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The greatest Ricci lower bound, conical Einstein metrics and Chern number inequality

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We partially confirm a conjecture of Donaldson relating the greatest Ricci lower bound $R(X)$ to the existence of conical Kähler–Einstein metrics on a Fano manifold X. In particular, if $D \in |-K_X|$ is a smooth divisor and the Mabuchi K–energy is bounded below, then there exists a unique conical Kähler–Einstein metric satisfying $\mathrm{Ric}(g) = \beta g + (1-\beta)[D]$ for any $\beta \in (0,1)$. We also construct unique conical toric Kähler–Einstein metrics with $\beta = R(X)$ and a unique effective $\mathbb{Q}$–divisor $D \in [-K_X]$ for all toric Fano manifolds. Finally we prove a Miyaoka–Yau-type inequality for Fano manifolds with $R(X) = 1$.

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1 Introduction

The existence of Kähler–Einstein metrics has been a central problem in Kähler geometry since Yau's celebrated solution [47] to the Calabi conjecture. In [47], Yau also successfully extended his study of complex Monge–Ampère equations to those admitting singularities. Constant scalar curvature metrics with conical singularities have been extensively studied by McOwen [25], Troyanov [43] and Luo and Tian [23] for Riemann surfaces. In general, we may consider a pair (X, D) for an n–dimensional compact Kähler manifold and a smooth complex hypersurface D of X. A conical Kähler metric g on X with cone angle $2\pi\beta$ along D is locally equivalent to the model edge metric

$$g = \sum_{j=1}^{n-1} dz_j \otimes d\bar{z}_j + |z_n|^{-2(1-\beta)} dz_n \otimes d\bar{z}_n$$

if D is defined by $z_n = 0$. Applications of conical Kähler metrics were proposed by Tian [37] to obtain various Chern number inequalities. Recently, Donaldson [13] developed a linear theory to study the existence of canonical conical Kähler metrics, and Brendle [5] solved Yau's Monge–Ampère equations for conical Kähler metrics

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with cone angle $2\pi\beta$ for $\beta \in (0, \frac{1}{2})$ along a smooth divisor D. The general case was settled by Jeffres, Mazzeo and Rubinstein [16] for all $\beta \in (0, 1)$. As an immediate consequence, there always exist conical Kähler–Einstein metrics with negative or zero constant scalar curvature with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0, 1)$. When X is a Fano manifold, Donaldson propose to study the conical Kähler–Einstein equation

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D], \tag{1-1}$$

where D is smooth divisor in the anticanonical class $[-K_X]$ and $\beta \in (0, 1)$. One of the motivations is that one can study the existence problem for smooth Kähler–Einstein metrics on X by deforming the cone angle. Such an approach can be regarded as a variant of the standard continuity method.

In particular, since Tian and Yau [41] have already established the existence of a complete Ricci-flat Kähler metric on the noncompact manifold $X \setminus D$, one would expect that (1-1) were solvable for β sufficient small. This was confirmed by Berman [2]. Now the question is how large β can be. The largest β is closely related to the holomorphic invariant known as the greatest Ricci lower bound, first introduced by Tian [36].

Definition 1.1 Let X be a Fano manifold. The greatest Ricci lower bound $R(X)$ is defined by

$$R(X) = \sup\{\beta \mid \mathrm{Ric}(\omega) \geq \beta\omega \text{ for some smooth Kähler metric } \omega \in c_1(X)\}. \tag{1-2}$$

Székelyhidi proved [34] that $[0, R(X))$ is the maximal interval on which one can use the continuity method to solve the Kähler–Einstein equation on a Fano manifold X. In particular, it is independent of the choice of the initial Kähler metric when applying the continuity method. The invariant $R(X)$ was explicitly calculated for $\mathbb{P}^2$ blown up at one point by Székelyhidi [34], and for all toric Fano manifolds by Li [18]. It is well-known that if the Mabuchi K–energy is bounded from below then $R(X) = 1$, and Munteanu and Székelyhidi proved [27] that $R(X) = 1$ implies that X is K–semistable. The following conjecture was proposed by Donaldson [13] to relate $R(X)$ to the existence of conical Kähler–Einstein metrics.

Conjecture 1.2 *There does not exist a conical Kähler–Einstein metric solving (1-1) if $\beta \in (R(X), 1]$, while one does exist if $\beta \in (0, R(X))$.*

This conjecture can be considered as a geometric interpretation of the invariant $R(X)$. The conjecture is also important because it gives a new approach to Yau's conjecture [48, Problem 65] of the equivalence of the existence of a Kähler–Einstein metric on

Fano manifolds and a certain algebro-geometric stability condition, which was refined and extended by Tian [38] and Donaldson [10]. The algebro-geometric aspect of Conjecture 1.2 has been studied by Li [20], Sun [33], Odaka-Sun [28] and Berman [3]. In particular, the notion of Log K–stability was introduced in [20] and [33] as the algebro-geometric obstruction to solving Equation (1-1). In particular, $R(X)$ can be applied to test the Log K–stability of X when it is toric Fano. In [3], Berman proves that Log K–stability is a necessary condition for the solution of (1-1). This naturally leads to the Log version of the Yau–Tian–Donaldson conjecture, that is, to establish the equivalence of the solvability of (1-1) and the Log K–stability of (X, D) (cf [20] and [28]). An interesting observation of Sun [33] is that K–stability implies Log K–stability.

Now let us fix our conventions.

Definition 1.3 Let (X, D) be a compact Kähler manifold together with a smooth divisor $D \subset X$. A Kähler current ω on X with bounded local potentials is said to be *a regular conical Kähler metric if ω is smooth on $X \setminus D$ and Hölder continuous in the sense of [13, Section 4.3] and [6, Section 3.2] (see also [16, Section 2.6.1]) on X.*

Now we describe the main results of the present work. The first one is to partially confirm Conjecture 1.2. We consider a more general class of conical Kähler–Einstein metrics with smooth divisors in any pluricanonical systems, and remove the assumption in Donaldson's [13, Theorem 2] on D by showing there exist no holomorphic vector fields tangential to D (cf Theorem 2.8).

Theorem 1.4 *Let X be a Fano manifold and $R(X)$ be the greatest lower bound of Ricci curvature of X.*

(1) *For any $\beta \in [R(X), 1]$ and any smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$, there does not exist a conical Kähler–Einstein metric ω satisfying*

$$\mathrm{Ric}(\omega) = \beta\omega + \frac{1-\beta}{m}[D] \tag{1-3}$$

if $R(X) < 1$.

(2) *For any $\beta \in (0, R(X))$, there exist a smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$ and a regular conical Kähler–Einstein metric ω satisfying (1-3).*

The second part of the theorem is not completely satisfactory in the sense that one would like to have an m that is independent of $\beta \in (0, R(X))$. In the case when the Mabuchi K–energy is bounded below, or more generally $R(X) = 1$, we show that D does not rely on β.

Theorem 1.5 *Let X be a Fano manifold. If the Mabuchi K–energy (cf Definition 2.6) is bounded below and if $D \in |-K_X|$ is a smooth divisor, then for any $\beta \in (0,1)$ there exists a regular conical Kähler–Einstein metric satisfying the conical Kähler–Einstein equation*

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D].$$

In general, suppose the paired Mabuchi K–energy $\mathcal{M}_{\omega,R(X)}$ (cf (2-7)) for a conical Kähler metric ω with cone angle $2\pi(1-(1-R(X))/m)$ along D is bounded below for a smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$. Then for any $\beta \in (0, R(X))$, there exists a regular conical Kähler–Einstein metric satisfying Equation (1-3).

We would like to remark that most results of Theorem 1.4 and 1.5 were independently obtained by Li and Sun [21]. Theorem 1.5 might have many applications. In particular, if the Mabuchi K–energy is bounded below, there exists a sequence of conical Kähler–Einstein metrics $\mathrm{Ric}(g_\epsilon) = (1-\epsilon)g + \epsilon[D]$ as $\epsilon \to 0$. (X, g_ϵ) might converge in Gromov–Hausdorff topology to a $\mathbb{Q}$–Fano variety X_∞ coupled with a canonical Kähler–Einstein metric. Theorem 1.5 also holds if $R(X) = 1$ and $D \in |-mK_X|$ for some $m \geq 2$ as in the following proposition. By Bertini's theorem, there always exists a smooth divisor $D \in |-mK_X|$ for m sufficiently large.

Proposition 1.6 *Let X be a Fano manifold and $D \in |-mK_X|$ be a smooth divisor for some $m \geq 2$. Then for any $\beta \in (0, (m-1)R(X)/(m-R(X)))$, there exists a regular conical Kähler–Einstein metric ω satisfying (1-3) for D. In particular, when $R(X) = 1$, Equation (1-3) with D is solvable for any $\beta \in (0,1)$.*

The invariant $R(X)$ can also be identified as the optimal constant for the nonlinear Moser–Trudinger inequality. Let X be a Fano manifold and $\omega \in c_1(X)$ be a smooth Kähler metric on X. Let us first recall a version of Ding's [9] F–functional,

$$F_{\omega,\beta} = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X e^{-\beta\varphi}\omega^n, \tag{1-4}$$

where

$$J_\omega(\varphi) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_X \partial\varphi\wedge\bar\partial\varphi\wedge\omega^i\wedge\omega_\varphi^{n-1-i}$$

is the Aubin–Yau functional, $\omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi > 0$ and $V = \int_X \omega^n$. As a corollary of Theorem 1.4, we can establish a connection between $R(X)$ and the Moser–Trudinger inequality.

(1) If $\beta \in (0, R(X))$, $F_{\omega,\beta}$ is bounded below and J–proper (see Definition 2.3) on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$, or equivalently, there exist $\epsilon, C_\epsilon > 0$ such that the Moser–Truding inequality

$$\int_X e^{-\beta\varphi}\omega^n \le C_\epsilon e^{(\beta-\epsilon)J_\omega(\varphi)-(\beta/V)\int_X \varphi\omega^n}$$

holds for $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$.

(2) If $\beta \in (R(X), 1)$, then

$$\inf_{\mathrm{PSH}(X,\omega)\cap L^\infty(X)} F_{\omega,\beta}(\cdot) = -\infty.$$

The properness of the F–functional on Fano Kähler–Einstein manifolds without holomorphic vector fields was first proved by Tian [38] and Tian and Zhu [42]. The J–properness of F was conjectured in this case in [38] and later proved by Phong, Song, Sturm and Weinkove [30]. The presence of the smooth divisor D eliminates the existence of holomorphic vector fields tangent to D, as will be shown in Theorem 2.8. It is interesting to ask whether or not $F_{\omega,\beta}$ is always bounded from below on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ if $\beta = R(X)$. In the case of toric Fano manifolds, $F_{\omega,\beta}$ is indeed bounded from below if $\beta = R(X)$ as a corollary of the following theorem (Corollary 3.15). A more interesting problem will be to understand the limiting behavior of the conical Kähler–Einstein metrics as $\beta \to R(X)$, since holomorphic vector fields will appear in the limiting space. The following theorem serves an example for the above speculation.

