

# Pseudo-Anosov mapping classes not arising from Penner's construction

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We show that Galois conjugates of stretch factors of pseudo-Anosov mapping classes arising from Penner's construction lie off the unit circle. As a consequence, we show that, for all but a few exceptional surfaces, there are examples of pseudo-Anosov mapping classes so that no power of them arises from Penner's construction. This resolves a conjecture of Penner.

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## 1 Introduction

Let  $S_{g,n}$  be the orientable surface of genus  $g$  with  $n$  punctures. The mapping class group  $\text{Mod}(S_{g,n})$  is the group of isotopy classes of orientation-preserving homeomorphisms of  $S_{g,n}$ . Thurston's classification theorem [18] states that each element of  $\text{Mod}(S_{g,n})$  is either periodic, reducible, or pseudo-Anosov. An element  $f \in \text{Mod}(S_{g,n})$  is pseudo-Anosov if there is a representative homeomorphism  $\psi$ , a number  $\lambda > 1$ , and a pair of transverse invariant singular measured foliations  $\mathcal{F}^u$  and  $\mathcal{F}^s$  such that

$$\psi(\mathcal{F}^u) = \lambda \mathcal{F}^u \quad \text{and} \quad \psi(\mathcal{F}^s) = \lambda^{-1} \mathcal{F}^s.$$

The number  $\lambda$  is called the stretch factor (or dilatation) of  $f$ .

Isotopy classes of orientation-preserving Anosov maps of the torus can easily be classified as actions of matrices  $M \in \text{SL}(2, \mathbb{Z})$  with  $|\text{tr}(M)| > 2$  on  $\mathbb{R}^2/\mathbb{Z}^2$ . However, it is much harder to give explicit examples of pseudo-Anosov maps on more complicated surfaces.

Thurston [18] gave the first general construction of pseudo-Anosov mapping classes in terms of Dehn twists. After Thurston's work, various other constructions have been developed; see Arnoux and Yoccoz [1], Kra [8], Long [12], Penner [13], Casson and Bleiler [4], and Bestvina and Handel [3]. In this paper, we study Penner's construction [13].

**Penner's construction** Let  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_m\}$  be a pair of multicurves on a surface  $S$ . Suppose that  $A$  and  $B$  are filling, that is,  $A$  and  $B$  are in minimal position and the complement of  $A \cup B$  is a union of disks and once-punctured disks. Then any product of positive Dehn twists about  $a_j$  and negative Dehn twists about  $b_k$  is pseudo-Anosov provided that all  $n + m$  Dehn twists appear in the product at least once.

Penner [14] used this construction to give examples of pseudo-Anosov mapping classes with small stretch factors. (See also Bauer [2] and Leininger [11] for more work on small stretch factors arising from Penner's construction.)

Pseudo-Anosov maps arising from Penner's construction fix the singularities and separatrices of their invariant foliations, and therefore not all pseudo-Anosov mapping classes arise from Penner's construction. However, since the construction is fairly general, Penner conjectured the following.

**Conjecture** (Penner, 1988) *Every pseudo-Anosov mapping class has a power that arises from Penner's construction.*

The conjecture is listed as Problem 4 in Penner [15] and also discussed briefly in Section 14.1.2 of Farb and Margalit [5].

It is a folklore theorem that Penner's construction is true for  $S_{1,0}$  and  $S_{1,1}$  and that it is false for  $S_{0,4}$ , but a modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is true. To the best of our knowledge, no proof of this has appeared in the literature. In the [appendix](#), we give a proof by considering the action of the mapping class group on the curve complex. The main result of this paper is the answer to Penner's conjecture in the remaining nontrivial cases.

We call a pseudo-Anosov mapping class and its stretch factor  $\lambda$  *coronal* if  $\lambda$  has a Galois conjugate on the unit circle.

**Main theorem** *A coronal pseudo-Anosov mapping class has no power coming from Penner's construction. Moreover, there exists a coronal pseudo-Anosov mapping class on  $S_{g,n}$  when  $3g + n \geq 5$ . In particular, Penner's conjecture is false for  $S_{g,n}$  when  $3g + n \geq 5$ .*

We remark that even the modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is false for  $S_{0,n}$  when  $n \geq 5$ .

