

## $E_n$ genera

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Let  $R$  be an  $E_2$  ring spectrum with zero odd-dimensional homotopy groups. Every map of ring spectra  $\mathrm{MU} \rightarrow R$  is represented by a map of  $E_2$  ring spectra. If 2 is invertible in  $\pi_0 R$ , then every map of ring spectra  $\mathrm{MSO} \rightarrow R$  is represented by a map of  $E_2$  ring spectra.

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### 1 Introduction

Genera (in the sense we use the word here) are multiplicative cobordism invariants of manifolds with extra structure. In the past 60 years, the study of various genera has led to stunning advances throughout mathematics, from algebraic geometry with the Hirzebruch–Riemann–Roch theorem [18; 19], to differential equations with the Atiyah–Singer index theorem [29], to mathematical physics with the Witten genus [31], in addition to innumerable advances inside topology. Because our perspective comes from stable homotopy theory, we will restrict attention to genera that extend to singular manifolds on pairs. With only minor additional hypotheses, such genera are precisely natural transformations of cohomology theories, or better, maps of ring spectra out of a cobordism spectrum or a related spectrum. These genera lie at the heart of modern stable homotopy theory, in particular, its organization in terms of chromatic phenomena, which derives from Quillen’s identification of genera of stably almost complex manifolds (ie ring spectrum maps out of  $\mathrm{MU}$ ) in terms of formal coordinates for formal group laws.

The three most basic cobordism spectra  $\mathrm{MO}$  (unoriented cobordism),  $\mathrm{MSO}$  (oriented cobordism), and  $\mathrm{MU}$  (complex cobordism) are all examples of  $E_\infty$  ring spectra (now usually called commutative  $S$ -algebras). These are ring spectra where the multiplication is not just associative, commutative, and unital in the stable category, but actually in a point-set symmetric monoidal category of spectra. The  $E_\infty$  structures on these cobordism spectra derive from products and powers of manifolds, and work of Ando, Hopkins, Rezk, and Strickland (and their collaborators, among others) shows that refining maps out of cobordism spectra and related spectra to  $E_\infty$  (or  $H_\infty$ ) ring

maps has implications in geometry as well as topology and stable homotopy theory (see, for example, [2; 3; 5]). An  $E_\infty$  ring structure brings with it many extra tools and much of the work of stable homotopy in the past two decades has involved producing  $E_\infty$  ring structures and  $E_\infty$  ring maps.

Recent work of Johnson and Noel [20], however, shows that maps out of MU that come from  $p$ -typical orientations usually do not commute with power operations. As a consequence, many of the maps of ring spectra out of MU that are fundamental in the chromatic picture of stable homotopy theory cannot be represented by  $E_\infty$  ring maps. This mandates consideration of less rigid structures than  $E_\infty$  ring structures, and an obvious place to start is the Boardman–Vogt hierarchy of  $E_n$  structures, of which  $E_\infty$  is the apex. An  $E_1$  ring structure is also called an  $A_\infty$  ring structure (or associative  $S$ -algebra structure) and retains all of the homotopy coherent associativity without the commutativity. An  $E_2$  ring spectrum is homotopy commutative and as  $n$  gets higher,  $E_n$  ring spectra become more coherently homotopy commutative and have more of the power operations in an  $E_\infty$  ring spectrum.

The purpose of this paper is to study which genera of oriented manifolds and stably almost complex manifolds are represented by maps of  $E_n$  ring spectra. For reasons explained at the end of Section 2, the easiest case is when  $n = 2$ , where we have the following results.

**Theorem 1.1** *Let  $R$  be an even  $E_2$  ring spectrum with  $1/2 \in \pi_0 R$ . Then every map of ring spectra  $\text{MSO} \rightarrow R$  lifts to a map of  $E_2$  ring spectra  $\text{MSO} \rightarrow R$ .*

**Theorem 1.2** *Let  $R$  be an even  $E_2$  ring spectrum. Then every map of ring spectra  $\text{MU} \rightarrow R$  lifts to a map of  $E_2$  ring spectra  $\text{MU} \rightarrow R$ .*

Here “even” means that the homotopy groups are concentrated in even degrees, ie  $\pi_q R = 0$  for  $q$  odd. Examples of  $E_2$  (or better) even ring spectra include the Brown–Peterson spectrum BP, the Lubin–Tate spectra  $E_n$ , and conjecturally, the truncated Brown–Peterson spectra  $\text{BP}\langle n \rangle$  and Johnson–Wilson spectra  $E(n)$ . Each of these spectra comes with a canonical map of ring spectra out of MU that is a  $p$ -typical orientation and that by the Johnson–Noel result [20, 1.3, 1.4] does not come from a map of  $E_\infty$  ring spectra (at small primes  $p$ , and conjecturally at all primes). Theorem 1.2 shows that these maps do come from maps of  $E_2$  ring spectra. The case of BP seems particularly worth highlighting as the coherence of the Quillen map  $\text{MU} \rightarrow \text{BP}$  has been an open question since the 1970s.

**Corollary 1.3** *The Quillen idempotent  $\text{MU}_{(p)} \rightarrow \text{MU}_{(p)}$  and the Quillen map  $\text{MU} \rightarrow \text{BP}$  are represented by maps of  $E_2$  ring spectra.*

In fact, since BP may be constructed as the telescope over the Quillen idempotent, this gives a new proof that BP is an  $E_2$  ring spectrum, independent of that of Basterra and Mandell [8]; this argument appears in detail in the first author's 2012 PhD thesis [13].

Our techniques extend to give information for  $E_n$  ring maps for  $n > 2$  as well, especially the case  $n = 4$ , but the picture is more complicated. For example, we prove the following result in Section 7.

**Theorem 1.4** *There exists a map of ring spectra  $MU \rightarrow MU$  that does not lift to a map of  $E_4$  ring spectra.*

Much of the work in this paper generalizes to study  $E_n$  ring maps out of any “ $E_n$  Thom spectrum”, an  $E_n$  ring spectrum that arises as the Thom spectrum of an  $E_n$  stable spherical (quasi)fibration  $X \rightarrow BF$ ; see especially Theorems 4.2 and 5.1. Partly for our work in the general case of  $E_n$  ring Thom spectra and partly to carry forward theorems from the 1970s and 1980s into the context of the modern categories of spectra (including symmetric spectra and orthogonal spectra), we prove some general results on model structures on categories of algebras over operads in various categories of spectra; see Sections 3 and 8 for specific statements.

## Conventions

Throughout this paper, the word “space” means compactly generated weak Hausdorff space and  $\mathcal{Top}$  denotes the category of such spaces. We work in one of the modern categories of spectra, either symmetric spectra (of spaces), orthogonal spectra, or EKMM  $S$ -modules and we have written the details so that they work in any one of these categories when no one is specified. The word “spectra” means (objects in) any one of these categories, and we write “LMS spectra” for (objects in) the category called spectra in the book by Lewis, May, Steinberger and McClure [21].

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## 2 Outline of the argument

To explain our approach to studying  $E_n$  ring maps, it is easier if we assume that the target ring spectrum  $R$  is at least  $E_{n+1}$ , and we begin in Section 4 with this hypothesis. For the source ring spectrum, the body of the paper studies an arbitrary  $E_n$  ring Thom spectrum (Definition 4.1), but for definiteness in this outline, we concentrate on the case of  $\text{MU}$ , which is the primary case of interest. In this case, we can take advantage of the fact that the Thom diagonal

$$\tau: \text{MU} \longrightarrow \text{MU} \wedge \text{BU}_+ = \text{MU} \wedge \Sigma_+^\infty \text{BU}$$

is an  $E_\infty$  ring map [21, p. 447] and that for any  $E_{n+1}$  ring spectrum  $R$ , the multiplication

$$\mu: R \wedge R \longrightarrow R$$

is an  $E_n$  ring map [12, 1.6]. Then for a fixed  $E_n$  ring map  $\sigma: \text{MU} \rightarrow R$  and a variable  $E_n$  ring map  $f: \Sigma_+^\infty \text{BU} \rightarrow R$ , the composite

$$\text{MU} \xrightarrow{\tau} \text{MU} \wedge \Sigma_+^\infty \text{BU} \xrightarrow{\sigma \wedge f} R \wedge R \xrightarrow{\mu} R$$

is an  $E_n$  ring map. This induces a map from the space of  $E_n$  ring maps  $\Sigma_+^\infty \text{BU} \rightarrow R$  to the space of  $E_n$  ring maps  $\text{MU} \rightarrow R$ ,

$$\mathcal{E}_n \text{Ring}(\Sigma_+^\infty \text{BU}, R) \longrightarrow \mathcal{E}_n \text{Ring}(\text{MU}, R).$$

The usual algebraic argument then shows that this map is an equivalence.

**Theorem 2.1** *Let  $R$  be an  $E_{n+1}$  ring spectrum. Then the space  $\mathcal{E}_n \text{Ring}(\text{MU}, R)$  of  $E_n$  ring maps from  $\text{MU}$  to  $R$  is either empty or weakly equivalent to the space  $\mathcal{E}_n \text{Ring}(\Sigma_+^\infty \text{BU}, R)$  of  $E_n$  ring maps from  $\Sigma_+^\infty \text{BU}$  to  $R$ .*

To be precise, the spaces of  $E_n$  ring maps  $\mathcal{E}_n \text{Ring}(-, R)$  in the previous theorem are the derived mapping spaces, ie the homotopy types of the mapping spaces in the homotopy categories, represented for example by the point set mapping space between a cofibrant replacement (in the domain) and a fibrant replacement (in the codomain) in a simplicial or topological model category of  $E_n$  ring spectra.

The analogous theorem also holds for  $\text{MSO}$  and  $\text{BSO}$ , and in general for any  $E_n$  Thom spectrum (see Theorem 4.2). In the general case considered in Section 4, the algebraic argument reduces to the existence of model structures with the expected properties on categories of algebras over operads; see Theorem 4.8 and Corollary 4.9 for details.

We can identify the space of  $E_n$  ring maps  $\Sigma_+^\infty \text{BU} \rightarrow R$  in more familiar terms. Let  $\text{SL}_1 R$  denote the component of the (right derived) zeroth space of  $R$  corresponding to the multiplicative identity element 1 in  $\pi_0 R$ . Since  $R$  is (in particular) an  $E_n$  ring spectrum, the  $E_n$  multiplication on  $R$  induces an  $E_n$  structure on  $\text{SL}_1 R$ . The space of  $E_n$  ring maps  $\Sigma_+^\infty \text{BU} \rightarrow R$  can be identified as the space of  $E_n$  maps from  $\text{BU}$  to  $\text{SL}_1 R$  [26, IV.1.8], which is just the space  $\mathcal{T}op_*(B^n \text{BU}, B^n \text{SL}_1 R)$  of based maps of topological spaces  $B^n \text{BU} \rightarrow B^n \text{SL}_1 R$  (where  $B^n$  denotes an  $n$ -fold delooping functor).

In the case when  $R$  is an  $E_\infty$  ring spectrum,  $\text{SL}_1 R$  and  $B^n \text{SL}_1 R$  are infinite loop spaces, and the Atiyah–Hirzebruch spectral sequence

$$E_2^{s,t} = H^s(B^n \text{BU}, \pi_t B^n \text{SL}_1 R) = H^s(B^n \text{BU}, \pi_{t-n}^+ R) \\ \implies \pi_{t-s} \mathcal{T}op_*(B^n \text{BU}, B^n \text{SL}_1 R) = \pi_{t-s} \mathcal{E}_n \mathcal{R}ing(\Sigma_+^\infty \text{BU}, R)$$

calculates the homotopy groups of  $\mathcal{E}_n \mathcal{R}ing(\Sigma_+^\infty \text{BU}, R)$ . Note that  $\pi_t B^n \text{SL}_1 R = \pi_{t-n} \text{SL}_1 R$  is  $\pi_{t-n} R$  for  $t - n > 0$  and 0 for  $t - n \leq 0$ , and we use the notation  $\pi_t^+ R := \pi_t \text{SL}_1 R$  for these groups.

When  $R$  is just an  $E_{n+1}$  ring spectrum as in Theorem 2.1,  $B^n \text{SL}_1 R$  is a loop space, and the Postnikov tower of  $B^n \text{SL}_1 R$  is a sequence of principal fibrations of loop spaces of the form

$$(B^n \text{SL}_1 R)_t \longrightarrow (B^n \text{SL}_1 R)_{t-1} \longrightarrow K(\pi_{t-n}^+ R, t + 1).$$

Mapping  $B^n \text{BU}$  into the tower  $B^n \text{SL}_1 R_t$  in the category of based spaces, we get a tower of principal fibrations of loop spaces

$$\mathcal{T}op_*(B^n \text{BU}, (B^n \text{SL}_1 R)_t) \longrightarrow \mathcal{T}op_*(B^n \text{BU}, (B^n \text{SL}_1 R)_{t-1}) \\ \longrightarrow \mathcal{T}op_*(B^n \text{BU}, K(\pi_{t-n}^+ R, t + 1))$$

with homotopy limit weakly equivalent to  $\mathcal{T}op_*(B^n \text{BU}, B^n \text{SL}_1 R)$ . This then again gives a spectral sequence for calculating the homotopy groups of  $\mathcal{E}_n \mathcal{R}ing(\Sigma_+^\infty \text{BU}, R)$ , whose  $E_2$  term is again

$$E_2^{s,t} = H^s(B^n \text{BU}, \pi_{t-n}^+ R)$$

and which generalizes the Atiyah–Hirzebruch spectral sequence displayed above.

In the case when  $R$  is just an  $E_n$  ring spectrum, Theorem 2.1 does not apply; nevertheless, we can identify  $\mathcal{E}_n \mathcal{R}ing(\text{MU}, R)$  as the homotopy limit of a tower of principal fibrations, using the Postnikov tower of  $R$  in the category of  $E_n$  ring spectra (after replacing  $R$  with its connective cover, if necessary). Basterra and the second author

studied this tower in [7; 8], and using the work there, we prove the following theorem in Section 5.

**Theorem 2.2** *Let  $R$  be an  $E_n$  ring spectrum, and if  $n = 1$ , assume that  $\pi_0 R$  is commutative. Then the space of  $E_n$  ring maps from  $\text{MU}$  to  $R$  is weakly equivalent to the homotopy limit of a tower of principal fibrations of the form*

$$\mathcal{E}_n\text{Ring}(\text{MU}, R_q) \longrightarrow \mathcal{E}_n\text{Ring}(\text{MU}, R_{q-1}) \longrightarrow \mathcal{T}op_*(B^n\text{BU}, K(\pi_q R, q + n + 1))$$

for  $q \geq 1$ .

We can think of the previous theorem as giving an “obstructed spectral sequence” (cf [11]) of the form

$$E_2^{s,t} = H^s(B^n\text{BU}, \pi_{t-n}^+ R) \implies \pi_{t-s} \mathcal{E}_n\text{Ring}(\text{MU}, R)$$

(for  $t = q + n$ ). In particular, it then gives an approach to calculating  $\pi_0 \mathcal{E}_n\text{Ring}(\text{MU}, R)$ , which we apply (in the generalized form of Theorem 5.1) in Section 6 to prove Theorems 1.1 and 1.2.

The discussion above shows why the cases of  $E_2$  and  $E_4$  maps are the most tractable: for  $n = 2$  and  $n = 4$ ,  $H^*(B^n\text{BU})$  consists of finitely generated free abelian groups and is concentrated in even degrees. In particular, when  $\pi_* R$  is concentrated in even degrees, the obstructions to lifting maps up the Postnikov tower vanish and we can compute  $\pi_0 \mathcal{E}_n\text{Ring}(\text{MU}, R)$  as

$$H^{n+2}(B^n\text{BU}, \pi_2 R) \times H^{n+4}(B^n\text{BU}, \pi_4 R) \times \cdots$$

(as a set;  $\pi_0 \mathcal{E}_n\text{Ring}(\text{MU}, R)$  has no natural structure). We cannot expect this to hold in general if  $n \neq 2, 4$ .

