Geometry & Topology 19 (2015) 3149–3191

The stable homology of congruence subgroups

FRANK CALEGARI

We relate the completed cohomology groups of $\mathrm{SL}_N(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of a number field, to K–theory and Galois cohomology. Various consequences include showing that Borel's stable classes become infinitely p–divisible up the p–congruence tower if and only if a certain p–adic zeta value is nonzero. We use our results to compute $H_2(\Gamma_N(p), \mathbf{F}_p)$ (for sufficiently large N), where $\Gamma_N(p)$ is the full level-p congruence subgroup of $\mathrm{SL}_N(\mathbf{Z})$.

11F75, 19F99; 11F80

1 Introduction

Let F be a number field, and let $\Gamma_N = \mathrm{SL}_N(\mathcal{O}_F)$. For an integer M, let $\Gamma_N(M)$ denote the principal congruence subgroup of level M, that is, the kernel of the mod-M reduction map $\Gamma_N \to \mathrm{SL}_N(\mathcal{O}_F/M)$. The cohomology and homology groups of Γ_N in any fixed degree are well known to be stable as $N \to \infty$; see Charney [14]. However, the cohomology and homology groups of $\Gamma_N(M)$ do not stabilize as $N \to \infty$ for trivial reasons (stability already fails for $H_1(\Gamma_N(p), \mathbf{F}_p)$). Given the important role that the cohomology of congruence subgroups plays in the Langlands program (see Ash [1], Scholze [38] and Calegari and Geraghty [12]), it is of interest to see whether the failure of stability can be repaired in some way. Various alternatives have been suggested, including the notion of *representation stability* by Church and Farb [18] (see also Church, Ellenberg and Farb [16; 17], and Putman [33]). The starting point for representation stability of arithmetic groups is the observation that the group $H_1(\Gamma_N(p), \mathbf{F}_p)$ for $F = \mathbf{Q}$ may be identified with the adjoint representation of $\mathrm{SL}_N(\mathbf{F}_p)$ for all $N > 2$, and that this description is, in some sense, independent of N. For example, one may ask whether $H_d(\Gamma_N(p), \mathbf{F}_p)$ admits a similar such description for all sufficiently large N (Church and Farb [18, Conjecture 8.2.1]). In contrast, the approach of Calegari and Emerton [9; 11] is to instead consider the *completed*

Published: 6 January 2016 DOI: 10.2140/gt.2015.19.3149

homology and cohomology groups

$$\begin{aligned}
\widetilde{H}_*(\mathrm{SL}_N,\mathbf{F}_p) &:= \varprojlim H_*(\Gamma_N(p^r),\mathbf{F}_p),\\
\widetilde{H}^*(\mathrm{SL}_N,\mathbf{F}_p) &:= \varinjlim H^*(\Gamma_N(p^r),\mathbf{F}_p),\\
\widetilde{H}_*(\mathrm{SL}_N,\mathbf{Z}_p) &:= \varprojlim H_*(\Gamma_N(p^r),\mathbf{Z}_p),\\
\widetilde{H}^*(\mathrm{SL}_N,\mathbf{Q}_p/\mathbf{Z}_p) &:= \varinjlim H^*(\Gamma_N(p^r),\mathbf{Q}_p/\mathbf{Z}_p).
\end{aligned}$$

There is a natural duality isomorphism $\widetilde{H}^*(\mathrm{SL}_N,\mathbf{F}_p)=\mathrm{Hom}(\widetilde{H}_*(\mathrm{SL}_N,\mathbf{F}_p),\mathbf{F}_p)$ from the universal coefficient theorem, and a corresponding isomorphism

$$\widetilde{H}^*(\mathrm{SL}_N,\mathbf{Q}_p/\mathbf{Z}_p)=\mathrm{Hom}(\widetilde{H}_*(\mathrm{SL}_N,\mathbf{Z}_p),\mathbf{Q}_p/\mathbf{Z}_p).$$

One has the following result from [9] (the generalization to number fields is immediate).

Theorem 1.1 *The modules $\widetilde{H}_d(\mathrm{SL}_N,\mathbf{Z}_p)$ stabilize as $N\to\infty$; the corresponding limits:*

$$\widetilde{H}_d(\mathrm{SL},\mathbf{Z}_p) := \lim_{N\to\infty}\widetilde{H}_d(\mathrm{SL}_N,\mathbf{Z}_p)$$

are finitely generated $\mathbf{Z}_p$–modules. Moreover, the action of $\mathrm{SL}_N(\mathcal{O}_p):=\prod_{v|p}\mathrm{SL}_N(\mathcal{O}_v)$ on $\widetilde{H}_d(\mathrm{SL}_N,\mathbf{Z}_p)$ is trivial for sufficiently large N.

(Here and elsewhere, $\mathcal{O}_p$ denotes $\prod_{v|p}\mathcal{O}_v$.) The perspective of this paper is that the groups $\widetilde{H}^*$ and $\widetilde{H}_*$ are not merely a convenient way to package information concerning the classical cohomology and homology groups, but are instead the correct object of study. From an arithmetic perspective, the classical cohomology groups $H^*(\Gamma_N(p^r),\mathbf{F}_p)$ should carry interesting information about the field $\mathcal{O}_F$. Let $G_N(p^r)$ be the kernel of the map $\mathrm{SL}_N(\mathcal{O}_p)\to\mathrm{SL}_N(\mathcal{O}/p^r)$. Note that there is an inflation map $H^*(G(p^r),\mathbf{F}_p)\to H^*(\Gamma_N(p^r),\mathbf{F}_p)$. The ultimate reason for the failure of stability of classical congruence subgroups is that the source of this map is unstable. On the other hand, the source only contains *local* information concerning F, in particular, it only depends on the decomposition of p in $\mathcal{O}_F$.[1] Hence, by taking the *direct limit over* r (or inverse limit in the case of homology), one excises all the local terms and arrives at a group that both contains all the interesting global information and is stable in N. From this optic, the relationship between representation stability (in this particular

[1] In this paper, we use *local* and *global* in the sense used by number theorists, namely, to distinguish quantities which only depend on the embeddings of F into either $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ from those that depend on more subtle invariants of F. For example, the ranks of the K–groups $K_n(\mathcal{O}_F)\otimes\mathbf{Q}$ are local because they are determined by the number of real and complex embeddings of F; see Borel [4]. In contrast, the torsion subgroups of $K_n(\mathcal{O}_F)$ are global.

context) and arithmetic disappears and, in particular, representation stability seems more to be a phenomenon related to the cohomology of p–adic Lie groups rather than arithmetic groups. Theorem 1.1 then says, in effect, that once one removes the uninteresting local terms from the cohomology of congruence subgroups, what remains is stable and finitely generated over $\mathbf{Z}_p$, and carries, as we shall demonstrate, interesting arithmetic information.

An instructive and elementary example of what information is contained in completed cohomology can already be seen in degree one, providing we consider the group GL_N rather than SL_N. Suppose that N is at least three. For convenience, let us also suppose that F admits a real embedding, so that the congruence subgroup property holds (exactly) for $\mathrm{SL}_N(\mathcal{O}_F)$ (see Bass, Milnor and Serre [2, Theorem 3.6]); more generally, by the same reference, if we assume that p is prime to the order w_F of the finite group of roots of unity in F, then the congruence kernel is abelian of order prime to p. The group $H_1(\mathrm{SL}_N(\mathcal{O}_F),\mathbf{Z})$ may be identified with the abelianization of $\mathrm{SL}_N(\mathcal{O}_F)$. The congruence subgroup property implies (under our assumptions) that any finite quotient of order prime to w_F must factor through the map $\mathrm{SL}_N(\mathcal{O}_F)\to\prod_S \mathrm{SL}_N(\mathcal{O}_{F,v})$ for some finite set of places $v\in S$. However, the abelianization of $\mathrm{SL}_N(\mathcal{O}_{F,v})$ is trivial for any finite place v (assuming that $N>2$), and so $H_1(\mathrm{SL}_N(\mathcal{O}_F),\mathbf{Z}_p)$ is trivial. The same argument shows that for the congruence subgroup $\Gamma_N(p^r)\subset\mathrm{SL}_N(\mathcal{O}_F)$, there is an isomorphism

$$H_1(\Gamma_N(p^r),\mathbf{Z}_p)=\Gamma_N(p^r)/\Gamma_N(p^{2r})\simeq G_N(p^r)/G_N(p^{2r}).$$

It follows that the natural maps $H_1(\Gamma_N(p^{2r}),\mathbf{Z}_p)\to H_1(\Gamma_N(p^r),\mathbf{Z}_p)$ will be zero, and so, in particular, the completed homology group $\widetilde{H}_1(\mathrm{SL}_N,\mathbf{Z}_p)$ vanishes. This implies that the completed homology group $\widetilde{H}_1(\mathrm{GL}_N,\mathbf{Z}_p)$ may be identified under the determinant map with the completed first homology groups of the unit group $\mathcal{O}_F^\times=\mathrm{GL}_1(\mathcal{O}_F)$. Let $\mathrm{GL}_N(\mathcal{O}_F,p^r)\subset\mathrm{GL}_N(\mathcal{O}_F)$ denote the principal congruence subgroup of level p^r. Then there is an equality

$$\widetilde{H}_1(\mathrm{GL}_N,\mathbf{Z}_p):=\varprojlim H_1(\mathrm{GL}_N(\mathcal{O}_F,p^r),\mathbf{Z}_p)=\ker\Bigl(\mathcal{O}_F^\times\otimes\mathbf{Z}_p\to\prod_{v|p}\mathcal{O}_v^\times\Bigr).$$

The claim that the right-hand side vanishes is exactly the statement of Leopoldt's conjecture, which is well known to be a deep open problem concerning the arithmetic of the field F. More precisely, our assumptions on roots of unity in F imply that $\mathcal{O}_F^\times\otimes\mathbf{Z}_p$ is torsion free; the usual statement of Leopoldt's conjecture is that the map

$$\mathcal{O}_F^\times\otimes\mathbf{Q}_p\longrightarrow\prod_{v|p}F_v^\times$$

is injective. For those unfamiliar with Leopoldt's conjecture, it may be helpful to imagine replacing the prime p by the prime ∞. The corresponding statement is then the injectivity of the map

$$\mathcal{O}_F^\times \otimes \mathbf{R} \longrightarrow \prod_{v|\infty} F_v^\times.$$

This map is exactly the classical regulator map ($u \mapsto (\log|u|_v)$) whose injectivity was established by Dirichlet as the first step in proving the unit theorem. In this paper, we shall argue how analogous questions and conjectures concerning the groups $\widetilde{H}_d(\mathrm{SL}, \mathbf{Z}_p)$ for higher d concern the (conjectural) injectivity of higher p–adic regulator maps, and bear the same relation to Borel's higher regulator maps as Leopoldt's conjecture bears to the classical regulator map. (Why the p–adic versions of these maps seem much harder to control than their real analogues is somewhat of a mystery.)

Theorem 1.1 gives a very good qualitative description of completed cohomology in the stable range. The main concern of this paper, which can be considered a sequel to Calegari and Emerton's [9], is then to address the groups $\widetilde{H}_*$ in a *quantitative* manner. One may also ask whether information concerning $\widetilde{H}_*$ or $\widetilde{H}^*$ can be translated into information concerning the classical cohomology groups. For example, Benson Farb asked the author whether Theorem 1.1 can be utilized for the *computation* of explicit cohomology groups. In order to answer the broader question of what it might mean to *compute* classical cohomology groups, we first recall what happens at full level. The stable homology of $\mathrm{GL}_N(\mathbf{Z})$ may be identified with the homology of $\mathrm{BGL}(\mathbf{Z})^+$, and the *homotopy* groups of $\mathrm{BGL}(\mathbf{Z})^+$ for $n \geq 1$ are the algebraic K–groups $K_n(\mathbf{Z})$, which are finitely generated over $\mathbf{Z}$ and are completely known, at least in terms of Galois cohomology groups (see Theorem 2.3). The ranks of the *rational* cohomology groups are determined by the signature of the field F by a theorem of Borel [4]. In this paper, we show that:

- The groups $\widetilde{H}_*$ are the continuous homology groups of a certain homotopy fiber Y_∞ (Theorem 2.17) whose homotopy groups with coefficients in $\mathbf{Z}_p$ we can calculate rationally (Lemma 2.4), and even integrally in many cases, in terms of Galois cohomology groups.
- Assuming, in addition, a generalization of Leopoldt's conjecture, we can also give (Theorem 3.2) a complete description of $\widetilde{H}_*(\mathrm{SL}, \mathbf{Q}_p) := \widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q}$; note that this is different from the inverse limit with coefficients in $\mathbf{Q}_p$, which, by Borel's theorem, coincides with the stable rational homology at level one.
- For $F = \mathbf{Q}$, we compute $H_2(\Gamma_N(p), \mathbf{F}_p)$ for sufficiently large N (Corollary 4.4), answering a question of Farb).
- For very regular primes (Definition 5.1) $p = \mathfrak{p}\overline{\mathfrak{p}}$ in an imaginary quadratic field, we compute $H_*(\Gamma_N(\mathfrak{p}^m), \mathbf{Z}_p)$ in the stable range explicitly (Theorem 5.7).

We give an example of a precise theorem we can state now. Let G_S denote the Galois group of the maximal Galois extension of F unramified away from primes dividing p and ∞. Let w_F denote the number of roots of unity in F.

Theorem 1.2 *If p does not divide $w_F \cdot |K_2(\mathcal{O}_F)|$, then there is an isomorphism*

$$\widetilde{H}^2(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \simeq H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1)).$$

For all p, the equality holds up to a finite group. If $F = \mathbf{Q}$ and $p \geq 3$, then $\widetilde{H}^2(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) = \mathbf{Q}_p/\mathbf{Z}_p$ and $\widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) \simeq \mathbf{Z}_p$.

We expect that the equality holds without any assumption on the order of $K_2(\mathcal{O}_F)$, and this would follow from Conjecture 2.5. (Note, however, that $K_2(\mathcal{O}_F)$ is always finite.) This theorem was inspired by the results of Calegari and Venkatesh [13], particularly Chapter 8, which suggests a link between classes in the cohomology of $H^2(\Gamma_N(M), \mathbf{Q}_p/\mathbf{Z}_p)$ of congruence subgroups and classes in $H^1(F, \mathbf{Q}_p/\mathbf{Z}_p(-1))$ unramified outside M (see, in particular, the discussion in [13, Section 8.3]).

Acknowledgements I would like to thank Benson Farb for asking the question that led to this paper. I would also like to thank Matthew Emerton, Eric Friedlander, Paul Goerss, Andrei Suslin, Akshay Venkatesh, and Chuck Weibel for useful conversations, in particular to Eric Friedlander to suggesting the use of the Zeeman comparison theorem, and Paul Goerss for suggesting that the Eilenberg–Moore spectral sequence was (in certain contexts) a better way to compute the cohomology of a fiber in a fibration than the Serre spectral sequence. I would also like to thank Toby Gee (and other members of the Imperial reading group) for some helpful comments on an earlier version of this manuscript. The influence of [13] on this paper should also be clear. Finally, I would like to thank the tremendous job of the referee, who, apart from a very thorough report which helped to correct a number of minor errors, lacunae, and other inaccuracies (both mathematical and pedagogical), also taught me some algebraic topology.

The author was supported in part by NSF Career Grant DMS-0846285 and NSF Grant DMS-1404620.

2 Homotopy

2.1 Completed K–theory

In this section, we define the completed K–groups $\widetilde{K}_*(\mathcal{O})$ of $\mathcal{O}$ with respect to a prime p, at least for the ring of integers $\mathcal{O} = \mathcal{O}_F$ of a number field F. Here the notion of "completion" refers (in principle) to replacing $\Gamma_N = \mathrm{GL}_N(\mathcal{O})$ by the congruence subgroups $\Gamma_N(p^m)$ for arbitrarily large m, rather than other possible forms of completion; the notation is chosen via the analogy with the completed homology groups $\widetilde{H}_*$. The groups $\widetilde{K}_n(\mathcal{O})$ are essentially, depending on the parity of n, either the kernel or cokernel of Soulé's p–adic regulator map [40].

For any prime p and integer m, define $X_m = X_m(\mathcal{O}, p^m)$ to be the homotopy fiber of the map

$$\mathrm{BGL}(\mathcal{O})^+ \longrightarrow \mathrm{BGL}(\mathcal{O}/p^m)^+.$$

The homotopy groups of X_m live in a long exact sequence with the K–groups of $\mathcal{O}$ and of $\mathcal{O}/p^m$ (at least for $n \geq 1$). In particular, by a result of Quillen [35], they are finitely generated (see also [14, Corollary 3.8]). The map $B\Gamma(p^m) \to X_m$ is *not*, however, a homological equivalence; this is the so called "failure of excision." (This remark was made in the introduction to [15].) The idea behind this paper is that, since the *limiting* groups $\widetilde{H}_*$ have a trivial action of $\mathrm{SL}(\mathcal{O}_p)$, then some analogue of excision will hold[2] for $m = \infty$.

