I') \text{ for } I'\subset J \text{ with } |I'|\le A, z\in G(k)_J(I) \text{ for } I\subset J \text{ with } |I|\le B_k\}. \tag{2-8}$$

In summary, $[G,G(k)]_J$ is generated by the G_J–conjugates of elements in (2-8). By Lemma 2.18, every element in (2-8) lies in $G(k+1)^{\le N}$, so this concludes the proof of Claim 1. □

Lemma 2.20 yields an FI–module $W(k):=G(k)/G(k+1)$; the assumption that the central filtration $\{G(k)\}_{k=1}^{\infty}$ is of finite rank says precisely that the FI–module $W(k)$ is of finite rank. Also, combining Claim 1 with Lemma 2.21, we obtain for any $N\ge A+B_k$ an FI–module $V^N(k):=G(k)/G(k+1)^{\le N}$. We warn the reader that we do *not* yet know that $V^N(k)$ is of finite rank.

Claim 2 *For $N\ge A+B_k$, both $V^N(k)$ and $W(k)$ are boundedly generated in degree B_k.*

Proof of claim Fix $N\ge A+B_k$. Since $W(k)$ is a quotient of $V^N(k)$, it suffices to prove that $V^N(k)$ is boundedly generated in degree B_k. Consider a finite set $J\subset\mathbb{N}$. There is a surjective map $\rho\colon G(k)_J\to V^N(k)_J$. Given $x\in G(k)_J$ and $y\in G_J$, Claim 1 implies that $[x,y]\in G(k+1)^{\le N}=\ker(\rho)$, so $\rho(x)=\rho(y^{-1}xy)$. Our inductive hypothesis says that $G(k)_J=G(k)_J^{\le B_k}$, ie that $G(k)_J$ is generated by the G_J–conjugates of $G(k)_J(I)$ for $|I|\le B_k$. We conclude that $V^N(k)_J=\rho(G(k)_J)$ is generated by

$$\{\rho(G(k)_J(I)) \mid I\subset J, |I|\le B_k\}=\{V^N(k)_J(I) \mid I\subset J, |I|\le B_k\},$$

as desired. This concludes the proof of Claim 2. □

The FI–module $W(k)$ is finite rank by assumption, and it is boundedly generated by Claim 2, so Proposition 2.22 implies that $W(k)$ is centrally stable. Choose B_{k+1} (which we may take to be at least $A + B_k$) such that $W(k)$ is centrally stable starting at B_{k+1}.

We have an FI–module morphism $\pi\colon V^{B_{k+1}}(k) \twoheadrightarrow W(k)$, since $G(k+1)_J^{\leq B_{k+1}} \subset G(k+1)_J$ for any finite set $J \subset \mathbb{N}$. Note that the kernel of $\pi_J\colon V^{B_{k+1}}(k)_J \to W(k)_J$ is isomorphic to $G(k+1)_J / G(k+1)_J^{\leq B_{k+1}}$.

If $|J| \leq B_{k+1}$, by definition we have that $G(k+1)_J^{\leq B_{k+1}} = G(k+1)_J$, so in this case $\pi_J\colon V^{B_{k+1}}(k)_J \to W(k)_J$ is an isomorphism. Moreover $V^{B_{k+1}}(k)$ is boundedly generated in degree $B_k \leq B_{k+1}$ by Claim 2. Applying Lemma 2.23, we conclude that $\pi^{B_{k+1}}\colon V^{B_{k+1}}(k) \twoheadrightarrow W(k)$ is an isomorphism.

We conclude that $\ker(\pi_J) \cong G(k+1)_J / G(k+1)_J^{\leq B_{k+1}}$ is trivial for all finite sets $J \subset \mathbb{N}$. In other words, we have $G(k+1) = G(k+1)^{\leq B_{k+1}}$; by (2-3), this means that $G(k+1)$ is boundedly normally generated in degree B_{k+1}. This finishes the proof of the inductive step, and thus concludes the proof of Theorem G.

Remark 2.24 Theorem G gives no bound whatsoever on the constants B_k, and it is not possible to obtain any such bounds from our proof. The reason is in our use of Proposition 2.22, which rests on the Noetherian property of FI–modules proved in [6, Corollary 2.11]. This property is nonconstructive, since it ultimately relies on the Noetherian property of the ring $\mathbb{Z}$. As a result we have no way to know how large the constant B_{k+1} must be taken in the inductive step.

3 Automorphism groups of free groups

We begin in Section 3.1 by showing how to assemble all the different automorphism groups of free groups into an FI–group. In Section 3.2 we discuss generators for IA_n and prove Theorem C.

3.1 Automorphism groups of free groups as an FI–group

In this section, we show how the automorphism groups of free groups fit together into an FI–group. We also show that a similar result holds for their Torelli subgroups and that the Johnson filtration gives a central filtration of this FI–group.

Automorphism groups of free groups We first define an FI–group AF which collects together the automorphism groups of free groups of different ranks as follows.

- For each finite set $I \subset \mathbb{N}$, let F_I be the free group on the set $\{x_i \mid i \in I\}$ and define $\mathrm{AF}_I = \mathrm{Aut}(F_I)$.
- For each injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, define an injection $\psi_f\colon F_I \hookrightarrow F_J$ via the formula $\psi_f(x_i) = x_{f(i)}$ for $i \in I$. We then define the homomorphism $\mathrm{AF}_f\colon \mathrm{AF}_I \to \mathrm{AF}_J$ via the formula

$$\mathrm{AF}_f(\varphi)(x_j) = \begin{cases} \psi_f \circ \varphi \circ \psi_f^{-1}(x_j) & \text{if } j \in f(I), \\ x_j & \text{if } j \notin f(I). \end{cases} \tag{3-1}$$

It is clear that these homomorphisms AF_f satisfy the compatibility condition in Definition 2.1(ii), so this defines an FI–group AF.

The Johnson filtrations For $k \geq 1$, we define $\mathrm{IA}(k) \lhd \mathrm{AF}$ as follows. For each finite set $I \subset \mathbb{N}$, define $\mathrm{IA}(k)_I \lhd \mathrm{AF}_I$ to be the kernel of the action of $\mathrm{AF}_I = \mathrm{Aut}(F_I)$ on $F_I/\gamma_{k+1}(F_I)$. The following lemma implies that $\mathrm{IA}(k) \lhd \mathrm{AF}$.

Lemma 3.1 *If $I, J \subset \mathbb{N}$ are finite sets and $f\colon I \hookrightarrow J$ is an injection, then we have $\mathrm{AF}_f(\mathrm{IA}(k)_I) \subset \mathrm{IA}(k)_J$.*

Proof We have a natural splitting $F_J = F_{f(I)} * F_{J-f(I)}$. Consider $\varphi \in \mathrm{IA}(k)_I$. Since the injection $\psi_f\colon F_I \to F_J$ takes $\gamma_{k+1}(F_I)$ into $\gamma_{k+1}(F_{f(I)}) \subset \gamma_{k+1}(F_J)$, the automorphism $\mathrm{AF}_f(\varphi)$ acts as the identity on the image of $F_{f(I)}$ in $F_J/\gamma_{k+1}(F_J)$. The automorphism $\mathrm{AF}_f(\varphi)$ also acts as the identity on $F_{J-f(I)}$, and thus certainly acts as the identity on its image in $F_J/\gamma_{k+1}(F_J)$. Since the images of $F_{f(I)}$ and $F_{J-f(I)}$ generate $F_J/\gamma_{k+1}(F_J)$, we conclude that $\mathrm{AF}_f(\varphi) \in \mathrm{IA}(k)_J$. □

Since AF is an FI–group (and not merely a weak FI–group), $\mathrm{IA}(k)$ is itself an FI–group. Note that for the set $[n] \subset \mathbb{N}$ we have $F_{[n]} = F_n$, so $\mathrm{AF}_{[n]} = \mathrm{Aut}(F_n)$, $\mathrm{IA}(1)_{[n]} = \mathrm{IA}_n$, and $\mathrm{IA}(k)_{[n]} = \mathrm{IA}_n(k)$.

Proposition 3.2 *The FI–subgroups $\{\mathrm{IA}(k)\}_{k=1}^{\infty}$ form a central filtration of $\mathrm{IA}(1)$ of finite rank.*

Proof Fix a finite set $I \subset \mathbb{N}$. Since $\gamma_2(F_I) \supset \gamma_3(F_I) \supset \gamma_4(F_I) \supset \cdots$, we have

$$\mathrm{IA}(1) \supset \mathrm{IA}(2) \supset \mathrm{IA}(3) \supset \cdots.$$

For $k \geq 1$, we must show that $[\mathrm{IA}(1)_I, \mathrm{IA}(k)_I] \subset \mathrm{IA}(k+1)_I$ and $\mathrm{IA}(k)_I/\mathrm{IA}(k+1)_I$ is a finite-rank abelian group. Setting $n = |I|$, the evident isomorphism $\mathrm{IA}(1)_I \cong \mathrm{IA}_n(1)$

takes $\mathrm{IA}(k)_I$ to $\mathrm{IA}_n(k)$ for all $k \geq 1$. Our claim is thus equivalent to showing for all $k \geq 1$ that $[\mathrm{IA}_n(1), \mathrm{IA}_n(k)] \subset \mathrm{IA}_n(k+1)$ and that $\mathrm{IA}_n(k)/\mathrm{IA}_n(k+1)$ is a finite-rank abelian group.

For this, we will need the higher Johnson homomorphisms. For all $k \geq 1$, let $\mathcal{L}_k(\mathbb{Z}^n)$ denote the k^{th} graded piece of the free Lie algebra on $\mathbb{Z}^n$. The k^{th} Johnson homomorphism is then a homomorphism $\tau_k\colon \mathrm{IA}_n(k) \to \mathrm{Hom}(\mathbb{Z}^n, \mathcal{L}_{k+1}(\mathbb{Z}^n))$. We will say more about τ_k in Section 6; right now, we only need the following properties (see Satoh [31] for a survey).

(i) The kernel of τ_k equals $\mathrm{IA}_n(k+1)$.