### 3 The homotopy theory of $E_n$ ring spectra

We use this section to review the background on the homotopy theory of  $E_n$  ring spectra that we need for later sections. Most of the review consists of recording facts about model categories of operadic algebras that are well-known to experts but scattered through the literature and difficult to find in the precise form we need. We claim no originality for theorems in this section.

For much of the work in this paper, we take “ $E_n$  ring spectrum” to mean an algebra over the Boardman–Vogt little  $n$ -cubes operad  $\mathcal{C}_n$  [10, 2.49; 25, Section 4] of spaces; however, in part of Section 4, we work instead with an  $E_{n+1}$  ring spectrum that is

an algebra over the tensor product operad  $C_n \otimes Ass$ . The first background result we need is therefore the well-known fact that we can model the homotopy category of  $E_n$  ring spectra using any  $E_n$  operad, ie any  $\Sigma$ -free operad (with paracompact Hausdorff underlying spaces) that is weakly equivalent through operad maps to  $C_n$ . The following two theorems proved in Section 8 establish this fact.

**Theorem 3.1** *Let  $\mathcal{M}$  denote either the model category  $\Sigma_*\mathcal{S}$  of symmetric spectra or the model category  $\mathcal{IS}$  of orthogonal spectra with their positive stable model structures [24, Section 14] or the model category  $\mathcal{M}_S$  of EKMM  $S$ -modules with its standard model structure [16, VII, Section 4]. Let  $\mathcal{O}$  be an operad in spaces. Then the category  $\mathcal{M}[\mathcal{O}]$  of  $\mathcal{O}$ -algebras in  $\mathcal{M}$  is a topological closed model category with fibrations and weak equivalences created in  $\mathcal{M}$ .*

**Theorem 3.2** *For  $\mathcal{M}$  as in Theorem 3.1, and  $\phi: \mathcal{O} \rightarrow \mathcal{O}'$  a map of operads, the pushforward (left Kan extension) and pullback functors*

$$L_\phi: \mathcal{M}[\mathcal{O}] \rightleftarrows \mathcal{M}[\mathcal{O}'] : R_\phi$$

*form a Quillen adjunction, which is a Quillen equivalence if (and only if) each  $\phi(n): \mathcal{O}(n) \rightarrow \mathcal{O}'(n)$  is a (non-equivariant) stable equivalence.*

In the course of proving the previous theorems, we develop the tools needed to deduce the following useful technical result. Note that the initial  $\mathcal{O}$ -algebra is  $\mathcal{O}(0)_+ \wedge S$ .

**Theorem 3.3** *Let  $\mathcal{M}$  and  $\mathcal{O}$  be as in Theorem 3.1, and assume that each  $\mathcal{O}(n)$  is a retract of a free  $\Sigma_n$ -cell complex. If  $A$  is a cofibrant  $\mathcal{O}$ -algebra, then  $\mathcal{O}(0)_+ \wedge S \rightarrow A$  is a cofibration in  $\mathcal{M}$ .*

We need two more results geared towards using the Thom diagonal in the context of  $E_n$  ring spectra. For a fibration of spaces  $f: B \rightarrow BF$  (where  $BF$  denotes the classifying space for stable spherical fibrations), Lewis constructed the Thom spectrum  $Mf$  as an LMS spectrum [21, IX.3.2] and showed that when  $f$  is a map of  $\mathcal{O}$ -spaces (for an operad  $\mathcal{O}$  with a map to the linear isometries operad  $\mathcal{L}$ ),  $Mf$  is naturally an  $\mathcal{O}$ -algebra in the category of LMS spectra [21, IX.7.1]. It follows that the Thom diagonal  $Mf \rightarrow Mf \wedge B$  is a map of  $\mathcal{O}$ -algebras. Instead of re-proving this in the context of a modern category of spectra, we just transport this construction and this map using the following well-known comparison theorems across the different categories of spectra.

**Theorem 3.4** *Let  $\mathcal{O}$  be an operad of spaces. In the Quillen equivalences*

$$\mathbb{P}: \Sigma_*\mathcal{S} \rightleftarrows \mathcal{S}\mathcal{S} : \mathbb{U}, \quad \mathbb{N}: \mathcal{S}\mathcal{S} \rightleftarrows \mathcal{M}_S : \mathbb{N}^\#$$

*of [24, page 442] and [23, I.1.1], all four functors preserve  $\mathcal{O}$ -algebras and induce Quillen equivalences*

$$\mathbb{P}: \Sigma_*\mathcal{S}[\mathcal{O}] \rightleftarrows \mathcal{S}\mathcal{S}[\mathcal{O}] : \mathbb{U}, \quad \mathbb{N}: \mathcal{S}\mathcal{S}[\mathcal{O}] \rightleftarrows \mathcal{M}_S[\mathcal{O}] : \mathbb{N}^\#$$

*on the categories of  $\mathcal{O}$ -algebras.*

**Theorem 3.5** *Let  $\mathcal{O}$  be an operad of spaces. Then the category  $\mathcal{S}_{LMS}[\mathcal{O} \times \mathcal{L}]$  of  $(\mathcal{O} \times \mathcal{L})$ -algebras in LMS spectra is a topological closed model category with fibrations and weak equivalences created in LMS spectra. Moreover:*

- (i)  $\mathcal{S}_{LMS}[\mathcal{O} \times \mathcal{L}]$  is equivalent to the category  $\mathcal{S}_{LMS}[\mathbb{L}][\mathcal{O}]$  of  $\mathcal{O}$ -algebras in EKMM  $\mathbb{L}$ -spectra [16, Chapter I].
- (ii) The forgetful functor from EKMM  $S$ -modules to EKMM  $\mathbb{L}$ -spectra and its right adjoint  $S \wedge_{\mathcal{L}} (-)$  both preserve  $\mathcal{O}$ -algebras; the unit and counit of this adjunction are both natural weak equivalences.
- (iii) The right adjoint  $F_{\mathcal{L}}(S, -): \mathcal{M}_S \rightarrow \mathcal{S}_{LMS}[\mathbb{L}]$  of  $S \wedge_{\mathcal{L}} (-)$  also preserves  $\mathcal{O}$ -algebras; the unit and counit of this adjunction are both natural weak equivalences.
- (iv) The adjunction

$$S \wedge_{\mathcal{L}} (-): \mathcal{S}_{LMS}[\mathbb{L}][\mathcal{O}] \rightleftarrows \mathcal{M}_S[\mathcal{O}] : F_{\mathcal{L}}(S, -)$$

*is a Quillen equivalence.*

The proof of the model structure in Theorem 3.5 is given in Section 8 with the proof of the model structures in Theorem 3.1. The proof of the remaining statements in Theorems 3.4 and 3.5 are now easy from the other theorems in the section.

**Proof of Theorem 3.4** All four functors are lax symmetric monoidal and therefore preserve operadic algebra structures. Since fibrations on the algebra categories are created in the underlying categories of spectra (ie in symmetric spectra, orthogonal spectra, or  $S$ -modules, as the case may be), the adjunctions on algebra categories are automatically Quillen adjunctions. To prove they are Quillen equivalences, by [24, A.2(ii)], it suffices to show that the derived functors are equivalences of homotopy categories. Applying Theorem 3.2, it suffices to consider the case when  $\mathcal{O}$  satisfies the hypothesis of Theorem 3.3, ie each  $\mathcal{O}(n)$  is a  $\Sigma_n$ -cell complex. In this case, every cofibrant  $\mathcal{O}$ -algebra is cofibrant in the underlying category of spectra lying under



$\mathcal{O}(0)_+ \wedge S$ . Since the unit of the adjunction is a weak equivalence for  $\mathcal{O}_+(0) \wedge S$ , the unit of the adjunction is a weak equivalence for the codomain of any cofibration with domain  $\mathcal{O}(0)_+ \wedge S$  (see for example the proof of [24, 10.3] and the proof of [23, I.3.5]). In particular the unit of the adjunction is a weak equivalence for cofibrant  $\mathcal{O}$ -algebras. It then follows from [24, A.2(iii)] that the Quillen adjunctions on algebra categories are Quillen equivalences.  $\square$

**Proof of Theorem 3.5.(i–iv)** As in [16, I§4], let  $\mathbb{L}$  denote the monad  $\mathcal{L}(1) \times (-)$  on the category of LMS spectra. If we write  $\mathcal{O}$  for the free  $\mathcal{O}$ -algebra functor on EKMM  $\mathbb{L}$ -spectra, then

$$\begin{aligned} \mathcal{O}\mathbb{L}X &= \bigvee_{n \geq 0} \mathcal{O}(n)_+ \wedge_{\Sigma_n} (\mathbb{L}X)^{\wedge \mathcal{L}n} \\ &= \bigvee_{n \geq 0} \mathcal{O}(n)_+ \wedge_{\Sigma_n} (\mathcal{L}(n) \times X^{\bar{\wedge}n}) \\ &= \bigvee_{n \geq 0} ((\mathcal{O}(n) \times \mathcal{L}(n)) \times X^{\bar{\wedge}n}) / \Sigma_n \end{aligned}$$

is the free  $(\mathcal{O} \times \mathcal{L})$ -algebra on  $X$ ; an easy composite of monads argument [16, II.6.1] then shows that the category of  $\mathcal{O}$ -algebras in  $\mathcal{S}_{\text{LMS}}[\mathbb{L}]$  is equivalent to the category of  $(\mathcal{O} \times \mathcal{L})$ -algebras in  $\mathcal{S}_{\text{LMS}}$ , which is (i). For (ii) and (iii), the adjunctions are [16, II.1.3]. The fact that all four functors are lax symmetric monoidal [16, II.1.1] shows that they preserve  $\mathcal{O}$ -algebra structures, and the remaining facts are [16, I.8.5(iii)] and [16, I.8.7]. For (iv), the adjunction is a Quillen adjunction because the adjunction on the underlying categories  $\mathcal{M}_S$  and  $\mathcal{S}_{\text{LMS}}[\mathbb{L}]$  is a Quillen adjunction [16, VII.4.6], and the adjunction is a Quillen equivalence since both the unit and counit are weak equivalences on all objects.  $\square$

### 4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1, which relates the space of  $E_n$  ring maps out of MU to the space of  $E_n$  ring maps out of  $\Sigma_+^\infty \text{BU}$ . We view this theorem as the  $E_n$  ring version of the Thom isomorphism: the usual Thom isomorphism relates maps of spectra out of MU to maps of spectra out of  $\Sigma_+^\infty \text{BU}$ . Indeed, our proof of Theorem 2.1 generalizes to any  $E_n$  ring Thom spectrum; see Theorem 4.2 below. Since the theorem concerns derived mapping spaces, the proof requires a certain amount of technical work in the model category of  $E_n$  ring spectra; however, for a statement about maps in the homotopy category ( $\pi_0$  of the derived mapping space) a simpler argument in the homotopy category suffices. We give the homotopy category argument first as an explanation and guide to the slightly more complicated model category argument.

We work in the context of an  $E_n$  ring Thom spectrum, defined as follows. Let  $G$  denote either  $O = \bigcup O(n)$ , the infinite orthogonal group, or  $F = \bigcup F(n)$ , the grouplike monoid of stable self-homotopy equivalences of spheres. (The monoid  $F(n)$  is the space of self-maps of  $S^n$  that are homotopy equivalences and that fix 0 and  $\infty$ .) Associated to any “good” map  $f: X \rightarrow BG$  is a Thom spectrum  $M = Mf$  [21, IX.3.2]. Here “good” is the technical condition of [21, page 423]: it is the empty condition when  $G = O$  and when  $G = F$  it is satisfied in particular when  $f$  is a Hurewicz fibration, which we can always assume without loss of generality [21, pages 411–412, 443]. The classifying space  $BG$  is an  $E_\infty$  space for the linear isometries operad  $\mathcal{L}$  [9]. When  $X$  is an  $\mathcal{O} \times \mathcal{L}$ -space for some operad  $\mathcal{O}$  and  $f$  is an  $\mathcal{O} \times \mathcal{L}$ -space map, the Thom spectrum  $M$  inherits the structure of an  $\mathcal{O} \times \mathcal{L}$ -spectrum. In the particular case when  $\mathcal{O}$  is the little  $n$ -cubes operad  $\mathcal{C}_n$ , we call this an  $E_n$  ring Thom spectrum.

**Definition 4.1** Let  $\mathcal{C}_n$  denote the Boardman–Vogt little  $n$ -cubes operad [10, 2.49; 25, Section 4]. An  $E_n$  ring Thom spectrum is the Thom spectrum of a  $\mathcal{C}_n \times \mathcal{L}$ -space map  $X \rightarrow BO$  or a “good”  $\mathcal{C}_n \times \mathcal{L}$ -space map  $X \rightarrow BF$ , viewed as an  $E_n$  ring spectrum.

An  $E_n$  ring Thom spectrum is then canonically a  $\mathcal{C}_n$ -algebra in EKMM  $\mathbb{L}$ -spectra and (by applying  $S \wedge_{\mathcal{L}} (-)$ ) canonically weakly equivalent to a  $\mathcal{C}_n$ -algebra in EKMM  $S$ -modules (Theorem 3.5), but up to weak equivalence, we can regard it as a  $\mathcal{C}_n$ -algebra in any of the modern categories of spectra (Theorem 3.4). We fix one of the modern categories of spectra, denoting it  $\mathcal{M}$  (calling its objects “spectra”), and write  $\mathcal{M}[\mathcal{C}_n]$  for  $\mathcal{C}_n$ -algebras in this category (calling its objects “ $E_n$  ring spectra”). As a topological model category (Theorem 3.1), the category of  $E_n$  ring spectra has a nice theory of derived mapping spaces, constructed for example as the mapping space out of a cofibrant replacement and into a fibrant replacement. We use  $\mathcal{E}_n\text{Ring}$  to denote the derived mapping spaces; the homotopy category of  $E_n$  ring spectra is then  $\pi_0 \mathcal{E}_n\text{Ring}$ . The main theorem of this section is the following generalization of Theorem 2.1.

**Theorem 4.2** Let  $M$  be the  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$  for some connected  $E_n$  space  $X$ , and let  $R$  be an  $E_{n+1}$  ring spectrum. Then the derived space of maps of  $E_n$  ring spectra from  $M$  to  $R$  is either empty or weakly equivalent to the derived space of maps of  $E_n$  ring spectra from  $\Sigma_+^\infty X$  to  $R$ ,

$$\mathcal{E}_n\text{Ring}(M, R) \simeq \mathcal{E}_n\text{Ring}(\Sigma_+^\infty X, R).$$

At its core, the argument is a straightforward algebraic argument, which gets somewhat obscured by technical details. To outline and explain the argument, we first prove the following easier theorem.

**Theorem 4.3** *Let  $M$  be the  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$  for some connected  $E_n$  space  $X$ , and let  $R$  be an  $E_{n+1}$  ring spectrum. If there exists a map  $M \rightarrow R$  in the homotopy category of  $E_n$  ring spectra, then the set of maps  $M \rightarrow R$  in the homotopy category of  $E_n$  ring spectra is in one-to-one correspondence with the set of maps  $\Sigma_+^\infty X$  to  $R$ , in the homotopy category of  $E_n$  ring spectra,*

$$\pi_0 \mathcal{E}_n \text{Ring}(M, R) \cong \pi_0 \mathcal{E}_n \text{Ring}(\Sigma_+^\infty X, R).$$

The proof of Theorem 4.3 is little more than an application of the Thom isomorphism theorem and an exercise with monoids and modules inside the homotopy category of  $E_n$  ring spectra. We note that if  $A$  and  $B$  are  $E_n$  ring spectra, then  $A \wedge B$  (point-set smash product in  $\mathcal{M}$ ) is canonically an  $E_n$  ring spectrum with action of  $\mathcal{C}_n$  induced by using the diagonal map  $\mathcal{C}_n \rightarrow \mathcal{C}_n \times \mathcal{C}_n$  and the actions on  $A$  and  $B$ . Since  $\mathcal{C}_n(0) = *$  and each space  $\mathcal{C}_n(m)$  is a free  $\Sigma_m$ -cell complex, cofibrant  $E_n$  ring spectra are cofibrant objects in spectra under  $S$  (Theorem 3.3). In particular, the smash product with a cofibrant  $E_n$  ring spectrum preserves weak equivalences in  $\mathcal{M}$ , and it follows that  $\wedge$  descends to a symmetric monoidal product on the homotopy category of  $E_n$  ring spectra, compatibly with the smash product in the stable category.

