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Dimer models and the special McKay correspondence

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We study the behavior of a dimer model under the operation of removing a corner from the lattice polygon and taking the convex hull of the rest. This refines an operation of Gulotta, and the special McKay correspondence plays an essential role in this refinement. As a corollary, we show that for any lattice polygon there is a dimer model such that the derived category of finitely generated modules over the path algebra of the corresponding quiver with relations is equivalent to the derived category of coherent sheaves on a toric Calabi–Yau 3–fold determined by the lattice polygon. Our proof is based on a detailed study of the relationship between combinatorics of dimer models and geometry of moduli spaces, and does not depend on the result of Bridgeland, King and Reid.

14F05; 14D20, 16G20, 14E16

1 Introduction

Dimer models were introduced in the 1960s as statistical mechanical models which include the two-dimensional Ising model as a special case. See eg Baxter [1] and Kenyon [27] and references therein for more on this aspect of dimer models. In this paper, a *dimer model* is a bicolored graph on a real 2–torus giving a polygon division of the torus. A fundamental object associated with a dimer model from the statistical mechanical point of view is its *characteristic polynomial*. It is a Laurent polynomial in two variables defined in a purely combinatorial way in terms of *perfect matchings*. The Newton polygon of the characteristic polynomial is called the *characteristic polygon*.

More recently, string theorists have discovered that dimer models encode the information of quivers with relations, and have used them to study supersymmetric quiver gauge theories in four dimensions (see eg Kennaway [26] and references therein). If a dimer model is *non-degenerate*, then the moduli space $\mathcal{M}_\theta$ of stable representations of the corresponding quiver with dimension vector $(1, \dots, 1)$ with respect to a generic stability parameter θ in the sense of King [28] is a smooth toric Calabi–Yau 3–fold; see Ishii and Ueda [20]. Here, a stability parameter is *generic* if all semi-stable objects are stable. The Calabi–Yau property of $\mathcal{M}_\theta$ implies that the convex hull Δ of the set

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Dimer model	Quiver	Moduli space
face	vertex	tautological line bundle
edge	arrow	morphism between tautological line bundles

Table 1: The correspondence among dimer models, quivers and moduli spaces

of primitive generators of one-dimensional cones of the fan describing $\mathcal{M}_\theta$ as a toric manifold is a lattice polygon (ie the generators all lie on a hyperplane). Moreover, this lattice polygon is known to coincide with the characteristic polygon of the dimer model; see Franco and Vegh [14] and Ishii and Ueda [20]. Although the structure of the fan is not determined by this lattice polygon, all fan structures give equivalent derived categories of coherent sheaves; see Bondal and Orlov [6] and Bridgeland [7].

The quiver associated with a dimer model is the dual graph of the dimer model, oriented in such a way that a white node is on the right of an arrow. A face of the dimer model gives a vertex v of the quiver, which in turn gives the corresponding tautological line bundle $\mathcal{L}_v$ on the moduli space $\mathcal{M}_\theta$. An edge of the dimer model gives an arrow $v \to w$ of the quiver, which corresponds to a morphism $\mathcal{L}_v \to \mathcal{L}_w$ of tautological bundles by the universal morphism $\mathbb{C}\Gamma \to \operatorname{End}(\bigoplus_v \mathcal{L}_v)$, where $\mathbb{C}\Gamma$ is the path algebra of the quiver with relations. These correspondences are summarized in Table 1.

Consider the following two conditions:

- **(T)** The tautological bundle $\bigoplus_v \mathcal{L}_v$ on the moduli space $\mathcal{M}_\theta$ is a tilting object.
- **(E)** The universal morphism $\mathbb{C}\Gamma \to \operatorname{End}\big(\bigoplus_v \mathcal{L}_v\big)$ is an isomorphism.

According to Morita theory for derived categories (see Bondal [5] and Rickard [32]), the conditions **(T)** and **(E)** imply that the functor

$$\Phi(-) = \mathbb{R}\Gamma\left(\left(\bigoplus_v \mathcal{L}_v\right) \otimes -\right)\colon D^b\operatorname{coh}\mathcal{M}_\theta \to D^b\operatorname{mod}\mathbb{C}\Gamma \tag{1-1}$$

is an equivalence of triangulated categories.

There is a notion of *consistency condition* on a dimer model (see Hanany and Vegh [17], Ishii and Ueda [21] and Bocklandt [4]), which ensures the Calabi–Yau property of the path algebra $\mathbb{C}\Gamma$ of the quiver with relations associated with the dimer model (see Mozgovoy and Reineke [29], Davison [12] and Broomhead [9]). An example of a consistent dimer model comes from a finite abelian subgroup A of $\mathrm{SL}(3, \mathbb{C})$, where the associated quiver is the McKay quiver.

The Calabi–Yau property of the path algebra $\mathbb{C}\Gamma$ implies that $\mathcal{M}_\theta$ is smooth and its canonical bundle is trivial, and the tautological bundle satisfies conditions **(T)** and **(E)**

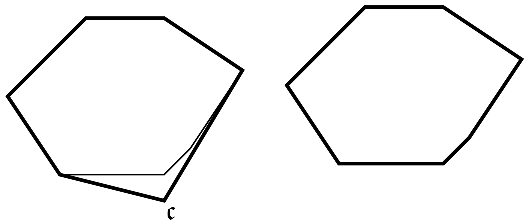

Figure 1: Left: the polygon Δ. Right: the polygon Δ'.

by a result of Bridgeland, King and Reid [8]; see also Van Den Bergh [2]. We do not rely on their results, and give an independent proof of these facts for any consistent dimer model.

In this paper, we study the behavior of a dimer model under the removal of a corner from the characteristic polygon. To make this precise, define a *corner* of a lattice polygon Δ to be an extremal point of Δ, and a *side* of Δ to be the interval between two neighboring corners. A side is divided into *primitive side segments*, defined as intervals between adjacent lattice points on the boundary of Δ. We reserve the words *edge* and *vertex* for an edge of a dimer model and a vertex of a quiver, respectively.

Theorem 1.1 *Let G be a consistent dimer model and Δ be the characteristic polygon of G. Let further $\mathfrak{c}$ be a corner of Δ and Δ' be the convex hull of the set of lattice points of Δ other than $\mathfrak{c}$. Assume that Δ' is not contained in a line. Then there is an explicit algorithm to remove some of the edges from G and produce another dimer model G' satisfying the following two conditions:*

(1) *G' is consistent.*

(2) *The characteristic polygon of G' coincides with Δ'.*

Theorem 1.1 is a combination of Algorithm 10.1 and Propositions 11.1 and 12.2. An example of a polygon Δ, a corner $\mathfrak{c}$ of Δ, and the polygon Δ' obtained from Δ by removing the corner $\mathfrak{c}$ is shown in Figure 1. This refines an operation of Gulotta [16] who studied the operation of removing a triangle from the characteristic polygon, in the sense that the removed part in our operation is smaller than or equal to the removed part in Gulotta's operation. These two operations are identical if the removed parts are equal. It is not clear if successive operations of our algorithm gives the same result as Gulotta's operation.

Since

- any lattice polygon can be embedded into a sufficiently large lattice triangle, and
- the McKay quiver gives a consistent dimer model for any lattice triangle,

Theorem 1.1 gives a constructive proof of the following:

Corollary 1.2 *For any lattice polygon Δ, there is a consistent dimer model whose characteristic polygon coincides with Δ.*

Corollary 1.2 also follows from a result of Gulotta [16, Theorem 6.1] which produces a properly ordered dimer model for any lattice polygon, and a result in Ishii and Ueda [21, Theorem 1.1] which shows that properly ordered dimer models are consistent.

Although the algorithm in Theorem 1.1 can be stated in a purely combinatorial way, its motivation comes from geometry of moduli spaces, where Wunram's special McKay correspondence [37] plays an essential role. Let A be a finite small subgroup of $\mathrm{GL}_2(\mathbb{C})$ and A–$\mathrm{Hilb}(\mathbb{C}^2)$ be the Hilbert scheme of A–orbits in $\mathbb{C}^2$ (see Nakamura [30]). The Hilbert–Chow morphism

$$\pi\colon A\text{–}\mathrm{Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/A = \operatorname{Spec} \mathbb{C}[x,y]^A$$

gives the minimal resolution of the quotient singularity; see Ishii [19]. The special McKay correspondence gives a description of the derived category of coherent sheaves on A–$\mathrm{Hilb}(\mathbb{C}^2)$ in terms of A; see Van Den Bergh [3], Craw [10] and Wemyss [35].

Let G be a consistent dimer model, Δ be its characteristic polygon, and G' be another consistent dimer model obtained from G by removing a corner $\mathfrak{c}$ from Δ as in Theorem 1.1. Let further $\mathcal{M}_\theta$ be the moduli space of the quiver Γ with relations associated with the consistent dimer model G and a generic stability parameter θ. Since $\mathcal{M}_\theta$ is a smooth toric variety and Δ is the convex hull of primitive generators of one-dimensional cones of the corresponding fan, any lattice point of Δ corresponds to a divisor in $\mathcal{M}_\theta$. A toric divisor $D_\mathfrak{c}$ of $\mathcal{M}_\theta$ corresponding to a corner $\mathfrak{c}$ of Δ will be called a *corner toric divisor*.

Proposition 1.3 *Let G be a consistent dimer model and $\mathfrak{c}$ be a corner of the characteristic polygon Δ. Then there is a generic stability parameter θ and a finite small abelian subgroup A of $\mathrm{GL}_2(\mathbb{C})$ satisfying the following:*

- *There is an open neighborhood $U_\mathfrak{c}$ of the corner toric divisor $D_\mathfrak{c}$ in $\mathcal{M}_\theta$ and a commutative diagram*

$$\begin{tikzcd} D_\mathfrak{c} \arrow[r] \arrow[d] & U_\mathfrak{c} \arrow[d, "\varphi"] \\ A\text{–}\mathrm{Hilb}(\mathbb{C}^2) \arrow[r] & A\text{–}\mathrm{Hilb}(\mathbb{C}^3) \end{tikzcd}$$

  where horizontal arrows are closed embeddings and vertical arrows are isomorphisms.

- *For any irreducible representation ρ of A, there is a vertex v of the quiver Γ such that the pull-back of the tautological bundle $\mathcal{L}_\rho$ on A–$\mathrm{Hilb}(\mathbb{C}^3)$ is isomorphic to the restriction of $\mathcal{L}_v$ on $\mathcal{M}_\theta$:*

$$\varphi^*\mathcal{L}_\rho \cong \mathcal{L}_v|_{U_{\mathfrak{c}}}. \tag{1-2}$$

Here $A \subset \mathrm{GL}_2(\mathbb{C})$ is embedded into $\mathrm{SL}_3(\mathbb{C})$ in a natural way.

The proof of Proposition 1.3 is given in Section 9. To prove Proposition 1.3, we introduce the notion of *large hexagons*. A large hexagon is the union of faces of a dimer model, which is cut out by a pair of zigzag paths. The tautological line bundles corresponding to faces of one large hexagon are isomorphic near the given corner divisor. A division of a dimer model into large hexagons gives a coarse graining of the associated quiver into the McKay quiver for some $A \subset \mathrm{GL}_2(\mathbb{C})$. The correspondence between combinatorics of dimer models and geometry of moduli spaces is summarized in Table 2.

An example of a lattice polygon Δ and a triangulation associated with a stability parameter θ as in Proposition 1.3 is shown in Figure 2. The sub-fan of Δ corresponding to A–$\mathrm{Hilb}(\mathbb{C}^3)$ is shown in green. The normal fan to the one-dimensional cone corresponding to the corner $\mathfrak{c}$ describes A–$\mathrm{Hilb}(\mathbb{C}^2)$. The relation between the inclusion of A–$\mathrm{Hilb}(\mathbb{C}^2)$ as a divisor in A–$\mathrm{Hilb}(\mathbb{C}^3)$ and the special McKay correspondence is explained in Section 3.

The main result in this paper is the following:

Theorem 1.4 *Let G be a consistent dimer model. Then for any generic stability parameter θ, the tautological bundle $\bigoplus_v \mathcal{L}_v$ on the moduli space $\mathcal{M}_\theta$ satisfies the conditions* (**T**) *and* (**E**).

The proof of Theorem 1.4 is given in Section 20. Theorem 1.4 contains the abelian case of the main result of Bridgeland, King and Reid [8]. Our proof is independent of theirs, and based on Theorem 1.5 below.

Let G be a consistent dimer model and G' be another consistent dimer model obtained from G by removing a corner from the characteristic polygon as in Theorem 1.1.

Dimer model	Characteristic polygon	Moduli space
perfect matching	lattice point	toric divisor
zigzag path	primitive side segment	non-compact torus-invariant curve

Table 2: The correspondence

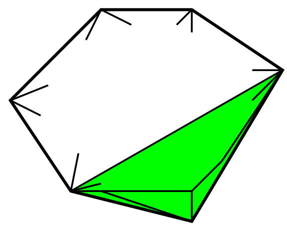

Figure 2: A triangulation of Δ

Choose a stability parameter θ for G described in Proposition 1.3. This stability parameter θ for G naturally induces a stability parameter θ' for G', and let $\mathcal{M}'_{\theta'}$ be the corresponding moduli space associated with the dimer model G'. Then $\mathcal{M}'_{\theta'}$ is naturally an open subscheme of $\mathcal{M}_\theta$, and the complement is exactly the divisor $D_{\mathfrak{c}}$:

$$\mathcal{M}'_{\theta'} = \mathcal{M}_\theta \setminus D_{\mathfrak{c}}.$$

A key to the proof of Theorem 1.4 is the following:

Theorem 1.5 *The conditions* (**T**) *and* (**E**) *hold for $\mathcal{M}_\theta$ if and only if they hold for $\mathcal{M}'_{\theta'}$.*

Theorem 1.5 is a combination of Propositions 14.1, 18.1, and 19.1. Their proofs are based on a detailed study of the interplay between combinatorics of dimer models and geometry of moduli spaces. Proposition 1.3 is an important step in reducing both Theorem 1.1 and Theorem 1.5 to the case where $\mathcal{M}_\theta = A$–Hilb$(\mathbb{C}^3)$. In the proof of Theorem 1.5, the special McKay correspondence plays an essential role again.

The proof of Theorem 1.4 also gives the following characterization of the edges removed in the operation in Theorem 1.1, which explains the geometric origin of the algorithm:

Proposition 1.6 *The edges removed from G in the operation in Theorem 1.1 are exactly those which correspond to morphisms between tautological bundles vanishing only on the toric divisor $D_{\mathfrak{c}} \subset \mathcal{M}_\theta$.*

Proposition 1.6 is proved in Section 13. The effect of the operation in Theorem 1.1 on various objects is summarized in Table 3.

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Object	Operation
characteristic polygon	removing a corner $\mathfrak{c} \in \Delta$
moduli space	removing the toric divisor $D_{\mathfrak{c}} \subset \mathcal{M}_\theta$
path algebra	inverting the arrows vanishing only on $D_{\mathfrak{c}}$
quiver	contracting the arrows as above
dimer model	removing the edges dual to the arrows as above

Table 3: The effect of the operation in Theorem 1.1

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2 The special McKay correspondence

Let $R := S^A$ be the invariant ring of the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ with respect to the natural action of a finite small subgroup A of $\mathrm{GL}_n(\mathbb{C})$. For any irreducible representation ρ of A, the invariant part $M_\rho := (S \otimes \rho^\vee)^A$ is an indecomposable Cohen–Macaulay (and hence reflexive) R–module, since it a direct summand of a Cohen–Macaulay R–module $S \otimes \rho^\vee$.

The *McKay quiver* Λ of A is a quiver with relations whose set of vertices is the set $\mathrm{Irrep}(A)$ of irreducible representations of A. The number $a_{\nu\mu}$ of arrows from a vertex $\mu \in \mathrm{Irrep}(A)$ to another vertex $\nu \in \mathrm{Irrep}(A)$ is given by the multiplicity in the irreducible decomposition of the tensor product

$$\mu \otimes \rho_{\mathrm{Nat}}^\vee = \bigoplus_{\nu \in \mathrm{Irrep}(A)} \nu^{\oplus a_{\nu\mu}},$$

where $\rho_{\mathrm{Nat}} \colon A \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is the natural representation of A and $(-)^\vee$ denotes the dual representation. The relations of Λ are such that the path algebra $\mathbb{C}\Lambda$ is isomorphic to $\mathrm{End}_R\big(\bigoplus_{\rho \in \mathrm{Irrep}(A)} M_\rho\big)$, which is Morita equivalent to

$$\mathrm{End}_R(S) \cong \mathrm{End}_R\Bigg(\bigoplus_{\rho \in \mathrm{Irrep}(A)} M_\rho^{\oplus \dim \rho}\Bigg) \cong S \rtimes A.$$

Now assume that A is a finite small subgroup of $\mathrm{GL}_2(\mathbb{C})$, and let $Y = A\text{–Hilb}(\mathbb{C}^2)$ be the Hilbert scheme of A–orbits in $\mathbb{C}^2$ [30]. The Hilbert–Chow morphism

$$\pi \colon Y \to X = \operatorname{Spec} \mathbb{C}[x, y]^A$$

gives the minimal resolution of the quotient singularity [19].

Definition-Lemma 2.1 (Esnault [13]) *Let $\mathcal{M}$ be a sheaf on Y and $\mathcal{M}^\vee$ be its dual sheaf. Then there exists a reflexive module M on X such that $\mathcal{M} \cong \widetilde{M} := \pi^* M/\mathit{torsion}$ if and only if the following three conditions are satisfied:*

(1) *$\mathcal{M}$ is locally free.*

(2) *$\mathcal{M}$ is generated by global sections.*

(3) *$H^1((\mathcal{M})^\vee \otimes \omega_Y) = 0$.*

In this case $\mathcal{M}$ is said to be full.

Let us recall the definition of a tilting object:

Definition 2.2 An object $\mathcal{E}$ in a triangulated category $\mathcal{T}$ is *acyclic* if

$$\operatorname{Ext}^k(\mathcal{E}, \mathcal{E}) = 0, \quad k \neq 0.$$

It is a *generator* if, for any object $\mathcal{F}$,

$$\operatorname{Ext}^k(\mathcal{E}, \mathcal{F}) = 0 \quad \text{for all } k \in \mathbb{Z}$$

implies that $\mathcal{F} \cong 0$. An acyclic generator is called a *tilting object*.