The proof of the first part of the main theorem is based on the fact that stretch factors of pseudo-Anosov mapping classes arising from Penner's construction appear as Perron-Frobenius eigenvalues of products of certain integral matrices, which depend only on

the intersection numbers of curves. We show that such matrix products may not have eigenvalues on the unit circle other than 1, which implies that pseudo-Anosov stretch factors arising from Penner's construction are not coronal.

The key idea is that an eigenvalue on the unit circle corresponds to a rotation on an invariant plane, which we consider as a dynamical system. Our topological setting provides a natural quadratic form  $h$  which, considered as a height function, plays a role similar to that of Lyapunov functions in stability theory. We show that the products of matrices arising from Penner's construction act by increasing the height, which prohibits rotations on subspaces.

To prove the second part of the main theorem, we use known coronal pseudo-Anosov mapping classes on  $S_{2,0}$  and  $S_{0,5}$  to construct coronal pseudo-Anosov mapping classes on the rest of the surfaces via introducing punctures and taking branched covers.

The main theorem provides a number-theoretic obstruction for pseudo-Anosov maps to arise from Penner's construction: if the stretch factor of  $f$  has a Galois conjugate on the unit circle, then no power of  $f$  can arise from Penner's construction. We do not know whether there are other obstructions.

**Question 1.1** Let  $f$  be a pseudo-Anosov mapping class whose stretch factor does not have Galois conjugates on the unit circle. Does  $f^n$  arise from Penner's construction for some  $n \in \mathbb{N}$ ?

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## 2 Proof of the main theorem

### 2.1 Stretch factors arising from Penner's construction

Let  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_m\}$  be a pair of multicurves on a surface. Introduce the notation

$$(e_1, \dots, e_{n+m}) = (a_1, \dots, a_n, b_1, \dots, b_m).$$

The *intersection matrix* of  $A$  and  $B$  is the symmetric  $(n+m) \times (n+m)$  nonnegative integral matrix  $\Omega = \Omega(A, B)$  whose  $(j, k)$ -entry is the geometric intersection number  $i(e_j, e_k)$ .

**The monoid  $\Gamma(\Omega)$**  Penner showed that actions of the Dehn twists  $T_{a_j}$  and  $T_{b_k}^{-1}$  on  $A \cup B$  can be described by the matrices

$$Q_i = I + D_i \Omega \quad (1 \leq i \leq n + m),$$

where  $I$  is the  $(n + m) \times (n + m)$  identity matrix, and  $D_i$  denotes the  $(n + m) \times (n + m)$  matrix whose  $i^{\text{th}}$  entry on the diagonal is 1 and whose other entries are zero. Any product of  $T_{a_j}$  and  $T_{b_k}^{-1}$ , where each  $a_j$  and each  $b_k$  appear at least once, is pseudo-Anosov, and its stretch factor is given by the Perron–Frobenius eigenvalue of the corresponding product of the matrices  $Q_i$ . Therefore one can study pseudo-Anosov stretch factors arising from Penner’s construction by studying the monoid

$$\Gamma(\Omega) = \langle Q_i : 1 \leq i \leq n + m \rangle$$

generated by the matrices  $Q_i$  depending on  $\Omega$ . For more details, see [13].

**The height function  $h$**  Define the quadratic form  $h: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$  by the equation

$$h(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \Omega \mathbf{v}.$$

Geometrically, the vector  $\mathbf{v}$  corresponds to assigning a real number to each curve in  $A$  and  $B$ . The function  $h$  is the sum of the products of the values of intersecting curves over all intersection points.

The multicurves  $A$  and  $B$  define two transverse cylinder decompositions of the surface. When  $\mathbf{v} > 0$ , the values assigned to the curves can be thought of as the widths of the cylinders. This way we get a singular flat metric on the surface with a rectangle corresponding to each intersection, and the area of this flat surface is  $h(\mathbf{v})$ . When  $\mathbf{v}$  is not positive, one can still think of  $h(\mathbf{v})$  as a signed area. However, it is not clear how this geometric interpretation explains the following interaction between the function  $h(\mathbf{v})$  and the matrices  $Q_i$ .