Let  $R$  be an  $E_{n+1}$  ring spectrum (a  $\mathcal{C}_{n+1}$ -algebra in  $\mathcal{M}$ ). We use the map of operads  $\ell: \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$  that sends a little  $n$ -cube  $a$  to the little  $n + 1$ -cube  $a \times [0, 1]$  to regard  $R$  as an  $E_n$  ring spectrum. We also have a map of operads  $r: \mathcal{C}_1 \rightarrow \mathcal{C}_{n+1}$  sending a little 1-cube  $b$  to the little  $n + 1$ -cube  $[0, 1]^n \times b$ ; using  $r$ , for any element  $c$  of  $\mathcal{C}_1(m)$ , we then get a map

$$r(c): R^{(m)} = R \wedge \cdots \wedge R \longrightarrow R,$$

where  $R^{(m)}$  denotes the  $m^{\text{th}}$  smash power of  $R$ . Because the actions induced by  $\ell$  and  $r$  on  $R$  satisfy the interchange law [12, 1-1], the map  $r(c)$  is a map of  $E_n$  ring spectra. In particular, working in the homotopy category of  $E_n$  ring spectra and taking  $c$  to be the element  $\mu$  in  $\mathcal{C}_1(2)$  representing the standard multiplication, we see that  $R$  is a monoid for the smash product in the homotopy category of  $E_n$  ring spectra. (We will henceforth omit the  $r$  and write  $\mu: R \wedge R \rightarrow R$  for this map.) We use the following terminology for modules.

**Definition 4.4** *Let  $R$  be an  $E_{n+1}$  ring spectrum. A homotopical  $R$ -module in  $E_n$  ring spectra is a left module for  $R$  in the homotopy category of  $E_n$  ring spectra: it consists of an  $E_n$  ring spectrum  $N$  together with an action map*

$$\xi: R \wedge N \longrightarrow N$$

in the homotopy category of  $E_n$  ring spectra such that the composite map

$$S \wedge N \longrightarrow R \wedge N \longrightarrow N$$

is the canonical isomorphism and the associativity diagram

$$\begin{array}{ccc} R \wedge R \wedge N & \xrightarrow{\text{id}_R \wedge \xi} & R \wedge N \\ \mu \wedge \text{id}_N \downarrow & & \downarrow \xi \\ R \wedge N & \xrightarrow{\xi} & N \end{array}$$

commutes, where  $\mu: R \wedge R \rightarrow R$  is the multiplication discussed above. A map of homotopical  $R$ -modules in  $E_n$  ring spectra is a map in the homotopy category of  $E_n$  ring spectra  $N \rightarrow N'$  that commutes with the action maps. We use the symbol  $\text{Mod}_{\text{Ho } R}^{E_n}$  to denote the category of homotopical  $R$ -modules in  $E_n$  ring spectra.

We omit “in  $E_n$  ring spectra” from the terminology for homotopical  $R$ -modules when it is clear from context. We have the usual free/forgetful adjunction for these modules.

**Proposition 4.5** *Let  $R$  be an  $E_{n+1}$  ring spectrum. The functor  $R \wedge (-)$  from the homotopy category of  $E_n$  ring spectra to the category of homotopical  $R$ -modules in  $E_n$  ring spectra is left adjoint to the forgetful functor: maps in the homotopy category of  $E_n$  ring spectra from an  $E_n$  ring spectrum  $E$  to a homotopical  $R$ -module  $N$  are in one-to-one correspondence with maps of homotopical  $R$ -modules from  $R \wedge E$  to  $N$ ,*

$$\pi_0 \mathcal{E}_n \text{Ring}(E, N) \cong \text{Mod}_{\text{Ho } R}^{E_n}(R \wedge E, N).$$

**Proof** The correspondence is the usual one: given a map  $h: E \rightarrow N$  in the homotopy category of  $E_n$  ring spectra, the composite

$$R \wedge E \xrightarrow{\text{id}_R \wedge h} R \wedge N \xrightarrow{\xi} N$$

is a map of homotopical  $R$ -modules and given a map  $k: R \wedge E \rightarrow N$  of homotopical  $R$ -modules, the map  $E \rightarrow N$  is the composite map in the homotopy category of  $E_n$  ring spectra

$$E \cong S \wedge E \longrightarrow R \wedge E \xrightarrow{k} N.$$

An easy check shows these are inverse correspondences. □

When  $M$  is the  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$ , it follows from [21, IX.7.1] (see in particular the top of page 447 in [21]) that the Thom diagonal

$$\tau: M \longrightarrow M \wedge X_+ = M \wedge \Sigma_+^\infty X$$

lifts to a natural map in the homotopy category of  $E_n$  ring spectra. The following is then the  $E_n$  ring spectrum version of the homology Thom isomorphism. Recall that a map  $\sigma$  from a Thom spectrum  $M$  to a ring spectrum  $R$  is an *orientation* when for every point  $x$  in  $X$ , the map  $S \rightarrow R$  obtained by restricting  $\sigma$  to the Thom spectrum of  $\{x\}$  represents a unit in the ring  $\pi_0 R$ . When  $X$  is connected, a map of ring spectra  $M \rightarrow R$  is always an orientation since the restriction  $S \rightarrow R$  represents the identity element in  $\pi_0 R$ .

**Proposition 4.6** *Let  $M$  be the  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$  and let  $R$  be an  $E_{n+1}$  ring spectrum. If  $\sigma: M \rightarrow R$  is a map in the homotopy category of  $E_n$  ring spectra and also an orientation, then the map*

$$M \xrightarrow{\tau} M \wedge \Sigma_+^\infty X \xrightarrow{\sigma \wedge \text{id}_{\Sigma_+^\infty X}} R \wedge \Sigma_+^\infty X$$

*induces an isomorphism of homotopical  $R$ -modules in  $E_n$ -ring spectra*

$$R \wedge M \longrightarrow R \wedge \Sigma_+^\infty X.$$

**Proof** As the composite map  $M \rightarrow R \wedge \Sigma_+^\infty X$  is a map in the homotopy category of  $E_n$  ring spectra, we get an induced map of homotopical  $R$ -modules as displayed above by the free/forgetful adjunction (Proposition 4.5). The question of it being an isomorphism is a question in the stable category (after forgetting the  $E_n$  ring structures and just remembering the homotopical ring spectrum structure on  $R$ ), and this is just the usual homology version of the Thom isomorphism, the map  $R \wedge M \rightarrow R \wedge \Sigma_+^\infty X$ , being the geometric cap product with the orientation  $\sigma$ .  $\square$

Theorem 4.3 is now an easy consequence. Propositions 4.5 and 4.6 give us bijections

$$\begin{aligned} \pi_0 \mathcal{E}_n \text{Ring}(M, R) &\cong \text{Mod}_{\text{Ho } R}^{E_n}(R \wedge M, R) \\ &\cong \text{Mod}_{\text{Ho } R}^{E_n}(R \wedge \Sigma_+^\infty X, R) \cong \mathcal{E}_n \text{Ring}(\Sigma_+^\infty X, R), \end{aligned}$$

under the hypothesis that a map  $\sigma: M \rightarrow R$  exists in the homotopy category of  $E_n$  ring spectra. This completes the proof of Theorem 4.3.

We can prove Theorem 4.2 by the same outline, but using stricter algebraic structures. Whereas an  $E_{n+1}$  ring spectrum is a monoid for the smash product in the homotopy category of  $E_n$  ring spectra, it is only an  $A_\infty$  monoid for the point-set smash product of  $E_n$  ring spectra. A monoid for the point-set smash product of  $E_n$  ring spectra is precisely an algebra over the operad  $\mathcal{C}_n \otimes \text{Ass}$  [12, Section 1.6], where  $\text{Ass}$  is the operad defining associative monoids. Theorem C of [12] shows that  $\mathcal{C}_n \otimes \text{Ass}$  is an  $E_{n+1}$  operad, and so given an  $E_{n+1}$  ring spectrum  $R$ , we can find an equivalent

$C_n \otimes \text{Ass}$ -algebra  $R'$ , which we can regard as a monoid for the point-set smash product of  $E_n$  ring spectra. We then have the following point-set category of point-set modules.

**Definition 4.7** Let  $R$  be a monoid for the point-set smash product of  $E_n$  ring spectra, or equivalently, an algebra over the operad  $C'_{n+1} := C_n \otimes \text{Ass}$ . An  $R$ -module in the category of  $E_n$  ring spectra consists of an  $E_n$  ring spectrum  $N$  together with an action map

$$\xi: R \wedge N \longrightarrow N$$

in the point-set category of  $E_n$ -ring spectra  $\mathcal{M}[C_n]$  such that the composite map

$$S \wedge N \longrightarrow R \wedge N \longrightarrow N$$

is the canonical isomorphism and the associativity diagram

$$\begin{array}{ccc} R \wedge R \wedge N & \xrightarrow{\text{id}_R \wedge \xi} & R \wedge N \\ \mu \wedge \text{id}_N \downarrow & & \downarrow \xi \\ R \wedge N & \xrightarrow{\xi} & N \end{array}$$

commutes (in  $\mathcal{M}[C_n]$ ). A map of  $R$ -modules in  $E_n$  ring spectra is a map  $N \rightarrow N'$  in  $\mathcal{M}[C_n]$  that commutes with the action maps. We denote the category of  $R$ -modules in  $E_n$  ring spectra as  $\text{Mod}_R^{E_n}$ .

The following theorem does the technical work in extending the outline above for the proof of Theorem 4.2.

**Theorem 4.8** Let  $R$  be a  $C'_{n+1}$ -algebra. Then the category  $\text{Mod}_R^{E_n}$  of  $R$ -modules in  $E_n$  ring spectra is a topological closed model category with the fibrations and weak equivalences created in  $\mathcal{M}[C_n]$ .

**Proof** The topological model structure is a consequence of Theorem 8.1 below, which generalizes Theorem 3.1 to operads in  $\mathcal{M}$ . Starting from  $\mathcal{M}$ , the free functor from  $\mathcal{M}$  to  $\text{Mod}_R^{E_n}$  is

$$R \wedge C_n X = R \wedge \left( \bigvee_{m \geq 0} C_n(m)_+ \wedge_{\Sigma_m} X^{(m)} \right).$$

This is the monad associated to the operad  $\mathcal{R}$  in  $\mathcal{M}$  defined by  $\mathcal{R}(m) = R \wedge C_n(m)_+$ , with identity

$$S \cong S \wedge \{*\}_+ \longrightarrow R \wedge C_n(1)_+,$$

equivariance from the equivariance of  $\mathcal{C}_n(m)$ , and multiplication

$$R \wedge \mathcal{C}_n(m)_+ \wedge ((R \wedge \mathcal{C}_n(j_1)_+) \wedge \cdots \wedge (R \wedge \mathcal{C}_n(j_m)_+)) \longrightarrow R \wedge \mathcal{C}_n(j)$$

induced by the operadic multiplication on  $\mathcal{C}_n$ , the  $\mathcal{C}_n$ -action on  $R$ ,

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} R^{(m)} \longrightarrow R,$$

and the multiplication  $\mu: R \wedge R \rightarrow R$  from the monoid structure on  $R$ . It follows that  $\text{Mod}_R^{E_n}$  is isomorphic to the category of  $\mathcal{R}$ -algebras, hence admits the topological model structure by Theorem 8.1.  $\square$

**Corollary 4.9** *The free functor  $R \wedge (-): \mathcal{M}[\mathcal{C}_n] \rightarrow \text{Mod}_R^{E_n}$  and forgetful functor  $\text{Mod}_R^{E_n} \rightarrow \mathcal{M}[\mathcal{C}_n]$  form a Quillen adjunction.*

As we have already noted, the smash product with a cofibrant  $E_n$  ring spectrum preserves all weak equivalences in  $\mathcal{M}$ ; it follows that the derived functor of the free functor  $R \wedge (-)$  is the derived smash product with  $R$  after forgetting down to the homotopy category of  $E_n$  ring spectra or all the way down to the stable category. Combining the previous corollary with Proposition 4.6, we then get the following corollary.

**Corollary 4.10** *Let  $M$  be the  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$  and let  $R$  be a  $\mathcal{C}'_{n+1}$ -algebra. If  $\sigma: M \rightarrow R$  is a map in the homotopy category of  $E_n$  ring spectra and also an orientation, then the map*

$$M \xrightarrow{\tau} M \wedge \Sigma_+^\infty X \xrightarrow{\sigma \wedge \text{id}_{\Sigma_+^\infty X}} R \wedge \Sigma_+^\infty X$$

*in the homotopy category of  $E_n$  ring spectra induces an isomorphism in the homotopy category of  $R$ -modules in  $E_n$ -ring spectra*

$$R \wedge M \longrightarrow R \wedge \Sigma_+^\infty X.$$

Corollary 4.10 is what we need to prove Theorem 4.2.

**Proof of Theorem 4.2** Let  $R$  be an  $E_{n+1}$  ring spectrum; we can then find an equivalent  $\mathcal{C}'_{n+1}$ -algebra  $R'$  (which is in particular weakly equivalent as an  $E_n$  ring spectrum). Without loss of generality, we can assume that  $R'$  is fibrant as a  $\mathcal{C}'_{n+1}$ -algebra and therefore also as an  $E_n$  ring spectrum. We choose cofibrant approximations  $M' \rightarrow M$  and  $A \rightarrow \Sigma_+^\infty X$ . Suppose there exists a map  $\sigma: M \rightarrow R \simeq R'$  in the homotopy of  $E_n$  ring spectra; then since  $X$  is connected,  $\sigma$  is an orientation and Corollaries 4.9 and 4.10 give us weak equivalences of mapping spaces

$$\mathcal{M}[\mathcal{C}_n](M', R') \cong \text{Mod}_R^{E_n}(R' \wedge M', R') \simeq \text{Mod}_R^{E_n}(R' \wedge A, R') \cong \mathcal{M}[\mathcal{C}_n](A, R').$$

The composite is then a weak equivalence

$$\mathcal{E}_n\mathcal{R}ing(M, R) \simeq \mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty X, R). \quad \square$$

## 5 Proof of Theorem 2.2

In this section we prove the following generalization of Theorem 2.2 from the introduction. (See Definition 4.1 for the definition of an  $E_n$  ring Thom spectrum.)

**Theorem 5.1** *Let  $M$  be an  $E_n$  ring Thom spectrum associated to an  $E_n$  space map  $f: X \rightarrow BG$  and assume that  $X$  is connected. Let  $R$  be an  $E_n$  ring spectrum, and if  $n = 1$ , assume that  $\pi_0 R$  is commutative. Then the space  $\mathcal{E}_n\mathcal{R}ing(M, R)$  of  $E_n$  ring maps from  $M$  to  $R$  is weakly equivalent to the homotopy limit of a tower of principal fibrations of the form*

$$\mathcal{E}_n\mathcal{R}ing(M, R_q) \longrightarrow \mathcal{E}_n\mathcal{R}ing(M, R_{q-1}) \longrightarrow \mathcal{T}op_*(B^n X, K(\pi_q R, q + n + 1))$$

for  $q \geq 1$ .

We fix  $X$ ,  $M$ , and  $R$  as in the theorem, and we assume without loss of generality that  $R$  is fibrant. Choose a cofibrant approximation  $M' \rightarrow M$  in the category of  $E_n$  ring spectra. Let  $c: \bar{R} \rightarrow R$  be a connective cover, ie  $\bar{R}$  is connective ( $\pi_q \bar{R} = 0$  for  $q < 0$ ) and  $c$  induces an isomorphism on non-negative homotopy groups. (The connective cover can be constructed by applying the small objects argument as if to construct a cofibrant approximation but only using the non-negative dimensional cells; alternatively it can be constructed using multiplicative infinite loop space theory applied to the zeroth space of  $R$  [27, §4].) We assume without loss of generality that  $\bar{R}$  is fibrant and also cofibrant in the category  $\mathcal{M}[\mathcal{C}_n]$  of  $E_n$  ring spectra. Then the derived mapping spaces  $\mathcal{E}_n\mathcal{R}ing(M, R)$  and  $\mathcal{E}_n\mathcal{R}ing(M, \bar{R})$  may be constructed as the point set mapping spaces  $\mathcal{M}[\mathcal{C}_n](M', R)$  and  $\mathcal{M}[\mathcal{C}_n](M', \bar{R})$ , respectively. The following observation reduces to the connective case.