A tilting object induces a derived equivalence:

Theorem 2.3 (Bondal [5], Rickard [32]) *Let $\mathcal{E}$ be a tilting object in the derived category $D^b \operatorname{coh} X$ of coherent sheaves on a smooth quasi-projective variety X. Then $D^b \operatorname{coh} X$ is equivalent to the derived category of finitely generated modules over the endomorphism algebra $\operatorname{Hom}(\mathcal{E}, \mathcal{E})$.*

The following theorem is the McKay correspondence as a derived equivalence for a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$:

Theorem 2.4 (Kapranov and Vasserot [25]; see also Bridgeland, King and Reid [8]) *When A is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, the direct sum of indecomposable full sheaves is a tilting object whose endomorphism ring is Morita equivalent to the crossed product algebra $\mathbb{C}[x, y] \rtimes A$.*

This is no longer true when $A \not\subset \mathrm{SL}_2(\mathbb{C})$, and one has to restrict the class of full sheaves. The following theorem is due to Wunram:

Theorem 2.5 [37, Main result] *Let $C = \bigcup_{i=1}^{r} C_i$ be the decomposition of the exceptional set C into irreducible components. Then for every curve C_i there exists exactly one indecomposable reflexive module M_i such that the corresponding full sheaf $\widetilde{M}_i = \pi^* M_i/torsion$ satisfies the conditions $H^1((\widetilde{M})^\vee) = 0$ and*

$$c_1(\widetilde{M}_i) \cdot C_j = \delta_{ij}.$$

A full sheaf is said to be *special* if there is an index $1 \le i \le r$ such that $\mathcal{M} = \mathcal{M}_i$ or it is isomorphic to the structure sheaf $\mathcal{O}_Y$. The special full sheaf $\mathcal{O}_Y$ corresponds to the trivial representation and is denoted by $\mathcal{M}_0$. Special full sheaves are characterized as follows:

Theorem 2.6 (Wunram [37, Theorem 1.2]) *An indecomposable full sheaf $\mathcal{M}$ is special if and only if $H^1(\mathcal{M}^\vee) = 0$.*

By Definition-Lemma 2.1, this condition is equivalent to the following:

Corollary 2.7 *An indecomposable full sheaf $\mathcal{M}$ is special if and only $\mathcal{M} \otimes \omega_Y$ is a full sheaf.*

An irreducible representation ρ of A is said to be special if the corresponding full sheaf

$$\mathcal{M}_\rho = \pi^*((\rho^\vee \otimes \mathbb{C}[x, y])^A)/\text{torsion}$$

is special.

Special full sheaves generate the derived category of coherent sheaves on Y:

Theorem 2.8 (Van den Bergh [3, Theorem B]) *The direct sum of indecomposable special full sheaves is a tilting object.*

Let $\mathcal{M}$ be the direct sum of indecomposable special full sheaves. It follows that the derived category $D^b \operatorname{coh} Y$ of coherent sheaves on Y is equivalent to the derived category $D^b \operatorname{mod}(\operatorname{End} \mathcal{M})$ of finitely generated right modules over $\operatorname{End} \mathcal{M}$. The special McKay correspondence as a derived equivalence is studied by Craw [10] and Wemyss [35]. The category $D^b \operatorname{coh} Y \cong D^b \operatorname{mod}(\operatorname{End} \mathcal{M})$ is an admissible subcategory of $D^b \operatorname{coh}[\mathbb{C}^2/A] \cong D^b \operatorname{mod}(\mathbb{C}[x, y] \rtimes A)$, whose semiorthogonal complement is generated by an exceptional collection [23].

3 Zero locus of the "multiplication by z" map

In this section, we explain why the special McKay correspondence should appear in the algorithm of Theorem 1.1. We include this section for motivational purposes only, and other parts of this paper are logically independent of this section. Unlike other parts of this paper, we use the results of [8] in this section.

Let A be a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$, which may not be abelian. Then we have a minimal resolution

$$Y = A\text{–Hilb}(\mathbb{C}^2) \to \mathbb{C}^2/A$$

as in the previous section. We can embed $\mathrm{GL}(2,\mathbb{C})$ into $\mathrm{GL}(3,\mathbb{C})$ by the map

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & \det(X)^{-1} \end{pmatrix},$$

and A becomes a subgroup of $\mathrm{SL}(3,\mathbb{C})$. Thus Y is embedded into

$$U = A\text{–Hilb}(\mathbb{C}^3),$$

which is a crepant resolution of $\mathbb{C}^3/A$ by [8]. The structure sheaf of the universal subscheme, if pushed forward to U, is decomposed as $\bigoplus_\rho \mathcal{R}_\rho \otimes \rho$, where $\mathcal{R}_\rho$ is the *tautological bundle* associated with an irreducible representation ρ of A. The restriction $\mathcal{R}_\rho|_Y$ of $\mathcal{R}_\rho$ to Y coincides with the full sheaf $\mathcal{M}_\rho$ associated with M_ρ.

Let z be the third coordinate of $\mathbb{C}^3$, so that $\mathbb{C}^2 \subset \mathbb{C}^3$ is defined by $z = 0$. Let ρ_{Nat} be the two-dimensional representation of A determined by the original embedding $A \subset \mathrm{GL}(2,\mathbb{C})$. Then multiplication by z induces a map

$$z_\rho\colon \mathcal{R}_\rho \to \mathcal{R}_{\rho\otimes\det\rho_{\mathrm{Nat}}}.$$

Let Z_ρ be the support of $\operatorname{coker} z_\rho \subset U$, which contains Y for any ρ.

Proposition 3.1 *$Z_\rho = Y$ if and only if ρ is special.*

Sketch of proof Note first that Z_ρ is a divisor since z_ρ is an injection of locally free sheaves of the same rank. The map z_ρ is zero at any point on Y and therefore it determines a map

$$z'_\rho\colon \mathcal{R}_\rho(Y) \to \mathcal{R}_{\rho\otimes\det\rho_{\mathrm{Nat}}}.$$

If Z'_ρ denotes the support of $\operatorname{coker} z'_\rho$, then we see $Z_\rho = Y \cup Z'_\rho$ and $Y \not\subset Z'_\rho$.

Assume $Z_\rho = Y$. Then Z'_ρ is empty and z'_ρ is an isomorphism. Then, by restricting it to Y, we see that $\mathcal{M}_\rho \otimes \mathcal{O}_Y(Y)$ is a full sheaf. Since U is a crepant resolution, $\mathcal{O}_Y(Y)$ is isomorphic to ω_Y and therefore ρ is special by Corollary 2.7.

Conversely, suppose ρ is special. Then by Corollary 2.7, we see $Y \cap Z'_\rho = \varnothing$. Since $\operatorname{coker} z_\rho$ corresponds via the Fourier–Mukai transform to a locally free sheaf on $\mathbb{C}^2$ which is indecomposable [8], Z_ρ must be connected. This implies that $Z'_\rho = \varnothing$. □

This proposition, together with Corollary 15.3, clearly explains the role of the special McKay correspondence in this paper. Namely, the restrictions of non-special tautological bundles form a tilting bundle on $U \setminus Y$, and to obtain the endomorphism algebra of the tilting bundle on $U \setminus Y$, the algebra for U should be "localized" by the maps z_ρ for the special representations ρ.

4 Specials and continued fractions

For relatively prime integers $0<q<n$, consider the small cyclic subgroup $A=\left\langle \frac{1}{n}(1,q)\right\rangle$ of $\mathrm{GL}_2(\mathbb{C})$ generated by

$$\frac{1}{n}(1,q)=\begin{pmatrix}\zeta & 0\\ 0 & \zeta^q\end{pmatrix},$$

where ζ is a primitive n^{th} root of unity. We label the irreducible representations of A by elements $a \in \mathbb{Z}/n\mathbb{Z}$ so that a sends the above generator to ζ^{-a}.

Remark 4.1 $\mathcal{M}_\rho$ in our notation corresponds to $\rho^\vee$ via the correspondence in [37]. So we dualize the labeling of the irreducible representations so that Theorem 4.2 is of the same form.

Define integers r, $b_1,\dots,b_r$ and $i_0,\dots,i_{r+1}$ as follows: put $i_0 := n$, $i_1 := q$ and define i_{t+2} and b_{t+1} inductively by

$$i_t = b_{t+1} i_{t+1} - i_{t+2} \quad (0 < i_{t+2} < i_{t+1}) \tag{4-1}$$

until we finally obtain $i_r = 1$ and $i_{r+1} = 0$. This gives a continued fraction expansion

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}} \tag{4-2}$$

and $-b_t$ is the self-intersection number of the t^{th} irreducible exceptional curve C_t.

For a general representation d, the degrees of the full sheaf $\mathcal{L}_d$ are given in the following way:

Theorem 4.2 (Wunram [36, Theorem]) *For an integer d with $0 \le d < n$, there is a unique expression*

$$d = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r,$$

where $d_i \in \mathbb{Z}_{\ge 0}$ are non-negative integers satisfying

$$0 \le \sum_{t > t_0} d_t i_t < i_{t_0}$$

for any t_0. Then one has

$$\deg \mathcal{M}_d|_{C_t} = d_t$$

for any $t = 1, \dots, r$.

Remark 4.3 The non-negative integers d_i in Theorem 4.2 can be computed by setting $e_0 = d$ and

$$e_t = d_{t+1} i_{t+1} + e_{t+1}, \quad 0 \le e_{t+1} < i_{t+1}$$

for $t = 0, \dots, r-1$.

Corollary 4.4 *Special representations are given by $i_0 \equiv i_{r+1}, i_1, \dots, i_r$, and the labeling of specials and irreducible components are related by*

$$\deg \mathcal{M}_{i_s}|_{C_t} = \delta_{st}.$$

Lemma 4.5 (Wunram [36, Lemma 1]) *A sequence $(d_1, \dots, d_r) \in (\mathbb{Z}_{\ge 0})^r$ is obtained from an integer $d \in [0, n-1]$ as in the previous theorem if and only if the following hold:*

- *$0 \le d_t \le b_t - 1$ for any t.*
- *If $d_s = b_s - 1$ and $d_t = b_t - 1$ for $s < t$, then there is an l with $s < l < t$ and $d_l \le b_l - 3$.*

Define the dual sequence $j_0, \dots, j_{r+1}$ by $j_0 = 0$, $j_1 = 1$, and

$$j_t = j_{t-1} b_{t-1} - j_{t-2}, \quad t \ge 2.$$

Then one has $j_{r+1} = n$.

Lemma 4.6 (Wunram [36, Lemma 2]) *Let $d = d_1 i_1 + \cdots + d_r i_r$ be as in Theorem 4.2, and put $f = d_1 j_1 + \cdots + d_r j_r$. Then one has $qf \equiv d \mod n$.*

In particular, special representations are given by

$$i_0 \equiv q j_0, \quad i_1 \equiv q j_1, \quad \dots, \quad i_r \equiv q j_r. \tag{4-3}$$

Note that $(i_t)_{t=0}^r$ is decreasing and $(j_t)_{t=0}^r$ is increasing.

Remark 4.7 A geometric interpretation of these numbers can be found in Lemma 16.2.

5 Dimer models and quivers

5.1 Dimer models

By a *graph*, we mean an abstract, unoriented graph, possibly with multiple edges and loops. To be more precise, a graph is a triple (N, E, ∂) consisting of

- a set N of nodes,
- a set E of edges, and
- the incidence relation $\partial\colon E \to N^{(2)}$, which is a map from E to the symmetric product $N^{(2)} = N^2/\mathfrak{S}_2$.

A graph is *bipartite* if one can divide the set N of nodes into the disjoint union of

- a set $B \subset N$ of black nodes, and
- a set $W \subset N$ of white nodes, so that
- no edge connects nodes with the same color.

A *bicolored graph* is a bipartite graph with a fixed choice of a coloring.

To a graph (N, E, ∂), one can associate a one-dimensional CW complex whose 0–cells and 1–cells correspond to nodes and edges, respectively. An *embedding* of a graph into a topological space T is a continuous injection from this CW complex to T. When a graph is embedded in a topological space, we often identify nodes and edges with their images under the embedding.

Let T be a real 2–torus. We fix an identification $T = \mathbb{R}^2/\mathbb{Z}^2$, which gives identifications $H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$ and $H^1(T, \mathbb{Z}) \cong \mathbb{Z}^2$. We equip T with the orientation coming from the standard orientation on $\mathbb{R}^2$.

A *dimer model* is a finite bicolored graph $G = (B, W, E)$ embedded in T such that

- G has no univalent node, and
- any connected component of the complement $T \setminus \bigcup_{e \in E} e$ of the graph is simply connected.

5.2 Perfect matchings and characteristic polygons

A *perfect matching* (or a *dimer configuration*) on a graph (N, E, ∂) is a subset D of E such that for any node $n \in N$ there is a unique edge $e \in D$ incident to n. A dimer model is said to be *non-degenerate* if for any edge $e \in E$ there is a perfect matching D such that $e \in D$.

Let $G = (B, W, E)$ be a dimer model, and consider the bicolored graph $\widetilde{G}$ on $\mathbb{R}^2$ obtained from G by pulling back to the universal cover $\mathbb{R}^2 \to T$. The set of perfect matchings on G is naturally identified with the set of periodic perfect matchings on the infinite graph $\widetilde{G}$ on the universal cover. Fix a perfect matching D_0 called the *reference matching*. For any perfect matching D, the union $D \cup D_0$ divides $\mathbb{R}^2$ into connected components. The *height function* h_{D,D_0} is a locally constant function on $\mathbb{R}^2 \setminus (D \cup D_0)$ which increases by 1 when one crosses an edge $e \in D$ with the black node on the right or an edge $e \in D_0$ with the white node on the right, and decreases by 1 when one crosses an edge $e \in D$ with the white node on the right or an edge $e \in D_0$ with the black node on the right. This rule determines the height function up to an addition of a constant. The height function may not be periodic even if D and D_0 are periodic, and the *height change* $h(D, D_0) = (h_x(D, D_0), h_y(D, D_0)) \in \mathbb{Z}^2$ of D with respect to D_0 is defined by the differences

$$h_x(D, D_0) = h_{D,D_0}(p + (1,0)) - h_{D,D_0}(p),$$
$$h_y(D, D_0) = h_{D,D_0}(p + (0,1)) - h_{D,D_0}(p)$$

of the height function, which does not depend on the choice of $p \in \mathbb{R}^2 \setminus (D \cup D_0)$. More invariantly, height changes can be considered as an element of $H^1(T, \mathbb{Z})$. The dependence of the height change on the choice of the reference matching is given by

$$h(D, D_1) = h(D, D_0) - h(D_1, D_0)$$

for any three perfect matchings D, D_0 and D_1. We often suppress the dependence of the height difference on the reference matching and just write $h(D) = h(D, D_0)$.

For a fixed reference matching D_0, the characteristic polynomial of G is defined by

$$Z(x, y) = \sum_{D \in \mathrm{Perf}(G)} x^{h_x(D)} y^{h_y(D)},$$

where $\mathrm{Perf}(G)$ is the set of perfect matchings on G. The characteristic polynomial is a Laurent polynomial in two variables, whose Newton polygon gives the *characteristic polygon*, defined as the convex hull

$$\Delta = \mathrm{Conv}\{(h_x(D), h_y(D)) \in \mathbb{Z}^2 \mid D \text{ is a perfect matching on } G\}$$

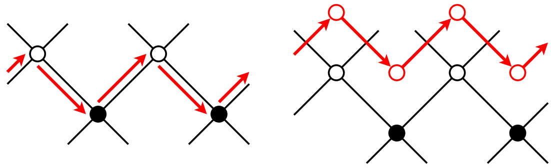

Figure 3: Left: a zigzag path. Right: a path on the quiver along a zigzag path.

of the set of height changes of perfect matchings on the dimer model.

A *corner* of Δ is an extremal point of Δ, and a *side* of Δ is the interval between two neighboring corners. A side is divided into *primitive side segments*, defined as intervals between two adjacent lattice points on the boundary of Δ. A perfect matching D is said to be a *corner perfect matching* if its height change $h(D)$ is on the corner of the characteristic polygon. The *multiplicity* of a perfect matching D is the number of perfect matchings whose height changes are the same as D.

5.3 Zigzag paths and their slopes

A *zigzag path* is a path on a bicolored graph in an oriented surface which makes a maximum turn to the right on a white node and a maximal turn to the left on a black node. We assume that a zigzag path does not have an endpoint, so that it is either periodic or infinite in both directions. Here, the latter can happen only if the graph is infinite. Figure 3, left, shows an example of a part of a dimer model and a zigzag path on it.

Let z be a zigzag path on a dimer model, and assume that there is a perfect matching D_0 which intersects half of the edges constituting z (ie every other edge of z belongs to D_0). Then the height change of any other perfect matching D with respect to D_0 in the direction of z is negative:

$$\langle h(D, D_0), [z] \rangle \leq 0. \tag{5-1}$$

Here, $[z] \in H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$ is the homology class of $[z]$, which is paired with the height change considered as an element of $H^1(T, \mathbb{Z})$. To show this, replace z by the path p on the quiver going along z (on the left side of z), which belongs to the class $[z]$ as shown in Figure 3, right. Then (5-1) follows from the fact that, as one goes around T along p, one crosses no edge in D_0 and every edge one crosses has a white node on the right. In this way, such a zigzag path gives an inequality which bounds the Newton polygon of the characteristic polynomial.

The homology class $[z] = (u, v) \in H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$ of a zigzag path z considered as an element of $\mathbb{Z}^2$ will be called its *slope*. If a zigzag path does not have a self-intersection, then $(u, v) \in \mathbb{Z}^2$ is a primitive element, and we sometimes think of the slope as an element

$$\frac{(u, v)}{\sqrt{u^2 + v^2}} \in S^1$$

of the unit circle. The set of slopes has the natural counterclockwise cyclic order as a subset of the unit circle.

5.4 Quivers

A *quiver* is an oriented graph, which is a quadruple (V, A, s, t) consisting of

- a set V of vertices,
- a set A of arrows, and
- two maps $s, t\colon A \to V$ from A to V.

For an arrow $a \in A$, the vertices $s(a)$ and $t(a)$ are called the *source* and the *target* of a, respectively.

A *path* on a quiver is an ordered set of arrows $(a_n, a_{n-1}, \dots, a_1)$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. We also allow for a path of length zero, starting and ending at the same vertex.