(ii) For $\psi \in \mathrm{IA}_n(1) = \mathrm{IA}_n$ and $\varphi \in \mathrm{IA}_n(k)$, we have $\tau_k(\psi\varphi\psi^{-1}) = \tau_k(\varphi)$.

Property (i) implies $\mathrm{IA}_n(k)/\mathrm{IA}_n(k+1)$ is a subgroup of $\mathrm{Hom}(\mathbb{Z}^n, \mathcal{L}_{k+1}(\mathbb{Z}^n))$, and in particular is a finite-rank abelian group. Property (ii) implies $\tau_k([\mathrm{IA}_n(1), \mathrm{IA}_n(k)]) = 0$, so Property (i) implies $[\mathrm{IA}_n(1), \mathrm{IA}_n(k)] \subset \mathrm{IA}_n(k+1)$. □

3.2 Generating the Torelli subgroup of $\mathrm{Aut}(F_n)$ and its Johnson filtration

As was discussed in the introduction, Magnus [22] gave a finite generating set for IA_n. We will need a corollary of his result. Given a splitting $F = A * B$, recall that an automorphism φ of F is *supported on the splitting* $A * B$ if $\varphi(A) = A$ and $\varphi|_B = \mathrm{id}$. Given a pair of finite sets $I \subset J \subset \mathbb{N}$, it is clear from (3-1) that $\mathrm{AF}_J(I)$ is exactly the subgroup of $\mathrm{AF}_J = \mathrm{Aut}(F_J)$ consisting of automorphisms that are supported on the splitting $F_J = F_I * F_{J-I}$. Similarly, $\mathrm{IA}(k)_J(I) = \mathrm{IA}(k)_J \cap \mathrm{AF}_J(I)$ consists of those automorphisms in $\mathrm{IA}(k)_J$ that are supported on this splitting.

In the case $J = [n]$, we write $\mathrm{Aut}(F_n, I)$ for $\mathrm{AF}_{[n]}(I)$, and we write $\mathrm{IA}_n(I)$ for $\mathrm{IA}(1)_{[n]}(I) = \mathrm{IA}_n \cap \mathrm{Aut}(F_n, I)$. For example, recall the automorphisms $c_{ij}, m_{ijk} \in \mathrm{IA}_n$ defined in the introduction:

$$c_{ij}(x_\ell) = \begin{cases} x_j^{-1} x_\ell x_j & \text{if } \ell = i, \\ x_\ell & \text{otherwise,} \end{cases} \qquad m_{ijk}(x_\ell) = \begin{cases} x_\ell [x_j, x_k] & \text{if } \ell = i, \\ x_\ell & \text{otherwise.} \end{cases}$$

Clearly c_{ij} is supported on the splitting $\langle x_i, x_j \rangle * \langle x_\ell \mid \ell \neq i, j \rangle$, so $c_{ij} \in \mathrm{IA}_n(\{i, j\})$. Similarly, the automorphism $m_{ijk} \in \mathrm{IA}_n$ is supported on the splitting $\langle x_i, x_j, x_k \rangle * \langle x_\ell \mid \ell \neq i, j, k \rangle$, so $m_{ijk} \in \mathrm{IA}_n(\{i, j, k\})$.

Since Magnus proved that the elements c_{ij} and m_{ijk} generate IA_n for all n, we have the following proposition.

Proposition 3.3 (Generators for IA_n) *For any $n \geq 0$, the group IA_n is generated by the subgroups*

$$\{\mathrm{IA}_n(I) \mid I \subset \{1, \dots, n\} \text{ satisfies } |I| \leq 3\}.$$

We are now ready to prove Theorem C.

Proof of Theorem C Since IA(1) is an FI–group, Lemma 2.13 and Proposition 3.3 together imply IA(1) is boundedly generated in degree $A = 3$. Proposition 3.2 states that $\{\mathrm{IA}(k)\}_{k=1}^{\infty}$ is a central filtration of bounded rank. Applying Theorem G, we conclude that for all $k \geq 1$, there exists $B_k \geq 0$ so that $\mathrm{IA}(k) \lhd \mathrm{IA}(1)$ is boundedly normally generated in degree B_k.

Let us apply this conclusion to $\mathrm{IA}(k)_{[n]} = \mathrm{IA}_n(k)$. The bounded normal generation of $\mathrm{IA}(k) \lhd \mathrm{IA}(1)$ states that $\mathrm{IA}(k)_{[n]}$ is generated by the IA_n–conjugates of its subgroups $\mathrm{IA}(k)_{[n]}(I)$ for those $I \subset [n]$ with $|I| \leq B_k$.

We saw above that $\mathrm{IA}(k)_{[n]}(I)$ consists of those automorphisms in $\mathrm{IA}_n(k)$ which are supported on the splitting $F_n = F_I * F_{[n]-I}$. The φ–conjugate of this subgroup thus consists of those automorphisms in $\mathrm{IA}_n(k)$ supported on the splitting

$$F_n = \varphi(F_I) * \varphi(F_{[n]-I}).$$

When $\varphi \in \mathrm{IA}_n$, this is a homologically standard splitting. Therefore $\mathrm{IA}_n(k)$ is generated by elements of $\mathrm{IA}_n(k)$ supported on homologically standard splittings of rank $\leq B_k$, as desired. □

4 Mapping class groups

We begin in Section 4.1 by showing how to assemble all the different mapping class groups for surfaces of different genus into a weak FI–group. We also show that we can do the same for their Torelli subgroups and that the Johnson filtration gives a central filtration of this weak FI–group. In Section 4.2 we establish a generating set for $\mathcal{I}_g^1$, prove Theorem H, and finally prove Theorem A.

4.1 Mapping class groups as a weak FI–group

Ideally, we would like to construct an FI–group Mod such that $\mathrm{Mod}_{[g]} \cong \mathrm{Mod}_g^1$ and such that $\mathrm{Mod}_{\{i\}}$ is the subgroup supported on the "i^{th} handle". Unfortunately, this is not possible, for the following reason.

Recall that any FI–group G has an action of the symmetric group $\mathfrak{S}_n$ on the group $G_{[n]}$. If there did exist an FI–group Mod as above, then the subgroups $\mathrm{Mod}_{\{i\}}$ would be permuted by the action of the symmetric group $\mathfrak{S}_g$ on Mod_g^1. Since these subgroups are disjoint, this action must be faithful. However, this is impossible. Indeed, for $g \geq 2$, it follows from work of Ivanov and McCarthy [14] that there is a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{Mod}_g^1) \to \mathrm{Mod}(\Sigma_g, *) \to 1,$$

where $\mathrm{Mod}(\Sigma_g, *)$ is the mapping class group of a closed genus-g surface relative to a marked point. Since every finite subgroup of $\mathrm{Mod}(\Sigma_g, *)$ is cyclic, every finite group of automorphisms of Mod_g^1 is cyclic or dihedral; in particular, $\mathfrak{S}_g$ cannot act faithfully on Mod_g^1 for $g \gg 0$. Even if we tried to work with closed surfaces, a faithful action of $\mathfrak{S}_g$ on Mod_g would contradict Hurwitz's classical theorem that finite subgroups of Mod_g have size at most $84(g-1)$; see Farb and Margalit [11, Theorem 7.4]. We will thus have to be content with constructing a weak FI–group Mod (this is our reason for introducing the notion of weak FI–groups).

Systems of subsurfaces To pin down the morphisms in our weak FI–group, it will be helpful to realize the surfaces supporting the various mapping class groups involved as subsurfaces of one infinite-genus surface. Let $S_{\mathbb{N}}$ be an infinite-genus surface with one end. As in Figure 2a, pick closed subsurfaces $X_1, X_2, \ldots$, a basepoint $*$, a ray α, and arcs $\delta_1', \delta_2', \ldots$ with the following properties.

- The X_i are disjoint and each is homeomorphic to a one-holed torus.
- The subsurface
$$Y := S_{\mathbb{N}} \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}(X_i)$$
has genus 0.
- The ray α lies in Y and starts at $*$.
- The arc δ_i' lies in Y, starts at a point p_i of α, and ends at a point $*_i \in \partial X_i$. Also, the arcs δ_i' are all disjoint from each other and their interiors are disjoint from α and the ∂X_j.
- The p_i appear on α in their natural order and have no accumulation points, and $p_1 = *$.

Define δ_i to be the arc that starts at $*$, travels along α to p_i, and then travels along δ_i'. For every finite set $I \subset \mathbb{N}$, let S_I be a closed regular neighborhood of $\bigcup_{i \in I}(\delta_i \cup X_i)$. Observe that S_I is a genus-$|I|$ surface with 1 boundary component, and contains each handle X_i for $i \in I$; see Figure 2b.

If $I, J \subset \mathbb{N}$ are finite sets such that $I \subset J$, then S_I is isotopic to a subsurface of S_J. For our convenience, we will assume that the S_I are chosen so that in fact $S_I \subset S_J$ whenever $I \subset J$. One way to achieve this is as follows. Pick a Riemannian metric on $S_{\mathbb{N}}$ such that for some $\epsilon > 0$, the closed neighborhood of radius ϵ around $\bigcup_{i=1}^{\infty}(\delta_i \cup X_i)$ is a regular neighborhood of $\bigcup_{i=1}^{\infty}(\delta_i \cup X_i)$. Letting $\eta\colon [0,\infty) \to (0,\epsilon)$ be a strictly increasing function, we define S_I to be the closed neighborhood of radius $\eta(|I|)$ around $\bigcup_{i \in I}(\delta_i \cup X_i)$. We let $\iota_I^J\colon S_I \hookrightarrow S_J$ denote the inclusion, so $\iota_J^K \circ \iota_I^J = \iota_I^K$.