**Proposition 2.1** 
$$h(Q_i \mathbf{v}) - h(\mathbf{v}) = \|Q_i \mathbf{v} - \mathbf{v}\|^2$$

**Proof** Since all entries on the diagonal of  $\Omega$  are zero, we have  $D_i \Omega D_i = 0$  for all  $i$ , and hence we have

$$\frac{1}{2} Q_i^T \Omega Q_i - \frac{1}{2} \Omega = \frac{1}{2} (I + \Omega D_i) \Omega (I + D_i \Omega) - \frac{1}{2} \Omega = \Omega D_i \Omega.$$

It follows that  $h(Q_i \mathbf{v}) - h(\mathbf{v}) = \|D_i \Omega \mathbf{v}\|^2 = \|Q_i \mathbf{v} - \mathbf{v}\|^2$ . □

**Corollary 2.2** *If  $M \in \Gamma(\Omega)$ , then  $h(M \mathbf{v}) \geq h(\mathbf{v})$  with equality if and only if  $M \mathbf{v} = \mathbf{v}$ .*

**Proposition 2.3** *If  $M \in \Gamma(\Omega)$ , then  $M$  cannot have eigenvalues on the unit circle except 1.*

**Proof** Assume for contradiction that  $M$  has an eigenvalue  $\mu \neq 1$  on the unit circle. Then there exists  $\mathbf{v} \in \mathbb{R}^{n+m}$  and a sequence  $p_i \rightarrow \infty$  of positive integer powers such that  $M\mathbf{v} \neq \mathbf{v}$  and  $M^{p_i}\mathbf{v} \rightarrow \mathbf{v}$ . (If  $\mu \neq -1$ , choose  $\mathbf{v}$  to be any nonzero vector in the two-dimensional invariant subspace on which  $M$  acts by a rotation. If  $\mu = -1$ , choose  $\mathbf{v}$  to be a corresponding eigenvector.) Therefore we have  $h(M\mathbf{v}) > h(\mathbf{v})$ , and hence  $h(M^{p_i}\mathbf{v}) \geq h(M(\mathbf{v}))$  for all  $p_i$  by [Corollary 2.2](#). However, we have  $h(M^{p_i}\mathbf{v}) \rightarrow h(\mathbf{v})$  by continuity, which is a contradiction.  $\square$

## 2.2 Coronal pseudo-Anosov mapping classes

Recall that a pseudo-Anosov mapping class and its stretch factor  $\lambda$  are *coronal* if  $\lambda$  has a Galois conjugate on the unit circle.

**Lemma 2.4** *If a pseudo-Anosov mapping class  $f$  is coronal, then each power of  $f$  is also coronal.*

**Proof** Let  $\lambda$  be the stretch factor of  $f$ . Let  $\sigma$  be an automorphism of the Galois extension  $L/\mathbb{Q}$  with  $|\sigma(\lambda)| = 1$ , where  $L$  is the splitting field of the minimal polynomial of  $\lambda$ . For all  $k \geq 1$ , we have  $|\sigma(\lambda^k)| = |\sigma(\lambda)^k| = 1$ . Therefore  $\lambda^k$ , the stretch factor of  $f^k$ , has a Galois conjugate  $\sigma(\lambda^k)$  on the unit circle.  $\square$

As a consequence of [Proposition 2.3](#) and [Lemma 2.4](#), we have the following.

**Corollary 2.5** (First part of the main theorem) *A coronal pseudo-Anosov mapping class has no power coming from Penner's construction.*

To complete the proof of the main theorem, we need to show that coronal pseudo-Anosov mapping classes exist on all but a few exceptional surfaces.