**Proposition 5.2** *The map  $c: \bar{R} \rightarrow R$  induces a weak equivalence  $\mathcal{E}_n\mathcal{R}ing(M, \bar{R}) \rightarrow \mathcal{E}_n\mathcal{R}ing(M, R)$ .*

**Proof** This can be deduced from the results in [27, Section 4]. A more modern approach is to observe that the cofibrant approximation  $M' \rightarrow M$  can be built starting from  $S$  entirely using “positive-dimensional cells”, ie cells of the form

$$C_n S_c^q \longrightarrow C_n C S^q \quad \text{or} \quad C_n F_m S_+^{m+q} \longrightarrow C_n F_m D_+^{m+q+1}$$

(the former when  $\mathcal{M}$  is EKMM  $S$ -modules, the latter when  $\mathcal{M}$  is symmetric or orthogonal spectra) for  $q \geq 0$ , where  $C_n$  denotes the free  $\mathcal{C}_n$  algebra functor. □



Let  $H = H\pi_0 R$  be a fibrant model of the Eilenberg–Mac Lane ring spectrum; the hypothesis of Theorem 5.1 allows us to choose the model  $H$  with the structure of an  $E_\infty$  ring spectrum (or commutative  $S$ –algebra). The following is an easy induction argument on the cell structure of a cofibrant approximation of the domain.

**Proposition 5.3** *If  $E$  is any connective  $E_n$  ring spectrum, then the mapping space  $\mathcal{E}_n\text{Ring}(E, H)$  is homotopy discrete with  $\pi_0$  the set of ring maps from  $\pi_0 E$  to  $\pi_0 H$ .*

The hypothesis that  $X$  is connected implies that  $\pi_0 M$  is either  $\mathbb{Z}$  or  $\mathbb{Z}/2$ , and so it follows that  $\mathcal{E}_n\text{Ring}(M, H)$  is either empty or weakly contractible. In the case when  $\mathcal{E}_n\text{Ring}(M, H)$  is empty, so is  $\mathcal{E}_n\text{Ring}(M, R)$  and Theorem 5.1 holds for trivial reasons. We henceforth restrict to the case when  $\mathcal{E}_n\text{Ring}(M, H)$  is weakly contractible and fix a map  $M' \rightarrow H$ . Likewise we fix a map  $\bar{R} \rightarrow H$  representing the identity map on  $\pi_0 \bar{R} = \pi_0 H$ . Writing  $\mathcal{E}_n\text{Ring}/_H$  for the derived mapping space in the category of  $E_n$  ring spectra lying over  $H$ , we then have the following result.

**Proposition 5.4** *The forgetful map*

$$\mathcal{E}_n\text{Ring}/_H(M', \bar{R}) \longrightarrow \mathcal{E}_n\text{Ring}(M', \bar{R}) \simeq \mathcal{E}_n\text{Ring}(M, R)$$

*is a weak equivalence.*

Thus, to study  $\mathcal{E}_n\text{Ring}(M, R)$ , we can study the space of maps in the category  $\mathcal{M}[\mathcal{C}_n]/H$  of  $E_n$  ring spectra lying over the  $E_\infty$  ring spectrum  $H$ . This is precisely the situation studied in [8, Section 4]. In particular, [8, Theorem 4.2] constructs a Postnikov tower for  $\bar{R}$  as a tower of principal fibrations in  $\mathcal{M}[\mathcal{C}_n]/H$ . Specifically, we start with  $\bar{R} \rightarrow R_0 \rightarrow H$  a cofibration followed by an acyclic fibration. Then for each  $q > 0$ , we can inductively construct  $R_q$  as (a cofibrant approximation of) the homotopy pullback of maps

$$R_{q-1} \xrightarrow[k_q^n]{} (H \vee \Sigma^{q+1} H\pi_q R)_f,$$

$$\begin{array}{c} H \\ \downarrow \end{array}$$

where  $(H \vee \Sigma^{q+1} H\pi_q R)_f$  denotes a fibrant approximation of the “square zero”  $E_\infty$  ring spectrum  $H \vee \Sigma^{q+1} H\pi_q R$  (meaning that the multiplication on the summand

$$\Sigma^{q+1} H\pi_q R \wedge \Sigma^{q+1} H\pi_q R$$

is the trivial map). Using the path space construction of the homotopy pullback, we can arrange that the map  $R_q \rightarrow R_{q-1}$  is a fibration. The map  $k_q^n$  is chosen so that there is

an induced map  $\bar{R} \rightarrow R_q$  that is an isomorphism on homotopy groups in dimension  $q$  and below; for formal reasons, the underlying map of spectra

$$R_{q-1} \longrightarrow H \vee \Sigma^{q+1} H\pi_q R \longrightarrow \Sigma^{q+1} H\pi_q R$$

is then the  $q^{\text{th}}$   $k$ -invariant  $k_q$  in the Postnikov tower for  $\bar{R}$ . Looking at the space of maps in  $\mathcal{M}[C_n]/H$  from  $M'$  into these squares and this tower, we get the following result.

**Theorem 5.5** *The space  $\mathcal{E}_n\mathcal{R}ing(M, R)$  of  $E_n$  ring maps from  $M$  to  $R$  is weakly equivalent to the homotopy limit of a tower of principal fibrations of the form*

$$\mathcal{E}_n\mathcal{R}ing(M, R_q) \longrightarrow \mathcal{E}_n\mathcal{R}ing(M, R_{q-1}) \longrightarrow \mathcal{E}_n\mathcal{R}ing(M, H \vee \Sigma^{q+1} H\pi_q R)$$

for  $q \geq 1$ .

Since  $H \vee H\pi_q R$  is an  $E_\infty$  ring spectrum, and we have a canonical map in the homotopy category of  $E_n$  ring spectra  $M \rightarrow H \rightarrow H \vee H\pi_q R$ , we can apply Theorem 4.2 to obtain a weak equivalence

$$\mathcal{E}_n\mathcal{R}ing(M, H \vee \Sigma^{q+1} H\pi_q R) \simeq \mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty X, H \vee \Sigma^{q+1} H\pi_q R).$$

Using primarily [7, Theorem 1.3], we prove the following theorem, which then completes the proof of Theorem 5.1.

**Theorem 5.6**  $\mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty X, H \vee \Sigma^{q+1} H\pi_q R) \simeq \mathcal{J}op_*(B^n X, K(\pi_q R, q + n + 1))$ .

**Proof** We note that  $\Sigma_+^\infty X$  comes with a canonical map to  $S$  induced by the map  $X \rightarrow *$ . From Proposition 5.3, we see that  $\mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty X, H)$  is weakly contractible and hence that the map

$$\mathcal{E}_n\mathcal{R}ing_{/H}(\Sigma_+^\infty X, H \vee \Sigma^{q+1} H\pi_q R) \longrightarrow \mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty X, H \vee \Sigma^{q+1} H\pi_q R)$$

is a weak equivalence. Pulling back along the map  $S \rightarrow H$  is the right adjoint in a Quillen adjunction between the category of  $E_n$  ring spectra lying over  $S$  and the category of  $E_n$  ring spectra lying over  $H$ , and so we get a weak equivalence

$$\mathcal{E}_n\mathcal{R}ing_{/S}(\Sigma_+^\infty X, S \vee \Sigma^{q+1} H\pi_q R) \longrightarrow \mathcal{E}_n\mathcal{R}ing_{/H}(\Sigma_+^\infty X, H \vee \Sigma^{q+1} H\pi_q R),$$

where  $S \vee \Sigma^{q+1} H\pi_q R$  has the square zero multiplication. In the notation of [7, Section 7],

$$S \vee \Sigma^{q+1} H\pi_q R = KZ(\Sigma^{q+1} H\pi_q R),$$

where  $KZ$  is the square zero multiplication functor from spectra to  $E_n$  ring spectra lying over  $S$ . The Quillen adjunctions of [7, 7.1, 7.2], then give us a weak equivalence

$$\begin{aligned} \mathcal{E}_n\text{Ring}/_S(\Sigma_+^\infty X, S \vee \Sigma^{q+1} H\pi_q R) &\simeq \mathcal{S}(\Sigma^{-n} I^{\mathbf{R}}(B^n \Sigma_+^\infty X), \Sigma^{q+1} H\pi_q R) \\ &\simeq \mathcal{S}(I^{\mathbf{R}}(B^n \Sigma_+^\infty X), \Sigma^{q+n+1} H\pi_q R), \end{aligned}$$

where  $\mathcal{S}$  denotes the derived space of maps of spectra,  $I^{\mathbf{R}}$  is the homotopy fiber of the augmentation, and  $B^n$  is the iterated bar construction for  $E_n$  ring spectra lying over  $S$  constructed in [7]. This bar construction commutes with the unbased suspension spectrum functor, so we get a weak equivalence

$$\mathcal{S}(I^{\mathbf{R}}(B^n \Sigma_+^\infty X), \Sigma^{q+n+1} H\pi_q R) \simeq \mathcal{S}(I^{\mathbf{R}} \Sigma_+^\infty B^n X, \Sigma^{q+n+1} H\pi_q R).$$

Since the augmentation  $\Sigma_+^\infty B^n X \rightarrow S$  is split by the unit  $S \rightarrow \Sigma_+^\infty B^n X$ , we can identify the homotopy fiber of the augmentation as the cofiber of the unit. This gives us a weak equivalence

$$\mathcal{S}(I^{\mathbf{R}} \Sigma_+^\infty B^n X, \Sigma^{q+n+1} H\pi_q R) \simeq \mathcal{S}(\Sigma^\infty B^n X, \Sigma^{q+n+1} H\pi_q R),$$

where  $B^n X$  has its usual basepoint. The usual suspension spectrum, zeroth space (ie underlying infinite loop space) adjunction then gives the weak equivalence

$$\begin{aligned} \mathcal{S}(\Sigma^\infty B^n X, \Sigma^{q+n+1} H\pi_q R) &\simeq \mathcal{T}op_*(B^n X, \Omega^\infty(\Sigma^{q+n+1} H\pi_q R)) \\ &\simeq \mathcal{T}op_*(B^n X, K(\pi_q R, q + n + 1)), \end{aligned}$$

completing the proof. □

## 6 Proof of Theorems 1.1 and 1.2

The entirety of this section is devoted to the proof of Theorems 1.1 and 1.2. We fix an even  $E_2$  ring spectrum  $R$  and carry over the notation  $\bar{R}$ ,  $R_q$ , and  $H$  from the last section:  $\bar{R} \rightarrow R$  is a connective cover, and

$$\bar{R} \longrightarrow \cdots \longrightarrow R_q \longrightarrow R_{q-1} \longrightarrow \cdots \longrightarrow R_0 \simeq H$$

is a Postnikov tower in the category of  $E_2$  ring spectra. In the case of MSO we assume that  $\pi_0 R$  contains  $\frac{1}{2}$ .

Our proof is an inductive argument up the Postnikov tower. Both arguments are essentially the same, so we do the case of MU in detail, with the changes necessary for MSO in Remark 6.7 below. We write  $Ho\mathcal{R}ing(\text{MU}, R_q)$  for the set of maps of ring spectra (in the stable category) from MU to  $R_q$ . The inductive hypothesis (indexed on integers  $q \geq 0$ ) is the following:

- (i) The forgetful map  $\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q) \rightarrow Ho\mathcal{R}ing(\mathbf{MU}, R_q)$  is surjective.
- (ii) For  $q > 0$ , the map from  $\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q)$  to the fiber product of the maps  $\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_{q-1}) \rightarrow Ho\mathcal{R}ing(\mathbf{MU}, R_{q-1}) \leftarrow Ho\mathcal{R}ing(\mathbf{MU}, R_q)$  is surjective.
- (iii)  $\pi_1(\mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q), f)$  is trivial for all basepoints  $f$ .

Under the hypothesis that  $R$  is even, we have

$$Ho\mathcal{R}ing(\mathbf{MU}, R) \cong \lim Ho\mathcal{R}ing(\mathbf{MU}, R_q).$$

Inductive hypothesis (iii) implies

$$\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R) \cong \lim \pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q),$$

and inductive hypotheses (i) and (ii) then imply that the map  $\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R) \rightarrow Ho\mathcal{R}ing(\mathbf{MU}, R)$  is surjective, which will complete the proof of Theorem 1.2.

In the base case  $q = 0$ ,  $R_0 \simeq H$  and both  $\pi_0 \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, H)$  and  $Ho\mathcal{R}ing(\mathbf{MU}, H)$  consist of a single point. Thus, inductive hypothesis (i) holds. Inductive hypothesis (ii) is empty in this case, and inductive hypothesis (iii) holds since  $\mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, H)$  is weakly contractible.

For  $q \geq 1$ , it suffices to consider the case when  $q$  is even since the map  $R_q \rightarrow R_{q-1}$  is a weak equivalence when  $q$  is odd. We look at the fiber sequence

$$(6.1) \quad \begin{aligned} \longrightarrow \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q) &\longrightarrow \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_{q-1}) \\ &\longrightarrow \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, H \vee \Sigma^{q+1} H\pi_q R) \end{aligned}$$

and use the identification of Theorems 5.1 and 5.6 of  $\mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, H \vee \Sigma^{q+1} H\pi_q R)$  with

$$\mathcal{T}op_*(B^2\mathbf{BU}, K(\pi_q R, q + 3)) \simeq \mathcal{T}op_*(\mathbf{BSU}, K(\pi_q R, q + 3)).$$

This then gives us a computation of the homotopy groups of the base space:

$$(6.2) \quad \begin{aligned} \pi_m \mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, H \vee \Sigma^{q+1} H\pi_q R) &\cong \pi_m \mathcal{T}op_*(\mathbf{BSU}, K(\pi_q R, q + 3)) \\ &= \tilde{H}^{q+3-m}(\mathbf{BSU}; \pi_q R). \end{aligned}$$

The integral cohomology of  $\mathbf{BSU}$  is well-known: it is a polynomial algebra on the Chern classes  $c_2, c_3, \dots$ . We see that the base space of the fibration (6.1) is therefore connected with non-zero homotopy groups only in odd degrees. The inductive hypothesis (iii) for  $q - 1$  that  $\pi_1(\mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_{q-1}), f)$  is trivial for all basepoints  $f$  now implies the inductive hypothesis (iii) for  $q$  that  $\pi_1(\mathcal{E}_2\mathcal{R}ing(\mathbf{MU}, R_q), g)$  is trivial for all basepoints  $g$ .