Since $\pi_0(\mathrm{BGL}(R)^+) = 0$ does not correspond to $K_0(R)$, it is technically more convenient to work with the homotopy fibrations of K–theory spaces:

$$K(\mathcal{O}, p^m) \longrightarrow K(\mathcal{O}) \longrightarrow K(\mathcal{O}_p/p^m),$$

$$K(\mathcal{O}, \mathcal{O}_p) \longrightarrow K(\mathcal{O}) \longrightarrow K(\mathcal{O}_p).$$

Note that $K(R, I)$ usually denotes the homotopy fiber of the map $K(R) \to K(R/I)$, and this is the meaning of $K(\mathcal{O}, p^m)$; by abuse of notation, we let $K(\mathcal{O}, \mathcal{O}_p)$ be the homotopy fiber of $K(\mathcal{O}) \to K(\mathcal{O}_p)$; hopefully no confusion will arise.

We make the following definition:

Definition 2.1 The completed K–groups $\widetilde{K}_n(\mathcal{O}) := K_n(\mathcal{O}, \mathcal{O}_p; \mathbf{Z}_p)$ with respect to a prime p are, for $n \geq 1$, the homotopy groups $\pi_n(K(\mathcal{O}, \mathcal{O}_p); \mathbf{Z}_p)$.

We make below (Definition 2.7) an ad hoc definition of $\widetilde{K}_0(\mathcal{O}, \mathcal{O}_p; \mathbf{Z}_p)$.

Lemma 2.2 *Let $\mathcal{O} = \mathcal{O}_F$ for a number field F. There is a long exact sequence*

$$\longrightarrow K_n(\mathcal{O}, \mathcal{O}_p; \mathbf{Z}_p) \longrightarrow K_n(\mathcal{O}) \otimes \mathbf{Z}_p \longrightarrow K_n(\mathcal{O}_p; \mathbf{Z}_p) \longrightarrow K_{n-1}(\mathcal{O}, \mathcal{O}_p; \mathbf{Z}_p) \longrightarrow \cdots.$$

[2]Literally, of course, this is not true. Since the homotopy fiber of $\mathrm{BGL}(\mathcal{O}) \to \mathrm{BGL}(\mathcal{O}/p^m)$ is $B\Gamma(p^m)$, one way of rephrasing the difficulty above is that the $+$ construction does not commute with taking homotopy fibers. The homotopy fiber of $\mathrm{BGL}(\mathcal{O}) \to \mathrm{BGL}(\mathcal{O}_p)$, however, is essentially the discrete set $\mathrm{GL}(\mathcal{O}_p)/\mathrm{GL}(\mathcal{O})$; the map on π_1 is injective and there are no higher homotopy groups. The plus construction applied to the discrete set $\mathrm{GL}(\mathcal{O}_p)/\mathrm{GL}(\mathcal{O})$ (whatever the plus construction would mean here), however, is certainly not the homotopy fiber of $\mathrm{BGL}(\mathcal{O})^+ \to \mathrm{BGL}(\mathcal{O}_p)^+$.

Proof We first note that all the relevant spaces $K(\mathcal{O},\mathcal{O}_p)$, $K(\mathcal{O})$, $K(\mathcal{O}_p)$, etc we consider are infinite loop spaces. We now recall some basic constructions, definitions, and theorems concerning K–theory with coefficients.

(1) [46, IV.2.1] The homotopy groups with coefficients in $\mathbf{Z}/p^m$ are defined to be

$$\pi_n(X;\mathbf{Z}/p^m) := [P^n(\mathbf{Z}/p^m), X],$$

where $P^n(\mathbf{Z}/p^m)$ is the space formed from the sphere S^{n-1} by attaching an n–cell via a degree p^m map. In general, this space is a group for $n \geq 3$ and a set for $n=2$, however, it is well defined as a group for all $n \geq 1$ when X is an infinite loop space. Given a Serre fibration, there is a naturally associated long exact sequence.

(2) [46, IV.2.9] The p–adically completed homotopy groups $\pi_n(\boldsymbol{E};\mathbf{Z}_p)$ of a spectrum $\boldsymbol{E}$ are defined to be the homotopy groups of the homotopy limit $\widehat{\boldsymbol{E}}$ of $\boldsymbol{E} \wedge P^\infty(\mathbf{Z}/p^m)$. If the $\pi_*(\boldsymbol{E};\mathbf{Z}/p^m)$ are finite for all m, then

$$\pi_*(\boldsymbol{E};\mathbf{Z}_p) \simeq \operatorname{proj\,lim} \pi_*(\boldsymbol{E};\mathbf{Z}/p^m).$$

(3) [46, IV.2.2] There is a universal coefficient sequence

$$0 \longrightarrow \pi_n(X) \otimes \mathbf{Z}/p^m \longrightarrow \pi_n(X;\mathbf{Z}/p^m) \longrightarrow \pi_{n-1}(X)[p^m] \longrightarrow 0.$$

Note that $K_n(\mathcal{O}) := \pi_n(K(\mathcal{O}))$ for all n. The groups $K_n(\mathcal{O})$ for $\mathcal{O} = \mathcal{O}_F$ the ring of integers of a number field are known to be finitely generated abelian groups, and hence the projective limit $\operatorname{proj\,lim} K_{n-1}(\mathcal{O})[p^m]$ vanishes. It follows from the universal coefficient sequence that

$$K_n(\mathcal{O};\mathbf{Z}_p) = \operatorname{proj\,lim} K_n(\mathcal{O};\mathbf{Z}/p^m) = K_n(\mathcal{O}) \otimes \mathbf{Z}_p.$$

The groups $K_n(\mathcal{O}_p;\mathbf{Z}/p^r)$ are similarly finite [25, Theorem A], and so

$$K_n(\mathcal{O}_p;\mathbf{Z}_p) = \operatorname{proj\,lim} K_n(\mathcal{O}_p;\mathbf{Z}/p^m).$$

The Serre long exact sequence gives a sequence

$$\cdots \longrightarrow \pi_n(K(\mathcal{O},\mathcal{O}_p);\mathbf{Z}/p^m)) \longrightarrow K_n(\mathcal{O};\mathbf{Z}/p^m) \longrightarrow K_n(\mathcal{O}_p;\mathbf{Z}/p^m) \longrightarrow \cdots .$$

We see that the $\mathbf{Z}_p$–modules $\pi_n(K(\mathcal{O},\mathcal{O}_p),\mathbf{Z}/p^m)$ are finite and also have bounded rank (as $\mathbf{Z}_p$–modules) for all m by comparison with the surrounding terms. Hence, taking an inverse limit in m and replacing $K_n(\mathcal{O};\mathbf{Z}_p)$ by $K_n(\mathcal{O}) \otimes \mathbf{Z}_p$, we obtain the desired exact sequence. (The finiteness properties guarantee that the relevant Mittag-Leffler conditions are automatically satisfied, and so there are no issues concerning $\lim^1$.) For the tail of the long exact sequence involving terms with $n=0$, see the remarks after Definition 2.7. □

It will also be useful to work with the homotopy fibrations

$$
\begin{aligned}
Y_m = SK(\mathcal{O}, p^m) &\longrightarrow SK(\mathcal{O}) \longrightarrow SK(\mathcal{O}/p^m), \\
Y_\infty = SK(\mathcal{O}, \mathcal{O}_p) &\longrightarrow SK(\mathcal{O}) \longrightarrow SK(\mathcal{O}_p).
\end{aligned} \tag{1}
$$

Since $\pi_n(\mathrm{BSL}(R)^+) \simeq \pi_n(\mathrm{BGL}(R)^+) = \pi_n(SK(R))$ for $n \geq 2$, one obtains isomorphisms $\pi_n(Y_\infty; \mathbf{Z}_p) \simeq K_n(\mathcal{O}, \mathcal{O}_p; \mathbf{Z}_p)$ for $n \geq 2$ and $m \in \mathbf{N} \cup \infty$. Working with Y_∞ allows us to apply the Hurewicz theorem more usefully, since, for many fields F, the fiber Y_∞ will be simply connected.

2.2 Tate global duality

Let $S = \{v \mid p\} \cup \{v \mid \infty\}$, and let G_S denote the Galois group of the maximal extension of F unramified outside S. If M is a finite G_S–module of odd order, then

$$H^*(G_S, M) \simeq H^*(\mathrm{Spec}(\mathcal{O}[1/p]), M)$$

is finite. If n is a positive integer, let $\mathbf{Z}_p(n)$ denote the n^{th} Tate twist of $\mathbf{Z}_p$, that is, the module $\mathbf{Z}_p$ such that the action of $G_{\mathbf{Q}}$ is via the n^{th} power of the cyclotomic character. Similarly, the module $\mathbf{F}_p(n)$ denotes the corresponding twist of the module $\mathbf{F}_p$. For a compact or discrete $\mathbf{Z}_p$–module A, let $A^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(A, \mathbf{Q}_p/\mathbf{Z}_p)$ denote the Pontryagin dual of A (so canonically $A^{\vee\vee} \simeq A$). For a compact or discrete G_S–module M, let M^* denote the twisted Pontryagin dual $\mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Q}_p/\mathbf{Z}_p(1))$. For a finite G_S–module M, recall [29; 41] that by Poitou–Tate duality there is an exact sequence

$$
\begin{aligned}
0 \longrightarrow H^0(G_S, M) &\longrightarrow \prod_S H^0(G_v, M) \longrightarrow H^2(G_S, M^*)^\vee \\
\longrightarrow H^1(G_S, M) &\longrightarrow \prod_S H^1(G_v, M) \longrightarrow H^1(G_S, M^*)^\vee \\
\longrightarrow H^2(G_S, M) &\longrightarrow \prod_S H^2(G_v, M) \longrightarrow H^0(G_S, M^*)^\vee \longrightarrow 0.
\end{aligned} \tag{2}
$$

By taking limits, one also obtains a corresponding sequence for compact M, where cohomology of compact or discrete modules is taken in the usual continuous sense. Specifically, let n be a positive integer and let $M = \mathbf{Z}_p(n)$. Then

$$M^* = \mathrm{Hom}(M, \mathbf{Q}_p/\mathbf{Z}_p(1)) = \mathbf{Q}_p/\mathbf{Z}_p(1-n).$$

Moreover, $H^0(G_S, M) = H^0(G_v, M) = 0$. We thus obtain the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^2(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee \longrightarrow H^1(G_S, \mathbf{Z}_p(n)) \longrightarrow \prod_S H^1(G_v, \mathbf{Z}_p(n)) \\ &\longrightarrow H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee \longrightarrow H^2(G_S, \mathbf{Z}_p(n)) \longrightarrow \prod_S H^2(G_v, \mathbf{Z}_p(n)) \\ &\longrightarrow H^0(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee \longrightarrow 0. \end{aligned}$$

The following result follows from work of many, including Soulé [39], Madsen and Hesselholt [25], Voevodsky, Rost, Suslin, Weibel and others [42; 45]:

Theorem 2.3 *Let $p > 2$. For $n > 1$, there are isomorphisms*

$$K_{2n-1}(\mathcal{O}) \otimes \mathbf{Z}_p \simeq H^1(G_S, \mathbf{Z}_p(n)), \qquad K_{2n-2}(\mathcal{O}) \otimes \mathbf{Z}_p \simeq H^2(G_S, \mathbf{Z}_p(n)),$$
$$K_{2n-1}(\mathcal{O}_p; \mathbf{Z}_p) \simeq \prod_S H^1(G_v, \mathbf{Z}_p(n)), \quad K_{2n-2}(\mathcal{O}_p; \mathbf{Z}_p) \simeq \prod_S H^2(G_v, \mathbf{Z}_p(n)).$$

Also, the maps $K_n(\mathcal{O}) \otimes \mathbf{Z}_p \to K_n(\mathcal{O}_p; \mathbf{Z}_p)$ induce the natural maps on cohomology.

For specific references, we refer to [25, Theorem A] for the K–theory of (rings of integers in) local fields (see also [45, Theorem 61]), and Weibel's survey [45, Theorem 70] for the K–theory of global fields. We note that, in the latter reference, the Galois cohomology groups have been excised from the statement for the benefit of topologists, but they are easily extracted from the argument. The compatibility of the these isomorphisms follows from the compatibility of the corresponding motivic spectral sequences. Note that for $p = 2$, the equalities hold up to a (known) finite group, and hold on the nose if F is totally imaginary. For the groups K_1, which are known classically, one must make a minor adjustment to these descriptions in terms of Galois cohomology: this amounts (for experts) to replacing the Galois cohomology groups H^1 with the Bloch–Kato groups H^1_f. More prosaically, there are Kummer isomorphisms

$$H^1(G_S, \mathbf{Z}_p(1)) = (\mathcal{O}_F[1/p])^\times \otimes \mathbf{Z}_p, \quad H^1(G_v, \mathbf{Z}_p(1)) = F_v^\times \otimes \mathbf{Z}_p,$$

whereas the K_1 groups we are interested in should be identified with the groups

$$H^1_f(G_S, \mathbf{Z}_p(1)) = (\mathcal{O}_F)^\times \otimes \mathbf{Z}_p, \quad H^1_f(G_v, \mathbf{Z}_p(1)) = \mathcal{O}_v^\times \otimes \mathbf{Z}_p$$

respectively. This discrepancy is related to the fact that $K_*(\mathcal{O}_F[1/p])$ and $K_*(\mathcal{O}_F)$ coincide in higher degrees but not in degree one. We now have:

Lemma 2.4 *Define the groups* $\widetilde{K}^{?}_n(\mathcal{O})$ *as*

$$\widetilde{K}^{?}_{2n-1}(\mathcal{O}) := H^2(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee \oplus H^0(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-n))^\vee,$$
$$\widetilde{K}^{?}_{2n-2}(\mathcal{O}) := H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee.$$

Then there is an exact sequence

$$\cdots \longrightarrow \widetilde{K}^{?}_n(\mathcal{O}) \longrightarrow K_n(\mathcal{O}) \otimes \mathbf{Z}_p \longrightarrow K_n(\mathcal{O}_p; \mathbf{Z}_p) \longrightarrow \widetilde{K}^{?}_{n-1}(\mathcal{O}) \longrightarrow \cdots.$$

There are rational isomorphisms $\widetilde{K}_n(\mathcal{O}) \otimes \mathbf{Q} = \widetilde{K}^{?}_n(\mathcal{O}) \otimes \mathbf{Q}$. *If either* $K_n(\mathcal{O})$ *is finite of order prime to* p *or* $K_{n+1}(\mathcal{O}_p; \mathbf{Z}_p) = 0$, *then there is an isomorphism*

$$\widetilde{K}_n(\mathcal{O}) = \widetilde{K}^{?}_n(\mathcal{O}).$$

Proof The exact sequence follows directly from the Poitou–Tate sequence (2) above (one also has to make an easy check for $n = 1$). It follows that *if* there existed maps

$$\widetilde{K}_n(\mathcal{O}) \to \widetilde{K}^{?}_n(\mathcal{O})$$

inducing a commutative map between long exact sequences, then $\widetilde{K}_n(\mathcal{O}) \simeq \widetilde{K}^{?}_n(\mathcal{O})$ by the 5–lemma. Such a map exists after tensoring with $\mathbf{Q}$ since $\mathbf{Q}_p$ is projective. If $K_n(\mathcal{O}) \otimes \mathbf{Z}_p = 0$, then both $\widetilde{K}_n(\mathcal{O})$ and $\widetilde{K}^{?}_n(\mathcal{O})$ are isomorphic to the quotient of $K_{n+1}(\mathcal{O}_p; \mathbf{Z}_p)$ by the image of $K_{n+1}(\mathcal{O}) \otimes \mathbf{Z}_p$. Similarly, if $K_{n+1}(\mathcal{O}_p; \mathbf{Z}_p) = 0$, then both $\widetilde{K}_n(\mathcal{O})$ and $\widetilde{K}^{?}_n(\mathcal{O})$ are isomorphic to the kernel of the map

$$K_n(\mathcal{O}) \otimes \mathbf{Z}_p \to K_n(\mathcal{O}_p; \mathbf{Z}_p). \qquad \square$$

We make the following conjecture.

Conjecture 2.5 *There is an isomorphism* $\widetilde{K}_n(\mathcal{O}) \simeq \widetilde{K}^{?}_n(\mathcal{O})$.

This seems natural enough for n even. For n odd, the conjecture also seems natural in light of the fact that we expect the term $H^2(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))^\vee$ to vanish (see also Remark 2.10).

Remark 2.6 There is some hope to prove this statement by constructing a natural map $\widetilde{K}_n(\mathcal{O}) \to \widetilde{K}^{?}_n(\mathcal{O})$. The author has had some discussions with Matthew Emerton regarding this question, and we hope to return to it in the future.