**Lemma 2.6** *If there exists a coronal pseudo-Anosov mapping class on a surface  $S$ , and there is a branched covering  $\tilde{S} \rightarrow S$ , then there exists a coronal pseudo-Anosov mapping class on  $\tilde{S}$  as well.*

**Proof** If  $f \in \text{Mod}(S)$  is a coronal pseudo-Anosov mapping class, then some power of  $f$  can be lifted to a pseudo-Anosov mapping class  $\tilde{f}$  on  $\tilde{S}$  with the same stretch factor as the power of  $f$  (see [[6](#), Exposé 13, II.1]). By [Lemma 2.4](#),  $\tilde{f}$  is also coronal.  $\square$

**Lemma 2.7** *If there exists a coronal pseudo-Anosov mapping class on a surface  $S$ , then there exists a coronal pseudo-Anosov mapping class on the surface  $S' = S \setminus \{p\}$  with one more puncture as well.*

**Proof** Let  $f \in \text{Mod}(S)$  be a coronal pseudo-Anosov mapping class with stretch factor  $\lambda$  and let  $\psi$  be its representative homeomorphism. Some power  $\psi^k$  has a fixed point  $p$  (see [6, Proposition 9.20] or [5, Theorem 14.19]), and hence  $\psi^k$  induces a pseudo-Anosov homeomorphism of  $S \setminus \{p\}$  with coronal stretch factor  $\lambda^k$ .  $\square$

**Proposition 2.8** (Second part of the main theorem) *There exists a coronal pseudo-Anosov mapping class on  $S_{g,n}$  when  $3g + n \geq 5$ .*

**Proof** On  $S_{2,0}$  there is a coronal pseudo-Anosov mapping class with stretch factor the Perron root of the polynomial  $x^4 - x^3 - x^2 - x + 1$  [19]. For each  $g \geq 3$  there is an unbranched covering of  $S_{2,0}$  by  $S_{g,0}$ . It follows from Lemma 2.6 and Lemma 2.7 that there exists a coronal pseudo-Anosov mapping class on all  $S_{g,n}$  with  $g \geq 2$  and  $n \geq 0$ .

For the genus 0 cases, start from a coronal pseudo-Anosov mapping class on  $S_{0,5}$  with stretch factor the Perron root of  $x^4 - 2x^3 - 2x + 1$  [9]. By Lemma 2.7, there exists a coronal pseudo-Anosov mapping class on  $S_{0,n}$  for each  $n \geq 5$ .

Finally, there is a branched covering  $S_{1,2} \rightarrow S_{0,5}$ , induced by the hyperelliptic involution of  $S_{1,2}$  exchanging the two punctures, which yields a coronal pseudo-Anosov mapping class on  $S_{1,2}$  by Lemma 2.6. (Technically, here  $S_{1,2}$  and  $S_{0,5}$  should be considered surfaces with marked points, not punctures. Because the theory of pseudo-Anosov maps and stretch factors is the same on surfaces with punctures and on surfaces with marked points, we can go back and forth between marked points and punctures as is convenient.) By Lemma 2.7, there exists a coronal pseudo-Anosov mapping class on  $S_{1,n}$  for  $n \geq 2$ .  $\square$

The main theorem immediately follows from Corollary 2.5 and Proposition 2.8.

### 3 Remarks on the Galois conjugates of stretch factors

**Examples of coronal pseudo-Anosov mapping classes** The set of coronal pseudo-Anosov mapping classes is presumably much larger than the set of examples constructed above. For example, the minimal pseudo-Anosov stretch factors tend to be coronal. In fact, when  $g = 2, 3, 4, 5, 7, 8$ , the minimal stretch factors on  $S_g$  among pseudo-Anosov mapping classes with orientable foliations are known, and they are all coronal [10]. The minimal pseudo-Anosov stretch factors on the surfaces  $S_{0,n}$  for  $5 \leq n \leq 9$  are also all coronal with the exception of  $n = 8$  [9].