Theorem 1.7 *Let X be a toric Fano manifold. Then there exist an effective toric $\mathbb{Q}$–divisor $D \in |-K_X|$, which is unique when $R(X) < 1$, and a* smooth toric *conical Kähler metric ω (cf Section 3.1) unique up to a holomorphic automorphism of X satisfying*

$$\mathrm{Ric}(\omega) = R(X)\omega + (1-R(X))[D]. \tag{1-5}$$

Moreover, $R(X)$ is the largest possible $\beta \in (0,1]$ such that

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D_\beta] \tag{1-6}$$

admits a regular conical toric solution ω_β for an effective *toric $\mathbb{R}$–divisor $D_\beta \in |-K_X|$.*

We remark that the divisor D cannot be smooth, instead it is a union of effective smooth toric $\mathbb{Q}$–divisors with simple normal crossings. Theorem 1.7 is closely related to the results of Li [19] with a different approach for the limiting behavior of the continuity method. The proof of Theorem 1.7 relies on the toric setting introduced by

Donaldson [11; 12] and the estimates in Wang and Zhu [45]. For $\beta > R(X)$, there still exists a regular conical solution for Equation (1-6), however, D_β won't be effective and so the Ricci current of the conical metric cannot be positive. In Theorem 1.7, $R(X)$ will be explicitly calculated as by Li [18] and D is determined by Lemma 3.8 and Lemma 3.9. For example, let X be $\mathbb{P}^2$ blown up at one point, which admits a $\mathbb{P}^1$–ruling $\pi\colon X \to \mathbb{P}^1$ with D_∞ being the section at the infinity. Then $R(X) = \frac{6}{7}$ and $D = 2D_\infty + (H_1 + H_2)/2$, where H_1 and H_2 are the two $\mathbb{P}^1$ fibers invariant under the torus action. This seems to suggest that Donaldson's conjecture might only hold for smooth divisors lying in the *pluri-anticanonical system*. In fact, it was shown by Li and Sun [21] that Theorem 1.7 can be applied to prove Conjecture 1.2 in the toric case when one is allowed to replace $|-K_X|$ by the linear system of a suitable power of $-K_X$.

Finally, we will give some applications of Theorem 1.5. To do that, let us define the conical Ricci curvature of a *regular* conical Kähler metric ω on (X, D) with angle $2\pi\beta$ along D by restricting $\mathrm{Ric}(\omega)$ to $X \setminus D$ (ie $\mathrm{Ric}(\omega) - (1-\beta)[D]$). In general, the conical Kähler metrics do not have bounded curvature tensors, as they might blow up near the divisor, particularly when the cone angle is greater than π. However, we have the following:

Proposition 1.8 *Let X be a Kähler manifold and D be a smooth divisor on X. Let g be a conical Kähler metric on X with cone angle $2\pi\beta$ along D with $\beta \in (0,1)$ satisfying the poly-homogenous expansion introduced in [16, Proposition 4.3] (cf Proposition 4.1). If the Ricci curvature of g is bounded, then the L^2–norm of the curvature tensors of g is also bounded.*

Here the Ricci curvature of a conical Kähler metric g being bounded means that the Ricci curvature of g is uniformly bounded on $X \setminus D$. Proposition 1.8 enables us to define Chern characters, and in particular the Chern numbers for those conical Kähler metrics and derive corresponding Gauss–Bonnet and signature formulas for Kähler surfaces with conical singularities along a smooth holomorphic curve Σ. This is related to recent results of Atiyah and Lebrun [1] for smooth Riemannian 4–folds. In fact, the bound on the L^2–norm of the curvature tensor only depend on the scalar curvature bound and topological invariants such as intersection numbers among D and the first and second Chern classes. In [16, Proposition 4.3], the authors prove that conical Kähler–Einstein metrics for smooth divisor $D \in |-mK_X|$ admit poly-homogenous expansions (cf Proposition 4.1). This in particular implies the following Miyaoka–Yau-type inequality (cf [26; 46]).

Theorem 1.9 *Let X be a Fano manifold. If $R(X)=1$, then the Miyaoka–Yau-type inequality*

$$c_2(X)\cdot c_1(X)^{n-2}\geq \frac{n}{2(n+1)}c_1(X)^n \tag{1-7}$$

holds. In general, if $D\in |-K_X|$ is a smooth divisor and if the paired Mabuchi K–energy $\mathcal{M}_{D,\beta}$ is bounded below, then

$$c_2(X)\cdot c_1(X)^{n-2}\geq \frac{n\beta^2}{2(n+1)}c_1(X)^n. \tag{1-8}$$

Remark 1.10 The above result can be obtained as a consequence of log K–stability as long as the equivalence between the existence of conical Kähler–Einstein metrics and log K–stability is established; in particular, the condition $R=1$ should be equivalent to K–semistability.

A parallel argument can be applied to give a complete proof of the Chern number inequality for smooth minimal models of general type by using conical Kähler–Einstein metrics. This approach was first proposed by Tsuji [44], while the analytic estimates seem missing. We remark that the first complete proof for smooth minimal models of general type is due to Zhang [49], who used the Kähler–Ricci flow.

2 $R(X)$ and conical Kähler–Einstein metrics

2.1 Paired energy functionals

We recall the paired energy functionals originally introduced in [2].

Definition 2.1 Let X be a Fano manifold and $\omega\in c_1(X)$ be a Kähler current with bounded local potential and Ω_θ be an integrable nonnegative real-valued (n,n)–current (hence a Hermitian metric on K_X^{-1}) on X whose curvature

$$\theta=-\sqrt{-1}\partial\overline{\partial}\log\Omega_\theta\in c_1(X)$$

is a nonnegative $(1,1)$–current. Let Ω_ω be the integrable nonnegative real-valued (n,n)–current satisfying

$$\sqrt{-1}\partial\overline{\partial}\log\Omega_\omega=\omega.$$

Suppose

$$\int_X(\Omega_\omega)^\beta(\Omega_\theta)^{1-\beta}=V=c_1(X)^n$$

for a fixed $\beta \in (0,1]$. We define the paired F–functional by

$$F_{\omega,\theta,\beta}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X \left(e^{-\varphi}\Omega_\omega\right)^\beta (\Omega_\theta)^{1-\beta} \tag{2-1}$$

for $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, where

$$J_\omega(\varphi) = \frac{\sqrt{-1}}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_X \partial\varphi\wedge\bar\partial\varphi\wedge\omega^i\wedge\omega_\varphi^{n-1-i}$$

is the Aubin–Yau J–functional and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi \geq 0$.

The Euler–Lagrangian equation for (2-1) is given by

$$(\omega+\sqrt{-1}\partial\bar\partial\varphi)^n = (e^{-\varphi}\Omega_\omega)^\beta(\Omega_\theta)^{1-\beta}, \tag{2-2}$$

and the corresponding curvature equation is

$$\mathrm{Ric}(\omega_\varphi) = \beta\omega_\varphi + (1-\beta)\theta. \tag{2-3}$$

When $\beta = 1$, $F_{\omega,\theta,\beta}(\varphi) = F_\omega$ is the original Ding's functional [9]. The paired F–functional also satisfies the cocycle condition by slightly modifying the proof for the original F–functional.

Lemma 2.2 *$F_{\omega,\theta,\beta}$ satisfies the cocycle condition*

$$F_{\omega,\theta,\beta}(\varphi) - F_{\omega_\psi,\theta,\beta}(\varphi-\psi) = F_{\omega,\theta,\beta}(\psi) \tag{2-4}$$

for any $\varphi,\psi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, where $\omega_\psi = \omega + \sqrt{-1}\partial\bar\partial\psi$.

Notice by letting $F(\omega,\omega_\varphi) = F_{\omega,\theta,\beta}(\varphi)$ with $\omega_\varphi = \omega+\sqrt{-1}\partial\bar\partial\varphi$, the above equation can be rewritten as

$$F(\omega,\omega_\varphi) + F(\omega_\varphi,\omega_\psi) + F(\omega_\psi,\omega) = 0.$$

Definition 2.3 We say a functional $G(\cdot)$ is J–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ if there exist $\delta, C_\delta > 0$ such that

$$G(\varphi) \geq \delta J_\omega(\varphi) - C_\delta$$

for all $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$.

Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$. Let s be a defining section of $[D]$. Since $s \in H^0(X, K_X^{\otimes -m})$,

$$\Omega_D = |s|^{-2/m} = (s\otimes\bar s)^{-1/m}$$

can be considered as a smooth nonnegative real (n,n)–form with poles along D of order m^{-1}. Obviously, $\mathrm{Ric}(\Omega_D) = -\sqrt{-1}\partial\bar{\partial}\log\Omega_D = m^{-1}[D]$. We then define the following notation for convenience.

Definition 2.4 Let $D \in |-mK_X|$ be a smooth divisor for some $m \in \mathbb{Z}^+$. We define

$$\begin{aligned}\mathcal{F}_{\omega,\beta}(\varphi) &= F_{\omega,m^{-1}[D],\beta}(\varphi)\\ &= J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X e^{-\beta\varphi}(\Omega_\omega)^\beta(\Omega_D)^{1-\beta},\end{aligned} \tag{2-5}$$

$$F_{\omega,\beta}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X e^{-\beta\varphi}\Omega_\omega. \tag{2-6}$$

To relate the Moser–Trudinger inequality to $R(X)$, we introduce the next definition.

Definition 2.5 Let X be a Fano manifold and $\omega \in c_1(X)$ be a smooth Kähler metric. We define the optimal *Moser–Trudinger constant* by

$$\mathfrak{mt}(X) = \sup\left\{\beta \in (0,1] \,\middle|\, \inf_{\mathrm{PSH}(X,\omega)\cap L^\infty(X)} F_{\omega,\beta}(\cdot) > -\infty\right\}.$$

It is straightforward to verify that the invariant $\mathfrak{mt}(X)$ does not depend on the choice of the Kähler metric $\omega \in c_1(X)$. We also define the paired Mabuchi K–energy for conical Kähler metrics, first introduced in [2], as follows.

Definition 2.6 Let X be a Fano manifold. Suppose ω and ω_φ are two regular conical Kähler metrics in $c_1(X)$ with cone angle $2\pi(1-(1-\beta)/m)$ along a smooth divisor $D \in |-mK_X|$. The paired Mabuchi K–energy for (X,D) is defined by

$$\begin{aligned}\mathcal{M}_{\omega,\beta}(\varphi) &= \mathcal{M}_{\omega,D,\beta}\\ &:= \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{\omega_n}\omega_\varphi^n - \beta(I_\omega - J_\omega)_\omega(\varphi) + \frac{1}{V}\int_X h_\omega(\omega^n - \omega_\varphi^n),\end{aligned} \tag{2-7}$$

where h_ω is the Ricci potential of ω defined by $\sqrt{-1}\partial\bar{\partial}h_\omega = \mathrm{Ric}(\omega) - \omega$, and

$$I_\omega(\varphi) = \sqrt{-1}\sum_{i=0}^{n-1}\int_X \partial\varphi\wedge\bar{\partial}\varphi\wedge\omega^i\wedge\omega_\varphi^{n-i-1}$$

is the Aubin–Yau I–functional. In particular, $\mathcal{M}_{\omega,1}$ is the original functional introduced by Mabuchi (cf Tian [38]).

It is proved in [2, Theorem 1.1] that if the conical Kähler–Einstein equation is solvable for the data (D,β), both $\mathcal{F}_{\omega,\beta}$ and $\mathcal{M}_{\omega,\beta}$ are bounded below $\mathrm{PSH}(X,\omega)\cap C^\infty(X)$. Furthermore, if one is bounded below, the other must also be bounded below, and conversely, if either one of the functionals is J–proper, the Monge–Ampère equation associated to the conical Kähler–Einstein equation admits a bounded solution [2]. Moreover, it follows from the work of Chen, Donaldson and Sun [6, Section 3] (see also [16]) that the solution is a regular conical Kähler–Einstein metric as in Definition 1.3.

2.2 Pluri-anticanonical system

In this section, we will remove the assumption on the nonexistence of holomorphic vector fields tangent to a smooth divisor D in [13, Theorem 2], when a conical Kähler–Einstein metric is constructed by deforming the angle along the divisor D. First, let us recall an elementary fact.