Definition 2.7 (The group $\widetilde{K}_0(\mathcal{O})$) We now extend our definitions to $n = 0$. For $n = 0$, let

$$\widetilde{K}_0(\mathcal{O}) := \widetilde{K}^{?}_0(\mathcal{O}) = H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p)^\vee.$$

There is an isomorphism $H^1(G_S,\mathbf{Q}_p/\mathbf{Z}_p)^\vee = \mathrm{Hom}(G_S,\mathbf{Q}_p/\mathbf{Z}_p)^\vee = (G_S)^{\mathrm{ab}}\otimes\mathbf{Z}_p$. There are isomorphisms $K_1(\mathcal{O})\simeq\mathcal{O}^\times$ and $K_1(\mathcal{O}_p;\mathbf{Z}_p)=\mathbf{Z}_p\otimes\mathcal{O}_p^\times=\mathbf{Z}_p\otimes\prod_{v|p}\mathcal{O}_v^\times$. Moreover, $K_0(\mathcal{O})\simeq\mathrm{Pic}(\mathcal{O})=\mathrm{Cl}(\mathcal{O})\oplus\mathbf{Z}$. From class field theory, there is an exact sequence

$$\mathcal{O}^\times\otimes\mathbf{Z}_p\longrightarrow\Big(\prod_{v|p}\mathcal{O}_p^\times\Big)\otimes\mathbf{Z}_p\longrightarrow(G_S)^{\mathrm{ab}}\otimes\mathbf{Z}_p\longrightarrow\mathrm{Cl}(\mathcal{O})\otimes\mathbf{Z}_p\longrightarrow 0.$$

Hence the long exact sequence of Lemma 2.2 continues as far as

$$\cdots\longrightarrow K_1(\mathcal{O})\otimes\mathbf{Z}_p\longrightarrow K_1(\mathcal{O}_p;\mathbf{Z}_p)\longrightarrow\widetilde{K}_0(\mathcal{O})\longrightarrow K_0(\mathcal{O})\otimes\mathbf{Z}_p\longrightarrow\mathbf{Z}_p\longrightarrow 0.$$

After tensoring with $\mathbf{Q}$, the long exact sequence of Poitou–Tate and hence of K–groups breaks up into exact sequences of length 6 (the $H^0\otimes\mathbf{Q}$ term vanishes for $n>1$). One immediately obtains the following:

Lemma 2.8 *There is an equality*

$$\dim\widetilde{K}_{2n-2}(\mathcal{O})\otimes\mathbf{Q}-\dim\widetilde{K}_{2n-1}(\mathcal{O})\otimes\mathbf{Q}=\begin{cases}r_1+r_2 & n>1\ \textit{even},\\ r_2 & n>1\ \textit{odd},\\ r_2+1 & n=1.\end{cases}$$

Proof By computing the Euler characteristic of the six term exact sequence, the result follows from Theorems 2.12 and 2.13 below. Alternatively, the result follows for $\widetilde{K}^?_*$ by the global Euler characteristic formula [29; 41]. □

A complete evaluation of the rank of $\widetilde{K}_*$ would follow (and is equivalent to) from the following conjecture, which is already implicit in the work of Soulé:

Conjecture 2.9 *For all n,* $\dim\widetilde{K}_{2n-1}(\mathcal{O})\otimes\mathbf{Q}=0$.

For $n=1$, this is Leopoldt's conjecture. For totally real fields, Conjecture 2.9 is equivalent to the nonvanishing of a certain p–adic zeta function. This equivalence follows from of [40, Theorem 3] (see also [40, Remark 3.4, page 399]). Actually, Soulé assumes that F is abelian, but the general case follows from the proof of the Quillen–Lichtenbaum conjecture (giving a cohomological description of the global and local K groups) together with the main conjecture for totally real fields proved by Wiles [47]. In the general case, it is equivalent to showing that the kernel of the p–adic regulator map on $K_{2n-1}(\mathcal{O}_F)\otimes\mathbf{Q}_p$ is zero. The first nontrivial case of this conjecture for $F=\mathbf{Q}$ is $n=3$, where (as noted above) it "reduces" to the question of the nonvanishing of the Kubota–Leopoldt zeta function $\zeta_p(3)$. One might actually make the stronger conjecture that this number is irrational [7]. For more discussion of injectivity of localization maps, see [3].

Remark 2.10 Since G_S has cohomological dimension 2 (with coefficients in $\mathbf{Z}_p$ for $p \neq 2$), the group $H^2(G_S, \mathbf{Q}_p/\mathbf{Z}_p(1-n))$ is divisible and hence its dual is torsion free. In particular, if Conjecture 2.9 is true, then this H^2 term vanishes, and

$$\widetilde{K}^?_{2n-1}(\mathcal{O}) = H^0(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-n))^\vee \simeq H_0(G_S, \mathbf{Z}_p(n)) = \mathbf{Z}_p/(a^n-1)\mathbf{Z}_p,$$

where a is any topological generator of $\mathbf{Z}_p^\times$. In particular, $\widetilde{K}^?_{2n-1}(\mathcal{O}) = 0$ unless $n \equiv 0 \bmod p-1$.

More generally, we have the following estimates.

Lemma 2.11 *Suppose that F is a number field of degree d and signature (r_1, r_2). If p does not divide the order of the class group of $F(\zeta_p)$, then, for $n > 0$, we have equalities*

$$\dim \widetilde{K}_n(\mathcal{O}) \otimes \mathbf{Q} = \begin{cases} r_2 & n \equiv 0 \bmod 4, \\ r_1 + r_2 & n \equiv 2 \bmod 4, \\ 0 & n \equiv 1 \bmod 2. \end{cases}$$

If $r_2 = 0$, so F is a totally real field of degree d, then unconditionally

$$\dim \widetilde{K}_{4n-1}(\mathcal{O}) \otimes \mathbf{Q} = 0, \quad \dim \widetilde{K}_{4n-2}(\mathcal{O}) \otimes \mathbf{Q} = d.$$

In particular, $\widetilde{K}_2(\mathcal{O}) \otimes \mathbf{Q} = \widetilde{H}_2 \otimes \mathbf{Q} = \mathbf{Q}_p^d$.

Proof We begin by recalling Borel's theorem [4, Proposition 12.2]:

Theorem 2.12 (Borel) *Suppose that F is a number field of degree d and signature (r_1, r_2). For $n > 0$, we have equalities*

$$\dim K_n(\mathcal{O}) \otimes \mathbf{Q} = \begin{cases} r_1 + r_2 - 1 & n = 1, \\ r_1 + r_2 & n \equiv 1 \bmod 4 \text{ and } n > 1 \\ r_2 & n \equiv 3 \bmod 4, \\ 0 & n \equiv 0 \bmod 2. \end{cases}$$

We also have the following result [45, Theorem 61], which (as noted in [45]) is essentially due to Wagoner and Milgram [43] and Panin [32]. (Alternatively, this result follows directly from local Tate duality and the Euler characteristic formula given the identification of these groups with Galois cohomology in Theorem 2.3.)

Theorem 2.13 *Let p be a rational prime, and let F be a number field of degree d and signature (r_1, r_2). For $n > 0$, we have equalities*

$$\dim K_n(\mathcal{O}_p; \mathbf{Z}_p) \otimes \mathbf{Q} = \begin{cases} d = r_1 + 2r_2 & n \equiv 1 \bmod 2 \\ 0 & n \equiv 0 \bmod 2. \end{cases}$$

All the equalities now follow from a diagram chase, assuming that the maps

$$H^1(G_S, \mathbf{Z}_p(n)) \longrightarrow \prod_{v|p} H^1(G_S, \mathbf{Z}_p(n))$$

are injective after tensoring with $\mathbf{Q}$. Specifically, the groups $\widetilde{K}_n(\mathcal{O}) \otimes \mathbf{Q}$ will then identified with the cokernel of the map from $K_{n+1}(\mathcal{O};\mathbf{Z}_p)\otimes\mathbf{Q}$ to $K_{n+1}(\mathcal{O}_p;\mathbf{Z}_p)\otimes\mathbf{Q}$, and hence have dimension zero when n is odd and, for $n > 1$, dimension

$$\begin{cases} (r_1 + 2r_2) - (r_1 + r_2) & n \equiv 0 \bmod 4, \\ (r_1 + 2r_2) - r_2 & n \equiv 2 \bmod 4. \end{cases}$$

The injectivity of the map of Galois cohomology groups is trivial when the group on the left is actually zero after tensoring with $\mathbf{Q}$, which accounts for the unconditional cases when F is totally real and $r_2 = 0$. Thus we may assume that p does not divide the class number of $F(\zeta_p)$. But any element in the kernel of the map on cohomology will give rise to unramified nontrivial extension classes of $\mathbf{Z}_p$ by $\mathbf{Z}_p(n)$. In particular, the first layer will generate an unramified extension of $F(\mathbf{F}_p(n)) \subseteq F(\zeta_p)$ of degree p, which would contradict the assumption on the class number. □

Note that Conjecture 2.9 would imply that the equality of ranks holds for all primes, not just regular ones. We also note that these regulator maps in the optic of Galois cohomology were also studied by Schneider [36; 37]. In particular, by [36, Satz 3, Section 6], one has:

Proposition 2.14 *For any fixed number field field F and prime p, Conjecture 2.9 is true for all but finitely many n.*

2.3 Comparison of homologies

Lemma 2.15 *Let A_j and B_j be a sequence of $\mathbf{F}_p$–modules with trivial G–action, and suppose that $A_0 = B_0 = \mathbf{F}_p$. Suppose that one has spectral sequences over $\mathbf{F}_p$,*

$$EA^2_{i,j} := H_i(G, A_j) \Rightarrow C_{i+j}, \quad EB^2_{i,j} = H_i(G, B_j) \Rightarrow C_{i+j},$$

together with compatible maps $EA^m_{i,j} \to EB^m_{i,j}$ for all $m \geq 2$ inducing an automorphism of C_{i+j}. Suppose that the maps

$$H_i(G, A_j) = EA^2_{i,j} \longrightarrow EB^2_{i,j} = H_i(G, B_j)$$

are induced from the maps

$$A_j = EA^2_{0,j} \longrightarrow EB^2_{0,j} = B_j.$$

Then there is an isomorphism $A_j \simeq B_j$ for all j.

Proof Since A_q is a trivial G–module, there is a canonical isomorphism

$$EA^2_{p,q} = H_p(G, A_q) = H_p(G, \mathbf{F}_p) \otimes A_q = H_p(G, A_0) \otimes H_0(G, A_q) = EA^2_{0,q} \otimes EA^2_{p,0}.$$

The assumed compatibility implies that there is a commutative diagram as follows:

$$\begin{tikzcd} EA^2_{0,q} \otimes EA^2_{p,0} \arrow[r, equal] \arrow[d] & EA^2_{p,q} \arrow[d] \\ EB^2_{0,q} \otimes EB^2_{p,0} \arrow[r, equal] & EB^2_{p,q} \end{tikzcd}$$

The result then follows from the Zeeman comparison theorem [28, Theorem 3.26]. (Note that, since the coefficient ring is a field, one may disregard the Tor_1 terms.) □

Definition 2.16 For a space X, define the continuous homology groups $H^{\mathrm{cont}}_*(X, \mathbf{Z}_p)$ to be $H^{\mathrm{cont}}_*(X, \mathbf{Z}_p) := \operatorname{proj}\lim H_*(X, \mathbf{Z}/p^n)$.

Theorem 2.17 *There are isomorphisms*

$$\widetilde{H}_*(\mathrm{SL}, \mathbf{F}_p) \simeq \varprojlim H_*(Y_m, \mathbf{F}_p) \simeq H_*(Y_\infty, \mathbf{F}_p),$$
$$\widetilde{H}_*(\mathrm{SL}, \mathbf{Z}/p^r\mathbf{Z}) \simeq \varprojlim H_*(Y_m, \mathbf{Z}/p^r\mathbf{Z}) \simeq H_*(Y_\infty, \mathbf{Z}/p^r\mathbf{Z}),$$
$$\widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p) \simeq \varprojlim H^{\mathrm{cont}}_*(Y_m, \mathbf{Z}_p) \simeq H^{\mathrm{cont}}_*(Y_\infty, \mathbf{Z}_p).$$

Proof Consider the following diagram:

$$\begin{tikzcd} \mathrm{B}\Gamma(p^m) \arrow[r] \arrow[d] & \mathrm{BSL}(\mathcal{O}) \arrow[r] \arrow[d] & \mathrm{BSL}(\mathcal{O}/p^m) \arrow[d] \\ Y_m \arrow[r] & \mathrm{BSL}(\mathcal{O})^+ \arrow[r] & \mathrm{BSL}(\mathcal{O}/p^m)^+ \\ Y_\infty \arrow[r] \arrow[u] & \mathrm{BSL}(\mathcal{O})^+ \arrow[r] \arrow[u, equal] & \mathrm{BSL}(\mathcal{O}_p)^+ \arrow[u] \end{tikzcd}$$

We obtain corresponding maps

$$H_*(\Gamma(p^m), \mathbf{F}_p) \longrightarrow H_*(Y_m, \mathbf{F}_p) \longleftarrow H_*(Y_\infty, \mathbf{F}_p).$$

Moreover, we have a natural map of spectral sequences:

$$\begin{tikzcd} H_i(\mathrm{SL}(\mathcal{O}/p^m), H_j(\Gamma(p^m), \mathbf{F}_p)) \arrow[r, Rightarrow] \arrow[d] & H_{i+j}(\Gamma, \mathbf{F}_p) \arrow[d, equal] \\ H_i(\mathrm{SL}(\mathcal{O}/p^m), H_j(Y_m, \mathbf{F}_p)) \arrow[r, Rightarrow] & H_{i+j}(\Gamma, \mathbf{F}_p) \end{tikzcd} \tag{3}$$

Sublemma 2.18 *The $E^2_{i,j}$ and hence $E^n_{i,j}$ terms of the spectral sequence*

$$H_i(\mathrm{SL}(\mathcal{O}/p^m), H_j(Y_m, \mathbf{F}_p)) \Longrightarrow H_{i+j}(\Gamma, \mathbf{F}_p)$$

have uniformly bounded dimension as $m \to \infty$.

We suspect that a stronger claim holds, namely that the spectral sequence is constant for sufficiently large m, but the boundedness is sufficient for our purposes.

Proof We first assume that $E^2_{i,0} = H_i(\mathrm{SL}(\mathcal{O}/p^m), \mathbf{F}_p)$ is bounded for sufficiently large m. The claimed result then follows immediately for the first row. We now proceed by induction on the rows. The general row consists of dim $H_j(Y_m, \mathbf{F}_p)$ copies of the first row, so it suffices to show that this dimension is uniformly bounded. However, if $E^2_{j,0} = H_j(Y_m, \mathbf{F}_p)$ is unbounded and all the terms in lower rows are uniformly bounded, then $E^\infty_{j,0}$ and hence $H_j(\Gamma, \mathbf{F}_p)$ will also be unbounded, which is a contradiction. It thus remains to show that $H_i(\mathrm{SL}(\mathcal{O}/p^m), \mathbf{F}_p)$ is bounded as m increases. By classical stability [14], we may replace $\mathrm{SL}(\mathcal{O}/p^m)$ by $\mathrm{SL}_N(\mathcal{O}/p^m)$ for some N depending only on i. Let $\mathrm{SL}_N(\mathcal{O}/p^m, p) \subset \mathrm{SL}_N(\mathcal{O}/p^m)$ denote the kernel of the reduction map modulo p. There is a spectral sequence

$$H_i(\mathrm{SL}_N(\mathbf{F}_p), H_j(\mathrm{SL}_N(\mathcal{O}/p^m, p), \mathbf{F}_p)) \Longrightarrow H_{i+j}(\mathrm{SL}_N(\mathcal{O}/p^m), \mathbf{F}_p).$$

It suffices to note that the coefficient system $H_j(\mathrm{SL}_N(\mathcal{O}/p^m, p), \mathbf{F}_p)$ is independent of m for $m \geq 1$ by [6, Corollary 2.34]. $\square$

We can not deduce that $H_j(\Gamma(p^m), \mathbf{F}_p) \simeq H_j(Y_m, \mathbf{F}_p)$ from Equation (3), exactly because the action of $\mathrm{SL}(\mathcal{O}/p^m)$ on $H_j(\Gamma(p^m), \mathbf{F}_p)$ is nontrivial. The key point is thus that, in the limit, the action of $\mathrm{SL}(\mathcal{O}_p)$ on $\widetilde{H}_j$ *is* trivial by Theorem 1.1. The diagram above gives rise to a compatible map of spectral sequences

$$\begin{tikzcd}
EA^2_{i,j} = H_i(\mathrm{SL}(\mathcal{O}_p), \widetilde{H}_j(\mathrm{SL}, \mathbf{F}_p)) \arrow[r, Rightarrow] \arrow[d] & H_{i+j}(\Gamma, \mathbf{F}_p) \arrow[d, equal] \\
EB^2_{i,j} = H_i(\mathrm{SL}(\mathcal{O}_p), \varprojlim H_j(Y_m, \mathbf{F}_p)) \arrow[r, Rightarrow] & H_{i+j}(\Gamma, \mathbf{F}_p) \arrow[d, equal] \\
EC^2_{i,j} = H_i(\mathrm{SL}(\mathcal{O}_p), H_j(Y_\infty, \mathbf{F}_p)) \arrow[r, Rightarrow] \arrow[u] & H_{i+j}(\Gamma, \mathbf{F}_p).
\end{tikzcd}$$

The inverse limit on the second term commutes with the construction of the spectral sequence because all the terms involved are uniformly bounded vector spaces over $\mathbf{F}_p$

by Sublemma 2.18, and so all inverse limits satisfy the Mittag-Leffler condition. The action of $\mathrm{SL}(\mathcal{O}_p)$ on $H_*(Y_m, \mathbf{F}_p)$ and $H_j(Y_\infty, \mathbf{F}_p)$ is trivial by construction, and the action on $\widetilde{H}_*$ is trivial by Theorem 1.1. Hence, by Lemma 2.15, one obtains isomorphisms $\widetilde{H}_j(\mathrm{SL}, \mathbf{F}_p) \simeq H_j(Y_\infty, \mathbf{F}_p)$. By dévissage, we obtain isomorphisms

$$\widetilde{H}_j(\mathrm{SL}, \mathbf{Z}/p^r\mathbf{Z}) \simeq \varprojlim H_j(Y_m, \mathbf{Z}/p^r\mathbf{Z}) \simeq H_j(Y_\infty, \mathbf{Z}/p^r\mathbf{Z})$$

for all r. Namely, we apply induction and compare the long exact sequences of homology associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z}/p^{r-1}\mathbf{Z} \longrightarrow \mathbf{Z}/p^r\mathbf{Z} \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow 0,$$

and then apply the 5–lemma. The groups $\widetilde{H}_j(\mathrm{SL}, \mathbf{Z}_p)$ are finitely generated over $\mathbf{Z}_p$ (by Theorem 1.1), and hence coincide with the inverse limit of $\widetilde{H}_j(\mathrm{SL}, \mathbf{Z}/p^r\mathbf{Z})$. $\square$

Note that the homology groups $H_*(Y_\infty, \mathbf{Z})$ are presumably quite badly behaved, thus $H_*(Y_\infty, \mathbf{Z}) \otimes \mathbf{Z}_p$ presumably differs from $H_*^{\mathrm{cont}}(Y_\infty, \mathbf{Z}_p)$ in general.