The *path algebra* $\mathbb{C}Q$ of a quiver $Q = (V, A, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths:

$$(b_m, \dots, b_1) \cdot (a_n, \dots, a_1) = \begin{cases} (b_m, \dots, b_1, a_n, \dots, a_1) & \text{if } s(b_1) = t(a_n), \\ 0 & \text{otherwise.} \end{cases}$$

A *quiver with relations* is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q/\mathcal{I}$.

5.5 A quiver with relations associated with a dimer model

A dimer model (B, W, E) encodes the information of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations in the following way: The set V of vertices is the set of connected components of the complement $T \setminus (\bigcup_{e \in E} e)$, and the set A of arrows is the set E of edges of the graph. The orientations of the arrows are determined by the colors of the nodes of the graph, so that the white node $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by $-90°$.

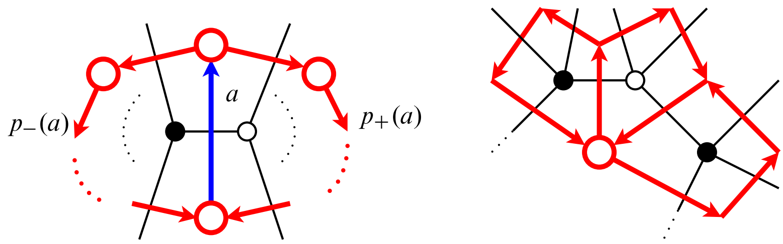

Figure 4: Left: relations on the quiver. Right: small cycles.

The relations of the quiver are described as follows: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from $t(a)$ to $s(a)$, the former going around the white node incident to $a \in E = A$ clockwise, and the latter going around the black node incident to a counterclockwise, as shown in Figure 4, left. Then the ideal $\mathcal{I}$ of the path algebra is generated by $p_+(a) - p_-(a)$ for all $a \in A$.

5.6 Small cycles, minimal paths and weak equivalence

A *small cycle* on a quiver associated with a dimer model is a path obtained as the product of arrows surrounding a node of the dimer model. Three small cycles are shown in Figure 4, right. A path p is said to be *minimal* if it is not equivalent to a path containing a small cycle.

Note that small cycles starting from a fixed vertex are equivalent to each other. It follows that the sum $\omega := \sum_{v \in V} \omega_v$ of small cycles over the set of vertices, where one picks one small cycle ω_v for each vertex v, is a well-defined element of the path algebra independent of the choice of ω_v. One can easily see that the element ω belongs to the center of the path algebra, and there is the universal map

$$\mathbb{C}\Gamma \to \mathbb{C}\Gamma[\omega^{-1}]$$

into the localization of the path algebra by the multiplicative subset generated by ω. Two paths are called *weakly equivalent* if they give the same element in $\mathbb{C}\Gamma[\omega^{-1}]$.

Suppose that there is a perfect matching D. Note that every small cycle contains exactly one arrow in D. Then [21, Lemma 2.1] implies that two paths with the same source and the target are weakly equivalent if and only if they have the same homology class and they contain the same number of arrows in D.

5.7 Moduli space of quiver representations

A *representation* of a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations is a module over the path algebra $\mathbb{C}\Gamma$. In other words, a representation of Γ is a collection $((V_v)_{v \in V}, (\psi_a)_{a \in A})$

of vector spaces V_v for $v \in V$ and linear maps $\psi_a\colon V_{s(a)} \to V_{t(a)}$ for $a \in A$ satisfying relations in $\mathcal{I}$. The *dimension vector* of a representation $((V_v)_{v\in V}, (\psi_a)_{a\in A})$ is given by $(\dim V_v)_{v\in V} \in \mathbb{Z}^V$. This allows us to think of $\mathbb{Z}^V$ as a quotient of the Grothendieck group of the abelian category of finite-dimensional representations of Γ. The *support* of a representation is the set of vertices $v \in V$ such that $\dim V_v \neq 0$.

A *stability parameter* θ is an element of $\operatorname{Hom}(\mathbb{Z}^V, \mathbb{Z})$. A $\mathbb{C}\Gamma$–module M is said to be θ*–stable* if $\theta(M) = 0$ and if for any non-trivial submodule $N \subsetneq M$ one has $\theta(N) > \theta(M)$. The module M is θ*–semistable* if $\theta(N) \geq \theta(M)$ holds instead of $\theta(N) > \theta(M)$. A stability parameter θ is said to be *generic* with respect to a fixed dimension vector if semistability implies stability. This stability condition was introduced by King [28] to construct the moduli space $\mathcal{M}_\theta$ representing (the sheafification of) the functor

$$(\mathcal{S}ch) \to (\mathcal{S}et), \quad T \mapsto (\text{a flat family over } T \text{ of } \theta\text{–stable representations of } \Gamma)/\sim$$

for a fixed dimension vector. Here, a *flat family of representations* of Γ over T is a collection $(\mathcal{L}_v)_{v\in V}$ of vector bundles on T for each vertex v of Γ and a collection $(\phi_a)_{a\in A}$ of morphisms $\phi_a\colon \mathcal{L}_{s(a)} \to \mathcal{L}_{t(a)}$ for each arrow a of Γ satisfying the relations $\mathcal{I}$ of Γ. Two families are defined to be equivalent if they are isomorphic up to tensor product $\mathcal{L}_v \mapsto \mathcal{L}_v \otimes \mathcal{L}$ by some line bundle $\mathcal{L}$ simultaneously for all vertices $v \in V$. If the dimension vector is a primitive vector, then we do not have to sheafify the functor, and there is a universal family over the moduli space. The bundles $\mathcal{L}_v$ in the universal family are called the *tautological bundles*. In the rest of this paper, $\mathcal{M}_\theta$ denotes the moduli space of θ–stable $\mathbb{C}\Gamma$–modules for the dimension vector $(1, 1, \ldots, 1)$. On the other hand, the moduli space $\overline{\mathcal{M}}_\theta$ of θ–semistable modules does not represent the moduli functor, but parametrizes S-equivalence classes of θ–semistable modules.

5.8 Perfect matchings and moduli spaces

The main theorem of [20] states that when a dimer model is non-degenerate the moduli space $\mathcal{M}_\theta$ is a smooth Calabi–Yau toric 3–fold for generic θ.

Lemma 5.1 *Let G be a non-degenerate dimer model. Then, for each arrow a of the associated quiver, the zero locus of $\phi_a\colon \mathcal{L}_{s(a)} \to \mathcal{L}_{t(a)}$ is a reduced subscheme of $\mathcal{M}_\theta$.*

Proof The proof of [20, Proposition 5.1] shows that the moduli space $\mathcal{M}_\theta$ is covered by open neighborhoods U_Ψ of the torus-fixed points $[\Psi] \in \mathcal{M}_\theta$. The proof of [20, Lemma 4.5] gives a coordinate description $U_\Psi \cong \operatorname{Spec} \mathbb{C}[t_1, t_2, t_3]$ of each U_Ψ, together with the trivialization of tautological line bundles $\mathcal{L}_v$ such that the homomorphism ϕ_a

for any arrow a is given by either 1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, or $t_1t_2t_3$. It follows that $\phi_a^{-1}(0) \cap U_\Psi$ is reduced. $\square$

It is also proved in [20, Section 6] that a toric divisor in $\mathcal{M}_\theta$ gives a perfect matching in such a way that the stabilizer group of the divisor is given by the height change of the perfect matching.

A perfect matching can be considered as a set of walls which block some of the arrows: for a perfect matching D, let Q_D be the subquiver of Q whose set of vertices is the same as Q and whose set of arrows consists of $A \setminus D$ (recall that $A = E$). The path algebra $\mathbb{C}Q_D$ of Q_D is a subalgebra of $\mathbb{C}Q$, and the ideal $\mathcal{I}$ of $\mathbb{C}Q$ defines an ideal $\mathcal{I}_D = \mathcal{I} \cap \mathbb{C}Q_D$ of $\mathbb{C}Q_D$. A path $p \in \mathbb{C}Q$ is said to be an *allowed path* with respect to D if $p \in \mathbb{C}Q_D$.

With a perfect matching, one can associate a representation of the quiver with dimension vector $(1, \dots, 1)$ by sending any allowed path to 1 and other paths to 0. A perfect matching is said to be *simple* if this representation is simple, ie has no non-trivial subrepresentation. This is equivalent to the condition that there is an allowed path starting and ending at any given pair of vertices.

5.9 Quivers as categories

To a quiver Γ with relations one can associate a $\mathbb{C}$–linear category $\mathcal{C}$, as follows:

- The set of objects of $\mathcal{C}$ is the set of vertices of Γ.
- The space of morphisms between two objects v and w is the vector space $e_w \cdot \mathbb{C}\Gamma \cdot e_v$, where e_v and e_w are the idempotents of the path algebra corresponding to the vertices v and w of Γ.
- The composition of morphisms comes from the product in the path algebra.

In terms of the category $\mathcal{C}$, a representation of Γ is just a linear functor from $\mathcal{C}$ to the category of vector spaces.

The advantage of working with categories rather than path algebras is the following: Let v and w be two vertices in a quiver $\Gamma = (V, A, s, t, \mathcal{I})$ with relations and $\{a_1, \dots, a_r\}$ be any subset of the set of arrows of Γ from v to w. Then we can define another quiver $\Gamma' = (V', A', s', t', \mathcal{I}')$ by setting $V' = V \setminus \{v\}$, $A' = A \setminus \{a_1, \dots, a_r\}$, and

$$s'(a) = \begin{cases} s(a) & \text{if } s(a) \neq v, \\ w & \text{if } s(a) = v, \end{cases} \qquad t'(a) = \begin{cases} t(a) & \text{if } t(a) \neq v, \\ w & \text{if } t(a) = v. \end{cases}$$

The relations of Γ' are determined by the condition that $\mathbb{C}\Gamma'$ is Morita equivalent to the localization of $\mathbb{C}\Gamma$ at the arrows $a_1, \dots, a_r$. This means that Γ' is obtained from Γ by inverting the arrows $a_1, \dots, a_r$ and identifying two vertices v and w which become isomorphic after the inversion of the arrows. There is a natural map $\pi\colon \mathbb{C}\Gamma \to \mathbb{C}\Gamma'$ between path algebras, which is *not* an algebra homomorphism since

$$\pi(e_w) \circ \pi(e_v) = e_w \circ e_w = e_w \neq 0 = \pi(0) = \pi(e_w \circ e_v).$$

Nevertheless, the map π induces a functor $\varpi\colon \mathcal{C} \to \mathcal{C}'$ from the category $\mathcal{C}$ associated with Γ to the category $\mathcal{C}'$ associated with Γ'. Since a representation of Γ is a functor from $\mathcal{C}$ to the category of vector spaces, the functor ϖ induces a functor $\varpi^*\colon \operatorname{mod} \mathbb{C}\Gamma' \to \operatorname{mod} \mathbb{C}\Gamma$ between categories of representations. The image of the functor ϖ^* consists of representations $((V_v)_{v \in V}, (\psi_a)_{a \in A})$ such that $V_v = V_w$ and $\psi_{a_1} = \cdots = \psi_{a_r} = \mathrm{id}_{V_v}$.

5.10 McKay quiver and hexagonal dimer models

Let $\widetilde{\mathbb{T}} \subset \mathrm{GL}(3, \mathbb{C})$ be the subgroup consisting of diagonal matrices and put $\widetilde{\mathbb{T}}_0 = \widetilde{\mathbb{T}} \cap \mathrm{SL}(3, \mathbb{C})$. For a finite subgroup $A \subset \widetilde{\mathbb{T}}_0$, the character group $A^* = \mathrm{Hom}(A, \mathbb{C}^\times)$ is a quotient of $\widetilde{\mathbb{T}}_0^* \cong \mathbb{Z}^2$, and hence a quotient of $\widetilde{\mathbb{T}}^* \cong \mathbb{Z}^3$. Let $\rho_x, \rho_y, \rho_z \in A^*$ be the images of the coordinate functions $x, y, z \in \widetilde{\mathbb{T}}^*$, respectively. The *McKay quiver* for A has A^* as the set of vertices, and there are three arrows starting from each vertex ρ, whose targets are $\rho\rho_x$, $\rho\rho_y$ and $\rho\rho_z$, respectively. We say that these arrows *correspond to multiplications by* x, y, z, respectively. If M_0 denotes the kernel of the surjection $\widetilde{\mathbb{T}}_0^* \to A^*$, then the McKay quiver can be embedded in the torus $T = (\widetilde{\mathbb{T}}_0^* \otimes \mathbb{R})/M_0$, and comes from a hexagonal dimer model on T as in [31] (see also [34, Section 5] and an example in Figure 24 below). The corresponding path algebra with relations is isomorphic to the crossed product algebra $\mathbb{C}[x, y, z] \rtimes A$. The Hilbert scheme A–Hilb$(\mathbb{C}^3)$ of A–orbits, parametrizing *A–clusters*, is isomorphic to the moduli space $\mathcal{M}_\theta$ for this quiver with respect to a stability parameter θ such that $\theta(\rho) > 0$ for every non-trivial $\rho \in A^*$ (see eg [24, Section 3]).

6 Consistency conditions on dimer models

6.1 Divalent node

Let $G = (B, W, E)$ be a non-degenerate dimer model. For a divalent node $n \in B \sqcup W$, one can contract two nodes adjacent to n and obtain another dimer model $G' = (B', W', E')$ as shown in Figure 5. Note that the two nodes adjacent to n must be distinct since the dimer model is non-degenerate. The numbers of black nodes and white nodes are reduced by one, and the number of edges is reduced by two under this operation. If G' still has a divalent node, then one can continue this process until the

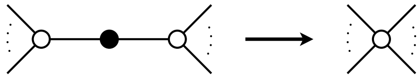

Figure 5: Removal of a divalent node

dimer model contains no divalent nodes. It is clear from the definition of the zigzag paths that there is a natural bijection between the sets of zigzag paths on dimer models before and after the removal of divalent nodes. It is also clear from the definition of the relations of the quiver associated with a dimer model that the isomorphism class of the path algebra does not change under the operation of removing divalent nodes.

Although divalent nodes do not cause any problem for the purpose of this paper, it is often convenient to assume that all the divalent nodes are removed to simplify the exposition.

6.2 Consistent dimer models

The following definition is taken from [21, Definition 3.5]. It originates from the work of Hanany and Vegh [17], and was also studied by Bocklandt [4].

Definition 6.1 A dimer model is *consistent* if

- there is no homologically trivial zigzag path,
- no zigzag path has a self-intersection on the universal cover, and
- no pair of zigzag paths on the universal cover intersect each other in the same direction more than once.

Here, two zigzag paths on a dimer model are said to *intersect* if they share an edge (not a node) after removing all the divalent nodes from the dimer model. One intersection consists of an odd number of consecutive edges connected by divalent nodes, which must be just one edge if the dimer model has no divalent node.

The third condition means that if a pair (z, w) of zigzag paths on the universal cover has two intersections a and b and the zigzag path z points from a to b, then the other zigzag path w must point from b to a.

6.3 Related notions

For a node in a dimer model, the set of zigzag paths going through the edges adjacent to it has a natural cyclic ordering given by the directions of the outgoing paths from the node. On the other hand, the homology classes of these zigzag paths determine another cyclic ordering if these classes are distinct. The following condition was introduced by Gulotta:

Definition 6.2 [16, Section 3.1] A dimer model is *properly ordered* if

- there is no homologically trivial zigzag path,
- no zigzag path has a self-intersection on the universal cover,
- no pair of zigzag paths in the same homology class have a common node, and
- for any node of the dimer model, the natural cyclic order on the set of zigzag paths going through that node coincides with the cyclic order determined by their homology classes.

Mozgovoy and Reineke [29, Condition 4.12] introduced the following condition:

Definition 6.3 A dimer model is said to be cancellative if weakly equivalent paths are equivalent.

Mozgovoy and Reineke called this condition the first consistency condition.

Proposition 6.4 [21, Proposition 4.4 and Lemma 3.1] *A dimer model is consistent if and only if it is properly ordered. Moreover, a consistent dimer model is cancellative.*

Mozgovoy and Reineke [29] proved that the path algebra of the quiver with relations coming from a dimer model is a Calabi–Yau–3 algebra in the sense of Ginzburg [15] if the dimer model is cancellative and one extra condition which they call the second consistency condition. The latter condition was shown to be redundant by Davison [12]. Broomhead has proved the Calabi–Yau–3 property of the path algebra for isoradial dimer models [9]. The proof of Theorem 1.4 in this paper does not rely on any of these results, and gives an independent proof of the Calabi–Yau–3 property of the path algebra of the quiver with relations associated with a consistent dimer model through the derived equivalence $D^b \operatorname{coh} \mathcal{M}_\theta \cong D^b \operatorname{mod} \mathbb{C}\Gamma$.

7 Adjacent zigzag paths and large hexagons

In this section, we assume for simplicity that all divalent nodes are removed from the dimer model. In this case, a pair of zigzag paths intersect each other if and only if they share a common edge, and one intersection consists of exactly one edge.

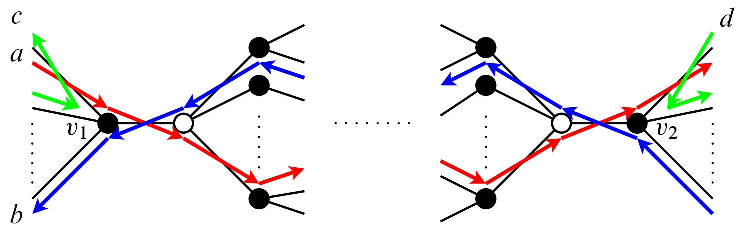

Figure 6: A pair of zigzag paths intersecting twice

7.1 Adjacent zigzag paths

Recall from Section 5.3 that the *slope* of a zigzag path on a dimer model is its homology class considered as an element in $\mathbb{Z}^2$. The lack of self-intersection of a zigzag path in a consistent dimer model implies the primitivity of its slope. There may be several zigzag paths with a given slope. The set of slopes naturally has a cyclic order, and a pair of zigzag paths are said to have *adjacent slopes* if their slopes are adjacent with respect to this cyclic order.