Not only is the set of coronal pseudo-Anosov mapping classes infinite, but so is the set of coronal stretch factors (even modulo taking powers). This follows from the

first author's examples of pseudo-Anosov mapping classes on  $S_g$  with stretch factor a degree- $2g$  Salem number [16]. Hironaka's infinite family of pseudo-Anosov mapping classes coming from the fibration of a single 3-manifold [7] also seem to consist mostly of coronal pseudo-Anosov mapping classes whose stretch factors can have arbitrarily high algebraic degree.

The abundance of coronal pseudo-Anosov mapping classes are also suggested by computer experiments of Nathan Dunfield and Giulio Tiozzo on random walks in the group of braids with 10 and 14 strands.<sup>1</sup> Using the standard Artin generators, mean length 25, variance 9, and a sample of 100,000 pseudo-Anosov mapping classes, 94% of the stretch factors had Galois conjugates on the unit circle. Computer experiments also show that a random reciprocal polynomial is very likely to have a root on the unit circle. This may suggest that pseudo-Anosov mapping classes arising from Penner's construction are actually rare.

**The location of Galois conjugates** The main theorem of this paper raises the question whether Galois conjugates of Penner stretch factors can at least approach the unit circle. The second author [17] has recently showed that the answer is yes. In fact, it is proved that Galois conjugates of Penner stretch factors are dense in the complex plane. However, the proof uses infinite sequences of pairs of multicurves, and the following question remains open.

**Question 3.1** Can the Galois conjugates of Penner stretch factors arising from a fixed pair of multicurves  $A$  and  $B$  approach the unit circle?

We conjecture that the answer is no, in which case it would be interesting to relate the size of the largest annulus around the unit circle missed by the Galois conjugates to the intersection matrix of  $A$  and  $B$ . Given a pseudo-Anosov mapping class  $f$ , such a result could yield constraints on the pairs  $(A, B)$  that may be used to construct  $f$  using Penner's construction.

## Appendix: Penner's conjecture for the exceptional surfaces

In this appendix, we show that Penner's construction is true for  $S_{1,0}$  and  $S_{1,1}$  and that it is false for  $S_{0,4}$ , but a modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is true.

<sup>1</sup>Personal communication.

**The curve complex** Let  $S$  be one of these three surfaces. The modified curve complex  $\mathcal{C}(S)$  is a graph with vertices the isotopy classes of simple closed curves on  $S$ , where two vertices are connected by an edge if they have minimal intersection number (one for  $S_{1,0}$  and  $S_{1,1}$ , and two for  $S_{0,4}$ ). In all three cases,  $\mathcal{C}(S)$  is isomorphic to the 1-skeleton of the Farey tessellation  $\mathcal{F}$  of the hyperbolic plane (Figure 1). For more details, see [5, Section 4.1.1].

**The action of  $\text{Mod}(S)$**  Let us consider the action of  $\text{Mod}(S)$  on  $\mathcal{C}(S)$ , which gives rise to a homomorphism

$$A: \text{Mod}(S) \rightarrow \text{Isom}^+(\mathcal{F}) \cong \text{PSL}(2, \mathbb{Z})$$

of  $\text{Mod}(S)$  to the orientation-preserving isometries of  $\mathcal{F}$ , once an identification of  $\mathcal{C}(S)$  with  $\mathcal{F}$  is chosen. We denote the image of an element  $f \in \text{Mod}(S)$  by  $A_f$ .

**Actions of Dehn twists** We call an ideal triangle in the complement of  $\mathcal{F}$  a *tile*. A rotation of  $\mathcal{F}$  about a vertex  $v$  of  $\mathcal{F}$  to the left by  $k$  tiles is defined as the parabolic element of  $\text{Isom}^+(\mathcal{F})$  that fixes  $v$  and shifts the tiles adjacent to  $v$  in counterclockwise direction by  $k$ . Rotations to the right are defined analogously.

For a Dehn twist  $T_c$  about a curve  $c$  in  $S$ , the isometry  $A_{T_c}$  is parabolic, and it fixes the vertex of  $\mathcal{F}$  corresponding to  $c$ . Depending on the choice of identification of  $\mathcal{C}(S)$  with  $\mathcal{F}$ , positive Dehn twists can act by rotating  $\mathcal{F}$  to the left or to the right. We choose the identification so that positive Dehn twists correspond to rotations to the right and negative Dehn twists correspond to rotations to the left. Note that Dehn twists on  $S_{1,0}$  and  $S_{1,1}$  act by rotations by one tile, but Dehn twists on  $S_{0,4}$  act by rotations by two tiles. It is the half-twists on  $S_{0,4}$  that correspond to rotations by one tile.