Lemma 2.7 *Let X be a Fano manifold of $\dim X\geq 2$. For any sufficiently large $m\in\mathbb{Z}^+$, there exists a smooth divisor $D\in|-mK_X|$. Moreover, we have*

$$c_1(D)=(1-m)c_1(X)\,|_D=\frac{1-m}{m}[D]\Big|_D.$$

Theorem 2.8 *Let X be a Fano manifold of $\dim X\geq 2$ and D be a smooth divisor in $|-mK_X|$ for some $m\in\mathbb{Z}^+$. Then there is no holomorphic vector field tangent to D.*

Proof First, we claim there that no holomorphic vector field vanishes along D. To achieve this, it suffices to show that

$$H^0(X,TX\otimes K_X^{\otimes m})=0 \tag{2-8}$$

thanks to the exact sequence

$$0\longrightarrow TX\otimes K_X^{\otimes m}\longrightarrow TX\longrightarrow TX\,|_D\longrightarrow 0.$$

Since $TX\otimes K_X\cong\Omega_X^{n-1}$, we have

$$H^0(X,TX\otimes K_X^{\otimes m})=H^0(X,\Omega_X^{n-1}\otimes K_X^{\otimes(m-1)})).$$

If $m>1$, then the right-hand side is 0 by the Kodaira–Akizuki–Nakano vanishing theorem and the fact that K_X is negative. For $m=1$, Equation (2-8) follows from $H^0(X,\Omega_X^{n-1})\cong\overline{H^{n-1}(X,\mathcal{O}_X)}=0$, which is a consequence of the Kodaira vanishing theorem and X being Fano.

Second, we claim that any holomorphic vector field tangent to D must vanish along D. Let us start with $\dim X=2$; by classification we know that any Fano surface X

admitting a nontrivial holomorphic vector field must be isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at $0, 1, 2$ or 3 points. If X is isomorphic to $\mathbb{P}^2$ blown up at $0, 1, 2$ or 3 points, any holomorphic vector field on X is the lifting of a holomorphic vector field on $\mathbb{P}^2$ fixing the blown-up points. So any smooth invariant divisor with *nontrivial* restriction of the holomorphic vector field on X must be $\mathbb{P}^1$, hence $g(D) = 0$. But by Lemma 2.7, $D \in |-mK_X|$ implies that $g(D) \geq 1$, which is a contradiction. Hence the holomorphic vector field tangent to D must vanish along D. The same argument applies to $X = \mathbb{P}^1 \times \mathbb{P}^1$.

So from now on, let us assume that $\dim X \geq 3$. Since X is Fano, we have $\pi_1(X) = 0$ and hence $\pi_1(D) = 0$ by the Lefschetz hyperplane theorem and our assumption $\dim X \geq 3$. Since $m > 0$, either $c_1(D) < 0$ or D is a simply connected Calabi–Yau manifold by Lemma 2.7. In both cases D does not admit any nontrivial holomorphic vector field. So our proof is completed. □

Remark 2.9 Theorem 2.8 was speculated by Donaldson [13] and was first proved in [2] in the case when the holomorphic vector field is Hamiltonian and $m = 1$.

Combined with the openness result in [13], we immediately have the following corollary.

Corollary 2.10 *Let X be a Fano manifold and $D \in |-mK|$ be a smooth divisor for some $m \in \mathbb{Z}^+$. If there exists a regular conical Kähler–Einstein metric satisfying*

$$\mathrm{Ric}(g) = \beta g + \frac{1-\beta}{m}[D]$$

for some $\beta \in (0, 1)$, then there exists $\epsilon > 0$ such that for any β' with $|\beta - \beta'| < \epsilon$, there exists a regular conical Kähler–Einstein metric g' satisfying

$$\mathrm{Ric}(g') = \beta' g + \frac{1-\beta'}{m}[D].$$

2.3 The α–invariant and the Moser–Trudinger inequality

Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$ for some $m \in \mathbb{Z}^+$. Let $\omega' \in c_1(X)$ be a smooth Kähler form and let $\Omega_{\omega'}$ be a smooth volume form on X such that

$$\mathrm{Ric}(\Omega_{\omega'}) = -\sqrt{-1}\partial\bar{\partial}\log\Omega_{\omega'} = \omega'.$$

We now apply the continuity method and consider the following family of equations for $\beta \in [0, 1]$:

$$(\omega' + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{-t\varphi_t}(\Omega_{\omega'})^{\beta}(\Omega_D)^{1-\beta}, \quad t \in [0, \beta]. \tag{2-9}$$

We let

$$S = \{t \in [0, \beta] \,|\, \text{(2-9) is solvable for some } t \text{ with } \omega_t \text{ a regular conical Kähler metric}\}.$$

By the results in [16], $0 \in S$ and S is open. Let $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ for any $t \in S$. The curvature equation of (2-9) is given by

$$\mathrm{Ric}(\omega_t) = t\omega_t + (\beta - t)\omega' + \frac{1-\beta}{m}[D] \geq t\omega_t.$$

Hence the Green function for ω_t is uniformly bounded below by t for all $t \in S$ [16]. Furthermore, let Δ_t be the Laplace operator associated to ω_t. Then

$$\Delta_t \dot{\varphi}_t = -\varphi_t - t\dot{\varphi}_t.$$

Following the argument for the smooth case with slight modification to the conical Kähler metrics, one can show the following proposition. It is proved in a more general setting in [2].

Proposition 2.11 *Let X be a Fano manifold and $D \in |-mK_X|$ be a smooth divisor.*

(1) *If there exists $\beta \in (0, 1]$ and a regular conical Kähler–Einstein metric ω_{KE} satisfying*

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \beta\omega_{\mathrm{KE}} + \frac{1-\beta}{m}[D],$$

then the paired F–functional

$$\mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi) = J_{\omega_{\mathrm{KE}}}(\varphi) - \frac{1}{V}\int_X \varphi\omega_{\mathrm{KE}}^n - \frac{1}{\beta}\log\frac{1}{V}\int_X \left(e^{-\varphi}\Omega_{\omega_{\mathrm{KE}}}\right)^{\beta}(\Omega_D)^{1-\beta}$$

is uniformly bounded below for all $\varphi \in \mathrm{PSH}(X, \omega_{\mathrm{KE}}) \cap L^\infty(X)$.

(2) *If $\omega \in c_1(X)$ is a smooth Kähler metric and the functional $\mathcal{F}_{\omega,\beta}(\varphi)$ is J–proper on $\mathrm{PSH}(X, \omega) \cap L^\infty(X)$ for some $\beta \in (0, 1]$, then there exists a unique regular conical Kähler metric ω_{KE} solving*

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \beta\omega_{\mathrm{KE}} + \frac{1-\beta}{m}[D].$$

The same argument as in the proof of Proposition 2.11 can be applied to prove the following lemma if one replaces $m^{-1}[D]$ by a *smooth* Kähler metric $\theta \in c_1(X)$.

Lemma 2.12 *Let X be a Fano manifold and θ be a smooth Kähler metric in $c_1(X)$.*

(1) *If there exists a smooth Kähler metric ω_θ on X satisfying*

$$\mathrm{Ric}(\omega_\theta) = \beta\omega_\theta + (1-\beta)\theta$$

for some $\beta \in (0,1]$, then

$$F_{\omega_\theta,\theta,\beta}(\varphi) = J_{\omega_\theta}(\varphi) - \frac{1}{V}\int_X \varphi\omega_\theta^n - \frac{1}{\beta}\log\frac{1}{V}\int_X e^{-\beta\varphi}(\Omega_{\omega_\theta})^\beta(\Omega_\theta)^{1-\beta}$$

is uniformly bounded below on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$.

(2) *If $\omega \in c_1(X)$ is a smooth Kähler metric and the functional $F_{\omega,\theta,\beta}(\varphi)$ is J–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ for some $\beta\in(0,1]$, then there exists a unique smooth Kähler metric ω_θ solving*

$$\mathrm{Ric}(\omega_\theta) = \beta\omega_\theta + (1-\beta)\theta.$$

The α–invariant was introduced by Tian [35] to obtain a sufficient condition for the existence of Kähler–Einstein metrics on Fano manifolds. Demailly showed [8] that the α–invariant coincides with the log canonical threshold in birational geometry. It is natural to relate the log canonical threshold for pairs to the paired α–invariant, which was first introduced in [2] as a generalization of the α–invariant.

Definition 2.13 Let X be a Fano manifold and $D \in |-mK_X|$ be a smooth divisor. Let s be a defining section of $[D]$ and h be a smooth Hermitian metric on $-mK_X$. Let $\omega \in c_1(X)$ be a smooth Kähler metric. Then we define the paired α–invariant for $\beta\in(0,1]$ by

$$\begin{aligned}\alpha_{D,\beta}(X) &= \sup\left\{\alpha>0 \;\middle|\; \sup_{\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)}\int_X |s|_h^{-\frac{2(1-\beta)}{m}} e^{-\alpha\beta(\varphi-\sup\varphi)}\omega^n < \infty\right\}.\end{aligned} \tag{2-10}$$

It is straightforward to check that the invariant $\alpha_{D,\beta}$ does not depend on the choice of *smooth* Hermitian metric h and Kähler metric $\omega\in c_1(X)$. The following existence theorem was first proved by Berman in [2, Section 6], where he constructed a unique Hölder continuous conical Kähler–Einstein metrics via an effective estimate of $\alpha_{D,\beta}$. By [6, Section 3] (see also [16, Section 8]), we know that this conical Kähler–Einstein metric is in fact *regular* in the sense of Definition 1.3.

Theorem 2.14 *There exists $\beta_D \in (0,1]$ such that for all $\beta \in (0,\beta_D]$ we have*

$$\alpha_{D,\beta}(X) > \frac{n}{n+1}. \tag{2-11}$$

In particular, there exists a regular conical Kähler–Einstein metric $\omega \in c_1(X)$ satisfying

$$\mathrm{Ric}(\omega) = \beta\omega + \frac{1-\beta}{m}[D]$$

for $\beta \in (0,\beta_D)$.

In [32], the first author proves that if the α–invariant on an n–dimensional Fano manifold is greater $n/(n+1)$, then the F–functional is J–proper. The following theorem is a generalization to the conical case.

Theorem 2.15 *Let X be a Fano manifold and $\omega \in c_1(X)$ be a smooth Kähler metric. If $D \in |-mK_X|$ is a smooth divisor and if there exists $\beta \in (0,1]$ such that*

$$\alpha_{D,\beta}(X) > \frac{n}{n+1},$$

then the functional

$$\mathcal{F}_{\omega,\beta}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X \left(e^{-\varphi}\Omega_\omega\right)^\beta (\Omega_D)^{1-\beta}$$

as in Definition 2.4 is J_ω–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$.

Proof We break the proof into three steps.