The following three lemmas are immediate consequences of Proposition 6.4:

Lemma 7.1 *If a pair of zigzag paths in a consistent dimer model intersect each other more than once on the universal cover, then their slopes are not adjacent.*

Proof Assume that there is a pair (a, b) of zigzag paths intersecting twice in the opposite direction as in Figure 6. Let v_1 and v_2 be the vertices adjacent to the first and the last edges where a and b intersect. Then there are two other zigzag paths c and d such that c intersects with a at the edge adjacent to the vertex v_1 and d intersects with a at the edge adjacent to the vertex v_2. Then the slopes of c and d must come in between a and b by Proposition 6.4, preventing them from being adjacent. □

Lemma 7.2 *If a pair of zigzag paths in a consistent dimer model have a common node other than their intersection, then the slopes of this pair of zigzag paths are not adjacent.*

Proof Since the dimer model is consistent, it is properly ordered by Proposition 6.4. If a pair of zigzag paths have a common node other than their intersection, then they are not adjacent with respect to the cyclic order around that node. Now it follows from Definition 6.2 that their slopes are not adjacent. □

Lemma 7.3 *If a dimer model is consistent, then there is a pair of zigzag paths with linearly independent slopes.*

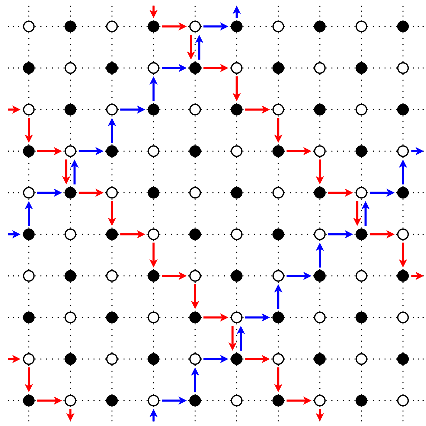

Figure 7: A pair of zigzag paths with adjacent slopes

Proof A dimer model always has a node with valence greater than two. Then there are at least three zigzag paths at the node whose slopes are different, since the dimer is properly ordered. □

7.2 Large hexagons

Lemmas 7.1 and 7.2 show that a pair of zigzag paths with adjacent slopes in a consistent dimer model behaves like a pair of lines: they have no self-intersection, and any pair of lifts to the universal cover intersect exactly once. Any pair of lines on a torus divides the torus into parallelograms. Since an intersection of a pair of zigzag paths in a consistent dimer model consists of an edge instead of a point, they divide the torus into hexagons instead of parallelograms.

Definition 7.4 Let $G=(B,W,E)$ be a consistent dimer model on a torus T and (z,w) be a pair of zigzag paths on G with adjacent slopes. A *large hexagon* is a connected component of the complement $T\setminus(z\cup w)$ of the union of the pair of zigzag paths.

Figure 7 shows an example of a collection of zigzag paths with adjacent slopes in a large square tiling. One can see that these zigzag paths divide the torus into large hexagons, as shown in Figure 8.

By removing arrows dual to edges in the pair of zigzag paths, the quiver associated with the dimer model is divided into a disjoint union of subquivers, each of whose connected components are in one-to-one correspondence with a large hexagon. Inside each such connected subquiver, there is a pair of distinguished vertices called the *source* and the *sink*. The source vertex is characterized by the existence of a path from that

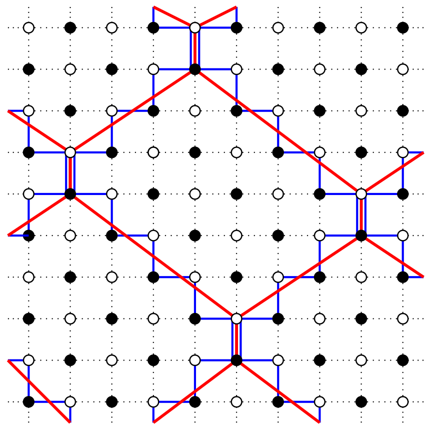

Figure 8: Large hexagons

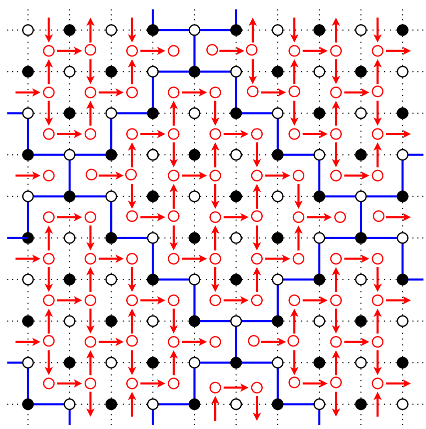

Figure 9: Sources and sinks

vertex to any other vertex in the subquiver, and the sink vertex is characterized by the dual property that there is a path of the subquiver from any other vertex to the sink vertex. The arrow dual to an intersection of the pair of zigzag paths goes from the source vertex of one large hexagon to the sink vertex of an adjacent large hexagon. See Figure 9 for an example of the subquivers and their source and sink vertices.

7.3 Large hexagons and the McKay quiver

The tessellation by large hexagons forms a new dimer model, and, as in Section 5.10, the resulting quiver Λ with relations can be identified with the McKay quiver for a suitable finite subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$ acting on $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ in the following way:

- Choose any vertex of Λ and identify it with the trivial representation.

- The arrow dual to an intersection of the two zigzag paths is identified with multiplication by z.
- The cyclic order of three arrows starting from a vertex of Λ coming from the orientation of the torus is given by (x, y, z).

The fact that we have taken only a pair of zigzag paths with adjacent slopes, so that there is only one zigzag path in each slope, implies that ρ_x generates the character group A^*, and so does ρ_y under the notation in Section 5.10. Hence the subgroup $A \subset \mathrm{SL}(3, \mathbb{C})$ is obtained by embedding a finite small subgroup $A \subset \mathrm{GL}(2, \mathbb{C})$ into $\mathrm{SL}(3, \mathbb{C})$.

8 Consistent dimer models are non-degenerate

Proposition 8.1 *A consistent dimer model is non-degenerate.*

Proof We may assume there is no divalent node. For an edge e in a consistent dimer model, choose a zigzag path z containing the edge. Choose another zigzag path w whose slope is adjacent to that of z. Then z and w divide the torus into large hexagons. In each large hexagon, there are two paths p and q from the source to the sink along $z \cup w$, as shown in Figure 10. One path p starts from the source vertex, goes along the

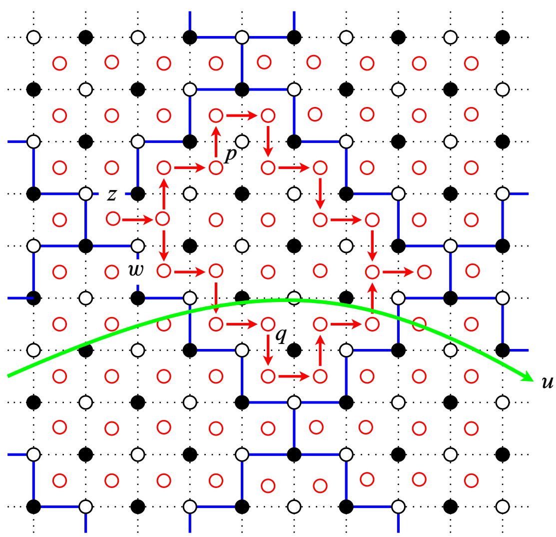

Figure 10: Two minimal paths p and q inside a large hexagon

zigzag path z until z intersects with w, from which point the path goes along w and arrives at the sink vertex. The other path q starts from the source vertex, goes along the zigzag path w until w intersects with z, from which point the path goes along z. Both p and q are minimal: if one is not minimal, then there is another zigzag path u which intersects this path in the same direction more than once by [21, Lemma 3.11]. This implies that the slope of the zigzag path u comes in between the slopes of z and w with respect to the natural cyclic order on the set of slopes, as shown in Figure 10. This contradicts the adjacency of the slopes of z and w.

Let D_1 be the set of edges formed by starting with e and taking every other edge in the union of z and w. Let further D_2 be the set of edges in the interiors of the large hexagons which are not crossed by any minimal path from the source to the sink. We show that the union $D = D_1 \cup D_2$ is a perfect matching. See Figure 11 for an example, where e is at the intersection of z and w.

Let n be a node on the union $z \cup w$ of the zigzag paths. Then it is clear from the construction that there is a unique edge in D_1 adjacent to n and no edge in D_2 is adjacent to n.

Take a node n in the interior of a large hexagon. We show that there is a unique edge in D_2 connected to n. Since p_1 and p_2 are minimal paths with the same source and target, they are equivalent. Since p_1 and p_2 are not homotopic in $T \setminus \{n\}$, there are two minimal paths q_1 and q_2 from the source vertex to the sink vertex inside the large hexagon such that p_1 is homotopic to q_1 in $T \setminus \{n\}$, p_2 is homotopic to q_2 in $T \setminus \{n\}$, and q_2 is obtained from q_1 by replacing $p_+(a)$ by $p_-(a)$ for an arrow $a \in A = E$. Then a must be adjacent to n, and either q_1 or q_2 passes through all edges incident to n except a. Hence it suffices to show $a \in D_2$. Let r be a minimal path from the source to the sink. Then r intersects neither z nor w by [21, Lemma 3.11] and hence r stays inside the large hexagon. Take a zigzag path y which passes through a. Since the dimer model is consistent and z and w have adjacent slopes, y divides the large hexagon into two connected components such that the source and the sink are not in the same component. By [21, Lemma 3.7], the number of intersections of y with r coincides with that of y with $p_i \equiv q_i$, which is 1. If r passes through a, the direction of the intersection with y is different from that of the intersection of q_i with y, which is a contradiction. This shows $a \in D_2$, and Proposition 8.1 is proved. □

Definition 8.2 For a pair (z, w) of zigzag paths with adjacent slopes, the perfect matching obtained as in the proof of Proposition 8.1 containing the edge at the intersection is said to *come from a pair of zigzag paths with adjacent slopes*.

Recall from Section 5.8 that a path p on a quiver is *allowed* by a perfect matching D if the path a does not contain any arrow dual to an edge in D. The proof of Proposition 8.1 also shows the following:

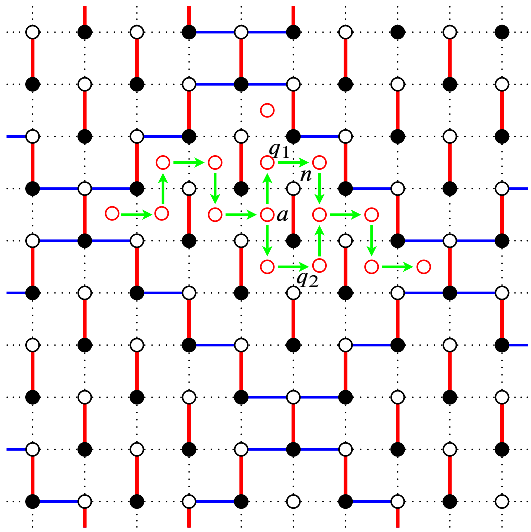

Figure 11: The perfect matching associated with a pair of zigzag paths

Lemma 8.3 *Let (z, w) be a pair of zigzag paths with adjacent slopes and D be the corresponding perfect matching. Then for any large hexagon and for any vertex v in the large hexagon, there is a path allowed by D inside the large hexagon from the source vertex to the sink vertex that passes through the vertex v.*

9 Corner perfect matchings

We prove Proposition 1.3. in this section. We first prove the following as an application of large hexagons:

Proposition 9.1 *Let G be a consistent dimer model. Then any choice of a pair of zigzag paths with adjacent slopes determines a corner $\mathfrak{c}$ of the characteristic polygon Δ and induces the following*:

(1) *The division of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ into large hexagons.*

(2) *A functor*

$$\phi_{\mathfrak{c}}^*\colon \operatorname{mod}(\mathbb{C}[x, y, z] \rtimes A) \to \operatorname{mod} \mathbb{C}\Gamma$$

from the category of representations of the McKay quiver of some finite small abelian subgroup $A \subset \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{SL}_3(\mathbb{C})$ to that of the path algebra of the quiver Γ with relations associated with the dimer model G.

(3) *An embedding*

$$\varphi_{\mathfrak{c}}\colon A\text{–Hilb}(\mathbb{C}^3) \hookrightarrow \mathcal{M}_\theta$$

of the A–Hilbert scheme as an open subscheme of the moduli space $\mathcal{M}_\theta$ for some generic stability parameter θ.

We also show the following characterization of corner perfect matchings in this section:

Proposition 9.2 *The following are equivalent for a perfect matching D in a consistent dimer model*:

(1) *D is simple.*

(2) *D is multiplicity one.*

(3) *D is a corner perfect matching.*

(4) *D comes from a pair of zigzag paths with adjacent slopes.*

The definitions of simplicity and multiplicity of a perfect matching are given in the ends of Sections 5.8 and 5.2, respectively. We first prove Proposition 9.2.

Proof The proof is divided into four steps:

Step 1 *A perfect matching is a corner perfect matching if and only if it comes from a pair of zigzag paths with adjacent slopes.*

Proof The "if" part follows from the fact that the height change of a perfect matching coming from a pair of zigzag paths with adjacent slopes satisfies the equality in the inequality (5-1) coming from both of these zigzag paths.

To show the "only if" part, consider three zigzag paths z_1, z_2, z_3 with consecutive slopes. Let D_1 and D_2 be the perfect matchings coming from z_1, z_2 and z_2, z_3, respectively. Then (5-1) implies $\langle h(D, D_1), [z_2] \rangle \leq 0$ for any D, where the equality holds for $D = D_1, D_2$. This shows that the line segment connecting the height changes of the corner perfect matchings D_1 and D_2 is on the boundary of the Newton polygon. In this way, we see that every corner perfect matching comes from a pair of zigzag paths with adjacent slopes. □

Step 2 *A perfect matching coming from a pair of zigzag paths with adjacent slopes is simple.*

Proof We have to show that the corresponding quiver representation M is simple, ie has no non-trivial submodule. This follows from the fact that, in a perfect matching coming from a pair of zigzag paths with adjacent slopes, one can find an allowed path from any vertex to any other vertex in the quiver. Indeed, starting from any vertex, one can first go to the sink of the large hexagon h_1 where the vertex belongs, and then to the adjacent vertex which is the source of the adjacent large hexagon h_2 by the path going around one of the nodes on the edge separating h_1 and h_2. Recall that one can go from the source of a large hexagon to any other vertex in the same large hexagon only through an allowed path. Note also that one can go from the source of one large hexagon to the source of another large hexagon adjacent in the x– and y–direction. Since one can go from one large hexagon to any other large hexagon by multiplying sufficiently many x and y, Step 2 is proved. □

Step 3 *A simple perfect matching is a corner perfect matching.*

Proof Since simple modules are θ–stable for any θ, the divisor corresponding to a simple perfect matching is not contracted in the affine quotient $\overline{\mathcal{M}}_0$. Hence it must be a corner perfect matching. □

Step 4 *A perfect matching is multiplicity-free if and only if it is simple.*

Proof Let us first prove the only if part. Assume M has a non-trivial submodule. Then one can find a stability parameter θ such that M is not θ–semistable. Since M is 0–semistable and the map $\mathcal{M}_\theta \to \overline{\mathcal{M}}_0$ is projective, there is another θ–semistable representation N with the same height change.

Now we prove the if part. Assume that M is simple, and take any module N with the same height change as M. Choose a stability parameter θ such that semistability implies stability and N is θ–stable [20, Lemma 6.2]. Since M is also θ–stable with the same height change as N, the modules N and M must belong to the same $\mathbb{T}$–orbit, so that the corresponding perfect matchings are identical.

This completes the proof of Proposition 9.2. □

The proof of Step 1 also shows the following:

Corollary 9.3 *The set of slopes of zigzag paths in a consistent dimer model is in one-to-one correspondence with the set of sides of the characteristic polygon, so that each slope is normal to the corresponding side.*

Proof of Proposition 9.1 Let $A \subset \mathrm{GL}(2, \mathbb{C})$ be the finite small subgroup whose McKay quiver Λ is identified with the tessellation by large hexagons as in Section 7.3. We discuss the embedding of A–$\mathrm{Hilb}(\mathbb{C}^3)$ into $\mathcal{M}_\theta$ for a suitable choice of θ. Let D be the perfect matching coming from a pair of zigzag paths with adjacent slopes. We regard quivers as categories as in Section 5.9, and define a functor

$$\phi_{\mathfrak{c}}\colon \Gamma \to \Lambda$$

as follows:

- A vertex of Γ is sent to the large hexagon containing it.
- An arrow inside a large hexagon that is not contained in D goes to the identity of the large hexagon.
- An arrow inside a large hexagon that is contained in D goes to the small cycle starting from the large hexagon.
- Suppose an arrow a of Γ is on the boundary of two large hexagons. Let b be the arrow of Λ connecting the two large hexagons. If a is in the same direction as b, then a goes to b. If a is in the opposite direction, then a goes to the path of length two that connects the two large hexagons in the same direction as a.

Recall that a representation of a quiver is regarded as a functor from the quiver as a category to the category of vector spaces. Thus $\phi_{\mathfrak{c}}$ induces a functor

$$\phi_{\mathfrak{c}}^*\colon \operatorname{mod} \mathbb{C}\Lambda \to \operatorname{mod} \mathbb{C}\Gamma.$$

The functor $\phi_{\mathfrak{c}}^*$ sends a G–cluster to a representation of Γ with dimension vector $(1, \dots, 1)$.

Let h_0 be the large hexagon identified with the trivial representation in the McKay quiver Λ of A. Choose a parameter $\eta \in \mathrm{Hom}(\mathbb{Z}^V, \mathbb{Q})$ satisfying the following:

- If a vertex v is not the source of a large hexagon, then $\eta(v) = 1$.
- The sum of $\eta(v)$ inside a fixed large hexagon is 0.

Then take a sufficiently small $\epsilon > 0$ and define a stability parameter $\theta \in \mathrm{Hom}(\mathbb{Z}^V, \mathbb{Q})$ as follows:

- If v is the source of a large hexagon other than h_0, then $\theta(v) = \eta(v) + \epsilon$.
- If v is the source of h_0, then $\theta(v) = \eta(v) - (\#A - 1)\epsilon$.
- For other vertices v, we set $\theta(v) = \eta(v)$.