For the inductive steps (i) and (ii), we need to relate our fiber sequence (6.1) and the map  $\text{HoRing}(\text{MU}, R_q) \rightarrow \text{HoRing}(\text{MU}, R_{q-1})$ . For this, we use the well-known fact that a ring spectrum map  $f: \text{MU} \rightarrow R_q$  is completely determined by its restriction to  $\text{MU}(1)$  as a map in the stable category (qv [1, II.4.6; 15, 10.10]), where  $\text{MU}(1) \simeq \Sigma^{-2}\mathbb{C}P^\infty$  is the Thom spectrum associated to the inclusion of  $\text{BU}(1)$  in  $\text{BU}$  (the Thom spectrum of the 0-dimensional virtual bundle  $\gamma^1 - 1$ , where  $\gamma^1$  denotes the universal complex line bundle). Writing  $\mathcal{S}(\text{MU}(1), R_q)$  for the derived space of maps in  $\mathcal{M}$  from  $\text{MU}(1)$  to  $R_q$ , let  $\mathcal{S}(\text{MU}(1), R_q)_u$  denote the subspace of components that map to the component of the unit map  $S \rightarrow R$  in  $\mathcal{S}(S, R_q)$  (via the inclusion of  $S$  in  $\text{MU}(1)$ ). Then the map

$$\text{HoRing}(\text{MU}, R_q) \longrightarrow \pi_0 \mathcal{S}(\text{MU}(1), R_q)_u$$

is a natural bijection and we can think of  $\mathcal{S}(\text{MU}(1), R_q)_u$  as an enrichment of the set  $\text{HoRing}(\text{MU}, R_q)$  into  $\pi_0$  of a space. Indeed, the map

$$\text{HoRing}(\text{MU}, R_q) \longrightarrow \text{HoRing}(\text{MU}, R_{q-1})$$

is compatible with the fiber sequence

$$\longrightarrow \mathcal{S}(\text{MU}(1), R_q)_u \longrightarrow \mathcal{S}(\text{MU}(1), R_{q-1})_u \longrightarrow \mathcal{S}(\text{MU}(1), H \vee \Sigma^{q+1} H \pi_q R)_u$$

induced by the principal fibration constructing  $R_q$ . We then have a map of fiber sequences

$$(6.3) \quad \begin{array}{ccccc} \rightarrow \mathcal{E}_2 \text{Ring}(\text{MU}, R_q) & \rightarrow & \mathcal{E}_2 \text{Ring}(\text{MU}, R_{q-1}) & \rightarrow & \mathcal{E}_2 \text{Ring}(\text{MU}, H \vee \Sigma^{q+1} H \pi_q R) \\ & & \downarrow & & \downarrow \\ \rightarrow \mathcal{S}(\text{MU}(1), R_q)_u & \longrightarrow & \mathcal{S}(\text{MU}(1), R_{q-1})_u & \longrightarrow & \mathcal{S}(\text{MU}(1), H \vee \Sigma^{q+1} H \pi_q R)_u. \end{array}$$

We can easily calculate the homotopy groups of  $\mathcal{S}(\text{MU}(1), H \vee \Sigma^{q+1} H \pi_q R)_u$  using the Thom isomorphism:

$$(6.4) \quad \begin{aligned} \pi_m \mathcal{S}(\text{MU}(1), H \vee \Sigma^{q+1} H \pi_q R)_u &\cong \pi_m \mathcal{S}(\Sigma^\infty \text{BU}(1)_+, H \vee \Sigma^{q+1} H \pi_q R)_u \\ &\cong \tilde{H}^{q+1-m}(\text{BU}(1); \pi_q R). \end{aligned}$$

The next task is to understand the comparison map relating the homotopy groups in (6.2) and the homotopy groups in (6.4). We prove that it is the obvious one.

**Lemma 6.5** *The induced map on the homotopy groups of the base spaces in (6.3) is the map  $\tilde{H}^{q+3-m}(\text{BSU}; \pi_q R) \rightarrow \tilde{H}^{q+1-m}(\text{BU}(1); \pi_q R)$  induced by the map*

$$\Sigma^2 \text{BU}(1) \longrightarrow \Sigma^2 \text{BU} \longrightarrow B^2 \text{BU} \simeq \text{BSU},$$

where  $\text{BU}(1) \rightarrow \text{BU}$  is the inclusion and  $\Sigma^2 \text{BU} \rightarrow B^2 \text{BU}$  is the adjoint of the canonical delooping equivalence  $\text{BU} \rightarrow \Omega^2 B^2 \text{BU}$ .

**Proof** The weak equivalence

$$\mathcal{E}_2\text{Ring}(\text{MU}, H \vee \Sigma^{q+1} H\pi_q R) \simeq \mathcal{E}_2\text{Ring}(\Sigma_+^\infty \text{BU}, H \vee \Sigma^{q+1} H\pi_q R)$$

of Theorem 4.2 is induced by a map which is just the usual Thom isomorphism map on the underlying spectra, and so we have a commuting diagram

$$\begin{array}{ccc} \mathcal{E}_2\text{Ring}(\text{MU}, H \vee \Sigma^{q+1} H\pi_q R) & \simeq & \mathcal{E}_2\text{Ring}(\Sigma_+^\infty \text{BU}, H \vee \Sigma^{q+1} H\pi_q R) \\ \downarrow & & \downarrow \\ \mathcal{S}(\text{MU}, H \vee \Sigma^{q+1} H\pi_q R)_u & \simeq & \mathcal{S}(\Sigma_+^\infty \text{BU}, H \vee \Sigma^{q+1} H\pi_q R)_u \\ \downarrow & & \downarrow \\ \mathcal{S}(\text{MU}(1), H \vee \Sigma^{q+1} H\pi_q R)_u & \simeq & \mathcal{S}(\Sigma_+^\infty \text{BU}(1), H \vee \Sigma^{q+1} H\pi_q R)_u \end{array}$$

with the bottom pair of vertical maps just induced by the inclusions  $\text{MU}(1) \rightarrow \text{MU}$  and  $\text{BU}(1) \rightarrow \text{BU}$ . This gives the first step in the lemma, factoring the map in the statement through the map  $\tilde{H}^{q+1-m}(\text{BU}, \pi_q R) \rightarrow \tilde{H}^{q+1-m}(\text{BU}(1), \pi_q R)$ .

We next need to bring in the equivalence in Theorem 5.6 and this requires a detour into the category of spectra lying over  $S$ . If we write  $\mathcal{S}/_S$  for the derived space of maps in the category of spectra lying over  $S$  and similarly  $\mathcal{S}/_H$  for the derived space of maps in the category of spectra lying over  $H$ , then it is easy to see that each of the maps

$$\begin{aligned} \mathcal{S}/_S(\Sigma_+^\infty \text{BU}, S \vee \Sigma^{q+1} H\pi_q R) &\longrightarrow \mathcal{S}/_H(\Sigma_+^\infty \text{BU}, H \vee \Sigma^{q+1} H\pi_q R) \\ &\longrightarrow \mathcal{S}(\Sigma_+^\infty \text{BU}, H \vee \Sigma^{q+1} H\pi_q R)_u \end{aligned}$$

is a weak equivalence.

Let  $I^{\mathbf{R}}$  be the functor from spectra lying over  $S$  back to spectra that takes the homotopy fiber of the map to  $S$ . Then we have a weak equivalence

$$\mathcal{S}/_S(\Sigma_+^\infty \text{BU}, S \vee \Sigma^{q+1} H\pi_q R) \longrightarrow \mathcal{S}(I^{\mathbf{R}}\Sigma_+^\infty \text{BU}, \Sigma^{q+1} H\pi_q R).$$

The relevance of this that for any augmented  $E_2$  ring spectrum  $A$  and any spectrum  $N$ , the diagram

$$\begin{array}{ccc} \mathcal{S}(\Sigma^{-2} I^{\mathbf{R}} B^2 A, N) & \simeq & \mathcal{E}_2\text{Ring}/_S(A, S \vee N) \\ \downarrow & & \downarrow \\ \mathcal{S}(I^{\mathbf{R}} A, N) & \simeq & \mathcal{S}/_S(A, S \vee N) \end{array}$$

commutes, where the lefthand vertical arrow is induced by the map  $\Sigma^2 I^{\mathbf{R}} A \rightarrow I^{\mathbf{R}} B^2 A$  and the righthand vertical arrow is the forgetful map. The top horizontal map is the map from [7, 7.4] and the fact that the diagram commutes is clear from explicit construction given there (in the non-unital context), cf [7, 8.2] (and the discussion preceding it).

In the case of  $A = \Sigma^\infty \text{BU}_+$ , we use the unit  $S \rightarrow \Sigma_+^\infty \text{BU}$  to split the augmentation  $\Sigma_+^\infty \text{BU} \rightarrow S$ , and then just as in the proof of Theorem 5.6 we can identify  $I^{\mathbf{R}}\Sigma_+^\infty \text{BU}$  as  $\Sigma^\infty \text{BU}$  and  $I^{\mathbf{R}}B^2\Sigma^\infty \text{BU}_+$  as  $\Sigma^\infty B^2\text{BU}$ . Under these identifications

$$\Sigma^2 I^{\mathbf{R}}\Sigma_+^\infty \text{BU} \longrightarrow I^{\mathbf{R}}\Sigma_+^\infty B^2\text{BU}$$

becomes  $\Sigma^\infty$  applied to the based map  $\Sigma^2\text{BU} \rightarrow B^2\text{BU}$ . We then get a commuting diagram

$$\begin{array}{ccc} \mathcal{G}(\Sigma^{-2}\Sigma^\infty B^2\text{BU}, \Sigma^{q+1}H\pi_q R) & \simeq & \mathcal{E}_2\text{Ring}(\Sigma_+^\infty \text{BU}, H \vee H\pi_q R) \\ \downarrow & & \downarrow \\ \mathcal{G}(\Sigma^\infty \text{BU}, \Sigma^{q+1}H\pi_q R) & \simeq & \mathcal{G}(\Sigma^\infty \text{BU}_+, H \vee H\pi_q R)_u, \end{array}$$

completing the proof of the lemma. □

As an immediate consequence of the lemma, we get the following result.

**Proposition 6.6** *The map*

$$\mathcal{E}_2\text{Ring}(\text{MU}, H \vee H\Sigma^{q+1}\pi_q R) \longrightarrow \mathcal{G}(\text{MU}(1), H \vee \Sigma^{q+1}H\pi_q R)$$

*is a split surjection on homotopy groups.*

**Proof** Applying the lemma, we can compute the induced map on homotopy groups by computing the map on integral homology

$$H_*\text{BU}(1) \longrightarrow H_*\text{BU} \longrightarrow H_{*+2}\text{BSU}$$

and using universal coefficients. Showing that the map on homology is a split injection is an easy exercise using the calculation of the Bott map (see Proposition 7.3 below) or the edge homomorphism in the Rothenberg–Steenrod spectral sequence (applied twice, each spectral sequence degenerating at  $E_2$  for formal reasons). □

Finally, we can complete the proof of the inductive step. To simplify notation, we rewrite the diagram of fibration sequences (6.3) as

$$\begin{array}{ccccc} \longrightarrow & F & \xrightarrow{i} & E & \xrightarrow{g} & B \\ & \downarrow h_F & & \downarrow h_E & & \downarrow h_B \\ \longrightarrow & F' & \xrightarrow{i'} & E' & \xrightarrow{g'} & B'. \end{array}$$

We have that both base spaces  $B$  and  $B'$  are connected. For each basepoint  $e$  of  $E$ , let  $F_e$  denote the components of  $F$  that lie above the component of  $e$ , and similarly

for  $E'$  and  $F'$ . Recall that in the long exact sequence of a fibration, at the  $\pi_0$  level “exact” means that for each  $e$  in  $E$ ,  $\pi_0 F_e$  is a transitive  $\pi_1(B, g(e))$ -set with isotropy group at the component of  $e$  (in  $F_e$ ) the image of  $\pi_1(E, e)$ .

In our case, by the inductive hypothesis, we have that  $\pi_1(E, e)$  is trivial for all  $e$ , and by inspection, we see that

$$\pi_1(E', e') \cong \pi_1(\mathcal{S}(\Sigma_+^\infty \text{BU}(1), R_{q-1})_u, e')$$

is trivial for all  $e'$ . It follows that for each  $e$ ,  $\pi_0 F_e$  is a free transitive  $\pi_1(B, g(e))$ -set and for each  $e'$ ,  $\pi_0 F'_{e'}$  is a free transitive  $\pi_1(B', g'(e'))$ -set. By the previous proposition,  $\pi_1(B, g(e)) \rightarrow \pi_1(B', g'(h_E(e)))$  is a surjection for every  $e$  in  $E$ , and so  $\pi_0 F_e \rightarrow \pi_0 F'_{h_E(e)}$  is a surjection for every  $e$  in  $E$ . Letting  $e$  vary over a choice of basepoint in each component of  $E$ , we then see that the map

$$\pi_0 F \longrightarrow \pi_0 E \times_{\pi_0 E'} \pi_0 F'$$

is surjective, which proves the inductive step for (ii). Since  $\pi_0 E \rightarrow \pi_0 E'$  is surjective by inductive hypothesis (i), it follows that  $\pi_0 F \rightarrow \pi_0 F'$  is surjective, which proves the inductive step for (i). This completes the proof of Theorem 1.2.

**Remark 6.7** The proof for MSO follows the same outline as the proof for MU, taking advantage of the 2-local equivalence between BSO and BSp and MSO and MSp (the latter attributed to [28] in [30]). Since  $\frac{1}{2} \in \pi_0 R$ , every map of ring spectra from MSO to  $R$  extends uniquely to a map of ring spectra from MSp to  $R$  and is determined by its restriction to a map of spectra  $\text{MSp}(1) \rightarrow R$ , inducing a bijection  $\mathcal{E}_2 \text{Ring}(\text{MSO}, R) \rightarrow \mathcal{S}(\text{MSp}(1), R)_u$  (cf [14, 7.5]). We have here that  $H_*(B^2\text{BSO}; Z_{(2)}) \cong H_*(\text{Sp}/\text{SU}; Z_{(2)})$  is torsion free and concentrated in even degrees, and the rest of the argument goes through as above.

## 7 Proof of Theorem 1.4

In this section we show that not all ring spectrum maps  $\text{MU} \rightarrow \text{MU}$  are represented by  $E_4$  ring maps. We proceed by studying the forgetful map from the set  $\pi_0 \mathcal{E}_n \text{Ring}(\text{MU}, \text{MU})$  of self-maps of MU in the homotopy category of  $E_n$  ring spectra to the set  $\text{HoRing}(\text{MU}, \text{MU})$  of self-maps of MU in the category of ring spectra. Although we do not obtain complete results, we do obtain enough to see that the map is not surjective for  $n \geq 4$ .

We begin with a refinement of the work in the previous section. Since the target MU is an  $E_\infty$  ring spectrum and comes with a canonical  $E_\infty$  ring map  $\text{MU} \rightarrow \text{MU}$  (namely,



the identity), Theorem 2.1 gives us a canonical weak equivalence

$$\mathcal{E}_n\mathcal{R}ing(\mathbf{MU}, \mathbf{MU}) \simeq \mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty\mathbf{BU}, \mathbf{MU})$$

for all  $n$ . Using the adjunction of [26, IV.1.8], we can identify  $\mathcal{E}_n\mathcal{R}ing(\Sigma_+^\infty\mathbf{BU}, \mathbf{MU})$  as the derived mapping space

$$\mathcal{E}_n\mathcal{T}op(\mathbf{BU}, \Omega^\infty\mathbf{MU}^\times) = \mathcal{E}_n\mathcal{T}op(\mathbf{BU}, \mathbf{SL}_1\mathbf{MU})$$

in the category of  $E_n$  spaces, where we regard  $\Omega^\infty\mathbf{MU}$  as an  $E_n$  space via the multiplicative (rather than additive)  $E_\infty$  structure. Here  $\mathbf{SL}_1\mathbf{MU}$  denotes the 1–component of  $\Omega^\infty\mathbf{MU}$ ; since  $\mathbf{BU}$  is connected, any  $E_n$  map must land in the 1–component. As  $\mathbf{BU}$  and  $\mathbf{SL}_1\mathbf{MU}$  are both connected, the theory of iterated loop spaces gives us a weak equivalence

$$\mathcal{E}_n\mathcal{T}op(\mathbf{BU}, \mathbf{SL}_1\mathbf{MU}) \simeq \mathcal{T}op_*(B^n\mathbf{BU}, B^n\mathbf{SL}_1\mathbf{MU}).$$

Because  $\mathbf{SL}_1\mathbf{MU}$  is a connected  $E_\infty$  space, it is the zeroth space of a connective spectrum that we denote as  $sl_1\mathbf{MU}$ . We then have an identification of the homotopy groups of  $\mathcal{E}_n\mathcal{R}ing(\mathbf{MU}, \mathbf{MU})$  in terms of the cohomology theory  $sl_1\mathbf{MU}$ . Specifically,

$$(7.1) \quad \pi_q \mathcal{E}_n\mathcal{R}ing(\mathbf{MU}, \mathbf{MU}) \cong \pi_q \mathcal{T}op_*(B^n\mathbf{BU}, B^n\mathbf{SL}_1\mathbf{MU}) = (\widetilde{sl}_1\mathbf{MU})^{n-q}(B^n\mathbf{BU})$$

(where tilde indicates the reduced cohomology theory), and in particular

$$\pi_0 \mathcal{E}_n\mathcal{R}ing(\mathbf{MU}, \mathbf{MU}) \cong (\widetilde{sl}_1\mathbf{MU})^n(B^n\mathbf{BU}).$$

We may further identify  $(\widetilde{sl}_1\mathbf{MU})^n(B^n\mathbf{BU})$  as  $(\widetilde{sl}_1\mathbf{MU})^n(\mathbf{BU}(n+2))$  when  $n$  is even or  $(\widetilde{sl}_1\mathbf{MU})^n(\mathbf{U}(n+2))$  when  $n$  is odd, by Bott periodicity. We regard (7.1) as a refinement of Theorem 2.2, as indicated in the introduction.