Step 1 Since $\alpha_{D,\beta}(X) > n/(n+1)$, by Theorem 2.14 there exists a regular conical Kähler–Einstein metric ω_{KE} satisfying

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \beta\omega_{\mathrm{KE}} + \frac{1-\beta}{m}[D].$$

Let $\mathrm{PSH}(X,\omega_{\mathrm{KE}},K)$ be the set of all $\varphi \in \mathrm{PSH}(X,\omega_{\mathrm{KE}})\cap L^\infty(X)$ such that

$$\mathrm{osc}_X\varphi = \sup_X \varphi - \inf_X \varphi \le (n+1)J_{\omega_{\mathrm{KE}}}(\varphi) + K. \tag{2-12}$$

We claim that $\mathcal{F}_{\omega_{\mathrm{KE}},\beta}$ is $J_{\omega_{\mathrm{KE}}}$–proper for all $\varphi \in \mathrm{PSH}(X,\omega_{\mathrm{KE}},K)$. To see that, take α satisfying

$$\frac{n\beta}{n+1} < \alpha < \beta\min(\alpha_{D,\beta},1) \le \beta$$

and let $\Omega_D = |s_D|^{2/m}$ with $s_D \in H^0(K_X^{\otimes -m})$ being a section defining D. Then

$$\begin{aligned}\frac{1}{V}\int_X e^{-\beta\varphi}(\Omega_D)^{1-\beta}(\omega_{\mathrm{KE}}^n)^\beta &= \frac{1}{V}\int_X e^{-\alpha(\varphi-\sup\varphi)+(\alpha-\beta)\varphi-\alpha\sup\varphi}(\Omega_D)^{1-\beta}(\omega_{\mathrm{KE}}^n)^\beta\\ &\le \frac{C}{V}e^{(\alpha-\beta)\inf\varphi-\alpha\sup\varphi}\int_X e^{-\alpha(\varphi-\sup\varphi)}(\Omega_D)^{1-\beta}(\omega_{\mathrm{KE}}^n)^\beta\\ &\le \frac{C}{V}e^{(\alpha-\beta)\inf\varphi-\alpha\sup\varphi}\end{aligned}$$

by the definition of $\alpha_{D,\beta}$. By assumption (2-12), we have

$$\begin{aligned}\frac{1}{V}\int_X e^{-\beta\varphi}(\Omega_D)^{1-\beta}(\omega_{\mathrm{KE}}^n)^\beta &\le Ce^{(\beta-\alpha)(n+1)J_{\omega_{\mathrm{KE}}}(\varphi)-\beta\sup\varphi)}\\ &\le Ce^{(n+1)(\beta-\alpha)J_{\omega_{\mathrm{KE}}}(\varphi)-(\beta/V)\int_X\varphi\omega_{\mathrm{KE}}^n}\\ &= Ce^{\beta J_{\omega_{\mathrm{KE}}}(\varphi)-((n+1)\alpha-n\beta)J_{\omega_{\mathrm{KE}}}(\varphi)-(\beta/V)\int_X\varphi\omega_{\mathrm{KE}}^n}.\end{aligned}$$

By taking logarithm of both sides of the above inequality, we obtain

$$\mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi) \ge \left((n+1)\frac{\alpha}{\beta}-n\right)J_{\omega_{\mathrm{KE}}}(\varphi)-C,$$

hence our claim follows.

Step 2 Now we will remove the assumption (2-12) for $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$. We first consider all $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\omega' = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar\partial\varphi$ is a regular conical Kähler metric with cone angle $2\pi(1-(1-\beta)/m)$ along D.

Following [39, Section 6.2], we consider the following family of Monge–Ampère equations:

$$(\omega' + \sqrt{-1}\partial\bar\partial\varphi_t)^n = \left(e^{-t\varphi_t}\Omega_{\omega'}\right)^\beta \Omega_D^{1-\beta}, \quad t\in[0,1]. \tag{2-13}$$

Since Equation (2-13) can be uniquely solved for $t=1$ and there exists no nontrivial holomorphic vector field tangential to D by Theorem 2.8, by the implicit function theorem in [13, Theorem 2] we deduce that Equation (2-13) is solvable for all t in a neighborhood of 1. In particular, we have $\varphi_1 = -\varphi$. Then, by an argument completely parallel to [39, Section 6.2], we obtain that (2-13) is actually solvable for all $t\in[0,1]$.

Let $\omega_t = \omega' + \sqrt{-1}\partial\bar\partial\varphi_t$. Note that the Ricci curvature of $\omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar\partial(\varphi_t-\varphi_1) = \omega_t$ is no less than $\beta/2$ for $t\ge\frac{1}{2}$. Then the Green functions for both ω and ω_t are uniformly bounded from below by $-G$ for some positive number G by [16, Lemma 6.7], and

$$\Delta_{\omega_{\mathrm{KE}}}(\varphi-\varphi_1) = \mathrm{tr}_{\omega_{\mathrm{KE}}}(\omega_t-\omega_{\mathrm{KE}}) \ge -n, \quad \Delta_{\omega_t}(\varphi_t-\varphi_1)\le n.$$

Then by Green's formula, for $t \geq \frac{1}{2}$, we have

$$\frac{1}{V}\int_X (\varphi_t-\varphi_1)\omega_{\mathrm{KE}}^n - nG \leq (\varphi_t-\varphi_1) \leq \frac{1}{V}\int_X (\varphi_t-\varphi_1)\omega_t^n + nG. \tag{2-14}$$

Recall from [39, Chapter 6] that for any smooth Kähler form ω on X, one can define

$$I_\omega(\psi) := \frac{1}{V}\int_X \psi(\omega^n-\omega_\psi^n) \quad \text{with } \omega_\psi = \omega + \sqrt{-1}\partial\bar\partial\psi,$$

and that I_ω and J_ω satisfy

$$0 \leq I_\omega(\psi) - J_\omega(\psi) \leq nJ_\omega(\psi) \quad \text{for } \psi \in \mathrm{PSH}(X,\omega). \tag{2-15}$$

Plugging them into (2-14) we obtain

$$\mathrm{osc}_X(\varphi_t-\varphi_1) \leq I_{\omega_{\mathrm{KE}}}(\varphi_t-\varphi_1) + 2nG \leq (n+1)J_{\omega_{\mathrm{KE}}}(\varphi_t-\varphi_1) + 2nG.$$

This implies that $\varphi_t-\varphi_1 \in \mathrm{PSH}(X,\omega_{\mathrm{KE}},2nG)$, and then the $J_{\omega_{\mathrm{KE}}}$–properness holds for $\varphi_t-\varphi_1$, and there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}\mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi_t-\varphi_1) &\geq \left((n+1)\frac{\alpha}{\beta}-n\right)J_{\omega_{\mathrm{KE}}}(\varphi_t-\varphi_1) - C_1\\ &\geq \left(\frac{\alpha}{\beta}-\frac{n}{n+1}\right)\mathrm{osc}_X(\varphi_t-\varphi_1) - C_2.\end{aligned}$$

Consequently, there exist $C_3 > 0$ such that

$$\begin{aligned}n(1-t)J_{\omega_{\mathrm{KE}}}(\varphi) &= n(1-t)J_{\omega'}(\varphi_1)\\ \text{(using (2-15))}\quad &\geq (1-t)(I_{\omega'}(\varphi_1) - J_{\omega'}(\varphi_1))\\ &\geq \int_t^1 (I_{\omega'}(\varphi_s) - J_{\omega'}(\varphi_s))\,ds\\ &= \mathcal{F}_{\omega',\beta}(\varphi_t) - \mathcal{F}_{\omega',\beta}(\varphi_1)\\ \text{(by Lemma 2.2)}\quad &= \mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi_t-\varphi_1)\\ &\geq \left(\frac{\alpha}{\beta}-\frac{n}{n+1}\right)\mathrm{osc}_X(\varphi_t-\varphi_1) - C_3,\end{aligned}$$

where the third inequality follows from the fact that $(I_{\omega'}-J_{\omega'})(\varphi_t)$ is monotonically increasing for $t \in [0,1]$. Then by applying the cocycle condition and the same argument

in the smooth case in [38; 42], we obtain

$$\begin{aligned}\mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi) &= -\mathcal{F}_{\omega',\beta}(\varphi_1) = \int_0^1 \big(I_{\omega'}(\varphi_t) - J_{\omega'}(\varphi_t)\big)dt \\ &\geq (1-t)\big(I_{\omega'}(\varphi_t) - J_{\omega'}(\varphi_t)\big) \geq \frac{1-t}{n} J_{\omega'}(\varphi_t) \\ &\geq \frac{1-t}{n} J_{\omega'}(\varphi_1) - \frac{2(1-t)}{n}\mathrm{osc}_X(\varphi_t - \varphi_1) - C_4 \\ &\geq \frac{1-t}{n^2} J_{\omega_{\mathrm{KE}}}(\varphi) - C_5(1-t)^2 J_{\omega_{\mathrm{KE}}}(\varphi) - C_6.\end{aligned}$$

Since C_5 and C_6 are independent of the choice for $t \geq \frac{1}{2}$, by choosing t sufficiently close to 1, we can find $\epsilon', C_{\epsilon'} > 0$, such that

$$\mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi) \geq \epsilon' J_{\omega_{\mathrm{KE}}}(\varphi) - C_{\epsilon'} \tag{2-16}$$

for all $\varphi \in \mathrm{PSH}(X, \omega_{\mathrm{KE}})$ satisfying that ω_φ is a *regular conical* Kähler metric with cone angle $2\pi(1-(1-\beta)/m)$ along D. We claim that the set of such φ is dense in $\mathrm{PSH}(X, \omega_{\mathrm{KE}}) \cap L^\infty(X)$, from which we deduce that the $J_{\omega_{\mathrm{KE}}}$–properness holds for $\mathrm{PSH}(X, \omega_{\mathrm{KE}}) \cap L^\infty(X)$.

To see that, notice that for any $\phi \in \mathrm{PSH}(X, \omega_{\mathrm{KE}}) \cap L^\infty(X)$, we have $\phi + \psi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ with ψ defined by $\omega_{\mathrm{KE}} = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ and $\sup_X \psi = 0$. Since $\mathrm{PSH}(X, \omega) \cap C^\infty(X)$ is dense in $\mathrm{PSH}(X, \omega) \cap L^\infty(X)$ and for any $\phi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$ we have

$$\phi_\epsilon := \phi + \epsilon\sqrt{-1}\partial\bar{\partial}|s|_h^{2(1-\beta)/m} \longrightarrow \phi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X) \quad \text{as } \epsilon \to 0,$$

and ω_{ϕ_ϵ} is a regular conical Kähler metric for each $0 < \epsilon \ll 1$. Hence our claim follows.