Remark 2.19 Combining Conjecture 2.5 with Lemmas 2.4, 2.8 and Theorem 2.17, we see that we have constructed an infinite loop space whose homotopy groups with coefficients in $\mathbf{Z}_p$ are given by $\widetilde{K}_*(\mathcal{O})$ and whose (continuous) homology groups are given by $\widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p)$. This should be thought of as completely analogous to the classical story, where the infinite loop space $K(\mathcal{O})$ has homotopy groups $K_n(\mathcal{O})$ and homology groups $H_*(\mathrm{SL}, \mathbf{Z})$.

Remark 2.20 Our methods may be extended in various natural ways. For example, one can take the completed cohomology groups $\widetilde{H}^*$ with respect to some subset of the primes dividing p in F (see Section 5). One may also add a tame level structure M for $(M, p) = 1$, that is, take the limit over the congruence subgroups $\Gamma(Mp^r)$. In the latter case, the answer will only depend on the radical of M (that is, the product of distinct primes dividing M), for reasons we now explain. Note that, by a result of Charney [15], the cohomology of the congruence subgroup $\Gamma(\mathfrak{q}^m)$ with coefficients in $\mathbf{Z}_p$ is stable if $(\mathfrak{q}, p) = 1$, and moreover that the concomitant action of $\mathrm{SL}(\mathcal{O}_\mathfrak{q})$ is trivial. Hence by (an easier version) of the argument above, the stable cohomology of $\Gamma(\mathfrak{q}^m)$ may be identified with the continuous $\mathbf{Z}_p$–cohomology of the homotopy fiber of $\mathrm{BSL}(\mathcal{O})^+$ mapping to $\mathrm{BSL}(\mathcal{O}/\mathfrak{q}^m)^+$. By Gabber's rigidity theorem [23], the map $K_n(\mathcal{O}/\mathfrak{q}^m; \mathbf{Z}_p) \to K_n(\mathcal{O}/\mathfrak{q}; \mathbf{Z}_p)$ is an isomorphism, and thus, by an application of Lemma 2.15, the maps $H_*(\Gamma(\mathfrak{q}^m), \mathbf{Z}_p) \to H_*(\Gamma(\mathfrak{q}), \mathbf{Z}_p)$ are isomorphisms in the stable range. (Alternatively, one can simply use the transfer map to see that $H_*(\Gamma(\mathfrak{q}), \mathbf{Z}/p^r\mathbf{Z}) \simeq H_*(\Gamma(\mathfrak{q}^m), \mathbf{Z}/p^r\mathbf{Z})_{G(\mathfrak{q})}$ for any $m \geq 1$ and r because $\Gamma(\mathfrak{q})/\Gamma(\mathfrak{q}^m) \simeq G(\mathfrak{q})/G(\mathfrak{q}^m)$ has order prime to p.)

2.4 Proof of Theorem 1.2

As explained in the introduction, the assumption that p does not divide w_F implies, by [2, Theorem 3.6], that $\widetilde{H}_1(\mathrm{SL}, \mathbf{Z}/p^r\mathbf{Z}) = 0$ for all r. It follows that $\pi_1(Y_\infty; \mathbf{Z}/p^r\mathbf{Z}) = 0$ for all r, and hence, via the Hurewicz map (for Hurewicz with coefficients, see [31, Theorem 9.7]), we obtain isomorphisms

$$\begin{aligned}\widetilde{K}_2(\mathcal{O}) := \pi_2(Y_\infty; \mathbf{Z}_p) &= \varprojlim \pi_2(Y_\infty; \mathbf{Z}/p^r\mathbf{Z}) \\ &= \varprojlim H_2(Y_\infty, \mathbf{Z}/p^r\mathbf{Z}) = \widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p),\end{aligned}$$

where the last equality follows from Theorem 2.17, and the second equality was established in the proof of Lemma 2.2. The isomorphism

$$\widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) = \widetilde{K}_2(\mathcal{O}) \stackrel{?}{=} \widetilde{K}_2^?(\mathcal{O}) := H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))^\vee$$

now follows rationally by Lemma 2.4, and also integrally under the assumption that p does not divide the order of $K_2(\mathcal{O})$. The main statement of Theorem 1.2 is the Pontryagin dual of this equality. Now suppose that $F = \mathbf{Q}$ and $p \geq 3$. Then $|K_2(\mathbf{Z})| = 2$ and $H^1(G_S, \mathbf{F}_p(-1)) = \mathbf{F}_p$ by Herbrand's theorem and class field theory. It follows that

$$\mathbf{F}_p = H^1(G_S, \mathbf{F}_p(-1)) = H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))[p],$$

and thus, by Pontryagin duality, that $H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p)^\vee/p = \mathbf{F}_p$. By Lemma 2.11, the group $H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))^\vee$ is infinite (it has rank $d = 1$). Hence, by Nakayama's lemma, we deduce that $H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))^\vee = \mathbf{Z}_p$, and thus $\widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) = \mathbf{Z}_p$. □

Remark 2.21 (Homology with coefficients) Let $\mathcal{L}_N$ be any algebraic local system (with $\mathbf{F}_p$ or $\mathbf{Z}_p$ coefficients) for $\mathrm{SL}_N(\mathcal{O}_F)$. Then one may also consider the completed homology groups $\widetilde{H}_*(\mathrm{SL}_N, \mathcal{L}_N) = \lim H_*(\Gamma(p^r), \mathcal{L}_N)$. Since $\mathcal{L}_N/p^m$ is trivial as a $\Gamma(p^m)$–module, the standard weight-level argument (Shapiro's lemma) implies that $\widetilde{H}_*(\mathrm{SL}_N, \mathcal{L}_N) \simeq \widetilde{H}_*(\mathrm{SL}_N, \mathbf{Z}_p) \otimes \mathcal{L}_N$. Hence, given any sequence of local systems $\mathcal{L}_N$ for SL_N such that $\lim \mathcal{L}_N = \mathcal{L}$, the corresponding sequence $\widetilde{H}_*(\mathrm{SL}_N, \mathcal{L}_N) = \mathcal{L}_N \otimes \widetilde{H}_*(\mathrm{SL}_N, \mathbf{Z}_p)$ converges to $\widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathcal{L}$.

3 Cohomology

3.1 The Hochschild–Serre spectral sequence, I

let us consider the completed cohomology groups with coefficients in $\mathbf{Q}_p/\mathbf{Z}_p$. It is easy to see that $\widetilde{H}^0(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) = \mathbf{Q}_p/\mathbf{Z}_p$. The congruence subgroup property [2] implies that if F does not contain any p^{th} roots of unity, then $\widetilde{H}^1(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ as explained in the introduction. However, for any number field F, the group $\widetilde{H}^1(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p)$ will always be finite (the obstruction to the full congruence subgroup property is a finite abelian group). For two $\mathbf{Z}_p$–modules A and B, let $A \approx B$ indicate that A and B are isomorphic up to a finite group, so $\widetilde{H}^1(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx 0$. Let $G = \mathrm{SL}(\mathcal{O}_p)$, and let $d := [F : \mathbf{Q}_p]$. We have an identification

$$H^*(G, \mathbf{Q}_p/\mathbf{Z}_p) \approx \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigotimes_{i=1}^{d} \bigwedge\nolimits_{\mathbf{Z}_p}[x_3, x_5, x_7, \dots],$$

where $\bigwedge$ denotes the exterior algebra, and the symbol $\approx$ indicates that in each degree we have equality up to a finite group [43, Proposition 1]. In particular, the first infinite cohomology group in degree bigger than zero is $H^3(G, \mathbf{Q}_p/\mathbf{Z}_p) \approx (\mathbf{Q}_p/\mathbf{Z}_p)^d$. The Hochschild–Serre spectral sequence for cohomology is the spectral sequence

$$H^i(G, \widetilde{H}^j(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p)) \Rightarrow H^{i+j}(\Gamma, \mathbf{Q}_p/\mathbf{Z}_p).$$

By Borel's computation of stable cohomology (Theorem 2.12), we also have isomorphisms

$$H^1(\Gamma, \mathbf{Q}_p/\mathbf{Z}_p) \approx 0, \quad H^2(\Gamma, \mathbf{Q}_p/\mathbf{Z}_p) \approx 0, \quad H^3(\Gamma, \mathbf{Q}_p/\mathbf{Z}_p) \approx K_3(\mathcal{O}) \otimes \mathbf{Q}_p/\mathbf{Z}_p.$$

Up to finite groups, the E_2 page of the spectral sequence therefore looks like:

$$\begin{array}{c|ccccc}
q & & & & & \\
2 & \widetilde{H}^2(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) & & & & \\
1 & 0 & 0 & 0 & & \\
0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & 0 & (\mathbf{Q}_p/\mathbf{Z}_p)^d & \\
\hline
 & 0 & 1 & 2 & 3 & p
\end{array}$$

Taking Pontryagin duals and noting that $\widetilde{H}^2(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee = \widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p)$, we obtain the exact sequence (up to finite groups)

$$K_3(\mathcal{O}) \otimes \mathbf{Z}_p \longrightarrow \mathbf{Z}_p^d \longrightarrow \widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) \longrightarrow 0.$$

We deduce the following corollary.

Corollary 3.1 *There are inequalities*

$$d = r_1 + 2r_2 \geq \dim \widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q},$$
$$\dim \widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q} \geq r_1 + r_2 = d - \dim K_3(\mathcal{O}) \otimes \mathbf{Q}.$$

By Theorem 1.2 (or rather the proof in Section 2.4), we obtain an isomorphism $\widetilde{H}_2(\mathbf{Z}_p) = \widetilde{K}_2(\mathcal{O})$, which therefore identifies $\widetilde{H}_2(\mathrm{SL}, \mathbf{Z}_p)$ with (the dual of) a certain Galois cohomology group. The reader unfamiliar with the complications in computing Galois cohomology groups may be surprised that this equality does not allow us to (greatly) improve the estimate of Corollary 3.1. The difficulty is that determining the ranks of these groups is effectively a generalization of Leopoldt's conjecture, which appears to be very difficult.

3.2 The Hochschild–Serre spectral sequence, II: Higher terms

Let us suppose that $F = \mathbf{Q}$. One may play the spectral sequence game to obtain information concerning $\widetilde{H}^d$ for higher d. Let $G = \mathrm{SL}(\mathbf{Z}_p)$, so

$$H^*(G, \mathbf{Q}_p/\mathbf{Z}_p) \approx \mathbf{Q}_p/\mathbf{Z}_p \otimes \bigwedge\nolimits_{\mathbf{Z}_p}[x_3, x_5, x_7, \dots].$$

We first consider $\widetilde{H}^3$. Let us work (as in the last section) in the Serre category of cofinitely generated $\mathbf{Z}_p$–modules up to cotorsion modules (so every term is equivalent to a finite number of copies of $\mathbf{Q}_p/\mathbf{Z}_p$). Since $\widetilde{H}^2 \approx \mathbf{Q}_p/\mathbf{Z}_p$, the relevant terms of the spectral sequence are

$$\begin{array}{c|ccccc}
q & & & & & \\
3 & \widetilde{H}^3 & & & & \\
2 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & 0 & & \\
1 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & 0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 \quad p
\end{array}$$

Yet $H^3(\Gamma, \mathbf{Q}_p/\mathbf{Z}_p)$ is trivial (since $K_3(\mathbf{Z}) = \mathbf{Z}/48\mathbf{Z}$ is finite [27]; alternatively, we can see this from the computation of stable cohomology given by Theorem 2.12), and thus $\widetilde{H}^3(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx 0$. Consider the next few terms. We obtain the following:

$$\begin{array}{c|ccccccc}
q & & & & & & & \\
5 & \widetilde{H}^5 & & & & & & \\
4 & \widetilde{H}^4 & 0 & 0 & & & & \\
3 & 0 & 0 & 0 & 0 & & & \\
2 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & 0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & 0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 & \mathbf{Q}_p/\mathbf{Z}_p & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \quad p
\end{array}$$

Since $K_4(\mathbf{Z}) \otimes \mathbf{Q}$ is trivial and $K_5(\mathbf{Z}) \otimes \mathbf{Q}$ has rank one, we deduce that either

$$\widetilde{H}^4(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx \mathbf{Q}_p/\mathbf{Z}_p \quad \text{and} \quad \widetilde{H}^5(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx 0$$

or

$$\widetilde{H}^4(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx (\mathbf{Q}_p/\mathbf{Z}_p)^2 \quad \text{and} \quad \widetilde{H}^5(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx \mathbf{Q}_p/\mathbf{Z}_p.$$

In particular, we have that

$$\dim \widetilde{H}_4(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q} - \dim \widetilde{H}_5(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q} = 1.$$

We can not rule out either possibility, just as we cannot rule out that $\widetilde{K}_5(\mathbf{Z}) \otimes \mathbf{Q} \neq 0$ or $\widetilde{K}_4(\mathbf{Z}) \otimes \mathbf{Q} \neq \mathbf{Q}_p$; however, assuming the conjectural part of Lemma 2.11, we deduce that $\widetilde{H}^4(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \approx \mathbf{Q}_p/\mathbf{Z}_p$ and $\widetilde{H}^5 \approx 0$.

3.3 Rational completed cohomology groups

We may define rational completed cohomology groups $\widetilde{H}^*(\mathrm{SL}, \mathbf{Q}_p)$ as follows. Let $\widetilde{H}^*(\mathrm{SL}, \mathbf{Z}/p^s\mathbf{Z}) := \lim_{\rightarrow} H^*(\Gamma(p^r), \mathbf{Z}/p^s\mathbf{Z})$ and

$$\widetilde{H}^*(\mathrm{SL}, \mathbf{Z}_p) = \varprojlim_s \varinjlim_r H^*(\Gamma(p^r), \mathbf{Z}/p^s\mathbf{Z}) = \varprojlim \widetilde{H}^*(\mathrm{SL}, \mathbf{Z}/p^s\mathbf{Z}).$$

This allows us to define rational completed cohomology groups as

$$\widetilde{H}^*(\mathrm{SL}, \mathbf{Q}_p) := \widetilde{H}^*(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q}_p, \quad \widetilde{H}_*(\mathrm{SL}, \mathbf{Q}_p) := \widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p) \otimes \mathbf{Q}_p.$$

We have (see [11, Theorem 1.1]) an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\widetilde{H}^*[1](\mathrm{SL}, \mathbf{Z}_p)[p^\infty], \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \widetilde{H}^*(\mathrm{SL}, \mathbf{Z}_p) \\ \longrightarrow \mathrm{Hom}(\widetilde{H}^*(\mathrm{SL}, \mathbf{Z}_p), \mathbf{Z}_p) \longrightarrow 0, \end{aligned}$$

where $M[1]$ denotes the usual shift of M. The modules $\widetilde{H}^*(\mathrm{SL}, \mathbf{Z}_p)$ are finitely generated and so the first term of this sequence is torsion; tensoring with $\mathbf{Q}$ we obtain

$$\widetilde{H}^*(\mathrm{SL}, \mathbf{Q}_p) \simeq \mathrm{Hom}(\widetilde{H}_*(\mathrm{SL}, \mathbf{Z}_p), \mathbf{Q}_p).$$

(Wagoner and Milgram considers similar completed cohomology groups when studying the *continuous* algebraic K–theory of local fields; see [43, page 244].) There is naturally an identification of $\widetilde{H}_*(\mathrm{SL}, \mathbf{Q}_p)$ with

$$H_*^{\mathrm{cont}}(Y_\infty, \mathbf{Z}_p) \otimes \mathbf{Q}_p = H_*^{\mathrm{cont}}(SK(\mathcal{O}, \mathcal{O}_p), \mathbf{Z}_p) \otimes \mathbf{Q}_p$$

by Theorem 2.17. We may denote the latter group by $H_*^{\mathrm{cont}}(Y_\infty, \mathbf{Q}_p)$.