The parallel (now classical) theory for maps of ring spectra  $\mathbf{MU}$  to  $\mathbf{MU}$ , qv [1, II, Section 4], provides the identification

$$\begin{aligned} Ho\mathcal{R}ing(\mathbf{MU}, \mathbf{MU}) &\cong \mathcal{H}\mathcal{T}op(\mathbf{BU}, \mathbf{SL}_1\mathbf{MU}) \cong \pi_0 \mathcal{T}op_*(\mathbf{BU}(1), \mathbf{SL}_1\mathbf{MU}) \\ &= (\widetilde{sl}_1\mathbf{MU})^0(\mathbf{BU}(1)), \end{aligned}$$

where  $\mathcal{H}\mathcal{T}op$  denotes the set of maps of  $H$ –spaces (maps in the homotopy category that respect the unit and multiplication in the homotopy category). As both the identification of  $\pi_0 \mathcal{E}_n\mathcal{R}ing(\mathbf{MU}, \mathbf{MU})$  and  $Ho\mathcal{R}ing(\mathbf{MU}, \mathbf{MU})$  in terms of reduced  $sl_1\mathbf{MU}$ –cohomology are induced by the Thom isomorphism for the identity map of  $\mathbf{MU}$  together with the  $(\Sigma^\infty, \Omega^\infty)$  adjunction, we immediately obtain the following comparison result.

**Proposition 7.2** *The map*

$$\begin{aligned}
 (\widetilde{sl}_1 \text{MU})^n(B^n \text{BU}) &\cong \pi_0 \mathcal{E}_n \text{Ring}(\text{MU}, \text{MU}) \\
 &\longrightarrow \text{HoRing}(\text{MU}, \text{MU}) \cong (\widetilde{sl}_1 \text{MU})^0(\text{BU}(1))
 \end{aligned}$$

induced by the forgetful map  $\pi_0 \mathcal{E}_n \text{Ring}(\text{MU}, \text{MU}) \rightarrow \text{HoRing}(\text{MU}, \text{MU})$  is the map on reduced  $sl_1 \text{MU}$  cohomology induced by the usual map

$$\Sigma^n \text{BU}(1) \longrightarrow \Sigma^n \text{BU} \longrightarrow B^n \text{BU}.$$

When  $n$  is even, we can take advantage of Bott periodicity  $B^n \text{BU} \simeq \text{BU}\langle n + 2 \rangle$  to identify  $\text{BU}(1) \rightarrow \text{BU}\langle n + 2 \rangle$  as the map induced by the Bott map, whose effect on complex oriented homology theories is well-understood [1, II§12] (at least after composing with the map  $\text{BU}\langle n + 2 \rangle \rightarrow \text{BU}$ ). Of course,  $sl_1 \text{MU}$  is not even a ring theory, so not complex oriented, but we can use the Atiyah–Hirzebruch spectral sequence to obtain some information. For example, the integral cohomology  $H^*(\text{BU}(1)) = H^*(\mathbb{C}P^\infty)$  is the polynomial ring  $\mathbb{Z}[x]$  and in particular is in each degree a finitely generated free abelian group and is concentrated in even degrees. The same is true of  $\pi_* sl_1 \text{MU}$ , and so the Atiyah–Hirzebruch spectral sequence has  $E_2 = E_\infty$ , with no extension problems, giving us a non-canonical isomorphism

$$(\widetilde{sl}_1 \text{MU})^q(\text{BU}(1)) \cong \bigoplus_{m>0} \widetilde{H}^{m+q}(\text{BU}(1)) \otimes \pi_m \text{MU},$$

noting that  $\pi_m sl_1 \text{MU} = \pi_m \text{MU}$  for  $m > 0$  whereas  $\pi_0 sl_1 \text{MU} = 0$ . Likewise, in the case  $n = 2$  and  $n = 4$ , we have that  $H^* B^2 \text{BU} \cong H^* \text{BSU}$  is the polynomial ring  $\mathbb{Z}[c_2, c_3, c_4, \dots]$  on Chern classes, and  $H^* B^4 \text{BU} \cong H^* \text{BU}\langle 6 \rangle$  is a polynomial ring  $\mathbb{Z}[y_3, y_4, y_5, \dots]$  on classes in degrees 6, 8, 10, ... (see [4, 4.7]). We then get non-canonical isomorphisms

$$\begin{aligned}
 (\widetilde{sl}_1 \text{MU})^q(\text{BSU}) &\cong \bigoplus_{m>0} \widetilde{H}^{m+q}(\text{BSU}) \otimes \pi_m \text{MU}, \\
 (\widetilde{sl}_1 \text{MU})^q(\text{BU}\langle 6 \rangle) &\cong \bigoplus_{m>0} \widetilde{H}^{m+q}(\text{BU}\langle 6 \rangle) \otimes \pi_m \text{MU}.
 \end{aligned}$$

Up to filtration (but only up to filtration), we can identify the maps

$$\begin{aligned}
 (\widetilde{sl}_1 \text{MU})^2(\text{BSU}) &\longrightarrow (\widetilde{sl}_1 \text{MU})^0(\text{BU}(1)), \\
 (\widetilde{sl}_1 \text{MU})^4(\text{BU}\langle 6 \rangle) &\longrightarrow (\widetilde{sl}_1 \text{MU})^0(\text{BU}(1))
 \end{aligned}$$

in terms of the maps  $\widetilde{H}^{*+2}(\text{BSU}) \rightarrow \widetilde{H}^*(\text{BU}(1))$  and  $\widetilde{H}^{*+4}(\text{BU}\langle 6 \rangle) \rightarrow \widetilde{H}^*(\text{BU}(1))$  on ordinary cohomology. We now compute the maps on ordinary cohomology.

**Proposition 7.3** *The map  $H^{*+2}(\text{BSU}) \rightarrow H^*(\text{BU}(1))$  kills decomposable elements and sends  $c_{m+1}$  to  $(-1)^m x^m$ .*

**Proof** The map clearly kills products as it is induced by the map of spaces

$$\Sigma^2\text{BU}(1) \longrightarrow \Sigma^2\text{BU} \longrightarrow \text{BSU},$$

and products in  $H^*(\Sigma^2\text{BU}(1))$  are zero. To see where the element  $c_{m+1}$  goes, we note that the composite map

$$\Sigma^2\text{BU} \longrightarrow \text{BSU} \longrightarrow \text{BU}$$

is the Bott map  $B$ , whose effect on homology was studied in [1, II, Section 12]. We write  $H_*(\text{BU}) = \mathbb{Z}[b_1, b_2, \dots]$ , where the  $b_m$  are the usual generators:  $b_m$  is the image of the usual generator of  $H_{2m}(\text{BU}(1))$  which is dual to  $x^m \in H^{2m}(\text{BU}(1))$ . Then on homology  $B_*: H_*(\text{BU}) \rightarrow H_*(\text{BU})$  kills decomposable elements and sends  $b_m$  to  $(-1)^m s_m$ , where  $s_m = q_m(b_1, \dots, b_m)$  and  $q_m$  is the  $m^{\text{th}}$  Newton polynomial defined by the relationship

$$q_m(\sigma_1, \dots, \sigma_m) = t_1^m + \dots + t_k^m$$

for  $\sigma_j$  the  $j^{\text{th}}$  elementary symmetric polynomial in  $t_1, \dots, t_k$ . Then

$$s_{m+1} = b_1^{m+1} + \text{terms involving } b_j \text{ for } j > 1.$$

On cohomology,  $H^*(\text{BU}) \rightarrow H^*(\text{BSU})$  is the quotient by the Chern class  $c_1$ , and so we can compute the map in the statement by means of the Bott map  $B^*: H^*(\text{BU}) \rightarrow H^*(\text{BU})$ . Using the Kronecker pairing of homology with cohomology, we see that

$$\begin{aligned} \langle B^*c_{m+1}, b_m \rangle &= \langle c_{m+1}, B_*b_m \rangle = \langle c_{m+1}, (-1)^m s_{m+1} \rangle \\ &= \langle c_{m+1}, (-1)^m b_1^{m+1} \rangle = (-1)^m, \end{aligned}$$

since  $c_{m+1}$  is the dual of  $b_1^{m+1}$  in the monomial basis of the  $b_m$ . □

**Proposition 7.4** *The map  $H^{*+4}(\text{BU}\langle 6 \rangle) \rightarrow H^*(\text{BU}(1))$  kills decomposable elements and sends the polynomial generator  $y_{m+2}$  in  $H^{2m+4}(\text{BU}\langle 6 \rangle)$  to*

$$\begin{cases} (-1)^{p^t} p^{t-1} x^{p^t-1} & m+1 = p^t \text{ for some prime } p, t > 0, \\ (-1)^{m+1} (m+1)x^m & \text{otherwise.} \end{cases}$$

**Proof** As in the proof of the previous proposition, the map clearly kills decomposables, and we approach the problem using the Bott map  $B^2: \Sigma^4\text{BU} \rightarrow \text{BU}$ . Since the Bott map on homology  $B_*$  kills decomposables and  $B_*b_m = (-1)^m s_{m+1}$ , using

$$s_{m+1} = (-1)^m (m+1)b_{m+1} + \text{decomposables}$$

we see that  $B_*^2 b_m = (-1)^{m+1}(m+1)s_{m+2}$  and that the composite map  $H^{*+4}(\text{BU}) \rightarrow H^*(\text{BU}(1))$  takes  $c_{m+2}$  to  $(-1)^{m+1}(m+1)x^m$ . Unlike in the previous proposition, the map  $H^*(\text{BU}) \rightarrow H^*(\text{BU}\langle 6 \rangle)$  is not onto. By [4, 4.6], for  $m+1 \neq p^t$ , we can take the generator in  $H^{2(m+2)}(\text{BU}\langle 6 \rangle)$  to be the image of  $c_{m+2}$ . By [4, 4.5], for  $m+1 = p^t$ , the image of  $c_{m+2}$  is up to decomposables  $p$  times a generator in  $H^{2(m+2)}(\text{BU}\langle 6 \rangle)$ . (It is  $p^1$  times a generator rather than some higher power of  $p$  since the transgressive elements  $u_k$  (in the notation of [4, 4.5]) all have non-trivial Bocksteins.) This completes the proof.  $\square$

In the  $n = 4$  case, we have that the maps  $H^{2m+4}(\text{BU}\langle 6 \rangle) \rightarrow H^{2m}(\text{BU}(1))$  are surjective for  $m = 1$  and  $m = 2$ . This says that for any  $a_1 \in \pi_2\text{MU}$  and  $a_2 \in \pi_4\text{MU}$ , there exist  $E_4$  ring maps  $\text{MU} \rightarrow \text{MU}$  whose coordinates are of the form

$$x + a_1x^2 + \dots \quad \text{and} \quad x + a_2x^3 + \dots,$$

but because of the filtration issue above, we cannot be sure exactly which coordinates of these forms represent  $E_4$  ring maps without further work. On the other hand, the map  $H^{2m+4}(\text{BU}\langle 6 \rangle) \rightarrow H^{2m}(\text{BU}(1))$  is not surjective for  $m = 3$  but has image divisible by 2. This has the consequence that if we look at any map of ring spectra  $f: \text{MU} \rightarrow \text{MU}$  corresponding to a coordinate of the form

$$x + a_3x^4 + \dots,$$

where  $a_3 \in \pi_6\text{MU}$  is not divisible by 2, then  $f$  cannot be represented by an  $E_4$  ring map  $\text{MU} \rightarrow \text{MU}$ . (Similar arguments can obviously be made at other primes.)

### 8 Proofs for Section 3

In this section we prove Theorems 3.1–3.3 and construct the model structure in Theorem 3.5. We base our approach on [17, Sections 11–12], which worked in the context of simplicial sets, but which generalizes to the current context. For convenience and to make this section more self-contained for future reference, we restate (and generalize) the results as Theorems 8.1, 8.2, and 8.5 below.

Because we have already proved the theorems in Section 3 that involve functors between different categories of spectra, we can now work with a single model category of spectra throughout this section. We let  $\mathcal{M}$  denote one of the following model categories:

- (i) The category  $\Sigma_*\mathcal{S}$  of symmetric spectra with its positive stable model structure [24, Section 14].
- (ii) The category  $\mathcal{IS}$  of orthogonal spectra with its positive model structure [24, Section 14].

- (iii) The category  $\mathcal{M}_S$  of EKMM  $S$ -modules with its standard model structure [16, VII, Section 4].
- (iv) The category  $\mathcal{G}_{\text{LMS}}[\mathbb{L}]$  of EKMM  $\mathbb{L}$ -spectra with its standard model structure [16, VII, Section 4].

(We extend the convention used throughout the paper that the unmodified word “spectrum” means precisely an object of  $\mathcal{M}$ .) We regard the category  $\mathcal{M}$  as a cofibrantly generated model category in its standard way, and in the arguments below we use  $I$  to denote the standard set of generating cofibrations and  $J$  to denote the standard set of generating acyclic cofibrations.

We will actually prove mild generalizations of the theorems of Section 3 partly because the extra generality may be useful in future papers, but mainly because the proofs require the extra generality anyway. In Section 3, we worked in the context of an operad  $\mathcal{O}$  of (unbased) spaces; here we let  $\mathcal{O}$  be an operad of based spaces or an operad in  $\mathcal{M}$ . Indeed, for  $\mathcal{O}$  an operad in unbased spaces, the category  $\mathcal{M}[\mathcal{O}]$  is the same as the category of algebras over the operad  $\mathcal{O}_+$  of based spaces, and for the true symmetric monoidal categories of spectra, it is isomorphic (not just equivalent) to the category of algebras over the operad  $\mathcal{O}_+ \wedge S$  in  $\mathcal{M}$ . In the category of EKMM  $\mathbb{L}$ -spectra (which we needed for the work involving the Thom isomorphism), operads in  $\mathcal{M}$  do not generalize operads in spaces. With this generalization in mind, we have written the statements and arguments below in the based context: in what follows,  $\mathcal{O}$  denotes either an operad in  $\mathcal{M}$  or an operad in based spaces.

We now need to prove three theorems generalizing the statements in Section 3. The first establishes the model structures, proving Theorem 3.1 and finishing the proof of Theorem 3.5. (We also used it in the proof of Theorem 4.8.)

**Theorem 8.1** *Let  $\mathcal{O}$  be an operad. Then the category  $\mathcal{M}[\mathcal{O}]$  of  $\mathcal{O}$ -algebras in  $\mathcal{M}$  is a topological closed model category with fibrations and weak equivalences created in  $\mathcal{M}$ .*

The next proves Theorem 3.2. In the statement  $\mathbf{O}$  denotes the free  $\mathcal{O}$ -algebra functor, which is

$$\mathbf{O}X = \bigvee_{n \geq 0} \mathcal{O}(n) \wedge_{\Sigma_n} X^{(n)} = (\mathcal{O}(0) \wedge S) \vee (\mathcal{O}(1) \wedge X) \vee (\mathcal{O}(2) \wedge X \wedge X) / \Sigma_2 \vee \dots$$

when  $\mathcal{O}$  is an operad of based spaces or an operad in  $\mathcal{M}$  when  $\mathcal{M}$  is one of the true symmetric monoidal categories. Here we have written  $\wedge$  both for the smash product in  $\mathcal{M}$  and the smash product of a based space with a spectrum, and we have used the parenthetical exponent

$$X^{(n)} = X \wedge \dots \wedge X$$

as an abbreviation for smash powers. In the case when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra and  $\mathcal{O}$  is an operad in  $\mathcal{M}$ , the free functor needs the following modifications: in homogeneous degree 0, we need to use  $\mathcal{O}(0) \triangleright S \cong \mathcal{O}(0)$  in place of  $\mathcal{O}(0) \wedge S$  and in homogeneous degree 1, we need to use  $\mathcal{O}(1) \triangleleft X$  in place of  $\mathcal{O}(1) \wedge X$ , where  $\triangleleft$  and  $\triangleright$  denote the one-sided unital products of [16, XIII.1.1]. (In general, in the case of EKMM  $\mathbb{L}$ -spectra, we need to use a unital product  $\triangleleft$ ,  $\triangleright$ , or  $\star$  in place of a smash product whenever one or both factors comes with a structure map from  $S$ ; in what follows, we refer to this as “the usual modifications”.)