Step 3 Finally, to prove the J_ω–properness for any smooth ω, all we need is to replace $J_{\omega_{\mathrm{KE}}}$ by J_ω for smooth ω in the definition of $\mathcal{F}_{\omega,\beta}$. To do that, let us write $\omega_{\mathrm{KE}} = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ and recall that

$$F^0_\omega(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n$$

satisfies the cocycle condition $F^0_\omega(\varphi) - F^0_{\omega_{\mathrm{KE}}}(\varphi - \psi) = F^0_\omega(\psi)$, which implies that

$$\begin{aligned}J_\omega(\varphi) - J_{\omega_{\mathrm{KE}}}(\varphi - \psi) &= J_\omega(\psi) + \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{V}\int_X (\varphi - \psi)\omega^n_{\mathrm{KE}} - \frac{1}{V}\int_X \psi\omega^n \\ &= J_\omega(\psi) + \frac{1}{V}\int_X (\varphi - \psi)(\omega^n - \omega^n_{\mathrm{KE}}),\end{aligned} \tag{2-17}$$

since by our choice, $\omega + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}(\varphi - \psi) > 0$. Again by [16, Lemma 6.7], the Green function with respect to ω_{KE} is bounded from below, so we obtain

$$\sup(\varphi - \psi) \leq \frac{1}{V}\int_X (\varphi - \psi)\omega_{\mathrm{KE}}^n + C,$$

and hence

$$\sup \varphi - C \leq \frac{1}{V}\int_X (\varphi - \psi)\omega_{\mathrm{KE}}^n \tag{2-18}$$

since ψ is fixed. So by Lemma 2.2, we obtain

$$\begin{aligned}
\mathcal{F}_{\omega,\beta}(\varphi) &= \mathcal{F}_{\omega_{\mathrm{KE}},\beta}(\varphi - \psi) + \mathcal{F}_{\omega,\beta}(\psi)\\
\text{(by (2-16))} \quad &\geq \epsilon' J_{\omega_{\mathrm{KE}}}(\varphi - \psi) - C_{\epsilon'}\\
\text{(by (2-17))} \quad &= \epsilon'\left(J_\omega(\varphi) - J_\omega(\psi) + \frac{1}{V}\int_X (\varphi - \psi)(\omega_{\mathrm{KE}}^n - \omega^n)\right) - C_{\epsilon'}\\
(\psi \text{ is fixed}) \quad &\geq \epsilon'\left(J_\omega(\varphi) + \frac{1}{V}\int_X (\varphi - \psi)\omega_{\mathrm{KE}}^n - \frac{1}{V}\int_X \varphi\omega^n\right) - C\\
\text{(by (2-18))} \quad &\geq \epsilon'\left(J_\omega(\varphi) + \sup\varphi - \frac{1}{V}\int_X \varphi\omega^n\right) - C\\
&\geq \epsilon'\left(J_\omega(\varphi) + \frac{1}{V}\int_X (\sup\varphi - \varphi)\omega^n\right) - C \geq \epsilon' J_\omega(\varphi) - C,
\end{aligned}$$

where the constant $C = C(\omega, \omega_{\mathrm{KE}}, \psi, C_{\epsilon'})$. □

2.4 An interpolation formula

In this section, we will prove the following interpolation formula for the F–functional to obtain J–properness.

Proposition 2.16 *Let X be a Fano manifold and D a smooth divisor in $|-mK_X|$ for some $m \in \mathbb{Z}^+$. Let ω be a smooth Kähler metric in $c_1(X)$. If there exists $\alpha \in (0, 1]$ such that*

$$\inf_{\mathrm{PSH}(X,\omega)\cap L^\infty(X)} \mathcal{F}_{\omega,\alpha}(\,\cdot\,) > -\infty,$$

then $\mathcal{F}_{\omega,\beta}(\varphi)$ is J–proper on $\mathrm{PSH}(X, \omega) \cap L^\infty(X)$ for all $\beta \in (0, \alpha)$.

Proof We want to show that

$$\mathcal{F}_{\omega,\beta}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\frac{1}{V}\int_X (e^{-\varphi}\Omega_\omega)^\beta(\Omega_D)^{1-\beta}$$

is J–proper for all $\beta \in (0, \alpha)$.

First for $0<\tau<\beta<\alpha$, we write $\beta=\tau/p+\alpha/q$ for some $1/p+1/q=1$. Then the Hölder inequality implies the interpolation

$$\begin{aligned}\mathcal{F}_{\omega,\beta}(\varphi)&=J_\omega(\varphi)-\frac{1}{\beta}\log\frac{1}{V}\int_X(e^{-\varphi}\Omega_\omega)^\beta\Omega_D^{1-\beta}\\&=\left(\frac{\tau}{\beta p}+\frac{\alpha}{\beta q}\right)J_\omega(\varphi)-\frac{1}{\beta}\log\frac{1}{V}\int_X\left(e^{-\varphi}\frac{\Omega_\omega}{\Omega_D}\right)^{\tau/p+\alpha/q}\cdot\Omega_D\\&\geq\frac{\tau}{\beta p}\left(J_\omega(\varphi)-\frac{1}{\tau}\log\frac{1}{V}\int_X(e^{-\varphi}\Omega_\omega)^\tau\Omega_D^{1-\tau}\right)\\&\qquad+\frac{\alpha}{\beta q}\left(J_\omega(\varphi)-\frac{1}{\alpha}\log\frac{1}{V}\int_X(e^{-\varphi}\Omega_\omega)^\alpha\Omega_D^{1-\alpha}\right)\\&=\frac{\tau}{\beta p}\cdot\mathcal{F}_{\omega,\tau}(\varphi)+\frac{\alpha}{\beta q}\cdot\mathcal{F}_{\omega,\alpha}(\varphi)\geq\frac{\tau}{\beta p}\cdot\mathcal{F}_{\omega,\tau}(\varphi)-C_1\geq\epsilon J_\omega(\varphi)-C.\end{aligned}$$

The last inequality follows from Theorem 2.15 by choosing τ sufficiently small so that $\alpha_{D,\tau}>n/(n+1)$. □

The same argument in the proof of Proposition 2.16 can be applied to prove the following lemma after replacing $m^{-1}[D]$ by a smooth Kähler metric $\theta\in c_1(X)$.

Lemma 2.17 *Let X be a Fano manifold and θ be a smooth Kähler metric in $c_1(X)$. Let ω be a smooth Kähler metric in $c_1(X)$. If there exists $\alpha\in(0,1]$ such that*

$$F_{\omega,\theta,\alpha}(\varphi)=J_\omega(\varphi)-\frac{1}{V}\int_X\varphi\omega^n-\frac{1}{\alpha}\log\frac{1}{V}\int_X e^{-\alpha\varphi}(\Omega_\omega)^{1-\alpha}(\Omega_\theta)^{1-\alpha}$$

is bounded below on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$, then $F_{\omega,\beta}(\varphi)$ is J–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$ for all $\beta\in(0,\alpha)$.

We remark that Lemma 2.17 also serves as an alternative proof for Theorem 1.1 in [34] relating $R(X)$ and the continuity method.

2.5 Proof of Theorem 1.4

Proposition 2.18 (First part of Theorem 1.4) *Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$ for some $m\in\mathbb{Z}^+$. If there exist $\beta\in(0,1]$ and a regular conical Kähler–Einstein metric ω satisfying*

$$\mathrm{Ric}(\omega)=\beta\omega+\frac{1-\beta}{m}[D],$$

then

$$\beta\leq R(X).$$

In particular, the inequality holds if and only if $\beta=1$.

Proof Let ω_{KE} be a regular conical Kähler–Einstein metric on X satisfying

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \beta\omega_{\mathrm{KE}} + \frac{1-\beta}{m}[D].$$

By Proposition 2.11, we know that $\mathcal{F}_{\omega_{\mathrm{KE}},\beta}$ is bounded below. By Proposition 2.16, $\mathcal{F}_{\omega,\beta'}$ is J–proper for all $\beta' \in (0,\beta)$.

Let $\omega, \theta \in c_1(X)$ be two smooth Kähler metrics on X. The J–properness of $\mathcal{F}_{\omega,\beta'}$ immediately implies the J–properness of

$$F_{\omega,\theta,\beta'}(\varphi) = J_\omega(\varphi) - \frac{1}{v}\int_X \varphi\omega^n - \frac{1}{\beta'}\log\frac{1}{V}\int_X e^{-\beta'\varphi}(\Omega_\omega)^{\beta'}(\Omega_\theta)^{1-\beta'}$$

because Ω_D is strictly bounded below from 0. Then by Lemma 2.12, there exists a minimizer of $F_{\omega,\theta,\beta'}$ which solves the equation

$$\mathrm{Ric}(\omega) = \beta'\omega + (1-\beta')\theta \geq \beta'\omega.$$

This shows that $R(X) \geq \beta'$ and so $R(X) \geq \beta$.

If $\beta = R(X) < 1$, there must exist $\epsilon > 0$ and a regular conical Kähler–Einstein metric g such that $\mathrm{Ric}(g) = (\beta+\epsilon)g + (1-\beta-\epsilon)m^{-1}[D]$ by Corollary 2.10. We get a contradiction to the definition of $R(X)$ by repeating the previous argument. □

Proposition 2.19 (Theorem 1.4) *Let X be a Fano manifold. Then for any $\beta \in (0, R(X))$, there exist a smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$ and a regular conical Kähler–Einstein metric g satisfying*

$$\mathrm{Ric}(\omega) = \beta\omega + \frac{1-\beta}{m}[D].$$

Proof We break the proof into the following steps.

Step 1 Let ω and θ be two smooth Kähler metrics in $c_1(X)$. For any $\beta \in (0, R(X))$, by Szeklyhidi's result [34], the family

$$(\omega + \sqrt{-1}\partial\bar\partial\varphi_t)^n = \left(e^{-t\varphi}\Omega_\omega\right)^\beta(\Omega_\theta)^{1-\beta}, \quad \int_X(\Omega_\omega)^\beta(\Omega_\theta)^{1-\beta} = V$$

of Monge–Ampère equations is solvable for all $t \in [0,1]$. Then by Lemma 2.12,

$$F_{\omega,\beta}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\beta}\log\int_X e^{-\beta\varphi}(\Omega_\omega)^\beta(\Omega_\theta)^{1-\beta}$$

is bounded below on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$. By Lemma 2.17, for any $\beta'\in(0,\beta)$, $F_{\omega,\beta'}(\varphi)$ is J–proper. It immediately follows that for any $\beta\in(0,R(X))$, there exist $\epsilon, C_\epsilon>0$ such that for all $\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)$,

$$\int_X e^{-\beta\varphi}\omega^n \le C_\epsilon e^{(\beta-\epsilon)J_\omega(\varphi)-(\beta/V)\int_X\varphi\omega^n}.$$

Step 2 Let D be a smooth divisor in $|-mK_X|$ for some $m\in\mathbb{Z}^+$ to be determined later. We will later choose m sufficiently large. Let s be a defining section of D and h be a smooth hermitian metric on $-mK_X$. For any $\beta\in(0,R(X))$, there exists $\delta>0$ such that $\beta+\delta<R(X)$. Then

$$\begin{aligned}\int_X |s|_h^{-\frac{2(1-\beta)}{m}} e^{-\beta\varphi}\omega^n &\le \left(\int_X e^{-(\beta+\delta)\varphi}\omega^n\right)^{\frac{\beta}{\beta+\delta}}\left(\int_X |s|_h^{-\frac{2(1-\beta)(\beta+\delta)}{m\delta}}\omega^n\right)^{\frac{\delta}{\beta+\delta}}\\ &\le C_\delta\left(\int_X e^{-(\beta+\delta)\varphi}\omega^n\right)^{\frac{\alpha}{\alpha+\delta}}\end{aligned}$$

if we choose $m>(1-\beta)(\beta+\delta)/\delta$. By the conclusion in Step 1, there exist $\epsilon, C_\epsilon>0$ such that

$$\int_X |s|^{-\frac{2(1-\beta)}{m}} e^{-\beta\varphi}\omega^n \le C_\epsilon e^{(\beta-\epsilon)J_\omega(\varphi)-(\beta/V)\int_X\varphi\omega^n}.$$

Equivalently, $\mathcal{F}_{\omega,\beta}(\varphi)$ is J–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$. By Proposition 2.11, there exists a unique smooth conical Kähler–Einstein metric ω_β solving

$$\mathrm{Ric}(\omega_\beta)=\beta\omega_\beta+\frac{1-\beta}{m}[D].$$ □

Remark 2.20 Since $(1-\beta)(\beta+\delta)/\delta$ is decreasing with respect to δ, there is a $\delta\in(0,R-\beta)$ such that the proof above works as long as $m>(1-\beta)R(X)/(R(X)-\beta)$, which is equivalent to $\beta<R(m-1)/(m-R)$.

Now we can relate the optimal Moser–Trudinger constant to the invariant $R(X)$.