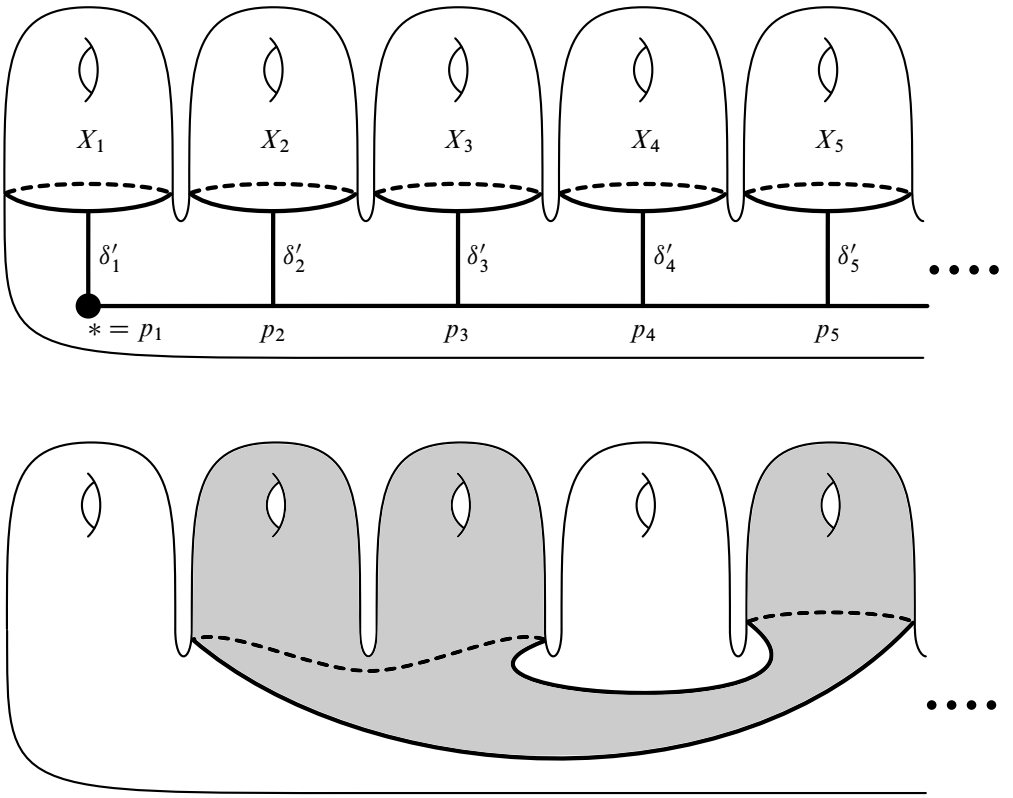


Figure 2: The top shows the one-ended infinite-genus surface $S_{\mathbb{N}}$. The long ray shown is α. The bottom shows the subsurface $S_{\{2,3,5\}}$ inside $S_{\mathbb{N}}$. We have perturbed the subsurface by an isotopy to make its structure clear.

The weak FI–group Mod We now define the weak FI–group Mod. For any surface S, let $\mathrm{Mod}(S)$ denote the mapping class group of S, ie the group of isotopy classes of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S. For each finite set $I \subset \mathbb{N}$, define $\mathrm{Mod}_I = \mathrm{Mod}(S_I)$.

We next define the distinguished homomorphisms $\mathrm{Mod}_I^J \colon \mathrm{Mod}_I \to \mathrm{Mod}_J$. Consider a pair of finite sets $I \subset J \subset \mathbb{N}$. By our assumption above we have $S_I \subset S_J$, so we can define a homomorphism $\mathrm{Mod}_I^J := (\iota_I^J)_* \colon \mathrm{Mod}_I \to \mathrm{Mod}_J$ by extending mapping classes on S_I to S_J by the identity. Since $\iota_J^K \circ \iota_I^J = \iota_I^K$, these homomorphisms satisfy the compatibility conditions of Definition 2.4(iii).

We now define homomorphisms-modulo-conjugacy $\mathrm{Mod}_f \colon \mathrm{Mod}_I \to \mathrm{Mod}_J$ for each injection $f \colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$. Choose an arbitrary orientation-preserving embedding $S_I \to S_J$. This induces a homomorphism $\mathrm{Mod}(S_I) \to \mathrm{Mod}(S_J)$;

we define $\mathrm{Mod}_f\colon \mathrm{Mod}_I \to \mathrm{Mod}_J$ to be the induced homomorphism-modulo-conjugacy. This definition may not seem very canonical, and we still need to check the compatibility conditions of Definition 2.4(ii). This requires the following lemma.

Lemma 4.1 *Let S_0, S be surfaces with one boundary component, and let $\phi, \phi'\colon S_0 \hookrightarrow S$ be two orientation-preserving embeddings of S_0 into S.*

(i) *The induced homomorphisms $\phi_*, \phi'_*\colon \mathrm{Mod}(S_0) \hookrightarrow \mathrm{Mod}(S)$ are conjugate by an element of $\mathrm{Mod}(S)$.*

(ii) *If ϕ and ϕ' induce the same map $\mathrm{H}_1(S_0;\mathbb{Z}) \to \mathrm{H}_1(S;\mathbb{Z})$ on homology, the homomorphisms $\phi_*, \phi'_*\colon \mathrm{Mod}(S_0) \hookrightarrow \mathrm{Mod}(S)$ are conjugate by an element of $\mathcal{I}(S)$.*

We prove Lemma 4.1 below. Part (i) of it shows that the homomorphism-modulo-conjugacy Mod_f does not depend on the choice of embedding $S_I \hookrightarrow S_J$, so our definition was canonical after all. In particular, Mod_I^J represents $\mathrm{Mod}_{i_I^J}$. It also guarantees that for all finite sets $I, J, K \subset \mathbb{N}$ and all injections $f\colon I \hookrightarrow J$ and $g\colon J \hookrightarrow K$, we have $\mathrm{Mod}_{g\circ f}$ equal to $\mathrm{Mod}_g \circ \mathrm{Mod}_f$ in $\mathtt{CGrp}$, so the conditions of Definition 2.4(ii) are satisfied. This completes the construction of the weak FI–group Mod.

Proof of Lemma 4.1 By perturbing ϕ and ϕ' by an isotopy, we can assume that their images lie in $\mathrm{Int}(S)$. Let $T := S \setminus \mathrm{Int}(\phi(S_0))$ and $T' := S \setminus \mathrm{Int}(\phi'(S_0))$. An Euler characteristic calculation shows that there exists an orientation-preserving homeomorphism $\psi_T\colon T \to T'$; moreover we may assume that ψ_T agrees with $\phi' \circ \phi^{-1}$ on $\partial(\phi(S_0)) = \partial T$. Let $\psi\colon S \to S$ be the orientation-preserving homeomorphism that restricts to $\phi' \circ \phi^{-1}$ on $\phi(S_0)$ and to ψ_T on T. We then have $\psi \circ \phi = \phi'$, so the mapping class defined by ψ conjugates ϕ_* to ϕ'_*, proving (i).

Let V (resp. V') be the image in $\mathrm{H}_1(S;\mathbb{Z})$ of $\mathrm{H}_1(T;\mathbb{Z})$ (resp. $\mathrm{H}_1(T';\mathbb{Z})$) under the map induced by the inclusion $T \hookrightarrow S$ (resp. $T' \hookrightarrow S$). We have orthogonal decompositions $\mathrm{H}_1(S;\mathbb{Z}) = \phi_*(\mathrm{H}_1(S_0;\mathbb{Z})) \oplus V$ and $\mathrm{H}_1(S;\mathbb{Z}) = \phi'_*(\mathrm{H}_1(S_0;\mathbb{Z})) \oplus V'$. If we assume as in (ii) that ϕ and ϕ' induce the same map $\mathrm{H}_1(S_0;\mathbb{Z}) \to \mathrm{H}_1(S;\mathbb{Z})$, so that $\phi_*(\mathrm{H}_1(S_0;\mathbb{Z})) = \phi'_*(\mathrm{H}_1(S_0;\mathbb{Z}))$, it follows that the complementary subspaces V and V' are equal. Recalling that ψ_T is an orientation-preserving homeomorphism from T to T', the map ψ_T induces a symplectic automorphism M of V. We can realize M by a homeomorphism ζ from T' to itself (see [11, Chapter 6]). Therefore replacing ψ_T by $\zeta^{-1} \circ \psi_T$ in the previous paragraph, we may assume that ψ_T acts trivially on V. The assumption on ϕ and ϕ' means that $\phi' \circ \phi^{-1}$ acts trivially on $\mathrm{H}_1(\phi(S_0);\mathbb{Z}) = \phi_*(\mathrm{H}_1(S_0;\mathbb{Z}))$. It follows that $\psi \in \mathcal{I}(S)$, proving (ii). □

The weak FI–group $\mathcal{I}$ We would like to define $\mathcal{I}$ in the same way. However, to ensure that $\mathcal{I}$ forms a weak FI–group we will need to be more careful with the homomorphisms-modulo-conjugacy $\mathcal{I}_f$.

For each $i \in \mathbb{N}$, fix once and for all a symplectic basis $\{a_i, b_i\}$ for $\mathrm{H}_1(X_i;\mathbb{Z})$. For any finite set $I \subset \mathbb{N}$, the map $\mathrm{H}_1(S_I;\mathbb{Z}) \to \mathrm{H}_1(S_{\mathbb{N}};\mathbb{Z})$ is injective, and we will identify $\mathrm{H}_1(S_I;\mathbb{Z})$ with its image. Therefore $\{a_i, b_i \mid i \in \mathbb{N}\}$ is a symplectic basis for $\mathrm{H}_1(S_{\mathbb{N}};\mathbb{Z})$, and $\{a_i, b_i \mid i \in I\}$ is a symplectic basis for $\mathrm{H}_1(S_I;\mathbb{Z})$.