3.4 The Eilenberg–Moore spectral sequence

A more direct way to compute the cohomology of the fiber from the cohomology of the total space and the cohomology of the base is by using the Eilenberg–Moore spectral sequence. This is especially practical in this case since we know the cohomology of $\mathrm{BSL}(\mathbf{Z}_p)^+$ and $\mathrm{BSL}(\mathbf{Z})^+$ with coefficients in $\mathbf{Q}_p$, namely, as exterior algebras $\bigwedge_{\mathbf{Q}_p}[x_3, x_5, x_7, \dots]$ and $\bigwedge_{\mathbf{Q}_p}[\eta_5, \eta_9, \eta_{13}, \dots]$ respectively. (In the former case, we are taking the continuous cohomology as defined at the end of the previous section.) In particular, we take the inverse limits of the Eilenberg–Moore spectral sequences associated to Equation (1)

$$SK(\mathcal{O}, p^m) \longrightarrow SK(\mathcal{O}) \longrightarrow SK(\mathcal{O}/p^m)$$

with coefficients in $\mathbf{Z}/p^r\mathbf{Z}$ take the corresponding inverse limit in r, and then tensor with $\mathbf{Q}$. (Relevant here is [43, Proposition 1].) Here we use the fact that the cohomology of $SK(\mathcal{O}, p^m)$ with coefficients in $\mathbf{Z}/p^r\mathbf{Z}$ is uniformly bounded and that the inverse limit over $\mathbf{Z}/p^r\mathbf{Z}$ recovers the homology groups $\widetilde{H}(\mathrm{SL}, \mathbf{Z}_p)$ by Theorem 2.17. The uniform boundedness of these groups ensures that all inverse limits are Mittag-Leffler and so there are no issues with the derived functor $\lim^1$, which will always vanish. What is less obvious, however, is the structure of $\bigwedge_{\mathbf{Q}_p}[\eta_5, \eta_9, \eta_{13}, \dots]$ as a module for $\bigwedge_{\mathbf{Q}_p}[x_3, x_5, x_7, \dots]$. The natural supposition is (up to scaling by a nonzero constant depending on n) that x_{4n-1} acts as zero and x_{4n+1} acts as η_{4n+1}. The latter claim, however, is equivalent to showing that the p–adic regulator map

$$K_{4n+1}(\mathbf{Z}; \mathbf{Z}_p) \otimes \mathbf{Q}_p \longrightarrow K_{4n+1}(\mathbf{Z}_p; \mathbf{Z}_p) \otimes \mathbf{Q}_p$$

is an isomorphism. To see this equivalence, note that $K(\mathbf{Z})$ and $K(\mathbf{Z}_p)$ are infinite loop spaces (and hence H–spaces), and so the Milnor–Moore theorem [30, Appendix] identifies the rational classes in K–theory with the rational primitive classes in homology under the Hurewicz map. More precisely, for $K(\mathbf{Z}_p)$, we use the p–adic Milnor–Moore theorem, which gives the corresponding relationship between the classes in $K_*(\mathbf{Z}_p; \mathbf{Z}_p) \otimes \mathbf{Q}$ and the rational primitive classes in the continuous homology of $G = \mathrm{SL}(\mathbf{Z}_p)$ (see [30, Proposition 3] and the subsequent arguments; indeed, this is how Wagoner and Milgram computed the groups $K_*(\mathbf{Z}_p; \mathbf{Z}_p) \otimes \mathbf{Q}$ in the first place.) As previously noted, however, showing that maps between global and local K–groups are injective is a problem whose difficulty may be of a similar level to Leopoldt's conjecture (see Conjecture 2.9 and the subsequent remarks).

One context in which we know this map is an isomorphism is for regular primes, by Lemma 2.11. Thus we make the following Ansatz:

$(*)$ Either p is regular, or $F = \mathbf{Q}$ and Conjecture 2.9 holds.

Under this assumption, the Eilenberg–Moore spectral sequence allows for a complete computation of $\widetilde{H}^*(\mathbf{Q}_p)$. Specifically, we have a spectral sequence

$$\begin{aligned} E^2_{*,*} &= \mathrm{Tor}_{*,*}^{\bigwedge_{\mathbf{Q}_p}[x_3,x_5,x_7,\ldots]}\Big(\mathbf{Q}_p, \bigwedge\nolimits_{\mathbf{Q}_p}[\eta_5,\eta_9,\eta_{13},\ldots]\Big) \Rightarrow \widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p) \\ &\simeq \mathrm{Tor}_{*,*}^{\bigwedge_{\mathbf{Q}_p}[x_3,x_7,x_{11},\ldots]}(\mathbf{Q}_p,\mathbf{Q}_p) \Rightarrow \widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p) \\ &\simeq \bigotimes \mathrm{Tor}_{*,*}^{\bigwedge_{\mathbf{Q}_p}[x_{4n-1}]}(\mathbf{Q}_p,\mathbf{Q}_p) \Rightarrow \widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p). \end{aligned}$$

This sequence converges because the fundamental group of the base is abelian (since it comes from a $+$ construction). The first isomorphism requires the assumption that the module structure arises from a surjection of exterior algebras $\bigwedge_{\mathbf{Q}_p}[x_*] \to \bigwedge_{\mathbf{Q}_p}[\eta_*]$, which is exactly the content of assumption $(*)$. On the other hand, the final sequence degenerates and may be computed explicitly; the limit is given by a tensor product of the polynomial algebras on x_{4n-1}, shifted by 1 degree. In particular, we have the following[3]:

Theorem 3.2 *Let $F=\mathbf{Q}$, and assume that either p is regular or Conjecture 2.9 holds. Then there is an isomorphism $\widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p) \simeq \mathbf{Q}_p[x_2,x_6,x_{10},\ldots]$. In particular, there is an equality* $\dim \widetilde{H}_n(\mathrm{SL},\mathbf{Z}_p)\otimes\mathbf{Q}_p = 0$ *unless n is even, and* $\dim \widetilde{H}_n(\mathrm{SL},\mathbf{Z}_p)\otimes\mathbf{Q}_p$ *is the coefficient of q^n in*

$$\begin{aligned} \prod_{k=1}^{\infty}\frac{1}{(1-q^{4k-2})} &= \prod_{k=1}^{\infty}(1+q^{2k}) \\ &= 1+q^2+q^4+2q^6+2q^8+3q^{10}+4q^{12}+5q^{14}+6q^{16}+8q^{18}+\cdots. \end{aligned}$$

Remark 3.3 (Where do the Borel classes go?) The maps

$$H^*(\Gamma,\mathbf{Q}_p) \to H^*(\Gamma(p^r),\mathbf{Q}_p)$$

are isomorphisms (in the stable range) for all r, and yet, under the assumptions of Theorem 3.2, $\widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p)=0$ in all odd degrees; how do we reconcile these two statements? The explanation is that the classes must become infinitely p–divisible up the congruence tower. In general, the divisibility of Borel classes in the limit is

[3] In particular, the rational cohomology conjecturally coincides with the stable rational cohomology of the classical group $\mathrm{Sp}_{2n}(\mathbf{Z})$. This is most likely a coincidence; the analogous computation for a general number field F of signature (r_1,r_2) yields (conditionally on the analogue of $(*)$)

$$\widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p) = \bigotimes_{r_1}\mathbf{Q}_p[x_2,x_6,x_{10},\ldots]\otimes\bigotimes_{r_2}\mathbf{Q}_p[x_2,x_4,x_6,\ldots],$$

which is *not* the stable cohomology of $\mathrm{Sp}_{2n}(\mathcal{O}_F)$ as soon as F is not totally real, and nor is it the stable rational cohomology of any classical group.

equivalent to Conjecture 2.9, by Proposition 2.14, this holds for all but finitely many of the primitive classes. This is in contrast to what happens for $\mathrm{Sp}_{2n}(\mathbf{Z})$. According to Deligne [21], the stable class in degree two does not become p–divisible for any p up the congruence tower. A back of the envelope calculation suggested by the methods of this paper indicates that the same should be true for all stable classes in $\mathrm{Sp}_{2n}(\mathbf{Z})$.

Remark 3.4 Since $SK(\mathcal{O};\mathcal{O}_p)$ is an infinite loop space and thus an H–space, one way to interpret Theorem 3.2 is simply to note that for H–spaces, that the homology is rationally a polynomial algebra on its homotopy. However, one must be slightly careful with such a statement, since our homotopy groups are with respect to coefficients in $\mathbf{Z}_p$, and are homology and cohomology groups with coefficients in $\mathbf{Q}_p$ are defined in terms of inverse limits (and are thus "continuous" (co)homology groups). A similar remark applies to the cohomology ring of $\mathrm{SL}(\mathcal{O}_p)$ and the K–theory of local fields with coefficients in $\mathbf{Z}_p$.

3.5 The Hochschild–Serre spectral sequence, III: Determining the differentials

We have an isomorphism $H^*(G)\otimes\mathbf{Q}_p=\bigwedge[x_3,x_5,x_7,\dots]$. We assume in this section that there is an isomorphism $\widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p)\simeq\mathbf{Q}_p[x_2,x_6,x_{10},\dots]$ as in Section 3.4. However, in order to avoid confusion, we shall use different notation, and in particular we shall write

$$\widetilde{H}^*(\mathrm{SL},\mathbf{Q}_p)\simeq\mathbf{Q}_p[\hat{x}_2,\hat{x}_6,\hat{x}_{10},\dots].$$

It follows that the second page of the Hochschild–Serre spectral sequence $E^2_{*,*}$ is given by

$$\mathbf{Q}_p[\hat{x}_2,\hat{x}_6,\hat{x}_{10},\dots]\otimes\bigwedge[x_3,x_5,x_7,\dots].$$

We may write this out on page two as in Table 1.

The Koszul complex associated to a polynomial algebra induces a spectral sequence

$$\begin{aligned}\mathbf{Q}_p[\hat{x}_2,\hat{x}_6,\hat{x}_{10},\dots]&\otimes\bigwedge[x_3,x_5,x_7,\dots]\\ &=\bigwedge[x_5,x_9,x_{13},\dots]\otimes\Big(\mathbf{Q}_p[\hat{x}_2,\hat{x}_6,\hat{x}_{10}\dots]\otimes\bigwedge[x_3,x_7,x_{11}\dots]\Big)\\ &\Rightarrow\bigwedge[x_5,x_9,x_{13},\dots],\end{aligned}$$

given explicitly by $d^r=0$ unless $r=4n-1$, in which case $d^{4n-1}(\hat{x}_{4n-2})=x_{4n-1}$ and is zero otherwise. For example, page 4 is given in Table 2.

q											
10	$\widehat{x}_2^5, \widehat{x}_{10}, \widehat{x}_2^2\widehat{x}_6$										
9	0	0									
8	$\widehat{x}_2^4, \widehat{x}_2\widehat{x}_6$	0	0								
7	0	0	0	0							
6	$\widehat{x}_2^3, \widehat{x}_6$	0	0	$\widehat{x}_2^3 x_3, \widehat{x}_6 x_3$	0						
5	0	0	0	0	0	0					
4	$\widehat{x}_2^2$	0	0	$\widehat{x}_2^2 x_3$	0	$\widehat{x}_2^2 x_5$	0				
3	0	0	0	0	0	0	0	0			
2	$\widehat{x}_2$	0	0	$\widehat{x}_2 x_3$	0	$\widehat{x}_2 x_5$	0	$\widehat{x}_2 x_7$	$\widehat{x}_2 x_3 x_5$		
1	0	0	0	0	0	0	0	0	0	0	
0	1	0	0	x_3	0	x_5	0	x_7	$x_3 x_5$	x_9	$x_3 x_7$
	0	1	2	3	4	5	6	7	8	9	10 p

Table 1: Page two

Under our assumptions, there is a natural map of spectral sequences from this to Hochschild–Serre provided that one knows the maps $d^{4n-1}(\widehat{x}_{4n-2}) = x_{4n-1}$ coincides with the corresponding maps in Hochschild–Serre (equivalently, the y_{4n-2} are *transgressive*). By induction, this reduces to showing that the maps

$$K_{4n-1}(\mathbf{Z}_p;\mathbf{Z}_p) \to \widetilde{K}_{4n-2}(\mathbf{Z},\mathbf{Z}_p;\mathbf{Z}_p)$$

are rational isomorphisms, which we certainly assumed in Section 3.4 in order to compute $\widetilde{H}^*(\mathbf{Q}_p)$ in the first place. Thus, by the Zeeman comparison theorem, these two spectral sequences coincide ([49]; see also [28, Theorem 3.27]).

4 Classical cohomology groups

We turn in this section to some explicit computations. Let $\Gamma_N = \mathrm{SL}_N(\mathcal{O}_F)$ for some N which is sufficiently large so that $\widetilde{H}_*(\mathrm{SL},\mathbf{Z}_p) = \widetilde{H}_*(\mathrm{SL}_N,\mathbf{Z}_p)$ for $*$ in the range of computation (this is $* \le 3$ except in Section 4.2). Let $G_N = \mathrm{SL}_N(\mathbf{Z}_p)$. Benson Farb asked (personal communication) whether one can compute the homology groups $H_2(\Gamma(p),\mathbf{F}_p)$; we give an complete answer below. The information about such groups is encoded in the completed cohomology groups $\widetilde{H}^*$ together with the differentials of the Hochschild–Serre spectral sequence. This exercise probably only serves to indicate why the completed homology groups $\widetilde{H}_*$ and completed K–groups $\widetilde{K}_*$ are more natural objects of study than their finite (unstable) analogues.

q															
13	0														
12	$\widehat{x}_6^2$	0													
11	0	0	0												
10	$\widehat{x}_{10}$	0	0	0											
9	0	0	0	0	0										
8	0	0	0	0	0	0									
7	0	0	0	0	0	0	0								
6	$\widehat{x}_6$	0	0	0	0	$\widehat{x}_6 x_5$	0	$\widehat{x}_6 x_7$							
5	0	0	0	0	0	0	0	0	0						
4	0	0	0	0	0	0	0	0	0	0					
3	0	0	0	0	0	0	0	0	0	0	0				
2	0	0	0	0	0	0	0	0	0	0	0	0			
1	0	0	0	0	0	0	0	0	0	0	0	0	0		
0	1	0	0	0	0	x_5	0	x_7	0	x_9	0	x_{11}	$x_5 x_7$	x_{13}	$\dots$
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	p

Table 2: Page four

We make the following assumptions on p:

$(**)$ $\quad p$ does not divide w_F, and $\widetilde{H}^2(\mathrm{SL}, \mathbf{Q}_p/\mathbf{Z}_p) \simeq H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))$.

By Theorem 1.2, this holds if p does not divide $w_F|K_2(\mathcal{O}_F)|$, and conjecturally always holds for $p > 2$. In fact, we shall state the theorems below with the stronger hypothesis $p \nmid w_F|K_2(\mathcal{O}_F)|$, but we only use the assumption above. Let $d = [F : \mathbf{Q}]$. Because p does not divide w_F, we have $\widetilde{H}^1(\mathrm{SL}_N, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ (once more using [2]), and thus, from the long exact sequence of completed cohomology corresponding to the short exact sequence of modules

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\times p} \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0,$$

we find that

$$\begin{aligned} \widetilde{H}^2(\mathrm{SL}_N, \mathbf{F}_p) = \widetilde{H}^2(\mathrm{SL}_N, \mathbf{Q}_p/\mathbf{Z}_p)[p] &= H^1(G_S, \mathbf{Q}_p/\mathbf{Z}_p(-1))[p] \\ &\simeq H^1(G_S, \mathbf{F}_p(-1)). \end{aligned}$$

The Hochschild–Serre spectral sequence

$$H^i(G_N(p^m), \widetilde{H}^j(\mathrm{SL}_N, \mathbf{F}_p)) \Rightarrow H^{i+j}(\Gamma_N(p^m), \mathbf{F}_p)$$

gives rise in the usual way (using $\widetilde{H}^1(\mathrm{SL}_N, \mathbf{F}_p) = 0$) to the exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(G_N(p^m), \mathbf{F}_p) \longrightarrow H^2(\Gamma_N(p^m), \mathbf{F}_p) \longrightarrow H^1(G_S, \mathbf{F}_p(-1)) \\ \longrightarrow H^3(G_N(p^m), \mathbf{F}_p) \longrightarrow H^3(\Gamma_N(p^m), \mathbf{F}_p). \end{aligned}$$

Consider the cohomology ring $H^*(G_N(p^m), \mathbf{F}_p)$. Since $p > 2$, we deduce that the group $G_N(p^m)$ is p–powerful, and (by a theorem of Lazard) that $G_N(p^m)$ is a Poincaré homology group of dimension $\dim(G_N) = d(N^2 - 1)$. More explicitly, by Lazard [26, Chapter V, 2.2.6.3 and 2.2.7.2, page 167], we deduce that the cohomology of $G_N(p^m)$ is an exterior algebra on the degree-one classes:

$$H^*(G_N(p^m), \mathbf{F}_p) \simeq \bigwedge\nolimits^* (H^1(G_N(p^m), \mathbf{F}_p)).$$

The group $G_N/G_N(p^m) = \mathrm{SL}_N(\mathcal{O}/p^m)$ acts naturally on this ring, the action factors through $\mathrm{PSL}_N(\mathcal{O}/p)$. Let us denote $H^1(G_N(p^m), \mathbf{F}_p)$ by M as an $\mathrm{SL}_N(\mathcal{O}/p)$–module. We have

$$M \simeq \mathrm{Hom}(H_1(G_N(p^m), \mathbf{F}_p), \mathbf{F}_p) \simeq \mathrm{Hom}(M_N^0(\mathbf{F}_p), \mathbf{F}_p),$$

where $M_N^0(\mathbf{F}_p)$ denotes the matrices of trace zero. We deduce that there is an exact sequence

$$0 \longrightarrow \bigwedge\nolimits^2 M \longrightarrow H^2(\Gamma_N(p^m), \mathbf{F}_p) \longrightarrow H^1(G_S, \mathbf{F}_p(-1)) \longrightarrow \bigwedge\nolimits^3 M. \tag{4}$$

Since the map $H^3(G_N(p^m), \mathbf{F}_p) \to H^3(G_N(p^{m+1}), \mathbf{F}_p)$ is zero for all $m \geq 1$, we also deduce that:

Lemma 4.1 *Suppose that p does not divide $w_F|K_2(\mathcal{O}_F)|$. For $m \geq 1$, there is an exact sequence*

$$0 \longrightarrow \bigwedge\nolimits^2 M \longrightarrow H^2(\Gamma_N(p^m), \mathbf{F}_p) \longrightarrow H^1(G_S, \mathbf{F}_p(-1)) \longrightarrow 0.$$

For $m \geq 2$, this sequence splits.