Corollary 2.21 *Let X be a Fano manifold and $\omega\in c_1(X)$ be a smooth Kähler metric.*

(1) *If $\beta\in(0,R(X))$, $F_{\omega,\beta}$ is J–proper on $\mathrm{PSH}(X,\omega)\cap L^\infty(X)$. Equivalently, there exist $\epsilon, C_\epsilon>0$ such that the following Moser–Trudinger inequality holds for all $\varphi\in\mathrm{PSH}(X,\omega)\cap L^\infty(X)$:*

$$\int_X e^{-\beta\varphi}\omega^n \le C_\epsilon e^{(\beta-\epsilon)J_\omega(\varphi)-(\beta/V)\int_X\varphi\omega^n}.$$

(2) *If $\beta\in(R(X),1)$, then*

$$\inf_{\mathrm{PSH}(X,\omega)\cap L^\infty(X)} F_{\omega,\beta}(\cdot)=-\infty.$$

Proof For $\beta \in (0, R(X))$ and a fixed smooth Kähler metric $\theta \in c_1(X)$, there exists a smooth Kähler metric ω satisfying $\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)\theta$. The corollary is an immediate consequence of by modifying the interpolation formula in Proposition 2.16, after replacing $m^{-1}[D]$ by θ. □

Immediately, we can show that $R(X)$ and $\mathfrak{mt}(X)$ take the same value for a Fano manifold X.

Corollary 2.22 *Let X be a Fano manifold. Then*

$$\mathfrak{mt}(X) = R(X) = \sup\{\beta \in (0,1) \mid F_{\omega,\beta} \text{ is } J\text{–proper on } \mathrm{PSH}(X,\omega) \cap L^\infty(X)\}, \tag{2-19}$$

where $\omega \in c_1(X)$ is a smooth Kähler metric.

2.6 Proof of Theorem 1.5

Before proving Theorem 1.5, we first quote the following proposition establishing the equivalence for the Mabuchi K–energy and the F–functional when either of them is bounded below, proved in [22] by applying the Kähler–Ricci flow and Perelman's estimates for the scalar curvature.

Proposition 2.23 *Let X be a Fano manifold and $\omega \in c_1(X)$ be a smooth Kähler metric. Then Ding's functional F_ω is bounded below on $\mathrm{PSH}(X,\omega) \cap C^\infty(X)$ if and only if the Mabuchi K–energy is bounded below.*

Proposition 2.23 holds for the paired Mabuchi K–energy and the paired F–functional as shown in [2]. One can also apply the continuity method for the conical Kähler metrics with positive Ricci curvature as in [31]. Let $\mathrm{PSH}(X,\omega) \cap C^\infty_{D,\beta}(X)$ be the set of all bounded φ such that $\omega + \sqrt{-1}\partial\bar\partial\varphi$ is a regular conical Kähler metric with cone angle $2\pi\beta$ along D.

Proposition 2.24 *Let X be a Fano manifold and $D \in |-mK_X|$ be a smooth divisor. Let ω be a regular conical Kähler metric in $c_1(X)$ with cone angle $2\pi(1-\beta)/m$ along D. Then*

$$\inf_{\mathrm{PSH}(X,\omega)\cap C^\infty_{D,(1-\beta)/m}(X)} \mathcal{M}_{\omega,\beta}(\cdot) > -\infty$$

is equivalent to

$$\inf_{\mathrm{PSH}(X,\omega)\cap L^\infty(X)} \mathcal{F}_{\omega,\beta}(\cdot) > t - \infty.$$

We can now prove Theorem 1.5.

Theorem 2.25 *Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$ for some $m \in \mathbb{Z}^+$. If the paired Mabuchi K–energy $\mathcal{M}_{\omega,R(X)}$ on X is bounded below, then for any $\beta \in (0, R(X))$ there exists a regular conical Kähler–Einstein metric satisfying*

$$\mathrm{Ric}(g) = \beta g + \frac{1-\beta}{m}[D]. \tag{2-20}$$

Proof Let $\omega \in c_1(X)$ be a smooth Kähler metric. By Proposition 2.24, $\mathcal{F}_{\omega,R(X)}(\varphi)$ is bounded below on $\mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Applying the interpolation formula in Proposition 2.16, $\mathcal{F}_{\omega,\beta}$ is J–proper on $\mathrm{PSH}(X,\omega) \cap L^\infty(X)$ for all $\beta \in (0, R(X))$. The theorem follows by Proposition 2.11. □

When the Mabuchi K–energy is bounded below on X, for any $\beta \in (0,1)$, there exists a conical Kähler–Einstein metric satisfying Equation (2-20) for $m = 1$. In this case, D is a smooth Calabi–Yau hypersurface of X. If we only assume $R(X) = 1$, we have the same conclusion in Theorem 2.25 for the linear systems $|-mK_X|$ with $m \geq 2$.

Proposition 2.26 *Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$ for some $m \geq 2$. Then for any $\beta \in (0, (m-1)R(X)/(m-R(X)))$, there exists a regular conical Kähler–Einstein metric ω satisfying*

$$\mathrm{Ric}(\omega) = \beta\omega + \frac{1-\beta}{m}[D].$$

In particular, when $R(X) = 1$, we have conical Kähler–Einstein metric for any $\beta \in (0,1)$.

Proof We will give a proof for the case $R(X) = 1$; the general case follows from the same argument and Remark 2.20. Let s be a defining section of $|-mK_X|$ and h be a smooth hermitian metric on $-mK_X$. Then $|s|_h^{-(2-\epsilon)}$ is integrable for any $\epsilon > 0$. Furthermore, $F_{\omega,(1-\beta)m^{-1}[D],\beta}$ is proper for any smooth Kähler forms $\omega, \theta \in c_1(X)$ if $\beta \in (0,1)$, Then the proposition can be proved by an interpolation argument similar to the proof of Theorem 2.25. □

3 Conical toric Kähler–Einstein metrics

3.1 Conical toric Kähler metrics

In this section, we will introduce toric conical Kähler metrics on toric Kähler manifolds and corresponding toric Kähler and symplectic potentials as in [11; 12]. We begin with some basic definitions for toric manifolds.

Definition 3.1 A convex polytope $P \subset \mathbb{R}^n$ is called a *Delzant polytope* if a neighborhood of any vertex of P is $\mathrm{SL}(n,\mathbb{Z})$ equivalent to $\{x_j \geq 0, j = 1,\dots,n\} \subset \mathbb{R}^n$. P is called an *integral Delzant polytope* if each vertex of P is a lattice point in $\mathbb{Z}^n$.

Let P be an integral Delzant polytope in $\mathbb{R}^n$ defined by

$$P = \{x \in \mathbb{R}^n \mid l_j(x) > 0, j = 1,\dots,N\}, \tag{3-1}$$

where

$$l_j(x) = v_j \cdot x + \lambda_j,$$

v_i is a primitive integral vector in $\mathbb{Z}^n$ and $\lambda_j \in \mathbb{Z}$ for all $j = 1,\dots,N$. Then P defines an n–dimensional nonsingular toric variety by the following observation.

For each n–dimensional integral Delzant polytope P, as in [11; 12], we consider the set of pairs $(p, \{v_{p,i}\}_{i=1}^n)$, where p is a vertex of P and the neighboring faces are given by $l_{p,i}(x) = v_{p,i} \cdot x - \lambda_{p,i} > 0$ for $i = 1,\dots,n$. For each p, there is an affine neighborhood $X \supset U_p \cong \mathbb{C}^n$ containing p with coordinate $z = (z_1,\dots,z_n)$ such that $z(p) = 0$. Then for any two vertices p and p', there exists $\sigma_{p,p'} \in \mathrm{GL}(n,\mathbb{Z})$ such that

$$\sigma_{p,p'} \cdot v_{p,i} = v_{p',i}.$$

Furthermore, we have

$$\sigma_{p,p'} \cdot \sigma_{p',p''} \cdot \sigma_{p'',p} = 1.$$

Therefore $\sigma_{p,p'}$ serves as the transition function for two coordinate charts over $(\mathbb{C}^*)^n$. More precisely, let $z = (z_1,\dots,z_n)$ and $z' = (z'_1,\dots,z'_n) \in \mathbb{C}^n$ be the coordinates for the chart associated to p and p' respectively. Suppose $\sigma_{p,p'} = (\alpha_{ij})$. Then

$$z'_i = \prod_j z_j^{\alpha_{ij}}.$$

Each integral Delzant polytope uniquely determines a nonsingular toric variety X_P by such a construction with the data $(p, \{v_{p,i}\}_{i=1}^n)$. The constant $\lambda_{p,i}$ determines an ample line bundle L over X_P, and moreover,

$$H^0(X_P, L) = \mathrm{span}\{z^\alpha\}_{\alpha \in \mathbb{Z}^n \cap \overline{P}}.$$

Let $\varphi_P = \log(\sum_{\alpha \in \mathbb{Z}^n \cap \overline{P}} |z|^{2\alpha})$. Then $\omega_P = \sqrt{-1}\partial\overline{\partial}\varphi_P$ is a smooth Kähler metric on $(\mathbb{C}^*)^n$ and it can be smoothly extended to a smooth global toric Kähler metric on X_P in $c_1(L)$. Then the space of toric Kähler metrics in $c_1(L)$ is equivalent to the set of all smooth plurisubharmonic function φ on $(\mathbb{C}^*)^n$ such that $\varphi - \varphi_P$ is bounded and $\sqrt{-1}\partial\overline{\partial}\varphi$ extends to a smooth Kähler metric on X_P. If we consider the toric Kähler

potential φ which is invariant under the real torus action, we can view φ as a function in $\mathbb{R}^n$ by

$$\varphi = \varphi(\rho), \quad \rho = (\rho_1, \dots, \rho_n), \quad \rho_i = \log |z_i|^2.$$

One can also define a symplectic potential u on P by

$$u(x) = \sum_{j=1}^{N} l_j(x) \log l_j(x) + f(x) \tag{3-2}$$

such that $f(x) \in C^\infty(\overline{P})$ and $u(x)$ is strictly convex in P. It is due to Guillemin [15] that the toric Kähler potential and the symplectic potential are related by the Legendre transform

$$\varphi(\rho) = \mathcal{L}u(\rho) = \sup_{x \in P} (x \cdot \rho - u(x)), \quad u(x) = \mathcal{L}\varphi(x) = \sup_{\rho \in \mathbb{R}^n} (x \cdot \rho - \varphi(\rho))$$

or equivalently

$$\varphi(\rho) = x \cdot \rho - u(x), \quad u(x) = x \cdot \rho - \varphi(\rho), \quad x = \nabla_\rho \varphi(\rho), \quad \rho = \nabla_x u(x).$$

We would like to generalize the Guillemin condition to toric conical Kähler metrics on X_P. This can be considered as a generalization of orbifold Kähler metrics by replacing the finite subgroup by a possibly infinite nondiscrete subgroup of $(S^1)^n$. Suppose that the integral Delzant polytope is defined by

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0,\ j = 1, \dots, N\}$$

with $v_j \in \mathbb{Z}^n$ being a primitive lattice point and $\lambda_j \in \mathbb{Z}$.