Lemma 4.2 *For any injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, there exists an embedding $\phi_f\colon S_I \hookrightarrow S_J$ which on homology induces the map*

$$\mathrm{H}_1(S_I;\mathbb{Z}) \to \mathrm{H}_1(S_J;\mathbb{Z}), \quad a_i \mapsto a_{f(i)}, \quad b_i \mapsto b_{f(i)} \quad \text{for all } i \in I. \tag{4-1}$$

Proof Let $\psi\colon S_I \to S_{f(I)}$ be an arbitrary orientation-preserving homeomorphism. Fix an arbitrary ordering on I. Then $\{\psi(a_i), \psi(b_i) \mid i \in I\}$ and $\{a_{f(i)}, b_{f(i)} \mid i \in I\}$ are both ordered symplectic bases for $\mathrm{H}_1(S_{f(I)};\mathbb{Z})$, so there is a symplectic automorphism M of $\mathrm{H}_1(S_{f(I)};\mathbb{Z})$ taking the former to the latter. We can realize M by $\xi_f \in \mathrm{Mod}(S_{f(I)})$ [11, Chapter 6], and $\phi_f := i^J_{f(I)} \circ \xi_f \circ \psi$ is the desired map. □

We are now ready to define the weak FI–group $\mathcal{I}$. For each finite set $I \subset \mathbb{N}$, define $\mathcal{I}_I$ to be the subgroup of Mod_I acting trivially on $\mathrm{H}_1(S_I;\mathbb{Z})$.

For each pair of finite sets $I \subset J \subset \mathbb{N}$, define $\mathcal{I}^J_I\colon \mathcal{I}_I \to \mathcal{I}_J$ to be the restriction of the map $\mathrm{Mod}^J_I\colon \mathrm{Mod}_I \to \mathrm{Mod}_J$ described above. The condition in Definition 2.4(iii) is automatically satisfied.

For each injection $f\colon I \hookrightarrow J$ between finite sets $I, J \subset \mathbb{N}$, choose an arbitrary embedding $\phi_f\colon S_I \hookrightarrow S_J$ inducing the map (4-1) on homology, as guaranteed by Lemma 4.2. We define $\mathcal{I}_f\colon \mathcal{I}_I \to \mathcal{I}_J$ to be the restriction of the induced map $(\phi_f)_*\colon \mathrm{Mod}_I \to \mathrm{Mod}_J$. By Lemma 4.1(ii), any two embeddings inducing the map (4-1) on homology are $\mathcal{I}_J$–conjugate, so this gives a well-defined homomorphism-modulo-conjugacy $\mathcal{I}_f$. Moreover since the maps (4-1) are preserved under composition, Lemma 4.1(ii) guarantees that these homomorphisms-modulo-conjugacy satisfy the compatibility condition in Definition 2.4(ii). This concludes the construction of the weak FI–group $\mathcal{I}$.

The Johnson filtration For $k \geq 1$, we define a normal weak FI–subgroup $\mathcal{I}(k)$ of $\mathcal{I}$ as follows. For each finite set $I \subset \mathbb{N}$, choose a basepoint $*_I \in \partial S_I$ and let $\pi_1(S_I) := \pi_1(S_I, *_I)$. We define $\mathcal{I}(k)_I$ to be the kernel of the action of Mod_I on $\pi_1(S_I)/\gamma_{k+1}(\pi_1(S_I))$. This kernel does not depend on the choice of basepoint. Note that $\mathcal{I}(1) = \mathcal{I}$. The following lemma guarantees that $\mathcal{I}(k) \lhd \mathcal{I}$.

Lemma 4.3 *For any $k \geq 1$, if $f\colon I \hookrightarrow J$ is an injection between finite sets $I, J \subset \mathbb{N}$, then $\mathcal{I}_f(\mathcal{I}(k)_I) \subset \mathcal{I}(k)_J$.*

Proof Let λ be an arc in $S_J \setminus \mathrm{Int}(S_{f(I)})$ joining the basepoint $*_J \in \partial S_J$ to the basepoint $*_{f(I)} \in \partial S_{f(I)}$. There is an injection $\pi_1(S_{f(I)}) \hookrightarrow \pi_1(S_J)$ that takes $\delta \in \pi_1(S_{f(I)})$ to $\lambda \cdot \delta \cdot \lambda^{-1}$; we will identify $\pi_1(S_{f(I)})$ with its image in $\pi_1(S_J)$. The free group $\pi_1(S_J)$ can then be decomposed as a free product $\pi_1(S_{f(I)}) * U$, where U is a subgroup generated by loops that lie entirely in $S_J \setminus \mathrm{Int}(S_{f(I)})$.

Consider $\varphi \in \mathcal{I}(k)_I$. Since the embedding $\psi_f\colon S_I \hookrightarrow S_J$ induces a map taking $\gamma_{k+1}(\pi_1(S_I))$ into $\gamma_{k+1}(\pi_1(S_{f(I)})) \subset \gamma_{k+1}(\pi_1(S_J))$, the mapping class $\mathcal{I}_f(\varphi)$ acts as the identity on the image of $\pi_1(S_{f(I)})$ in $\pi_1(S_J)/\gamma_{k+1}(\pi_1(S_J))$. The mapping class $\mathcal{I}_f(\varphi)$ also acts as the identity on $S_J \setminus \mathrm{Int}(S_{f(I)})$, and thus certainly acts as the identity on the image of U in $\pi_1(S_J)/\gamma_{k+1}(\pi_1(S_J))$. Thus, $\mathcal{I}_f(\varphi) \in \mathcal{I}(k)_J$. □

Proposition 4.4 *The weak FI–subgroups $\{\mathcal{I}(k)\}_{k=1}^{\infty}$ form a central filtration of $\mathcal{I} = \mathcal{I}(1)$ of finite rank.*

Proof Since $\gamma_2(\pi_1(S_I)) \supset \gamma_3(\pi_1(S_I)) \supset \gamma_4(\pi_1(S_I)) \supset \cdots$, we have

$$\mathcal{I} = \mathcal{I}(1) \supset \mathcal{I}(2) \supset \mathcal{I}(3) \supset \cdots.$$

We must show for $k \geq 1$ that $[\mathcal{I}(1)_I, \mathcal{I}(k)_I] \subset \mathcal{I}(k+1)_I$ and that $\mathcal{I}(k)_I/\mathcal{I}(k+1)_I$ is a finite-rank abelian group. Just as in Proposition 3.2, this is an immediate consequence of the higher Johnson homomorphisms for $\mathcal{I}_g^1(k)$ (see [31]). □

4.2 Generating the Torelli group and its Johnson filtration

Identify Σ_g^1 with $S_{[g]}$, so for all subsets $I \subset \{1, \ldots, g\}$ we have a subsurface $S_I \subset \Sigma_g^1$. As notation, if S is a subsurface of Σ_g^1, we denote by $\mathrm{Mod}_g^1(S)$ the subgroup of Mod_g^1 consisting of mapping classes that are supported on S. Also, define $\mathcal{I}_g^1(S) := \mathcal{I}_g^1 \cap \mathrm{Mod}_g^1(S)$. The following result is a strengthening of the main result of the second author in [29].

Proposition 4.5 (Torelli generators) *For $g \geq 3$, the group $\mathcal{I}_g^1$ is generated by the subgroups*

$$\{\mathcal{I}_g^1(S_I) \mid I \subset \{1, \ldots, g\} \text{ satisfies } |I| = 3\}.$$

Before proving Proposition 4.5, we deduce Theorem H from it.

Proof of Theorem H Johnson [17] proved that $\mathcal{I}_3^1$ is generated by 42 elements. There are $\binom{g}{3}$ subsurfaces S_I in Proposition 4.5, and each subgroup $\mathcal{I}_g^1(S_I)$ is isomorphic to $\mathcal{I}_3^1$, so we deduce that $\mathcal{I}_g^1$ is generated by $42\binom{g}{3}$ elements. There is a surjection $\mathcal{I}_g^1 \twoheadrightarrow \mathcal{I}_g$ obtained by gluing a disc to $\partial\Sigma_g^1$ and extending mapping classes over the disc by the identity, so $\mathcal{I}_g$ is also generated by $42\binom{g}{3}$ elements. □

Proof of Proposition 4.5 Let $\Gamma \subset \mathcal{I}_g^1$ be the subgroup generated by the subgroups $\mathcal{I}_g^1(S_I)$ for $|I| = 3$, or equivalently for $|I| \leq 3$; our goal is to prove that $\Gamma = \mathcal{I}_g^1$. We begin by describing some simple elements of $\mathcal{I}_g^1$ that lie in Γ.

First, choose $i \in \{1, \dots, g\}$. Recall the genus-1 subsurfaces $X_1, \dots, X_g$ of Σ_g^1, which satisfy $X_i \subset S_I$ if and only if $i \in I$. The boundary curve ∂X_i is a separating curve contained in $S_{\{i\}}$, so the Dehn twist $T_{\partial X_i}$ lies in $\mathcal{I}_g^1(S_{\{i\}}) \subset \Gamma$.

Next, choose $j \in \{1, \dots, g\}$ with $j \neq i$, and let γ be an embedded curve in X_j based at $*_j \in \partial X_j$. The regular neighborhood of $\partial X_i \cup \delta_i \cup \delta_j \cup \gamma$ is a genus-0 surface with 3 boundary components. These 3 boundary components are isotopic to γ, the separating curve ∂X_i, and a third curve γ' homologous to γ. The mapping class $T_\gamma T_{\gamma'}^{-1}$ has the effect of "sliding" the handle X_i around the curve $\delta_i^{-1}\delta_j\gamma\delta_j\delta_i^{-1}$ (though this notion is only well defined modulo powers of $T_{\partial X_i}$); see [11, Fact 4.7]. Since γ and γ' are homologous, $T_\gamma T_{\gamma'}^{-1}$ lies in $\mathcal{I}_g^1$. Since our regular neighborhood is contained in $S_{\{i,j\}}$, we have $T_\gamma T_{\gamma'}^{-1} \in \mathcal{I}_g^1(S_{\{i,j\}}) \subset \Gamma$.