Proof By the computation above, we have exact sequences and commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^2 M \arrow[r] \arrow[d] & H^2(\Gamma_N(p), \mathbf{F}_p) \arrow[r] \arrow[d] & H^1(G_S, \mathbf{F}_p(-1)) \arrow[r] \arrow[d, equal] & \bigwedge^3 M \arrow[d] \\ 0 \arrow[r] & \bigwedge^2 M \arrow[r] & H^2(\Gamma_N(p^m), \mathbf{F}_p) \arrow[r] & H^1(G_S, \mathbf{F}_p(-1)) \arrow[r] & \bigwedge^3 M, \end{tikzcd}$$

where $m > 1$. The equality in the third column follows from the fact that the invariants of $\widetilde{H}^2(\mathrm{SL}_N, \mathbf{F}_p)$ under $G_N(p^m)$ do not depend on m, because the entire group G_N acts trivially. On the other hand, the maps on the first and last columns are identically zero, because they are induced from the map of groups

$$G_N(p)/G_N(p^2) \to G_N(p^m)/G_N(p^{m+1}),$$

which is the zero map. In particular, the vanishing of the map in the last column implies by commutativity that the image of $H^1(G_S, \mathbf{F}_p(-1))$ in $\bigwedge^3 M$ must be trivial, which shows that the sequence of the lemma is exact. On the other hand, the vanishing of the map in the first row implies that the quotient $H^2(\Gamma_N(p), \mathbf{F}_p)/\bigwedge^2 M = H^1(G_S, \mathbf{F}_p(-1))$ maps into $H^2(\Gamma_N(p^m), \mathbf{F}_p)$ in a way that is compatible with the map to $H^1(G_S, \mathbf{F}_p(-1))$, which splits the sequence for $m > 1$. □

4.1 A mild improvement of Lemma 4.1

Lemma 4.2 *Suppose that $p > 3$ splits completely in F and does not divide the order of $K_2(\mathcal{O}_F)$ or $K_3(\mathcal{O}_F)$. Then, for all m, there is an isomorphism*

$$H^2(\Gamma_N(p^m), \mathbf{F}_p) \simeq \bigwedge^2 M \oplus (\mathbf{F}_p)^d.$$

Remark 4.3 The condition on p is never satisfied unless F is totally real, since otherwise $K_3(\mathcal{O}_F)$ is infinite. If F is totally real, however, then the assumptions hold for all but finitely many p that split completely in F. For example, if $F = \mathbf{Q}$, then the assumptions hold for $p > 3$. The splitting condition should not be necessary; we assume it in order to invoke a theorem of [22] concerning the homology of $\mathrm{GL}_N(\mathbf{F}_p)$ of some particular module; the methods of [22] should also apply to $\mathrm{GL}_N(\mathbf{F}_q)$ for a finite field $\mathbf{F}_q$, which would be sufficient provided that p is unramified in F.

Proof The assumptions on p imply that $H^3(\Gamma_N, \mathbf{F}_p) = H^2(\Gamma_N, \mathbf{F}_p) = 0$, and hence that $\widetilde{H}^2(\mathrm{SL}_N, \mathbf{F}_p) \simeq H^3(G_N, \mathbf{F}_p) \simeq (\mathbf{F}_p)^d$. Consider the *classical* Hochschild–Serre spectral sequence

$$H^i(\mathrm{SL}_N(\mathcal{O}/p), H^j(\Gamma_N(p), \mathbf{F}_p)) \Rightarrow H^{i+j}(\Gamma_N, \mathbf{F}_p).$$

As representations for $\mathrm{SL}_N(\mathcal{O}/p)$, we have

$$H^1(\Gamma_N(p),\mathbf{F}_p)=M \quad \text{and} \quad H^0(\Gamma_N(p),\mathbf{F}_p)=\mathbf{F}_p.$$

By Quillen's computation of $K_*(\mathbf{F}_q)$ in [34, Theorem 6], if p is unramified in F, then $K_n(\mathcal{O}/p)\otimes\mathbf{F}_p=0$ for $n>0$, or equivalently that $H^n(\mathrm{SL}_N(\mathcal{O}/p),\mathbf{F}_p)=0$ for all $n>0$. Since M has no $\mathrm{SL}_N(\mathcal{O}/p)$–invariants, the second page of the sequence is:

$$\begin{array}{c|ccccc}
q & & & & & \\
2 & H^0(\mathrm{SL}_N(\mathcal{O}/p),H^2(\Gamma_N(p),\mathbf{F}_p)) & & & & \\
1 & 0 & H^1(\mathrm{SL}_N(\mathcal{O}/p),M) & H^2(\mathrm{SL}_N(\mathcal{O}/p),M) & & \\
0 & \mathbf{F}_p & 0 & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Since $H^2(\Gamma_N,\mathbf{F}_p)=0$ and $H^3(\Gamma_N,\mathbf{F}_p)=0$ by assumption, we deduce that there is an isomorphism

$$H^0(\mathrm{SL}_N(\mathcal{O}/p),H^2(\Gamma_N(p),\mathbf{F}_p))\simeq H^2(\mathrm{SL}_N(\mathcal{O}/p),M).$$

By [22, Proposition 3.0], there is an isomorphism: $H_2(\mathrm{GL}_N(\mathbf{F}_p),M^*)\simeq\mathbf{Z}/p\mathbf{Z}$, which implies that $H^2(\mathrm{GL}_N(\mathbf{F}_p),M)=\mathbf{Z}/p\mathbf{Z}$, and hence that $H^2(\mathrm{SL}_N(\mathbf{F}_p),M)$ has dimension *at least* one. Explicitly, since $\mathrm{GL}_N(\mathbf{F}_p)/\mathrm{SL}_N(\mathbf{F}_p)=\mathbf{F}_p^\times$ has order prime to p, the Hochschild–Serre spectral sequence degenerates and one has an isomorphism

$$H^*(\mathrm{GL}_N(\mathbf{F}_p),M)=H^*(\mathrm{SL}_N(\mathbf{F}_p),M)^{\mathbf{F}_p^\times},$$

and so a lower bound for the former group gives a lower bound for the latter. Since p is totally split in F, we have $\mathrm{SL}_N(\mathcal{O}/p)=\prod_{v|p}\mathrm{SL}_N(\mathbf{F}_p)$ and thus $H^2(\mathrm{SL}_N(\mathcal{O}/p),M)$ has dimension at least $[F:\mathbf{Q}]$. Since

$$H^0\Big(\mathrm{SL}_N(\mathcal{O}/p),\bigwedge^2 M\Big)=0,$$

from the exact sequence (4), we deduce that there is an injection

$$H^0(\mathrm{SL}_N(\mathcal{O}/p),H^2(\Gamma_N(p),\mathbf{F}_p))\to H^1(G_S,\mathbf{F}_p(-1))\simeq(\mathbf{F}_p)^{[F:\mathbf{Q}]}.$$

The latter equality relies on the fact that p does not divide $w_F|K_2(\mathcal{O}_F)|$; however, we are assuming that p does not divide the order of $K_2(\mathcal{O}_F)$, and $p>2$ cannot divide w_F because F is totally real. As we just proved the left-hand side has dimension at least $[F:\mathbf{Q}]$, this map is an isomorphism, which therefore splits (4), proving the lemma. □

This allows us to answer the question of Farb:

Corollary 4.4 *Let $p > 3$, and let $\Gamma_N(p)$ denote the congruence subgroup of $\mathrm{SL}_N(\mathbf{Z})$. Then for sufficiently large N, there are isomorphisms of $\mathrm{SL}_N(\mathbf{F}_p)$–modules*

$$H^2(\Gamma_N(p), \mathbf{F}_p) \simeq \bigwedge^2 M \oplus \mathbf{F}_p, \quad H_2(\Gamma_N(p), \mathbf{F}_p) \simeq \bigwedge^2 M^* \oplus \mathbf{F}_p,$$

where M^* is isomorphic to $M_N^0(\mathbf{F}_p)$. In particular,

$$\dim H^2(\Gamma_N(p), \mathbf{F}_p) = \binom{N^2-1}{2} + 1.$$

4.2 Remarks on cohomology in higher degrees

One may continue computations as above in higher degrees, although the analysis becomes more and more intricate. Consider, for example, the group $H^3(\Gamma_N(p^m), \mathbf{F}_p)$ when $F = \mathbf{Q}$. For $p > 3$, we should have $\widetilde{H}_3(\mathbf{F}_p) = 0$. The Hochschild–Serre spectral sequence

$$H_i(G_N(p^m), \widetilde{H}_j(\mathrm{SL}_N, \mathbf{F}_p)) \Rightarrow H_{i+j}(\Gamma_N(p^m), \mathbf{F}_p)$$

would then yield:

$$\begin{array}{c|ccccc}
q & & & & & \\
3 & 0 & 0 & & & \\
2 & \mathbf{F}_p & M & \bigwedge^2 M & & \\
1 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{F}_p & M & \bigwedge^2 M & \bigwedge^3 M & \bigwedge^4 M \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Since

$$H^2(\Gamma_N(p^m), \mathbf{F}_p) = \bigwedge^2 M \oplus \mathbf{F}_p,$$

this would give an exact sequence

$$0 \longrightarrow \bigwedge^3 M \longrightarrow H^3(\Gamma_N(p^m), \mathbf{F}_p) \longrightarrow M \longrightarrow \bigwedge^4 M.$$

In this case, the map

$$H^3(\Gamma_N(p^m), \mathbf{F}_p) \longrightarrow H^3(\Gamma_N(p^{m+1}), \mathbf{F}_p)$$

is zero. Hence the order of $H^3(\Gamma_N(p^m), \mathbf{F}_p)$ should be the order of $\bigwedge^3 M$, up to an error that is bounded by the order of M. More generally, we have:

Lemma 4.5 *For $F=\mathbf{Q}$, all primes p, and all sufficiently large N, the natural edge map*

$$\bigwedge^k M = H^k(G_N(p^m),\mathbf{F}_p) \longrightarrow H^k(\Gamma_N(p^m),\mathbf{F}_p)$$

has kernel and cokernel whose dimensions are $O(N^{2(k-2)})$, where the implied constant does not depend on N. In particular,

$$\begin{aligned}\dim H^k(\Gamma_N(p^m),\mathbf{F}_p) &= \binom{N^2-1}{k} + O(N^{2(k-2)})\\ &= \frac{N^{2k}}{k!} - \binom{k+1}{2}\frac{N^{2k-2}}{k!} + O(N^{2k-4}).\end{aligned}$$

Proof This follows from the Hochschild–Serre spectral sequence, noting that the first row is zero (since $\widetilde{H}^1(\mathrm{SL}_N,\mathbf{F}_p)=0$). □

Remark 4.6 It follows from [17, Theorem 1.5] that, for all F and p, the quantity $\dim H^k(\Gamma_N(p^m),\mathbf{F}_p)$ (in the stable range) is actually a *polynomial* in N.

Remark 4.7 One can obtain a corresponding result for number fields, except that $\widetilde{H}^1(\mathrm{SL}_N,\mathbf{F}_p)$ does not vanish in general if F contains p^{th} roots of unity; and thus one only obtains the estimate (without any assumption on p or F)

$$\dim H^k(\Gamma_N(p^m),\mathbf{F}_p) = \frac{N^{2kd}}{k!} + O(N^{2kd-2}).$$

Remark 4.8 One may sensibly define the stable homology groups

$$H_*^{\mathrm{stab}}(\Gamma_N(p^m),\mathbf{Z}_p) \quad \text{and} \quad H_*^{\mathrm{stab}}(\Gamma_N(p^m),\mathbf{F}_p)$$

for sufficiently large N to be the images of $\widetilde{H}_*(\mathrm{SL}_N,\mathbf{Z}_p)$ and $\widetilde{H}_*(\mathrm{SL}_N,\mathbf{F}_p)$ in $H_*(\Gamma_N(p^m),\mathbf{Z}_p)$ and $H_*(\Gamma_N(p^m),\mathbf{F}_p)$ respectively. Our results certainly imply that the action of $G_N/G_N(p^m)$ on $H_*^{\mathrm{stab}}(\Gamma_N(p^m),\mathbf{Z}_p)$ is trivial in the stable range. Moreover, one expects that these groups should be directly related (perhaps even equal) to the continuous homology groups $H_*^{\mathrm{cont}}(Y_m,\mathbf{Z}_p)$, which then relate to the K–theory (with coefficients) of the rings $\mathbf{Z}/p^m\mathbf{Z}$. Indeed, one can see a reflection of the calculation of these groups (in the simplest situations) in the computations above.

5 Partially completed K–groups

Suppose that F is an number field, and that $\mathfrak{p} \mid p$ is a prime in $\mathcal{O}_F$. Then one can apply the analysis above to the partially completed homology groups

$$\widetilde{H}_*(\mathrm{SL}_N,\mathfrak{p},\mathbf{Z}_p) := \varprojlim H_*(\Gamma_N(\mathfrak{p}^r),\mathbf{Z}_p).$$

(More generally, one can do this for any set of primes dividing p.) The arguments of [9] show that these groups do not depend on N for sufficiently large N (and have a trivial action of $\mathrm{SL}_N(\mathcal{O}_\mathfrak{p})$) and for such N we write $\widetilde{H}_*(\mathrm{SL}, \mathfrak{p}, \mathbf{Z}_p)$. Although we no longer have recourse to the Poitou–Tate sequence (which requires that S contain all places above p), it turns out that these groups are even simpler to understand than the standard completed cohomology groups under the following favorable circumstances. In particular, we shall find contexts in which all the partially completed homology groups (beyond $\widetilde{H}_0$) vanish identically.

Let $F/\mathbf{Q}$ be an imaginary quadratic field. Suppose that p splits in F, and let $v \mid p$ be a place above p. Let Φ be the Galois group of the maximal pro-p extension of $F(\zeta_p)$ over F unramified outside the places S above p. Let D_v be the decomposition group at v.

Definition 5.1 Let p be a prime that splits in an imaginary quadratic field $F/\mathbf{Q}$. The prime p is *very regular* if the map

$$\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to D_v \subseteq \Phi$$

is surjective for either $v \mid p$.

If the result is true for one $v \mid p$, then it is also true for the other (the images of D_v for $v \mid p$ are permuted by the action of $\mathrm{Gal}(F/\mathbf{Q})$.)

Lemma 5.2 *Suppose that p is very regular. Then the map $H^i(G_S, M) \to H^i(D_v, M)$ is an isomorphism for all i and all G_S–modules M that are subquotients of the Tate twists $\mathbf{Z}_p(n)$ for any $n \in \mathbf{Z}$.*

Before proving this lemma, we note the following consequence, which is the reason for considering very regular primes.

Corollary 5.3 *If p is a very regular prime, then the map $K_n(\mathcal{O}_F) \otimes \mathbf{Z}_p \to K_n(\mathcal{O}_\mathfrak{p}; \mathbf{Z}_p)$ is an isomorphism for $n > 1$.*

This follows from the description of these groups in terms of Galois cohomology (Theorem 2.3) and the fact that the corresponding map on Galois cohomology is an isomorphism.