**Theorem 8.2** *Let  $\phi: \mathcal{O} \rightarrow \mathcal{O}'$  be a map of operads. The pushforward (left Kan extension) and pullback functors*

$$L_\phi: \mathcal{M}[\mathcal{O}] \rightleftarrows \mathcal{M}[\mathcal{O}'] : R_\phi$$

*form a Quillen adjunction, which is a Quillen equivalence if (and only if) the induced map on free algebras*

$$\mathbf{O}X \longrightarrow \mathbf{O}'X$$

*is a weak equivalence for all cofibrant objects  $X$ .*

To deduce Theorem 3.2, we need to show that in the context of operads of unbased spaces,  $\phi$  induces a weak equivalence on free algebras as in the statement if and only if each  $\phi(n)$  is a (non-equivariant) stable equivalence. The “if” direction is a straightforward generalization of [24, 15.5] (in the case of symmetric spectra and orthogonal spectra) or [16, III.5.1] (in the case of EKMM  $S$ -modules or  $\mathbb{L}$ -spectra) that follows by essentially the same argument. The “only if” direction follows by taking  $X$  to be a wedge of cofibrant 0-spheres  $\bigvee S_c^0$ ; for a wedge of  $n$  or more,  $\mathbf{O}X$  and  $\mathbf{O}'X$  contain

$$\mathcal{O}(n)_+ \wedge (S_c^0)^{(n)} \quad \text{and} \quad \mathcal{O}'(n)_+ \wedge (S_c^0)^{(n)}$$

(respectively) as wedge summands. With an eye toward more generality, we offer the following additional remark on the criterion in the theorem above in the case when  $\mathcal{O}$  is an operad in  $\mathcal{M}$ .

**Remark 8.3** *The criterion that  $\mathbf{O}X \rightarrow \mathbf{O}'X$  is a weak equivalence for every cofibrant object  $X$  is satisfied in particular in the following cases.*

- (i) *In the case when  $\mathcal{M}$  is the category of symmetric spectra or orthogonal spectra, the criterion is satisfied whenever each map  $\phi(n): \mathcal{O}(n) \rightarrow \mathcal{O}'(n)$  is a (non-equivariant) weak equivalence. This follows from the observation that the proof of [24, 15.5] still works when (in the notation there)  $X$  (our  $\mathcal{O}(n)$ ) also has*

a  $\Sigma_n$  action: the  $\Sigma_n$  action remains free on  $O(q)$  (or  $\Sigma_q$ ). Induction up the cellular filtration of  $E\Sigma_n$  shows that when  $X$  is cofibrant,

$$E\Sigma_{n+} \wedge_{\Sigma_n} (\mathcal{O}(n) \wedge X^{(n)})$$

preserves (non-equivariant) weak equivalences in (equivariant) maps of  $\mathcal{O}(n)$ .

- (ii) In the case when  $\mathcal{M}$  is EKMM  $S$ -modules, the criterion is satisfied whenever each  $\phi(n)$  is a (non-equivariant) weak equivalence and each  $\mathcal{O}(n)$  and  $\mathcal{O}'(n)$  has underlying non-equivariant object in the class  $\overline{\mathcal{E}}$  of [16, VII.6.4] (or Basterra's generalization  $\overline{\mathcal{F}}$  of [7, 9.3]), or more generally, the closure of  $\mathcal{E}$  (or  $\mathcal{F}$ ) under also the additional operation of smash product with a based space. The proof is essentially the same as the proof of [16, III.5.1]. In the case of EKMM  $\mathbb{L}$ -spectra it also works for the analogous class (where we allow omitting  $S \wedge_{\mathcal{L}} (-)$ ).

Both cases include in particular operads of the form  $\mathcal{O} \wedge S$  when  $\mathcal{O}$  is an operad of based spaces. For based spaces with non-degenerate basepoints (eg disjoint basepoints), a map  $X \rightarrow X'$  is a stable equivalence if and only if  $X \wedge S \rightarrow X' \wedge S$  is a weak equivalence; however, the same is not necessarily true for based spaces with degenerate basepoints, so some caution is in order when applying the remarks above in the context of operads of based spaces.

Finally, the third result generalizes Theorem 3.3. To state it, we need to generalize the hypothesis on  $\mathcal{O}$ . We use the following terminology.

**Definition 8.4** An operad (or symmetric sequence)  $\mathcal{O}$  of based spaces is a  $\Sigma$ -free cell retract if for each  $n > 0$ ,  $\mathcal{O}(n)$  is the retract of a free based  $\Sigma_n$ -cell complex. An operad (or symmetric sequence)  $\mathcal{O}$  in  $\mathcal{M}$  is a  $\Sigma$ -free cell retract if each  $\mathcal{O}(n)$  is equivariantly the retract of a  $\Sigma_n$ -equivariant spectrum built equivariantly as a complex with cells of the form

$$\Sigma_{n+} \wedge X \longrightarrow \Sigma_{n+} \wedge Y,$$

where  $X \rightarrow Y$  is a wedge of maps in  $I$  and/or maps of the form

$$S_+^{j-1} \wedge S \longrightarrow D_+^j \wedge S,$$

where  $S^{j-1} \rightarrow D^j$  is the inclusion of the boundary of the standard  $j$ -dimensional disk (or the inclusion of the empty set in the one-point space for  $j = 0$ ). Note that there is no condition on  $\mathcal{O}(0)$ .

The previous definition is adapted for ease of use in the proof of the following theorem that directly generalizes Theorem 3.3.

**Theorem 8.5** *Assume either that  $\mathcal{O}$  is an operad of based spaces or that  $\mathcal{M}$  is symmetric spectra, orthogonal spectra, or EKMM  $S$ -modules. If  $\mathcal{O}$  is a  $\Sigma$ -free cell retract, then every cofibrant object in  $\mathcal{M}[\mathcal{O}]$  is cofibrant in the category of  $\mathcal{M}$  under  $\mathcal{O}(\ast)$ .*

Before going on to the proofs, we make the following remark about generalizing Theorem 3.4.

**Remark 8.6** In order to keep  $\mathcal{M}$  fixed, we did not restate Theorem 3.4 in this section, nor do we prove it below; nevertheless, Theorem 3.4 does generalize to the case when  $\mathcal{O}$  is an operad in the domain category of the left adjoint, provided we add the hypothesis that the unit of the adjunction is a weak equivalence for  $\mathcal{O}X$  for all cofibrant  $X$ . The proof follows the same outline as the proof of Theorem 8.2.

We now move on to the proofs. We fix  $\mathcal{O}$  an operad of based spaces or an operad in  $\mathcal{M}$ , and we write  $\mathcal{O}I$  and  $\mathcal{O}J$  for the sets of maps in  $\mathcal{M}[\mathcal{O}]$  obtained by applying  $\mathcal{O}$  to  $I$  and  $J$ , respectively (where as indicated above,  $I$  and  $J$  are the canonical sets of generating cofibrations and generating acyclic cofibrations, respectively). According to [24, 5.13], to show that  $\mathcal{M}[\mathcal{O}]$  is a cofibrantly generated topological model category with generating cofibrations  $\mathcal{O}I$  and generating acyclic cofibrations  $\mathcal{O}J$ , it suffices to prove the following lemma.

**Lemma 8.7** *Let  $C$  be an object in  $\mathcal{M}[\mathcal{O}]$ .*

- (i) *For  $A \rightarrow B$  any coproduct (in  $\mathcal{M}[\mathcal{O}]$ ) of maps in  $\mathcal{O}I$  and any map of  $\mathcal{O}$ -algebras  $A \rightarrow C$ , the map  $C \rightarrow C \amalg_A B$  from  $C$  to the pushout in  $\mathcal{M}[\mathcal{O}]$  is an  $h$ -cofibration in  $\mathcal{M}$ .*
- (ii) *For  $A \rightarrow B$  any coproduct (in  $\mathcal{M}[\mathcal{O}]$ ) of maps in  $\mathcal{O}J$  and any map of  $\mathcal{O}$ -algebras  $A \rightarrow C$ , the map  $C \rightarrow C \amalg_A B$  from  $C$  to the pushout in  $\mathcal{M}[\mathcal{O}]$  is a weak equivalence in  $\mathcal{M}$ .*

In the statement above, an  $h$ -cofibration is a map  $X \rightarrow Y$  satisfying the homotopy extension property, or in other words, such that the map  $Y \cup_X (X \wedge I_+) \rightarrow Y \wedge I_+$  admits a retraction.

The key to proving the lemma is understanding pushouts in  $\mathcal{M}[\mathcal{O}]$  of the form  $C \rightarrow C \amalg_{\mathcal{O}X} \mathcal{O}Y$ . We will show that the underlying spectrum has a filtration induced by powers of  $Y$ . To construct this, we use the universal enveloping operad of [6, 8.3] and [17, Section 12].



**Construction 8.8** For an  $\mathcal{O}$ -algebra  $C$ , define  $U_{\mathcal{O}}C(n)$  to be the coequalizer in  $\mathcal{M}$ ,

$$\bigvee_k \mathcal{O}(n+k) \wedge_{\Sigma_k} (\mathcal{O}C)^{(k)} \rightrightarrows \bigvee_k \mathcal{O}(n+k) \wedge_{\Sigma_k} C^{(k)} \longrightarrow U_{\mathcal{O}}C(n)$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). Here one map is induced by the action  $\mathcal{O}C \rightarrow C$  and the other by the operadic multiplication. The spectra  $U_{\mathcal{O}}C(-)$  form an operad in  $\mathcal{M}$  with  $\Sigma_n$  action on  $U_{\mathcal{O}}C(n)$  induced by the unused  $\Sigma_n$  action on  $C(k+n)$ , identity  $S \rightarrow U_{\mathcal{O}}C(1)$  induced by the identity of  $\mathcal{O}$  and operadic multiplication induced by the operadic multiplication of  $\mathcal{O}$ .

(In what follows, we never actually use the operadic multiplication, just the identity and equivariance.)

An easy check of universal properties shows that the coproduct of  $\mathcal{O}$ -algebras  $C \amalg \mathcal{O}Y$  has

$$\bigvee_n U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Y^{(n)}$$

as its underlying spectrum when  $\mathcal{M}$  is one of the true symmetric monoidal categories of spectra. For EKMM  $\mathbb{L}$ -spectra, we have the usual modification discussed above

$$C \amalg \mathcal{O}Y = (U_{\mathcal{O}}C(0)) \vee (U_{\mathcal{O}}C(1) \triangleleft Y) \vee \bigvee_{n>1} (U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Y^{(n)})$$

when  $\mathcal{O}$  is an operad in  $\mathcal{M}$ , but when  $\mathcal{O}$  is an operad of based spaces, we have a slightly different formula. (Here for clarity we temporarily break our convention and write  $\wedge_{\mathcal{L}}$  for the smash product of  $\mathbb{L}$ -spectra, reserving  $\wedge$  for the smash product with a space.) For  $\mathcal{O}$  an operad of based spaces, the summands  $k = 0$  in Construction 8.8 induce a map of  $\Sigma_n$ -equivariant  $\mathbb{L}$ -spectra  $\mathcal{O}(n) \wedge S \rightarrow U_{\mathcal{O}}C$ . For  $Z$  an  $\mathbb{L}$ -spectrum, define  $U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Z$  to be the following pushout:

$$\begin{array}{ccc} \mathcal{O}(n) \wedge S \wedge_{\mathcal{L}} Z & \longrightarrow & U_{\mathcal{O}}C(n) \wedge_{\mathcal{L}} Z \\ \downarrow & & \downarrow \\ \mathcal{O}(n) \wedge Z & \longrightarrow & U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Z \end{array}$$

We note that both vertical maps are weak equivalences, as they become isomorphisms after smashing with  $S$  (applying  $S \wedge_{\mathcal{L}} (-)$ ). If  $H < \Sigma_n$  and  $Z$  is an  $H$ -equivariant  $\mathbb{L}$ -spectrum, then  $U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Z$  has a right action of  $H$  from  $U_{\mathcal{O}}C(n)$  (and  $\mathcal{O}(n)$ ) and a left action from  $Z$ , and we let  $(U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Z)_H$  denote the coequalizer of these actions. Then the universal property of the coproduct gives us

$$C \amalg \mathcal{O}Y = \bigvee_n (U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Y^{(n)})_{\Sigma_n}.$$

In general, when  $\mathcal{M}$  is EKMM  $\mathbb{L}$ -spectra and  $\mathcal{O}$  is an operad of based spaces, whenever we encounter a formula involving the universal enveloping operad, we must replace  $U_{\mathcal{O}}C(n) \wedge_H Z$  with  $(U_{\mathcal{O}}C(n) \triangleleft_{\mathcal{O}(n)} Z)_H$ ; we add this to our list of “usual modifications” for the case of EKMM  $\mathbb{L}$ -spectra.

The discussion above gives us a (split) filtration on  $C \amalg_{\mathcal{O}X} \mathcal{O}Y$ . To get the filtration on  $C \amalg_{\mathcal{O}X} \mathcal{O}Y$ , we also need the following construction from [17, Section 12].

**Construction 8.9** For  $g: X \rightarrow Y$  a map in  $\mathcal{M}$ , define  $Q_i^n(g)$  inductively as follows. Let  $Q_0^n(g) = X^{(n)}$  and for  $i > 0$ , define  $Q_i^n(g)$  to be the following pushout:

$$\begin{array}{ccc} \Sigma_{n+} \wedge_{\Sigma_{n-i} \times \Sigma_i} X^{(n-i)} \wedge Q_{i-1}^i(g) & \longrightarrow & \Sigma_{n+} \wedge_{\Sigma_{n-i} \times \Sigma_i} X^{(n-i)} \wedge Y^{(i)} \\ \downarrow & & \downarrow \\ Q_{i-1}^n(g) & \longrightarrow & Q_i^n(g) \end{array}$$

The basic idea is that (when  $g$  is an inclusion)  $Q_i^n(g)$  is the  $\Sigma_n$ -equivariant subspectrum of  $Y^{(n)}$  with  $i$  factors of  $Y$  and  $n-i$  factors of  $X$ . Mainly we need  $Q_{n-1}^n(g)$  and we see that  $Y^{(n)}/Q_{n-1}^n(g) \cong (Y/X)^{(n)}$ . Just as in [17, 12.6], we have the following proposition.

**Proposition 8.10** Let  $C$  be an  $\mathcal{O}$ -algebra and let  $g: X \rightarrow Y$  be a map in  $\mathcal{M}$ . For any map  $X \rightarrow C$  in  $\mathcal{M}$ , let  $Fil_0(C, g) = C$  and inductively define  $Fil_n(C, g)$  to be the pushout

$$\begin{array}{ccc} U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Q_{n-1}^n(g) & \longrightarrow & U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Y^{(n)} \\ \downarrow & & \downarrow \\ Fil_{n-1}(C, g) & \longrightarrow & Fil_n(C, g) \end{array}$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). Then  $\text{colim } Fil_n(C, g)$  is the underlying spectrum of the pushout  $C \amalg_{\mathcal{O}X} \mathcal{O}Y$  in  $\mathcal{M}[\mathcal{O}]$ .