**Actions of pseudo-Anosov elements** For a pseudo-Anosov element  $f \in \text{Mod}(S)$ , the isometry  $A_f$  is hyperbolic in  $\text{PSL}(2, \mathbb{Z})$  and hence  $A_f$  has an invariant geodesic on the hyperbolic plane, called the *axis*  $\gamma$  of  $A_f$ . Since  $f$  does not fix any curve on  $S$ ,  $A_f$  does not fix any vertex of  $\mathcal{F}$ . In particular, the endpoints of the axis  $\gamma$  of  $A_f$  are not vertices of  $\mathcal{F}$ . Therefore  $\gamma$  traverses a bi-infinite sequence of triangles in the Farey tessellation, and it cuts two sides of each triangle.

Associated to  $f$ , there is a bi-infinite sequence of letters  $L$  and  $R$  obtained as follows: travel along  $\gamma$  in the direction of the translation, and for each triangle record if the common vertex of the cut sides is on the left or the right side of  $\gamma$ . This sequence is periodic, because  $A_f$  is a translation along  $\gamma$ . As the following lemma shows, this bi-infinite sequence encodes how the hyperbolic isometry  $A_f$  can be written as a composition of parabolic isometries.



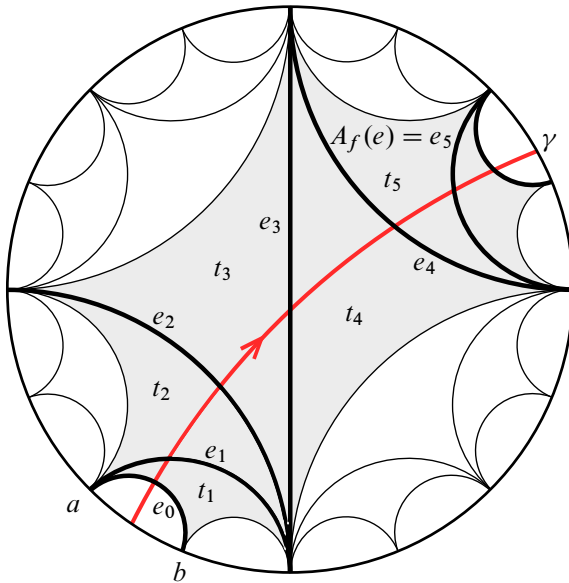


Figure 1: The action  $A_f$  of a pseudo-Anosov mapping class  $f$  on the Farey tessellation

**Lemma A.1** Let  $e_0$  be an edge of  $\mathcal{F}$  intersecting  $\gamma$ . Let  $a$  and  $b$  be the endpoints of  $e_0$  on the left- and right-hand sides of  $\gamma$ , respectively. Let  $e_1, e_2, \dots, e_n = A_f(e_0)$  be edges of  $\mathcal{F}$  intersected by  $\gamma$  such that  $e_{k-1}$  and  $e_k$  are different sides of an ideal triangle  $t_k$  of  $\mathcal{F}$  for all  $1 \leq k \leq n$ . (See Figure 1 for an illustration when  $n = 5$ .) For all  $1 \leq k \leq n$ , define  $s_k$  to be the letter  $L$  or the letter  $R$  depending on whether the common vertex of  $e_{k-1}$  and  $e_k$  is on the left or right side of  $\gamma$ .

Let  $\tau_a$  and  $\tau_b$  be the rotations of  $\mathcal{F}$  by one tile to the right about the points  $a$  and  $b$ , respectively, and introduce the notation

$$\tau(s) = \begin{cases} \tau_a^{-1} & \text{if } s = L, \\ \tau_b & \text{if } s = R. \end{cases}$$

Then

$$A_f = \tau(s_1) \circ \dots \circ \tau(s_n).$$

(By the usual convention for composition of functions, the rotations are applied in right-to-left order.)