Proof of Lemma 5.2 For such modules M as in the statement of the lemma, there is a canonical isomorphism $H^i(G_S, M) \simeq H^i(\Phi, M)$, because the cohomology groups will only depend on the pro-p completion of the fixed field of any relevant module. There is a natural isomorphism $H^0(G_S, M) = H^0(D_v, M_v)$, and the map

$$H^1(G_S, M) \to H^1(D_v, M_v)$$

is injective by inflation-restriction. Let Σ denote the set of Selmer conditions where no condition is imposed at $v \mid p$, and the dual Selmer condition Σ^* consists of classes that are totally trivial locally at $v \mid p$, and the classes are unramified outside p. We have $H^1_{\Sigma}(F, M) = H^1(G_S, M)$ by definition, where $H^1_{\Sigma}(F, M)$ as usual denotes classes in $H^1(F, M)$ which satisfy the local conditions corresponding to Σ. Since p is very regular, the group $H^1_{\Sigma^*}(F, M^*) = 0$ because any such class is trivial in $H^1(D_v, M^*)$. To this point, we have not used the fact that F is an imaginary quadratic field. This condition arises in the numerical computation of Selmer groups via the formula of Greenberg and Wiles [24; 48; 20] and the global Euler characteristic formula [29; 41]. From the Greenberg–Wiles formula, we have

$$|H^1(G_S, M)| = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|^2}{|H^0(D_v, M)|^2} \cdot \frac{1}{|M|}.$$

From the local Euler characteristic formula,

$$\begin{aligned} |H^1(D_v, M)| &= |H^0(D_v, M)| \cdot |H^2(D_v, M)| \cdot |M| \\ &= |H^0(D_v, M)| \cdot |H^0(D_v, M^*)| \cdot |M|, \end{aligned}$$

and thus

$$\begin{aligned} &|H^1(G_S, M)| \\ &\qquad = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|^2}{|H^0(D_v, M)|^2} \cdot \frac{1}{|M|} \\ &\qquad = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot \frac{|H^1(D_v, M)|}{|H^0(D_v, M)|} \cdot \frac{|H^0(D_v, M)| \cdot |H^0(D_v, M^*)| \cdot |M|}{|H^0(D_v, M)|} \cdot \frac{1}{|M|} \\ &\qquad = \frac{|H^0(F, M)|}{|H^0(F, M^*)|} \cdot |H^1(D_v, M)| \cdot \frac{1}{|H^0(D_v, M)|} \cdot |H^0(D_v, M^*)| \\ &\qquad = \frac{|H^0(F, M)|}{|H^0(D_v, M)|} \cdot \frac{|H^0(D_v, M^*)|}{|H^0(F, M^*)|} \cdot |H^1(D_v, M)| \\ &\qquad = |H^1(D_v, M)|. \end{aligned}$$

It follows that $H^1(G_S, M) \to H^1(D_v, M)$, which we already showed was an injection, is actually an isomorphism. On the other hand, by the global and local Euler characteristic formulae,

$$|H^2(G_S, M)| = \frac{|H^1(G_S, M)|}{|H^0(G_S, M)|} \cdot \frac{H^0(G_{\mathbf{C}}, M)}{|M|^2} = \frac{|H^1(D_v, M)|}{|H^0(D_v, M)|} \cdot \frac{1}{|M|} = |H^2(D_v, M)|.$$

By the 5–lemma and dévissage, to prove the lemma it suffices to show that the maps

$$H^2(G_S, \mathbf{F}_p(n)) \to H^2(D_v, \mathbf{F}_p(n))$$

are injective (equivalently, isomorphisms) for any $n \in \mathbf{Z}$ (we use here the fact that the local and global cohomology groups have cohomological dimension 2 in this context). The only nontrivial case (ie when both groups are not trivial) is $n = 1$. In this case, there is a commutative diagram

$$\begin{tikzcd} H^1(G_S, V) \arrow[r] \arrow[d, equal] & H^1(G_S, \mathbf{F}_p) \arrow[r] \arrow[d, equal] & H^1(G_S, \mathbf{F}_p(1)) \arrow[d, equal] \\ H^1(D_v, V) \arrow[r] & H^1(D_v, \mathbf{F}_p) \arrow[r] & H^1(D_v, \mathbf{F}_p(1)), \end{tikzcd}$$

where V is any nonsplit extension of $\mathbf{F}_p$ by $\mathbf{F}_p(1)$. Since V is nonsplit, $H^0(V^*) = 0$, and hence $H^2(V) = 0$. Thus the maps $H^1(\mathbf{F}_p) \to H^1(\mathbf{F}_p(1))$ are surjective, and it follows (by commutativity) that map $H^1(G_S, \mathbf{F}_p(1)) \to H^1(D_v, \mathbf{F}_p(1))$ must be surjective, hence an isomorphism. □

We now give a numerical characterization of very regular primes.

Lemma 5.4 *Let F be an imaginary quadratic field such that $p > 2$ splits completely, and let $v \mid p$ be a place above p. Let w be the unique place above v in $F(\zeta_p)$. Suppose that:*

(1) *The maximal exponent p abelian extension of $F(\zeta_p)$ that is unramified outside w is cyclic.*

(2) *If $p = \mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_F$ and $h = |\mathrm{Cl}(\mathcal{O}_F)|$, then the projection of $\mathfrak{p}^h$ generates*

$$(1 + \overline{\mathfrak{p}})/(1 + \overline{\mathfrak{p}}^2).$$

Then p is very regular. Conversely, if either of these conditions fail, then p is not very regular.

Proof To show the map of pro-p groups $D_w \to \Phi$ is surjective, it suffices to prove that D_w surjects onto the Frattini quotient of Φ. The cokernel of this map corresponds to an abelian exponent p extension of $F(\zeta_p)$ that is unramified at w and at all primes away from p. If the cokernel is not cyclic, then there exists a quotient where w splits completely, and p is not very regular. Let us assume it is cyclic. By class field theory, F itself admits an abelian p–extension H/F unramified at v and all primes away from p, and hence the extension over $F(\zeta_p)$ descends to F. It suffices to show that this extension is inert at v if and only if the second condition holds. Equivalently, it suffices to show that the maximal abelian p–extension of F completely split at $v \mid p$ and unramified outside p is trivial. Let $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ be the primes in $\mathcal{O}_F$ corresponding to the place $v \mid p$ and its conjugate. Recall that the p–part of the ray class group of conductor $\overline{\mathfrak{p}}^2$ lives in an exact sequence

$$\mathcal{O}_F^\times \cap (1+\overline{\mathfrak{p}}) \longrightarrow (1+\overline{\mathfrak{p}})/(1+\overline{\mathfrak{p}}^2) \longrightarrow \mathrm{RCl}(\overline{\mathfrak{p}}^2)\otimes \mathbf{Z}_p \longrightarrow \mathrm{Cl}(\mathcal{O}_F)\otimes \mathbf{Z}_p \longrightarrow 0.$$

Since F is an imaginary quadratic field, $\mathcal{O}_F^\times$ consists entirely of roots of unity, and so the first term $\mathcal{O}_F^\times \cap (1+\overline{\mathfrak{p}})$ vanishes because we are assuming that p is prime to w_F (this follows because $p>2$ is unramified in F). It suffices to show that $\mathrm{RCl}(\overline{\mathfrak{p}}^2)\otimes \mathbf{Z}_p$ is generated by $\mathfrak{p}$, since, by Nakayama, any generator of $\mathrm{RCl}(\overline{\mathfrak{p}}^2)\otimes \mathbf{Z}_p$ lifts to a generator of $\mathrm{RCl}(\overline{\mathfrak{p}}^m)\otimes \mathbf{Z}_p$ for any m. Since $\mathrm{RCl}(\overline{\mathfrak{p}}^2)\otimes \mathbf{Z}_p$ is an abelian p–group of order $p\cdot h_p$ (where h_p is the p–part of h), it is cyclic with generator $\mathfrak{p}$ if and only if $\mathfrak{p}^h$ has order p. Yet $\mathfrak{p}^h$ is nontrivial in $\mathrm{RCl}(\overline{\mathfrak{p}}^2)\otimes \mathbf{Z}_p$ if and only if its projection to $(1+\overline{\mathfrak{p}})/(1+\overline{\mathfrak{p}}^2)$ is nontrivial. □

Note that the projection of any principal ideal (α) onto $(1+\overline{\mathfrak{p}})/(1+\overline{\mathfrak{p}}^2)$ is given by the image of α^{p-1}.

We give an alternate numerical formulation of the previous lemma:

Lemma 5.5 *Let F be an imaginary quadratic field such that $p>2$ splits completely, and let $v \mid p$ be a place above p. Let χ be the odd quadratic character corresponding to F. Then p is very regular if and only if the following hold:*

(1) *p is a regular prime, that is, $p>3$ does not divide*

$$\zeta(-1), \quad \zeta(-3), \quad \zeta(-5), \quad \dots, \quad \zeta(4-p).$$

(2) *p does not divide any of the L–values*

$$L(\chi,-2), \quad L(\chi,-4), \quad L(\chi,-6), \quad \dots, \quad L(\chi,3-p).$$

(3) *If $p=\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_F$ and $h=|\mathrm{Cl}(\mathcal{O}_F)|$, then the projection of $\mathfrak{p}^h$ generates*

$$(1+\overline{\mathfrak{p}})/(1+\overline{\mathfrak{p}}^2).$$

Proof It suffices to show that the conditions are equivalent to those of Lemma 5.4. If either p is not regular or divides one of the L–values, then, by the main conjecture of Iwasawa theory [47], there exists an unramified extension of $\mathbf{F}_p$ by either $\mathbf{F}_p(n)$ or $\mathbf{F}_p(n) \otimes \chi$ for some $n \neq 0, 1 \bmod (p-1)$. Since $H^1(\mathbf{Q}_p, \mathbf{F}_p(n)) = 0$ for such n, these extensions necessarily split completely at primes above p, implying that p is not very regular. By Lemma 5.4 and its proof the same is true of the final condition.

Now suppose that p is not very regular. Equivalently, the subspace of $H^1(F(\zeta_p), \mathbf{F}_p)$ consisting of classes that are trivial away from p and for some w with $w \mid p$ is nonzero. The group $\mathrm{Gal}(F(\zeta_p)/F)$ acts on this space, and the third condition above implies that the projection of this space onto $H^1(F, \mathbf{F}_p)$ is trivial. This implies the same for $H^1(F, \mathbf{F}_p(1))$ by the Greenberg–Wiles formula. Hence there exists a nontrivial $n \neq 0, 1 \bmod (p-1)$ such that

$$H^1(G_{F,S}, \mathbf{F}_p(n)) \simeq H^1(G_{\mathbf{Q},S}, \mathbf{F}_p(n)) \oplus H^1(G_{\mathbf{Q},S}, \mathbf{F}_p(n) \otimes \chi)$$

contains a class η that is unramified at $v \mid p$. Suppose that $\dim H^1(G_{F,S}, \mathbf{F}_p(n)) = 1$. Then $\eta \in H^1(\mathbf{Q}, V)$ where V is either $\mathbf{F}_p(n)$ or $\mathbf{F}_p(n) \otimes \chi$. As these spaces are $\mathrm{Gal}(F/\mathbf{Q})$–invariant, it follows that η is also unramified at the other prime dividing p. Now one of the following holds:

(1) Complex conjugation acts on V by -1, in which case by the main conjecture one obtains a divisibility of L–values as above.

(2) Complex conjugation acts by $+1$ on V, in which case, by the Greenberg–Wiles formula, there exists an everywhere unramified class in $H^1(\mathbf{Q}, V^*)$, that also implies a divisibility of L–values (again by the main conjecture).

Suppose now that $\dim H^1(G_{F,S}, \mathbf{F}_p(n)) > 1$. If either

$$H^1(G_{\mathbf{Q},S}, \mathbf{F}_p(n)) \quad \text{or} \quad H^1(G_{\mathbf{Q},S}, \mathbf{F}_p(n) \otimes \chi))$$

is zero, the same argument as above applies. Hence we may assume that

$$H^1(G_{\mathbf{Q},S}, V) \neq 0,$$

where complex conjugation acts by 1 on V, and the result follows as in the previous case. □

The following result shows that the partially completed homology groups at very regular primes are particularly simple:

Lemma 5.6 *Suppose that p is very regular in a imaginary quadratic field $F/\mathbf{Q}$. Let $\mathfrak{p}$ denote a prime above p. Then the stable $\mathfrak{p}$–completed homology groups*

$$\widetilde{H}_n(\mathrm{SL}, \mathfrak{p}, \mathbf{Z}_p) = \varprojlim H_n(\mathrm{SL}, \Gamma(\mathfrak{p}^r), \mathbf{Z}_p)$$

are trivial for all $n > 0$, and equal to $\mathbf{Z}_p$ for $n = 0$.

Proof The analogues of the completed K–groups in these contexts are the homotopy groups with coefficients in $\mathbf{Z}_p$ of the homotopy fibers $K(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}; \mathbf{Z}_p)$ and $SK(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}; \mathbf{Z}_p)$. Let

$$\begin{aligned} \widetilde{K}_n(\mathcal{O}, \mathfrak{p}) &:= \widetilde{K}_n(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}; \mathbf{Z}_p) := \pi_n(K(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}); \mathbf{Z}_p), \\ \widetilde{SK}_n(\mathcal{O}, \mathfrak{p}) &:= \widetilde{SK}_n(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}; \mathbf{Z}_p) := \pi_n(SK(\mathcal{O}, \mathcal{O}_{\mathfrak{p}}); \mathbf{Z}_p) \end{aligned}$$

for $n > 0$. By Corollary 5.3, the natural map

$$SK_n(\mathcal{O}_F; \mathbf{Z}_p) \longrightarrow SK_n(\mathcal{O}_{\mathfrak{p}}; \mathbf{Z}_p) = SK_n(\mathbf{Z}_p; \mathbf{Z}_p)$$

is an isomorphism for all $n \geq 2$, and (because both vanish) the map is also an isomorphism for $n = 1$. Hence, from the Serre exact sequence, the completed homotopy groups $\widetilde{SK}_n(\mathcal{O}, \mathfrak{p})$ vanish for all n. The result follows by the Hurewicz theorem. □

This lemma allows us to explicitly compute all the classical congruence homology groups at powers of a very regular prime in the stable range.

Theorem 5.7 *Suppose that p is very regular in a imaginary quadratic field $F/\mathbf{Q}$, and fix an integer n. Let $\mathfrak{p}$ denote a prime above p, let $\Gamma_N = \mathrm{SL}_N(\mathcal{O}_F)$, let $G_N = \mathrm{SL}_N(\mathcal{O}_{\mathfrak{p}}) = \mathrm{SL}_N(\mathbf{Z}_p)$. Then, for all sufficiently large N, there are, for all positive integers m, isomorphisms*

$$\begin{aligned} &H_n(\Gamma_N(\mathfrak{p}^m), \mathbf{Z}_p) \simeq H_n(G_N(p^m), \mathbf{Z}_p), \\ &H_n(\Gamma_N(\mathfrak{p}^m), \mathbf{F}_p) \simeq H_n(G_N(p^m), \mathbf{F}_p) \simeq \bigwedge^n M^0_N(\mathbf{F}_p), \end{aligned}$$

where $M^0_N(\mathbf{F}_p)$ denotes the trace zero $N \times N$ matrices with coefficients in $\mathbf{F}_p$. Moreover, these isomorphisms respect the $G_N/G_N(\mathfrak{p}^m)$ and $G_N/G_N(\mathfrak{p}) = \mathrm{SL}_N(\mathbf{F}_p)$–module structures, respectively.

Proof This follows immediately from the Hochschild–Serre spectral sequence, since $E^2_{ij} = 0$ unless $i = 0$, and hence the sequence degenerates immediately. Note that the vanishing of $\widetilde{H}_*(\mathrm{SL}_N, \mathfrak{p}, \mathbf{Z}_p)$ implies the vanishing of $\widetilde{H}_*(\mathrm{SL}_N, \mathfrak{p}, \mathbf{F}_p)$. □

Remark 5.8 Algebraic local systems of GL for an imaginary quadratic field are algebraic representations of $\mathrm{GL}_N(\mathbf{C})$ as a real group, in particular, they are direct sums of representations of the form $V_\mu \otimes \overline{V_\lambda}$, where V_μ and V_λ are algebraic representations of GL_N of highest weight μ and λ respectively, and where the bar indicates the action of $\mathrm{GL}_N(\mathbf{C})$ is composed with complex conjugation. If $\mathscr{L}$ corresponds to such a representation with $\lambda = 0$, then $\mathscr{L}$ becomes locally trivial up the $\mathfrak{p}$–adic tower, and hence the coefficients can be pulled out as in Remark 2.21. In particular, for such local systems, we have, under the conditions of Theorem 5.7, isomorphisms

$$H_n(\Gamma_N(\mathfrak{p}^m), \mathscr{L}) \simeq H_n(G_N(p^m), \mathscr{L} \otimes_{\mathcal{O}_F} \mathbf{Z}_p).$$

We note that (as it must) the algebra $H^*(G_N, \mathbf{Q}_p)$ as $N \to \infty$ coincides with the stable cohomology groups as computed by Borel. The groups $H^*(G_N, \mathscr{L} \otimes \mathbf{Q}_p)$ vanish for nontrivial $\mathscr{L}$ as can be seen by considering the action of the infinitesimal character. This is consistent with the vanishing of $H^*(\Gamma_N, \mathscr{L} \otimes \mathbf{Q}_p)$ in the stable range, as follows (essentially) from a similar computation.