One can easily see that every A–cluster goes to a θ–stable representation of Γ. This gives an embedding $\varphi_{\mathfrak{c}}\colon A\text{–Hilb}(\mathbb{C}^3) \hookrightarrow \mathcal{M}_\theta$, and Proposition 9.1 is proved. □

Proof of Proposition 1.3 The open neighborhood $U_{\mathfrak{c}}$ is given by the image of the embedding $\varphi_{\mathfrak{c}}\colon A\text{–Hilb}(\mathbb{C}^3) \hookrightarrow \mathcal{M}_\theta$. The closed subscheme $A\text{–Hilb}(\mathbb{C}^2) \subset A\text{–Hilb}(\mathbb{C}^3)$ consists of stable representations of the McKay quiver Λ of A as a subgroup of $\mathrm{SL}(3,\mathbb{C})$ such that all the arrows corresponding to multiplication by z go to zero. It follows from the definition of the functor $\phi_{\mathfrak{c}}$ that such representations go to representations of Γ such that any arrow contained in the corner perfect matching D goes to zero. Hence the divisor $A\text{–Hilb}(\mathbb{C}^2) \subset A\text{–Hilb}(\mathbb{C}^3)$ goes to the corner toric divisor $D_{\mathfrak{c}} \subset U_{\mathfrak{c}} \subset \mathcal{M}_\theta$ under the embedding $\varphi_{\mathfrak{c}}\colon A\text{–Hilb}(\mathbb{C}^3) \hookrightarrow \mathcal{M}_\theta$. The isomorphism (1-2) of tautological bundles is clear from the construction of the embedding $\varphi_{\mathfrak{c}}\colon A\text{–Hilb}(\mathbb{C}^3) \hookrightarrow \mathcal{M}_\theta$ as a morphism between moduli spaces coming from the universal property of the moduli space, and Proposition 1.3 is proved. □

10 Description of the algorithm

10.1 Removal of edges

Let $G = (B, W, E)$ be a consistent dimer model. The algorithm to remove the corner $\mathfrak{c}$ from the characteristic polygon Δ is the following:

Algorithm 10.1

(0) *Remove all divalent nodes. This step in fact is not necessary but simplifies the exposition below.*

(1) *Choose a pair of zigzag paths with adjacent slopes corresponding to the corner* $\mathfrak{c}$.

(2) *Choose an identification of the resulting large hexagons with vertices of the McKay quiver for a finite small abelian group* $A \subset \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{SL}_3(\mathbb{C})$ *by choosing the large hexagon corresponding to the trivial representation.*

(3) *Remove the edges of the dimer corresponding to the arrows of the quiver going from the sources of the large hexagons corresponding to special representations to the sinks of the adjacent large hexagons related by multiplication by* z.

One has a choice in Steps (1) and (2), and the result of the operation depends on this choice. See Section 10.3 below for examples.

10.2 Inversion of arrows

Removing an edge of a dimer model corresponds to merging adjacent vertices into a single vertex. It also corresponds to adding an inverse arrow under a mild condition, which is always satisfied when we remove a corner from the characteristic polygon of a consistent dimer model:

Lemma 10.2 *Let $G = (B, W, E)$ be a dimer model without divalent nodes, and let Γ be the associated quiver with relations. Let further S be a subset of E. Then $G' = (B, W, E \setminus S)$ is again a dimer model if and only if the following conditions are satisfied:*

(1) *Every node is contained in at least two edges in $E \setminus S$.*

(2) *There are no (not necessarily oriented) cycles of Γ consisting of arrows in S.*

Moreover, if G' is a dimer model, then the path algebra of the quiver with relations associated with G' is Morita equivalent to the path algebra of the quiver with relations obtained from Γ by adding the inverses a^{-1} of arrows $a \in S$ together with relations $aa^{-1} = e_{t(a)}$ and $a^{-1}a = e_{s(a)}$. Here e_v is the idempotent element associated with a vertex v of a quiver.

Proof Condition (1) holds if and only if all the nodes in G' have valence at least two. Condition (2) holds if and only if all the connected components of $T \setminus \bigcup_{e \in E \setminus S} e$ are simply connected. It is easy to see that the categories of representations of the above two quivers with relations are equivalent to each other. □

We show in Section 11 that the consistency condition is preserved by Algorithm 10.1, so that Lemma 10.2 can be applied. The point of view of adding inverse arrows will be used in Sections 16, 18 and 19 to prove the derived equivalence inductively.

10.3 Examples

As an example, consider the construction of dimer models for the hexagon in Figure 12, right, starting from the dimer model in Figure 13 corresponding to the square lattice polygon in Figure 12, left, by removing two vertices.

To remove the top left corner from the square lattice polygon in Figure 12, left, we have to choose a pair of zigzag paths, one from each of those with homology classes $(-1, 0)$ (shown in red in Figure 14) and $(0, 1)$ (shown in blue in Figure 14). There are four choices in Step (1), which actually do not matter for symmetry reasons. There is no choice in Step (2), and Figure 15 shows the resulting dimer model.

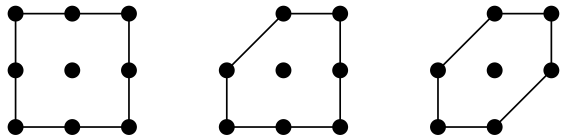

Figure 12: Left: a square. Middle: a pentagon. Right: a hexagon.

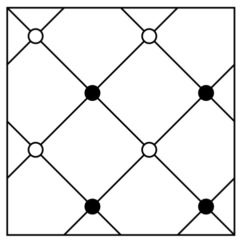

Figure 13: A dimer model for the square lattice polygon

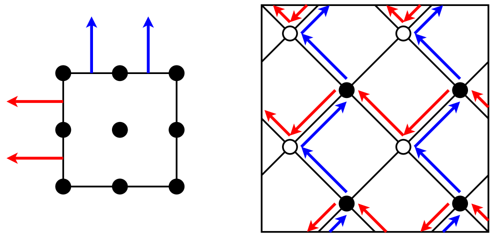

Figure 14: Zigzag paths

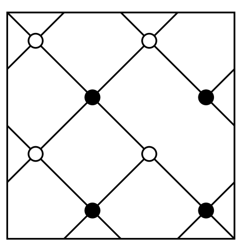

Figure 15: The dimer model for the pentagonal lattice polygon

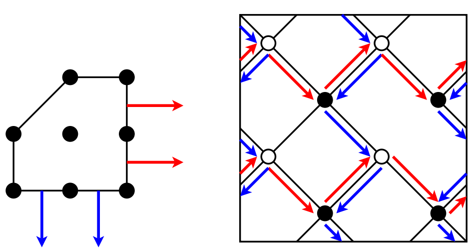

Figure 16: Zigzag paths

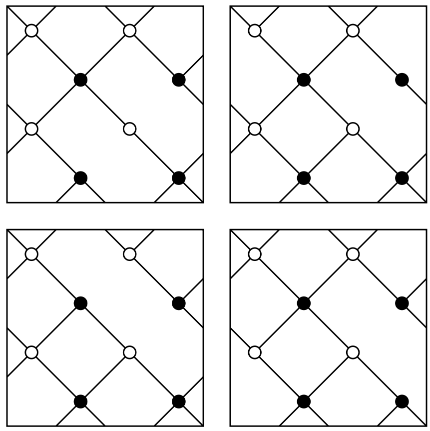

Figure 17: Dimer models for the hexagonal lattice polygon

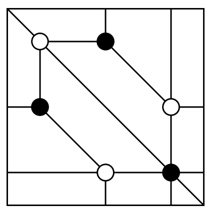

Figure 18: A dimer model equivalent to the dimer model in Figure 17, top left

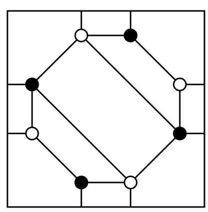

Figure 19: A dimer model equivalent to the dimer model in Figure 17, bottom left

Now consider the removal of the lower right corner from the pentagonal lattice polygon in Figure 12, middle. In this case there are four choices in Step (1), which lead to the dimer models shown in Figure 17. Note that the dimer models in the top right and bottom right are obtained from the dimer models in the top left and bottom left, respectively, by changing the colors of the nodes, so that the corresponding quivers are related by the reversal of arrows.

The dimer model in Figure 17, top left, has a divalent white node, and one obtains the dimer model in Figure 18 by removing it. The dimer model in Figure 17, bottom left, is equivalent to the dimer model shown in Figure 19.

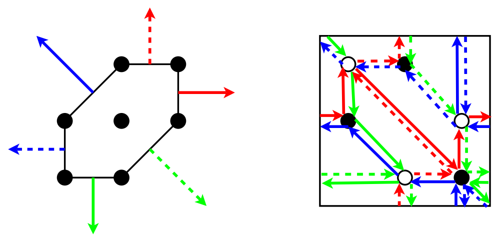

Figure 20: Zigzag paths

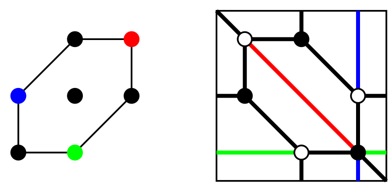

Figure 21: From a hexagon to a triangle

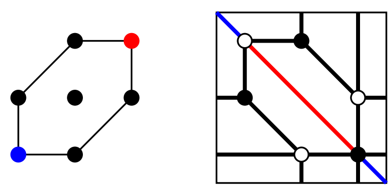

Figure 22: From a hexagon to a square

Figure 23: Two characteristic polygons

The zigzag paths on the dimer model in Figure 18 are shown in Figure 20.

From the dimer model in Figure 18, one can construct the dimer model for $\mathbb{P}^2$ by removing three vertices from the lattice polygon as in Figure 21.

Similarly, from the dimer model in Figure 18, one can construct the dimer model for $\mathbb{P}^1 \times \mathbb{P}^1$ by removing two vertices from the lattice polygon as in Figure 22.

Next we discuss a simple example where the special McKay correspondence plays a role. Let $A = \left\langle \frac{1}{5}(1,2) \right\rangle$ be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by $\mathrm{diag}(\zeta, \zeta^2)$ for $\zeta = \exp(2\pi\sqrt{-1}/5)$. Recall from Section 4 that the integers r, $b_1, \dots, b_r$ and $i_0, \dots, i_{r+1}$

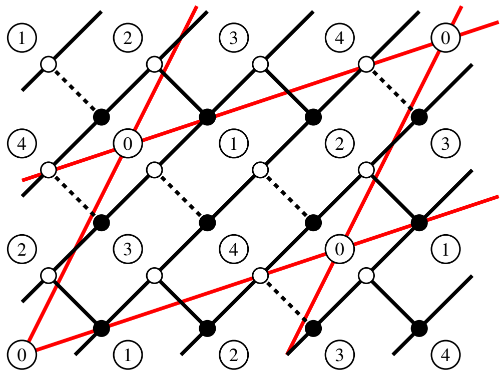

Figure 24: The dimer model associated with Δ

are defined inductively by $i_0 := n$, $i_1 := q$, and

$$i_t = b_{t+1} i_{t+1} - i_{t+2} \quad (0 < i_{t+2} < i_{t+1}),$$

until we finally obtain $i_r = 1$ and $i_{r+1} = 0$. This gives

$$\begin{aligned} 5 &= 3 \cdot 2 - 1, \\ 2 &= 2 \cdot 1 - 0, \end{aligned}$$

so that $r = 2$, $(b_1, b_2) = (3, 2)$, and $(i_0, i_1, i_2) = (5, 2, 1)$. The continued fraction expansion (4-2) is given by

$$\frac{n}{q} = \frac{5}{2} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}} = 3 - \frac{1}{2},$$

and the special representations are given by $\rho_5 = \rho_0$, ρ_1 and ρ_2. The McKay quiver for A as a subgroup of $\mathrm{SL}_3(\mathbb{C})$ is the quiver associated with the dimer model shown in Figure 24, where the parallelogram shows a fundamental region of the torus. To remove the top right corner from the characteristic polygon shown in Figure 23, left, we have to remove edges corresponding to multiplication by z from special representations. These edges are shown in dotted lines in Figure 24, and by removing them, one obtains the dimer model shown in Figure 25. This dimer model contains divalent nodes, and by removing them, one obtains the dimer model shown in Figure 26, which is exactly the dimer model corresponding to the characteristic polygon shown in Figure 23, right.

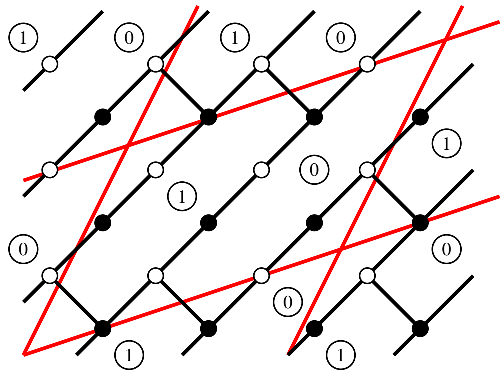

Figure 25: The dimer model after the operation

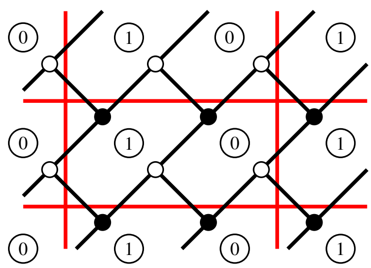

Figure 26: The dimer model after removing divalent nodes

11 Preservation of the consistency

We use the same notation as in Section 4. We prove the following in this section:

Proposition 11.1 *A consistent dimer model remains consistent after performing Algorithm 10.1, if the lattice points of the polygon other than the removed one do not lie on a line.*

We need the following lemma to prove Proposition 11.1:

Lemma 11.2 *Let $t \in [1, r+1]$, $a \in (0, i_{t-1} - i_t)$ and $b \in (0, j_t - j_{t-1})$ be integers. Then $i_t + a + bq$ is special if and only if $a = b = 0$.*

Proof Write

$$a = d_t i_t + d_{t+1} i_{t+1} + \cdots + d_r i_r,$$

as in Theorem 4.2. Using the same theorem for the dual sequence, we can write

$$b = d_{t-1} j_{t-1} + d_{t-2} j_{t-2} + \cdots + d_1 j_1.$$

Then Lemma 4.6 implies

$$i_t + a + bq \equiv d_1 i_1 + \cdots + d_{t-1} i_{t-1} + (d_t + 1) i_t + d_{t+1} i_{t+1} + \cdots + d_r i_r.$$

Therefore, if the sequence $(d_1, \dots, d_{t-1}, d_t + 1, d_{t+1}, \dots, d_r)$ satisfies the condition in Lemma 4.5, then $i_t + a + bq$ is special if and only if $d_1 = \cdots = d_r = 0$ by the uniqueness of the expression in Theorem 4.2.

By using $b_t i_t = i_{t-1} + i_{t+1}$ and the assumption $a < i_{t-1} - i_t$, we obtain

$$(d_t + 1 - b_t) i_t + (d_{t+1} + 1) i_{t+1} + d_{t+2} i_{t+2} + \cdots + d_r i_r < 0,$$

which implies $d_t \le b_t - 2$. Moreover, if the equality $d_t = b_t - 2$ holds, then we have

$$d_{t+1} i_{t+1} + d_{t+2} i_{t+2} + \cdots + d_r i_r < i_t - i_{t+1},$$

which is of the same form as the assumption $a < i_{t-1} - i_t$ with t increased by 1 so that we obtain $d_{t+1} \le b_{t+1} - 2$. Thus we can inductively show that if $d_k = b_k - 1$ for some $k > t$ then there is an integer $l \in (t, k)$ with $d_l \le b_l - 3$.

We can argue in the same way to conclude that $d_{t-1} \le b_{t-1} - 2$, and if $d_k = b_k - 1$ for some $k < t - 1$ then there is an integer $l \in (k, t-1)$ with $d_l \le b_l - 3$.

Thus we have shown that the sequence $(d_1, \dots, d_{t-1}, d_t + 1, d_{t+1}, \dots, d_r)$ satisfies the condition in Lemma 4.5. □

Now we prove Proposition 11.1:

Proof of Proposition 11.1 We first note that if the zigzag paths of the bicolored graph obtained by Algorithm 10.1 satisfy the consistency condition then the assumption of Proposition 11.1 implies that the bicolored graph satisfies the conditions in Lemma 10.2 and hence is actually a dimer model. We prove the consistency conditions in two steps.

Step 1 *The case* A–Hilb$(\mathbb{C}^3) \setminus A$–Hilb$(\mathbb{C}^2)$ *for* $A = \left\langle \frac{1}{n}(1, q) \right\rangle \subset \mathrm{GL}_2(\mathbb{C})$.

Let Λ be the hexagonal dimer model for A–Hilb$(\mathbb{C}^3)$. The associated quiver is the McKay quiver of A, where the vertices are the irreducible representations of A and there are three arrows from each vertex corresponding to the multiplications by the coordinate functions x, y, z. Regard the set V of vertices as $V = (\mathbb{Z}/n\mathbb{Z})^* = \mathbb{Z}/n\mathbb{Z}$ and let $\alpha_i, \beta_i, \gamma_i$ be the three arrows with the source $i \in V$ whose targets are $i+1$, $i+q$, $i-q-1$, respectively. A zigzag path of Λ is of one of the following three forms according to its homology class: $(\dots, \beta_{i+1}, \alpha_i, \beta_{i-q}, \alpha_{i-q-1}, \dots)$, $(\dots, \gamma_{i+q}, \beta_i, \gamma_{i+q+1}, \beta_{i+1}, \dots)$ or $(\dots, \alpha_{i-q-1}, \gamma_i, \alpha_{i-1}, \gamma_{i+q}, \dots)$.