Now we introduce the function spaces we will work with. Let $p \in P$ be a vertex, whose neighboring faces are determined by vectors $\{v_{p,i}\}_{i=1}^n$. Let $U_p \subset X$ be the affine neighborhood corresponding to p with coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$. Then for each $1 \le i \le n$, $[z_i = 0]$ (corresponding to the faces determined by $v_{p,i}$) extends to a smooth toric divisor of X_P. Let D be a toric divisor of X_P and suppose D restricted to U_p is given by

$$\sum_{i=1}^{n} a_i [z_i = 0].$$

We can lift any function $f(z)$ on U_p invariant under the $(S^1)^n$–action to a function

$$\widetilde{f}(w) = f(z)$$

by letting

$$|w_i| = |z_i|^{\beta_i(p)}, \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n,$$

and clearly $\widetilde{f}(w)$ is also $(S^1)^n$–invariant. In particular, we can regard the map w as a $\beta(p)=(\beta(p)_1,\dots,\beta(p)_n)$–covering of $z\in\mathbb{C}^n$. Then for any $k\in\mathbb{Z}^{\geq 0}$ and $\alpha\in(0,1)$, we define a space of $(S^1)^n$–invariant functions on U_p by

$$C^{k,\alpha}_{\beta(p),p}:=\{f(z)=f(|z_1|,\dots,|z_n|)\mid \widetilde{f}(w)\in C^{k,\alpha}(\mathbb{C}^n)\}.$$

This allows us to define the *weighted function space*

$$C^{k,\alpha}_{\beta}(X_P):=\{f\in C^0(X)\mid f|_{U_p}\in C^{k,\alpha}_{\beta(p),p} \text{ for every vertex } p\in P\},$$

where $\beta=(\beta_1,\dots,\beta_N)\in(\mathbb{R}^+)^N$.[1] Now we define the space of *weighted toric Kähler metrics* on X_P by considering a Kähler current ω whose restriction to each chart is given by

$$\omega=\sqrt{-1}\partial\overline{\partial}\varphi_p$$

such that $\varphi_p\in C^\infty_{\beta,p}$. Such a weighted toric Kähler metric is naturally a regular conical Kähler metric with cone angle $2\pi\beta_i$ along $[z_i=0]$ and is called a *smooth β–weighted Kähler metric*. The local lifting $\widetilde{\varphi}(w)$ is a smooth plurisubharmonic function on the covering space $w\in\mathbb{C}^n$.

We can also define the space of weighted toric Kähler potential φ on $(\mathbb{C}^*)^n$ such that $\varphi-\varphi_P$ is bounded and $\sqrt{-1}\partial\overline{\partial}\varphi$ extends to a smooth weighted Kähler metric on X_P.

We now define a weighted C^∞_β *symplectic potential*

$$u(x)=\sum_{j=1}^N\beta_j^{-1}l_j(x)\log l_j(x)+f(x)$$

for $f\in C^\infty(\overline{P})$ for $j=1,\dots,n$ such that $f\in C^\infty(\overline{P})$ and u is strictly convex in P. Then the weighted Kähler potentials and the weighted symplectic potential determine each other uniquely. In particular, we have following straightforward generalization of the Guillemin condition for conical toric Kähler metrics.

Proposition 3.2 *The weighted C^∞_β toric potential φ and the weighted C^∞_β symplectic potential are related by the Legendre transform*

$$\varphi(\rho)=\mathcal{L}u(\rho)=\sup_{x\in P}(x\cdot\rho-u(x)),\quad u(x)=\mathcal{L}\varphi(x)=\sup_{\rho\in\mathbb{R}^n}(x\cdot\rho-\varphi(\rho))$$

or equivalently

$$\varphi(\rho)=x\cdot\rho-u(x)\ \text{with}\ x=\nabla_\rho\varphi(\rho)\quad\text{and}\quad u(x)=x\cdot\rho-\varphi(\rho)\ \text{with}\ \rho=\nabla_x u(x).$$

[1] Notice that $\{\beta(p)_1,\dots,\beta(p)_n\}\subset\{\beta_1,\dots,\beta_N\}$.

In particular, if $u(x)=\beta_j^{-1}l_j(x)\log l_j(x)+f(x)$, *then the cone angle of the corresponding conical toric Kähler metric is* $2\pi\beta_j$ *along the toric divisor determined by* $l_j(x)=0$.

Let $\omega=\sqrt{-1}\partial\bar{\partial}\phi$ be a smooth β–weighted Kähler metric and let $u=\mathcal{L}\phi$. Then $u(x)=\sum_j \beta_j^{-1}l_j(x)\log l_j(x)+f(x)$ for some $f\in C^\infty(\overline{P})$.

Example 3.3 Let $P=[0,1]$. Then the associated toric manifold is $X=\mathbb{P}^1$ with the polarization $\mathcal{O}(1)$. We consider the symplectic potential

$$u=(\beta_1)^{-1}x\log x+\beta_2^{-1}(1-x)\log(1-x).$$

Then

$$\rho=\log|z|^2=u'(x)=(\beta_1^{-1}-\beta_2^{-1})+\log\frac{x^{\beta_1^{-1}}}{(1-x)^{\beta_2^{-1}}},\quad |z|^2=\frac{x^{\beta_1^{-1}}}{(1-x)^{\beta_2^{-1}}}e^{\beta_1^{-1}-\beta_2^{-1}}$$

and so

$$x\sim|z|^{2\beta_1}\text{ near }0,\quad (1-x)\sim|z|^{-2\beta_2}\text{ near }\infty.$$

In particular, x is a smooth function of $|z|^{2\beta_1}$ near $z=0$ and $(1-x)$ is a smooth function of $|z|^{-2\beta_2}$ near $z=\infty$. The Kähler potential φ is given by

$$\varphi(\rho)=x(\beta_1^{-1}-\beta_2^{-1})-\beta_2^{-1}\log(1-x).$$

Hence $\sqrt{-1}\partial\bar{\partial}\varphi$ extends to a conical metric with cone angle $2\pi\beta_1$ and $2\pi\beta_2$ at $[z=0]$ and $[z=\infty]$ respectively. In particular, when $\beta=\beta_1=\beta_2$, $\varphi=\beta^{-1}\log(1+|z|^{2\beta})$ and $\omega=2\sqrt{-1}\partial\bar{\partial}\varphi$ is a smooth β–weighted Kähler–Einstein metric in $c_1(\mathbb{P}^1)$ satisfying

$$\mathrm{Ric}(\omega)=\beta\omega+(1-\beta)([z=0]+[z=\infty]).$$

Lemma 3.4 *Let* g *be a* smooth β–weighted toric Kähler metric *on a toric manifold* X_P. *Let* D *be the toric divisor such that* g *is a smooth toric Kähler metric on* $X\setminus D$. *Then for any* $k\geq 0$, *there exists* $C_k>0$ *such that for any point* $p\in X\setminus D$,

$$|\nabla_g^k Rm(g)|_g(p)\leq C_k. \tag{3-3}$$

Proof The calculation of $|\nabla_g^k Rm(g)|_g(p)$ can be locally carried out on the β–covering space for each coordinate chart $(p,\{v_i\}_{i=1}^n)$ as in the orbifold case. All the quantities must be bounded because the g is a smooth toric Kähler metric after being lifted to the covering space. □

We now can solve a Monge–Ampère equation with smooth β–weighted data.

Proposition 3.5 *Let ω be a smooth β–weighted toric Kähler metric on a toric manifold X_P. Then for any smooth β–weighted function f on X_P with $\int_{X_P} e^{-f}\omega^n = \int_{X_P}\omega^n$, there exists a unique β–weighted smooth function φ satisfying*

$$(\omega+\sqrt{-1}\partial\bar\partial\varphi)^n = e^f\omega^n, \quad \sup_{X_P}\varphi = 0. \tag{3-4}$$

Proof We use the continuity method for $t\in[0,1]$ to solve the equation

$$(\omega+\sqrt{-1}\partial\bar\partial\varphi_t)^n = e^{tf+c_t}\omega^n, \tag{3-5}$$

where c_t is determined by $\int_{X_P} e^{tf+c_t}\omega^n = \int_{X_P}\omega^n$. Let

$$S = \{t\in[0,1] \mid \text{(3-5) is solvable for } t \text{ with } \varphi_t\in C^\infty_\beta(X_P)\}.$$

Obviously, $0\in S$. S is open by applying the implicit function theorem for the linearized operator

$$\Delta_{\beta,t}\colon C^{k+2,\alpha}_\beta(X_P)\to C^{k,\alpha}_\beta(X_P).$$

All the local calculation can be carried out in the β–covering space on each coordinate chart $(p,\{v_i\}_{i=1}^n)$ because all the data involved are invariant under the $(S^1)^n$–action. It suffices to prove uniform a priori estimates for φ_t in $C^k_\beta(X_P)$ for $t\in[0,1]$.

C^0–estimates Let Ω be a smooth volume form on X_P. Then $e^{tf+c_t}\omega^n/\Omega$ lies in $L^{1+\epsilon}(X_P,\Omega)$ for some $\epsilon>0$. By Yau's Moser iteration [47] adapted to the conical case or by Kolodziej's L^∞–estimate [17], there exists $C>0$ such that for all $t\in[0,1]$, if $\varphi_t\in C^\infty_\beta(X)$ solves (3-5), then

$$\|\varphi_t\|_{L^\infty(X_P)}\le C.$$

Second-order estimates We consider

$$H_t = \log \mathrm{tr}_\omega(\omega_t) - A\varphi_t.$$

Suppose at $t\in[0,1]$, $\sup_{X_P} H_t = H_t(q)$. We lift all the calculation on the $(S^1)^n$–invariant β–covering space in a fixed local coordinate chart $w\in\mathbb{C}^n$. Standard calculations show that near $\tilde q$, there exists $C>0$ such that

$$\tilde\Delta_{t,\beta}\tilde H_t \ge -C\,\mathrm{tr}_{\tilde\omega_t}(\tilde\omega) - An + A\cdot\mathrm{tr}_{\tilde\omega_t}(\tilde\omega) \ge \frac{A}{2}\,\mathrm{tr}_{\tilde\omega_t}(\tilde\omega) - C,$$

where $\tilde q$, $\tilde\omega$ and $\tilde\omega_t$ are the lifting of q, ω and ω_t. By the maximum principle, at $\tilde q$, $\mathrm{tr}_{\tilde\omega_t}(\tilde\omega)$ is bounded above by a constant independent of $t\in[0,1]$. Combining the Equation (3-4), $\mathrm{tr}_{\tilde\omega}(\tilde\omega_t)$ is also bounded above by a constant independent of t. Hence there exists $C>0$ such that for all $t\in[0,1]$, if $\varphi_t\in C^\infty_\beta(X)$ solves (3-5),

$$C^{-1}\omega\le\omega_t\le C\omega.$$

Higher-order estimates Calabi's third-order estimates can be applied in the same the way as in [47; 29] by using the maximum principle after lifting all the local calculations on the $(S^1)^n$–invariant covering space. The Schauder estimates can also be applied by the bootstrap argument. Eventually, for any $k > 0$, there exists C_k such that for all $t \in [0, 1]$, if $\varphi_t \in C^\infty_\beta(X)$ solves (3-5),

$$\|\varphi_t\|_{C^k_\beta(X_P)} \leq C_k. \qquad \square$$

3.2 Proof of Theorem 1.7

An n–dimensional integral polytope P is called *Fano* if it is a Delzant polytope and $\lambda_i = 1$ for each defining function $l_i(x) = v_i \cdot x + \lambda_i$, from which we deduce $0 \in P$. The toric manifold X_P associated to P is a Fano manifold. Each $(n-1)$–face of P corresponds to a toric divisor of P. Then the union $D_P = \sum_j D_j$ for all the boundary divisors lies in $c_1(X) = c_1(-K_X)$, where D_j is the toric divisor induced by the face $\{l_j(x) = 0\}$. In particular, $[D]$ is very ample.

The Futaki invariant of X_P with respect to $(S^1)^n$–action is shown in [24] exactly the barycenter of P defined by

$$P_c = \frac{\int_P x dV}{\int_P dV},$$

where $dV = dx_1 dx_2 \cdots dx_n$ is the standard Euclidean volume form.