We observe that very regular primes do exist. For any particular fixed field F, we can give a relatively fast and explicit algorithm for computing regular primes using generalized Bernoulli numbers. We give some details in the case $F = \mathbf{Q}(\sqrt{-1})$. Recall that the Bernoulli numbers B_{2n} and Euler numbers E_{2n} are defined by the Taylor series

$$\frac{t}{e^t - 1} = \sum \frac{B_n}{n!} \cdot t^n, \qquad \frac{2}{e^t + e^{-t}} = \sum \frac{E_n}{n!} \cdot t^n.$$

Lemma 5.9 *If $F = \mathbf{Q}(\sqrt{-1})$, then $p \equiv 1 \bmod 4$ is very regular if and only if the following conditions are satisfied:*

(1) *The prime p divides neither B_{2n} nor E_{2n} for $2n$ less than $p-1$.*
(2) *If $p = a^2 + b^2$, then $(4ab)^{p-1} \not\equiv 1 \bmod p^2$.*

Proof The first condition follows from the formulae

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}, \qquad L(\chi_4, -2n) = \frac{E_{2n}}{2},$$

the second is equivalent to the condition $\alpha^{p-1} \not\equiv 1 \bmod \overline{\alpha}^2$, where $\alpha = a + bi$. □

The nonregular primes p less than 100 that split in $\mathbf{Q}(\sqrt{-1})$ are $p = 37$ (which divides B_{32}) and $p = 61$, which divides

$$\frac{-\pi^7}{6! \cdot 2^7} \cdot L(\chi_4, -6) = L(\chi_4, 7) = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \cdots = 61 \cdot \frac{\pi^7}{6! \cdot 2^6}.$$

Using Lemmas 5.4, 5.5 and 5.9, we compute the following examples for $p=3$ and $|d_F|<200$ as well as primes less than 100 for the ten smallest imaginary quadratic fields.

Example 5.10 If $F=\mathbf{Q}(\sqrt{-|d_F|})$, then $p=3$ is very regular for a fundamental discriminant $|d_F|<200$ if and only if $-d_F$ is one of the following integers:

$$8,\ 11,\ 20,\ 23,\ 59,\ 68,\ 71,\ 83,\ 95,\ 104,\ 116,\ 119,\ 131,\ 143,\ 152,\ 155,\ 167,\ 179,\ 191.$$

Remark 5.11 Following [19], for an imaginary quadratic field F, one may consider a heuristic model of the p–class group $\mathrm{Cl}(\mathcal{O}_F)\otimes\mathbf{Z}_p$ as the quotient of a free $\mathbf{Z}_p$–module with N generators by N relations, where N goes to infinity. Correspondingly, one might model the ray class group of conductor $(\overline{\mathfrak{p}})^{\infty}$ for a $\mathfrak{p}$ which is completely split by allowing an extra generator for ramification at $\overline{\mathfrak{p}}$, and imposing a further relation by demanding local triviality at $\mathfrak{p}$ (via the image of the Artin map). In this case, one would predict that the probability that p fails to satisfy the given condition is exactly the same as the probability that the p–part of the class group is trivial, namely

$$\prod_{n=0}^{\infty}\left(1-\frac{1}{p^n}\right).$$

For example, if $p=3$, then p is very regular if and only if it satisfies the final condition of Lemma 5.5, and this calculation suggests that the density of such quadratic fields (amongst those in which 3 splits) is

$$\prod_{n=0}^{\infty}\left(1-\frac{1}{3^n}\right)\sim 0.560126\ldots.$$

For such fields of fundamental discriminant $|d_F|<$ 1,000, 10,000, 100,000, and 300,000, the corresponding percentage of fields in which 3 is very regular is 68.1%, 64.6%, 61.2%, and 59.8% respectively. (This is consistent with the slow monotone rate of convergence often observed in Cohen–Lenstra phenomena, and also with the fact that the apparent error term is positive.) There are natural maps

$$\mathrm{RCl}((\overline{\mathfrak{p}})^{\infty})/\mathfrak{p}\otimes\mathbf{Z}_p\to\mathrm{Cl}(\mathcal{O}_F)/\mathfrak{p}\otimes\mathbf{Z}_p\leftarrow\mathrm{Cl}(\mathcal{O}_F)\otimes\mathbf{Z}_p.$$

From our discussion, the two outer groups should have a similar distribution. The group in the middle, however, having one further relation, should behave like the p–part of the class group of a *real* quadratic field (this exact remark is made in the last paragraph of Section 8 of [19]).

Example 5.12 We have the following table for the 10 smallest imaginary quadratic fields.

$-d_F$	3	4	7	8	11	15	19	20	23	24
3				✓	✓			✓	✓	
5		✓			✗		✓			✓
7	✓						✓	✓		✓
11			✓	✓			2, ✗			✓
13	✗	✓							✓	
17		✓		✓		14	✓			
19	✓			4		✓				
23			✓		✓	✓	✓	✓	✓	
29		✓	18					16	✓	✓
31	✓				✓	4			✓	✓
37	31	31	31		31					
41		✓		✓				24	8	
43	✓		✓	✓			✓	✓		
47					✓	✓	✓	24	✓	
53		✓	18, 42		✓	✓				✓
59				18, 43	43				43	43, 52
61	✓	6				54	56	42		
67	46, 57		57	57	57			57		
71			✓		6				4	
73	✓	✓		30			58		✓	28
79	✓		✓			✓				✓
83				14		✓	✓	8, 24		✓
89		✓		32	✓			✓		
97	✓	✓		✓	✓					✓

A tick indicates that p is very regular for F, a blank indicates that p is either inert or ramified in F, an even integer $2n$ means that p fails to be regular because p divides $L(\chi,-2n)$, an odd integer $2n-1$ means that p divides $\zeta(1-2n)$, and the crosses indicate the failure of condition 3 of Lemma 5.5, explicitly, $\alpha^{p-1} \equiv 1 \bmod \overline{\alpha}^2$, where α is $(7+\sqrt{-3})/2$, $(3+\sqrt{-11})/2$, $(5+\sqrt{-19})/2$ respectively; note that $h_F = 1$ in each case.

It is interesting to compare the case $F = \mathbf{Q}(\sqrt{-2})$ and $3 = \mathfrak{p}\overline{\mathfrak{p}}$ to computations of *nonstable* completed cohomology groups in the work of the author and Dunfield [8] (see also [5]). In particular, for $N = 2$, one may show in this case that $\widetilde{H}_n(\mathfrak{p}, \mathbf{Z}_p) = 0$ for all $n > 0$. Moreover, in the context of [10], it is theoretically possible that $\widetilde{H}_n(\mathfrak{p}, \mathbf{Z}_p) = 0$

for very regular primes for the same F and $\mathfrak{p}$ for other values of N, since one has the necessary "numerical coincidence" between dimensions of locally symmetric spaces and p–adic lie groups:

$$\dim \mathrm{SL}_N(\mathcal{O}_F)\backslash \mathrm{SL}_N(\mathbf{C})/\mathrm{SU}_N(\mathbf{C}) = N^2 - 1 = \dim \mathrm{SL}_N(\mathbf{Z}_p).$$

However, under the assumption that for sufficiently large N there will exist weight 0 regular cuspidal automorphic representations of level 1 for $F = \mathbf{Q}$, one would anticipate the existence of nonstable characteristic zero classes, which would imply that $\widetilde{H}_*(\mathfrak{p}, \mathbf{Z}_p)$ can only vanish over the entire nonstable range only for finitely many N (for all F and $\mathfrak{p}$).

Remark 5.13 Natural heuristics suggest that for any imaginary quadratic field F, there are infinitely many very regular primes (possibly with density $e^{-1} = 0.367879\ldots$). This seems hard to prove, however, since being very regular implies that p is regular in $\mathbf{Q}$, and the infinitude of regular primes is a well-known open question.

Remark 5.14 There is a natural generalization of very regular to a prime p that splits completely in a totally imaginary CM field F/F^+. Here one replaces a single place $v \mid p$ by a collection T of $[F^+ : \mathbf{Q}]$ places $v \mid p$, and asks that the map

$$\prod_{[F^+:\mathbf{Q}]} \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow \prod_{v \mid T} D_v \subseteq \Gamma$$

is surjective, where Γ is the Galois group of the pro-p extension of $F(\zeta_p)$ unramified outside p.

Remark 5.15 In contrast to the case of regular primes, it seems difficult to control the module $\widetilde{H}_d(\mathrm{SL}, \mathfrak{p}, \mathbf{Z}_p)$ for a prime p that splits in F, even for $d = 2$. An elementary computation shows that $\widetilde{H}_2(\mathrm{SL}, \mathfrak{p}, \mathbf{Z}_p)$ is infinite if and only if $L_p(\chi, 2) \neq 0$. (Wagoner computed that the associated regulator map was nonzero for $F = \mathbf{Q}(\sqrt{-3})$ and primes $\mathfrak{p}$ of prime norm ≤ 73 in [44].) As with $\zeta_p(3) \in \mathbf{Q}_p$, it seems difficult to prove that $L_p(\chi, 2) \neq 0$ in general, even for a fixed p (and varying imaginary quadratic character χ).

References

[1] **A Ash**, *Galois representations attached to mod p cohomology of* $\mathrm{GL}(n, \mathbf{Z})$, Duke Math. J. 65 (1992) 235–255 MR1150586

[2] **H Bass**, **J Milnor**, **J-P Serre**, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. (1967) 59–137 MR0244257

[3] **A Besser**, **P Buckingham**, **R de Jeu**, **X-F Roblot**, *On the p–adic Beilinson conjecture for number fields*, Pure Appl. Math. Q. 5 (2009) 375–434 MR2520465

[4] **A Borel**, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. 7 (1974) 235–272 MR0387496

[5] **N Boston**, **J S Ellenberg**, *Pro-p groups and towers of rational homology spheres*, Geom. Topol. 10 (2006) 331–334 MR2224459

[6] **W Browder**, **J Pakianathan**, *Cohomology of uniformly powerful p–groups*, Trans. Amer. Math. Soc. 352 (2000) 2659–2688 MR1661313

[7] **F Calegari**, *Irrationality of certain p–adic periods for small p*, Int. Math. Res. Not. 2005 (2005) 1235–1249 MR2144087

[8] **F Calegari**, **N M Dunfield**, *Automorphic forms and rational homology 3–spheres*, Geom. Topol. 10 (2006) 295–329 MR2224458

[9] **F Calegari**, **M Emerton**, *Hecke operators on stable cohomology* `arXiv:1311.5183`

[10] **F Calegari**, **M Emerton**, *Mod–p cohomology growth in p–adic analytic towers of 3-manifolds*, Groups Geom. Dyn. 5 (2011) 355–366 MR2782177

[11] **F Calegari**, **M Emerton**, *Completed cohomology — a survey*, from: "Non-abelian fundamental groups and Iwasawa theory", (J Coates, M Kim, F Pop, M Saidi, P Schneider, editors), London Math. Soc. Lecture Note Ser. 393, Cambridge Univ. Press (2012) 239–257 MR2905536

[12] **F Calegari**, **D Geraghty**, *Modularity lifting beyond the Taylor–Wiles method* `arXiv:1207.4224`

[13] **F Calegari**, **A Venkatesh**, *A torsion Jacquet–Langlands correspondence* `arXiv:1212.3847`

[14] **R M Charney**, *Homology stability for GL_n of a Dedekind domain*, Invent. Math. 56 (1980) 1–17 MR557579

[15] **R Charney**, *On the problem of homology stability for congruence subgroups*, Comm. Algebra 12 (1984) 2081–2123 MR747219

[16] **T Church**, **J S Ellenberg**, **B Farb**, *FI–modules and stability for representations of symmetric groups*, Duke Math. J. 164 (2015) 1833–1910

[17] **T Church**, **J S Ellenberg**, **B Farb**, **R Nagpal**, *FI–modules over Noetherian rings*, Geom. Topol. 18 (2014) 2951–2984 MR3285226

[18] **T Church**, **B Farb**, *Representation theory and homological stability*, Adv. Math. 245 (2013) 250–314 MR3084430

[19] **H Cohen**, **H W Lenstra, Jr**, *Heuristics on class groups of number fields*, from: "Number theory, Noordwijkerhout 1983", (H Jager, editor), Lecture Notes in Math. 1068, Springer, Berlin (1984) 33–62 MR756082

[20] **H Darmon**, **F Diamond**, **R Taylor**, *Fermat's last theorem*, from: "Elliptic curves, modular forms & Fermat's last theorem", (J Coates, S T Yau, editors), Int. Press, Cambridge, MA (1997) 2–140 MR1605752

[21] **P Deligne**, *Extensions centrales non résiduellement finies de groupes arithmétiques*, C. R. Acad. Sci. Paris Sér. A-B 287 (1978) A203–A208 MR507760

[22] **L Evens**, **E M Friedlander**, *On $K_*(\mathbf{Z}/p^2\mathbf{Z})$ and related homology groups*, Trans. Amer. Math. Soc. 270 (1982) 1–46 MR642328

[23] **O Gabber**, *K–theory of Henselian local rings and Henselian pairs*, from: "Algebraic K–theory, commutative algebra, and algebraic geometry", (R K Dennis, C Pedrini, M R Stein, editors), Contemp. Math. 126, Amer. Math. Soc. (1992) 59–70 MR1156502

[24] **R Greenberg**, *Iwasawa theory for p–adic representations*, from: "Algebraic number theory", (J Coates, R Greenberg, B Mazur, I Satake, editors), Adv. Stud. Pure Math. 17, Academic Press, Boston, MA (1989) 97–137 MR1097613

[25] **L Hesselholt**, **I Madsen**, *On the K–theory of local fields*, Ann. of Math. 158 (2003) 1–113 MR1998478

[26] **M Lazard**, *Groupes analytiques p–adiques*, Inst. Hautes Études Sci. Publ. Math. (1965) 389–603 MR0209286

[27] **R Lee**, **R H Szczarba**, *The group $K_3(Z)$ is cyclic of order forty-eight*, Ann. of Math. 104 (1976) 31–60 MR0442934

[28] **J McCleary**, *A user's guide to spectral sequences*, 2nd edition, Cambridge Studies in Advanced Mathematics 58, Cambridge Univ. Press (2001) MR1793722

[29] **J S Milne**, *Arithmetic duality theorems*, 2nd edition, BookSurge, Charleston, SC (2006) MR2261462

[30] **J W Milnor**, **J C Moore**, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965) 211–264 MR0174052

[31] **J A Neisendorfer**, *Homotopy groups with coefficients*, J. Fixed Point Theory Appl. 8 (2010) 247–338 MR2739026

[32] **I A Panin**, *The Hurewicz theorem and K–theory of complete discrete valuation rings*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986) 763–775, 878 MR864175 In Russian; translated in Math. USSR-Ivz. 29 (1987) 119–131

[33] **A Putman**, *Stability in the homology of congruence subgroups* `arXiv:1201.4876`

[34] **D Quillen**, *On the cohomology and K–theory of the general linear groups over a finite field*, Ann. of Math. 96 (1972) 552–586 MR0315016

[35] **D Quillen**, *Finite generation of the groups K_i of rings of algebraic integers*, from: "Algebraic K–theory, I: Higher K–theories", Lecture Notes in Math. 341, Springer, Berlin (1973) 179–198 MR0349812

[36] **P Schneider**, *Über gewisse Galoiscohomologiegruppen*, Math. Z. 168 (1979) 181–205 MR544704

[37] **P Schneider**, *Über die Werte der Riemannschen Zetafunktion an den ganzzahligen Stellen*, J. Reine Angew. Math. 313 (1980) 189–194 MR552472

[38] **P Scholze**, *On torsion in the cohomology of locally symmetric varieties* arXiv:1306.2070

[39] **C Soulé**, *K–théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. 55 (1979) 251–295 MR553999

[40] **C Soulé**, *On higher p–adic regulators*, from: "Algebraic K–theory", (E M Friedlander, M R Stein, editors), Lecture Notes in Math. 854, Springer, Berlin (1981) 372–401 MR618313

[41] **J Tate**, *Duality theorems in Galois cohomology over number fields*, from: "Proc. Internat. Congr. Mat. (Stockholm, 1962)", Inst. Mittag-Leffler, Djursholm (1963) 288–295 MR0175892

[42] **V Voevodsky**, *On motivic cohomology with $\mathbf{Z}/l$–coefficients*, Ann. of Math. 174 (2011) 401–438 MR2811603

[43] **J B Wagoner**, *Continuous cohomology and p–adic K–theory*, from: "Algebraic K–theory", (M R Stein, editor), Lecture Notes in Math. 551, Springer, Berlin (1976) 241–248 MR0498502

[44] **J B Wagoner**, *A p–adic regulator problem in algebraic K–theory and group cohomology*, Bull. Amer. Math. Soc. 10 (1984) 101–104 MR722861

[45] **C Weibel**, *Algebraic K–theory of rings of integers in local and global fields*, from: "Handbook of K–theory, Vol. 1, 2", (E M Friedlander, D R Grayson, editors), Springer, Berlin (2005) 139–190 MR2181823

[46] **C A Weibel**, *The K–book: An introduction to algebraic K–theory*, Graduate Studies in Mathematics 145, Amer. Math. Soc. (2013) MR3076731

[47] **A Wiles**, *The Iwasawa conjecture for totally real fields*, Ann. of Math. 131 (1990) 493–540 MR1053488

[48] **A Wiles**, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. 141 (1995) 443–551 MR1333035

[49] **E C Zeeman**, *A note on a theorem of Armand Borel*, Proc. Cambridge Philos. Soc. 54 (1958) 396–398 MR0105099

Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208, USA

fcale@math.northwestern.edu

Proposed: Benson Farb — Received: 7 November 2013
Seconded: Danny Calegari, Walter Neumann — Revised: 27 December 2014