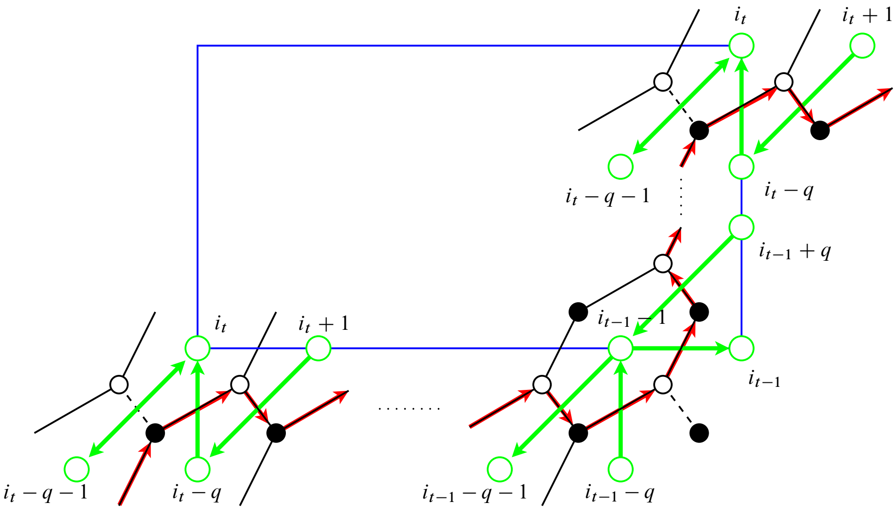

Figure 27: New zigzag paths intersect at most once on the universal cover

Let Λ' be the bicolored graph obtained from Λ by Algorithm 10.1 (ie by removing the edges γ_i for special i). Of three homology classes of zigzag paths on Λ, only the ones consisting of α and β survive in Λ'. The other two zigzag paths will be transformed into new zigzag paths on Λ', indexed by t with $t \in \mathbb{Z}/r\mathbb{Z}$, as follows: Start with the edge $\beta_{i_{t-1}-q}$ whose target is the special hexagon i_{t-1}, next choose the adjacent edge $\gamma_{i_{t-1}+1}$ if $i_{t-1}+1$ is not special, and go along the old zigzag path consisting of β and γ until one arrives at the next special hexagon i_t, where one is blocked by the removed edge γ_{i_t}. Then one changes the direction and go along the old zigzag path consisting of α and γ. By virtue of (4-3), one comes back to the starting point without meeting any other removed edges.

Now let us check the consistency of the new dimer model. It is obvious that a new zigzag path has no self-intersection on the universal cover. Choose two zigzag paths on Λ'. If they are both old, ie zigzag paths of Λ, then they do not meet at all. If one is old and the other is new, then they meet more than once in general but always in the opposite direction. If they are both new, then they meet at most once on the universal cover, since there are no special representations in the rectangular region in Figure 27, by Lemma 11.2.

Step 2 *The general case.*

Old zigzag paths except the chosen two survive, and new zigzag paths are described in the same way as above using the large hexagons. Let us analyze intersections of two zigzag paths in the new dimer model. If two zigzag paths are both old or new, then the same reasoning as Step 1 shows that they do not intersect in the same direction twice. Take one new zigzag path and one survivor from the old one, and suppose they meet twice in the same direction. Since we have chosen two zigzag paths with adjacent slopes to perform the operation, the slope of the survivor cannot be in between the slopes of these two zigzag paths. This implies that the survivor must meet either of the two zigzag paths twice in the same direction, thus contradicting the consistency of the old dimer model. □

12 Zigzag paths and characteristic polygons

We use the relation between zigzag paths and the characteristic polygon to show that the characteristic polygon changes as expected under the operation.

Let $([z_i])_{i=1}^k$ be the sequence of slopes of zigzag paths ordered cyclically starting from any zigzag path. Here k is the number of zigzag paths, and some of the slopes may coincide, in general. Define another sequence $(w_i)_{i=1}^r$ in $\mathbb{Z}^2$ by $w_0 = 0$ and

$$w_{i+1} = w_i + [z_{i+1}]' \quad (i = 0, \dots, k-1),$$

where $[z_{i+1}]'$ is obtained from $[z_{i+1}]$ by rotating by $90°$ in the positive direction. Note that one has $w_r = 0$, since every edge is contained in exactly two zigzag paths whose directions on that edge are opposite, and hence the homology classes of the zigzag paths add up to zero. The convex hull of $(w_i)_{i=1}^r$ is called the *zigzag polygon*.

The following theorem was proved by Gulotta [16, Theorem 3.3] for properly ordered dimer models and therefore it holds for consistent dimer models by [21, Proposition 4.4].

Theorem 12.1 *For a consistent dimer model, the characteristic polygon Δ coincides with the zigzag polygon up to translation.*

Theorem 12.1 yields the second part of Theorem 1.1:

Proposition 12.2 *The characteristic polygon of the dimer model after performing Algorithm 10.1 is obtained by removing the chosen corner and taking the convex hull of the rest.*

Proof In the proof of Proposition 11.1, we have described the change of zigzag paths under the operation. Theorem 12.1 shows that this induces the desired change in the characteristic polygon. □

Theorem 12.1 and Lemma 7.1 gives the following uniqueness result in the case of lattice triangles:

Proposition 12.3 *For any lattice triangle Δ, there is a unique consistent dimer model whose characteristic polygon coincides with Δ.*

Proof In the case of a triangle, any pair of zigzag paths are adjacent or have the same slope. If they have the same slope, then the consistency condition prevents them from intersecting at all. If they are adjacent, then they can intersect only once on the universal cover, by Lemma 7.1. These two conditions suffice to show that the resulting dimer model gives a hexagonal tiling of the 2–torus, and the corresponding quiver is the McKay quiver for some abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. □

See also [34, Theorem 1.2] for a closely related uniqueness result. The uniqueness fails even for squares; see eg [33] for a discussion of an example.

Another corollary is the following statement, which is stronger than Proposition 8.1:

Corollary 12.4 *Let G be a consistent dimer model. Then for any edge of G there is a corner perfect matching containing it.*

Proof For an edge e, choose a zigzag path z containing e. Then we can construct a corner perfect matching D of G which contains half of the edges of z as in the proof of Proposition 8.1. If D contains e, we are done. If D does not contain e, then the other corner perfect matching D' in the proof of Theorem 12.1 contains e. □

13 Effect of Algorithm 10.1 on the moduli space

Suppose a dimer model G' is obtained from a consistent dimer model $G = (B, W, E)$ by performing Algorithm 10.1. Let Γ and Γ' be the quivers associated with G and G', respectively, and let $S \subset E$ be the set of removed edges. A vertex of Γ' is the union of vertices of Γ' connected by arrows in S.

Regarding quivers as categories, we can define a functor $\phi\colon \Gamma \to \Gamma'$ as follows: A vertex v of Γ is sent to the vertex of Γ' containing v. An arrow a is sent to itself if $a \notin S$, and to the identity of the vertex containing a if $a \in S$. The functor ϕ induces the functor

$$\phi^*\colon \operatorname{mod}\Gamma' \to \operatorname{mod}\Gamma.$$

For the stability parameter θ in Proposition 9.1, we define a stability parameter θ' for Γ' such that $\theta'(v')$ is the sum of $\theta(v)$ for vertices $v \subset v'$. Then the above functor gives an open embedding $\mathcal{M}'_{\theta'} \to \mathcal{M}_\theta$. In terms of the moduli spaces, Proposition 12.2 is interpreted as follows:

Proposition 13.1 *The image of $\mathcal{M}'_{\theta'}$ is the complement of the toric divisor $D_{\mathfrak{c}} \subset \mathcal{M}_\theta$ corresponding to the removed corner $\mathfrak{c}$.*

As a corollary, we obtain Proposition 1.6:

Corollary 13.2 *The edges in S are exactly those which correspond to morphisms between tautological bundles vanishing only on the toric divisor $D_{\mathfrak{c}} \subset \mathcal{M}_\theta$.*

Proof For an edge e, let $\Psi(e)$ denote the corresponding morphism between tautological bundles on $\mathcal{M}_\theta$. First consider an edge $e \in S$. By the construction of the corner perfect matching $D_{\mathfrak{c}}$, the morphism $\Psi(e)$ vanishes on $D_{\mathfrak{c}}$. On the other hand, Proposition 13.1 shows that $\Psi(e)$ does not vanish on any other toric divisor. Next suppose $e \notin S$. Then e survives as an edge e' in the consistent dimer model G', and $\Psi(e)$ restricts to a morphism $\Psi'(e')$ of tautological bundles on $\mathcal{M}'_{\theta'}$. By Corollary 12.4, there is a corner perfect matching D' of G' containing e. Then the simplicity of D' shows that $\Psi'(e')$ vanishes along the divisor corresponding to D'. □

14 Injectivity of the universal morphism

Let G be a dimer model and $\bigoplus_v \mathcal{L}_v$ be the tautological bundle on the moduli space $\mathcal{M}_\theta$ of quiver representations with respect to a generic stability parameter θ.

Proposition 14.1 *If G is consistent, then the universal morphism*

$$\mathbb{C}\Gamma \to \operatorname{End}\left(\bigoplus_v \mathcal{L}_v\right)$$

is injective.

Proof A consistent dimer model is non-degenerate by Proposition 8.1. Therefore, the moduli space contains a three-dimensional algebraic torus $\mathbb{T}$ as an open set by [20, Proposition 5.1]. Fix a $\mathbb{T}$–fixed point $[\Psi]$ on $\mathcal{M}_\theta$ which is the isomorphism class of a representation Ψ of Γ. Then the toric affine open neighborhood U_Ψ of $[\Psi]$ is isomorphic to a closed subscheme of the space $\widetilde{\mathcal{M}}$ of all the representations of Γ by [20, Lemma 4.3]. Then $\mathbb{T} \subset U_\Psi$ is lifted to a subgroup of the group $\widetilde{\mathbb{T}}$ of $\mathbb{C}^\times$–valued representations of Γ. Thus $\mathbb{T}$ acts on both $\mathbb{C}\Gamma$ and $\operatorname{End}(\bigoplus_v \mathcal{L}_v)$ in such a way that the homomorphism is equivariant. If two paths p and q from u to v are not equivalent, they are not weakly equivalent by the cancellativity, and hence they have different weights with respect to the $\mathbb{T}$–action. Since equivalence classes of paths form a basis of $\mathbb{C}\Gamma$ and any path goes to a non-zero element in $\operatorname{End}(\bigoplus_v \mathcal{L}_v)$, the homomorphism is injective. □

15 Preservation of the tilting condition: A–Hilb($\mathbb{C}^3$) versus A–Hilb($\mathbb{C}^3$) \ A–Hilb($\mathbb{C}^2$)

Let A be a finite small subgroup of $\mathrm{GL}_2(\mathbb{C})$ and set

$$Y = A\text{–Hilb}(\mathbb{C}^2), \quad U = A\text{–Hilb}(\mathbb{C}^3) \quad \text{and} \quad U' = U \setminus Y.$$

Let $\mathcal{R}_\rho$ be the tautological bundle on $U = A\text{–Hilb}(\mathbb{C}^3)$ corresponding to an irreducible representation ρ of A, and $\mathcal{R}'_\rho = \mathcal{R}_\rho|_{U'}$ be its restriction to U'. In this section, we compare the tilting conditions of $\bigoplus_\rho \mathcal{R}_\rho$ and $\bigoplus_\rho \mathcal{R}'_\rho$ and prove two lemmas which will be used later in a more general setting. We first prove a general result that the restriction of a tilting object to an open subset is also a generator:

Lemma 15.1 *Let $\mathcal{E}$ be a tilting object in $D^b \operatorname{coh} U$. Then the pull-back of $\mathcal{E}$ by an open immersion $\iota\colon U' \to U$ is a generator in $D^b \operatorname{coh} U'$.*

Proof For any coherent sheaf $\mathcal{F}$ on U', there is a coherent sheaf $\widetilde{\mathcal{F}}$ on U such that $\iota^* \widetilde{\mathcal{F}} = \mathcal{F}$; see eg [18, Exercise 5.15]. Since $\mathcal{E}$ is a tilting object, $\mathcal{F}$ is a direct summand of an object in $D^b \operatorname{coh} U$ obtained from $\mathcal{E}$ by taking mapping cones. Since derived restriction commutes with the operation of taking mapping cones, this shows that $\mathcal{F}$ is obtained from $\iota^* \mathcal{E}$ by taking direct summands and mapping cones. This implies that $\iota^* \mathcal{E}$ is a generator in $D^b \operatorname{coh} U'$. □

To compare tilting properties of $\bigoplus_\rho \mathcal{R}_\rho$ and $\bigoplus_\rho \mathcal{R}'_\rho$, we use the exact sequence

$$\cdots \to H^i_Y(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \to H^i(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \to H^i(U', {\mathcal{R}'_\rho}^\vee \otimes \mathcal{R}'_\tau) \to \cdots . \tag{15-1}$$

In this exact sequence, we have the following vanishing result.

Lemma 15.2 *The local cohomology $H^i_Y(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau)$ vanishes for $i \geq 2$.*

Proof We use

$$H^i_Y(U, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \cong \varinjlim_n \operatorname{Ext}^i_U(\mathcal{O}_{nY}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau)$$

to compute the local cohomology. One has

$$\begin{aligned} \operatorname{Ext}^i_U(\mathcal{O}_{nY}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) &\cong \operatorname{Ext}^i_U(\{\mathcal{O}_U(-nY) \to \mathcal{O}_U\}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \\ &\cong H^i(\{\mathcal{O}_U \to \mathcal{O}_U(nY)\} \otimes \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) \\ &\cong H^{i-1}(\mathcal{O}_U(nY)|_{nY} \otimes \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau). \end{aligned} \tag{15-2}$$

Since U has the trivial canonical bundle, the adjunction formula gives an isomorphism

$$\mathcal{O}_U(nY)|_{nY} \cong \omega_{nY}$$

with the dualizing sheaf ω_{nY} of nY. Since Y is a resolution of an affine surface, one has $H^2(\mathcal{E}) = 0$ for any coherent sheaf $\mathcal{E}$ on Y. It follows that any surjection $\mathcal{F} \to \mathcal{G} \to 0$ of coherent sheaves on Y induces a surjection $H^1(\mathcal{F}) \to H^1(\mathcal{G}) \to 0$ of cohomology groups. By definition of full sheaves, one has $H^1(\mathcal{R}_\rho^\vee \otimes \omega_Y) = 0$ and $\mathcal{R}_\tau|_Y$ is generated by global sections. The latter shows the existence of a surjection $\mathcal{O}_U^{\oplus N} \to \mathcal{R}_\tau|_Y \to 0$ for some $N \in \mathbb{N}$, which gives a surjection

$$\mathcal{R}_\rho^\vee \otimes \omega_Y^{\oplus N} \to \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_Y \to 0,$$

which, combined with $H^1(\mathcal{R}_\rho^\vee \otimes \omega_Y) = 0$, gives

$$H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_Y) = 0.$$

This proves

$$\operatorname{Ext}_U^i(\mathcal{O}_{nY}, \mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau) = 0$$

for $n = 1$.

For $n > 1$, note the exact sequence

$$0 \to \mathcal{O}_U(D - Y)|_{(n-1)Y} \to \mathcal{O}_U(D)|_{nY} \to \mathcal{O}(D)|_Y \to 0,$$

which holds for any divisor D on Y. By substituting $D = nY$, one obtains

$$0 \to \mathcal{O}_U((n-1)Y)|_{(n-1)Y} \to \mathcal{O}_U(nY)|_{nY} \to \mathcal{O}(nY)|_Y \to 0,$$

which is the same as

$$0 \to \omega_{(n-1)Y} \to \omega_{nY} \to \omega_Y^{\otimes n} \to 0.$$

Since Y is the minimal resolution, $\omega_Y^{\otimes n}$ is generated by global sections and one has

$$H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_Y^{\otimes n}) = 0$$

by the same argument as above. Together with the exact sequence

$$H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_{(n-1)Y}) \to H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_{nY}) \to H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_Y^{\otimes n}),$$

one can inductively show

$$H^1(\mathcal{R}_\rho^\vee \otimes \mathcal{R}_\tau \otimes \omega_Y^{\otimes n}) = 0$$

for any positive integer n. □

We obtain the following corollary, which we will not use. Note that its assumption follows from [8].

Corollary 15.3 *If the condition* (**T**) *holds for* U, *then the direct sum* $\bigoplus_\rho \mathcal{R}'_\rho$ *over the set of irreducible representations of* A *is a tilting object.*

Proof The restriction $\bigoplus_\rho \mathcal{R}'_\rho$ is a generator by Lemma 15.1. The vanishing of $H^i({\mathcal{R}'_\rho}^\vee \otimes \mathcal{R}'_\tau)$ for $i \geq 1$ follows from the long exact sequence (15-1) together with Lemma 15.2. □

16 Preservation of surjectivity: A–Hilb($\mathbb{C}^3$) versus A–Hilb($\mathbb{C}^3$) \ A–Hilb($\mathbb{C}^2$)

We use the same notation as in Section 15. Let Λ be the McKay quiver of A, and Λ' be the quiver obtained from Λ by adding inverse arrows to the arrows starting from special representations corresponding to multiplication by z.

We prove the following in this section:

Proposition 16.1 *The natural map from* $\mathbb{C}\Lambda'$ *to the endomorphism algebra of* $\bigoplus_i \mathcal{R}'_i$ *is surjective.*

Let $\widetilde{N} = \mathbb{Z}^3$ be the group of one-parameter subgroups of the dense torus in $\mathbb{C}^3$. The group $N \supset \widetilde{N}$ of one-parameter subgroups of the dense torus in $U = A$–Hilb($\mathbb{C}^3$) is given by

$$N = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{n}(1, q, n-(1+q)),$$

and the fan describing the quotient $\mathbb{C}^3/A$ has the unique three-dimensional cone given by the first quadrant $(\mathbb{R}_{\geq 0})^3 \subset \widetilde{N}_\mathbb{R} = N_\mathbb{R}$.

Lemma 16.2 (Craw and Reid [11]) *One-dimensional cones in the fan describing* $U = A$–Hilb($\mathbb{C}^3$) *which are adjacent to* $\mathbb{R}_{\geq 0}(0,0,1)$ *are generated by*

$$\frac{1}{n}(j_t, i_t, n-(i_t+j_t)) \in N$$

for $0 \leq t \leq r+1$. *Here we say two one-dimensional cones are adjacent if they are contained in a common two-dimensional cone.*

Now let us express the tautological bundles as $\mathbb{Q}$–linear combinations of exceptional divisors, ie toric divisors except the three which correspond to the corners of the junior simplex. Let $x, y, z \in \mathbb{C}[\widetilde{M}]$ be the coordinates of $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ corresponding to the standard basis of $\widetilde{M} = \operatorname{Hom}(\widetilde{N}, \mathbb{Z}) \cong \mathbb{Z}^3$. Then rational sections of $\mathcal{R}_d$ form a vector space with a basis consisting of Laurent monomials $x^a y^b z^c$ with

$$a + bq - (1+q)c \equiv d \mod n.$$

On the other hand, the coordinate ring of the dense torus in $\mathbb{C}^3/A$ is given by $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]^A = \mathbb{C}[M]$, where

$$M = \operatorname{Hom}(N, \mathbb{Z}) = \{(a, b, c) \in \widetilde{M} \mid a + bq - (1+q)c \equiv 0 \bmod n\}.$$

It follows that one can embed the line bundle $\mathcal{R}_d^{\otimes n}$ into $\mathcal{O}_U$ in a natural way and it defines an effective exceptional divisor E_d on U with $\mathcal{R}_d^{\otimes n} = \mathcal{O}_U(-E_d)$.