**Proof** For all  $1 \leq k \leq n$ , there is a unique  $\phi_k \in \text{Isom}^+(\mathcal{F})$  that maps  $e_0$  to  $e_k$  and  $a$  to the endpoint of  $e_k$  lying on the left-hand side of  $\gamma$ . We have  $\phi_n = A_f$ , so we need to prove that  $\phi_n = \tau(s_1) \circ \dots \circ \tau(s_n)$ . We will prove by induction that  $\phi_k = \tau(s_1) \circ \dots \circ \tau(s_k)$  for all  $1 \leq k \leq n$ .

For  $k = 1$ , we can easily see that the isometry mapping  $e_0$  to  $e_1$  is  $\tau_a^{-1}$  or  $\tau_b$ , depending on whether the common vertex of  $e_0$  and  $e_1$  is  $a$  or  $b$ .

Now assume that the claim is true for  $k$  where  $1 \leq k < n$ , that is,  $\phi_k = \tau(s_1) \circ \dots \circ \tau(s_k)$ . We want to show that  $\phi_{k+1} = \phi_k \circ \tau(s_{k+1})$ . Note that the edges  $e_0$  and  $\tau(s_{k+1})(e_0)$  of  $t_1$  meet on the same side of  $\gamma$  as the edges  $e_k$  and  $e_{k+1}$  of  $t_{k+1}$ . Since we have  $\tau(s_k)(e_0) = e_k$  by the induction hypothesis, this implies

$$\phi_k(\tau(s_{k+1})(e_0)) = e_{k+1}.$$

The right-hand side can also be written as  $\phi_{k+1}(e_0)$ , therefore  $\phi_k \circ \tau(s_{k+1})$  and  $\phi_{k+1}$  map  $e_0$  to the same edge, and their actions on the endpoints also agree. Hence we have  $\phi_{k+1} = \phi_k \circ \tau(s_{k+1})$ , as claimed. □

**Proof of Penner’s conjecture for  $S_{1,0}$ ,  $S_{1,1}$  and  $S_{0,4}$**  Let  $S$  be one of these three surfaces and let  $f$  be any pseudo-Anosov element of  $\text{Mod}(S)$ . We want to show that  $f$  has a power arising from Penner’s construction.

Let  $A_f, \gamma, e_0, a$  and  $b$  be as above. Choose  $a$  and  $b$  for the role of the filling curves in the construction. By Lemma A.1, some product of  $T_a^{-1}$  and  $T_b$  defines an element  $h \in \text{Mod}(S)$  such that  $A_f = A_h$ . Since  $A_h$  is hyperbolic, both Dehn twists must appear in this product. Therefore  $h$  is a pseudo-Anosov mapping class arising from Penner’s construction.

When  $S$  is the torus or the once-punctured torus, we have  $\text{Mod}(S) \cong \text{SL}(2, \mathbb{Z})$ . So  $f = \pm h$  and hence  $f^2 = h^2$ . When  $S = S_{0,4}$ , then  $A$  is surjective with kernel  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  [5, Proposition 2.7]. The kernel is generated by two hyperelliptic involutions and only its identity element fixes all four punctures. Thus two elements of  $\text{Mod}(S_{0,4})$  that project to the same element of  $\text{PSL}(2, \mathbb{Z})$  are equal if they permute the four punctures in the same way. Therefore  $f^{12} = h^{12}$ , because both maps act trivially on the punctures. (The number 12 is the least common multiple of the orders of elements of the symmetric group on 4 points.) Hence  $f$  has a power arising from Penner’s construction.

**Remark** Note that, without allowing half-twists, the conjecture is false for  $S_{0,4}$ . Indeed, an  $LR$ -sequence corresponding to a product of Dehn twists is a sequence of  $LL$  and  $RR$  blocks. So any pseudo-Anosov mapping class that contains the block  $LRL$  in its sequence does not have a power arising from the construction.

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