For any subset $I \subset \{1, \dots, g\}$, define the subsurface

$$Y_I := \Sigma_g^1 \setminus \left(\bigcup_{i \notin I} \operatorname{Int}(X_i) \right),$$

so Y_I is a genus-$|I|$ surface with $g - |I| + 1$ boundary components. See Figure 3 for an example. For $i \notin I$, let $Z_I^{(i)}$ be the genus-$|I|$ surface with $g - |I|$ boundary components obtained from Y_I by attaching a single disk to the boundary component ∂X_i. We will next show that the kernel of the corresponding map $\pi^{(i)}\colon \mathcal{I}_g^1(Y_I) \twoheadrightarrow \mathcal{I}_g^1(Z_I^{(i)})$ is contained in Γ.

Certainly $T_{\partial X_i}$ lies in $\ker(\pi^{(i)})$. Birman proved [3] that the quotient $\ker(\pi^{(i)})/\langle T_{\partial X_i} \rangle$ is isomorphic to $\pi_1(Z_I^{(i)})$, with a loop in $\pi_1(Z_I^{(i)})$ corresponding to the mapping class that slides the handle X_i around that loop. The fundamental group $\pi_1(Z_I^{(i)})$ can be generated by elements of the form $\delta_i^{-1}\delta_j\gamma\delta_j\delta_i^{-1}$ where γ is an embedded curve in X_j: for each $j \notin I$ we take $\gamma = \partial X_j$, and for each $j \in I$ we take two embedded curves generating $\pi_1(X_i)$. We saw earlier that Γ contains the mapping class $T_\gamma T_{\gamma'}^{-1}$ which slides the handle X_i along any such loop, and so we conclude that $\ker(\pi^{(i)}) \subset \Gamma$.

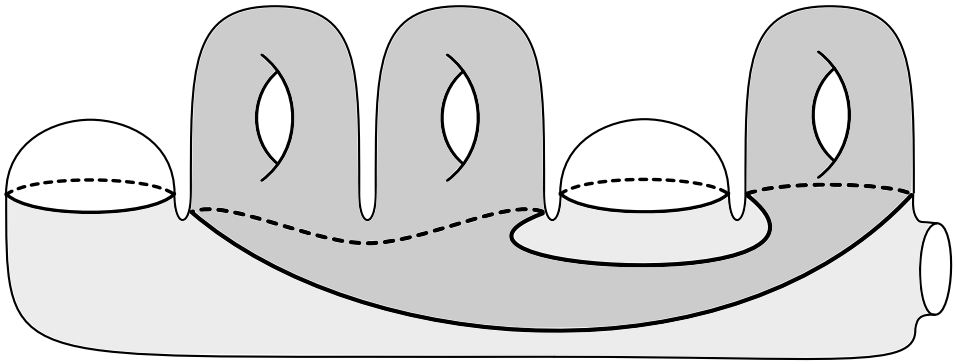


Figure 3: For $I = \{2, 3, 5\}$, the surfaces S_I (dark gray), Y_I (light and dark gray), and Z_I

Let Z_I be the abstract surface obtained from Y_I by attaching disks to each of the boundary components ∂X_i for $i \notin I$, so Z_I is a genus-$|I|$ surface with one boundary component. We can find an identification of Z_I with S_I so that the composition $S_I \hookrightarrow Y_I \hookrightarrow Z_I \cong S_I$ is isotopic to the identity. It follows that the resulting homomorphism $\pi\colon \mathcal{I}_g^1(Y_I) \twoheadrightarrow \mathcal{I}_g^1(S_I)$ is a split surjection, with section given by the inclusion $\mathcal{I}_g^1(S_I) \hookrightarrow \mathcal{I}_g^1(Y_I)$. It follows from the classical Fadell–Neuwirth exact sequences [10] that the kernel $\ker(\pi) \subset \mathcal{I}_g^1(Y_I)$ is generated by the subgroups $\ker(\pi^{(i)})$ for all $i \notin I$ (in fact, $\ker(\pi)$ is isomorphic to the $(g - |I|)$–strand pure framed braid group on the surface S_I, though we will not use this directly). We conclude from the previous paragraph that $\ker(\pi) \subset \Gamma$.

When $|I| = 3$ we have $\mathcal{I}_g^1(S_I) \subset \Gamma$ by definition, so $\mathcal{I}_g^1(Y_I) \subset \Gamma$ as well. The second author proved in [29] that $\mathcal{I}_g^1$ is generated by the set

$$\{\mathcal{I}_g^1(Y_I) \mid I \subset \{1, \dots, g\} \text{ satisfies } |I| = 3\},$$

so we conclude that $\Gamma = \mathcal{I}_g^1$, as desired. □

We would like to conclude from Proposition 4.5 that the weak FI–group $\mathcal{I}$ is boundedly generated. However since $\mathcal{I}$ is only a weak FI–group, this conclusion is not at all automatic (cf Remark 2.14). To do this, we need the following lemma. Along with Proposition 4.5, this lemma is the reason for our care in Section 4.1 when defining the system of subsurfaces S_I.

Lemma 4.6 *Let $J, J' \subset \mathbb{N}$ be finite sets such that $|J| = |J'|$. There exists a bijection $\sigma\colon J \to J'$ and an orientation-preserving homeomorphism $\phi\colon S_J \to S_{J'}$ such that for all $I \subset J$ the subsurface $\phi(S_I)$ of $S_{J'}$ is isotopic to the subsurface $S_{\sigma(I)}$.*

Proof Let $\sigma\colon J \to J'$ be the unique order-preserving bijection. Recall from Section 4.1 the basepoint $*$, the genus-1 subsurfaces $X_1, X_2, \ldots$, and the arcs $\delta_1, \delta_2, \ldots$ used to define the surfaces S_I. Using the standard "change of coordinates principle" (see [11, Section 1.3.2]), there exists a homeomorphism $\phi\colon S_J \to S_{J'}$ with the following three properties:

- $\phi(*) = *$.
- For all $i \in J$, we have $\phi(X_i) = X_{\sigma(i)}$.
- For all $i \in J$, we have $\phi(\delta_i) = \delta_{\sigma(i)}$.

From the definition of the surface S_I we see that ϕ has the desired properties. □

We emphasize that Lemma 4.6 depends in an essential way on the precise details of our construction of the surfaces S_I (unlike Lemmas 4.1 and 4.2 above, which were rather tautological). We are now ready to prove Theorem A.

Proof of Theorem A We begin by showing that the weak FI–group $\mathcal{I}$ is boundedly generated in degree 3. Fix a finite set $J \subset \mathbb{N}$. If $|J| \leq 3$ the condition (2-1) is vacuous, so assume that $|J| > 3$. Taking $g := |J|$, let $\sigma\colon J \to [g]$ be the bijection given by Lemma 4.6, and $\phi\colon S_J \to S_{[g]} = \Sigma_g^1$ the corresponding homeomorphism.

Consider $I \subset J$ with $|I| = 3$. By construction, ϕ takes S_I to the subsurface $S_{\sigma(I)}$ of $S_{[g]}$. Therefore the isomorphism $\phi_*\colon \mathcal{I}_J \to \mathcal{I}_g^1$ takes the subgroup $\mathcal{I}_J(I)$ supported on S_I to the subgroup $\mathcal{I}_g^1(\sigma(I))$ supported on $S_{\sigma(I)}$. Proposition 4.5 states that $\mathcal{I}_g^1$ is generated by the subgroups $\mathcal{I}_g^1(S_{\sigma(I)})$. We conclude that $\mathcal{I}_J$ is generated by the subgroups $\mathcal{I}_J(I)$ for $I \subset J$ satisfying $|I| = 3$. Therefore (2-1) is satisfied, and the weak FI–group $\mathcal{I}$ is boundedly generated in degree 3.

Proposition 4.4 states that $\{\mathcal{I}(k)\}_{k=1}^{\infty}$ is a central filtration of bounded rank. Applying Theorem G, we conclude that for all $k \geq 1$, there exists $B_k \geq 0$ so that $\mathcal{I}(k) \lhd \mathcal{I}$ is boundedly normally generated in degree B_k.

Fix $g \geq 0$, and let us apply this conclusion to $\mathcal{I}(k)_{[g]} = \mathcal{I}_g^1(k)$. The bounded normal generation of $\mathcal{I}(k) \lhd \mathcal{I}$ states that $\mathcal{I}(k)_{[g]}$ is generated by the $\mathcal{I}_g^1$–conjugates of its subgroups $\mathcal{I}(k)_{[g]}(I)$ for those $I \subset [n]$ with $|I| \leq B_k$. The subgroup $\mathcal{I}(k)_{[g]}(I)$ consists of those elements of $\mathcal{I}_g^1(k)$ supported on the genus $|I|$ subsurface $S_I \subset \Sigma_g^1$, so its φ–conjugate consists of those elements of $\mathcal{I}_g^1(k)$ supported on the subsurface $\varphi(S_I)$. If $\varphi \in \mathcal{I}_g^1$, the subsurface $\varphi(S_I)$ is homologically standard. Therefore $\mathcal{I}_g^1(k)$ is generated by elements of $\mathcal{I}_g^1(k)$ supported on homologically standard subsurfaces of genus at most B_k, as desired. □

5 Mod-p filtrations

Fix a prime $p \geq 2$. In this section we discuss the modifications that must be done to our proofs of Theorems A and C to obtain proofs of Theorems E and F. Almost everything goes through verbatim. There are only two places where additional work is necessary.

The first occurs in the proofs of Propositions 3.2 and 4.4, where the higher Johnson homomorphisms are invoked. These should be replaced with the higher mod-p Johnson homomorphisms constructed by Cooper in [7]. The second place where a new idea is needed is in the analogues of Propositions 3.3 and 4.5, which give generators for IA_n and $\mathcal{I}_g^1$. We need generators for the level p congruence subgroups $\mathrm{Aut}(F_n, p)$ and $\mathrm{Mod}_g^1(p)$. These are given in Propositions 5.1 and 5.3 below. Given these results, the proofs of Theorems E and F parallel exactly the proofs of Theorems A and C.

5.1 Generators for $\mathrm{Aut}(F_n, p)$

For a subset $I \subset \{1, \ldots, n\}$, let $\mathrm{Aut}(F_n, I)$ consist of automorphisms supported on the splitting $F_n = F_I * F_{[n]-I}$, as defined in Section 3.2. Define $\mathrm{Aut}(F_n, p, I) = \mathrm{Aut}(F_n, p) \cap \mathrm{Aut}(F_n, I)$.