Let $C = (c_{st})_{s,t=1}^r$ be the negative of the intersection matrix of the resolution $Y \to \mathbb{C}^2/A$:

$$c_{st} = \begin{cases} b_s & \text{if } s = t, \\ -1 & \text{if } |s-t| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The lower-right principal minors

$$i_t = \begin{vmatrix} b_{t+1} & -1 & & & \\ -1 & b_{t+2} & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & b_{r-1} & -1 \\ & & & -1 & b_r \end{vmatrix}$$

give the integers appearing in the continued fraction expansion in Section 4, since they satisfy (4-1). In particular, one has $\det C = i_0 = n$. Let η_{st} be the $(s,t)^{\text{th}}$ entry of the integer matrix nC^{-1}. Since

$$\begin{pmatrix} b_1 & -1 & & & \\ -1 & b_2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & b_r \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_r \end{pmatrix} = \begin{pmatrix} b_1 i_1 - i_2 \\ -i_1 + b_2 i_2 - i_3 \\ -i_2 + b_3 i_3 - i_4 \\ \vdots \\ -i_{r-1} + b_r i_r \end{pmatrix} = \begin{pmatrix} i_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} b_1 & -1 & & & \\ -1 & b_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b_{r-1} & -1 \\ & & & -1 & b_r \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ \vdots \\ j_{r-1} \\ j_r \end{pmatrix} = \begin{pmatrix} b_1 j_1 - j_2 \\ -j_1 + b_2 j_2 - j_3 \\ \vdots \\ -j_{r-2} + b_{r-1} j_{r-1} - j_r \\ -j_{r-1} + b_r j_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ j_{r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ n \end{pmatrix},$$

one has

$$i_t = \eta_{t1} \quad \text{and} \quad j_t = \eta_{tr} \tag{16-1}$$

for $1 \le t \le r$.

Let D_t be the divisor on U corresponding to the ray $\mathbb{R}_{\ge 0}(j_t, i_t, n-(i_t+j_t))$ in $N_{\mathbb{R}}$. Since a line bundle on Y is determined by the degrees of the restrictions to the exceptional curves, the fact that

$$\begin{aligned} \deg \mathcal{O}(-E_{i_s})|_{Y \cap D_t} &= \deg \mathcal{R}_{i_s}^{\otimes n}|_{Y \cap D_t} \\ &= n \deg \mathcal{R}_{i_s}|_{Y \cap D_t} \\ &= n \deg \mathcal{M}_{i_s}|_{C_t} \\ &= n\delta_{st} \end{aligned}$$

implies the following:

Lemma 16.3 *We can write*

$$E_{i_s} = \sum_{t=1}^{r} \eta_{st} D_t + (\text{sum of other exceptional divisors}).$$

Therefore, for an integer $d = \sum_t d_t i_t$ as in Theorem 4.2, the coefficient of D_t in E_d is $\sum_s d_s \eta_{st}$.

For integers $f, g \in [0, n-1]$, the space of rational sections of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ has

$$\{x^a y^b z^c \mid a + bq - c(1+q) \equiv g - f \mod n\}$$

as a basis. Write $f = \sum_t f_t i_t$ and $g = \sum_t g_t i_t$ as in Theorem 4.2.

Corollary 16.4 *For integers a, b, c with $a + bq - c(1+q) \equiv g - f \mod n$, the order of zero of the rational section $x^a y^b z^c$ of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ along D_t is given by the integer*

$$e_t := \frac{1}{n}\left(a j_t + b i_t + c(n - (i_t + j_t)) - \sum_{s=1}^{r} (g_s - f_s)\eta_{st}\right). \tag{16-2}$$

Proof The order of zero of $x^a y^b z^c$ along D_t as a section of $\mathcal{O}_U$ is given by

$$aj_t + bi_t + c(n - (i_t + j_t)),$$

and the difference between the order of zero as a section of $\mathcal{O}_U(-E_d) \cong \mathcal{R}_d^{\otimes n}$ and that of $\mathcal{O}_U$ is given by $\sum_s (g_s - f_s)\eta_{st}$. □

It follows from Corollary 16.4 that a rational section $x^a y^b z^c$ of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ is holomorphic on U' only if

$$a \geq 0, \quad b \geq 0, \quad \text{and} \quad e_t \geq 0 \quad (1 \leq t \leq r). \tag{16-3}$$

By substituting $t = 1$ in (16-2), one obtains

$$\begin{aligned} e_1 &= \frac{1}{n}\left(aj_1 + bi_1 + c(n - (i_1 + j_1)) - \sum_{s=1}^{r}(g_s - f_s)\eta_{s1}\right) \\ &= \frac{1}{n}\left(a + bq + c(n - (q+1)) - \sum_{s=1}^{r}(g_s - f_s)i_s\right) \\ &= \frac{1}{n}(a + bq + c(n - 1 - q) - (g - f)), \end{aligned}$$

and the condition $a + bq - c(1+q) \equiv g - f \mod n$ is satisfied if e_1 is an integer.

By multiplying (16-2) by the matrix C, one obtains

$$\begin{aligned} \sum_{t=1}^{r} c_{st}e_t &= \frac{1}{n}\sum_{t=1}^{r} c_{st}\left(aj_t + bi_t + c(n - (i_t + j_t)) - \sum_{u=1}^{r}(g_u - f_u)\eta_{ut}\right) \\ &= \frac{1}{n}\sum_{t=1}^{r} c_{st}\left(a\eta_{tr} + b\eta_{t1} + c(n - (\eta_{t1} + \eta_{tr})) - \sum_{u=1}^{r}(g_u - f_u)\eta_{ut}\right) \\ &= a\delta_{sr} + b\delta_{s1} + c\left(\sum_{t=1}^{r} c_{st} - \delta_{s1} - \delta_{sr}\right) - (g_s - f_s), \end{aligned}$$

which gives

$$\begin{cases} b - b_1e_1 + e_2 = g_1 - f_1 - (b_1 - 2)c, \\ e_{t-1} - b_te_t + e_{t+1} = g_t - f_t - (b_t - 2)c \qquad (2 \leq t \leq r-1), \\ e_{r-1} - b_re_r + a = g_r - f_r - (b_r - 2)c. \end{cases} \tag{16-4}$$

If $x^a y^b z^c$ is a holomorphic section of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ on U', then the solution $(e_t) \in \mathbb{Z}^r$ to (16-4) must satisfy (16-3). Putting $e_0 := b$ and $e_{r+1} := a$, we consider the second difference

$$e_t'' := e_{t-1} - 2e_t + e_{t+1}$$

for $1 \le t \le r$. Then (16-4) can be written as

$$e_t'' = g_t - f_t + (b_t - 2)(e_t - c) \quad (1 \le t \le r). \tag{16-5}$$

If $e_t'' \ge 0$ for all t, then the function $t \mapsto e_t$ is convex. This is not true in general, but the situation is very close, as we will see now. To estimate e_t'' from below, we use the following lemma:

Lemma 16.5 *Let $e \ge 0$, $b_t \ge 2$, $f_t \le b_t - 1$ and $c < 0$ be integers. Then:*

(1) $-f_t + (b_t - 2)(e - c) \ge -1$.

(2) *If $-f_t + (b_t - 2)(e - c) = -1$, then $f_t = b_t - 1$.*

(3) *If $-f_t + (b_t - 2)(e - c) = 0$, then $f_t \ge b_t - 2$.*

We omit the proof, which is elementary and straightforward. Since $(f_1, \dots, f_r)$ satisfies the condition in Lemma 4.5, this implies the following:

Corollary 16.6 *Suppose $(e_t)_{t=0}^{r+1} \in \mathbb{Z}^{r+2}$ is an integer solution to the difference equation* (16-5) *for $c < 0$, and $f = \sum_t f_t i_t$, $g = \sum_t g_t i_t$ as in Theorem 4.2. Then we have the following:*

(1) *For a fixed t, $e_t \ge 0$ implies $e_t'' \ge -1$.*

(2) *If $e_s'' = e_t'' = -1$ for $s < t$ and $e_u \ge 0$ for any $u \in [s, t]$, then there is an $l \in (s, t)$ with $e_l'' \ge 1$.*

(3) *If $e_{\alpha-1} > e_\alpha \ge 0$ for some $\alpha \ge 1$, then we have $e_0 \ge \dots \ge e_{\alpha-1} > e_\alpha$.*

(4) *If $0 \le e_\beta < e_{\beta+1}$ for some $\beta \le r$, then we have $e_\beta < e_{\beta+1} \le \dots \le e_{r+1}$.*

In particular, if $e_t \ge 0$ for all t, then there are integers p and p' with $0 \le p \le p' \le r+1$ such that

$$e_0 \ge \dots \ge e_{p-1} > e_p = \dots = e_{p'} < e_{p'+1} \le \dots \le e_{r+1}. \tag{16-6}$$

The following is the key to the proof of Proposition 16.1:

Lemma 16.7 *Let $x^a y^b z^c$ be a rational section of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ satisfying* (16-3)*. If c is negative, then there exist a special representation i_s and a rational section $x^{a'} y^{b'} z^c$ of $\mathcal{R}_f^\vee \otimes \mathcal{R}_{i_s}$ satisfying $0 \le a' \le a$, $0 \le b' \le b$, and*

$$h_t := \frac{1}{n}\left(a' j_t + b' i_t + c(n - (i_t + j_t)) - \sum_u (\delta_{us} - f_u)\eta_{ut}\right) \ge 0 \quad (1 \le t \le r).$$

Proof Since the claim is obvious if g is special, we assume that g is not special. First, note that it suffices to show that for a suitable choice of s there is a solution $(h_0, \dots, h_{r+1}) \in (\mathbb{Z}_{\geq 0})^{r+2}$ to

$$h_t'' = \delta_{ts} - f_t + (b_t - 2)(h_t - c) \quad (1 \leq t \leq r), \tag{16-7}$$

with $0 \leq h_t \leq e_t$ for $0 \leq t \leq r+1$. Indeed, if (h_t) is such a solution, then $a' := h_{r+1}$ and $b' := h_0$ determine a desired rational section $x^{a'} y^{b'} z^c$ of $\mathcal{R}_f^\vee \otimes \mathcal{R}_{i_s}$. Note also that an integer solution $(h_t) \in \mathbb{Z}^{r+2}$ satisfying (16-7) (without the assumption $h_t \geq 0$) is determined by any two consecutive values $h_\alpha, h_{\alpha+1}$. Thus all we have to do is to choose suitable s and values $h_\alpha, h_{\alpha+1}$ for some α such that the corresponding solution $(h_t) \in \mathbb{Z}^{r+2}$ to (16-7) satisfies $0 \leq h_t \leq e_t$.

Let $0 \leq p \leq p' \leq r+1$ be as in (16-6) and put

$$e := e_p (= e_{p'}),$$

which is the minimum value of e_t. We note that if $p < t < p'$, then $e_t'' = 0$ and

$$-f_t + (b_t - 2)(e - c) = e_t'' - g_t = -g_t \leq 0. \tag{16-8}$$

Let q be the integer determined by

$$q := \max \{t \in \mathbb{Z} \mid 1 \leq t \leq p \text{ and } -f_t + (b_t - 2)(e - c) > 0\}$$

if this set is non-empty, and put $q = 0$ otherwise. Similarly, let q' be the integer determined by

$$q' := \min \left\{t \in \mathbb{Z} \mid p' \leq t \leq r \text{ and } -f_t + (b_t - 2)(e - c) > 0\right\}$$

if this set is non-empty, and put $q' = r+1$ otherwise. Since we have (16-8) for $t \in (p, p')$, our choice of q and q' implies

$$-f_t + (b_t - 2)(e - c) \leq 0 \quad (q < t < q'). \tag{16-9}$$

We first consider the case where there is an integer $v \in (q, q')$ such that

$$-f_v + (b_v - 2)(e - c) < 0.$$

In this case, we have $f_v = b_v - 1$ and $-f_v + (b_v - 2)(e - c) = -1$ by Lemma 16.5. Such an integer $v \in (q, q')$ is unique by (16-9), Lemma 4.5 and Lemma 16.5. Thus, if $t \in (q, q')$ and $t \neq v$, then

$$-f_t + (b_t - 2)(e - c) = 0. \tag{16-10}$$

Now we choose s as follows:

(1) If $v \in [p, p']$, then $s := v$.

(2) If $v < p$, then $s := p$.

(3) If $v > p'$, then $s := p'$.

Note that $e_s = e$ and $q < s < q'$ in all cases. We have $e''_s \geq 0 > -f_s + (b_s - 2)(e_s - 2)$ in (1) and $e''_s > 0 = -f_s + (b_s - 2)(e_s - 2)$ in (2) and (3). Thus $e''_s > -f_s + (b_s - 2)(e_s - 2)$ holds in all cases and we obtain $g_s > 0$. This means that

$$\delta_{st} \leq g_t$$

holds for any t.

Now we define (h_t) satisfying (16-7) by the following two consecutive values:

(1) If $v \in [p, p']$, then $h_p = h_{p+1} = e$.

(2) If $v < p$, then $h_p = h_{p+1} = e$.

(3) If $v > p'$, then $h_{p'-1} = h_{p'} = e$.

Then, by (16-10) and by our choice of q, q' and s, it satisfies the following conditions in each case:

(1) $h_{q-1} > h_q = \cdots = h_{q'} < h_{q'+1}$.

(2) $h_{p-1} > h_p = \cdots = h_{q'} < h_{q'+1}$.

(3) $h_{q-1} > h_q = \cdots = h_{p'} < h_{p'+1}$.

By Corollary 16.6, we see that $h_t \geq e \geq 0$ for any t. To compare h_t and e_t, note that ($h_p = e_p$ and $h_{p+1} \leq e_{p+1}$) or ($h_{p'-1} \leq e_{p'-1}$ and $h_{p'} = e_{p'}$) hold. Moreover, by our choice of s, we have $\delta_{st} \leq g_t$ for any t. Therefore, we inductively obtain $h''_t \leq e''_t$ and $h_t \leq e_t$.

The case where there is no such v is similar and easier. If $q \neq q'$, we can take any s with $g_s > 0$ and we can define (h_t) by $h_q = h_{q+1} = e$. When $q = q'$, we have $e''_q = g_q - f_q + (b_q - 2)(e_q - 2) \geq 2$. If $-f_q + (b_q - 2)(e_q - 2) = 1$, then since $g_q > 0$, we can take $s = q$ and we can define (h_t) by $h_q = e$, $h_{q+1} = e + 1$. If $-f_q + (b_q - 2)(e_q - 2) \geq 2$, then take any s with $g_s > 0$ and define (h_t) by $h_q = e$, $h_{q+1} = e + 1$. □

Now we prove Proposition 16.1:

Proof of Proposition 16.1 Recall that a path in Λ' is obtained by concatenating paths in Λ and inverse arrows to the arrows in Λ corresponding to multiplication by z from special representations. We show that if $x^a y^b z^c$ is a rational section of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ satisfying (16-3), then there is a path in Λ' from f to g that is mapped to $x^a y^b z^c$. Since the assertion is obvious if c is non-negative, we assume that c is negative. Then we have s, a' and b' as in Lemma 16.7. We can regard $x^{a'} y^{b'} z^{c+1}$ as a rational map from $\mathcal{R}_f$ to $\mathcal{R}_{i_s+n-q-1}$, whose orders of zeros along the divisors D_t are the same as those of $x^{a'} y^{b'} z^c$ by Corollary 13.2. Therefore, we can represent the rational map $x^a y^b z^c \colon \mathcal{R}_f \to \mathcal{R}_g$ as the product of the rational maps

$$x^{a'} y^{b'} z^{c+1} \colon \mathcal{R}_f \to \mathcal{R}_{i_s+n-q-1}, \quad z^{-1} \colon \mathcal{R}_{i_s+n-q-1} \to \mathcal{R}_{i_s}, \quad x^{a-a'} y^{b-b'} \colon \mathcal{R}_{i_s} \to \mathcal{R}_g.$$

The last rational map corresponds to a path in the McKay quiver and we can prove the assertion by induction on $-c$. □

The proof of Proposition 16.1 also shows the following:

Corollary 16.8 *A rational section $x^a y^b z^c$ of $\mathcal{R}_f^\vee \otimes \mathcal{R}_g$ is holomorphic on U' if and only if* (16-3) *is satisfied.*

17 Some technical lemmas

This section is devoted to the proof of technical lemmas on the paths of the quiver associated with a dimer model, which will be needed later. Consider a pair of zigzag paths with adjacent slopes, which gives a corner perfect matching D as in Section 7. We have a functor

$$\phi_{\mathfrak{c}} \colon \Gamma \to \Lambda$$

with respect to the corner $\mathfrak{c}$ corresponding to D, as in Section 9, where Λ is the McKay quiver whose vertices are large hexagons. There is a corner perfect matching $\overline{D}$ of Λ corresponding to D, which consists of the arrows representing multiplications by z.

Lemma 17.1 *Let v be a vertex of Γ.*

(1) *Suppose v is the source of the large hexagon $\phi_{\mathfrak{c}}(v)$ and a path p of Λ starting from $\phi_{\mathfrak{c}}(v)$ does not intersect with $\overline{D}$. Then there is a path $\tilde{p}$ of Γ from v to any vertex in the large hexagon $t(p)$ such that $\phi_{\mathfrak{c}}(\tilde{p}) = p$ and $\tilde{p}$ does not intersect with D.*

(2) *Suppose v is the sink of the large hexagon $\phi_{\mathfrak{c}}(v)$ and a path p of Λ ending at $\phi_{\mathfrak{c}}(v)$ does not intersect with $\overline{D}$. Then there is a path $\tilde{p}$ of Γ from any vertex in the large hexagon $s(p)$ to v such that $\phi_{\mathfrak{c}}(\tilde{p}) = p$ and $\tilde{p}$ does not intersect with D.*

The first assertion follows from the following lemma. We can also show the dual statement, which implies the second assertion above.