**Proof** Using the constructions of  $U_{\mathcal{O}}C(n)$  and  $Q_{n-1}^n(g)$ , and commuting colimits, we see that  $\text{colim } Fil_n(C, g)$  can be identified as the coequalizer of the pair of arrows

$$\begin{array}{ccc} \bigvee_{i,k} \mathcal{O}(k+n) \wedge_{\Sigma_k \times \Sigma_{n-i} \times \Sigma_i} (\mathcal{O}C)^{(k)} \wedge X^{(n-i)} \wedge Y^{(i)} & & \\ & \rightrightarrows & \bigvee_k \mathcal{O}(k+n) \wedge_{\Sigma_k \times \Sigma_n} C^{(k)} \wedge Y^{(n)} \end{array}$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra), where one map is induced by the action map  $\mathcal{O}C \rightarrow C$  and the map  $g: X \rightarrow Y$  and the other is induced by the operadic multiplication on  $\mathcal{O}$  and the given map  $X \rightarrow C$ . Comparing

universal properties, the coequalizer above is easily identified as the pushout  $C \amalg_{\mathcal{O}X} \mathcal{O}Y$  in  $\mathcal{M}[\mathcal{O}]$ . □

The previous proposition leads to an interesting spectral sequence for computing the homotopy groups (or homology) of a pushout: its  $E^1$ -term looks like the homotopy groups (or homology) of the coproduct in the category of  $\mathcal{O}$ -algebras. The analogue of this spectral sequence in algebra for  $\mathcal{O}$  an  $E_\infty$  operad, where the homology of the coproduct can be easily calculated, formed an important part of the core argument for the proof of the Main theorem of [22] (qv Section 14).

Proposition 8.10 is what we need to prove Lemma 8.7 and therefore complete the proof of Theorem 8.1.

**Proof of Lemma 8.7** In both parts, we let  $g: X \rightarrow Y$  be a wedge of maps in  $I$  (for (i)) or  $J$  (for (ii)) such that  $A \rightarrow B$  is  $\mathcal{O}g$ .

For (i), by Proposition 8.10, it suffices to see that each map  $Fil_{n-1}(C, g) \rightarrow Fil_n(C, g)$  is an  $h$ -cofibration (for  $n > 1$ ), and for this it suffices to show that each map

$$(*) \quad U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Q_{n-1}^n(g) \longrightarrow U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} Y^{(n)}$$

is an  $h$ -cofibration (with the usual modification when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). In the case when  $\mathcal{M}$  is symmetric or orthogonal spectra,  $g$  is a wedge of maps of the form

$$F_i S^0 \wedge S_+^{j-1} \longrightarrow F_i S^0 \wedge D_+^j,$$

where  $S^{j-1} \rightarrow D^j$  is the inclusion of the boundary into the standard  $j$ -dimensional disk, and  $F_i S^0$  is the point-set model of the  $(-i)$ -sphere that represents the  $i^{\text{th}}$  space functor (see [24, 4.2] or [24, 4.4], where it is denoted  $F_{\mathbb{R}^i} S^0$ ). Then  $Y^{(n)}$  becomes the wedge of  $\Sigma_n$ -equivariant spectra of the form

$$\begin{aligned} \Sigma_{n+} \wedge_{\Sigma_{m_1, \dots, m_k}} ((F_{i_1} S^0 \wedge D_+^{j_1})^{(m_1)} \wedge \dots \wedge (F_{i_k} S^0 \wedge D_+^{j_k})^{(m_k)}) \\ \cong \Sigma_{n+} \wedge_{\Sigma_{m_1, \dots, m_k}} (F_{\mathbf{i}} S^0 \wedge ((D^{j_1})^{m_1} \times \dots \times (D^{j_k})^{m_k})_+), \end{aligned}$$

where  $\Sigma_{m_1, \dots, m_k} := \Sigma_{m_1} \times \dots \times \Sigma_{m_k}$ ,  $m_1 + \dots + m_k = n$ , and  $\mathbf{i} := m_1 i_1 + \dots + m_k i_k$ , and where we have written  $\mathbf{i}$  in bold to emphasize that  $\Sigma_{m_1, \dots, m_k}$  acts on  $F_{\mathbf{i}} S^0$  as well as on the product of the disks. We can then identify  $Q_{n-1}^n(g)$  as the wedge  $\Sigma_n$ -equivariant spectra of the form

$$\Sigma_{n+} \wedge_{\Sigma_{m_1, \dots, m_k}} (F_{\mathbf{i}} S^0 \wedge \partial((D^{j_1})^{m_1} \times \dots \times (D^{j_k})^{m_k})_+)$$

and the map  $Q_{n-1}^n(g) \rightarrow Y^{(n)}$  as the induced by the boundary inclusions. By inspection, this is a  $\Sigma_n$ -equivariant  $h$ -cofibration, and it then follows that the map  $(*)$  is an  $h$ -fibration. The case of EKMM  $S$ -modules and  $\mathbb{L}$ -spectra is analogous.

For (ii), the case of EKMM  $S$ -modules and  $\mathbb{L}$ -spectra is trivial as the map  $X \rightarrow Y$  is the inclusion of a deformation retraction, and therefore, the map  $OX \rightarrow OY$  is the inclusion of a deformation retract in the category  $\mathcal{M}[\mathcal{O}]$ ; it follows that  $C \rightarrow C \amalg_A B$  is the inclusion of a deformation retract in the category  $\mathcal{M}[\mathcal{O}]$  and therefore a homotopy equivalence. For the case of symmetric spectra or orthogonal spectra, applying Proposition 8.10, it suffices to show that the map  $Fil_{n-1}(C, g) \rightarrow Fil_n(C, g)$  is a stable equivalence for all  $n \geq 1$ . The argument above shows the map is an  $h$ -cofibration. Its cofiber is

$$(**) \quad U_{\mathcal{O}}C(n) \wedge_{\Sigma_n} (Y/X)^{(n)}.$$

Since  $Y/X$  is positive cofibrant and stably equivalent to the trivial spectrum, it follows from [24, 15.5] that  $(**)$  is weakly equivalent to the trivial spectrum, and hence that  $Fil_{n-1}(C, g) \rightarrow Fil_n(C, g)$  is a weak equivalence. □

Proposition 8.10 is also all we need for the proof of Theorem 8.2.

**Proof of Theorem 8.2** Since in both algebra categories fibrations and weak equivalences are created in  $\mathcal{M}$  and since the right adjoint does not change the underlying spectrum, the adjunction is a Quillen adjunction. Now assume that  $\phi$  induces a weak equivalence  $OX \rightarrow O'X$  for every cofibrant object  $X$ . To see that the adjunction is a Quillen equivalence, it suffices to show that for a cofibrant  $\mathcal{O}$ -algebra  $C$ , the unit map  $C \rightarrow R_{\phi}L_{\phi}C$  is a weak equivalence. Without loss of generality, we can assume that  $C$  is an  $OI$ -cell complex. Then  $C = \text{colim } C_k$ , where  $C_0 = O(*)$  and  $C_k = C_{k-1} \amalg_{A_k} B_k$  where  $A_k \rightarrow B_k$  is a coproduct of maps in  $OI$ , or equivalently, is  $Og_k$  for  $g_k: X_k \rightarrow Y_k$  a wedge of maps in  $I$ . We have that  $L_{\phi}C_0 = O'(*)$  and by hypothesis the map  $O(*) \rightarrow O'(*)$  is a weak equivalence. Likewise,  $C_1 = O(Y_1/X_1)$ ,  $L_{\phi}C_1 = O'(Y_1/X_1)$  and the unit map is a weak equivalence. This shows that for any  $OI$ -cell complex built in 0 stages or 1 stage, the unit map is a weak equivalence. Assume by induction that for any  $OI$ -cell complex built in  $k$  or fewer stages, the unit map is a weak equivalence, and consider  $C_{k+1} = C_k \amalg_{OX} OY$  for  $g: X \rightarrow Y$ . Proposition 8.10 writes  $C_{k+1}$  as  $\text{colim } Fil_n(C_k, g)$  and similarly  $L_{\phi}C_{k+1}$  is  $\text{colim } Fil_n(L_{\phi}C_k, g)$  constructed using the operad  $\mathcal{O}'$ . The proof of Lemma 8.7 showed that this is a filtration by  $h$ -cofibrations, and we note that the associated graded spectra are

$$\bigvee U_{\mathcal{O}}C_k(n) \wedge_{\Sigma_n} (Y/X)^{(n)} \quad \text{and} \quad \bigvee U_{\mathcal{O}'}(L_{\phi}C_k)(n) \wedge_{\Sigma_n} (Y/X)^{(n)}$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). These naturally form algebras: the first is the  $\mathcal{O}$ -algebra  $C_k \amalg O(Y/X)$  and the second is

the  $\mathcal{O}'$ -algebra

$$L_\phi C_k \amalg \mathcal{O}'(Y/X) = L_\phi(C_k \amalg \mathcal{O}(Y/X))$$

(with coproducts taken in the appropriate algebra category  $\mathcal{M}[\mathcal{O}]$  and  $\mathcal{M}[\mathcal{O}']$ , respectively). As  $C_k \amalg (Y/X)$  is an  $\mathcal{O}$ -algebra that can be built in  $k$  or fewer stages (as  $k \geq 1$ ), it follows that the unit map

$$C_k \amalg \mathcal{O}(Y/X) \longrightarrow R_\phi L_\phi(C_k \amalg \mathcal{O}(Y/X))$$

is a weak equivalence, which shows that each map on quotients

$$U_{\mathcal{O}} C_k(n) \wedge_{\Sigma_n} (Y/X)^{(n)} \longrightarrow U_{\mathcal{O}'} L_\phi C_k(n) \wedge_{\Sigma_n} (Y/X)^{(n)}.$$

is a weak equivalence and shows that each map

$$Fil_n(C_k, g) \longrightarrow Fil_n(L_\phi C_k, g)$$

is a weak equivalence. It follows that  $C_{k+1} \rightarrow R_\phi L_\phi C_{k+1}$  is a weak equivalence. Finally,  $C = \text{colim } C_k$  is the colimit of a sequence of  $h$ -cofibrations as is  $L_\phi C = \text{colim } L_\phi C_k$ , and so the unit  $C \rightarrow R_\phi L_\phi C$  is a weak equivalence.  $\square$

Finally, we need to prove Theorem 8.5. For this we need a slight generalization of Proposition 8.10 that handles the construction of the universal enveloping operad.

**Proposition 8.11** *Let  $C$  be an  $\mathcal{O}$ -algebra and let  $g: X \rightarrow Y$  be a map in  $\mathcal{M}$ . For a map  $X \rightarrow C$  in  $\mathcal{M}$ , let  $Fil_0(U_{\mathcal{O}}C, g)(m) = U_{\mathcal{O}}C(m)$ . Inductively define  $Fil_n(U_{\mathcal{O}}C, g)(m)$  as the pushout*

$$\begin{array}{ccc} U_{\mathcal{O}}C(m+n) \wedge_{\Sigma_n} Q_{n-1}^n(g) & \longrightarrow & U_{\mathcal{O}}C(m+n) \wedge_{\Sigma_n} Y^{(n)} \\ \downarrow & & \downarrow \\ Fil_{n-1}(U_{\mathcal{O}}C, g)(m) & \longrightarrow & Fil_n(U_{\mathcal{O}}C, g)(m) \end{array}$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). Then  $\text{colim}_n Fil_n(U_{\mathcal{O}}C, g)(m)$  is the underlying  $\Sigma_n$ -equivariant spectrum of  $U_{\mathcal{O}}(C \amalg_{\mathcal{O}X} \mathcal{O}Y)(m)$ .

**Proof** As in the proof of Proposition 8.10, unwinding the definitions and interchanging colimits identifies  $\text{colim}_n Fil_n(U_{\mathcal{O}}C, g)(m)$  as the coequalizer

$$\begin{array}{ccc} \bigvee_{i,k} \mathcal{O}(k+n+m) \wedge_{\Sigma_k \times \Sigma_{n-i} \times \Sigma_i} (\mathcal{O}C)^{(k)} \wedge X^{(n-i)} \wedge Y^{(i)} & & \\ \rightrightarrows & \bigvee_k \mathcal{O}(k+n+m) \wedge_{\Sigma_k \times \Sigma_n} C^{(k)} \wedge Y^{(n)} & \longrightarrow \text{colim}_n Fil_n(U_{\mathcal{O}}C, g)(m) \end{array}$$

(with the usual modifications when  $\mathcal{M}$  is the category of EKMM  $\mathbb{L}$ -spectra). We can rewrite this as the coequalizer

$$U_{\mathcal{O}}(\mathcal{O}((\mathcal{O}C) \vee X \vee Y))(m) \rightrightarrows U_{\mathcal{O}}(\mathcal{O}(C \vee Y))(m) \rightarrow \operatorname{colim}_n \operatorname{Fil}_n(U_{\mathcal{O}}C, g)(m),$$

which is the universal enveloping operad construction  $U_{\mathcal{O}}(-)(m)$  applied to the reflexive coequalizer

$$\mathcal{O}((\mathcal{O}C) \vee X \vee Y) \rightrightarrows \mathcal{O}(C \vee Y) \rightarrow C \amalg_{\mathcal{O}X} \mathcal{O}Y. \quad \square$$

**Proof of Theorem 8.5** First consider the case where  $\mathcal{M}$  is one of the true symmetric monoidal categories of spectra. It suffices to prove that  $OI$ -cell complexes are cofibrant in  $\mathcal{M}$  under  $\mathcal{O}(\ast)$ . Let  $C$  be an  $OI$ -cell complex; then  $C = \operatorname{colim} C_k$  with  $C_0 = \mathcal{O}(\ast)$  and  $C_{k+1} = C_k \amalg_{\mathcal{O}X_k} \mathcal{O}Y_k$  for  $X_k \rightarrow Y_k$  a wedge of maps in  $I$ . It therefore suffices to show that each map  $C_k \rightarrow C_{k+1}$  is a cofibration in  $\mathcal{M}$ . Since  $U_{\mathcal{O}}(C_0)(n) = \mathcal{O}(n) \wedge S$ , by hypothesis  $U_{\mathcal{O}}(C_0)$  is a  $\Sigma$ -free cell retract. Assume by induction that  $U_{\mathcal{O}}C_k$  is a  $\Sigma$ -free cell retract. The argument in Lemma 8.7 generalizes to show that each map  $Q_{n-1}^n(g_k) \rightarrow Y_k^{(n)}$  is a (non-equivariant) cofibration between cofibrant objects and each map

$$U_{\mathcal{O}}C_k(n+m) \wedge_{\Sigma_n} Q_{n-1}^n(g_k) \longrightarrow U_{\mathcal{O}}C_k(n+m) \wedge_{\Sigma_n} Y_k^{(n)}$$

is an  $h$ -cofibration. Each map above and therefore each map

$$\operatorname{Fil}_{n-1}(C_k, g_k) \longrightarrow \operatorname{Fil}_n(C_k, g_k)$$

(for  $m = 0$ ) and each map

$$\operatorname{Fil}_{n-1}(U_{\mathcal{O}}C_k, g_k)(m) \longrightarrow \operatorname{Fil}_n(U_{\mathcal{O}}C_k, g_k)(m)$$

is  $\Sigma_m$ -equivariantly a retract of a relative cell complex built out of cells of the form

$$\Sigma_{m+} \wedge X \longrightarrow \Sigma_{m+} \wedge Y,$$

where  $X \rightarrow Y$  is a wedge of maps in  $I$ . It follows that  $C_k \rightarrow C_{k+1}$  is a cofibration in  $\mathcal{M}$  and that  $U_{\mathcal{O}}C_{k+1}$  is a  $\Sigma$ -free cell retract.

The case when  $\mathcal{M}$  is EKMM  $\mathbb{L}$ -spectra and  $\mathcal{O}$  is an operad of spaces is similar except that the inductive hypothesis on  $U_{\mathcal{O}}C_k$  is replaced by the hypothesis that for each  $m$ ,  $\mathcal{O}(m) \wedge S \rightarrow U_{\mathcal{O}}C_k(m)$  is the retract of a relative cell complex built out of cells of the form

$$\Sigma_{m+} \wedge X \longrightarrow \Sigma_{m+} \wedge Y$$

where  $X \rightarrow Y$  is a wedge of maps in  $I$ . □

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