Proposition 5.1 (Generators for $\mathrm{Aut}(F_n, p)$) *For any $n \geq 0$ and any prime $p \geq 2$, the group $\mathrm{Aut}(F_n, p)$ is generated by the subgroups*

$$\{\mathrm{Aut}(F_n, p, I) \mid I \subset \{1, \ldots, n\} \text{ satisfies } |I| \leq 3\}.$$

For the proof of Proposition 5.1, we will need a generating set for the level-p congruence subgroup $\mathrm{SL}_n(\mathbb{Z}, p)$ of $\mathrm{SL}_n(\mathbb{Z})$, which is the kernel of the natural map $\mathrm{SL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_n(\mathbb{Z}/p)$. Given $r \in \mathbb{Z}$ and $1 \leq i, j \leq n$, let $\epsilon_{ij}^n(r)$ be the $n \times n$ matrix with (i, j) entry equal to r and all other entries equal to zero. For $1 \leq i < n$, let $\beta_i^n(r)$ be the $n \times n$ matrix with (i, i) and $(i, i+1)$ entries equal to r, with $(i+1, i)$ and $(i+1, i+1)$ entries equal to $-r$, and all other entries equal to zero.

Given $r \in \mathbb{Z}$ and $i \neq j$, let

$$\mathcal{E}_{ij}^n(r) := \mathbb{I}_n + \epsilon_{ij}^n(r) \in \mathrm{SL}_n(\mathbb{Z})$$

be the $n \times n$ elementary matrix whose diagonal entries are 1 and whose (i, j) entry is r. Similarly, given $r \in \mathbb{Z}$ and $1 \leq i < n$, let $\mathcal{B}_i^n(r) := \mathbb{I}_n + \beta_i^n(r)$. For instance,

$$\mathcal{B}_2^4(7) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 7 & 0 \\ 0 & -7 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then have the following theorem of Sury and Venkataramana.

Theorem 5.2 (Sury and Venkataramana [34]) *For $n \geq 3$ and $p \geq 2$, the group $\mathrm{SL}_n(\mathbb{Z}, p)$ is generated by the set*

$$\{\mathcal{E}_{ij}^n(p) \mid 1 \leq i, j \leq n, i \neq j\} \cup \{\mathcal{B}_i^n(p) \mid 1 \leq i < n\}.$$

Let $\mathrm{GL}_n(\mathbb{Z}, p)$ be the level-p congruence subgroup $\ker(\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/p))$. For any $M \in \mathrm{GL}_n(\mathbb{Z})$ we have $\det M = \pm 1$; moreover, if $M \in \mathrm{GL}_n(\mathbb{Z}, p)$, then $M \equiv \mathbb{I}_n \bmod p$ implies that $\det M \equiv 1 \bmod p$. For $p \geq 3$ these together imply that $\det M = 1$, and so $\mathrm{GL}_n(\mathbb{Z}, p) = \mathrm{SL}_n(\mathbb{Z}, p)$. However for $p = 2$ we have an extension

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{Z}, 2) \longrightarrow \mathrm{GL}_n(\mathbb{Z}, 2) \xrightarrow{\det} \{\pm 1\} \longrightarrow 1.$$

Let $\mathcal{N}_1 \in \mathrm{GL}_n(\mathbb{Z})$ be the matrix obtained from the identity matrix by replacing the 1 at position $(1,1)$ with a -1. Then $\mathcal{N}_1 \in \mathrm{GL}_n(\mathbb{Z}, 2)$ has $\det \mathcal{N}_1 = -1$, and $\mathrm{GL}_n(\mathbb{Z}, 2)$ is generated by $\mathrm{SL}_n(\mathbb{Z}, 2)$ together with $\mathcal{N}_1$.

Proof of Proposition 5.1 Let $\Gamma \subset \mathrm{Aut}(F_n, p)$ be the subgroup generated by the purported generators $\mathrm{Aut}(F_n, p, I)$ with $|I| \leq 3$; our goal is to prove that $\Gamma = \mathrm{Aut}(F_n, p)$. The map $\pi\colon \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ is known to be surjective, so we have a short exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n, p) \xrightarrow{\pi} \mathrm{GL}_n(\mathbb{Z}, p) \longrightarrow 1.$$

Since $\mathrm{IA}_n(I) \subset \mathrm{Aut}(F_n, p, I)$, Proposition 3.3 implies that $\mathrm{IA}_n \subset \Gamma$. It is therefore enough to show that $\pi(\Gamma)$ is all of $\mathrm{GL}_n(\mathbb{Z}, p)$.

Define automorphisms $\widetilde{\mathcal{E}}_{ij}^n(p) \in \mathrm{Aut}(F_n, p)$ for $1 \leq i, j \leq n$ with $i \neq j$, automorphisms $\widetilde{\mathcal{B}}_i^n(p) \in \mathrm{Aut}(F_n, p)$ for $1 \leq i < n$, and the automorphism $\widetilde{\mathcal{N}}_1 \in \mathrm{Aut}(F_n, 2)$ via the following formulas:

$$\widetilde{\mathcal{E}}_{ij}^n(p)(x_\ell) = \begin{cases} x_j x_i^p & \text{if } \ell = j, \\ x_\ell & \text{otherwise,} \end{cases}$$

$$\widetilde{\mathcal{B}}_i^n(p)(x_\ell) = \begin{cases} x_i (x_i x_{i+1}^{-1})^p & \text{if } \ell = i, \\ x_{i+1} (x_i x_{i+1}^{-1})^p & \text{if } \ell = i+1, \\ x_\ell & \text{otherwise,} \end{cases}$$

$$\widetilde{\mathcal{N}}_1(x_\ell) = \begin{cases} x_1^{-1} & \text{if } \ell = 1, \\ x_\ell & \text{otherwise.} \end{cases}$$

The automorphism $\widetilde{\mathcal{E}}_{ij}^n(p)$ is supported on the splitting $\langle x_i, x_j \rangle * \langle x_\ell \mid \ell \neq i, j \rangle$, so $\widetilde{\mathcal{E}}_{ij}^n(p) \in \mathrm{Aut}(F_n, p, \{i, j\})$. Similarly, we have $\widetilde{\mathcal{B}}_i^n(p) \in \mathrm{Aut}(F_n, p, \{i, i+1\})$

and $\widetilde{\mathcal{N}}_1 \in \mathrm{Aut}(F_n, 2, \{1\})$. These elements are therefore contained in Γ. Direct computations show that $\pi(\widetilde{\mathcal{E}}^n_{ij}(p)) = \mathcal{E}^n_{ij}(p)$ and $\pi(\widetilde{\mathcal{B}}^n_i(p)) = \mathcal{B}^n_i(p)$ and $\pi(\widetilde{\mathcal{N}}_1) = \mathcal{N}_1$. By Theorem 5.2 these elements generate $\mathrm{GL}_n(\mathbb{Z}, p)$, so we conclude that $\pi(\Gamma) = \mathrm{GL}_n(\mathbb{Z}, p)$, as desired. □

5.2 Generators for $\mathbf{Mod}^1_g(p)$

Recall from Section 4.2 that $\Sigma^1_g = S_{[g]}$, so for any $I \subset \{1, \dots, g\}$ we have a subsurface S_I of Σ^1_g. For any subsurface S of Σ^1_g, we denote by $\mathrm{Mod}^1_g(p, S)$ the subgroup $\mathrm{Mod}^1_g(p, S) := \mathrm{Mod}^1_g(p) \cap \mathrm{Mod}(S)$ consisting of mapping classes supported on S.

Proposition 5.3 (Level–p generators) *For $g \geq 3$ and $p \geq 2$, the group $\mathrm{Mod}^1_g(p)$ is generated by the set*

$$\{\mathrm{Mod}^1_g(p, S_I) \mid I \subset \{1, \dots, g\} \text{ satisfies } |I| = 3\}.$$

The level-p congruence subgroup $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ is the kernel of the natural map

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/p).$$

To prove Proposition 5.3, we will need a generating set for $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ analogous to Theorem 5.2.

For $g \geq 1$ let $\mathbb{I}_g$ and $\mathbb{O}_g$ be the $g \times g$ identity matrix and zero matrix, respectively. Recall from Section 3.2 that for any $r \in \mathbb{Z}$, we defined $\epsilon^g_{ij}(r)$ as the $g \times g$ matrix with (i, j) entry equal to r and zero otherwise, and $\beta^g_i(r)$ as the $g \times g$ matrix with (i, i) and $(i, i+1)$ entries equal to r, with $(i+1, i)$ and $(i+1, i+1)$ entries equal to $-r$, and zero otherwise. We define $s\epsilon^g_{ij}(r)$ to be the $g \times g$ matrix with (i, j) and (j, i) entries equal to r and zero otherwise; when $i \neq j$ this is just $\epsilon^g_{ij}(r) + \epsilon^g_{ji}(r)$, while when $i = j$ we have $s\epsilon^g_{ii}(r) = \epsilon^g_{ii}(r)$.