Lemma 17.2 *Suppose a vertex v of Γ is the source of the large hexagon $\phi_{\mathrm{c}}(v)$.*

(1) *For any vertex w of Γ in $\phi_{\mathrm{c}}(v)$, there is a path q from v to w with $\phi_{\mathrm{c}}(q) = e_{\phi_{\mathrm{c}}(v)}$ (the idempotent of $\phi_{\mathrm{c}}(v)$) which doesn't contain arrows in D.*

(2) *If a is an arrow of Λ with $s(a) = \phi_{\mathrm{c}}(v)$ and $a \notin \overline{D}$, then there is a path q' from v to the source w of the large hexagon $t(a)$ with $\phi_{\mathrm{c}}(q') = a$ which doesn't contain arrows in D.*

Proof For the first assertion, let w be a vertex in $\phi_{\mathrm{c}}(v)$ and take the minimal path q from v to w inside $\phi_{\mathrm{c}}(v)$. Then, by the construction of the corner perfect matching D, q doesn't contain arrows in D.

For the second assertion, one of the two zigzag paths used to construct the large hexagons contacts both v and w, and one can take the path from v to w on Γ parallel to this zigzag path as q'. □

Lemma 17.3 *Suppose a is an arrow of Γ contained in the perfect matching D. Then there is a path q of Γ with the following properties:*

- *q goes from $s(a)$ to the source w of the large hexagon that is adjacent to the sink u of $\phi_{\mathrm{c}}(t(a))$ by the arrow b in D with $s(b) = w$ and $t(b) = u$.*
- *$\phi_{\mathrm{c}}(bq)$ is equivalent to $\phi_{\mathrm{c}}(a)$.*
- *q doesn't contain arrows in D.*

Proof Recall from Section 5.6 that two paths are equivalent if and only if they have the same homology class and they contain the same number of arrows in D. First assume that a is inside a large hexagon (ie $\phi_{\mathrm{c}}(s(a)) = \phi_{\mathrm{c}}(t(a))$), as in Figure 28, left. Then there is a minimal path q' from $s(a)$ to u inside $\phi_{\mathrm{c}}(t(a))$. In this case, q is obtained by composing q' and the path from u to w that goes around a node. Next, consider the case where a is on one of the two zigzag paths determining large hexagons but not on the other one, as in Figure 28, middle. In this case, q is the path parallel to the zigzag path on which a is lying. Finally, suppose that a is on the intersection of the two zigzag paths, as in Figure 28, right. In this case, b coincides with a and we can put $q = e_{s(a)}$. □

Lemma 17.1 and Lemma 17.3 and its dual yield the following:

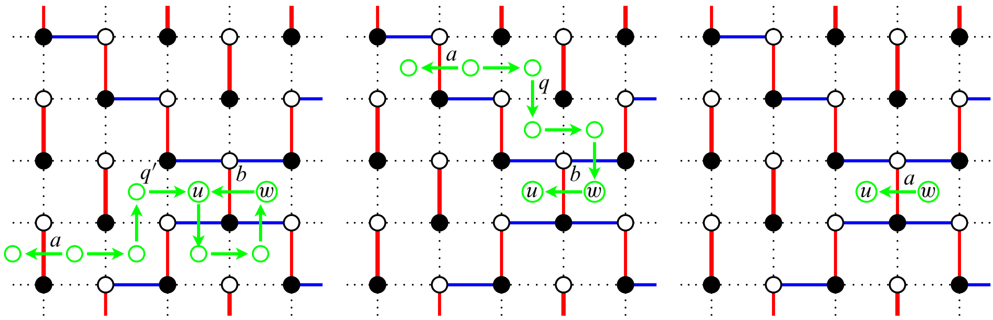

Figure 28: Left: case 1. Middle: case 2. Right: case 3.

Lemma 17.4 *Let a be an arrow of Γ contained in the perfect matching D.*

- *Suppose p is a path from $t(a)$ to the sink u of some large hexagon and p does not contain arrows in D. Let b be the arrow such that $t(b) = u$ and $s(b)$ is the source of the adjacent large hexagon. Then there is a path p' such that pa is equivalent to bp'.*
- *Suppose q is a path from the source u of some large hexagon to $s(a)$ and q does not contain arrows in D. Let c be the arrow such that $s(c) = u$ and $t(c)$ is the sink of the adjacent large hexagon. Then there is a path q' such that aq is equivalent to $q'c$.*

18 Preservation of the tilting condition: The general case

Let Γ be the quiver with relations associated with a consistent dimer model, and Γ' be another quiver obtained from Γ by adding the inverses to the arrows from the sources of special large hexagons to the sinks of the neighboring large hexagons corresponding to multiplication by z. Let $\mathcal{M}$ be the moduli space of representations of Γ with the stability parameter chosen in Section 9, so that $\mathcal{M}$ contains $U = A\text{–Hilb}(\mathbb{C}^3)$ as an open subscheme and $Y = A\text{–Hilb}(\mathbb{C}^2)$ as a closed subscheme for some finite abelian small subgroup A of $\mathrm{GL}_2(\mathbb{C})$. The McKay quiver of A as a subgroup of $\mathrm{SL}_3(\mathbb{C})$ is denoted by Λ. The moduli space $\mathcal{M}$ carries the tautological bundles $\mathcal{L}_v$ corresponding to vertices v of Γ. Let $\mathcal{M}'$ be the complement $\mathcal{M} \setminus Y$ and $\mathcal{L}'_v$ be the restriction of $\mathcal{L}_v$ to $\mathcal{M}'$. The restrictions of $\mathcal{L}_v$ and $\mathcal{L}'_v$ to U and $U' = U \setminus Y$ give the tautological bundle $\mathcal{R}_{\phi_c(v)}$ on $U = A\text{–Hilb}(\mathbb{C}^3)$ and its restriction $\mathcal{R}'_{\phi_c(v)}$ to $U' = A\text{–Hilb}(\mathbb{C}^3) \setminus A\text{–Hilb}(\mathbb{C}^2)$, respectively. We prove the following in this section:

Proposition 18.1 *The tautological bundle $\bigoplus_{v \in V} \mathcal{L}_v$ is a tilting object if and only if so is $\bigoplus_{v \in V} \mathcal{L}'_v$.*

Proof In both directions, we use the long exact sequence

$$\cdots \to H^i_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \to H^i(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \to H^i(\mathcal{M}', {\mathcal{L}'_v}^\vee \otimes \mathcal{L}'_w) \to \cdots. \tag{18-1}$$

Since Y is contained in U, one has $H^i_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w) \cong H^i_Y(U, \mathcal{R}^\vee_{\phi_c(v)} \otimes \mathcal{R}_{\phi_c(w)})$, and the "only if" part follows immediately from Lemma 15.1 and Lemma 15.2.

To show the "if" part, assume that $\bigoplus_v \mathcal{L}'_v$ is a tilting object. In this case, Lemma 15.2 and (18-1) imply the vanishing of $H^i(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w)$ for $i \geq 2$, and for acyclicity it suffices to show the surjectivity of

$$H^0(\mathcal{M}', {\mathcal{L}'_v}^\vee \otimes \mathcal{L}'_w) \to H^1_Y(\mathcal{M}, \mathcal{L}_v^\vee \otimes \mathcal{L}_w). \tag{18-2}$$

Put $\mathcal{L}_{vw} := \mathcal{L}_v^\vee \otimes \mathcal{L}_w$ and note that

$$\begin{aligned} H^1_Y(\mathcal{M}, \mathcal{L}_{vw}) &\cong H^1_Y(U, \mathcal{L}_{vw}|_U) \\ &\cong \varinjlim_l \operatorname{Ext}^1_{\mathcal{O}_U}(\mathcal{O}_{lY}, \mathcal{L}_{vw}|_U) \\ &\cong \varinjlim_l H^0(\mathcal{L}_{vw} \otimes \mathcal{O}_{lY}(lY)), \end{aligned}$$

where the last isomorphism follows from (15-2). Then the surjectivity of (18-2) follows from the surjectivity of

$$H^0(\mathcal{L}_{vw}(lY)) \to H^0(\mathcal{L}_{vw} \otimes \mathcal{O}_{lY}(lY))$$

for each $l > 0$, which is reduced to the surjectivity of

$$H^0(\mathcal{L}_{vw}(lY)) \to H^0(\mathcal{L}_{vw}(lY)|_Y)$$

by induction on l with the aid of the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{L}_{vw}((l-1)Y)) \arrow[r] \arrow[d] & H^0(\mathcal{L}_{vw}(lY)) \arrow[r] \arrow[d] & H^0(\mathcal{L}_{vw}(lY)|_Y) \arrow[d, equal] \\ 0 \arrow[r] & H^0(\mathcal{L}_{vw} \otimes \mathcal{O}_{(l-1)Y}((l-1)Y)) \arrow[r] & H^0(\mathcal{L}_{vw} \otimes \mathcal{O}_{lY}(lY)) \arrow[r] & H^0(\mathcal{L}_{vw}(lY)|_Y) \end{tikzcd}$$

Now, for a fixed l, $H^0(\mathcal{L}_{vw}(lY)|_Y)$ has a basis of the form $x^a y^b z^{-l}$ satisfying (16-3), where we replace c with $-l$. Then Corollary 16.8 shows that it can be lifted to a section of $\mathcal{L}_{vw}|_{U'}$ and therefore is given by a path of Λ' by Proposition 16.1. Moreover, by the proof of Proposition 16.1 and the assumption $l > 0$, the path can be chosen so that it contains an inverse arrow (corresponding to multiplication by z^{-1} to a special representation) but not arrows in the corner perfect matching $\overline{D}$. Since

- an inverse arrow in Λ' can be lifted to an inverse arrow of Γ' going from a sink to a source,
- a path to the source of an inverse arrow in Λ' can be lifted to a path from an arbitrary vertex in the large hexagon to the source of the corresponding inverse arrow in Γ' by the second statement of Lemma 17.1, and
- a path from the target of an inverse arrow in Λ' can be lifted to a path to an arbitrary vertex in the large hexagon from the source of the corresponding inverse arrow in Γ' by the first statement of Lemma 17.1,

the path can be lifted to a path of Γ' from v to w, and (18-2) is surjective.

Finally, we show that $\bigoplus_v \mathcal{L}_v$ is a generator. For an object α of $D^b\mathrm{coh}\,\mathcal{M}$, assume that $\mathbb{R}\mathrm{Hom}\big(\bigoplus_v \mathcal{L}_v, \alpha\big) = 0$. Let s be the source of the large hexagon corresponding to a special representation of A and let t be the sink of the adjacent large hexagon which is the target of multiplication by z from the special representation. Let ι denote the closed immersion $Y \to \mathcal{M}$. Lemma 18.2 below shows that

$$\iota_*\iota^*\mathcal{L}_s^\vee \cong \{\mathcal{L}_t^\vee \to \mathcal{L}_s^\vee\},$$

so that one has

$$\begin{aligned}\mathbb{R}\mathrm{Hom}(\iota^*\mathcal{L}_s, \iota^*\alpha) &= \mathbb{R}\Gamma((\iota^*\mathcal{L}_s)^\vee \otimes \iota^*\alpha) = \mathbb{R}\Gamma(\iota^*\mathcal{L}_s^\vee \otimes \iota^*\alpha)\\ &= \mathbb{R}\Gamma(\iota_*(\iota^*\mathcal{L}_s^\vee \otimes \iota^*\alpha)) = \mathbb{R}\Gamma(\iota_*\iota^*\mathcal{L}_s^\vee \otimes \alpha)\\ &= \mathbb{R}\Gamma(\{\mathcal{L}_t^\vee \to \mathcal{L}_s^\vee\} \otimes \alpha) = 0.\end{aligned}$$

Since $\bigoplus \iota^*\mathcal{L}_s$ is a tilting object on Y by Theorem 2.8, we have $\iota^*\alpha = 0$. It follows that $\mathrm{Supp}\,\alpha \subset \mathcal{M}'$, and we obtain $\alpha = 0$ by our assumption that $\bigoplus_v \mathcal{L}'_v$ is a tilting object. □

Lemma 18.2 *Let s be the source of the large hexagon corresponding to a special representation of A and let t be the target of multiplication by z into the adjacent large hexagon. Then we have an exact sequence*

$$0 \to \mathcal{L}_t^\vee \to \mathcal{L}_s^\vee \to \mathcal{L}_s^\vee|_Y \to 0.$$

Proof Since $\mathcal{M}'$ is the moduli of representations of Γ' by Proposition 13.1, the restriction of the map $\mathcal{L}_s \to \mathcal{L}_t$ to $\mathcal{M}'$ is an isomorphism. Then the assertion follows from Lemma 5.1. □

19 Preservation of surjectivity: The general case

We use the same notation as in Section 18. In particular, the quiver Γ' is obtained from Γ by inverting some of the arrows.

Proposition 19.1 *Assume that both $\bigoplus \mathcal{L}_v$ and $\bigoplus \mathcal{L}'_v$ are tilting objects. Then the map $\mathbb{C}\Gamma \to \mathrm{End}(\bigoplus \mathcal{L}_v)$ is surjective if and only if so is $\mathbb{C}\Gamma' \to \mathrm{End}(\bigoplus \mathcal{L}'_v)$.*

Proof Take a pair (v, w) of vertices of Γ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & e_v\mathbb{C}\Gamma e_w \arrow[r, "\gamma"] \arrow[d, "f"'] & e_v\mathbb{C}\Gamma' e_w \arrow[r, "\delta"] \arrow[d, "g"'] & Q \arrow[r] \arrow[d, "k"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{L}_v, \mathcal{L}_w) \arrow[r, "\alpha"] & \mathrm{Hom}(\mathcal{L}'_v, \mathcal{L}'_w) \arrow[r, "\beta"] & H^1_Y(\mathcal{L}_v^\vee \otimes \mathcal{L}_w) \arrow[r] & 0,
\end{tikzcd}$$

where Q is defined as the cokernel of γ. The second row is exact by (18-1) and our assumption. Moreover, f and g are injective by consistency and hence the first row is also exact. The map k is defined so that the diagram is commutative, and it suffices to show that k is an isomorphism.

In the proof of the surjectivity of (18-2) ($= \beta$), we show that $\beta \circ g$ is surjective and hence k is surjective. To see that k is injective, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & e_v\mathbb{C}\Gamma e_w \arrow[r, "\gamma"] \arrow[d] & e_v\mathbb{C}\Gamma' e_w \arrow[r, "\delta"] \arrow[d, "i"'] & Q \arrow[r] \arrow[d, "j"'] & 0 \\
0 \arrow[r] & e_{\phi_c(v)}\mathbb{C}\Lambda e_{\phi_c(w)} \arrow[r] \arrow[d, "f'"'] & e_{\phi_c(v)}\mathbb{C}\Lambda' e_{\phi_c(w)} \arrow[r, "\delta'"] \arrow[d, "g'"'] & Q' \arrow[r] \arrow[d, "k'"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathcal{R}_{\phi_c(v)}, \mathcal{R}_{\phi_c(w)}) \arrow[r] & \mathrm{Hom}(\mathcal{R}'_{\phi_c(v)}, \mathcal{R}'_{\phi_c(w)}) \arrow[r] & H^1_Y(\mathcal{R}^\vee_{\phi_c(v)} \otimes \mathcal{R}_{\phi_c(w)}) \arrow[r] & 0
\end{tikzcd}$$

where Λ is the McKay quiver whose vertices are large hexagons. Here k' is an isomorphism since f' and g' are isomorphisms.

By Lemma 17.4, any path in $\mathbb{C}\Gamma' \setminus \mathbb{C}\Gamma$ is equivalent to a path that contains an inverse arrow in the intersection of the two zigzag paths (corresponding to multiplication by z^{-1}) but not arrows in the corner perfect matching D. This implies that Q (resp. Q') is isomorphic to the subspace of $e_v\mathbb{C}\Gamma' e_w$ (resp. $e_{\phi_c(v)'}\mathbb{C}\Lambda' e_{\phi_c(w)'}$) spanned by (the classes of) paths that contain inverse arrows but not arrows contained in D. Therefore, the injectivity of j is reduced to the injectivity of i, which follows from Proposition 14.1. Now $H^1_Y(\mathcal{R}^\vee_{\phi_c(v)} \otimes \mathcal{R}_{\phi_c(w)})$ coincides with $H^1_Y(\mathcal{L}_v^\vee \otimes \mathcal{L}_w)$, and $k = k' \circ j$ is injective. □

20 Proof of Theorem 1.4

We prove Theorem 1.4 in this section. Let G be a consistent dimer model. Since any lattice polygon Δ can be turned into a triangle with unit area by successively removing corners, one can find a sequence

$$G = G_0 \mapsto G_1 \mapsto \cdots \mapsto G_k$$

of consistent dimer models, where each step is given by the operation in Theorem 1.1, and the characteristic polygon of G_k is the triangle with unit area.

By Proposition 12.3, the dimer model G_k is determined uniquely by its characteristic polygon. The corresponding quiver is the McKay quiver for the trivial group, and the path algebra is isomorphic to the polynomial algebra in three variables. In this case, the moduli space is the affine space and the tautological bundle is the trivial line bundle, so that the conditions **(T)** and **(E)** are clearly satisfied.

Assume the existence of a derived equivalence

$$\Phi(-) = \mathbb{R}\Gamma\left(\left(\bigoplus_v \mathcal{L}_v\right) \otimes -\right)\colon D^b\mathrm{coh}\,\mathcal{M}_{i,\theta} \to D^b\mathrm{mod}\,\mathbb{C}\Gamma_i$$

for some $i > 0$ between the quiver Γ_i associated with the dimer model G_i and the moduli space $\mathcal{M}_{i,\theta}$ of θ–stable representations of Γ_i for some generic θ. Then we change the stability parameter to the one described in Proposition 1.3. This preserves the conditions **(T)** and **(E)** by [22, Theorem 1.1].

Then we use the "if" part of Theorem 1.5 to show that conditions **(T)** and **(E)** hold for G_{i-1} for some generic stability parameter.

By repeating this process, we show that the conditions **(T)** and **(E)** hold for G with any generic stability parameter, and Theorem 1.4 is proved.

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