We can now describe our generating set. First, for $1 \leq i \leq j \leq g$, define

$$\mathcal{X}^g_{ij}(r) := \mathbb{I}_{2g} + \begin{pmatrix} \mathbb{O}_g & \mathbb{O}_g \\ s\epsilon^g_{ij}(r) & \mathbb{O}_g \end{pmatrix}, \quad \mathcal{Y}^g_{ij}(r) := \mathbb{I}_{2g} + \begin{pmatrix} \mathbb{O}_g & s\epsilon^g_{ij}(r) \\ \mathbb{O}_g & \mathbb{O}_g \end{pmatrix}.$$

Second, for $1 \leq i, j \leq g$ with $i \neq j$, define

$$\mathcal{Z}^g_{ij}(r) := \mathbb{I}_{2g} + \begin{pmatrix} \epsilon^g_{ij}(r) & \mathbb{O}_g \\ \mathbb{O}_g & -\epsilon^g_{ji}(r) \end{pmatrix}.$$

Third, for $1 \leq i < g$, define

$$\mathcal{W}^g_i(r) := \mathbb{I}_{2g} + \begin{pmatrix} \beta^g_i(r) & \mathbb{O}^g \\ \mathbb{O}_g & -\beta^g_i(r)^\top \end{pmatrix}.$$

Finally, define

$$\mathcal{U}_1^g(r) := \mathbb{I}_{2g} + \begin{pmatrix} \epsilon_{11}^g(r) & \epsilon_{11}^g(r) \\ -\epsilon_{11}^g(r) & -\epsilon_{11}^g(r) \end{pmatrix}.$$

Lemma 5.4 *For $g \geq 2$ and $p \geq 2$ the congruence group $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ is generated by the set*

$$\{\mathcal{X}_{ij}^g(p), \mathcal{Y}_{ij}^g(p) | 1 \leq i \leq j \leq g\} \cup \{\mathcal{Z}_{ij}^g(p) | 1 \leq i, j \leq g,\ i \neq j\} \cup \{\mathcal{W}_i^g(p) | 1 \leq i < g\} \cup \{\mathcal{U}_1^g(p)\}.$$

Proof of Lemma 5.4 Let $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z}, p)$ be the subgroup generated by the purported generating set. Let $\mathfrak{sp}_{2g}(\mathbb{Z}/p)$ and $\mathfrak{gl}_{2g}(\mathbb{Z}/p)$ be the symplectic Lie algebra and matrix Lie algebra over $\mathbb{Z}/p$, considered as abelian groups.

Let $\rho\colon \mathrm{Sp}_{2g}(\mathbb{Z}, p) \to \mathfrak{gl}_{2g}(\mathbb{Z}/p)$ be the map sending $\mathbb{I}_{2g} + pA \in \mathrm{Sp}_{2g}(\mathbb{Z}, p)$ to the mod-p reduction of A in $\mathfrak{gl}_{2g}(\mathbb{Z}/p)$. It was first proved by Newman and Smart (see [25, Theorem 7]) that the image $\rho(\mathrm{Sp}_{2g}(\mathbb{Z}, p))$ is precisely the subgroup

$$\mathfrak{sp}_{2g}(\mathbb{Z}/p) \subset \mathfrak{gl}_{2g}(\mathbb{Z}/p).$$

It is easy to see that ρ maps our purported generating set to a basis for $\mathfrak{sp}_{2g}(\mathbb{Z}/p)$ (the generators $\mathcal{W}_i^g(p)$ are needed to get matrices whose diagonal does not vanish, and $\mathcal{U}_g^1(p)$ is needed to get matrices whose trace is not zero in each block). Therefore $\rho(\Gamma) = \mathfrak{sp}_{2g}(\mathbb{Z}/p) = \rho(\mathrm{Sp}_{2g}(\mathbb{Z}, p))$.

It remains to show that $\ker(\rho) \subset \Gamma$. But the kernel of ρ is $\mathrm{Sp}_{2g}(\mathbb{Z}, p^2)$ by definition, and Tits [35, Proposition 4] proved that $\mathrm{Sp}_{2g}(\mathbb{Z}, p^2) \subset \Gamma$ (the generator $\mathcal{U}_1^g(p)$ is not necessary here). Therefore $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z}, p)$, as desired. □

Remark 5.5 Bass, Milnor and Serre [1, Theorem 12.4] proved that $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ is the normal closure in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of

$$\{\mathcal{X}_{ij}^g(p), \mathcal{Y}_{ij}^g(p) \mid 1 \leq i \leq j \leq g\}$$

for $g \geq 2$ and $p \geq 2$. However, one can show that these $g^2 + g$ generators do not suffice to generate $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$. Indeed, we saw above that $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ surjects to $\mathfrak{sp}_{2g}(\mathbb{Z}/p)$, an elementary abelian group of rank $2g^2 + g$, so $\mathrm{Sp}_{2g}(\mathbb{Z}, p)$ cannot be generated by fewer than $2g^2 + g$ elements. Since the generating set in Lemma 5.4 consists of exactly $2g^2 + g$ elements, it is in fact a *minimal* generating set.

Proof of Proposition 5.3 Let $\Gamma \subset \mathrm{Mod}_g^1(p)$ be the subgroup generated by the subgroups $\mathrm{Mod}_g^1(p, S_I)$ with $|I| = 3$, or equivalently with $|I| \leq 3$. We have a short exact sequence

$$1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathrm{Mod}_g^1(p) \xrightarrow{\pi} \mathrm{Sp}_{2g}(\mathbb{Z}, p) \longrightarrow 1.$$

Consider the image $\pi(\Gamma) \subset \mathrm{Sp}_{2g}(\mathbb{Z}, p)$. By examination we see that the generators $\mathcal{X}_{ij}^g(p)$, $\mathcal{Y}_{ij}^g(p)$, and $\mathcal{Z}_{ij}^g(p)$ are in the image of $\mathrm{Mod}_g^1(p, S_{\{i,j\}})$, the generator $\mathcal{W}_i^g(p)$ is in the image of $\mathrm{Mod}_g^1(p, S_{\{i,i+1\}})$, and the generator $\mathcal{U}_1^g(p)$ is in the image of $\mathrm{Mod}_g^1(p, S_{\{1\}})$. By Lemma 5.4, $\pi(\Gamma) = \mathrm{Sp}_{2g}(\mathbb{Z}, p)$. Since $\mathcal{I}_g^1(I) \subset \mathrm{Mod}_g^1(p, I)$, Proposition 4.5 implies that $\mathcal{I}_g^1 \subset \Gamma$. We conclude that $\Gamma = \mathrm{Mod}_g^1(p)$, as desired. □

6 Lower bounds on generators

Our goal now is to prove Theorems B and D. We begin by recalling some facts about the higher Johnson homomorphisms. See Satoh [31] for more details.

Automorphism groups of free groups Fix $n \geq 1$, and let $H := F_n^{\mathrm{ab}}$. Since $\gamma_k(F_n)$ is a central filtration, the graded quotients $\mathrm{gr}_k(F_n) := \gamma_k(F_n)/\gamma_{k+1}(F_n)$ form a graded Lie algebra $\mathrm{gr}(F_n)$ under the commutator bracket. Witt [36] proved that $\mathrm{gr}(F_n)$ is naturally isomorphic to the free Lie algebra $\mathcal{L}(H)$ on $H = \mathrm{gr}_1(F_n)$.

Similarly, from the central filtration $\mathrm{IA}_n(k)$ we obtain a graded Lie algebra $\mathrm{gr}(\mathrm{IA}_n)$ with $\mathrm{gr}_k(\mathrm{IA}_n) := \mathrm{IA}_n(k)/\mathrm{IA}_n(k+1)$. The action of IA_n on F_n induces an injective map of Lie algebras $\tau\colon \mathrm{gr}(\mathrm{IA}_n) \hookrightarrow \mathrm{Der}(\mathrm{gr}(F_n)) \cong \mathrm{Der}(\mathcal{L}(H)) \cong \mathrm{Hom}(H, \mathcal{L}(H))$. Traditionally one thinks of the k^{th} graded piece of τ as a homomorphism $\tau_k\colon \mathrm{IA}_n(k) \to \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$ with $\ker(\tau_k) = \mathrm{IA}_n(k+1)$; the map τ_k is known as the k^{th} Johnson homomorphism. Explicitly, given $\varphi \in \mathrm{IA}_n(k)$ and $x \in F_n$ we have $\varphi(x)x^{-1} \in \gamma_{k+1}(F_n)$, and $\tau_k(\varphi) \in \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$ is the map which takes $[x] \in H$ to $[\varphi(x)x^{-1}] \in \mathrm{gr}_{k+1}(F_n) \cong \mathcal{L}_{k+1}(H)$. Determining the image of τ is a fundamental and difficult problem which has a large literature (see [31] for a discussion; we especially would like to point out the papers by Satoh [32] and Enomoto and Satoh [9]).

The universal enveloping algebra of $\mathcal{L}(H)$ is the tensor algebra $T(H)$, that is, the free associative algebra on H. Since $\mathcal{L}(H)$ is a free $\mathbb{Z}$–module, the natural map $i\colon \mathcal{L}(H) \hookrightarrow T(H)$ to its universal enveloping algebra $T(H)$ is injective by the Poincaré–Birkhoff–Witt theorem ([4, Theorem I.2.7.1]; see especially [4, Corollary I.2.7.2]).

Proof of Theorem D Fix $k \geq 1$, and let $\rho\colon \mathcal{L}_{k+1}(H) \to H \otimes \bigwedge^k H$ be the composition

$$\rho\colon \mathcal{L}_{k+1}(H) \hookrightarrow H^{\otimes k+1} \twoheadrightarrow H \otimes \bigwedge^k H$$

of the injection $i\colon \mathcal{L}_{k+1}(H) \hookrightarrow H^{\otimes k+1}$ with the natural projection. Denote by $\widehat{\tau}_k\colon \mathrm{IA}_n(k) \to \mathrm{Hom}(H, H \otimes \bigwedge^k H)$ the composition

$$\widehat{\tau}_k\colon \mathrm{IA}_n(k) \xrightarrow{\tau_k} \mathrm{Hom}(H, \mathcal{L}_{k+1}(H)) \to \mathrm{Hom}(H, H \otimes \bigwedge^k H),$$

where the second map is induced by ρ.

Consider an automorphism $\varphi \in \mathrm{IA}_n(k)$ supported on the splitting $F_n = A * B$, and let $H_A := A^{\mathrm{ab}} \subset H$. From the explicit description of $\tau_k(\varphi)$ above, it is easy to see that $\tau_k(\varphi)$ lies in the subspace $\mathrm{Hom}(H_A, \mathcal{L}_{k+1}(H_A)) \subset \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$. From the naturality of the Poincaré–Birkhoff–Witt injection, $\widehat{\tau}_k(\varphi)$ lies in $\mathrm{Hom}(H_A, H_A \otimes \bigwedge^k H_A)$. If the splitting $F_n = A * B$ has rank $r < k$, then since $H_A \cong \mathbb{Z}^r$ we have $\bigwedge^k H_A = 0$, so $\widehat{\tau}_k(\varphi) = 0$. This shows that any automorphism $\varphi \in \mathrm{IA}_n(k)$ supported on a splitting of rank less than k has $\widehat{\tau}_k(\varphi) = 0$.

To complete the proof of Theorem D, it thus suffices to show that $\widehat{\tau}_k(\mathrm{IA}_n(k)) \neq 0$ when $n > k$. Since F_n is centerless, conjugation gives an injection $\mathrm{InnerAut}\colon F_n \hookrightarrow \mathrm{IA}_n$. This corresponds under τ to the injection $\mathrm{InnerDer}\colon \mathcal{L}(H) \hookrightarrow \mathrm{Der}(\mathcal{L}(H))$:

$$\begin{array}{ccccc} F_n & & \mathrm{gr}(F_n) & \xrightarrow{\cong} & \mathcal{L}(H) \\ {\scriptstyle \mathrm{InnerAut}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{InnerDer}} \\ \mathrm{IA}_n & & \mathrm{gr}(\mathrm{IA}_n) & \xrightarrow[\tau]{} & \mathrm{Der}(\mathcal{L}(H)) \end{array}$$

Explicitly, the inner derivation corresponding to an element $\lambda \in \mathcal{L}_k(H)$ is the map $\eta_\lambda \in \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$ defined by $\eta_\lambda(h) = [\lambda, h]$ for $h \in H$.

Let $\{a_1, \dots, a_n\}$ be a free basis for H, and set

$$\lambda := [[\cdots[[a_1, a_2], a_3], \cdots], a_k] \in \mathcal{L}_k(H).$$

The commutativity of the diagram above implies that all inner derivations lie in the image of τ, so there exists some $\varphi \in \mathrm{IA}_n(k)$ with $\tau_k(\varphi) = \eta_\lambda$. It thus suffices to show that $\rho \circ \eta_\lambda \neq 0$; we do this by verifying that the element

$$\widehat{\tau}_k(\varphi)(a_{k+1}) = \rho(\eta_\lambda(a_{k+1})) = \rho([\lambda, a_{k+1}])$$

is nonzero.

The image $i([\lambda, a_{k+1}]) \in H^{\otimes k+1}$ is an alternating sum of 2^{k+1} monomials, each of the form $a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k+1)}$ for some permutation $\sigma \in \mathfrak{S}_{k+1}$. However, by induction on k we see that the only such permutation σ with $\sigma(1) = 1$ is the identity $\mathrm{id} \in \mathfrak{S}_{k+1}$. Accordingly, let $a_1^*\colon H \to \mathbb{Z}$ be the dual functional, and $(a_1^* \otimes \mathrm{id})\colon H^{\otimes k+1} \to H^{\otimes k}$ be the map that applies this functional to the first factor. We then have $(a_1^* \otimes \mathrm{id}) \circ i([\lambda, a_{k+1}]) = a_2 \otimes \cdots \otimes a_{k+1} \in H^{\otimes k}$. This projects to $a_2 \wedge \cdots \wedge a_{k+1} \in \bigwedge^k H$ under the natural projection, so $(a_1^* \otimes \mathrm{id}) \circ \rho([\lambda, a_{k+1}]) = a_2 \wedge \cdots \wedge a_{k+1} \neq 0$. This shows that $\rho([\lambda, a_{k+1}]) \neq 0 \in H \otimes \bigwedge^k H$, so $\rho \circ \eta_\lambda = \widehat{\tau}_k(\varphi) \neq 0$. This demonstrates that $\widehat{\tau}_k(\mathrm{IA}_n(k)) \neq 0$ when $n > k$, and thus completes the proof of the theorem. □

Mapping class groups We now turn to Theorem B, which requires introducing the higher Johnson homomorphisms for the mapping class group. Fix $g \geq 1$, and set $n = 2g$. Choosing an isomorphism $\pi_1(\Sigma_g^1, *) \cong F_{2g}$, we obtain an embedding of $\mathcal{I}_g^1$ into IA_n. The central filtration $\mathcal{I}_g^1(k)$ is taken to the central filtration $\mathrm{IA}_n(k)$, so we obtain an embedding $\mathrm{gr}(\mathcal{I}_g^1) \hookrightarrow \mathrm{gr}(\mathrm{IA}_n)$ of graded Lie algebras. Setting $H := F_{2g}^{\mathrm{ab}} \cong H_1(\Sigma_g^1; \mathbb{Z})$, we obtain from this embedding the k^{th} Johnson homomorphism $\tau_k \colon \mathcal{I}_g^1(k) \to \mathrm{Hom}(H, \mathcal{L}_{k+1}(H))$.

Proof of Theorem B Just like for IA_n, we define $\widehat{\tau}_k \colon \mathcal{I}_g^1(k) \to \mathrm{Hom}(H, H \otimes \bigwedge^k H)$ via the formula $\widehat{\tau}_k(\varphi) = \rho \circ \tau_k(\varphi)$, where $\rho \colon \mathcal{L}_{k+1}(H) \to H \otimes \bigwedge^k H$ is the same map as before.

Consider a subsurface $S \subset \Sigma_g^1$ such that $S \cong \Sigma_h^1$. Choose a disjoint subsurface $T \subset \Sigma_g^1$ with $T \cong \Sigma_{g-h}^1$. Fix a basepoint $*_S \in \partial S$ and an arc connecting $*_S$ to the basepoint $* \in \partial \Sigma_g^1$, and similarly for $*_T \in \partial T$. In the usual way, this determines inclusions $\pi_1(S, *_S) \hookrightarrow \pi_1(\Sigma_g^1, *)$ and $\pi_1(T, *_T) \hookrightarrow \pi_1(\Sigma_g^1, *)$. By van Kampen's theorem, we have a splitting $F_{2g} \cong \pi_1(\Sigma_g^1, *) = \pi_1(S, *_S) * \pi_1(T, *_T)$.

If $\varphi \in \mathcal{I}_g^1(k)$ is supported on the subsurface S, the induced automorphism of F_{2g} preserves this splitting, which is of rank $2h$. If $2h < k$, our computation in the proof of Theorem D thus shows that $\widehat{\tau}_k(\varphi) = 0$. Therefore $\widehat{\tau}_k$ vanishes on any element of $\mathcal{I}_g^1(k)$ supported on a subsurface Σ_h^1 of genus less than $k/2$. To complete the proof, it thus suffices to prove that $\widehat{\tau}_k(\mathcal{I}_g^1(k)) \neq 0$ when $g > k$.

In the proof of Theorem D, we made use of the map $\mathrm{InnerDer} \colon \mathcal{L}(H) \to \mathrm{Der}(\mathcal{L}(H))$, which is determined by $\mathrm{InnerDer}_1 \colon H \to \mathrm{Der}_1(\mathcal{L}(H))$. The image of $\tau_1 \colon \mathcal{I}_g^1 \to \mathrm{Der}_1(\mathcal{L}(H))$ does not contain $\mathrm{InnerDer}_1(H)$, but the work of Johnson [16] shows that $\tau_1(\mathcal{I}_g^1)$ does contain the image of another map $\mathrm{PP}_1 \colon H \to \mathrm{Der}_1(\mathcal{L}(H))$, defined as follows.

Fix a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for H and let $\omega \in \mathcal{L}_2(H)$ represent the algebraic intersection form $\widehat{\imath}$ on H, so $\omega = \sum_{i=1}^g [a_i, b_i]$. Given $x \in H$, we define

$$\mathrm{PP}_1(x) := [h \mapsto [x, h] + \widehat{\imath}(h, x)\omega] \in \mathrm{Hom}(H, \mathcal{L}_2(H)) \cong \mathrm{Der}_1(\mathcal{L}(H)).$$

The map PP_1 induces a map of Lie algebras $\mathrm{PP} \colon \mathcal{L}(H) \to \mathrm{Der}(\mathcal{L}(H))$. We remark that PP is not injective. The initials "PP" stand for "point-pushing", since the image of this map turns out to be the image under the Johnson homomorphism of the point-pushing subgroup of the mapping class group. However, neither of these facts will be necessary for our proof.

Let $L \subset H$ be the isotropic subspace $\langle a_1, \ldots, a_g \rangle$. For any $x, y \in L$ we have $\widehat{\imath}(x, y) = 0$, so

$$\mathrm{PP}_1(x)(y) = [x, y] + \widehat{\imath}(y, x)\omega = [x, y].$$

It follows by induction that for *any* $\mu_1, \mu_2 \in \mathcal{L}(L)$ we have

$$\mathrm{PP}(\mu_1)(\mu_2) = [\mu_1, \mu_2]. \tag{6-1}$$

Consider the element $\lambda := [[\cdots[a_1, a_2], \cdots], a_k] \in \mathcal{L}_k(L) \subset \mathcal{L}_k(H)$. The work of Johnson in [16, Section 6] shows that $\mathrm{Im}(\mathrm{PP}_1) \subset \tau_1(\mathcal{I}_g^1)$; indeed, generators for $\mathrm{Im}(\mathrm{PP}_1)$ can be realized by genus-$(g-1)$ bounding pairs that lie in the point-pushing subgroup. Since PP is a map of Lie algebras, it follows that there exists some $\varphi \in \mathcal{I}_g^1(k)$ with $\tau_k(\varphi) = \mathrm{PP}(\lambda)$.

As long as $g > k$ we can consider $a_{k+1} \in L$, and from (6-1) we have $\mathrm{PP}(\lambda)(a_{k+1}) = [\lambda, a_{k+1}]$. During the proof of Theorem D, we showed that $\rho([\lambda, a_{k+1}]) \neq 0$, so $\widehat{\tau}_k(\varphi)(a_{k+1}) = \rho(\mathrm{PP}(\lambda)(a_{k+1}))$ is nonzero. Thus $\widehat{\tau}_k(\mathcal{I}_g^1) \neq 0$ when $g > k$, completing the proof. □

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