The following theorem on the existence of Kähler–Einstein metrics on toric Fano manifolds is due to Wang and Zhu [45].

Theorem 3.6 *There exists a smooth toric Kähler–Einstein metric on a toric Fano manifold X_P if and only if the barycenter of P coincides with 0.*

If the barycenter is not at the origin, it is also proved in [45] that there exists a toric Kähler–Ricci soliton on X_P. The following theorem was proved by Li [18] to calculate the greatest Ricci lower bound $R(X)$.

Theorem 3.7 *Let X_P be a toric Fano manifold associated to a Fano Delzant polytope P. Let P_c be the barycenter of P and $Q \in \partial P$ such that the origin $O \in \overline{P_c Q}$. Then the greatest Ricci lower bound of X_P is given by*

$$R(X_P) = \frac{|\overline{OQ}|}{|\overline{P_c Q}|}. \tag{3-6}$$

For any $\tau \in \mathbb{R}^n$, we define the divisor $D(\tau)$ by

$$D(\tau) = \sum_{j=1}^{N} l_j(\tau) D_j. \tag{3-7}$$

$D(\tau)$ is a Cartier $\mathbb{R}$–divisor in $c_1(X)$ and it is effective if and only if $\tau \in \overline{P}$. The defining section s_τ of $D(\tau)$ is given by the monomial

$$s_\tau = z^\tau.$$

Although s_τ is only locally defined,

$$|s_\tau|^2 = |z|^{2\tau} = e^{\tau\cdot\rho}$$

is globally defined and $|s_\tau|^{-2}$ induces a singular hermitian metric on $-K_X$ and can be viewed as a singular volume form with poles along D_j of order $l_j(\tau)$. We deduce:

Lemma 3.8 *If $R(X) < 1$, then the $\mathbb{R}$–divisor $D(\tau)$ with $\tau = -\frac{\alpha}{1-\alpha} P_c$ is effective if and only if $\alpha \in [0, R(X)]$.*

We consider the real Monge–Ampère equation on $\mathbb{R}^n$ for a convex function ϕ,

$$\det(\nabla^2\phi) = e^{-\alpha\phi - (1-\alpha)\tau\cdot\rho}. \tag{3-8}$$

Let $u = \mathcal{L}\phi$ be the symplectic potential. Then $\det(\nabla^2 u) = (\det(\nabla^2\phi)^n)^{-1}$ and the dual Monge–Ampère equation for u is given by

$$\det(\nabla^2 u) = e^{-\alpha u + (\alpha x + (1-\alpha)\tau)\cdot\nabla u}. \tag{3-9}$$

If we let $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, the corresponding curvature equation is given by

$$\mathrm{Ric}(\omega) = \alpha\omega + (1-\alpha)[D(\tau)].$$

Lemma 3.9 *Suppose there exists a smooth α–weighted Kähler–Einstein metric $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ satisfying*

$$\mathrm{Ric}(\omega) = \alpha\omega + (1-\alpha)[D(\tau)]$$

for some $\alpha \in (0, R(X)]$ and $\tau \in \overline{P}$. Then

$$\tau = -\frac{\alpha}{1-\alpha} P_c \quad \text{for } \alpha \neq 1 \text{ and any vector in } P \text{ for } \alpha = 1. \tag{3-10}$$

Furthermore, there exists $f \in C^\infty(\overline{P})$ such that

$$\mathcal{L}\phi(x) = \sum_{j=1}^{N} \beta_j^{-1} l_j(x) \log l_j(x) + f(x), \quad \beta_j = \frac{l_j(P_c)}{l_j(0)}\alpha. \tag{3-11}$$

Proof Consider the corresponding Monge–Ampère equation

$$(\det \nabla^2 \phi)^n = e^{-\alpha\phi-(1-\alpha)\tau\cdot\rho}.$$

The right-hand side $e^{-\alpha\phi-(1-\alpha)\tau\cdot\rho}$ is integrable on $\mathbb{R}^n$ and in fact

$$\int_{\mathbb{R}^n} e^{-\alpha\phi-(1-\alpha)\tau\cdot\rho} d\rho = \int_{\mathbb{R}^n} \det(\nabla^2\phi) d\rho = \int_X \omega^n = c_1(X)^n.$$

Then the Monge–Ampère mass $\det(\nabla^2\phi)d\rho$ becomes dx by the moment map and

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla\big(e^{-\alpha\phi-(1-\alpha)\tau\cdot\rho}\big) d\rho \\ &= -\int_{\mathbb{R}^n} (\alpha\nabla\phi + (1-\alpha)\tau) e^{-\alpha\phi-(1-\alpha)\tau\cdot\rho} d\rho \\ &= -\int_P (\alpha x + (1-\alpha)\tau) dx = -(\alpha P_c + (1-\alpha)\tau) \int_P dx. \end{aligned}$$

Therefore $\tau = \frac{\alpha}{1-\alpha} P_c$ for $\alpha \neq 1$.

Suppose $u(x) = \mathcal{L}\phi(x) = \sum_{j=1}^N \beta_j^{-1} l_j(x) \log l_j(x) + f(x)$. The Monge–Ampère equations for ϕ and u are given by

$$\det(\nabla^2\phi) = e^{-\alpha(\phi + P_c\cdot\rho)}, \quad \det(\nabla^2 u) = e^{-\alpha(u-(x-P_c)\cdot\nabla u)}.$$

Without loss of generality, we assume that $l_1(x) = l_2(x) = \cdots = l_n(x) = 0$ with $l_i(x) = v_i \cdot x + 1$, $1 \le i \le n$ defines a vertex p of P. Then there exists a smooth positive function $F(x)$ on any compact subset U of $\overline{P}$ with $U \cap \{l_j(x) = 0\} = \phi$ for all $j > n$, such that

$$\det(\nabla^2 u) = \frac{F(x)}{l_1(x) l_2(x) \cdots l_n(x)}.$$

On the other hand,

$$\begin{aligned} &u(x) - (x - P_c)\nabla u(x) \\ &\qquad = \sum_{j=1}^N \beta_j^{-1} l_j(P_c) \log l_j(x) - \sum_{j=1}^N \beta_j^{-1} (x - P_c)\cdot v_j - (x - P_c)\cdot\nabla f(x) \end{aligned}$$

and so

$$e^{-\alpha(u-(x-P_c)\cdot\nabla u)} = \frac{e^{-\alpha(x-P_c)\cdot(\nabla f(x) + \sum_j \beta_j^{-1} v_j)}}{\prod_{j=1}^N (l_j(x)^{\alpha\beta_j^{-1} l_j(P_c)})}.$$

Comparing the powers of $l_j(x)$, we have

$$\beta_j = l_j(P_c)\alpha = \frac{l_j(P_c)}{l_j(0)}\alpha$$

since $l_j(0) = 1$. This completes the proof of the lemma. □

Remark 3.10 Notice that it follows from [7] that the log Donaldson–Futaki invariant defined in [13] for the toric Fano pair (X, D_τ), with D_τ being the toric divisor determined by the vector $\tau \in P$, is

$$\mathrm{DF}(X, D_\tau; L) = -V(P)(\alpha P_C + \tau) \in \mathbb{R}^n, \tag{3-12}$$

where $V(P)$ is the volume and $P_C \in \mathbb{R}^n$ is the barycenter of the polytope P.

Lemma 3.9 leads us to consider the Monge–Ampère equation

$$\det(\nabla^2 \phi)^n = e^{-\alpha(\phi - P_c \cdot \rho)}. \tag{3-13}$$

The right-hand side of Equation (3-13) is always integrable since P_c lies in P and $\phi - \log(\sum_k e^{p_k \cdot \rho})$ is bounded on $\mathbb{R}^n$, where we are summing over all vertices $\{p_k\}$ of P.

For each $\alpha \in (0, 1]$, we define $\beta = \beta(\alpha) = (\beta_1, \beta_2, \ldots, \beta_N)$ by $\beta_j = (l_j(P_c)/l_j(0))\alpha$.

Lemma 3.11 *For any $\alpha \in (0, 1]$, there exists a $C^\infty_{\beta(\alpha)}$ conical toric Kähler metric ω such that*

$$\mathrm{Ric}(\omega) = \alpha\theta + (1-\alpha)[D(\tau)],$$

where $\theta \in c_1(X)$ is a fixed $C^\infty_{\beta(\alpha)}$ toric Kähler metric and $\tau = \frac{\alpha}{1-\alpha} P_c$. In particular, $\mathrm{Ric}(\omega) > 0$, if $\alpha \in (0, R(X))$.

Proof It suffices to prove for $\alpha \in (0, 1)$. Let $\hat{u}(x) = \sum_j \beta_j^{-1} l_j(x) \log l_j(x)$ for $\beta_j = (l_j(P_c)/l_j(0))\alpha$. Let $\hat{\phi} = \mathcal{L}\hat{u}$ and $\hat{\omega} = \sqrt{-1}\partial\bar{\partial}\hat{\phi}$. Then there exists a $C^\infty_{\beta(\alpha)}$ real valued $(1,1)$–form $\eta \in c_1(X)$ and a divisor $D(\tau)$ with $\tau = -\frac{\alpha}{1-\alpha} P_c$ such that

$$\sqrt{-1}\partial\bar{\partial} \log \hat{\omega}^n = \alpha\eta + (1-\alpha)[D(\tau)].$$

This is because along each D_j defined by $l_j(x) = 0$, $\hat{\omega}^n$ has a pole of order

$$1 - \beta_j = 1 - l_j(P_c)\alpha$$

and

$$(1-\alpha)D(\tau) = (1-\alpha)\left(\sum_j l_j\left(\frac{\alpha}{1-\alpha} P_c\right) D_j\right) = \sum_j (1 - l_j(P_c)\alpha) D_j.$$

Since $\eta \in c_1(X)$ is $C^\infty_{\beta(\alpha)}$, there exists $\psi \in C^\infty_{\beta(\alpha)}$ such that $\eta + \sqrt{-1}\partial\bar{\partial}\psi$ is a $C^\infty_{\beta(\alpha)}$ toric Kähler metric. Without loss of generality, we may assume that

$$\int_{X_P} e^\psi \hat{\omega}^n = \int_{X_P} \hat{\omega}^n$$

after a constant translation.

By Proposition 3.5, the equation

$$(\widehat{\omega}+\sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\psi}\widehat{\omega}^n, \quad \sup_{X_P}\varphi = 0$$

admits a unique $C^{\infty}_{\beta(\alpha)}$ solution φ. Let $\omega = \widehat{\omega}+\sqrt{-1}\partial\bar{\partial}\varphi$ and $\theta = \eta+\sqrt{-1}\partial\bar{\partial}\psi$, then we obtain

$$\mathrm{Ric}(\omega) = \alpha\theta + (1-\alpha)[D_{(}\tau)]. \qquad \square$$

By Lemma 3.11, there is a $C^{\infty}_{\beta(\alpha)}$ Kähler potential ϕ_0 with $\omega_0 = \sqrt{-1}\partial\bar{\partial}\phi_0$ such that

$$\mathrm{Ric}(\omega_0) = \alpha\theta + (1-\alpha)D(\tau)$$

for a $C^{\infty}_{\beta(\alpha)}(X_P)$ Kähler metric $\theta \in c_1(X_P)$ and $\tau = \frac{\alpha}{1-\alpha}P_c$ if $\alpha \neq 0$. Let

$$w = \frac{1}{\alpha}\big(-\alpha P_c\cdot\rho - \log\det(\nabla^2\phi_0)\big).$$

Then

$$\sqrt{-1}\partial\bar{\partial}w = \theta$$

and $|w-\widehat{\phi}|$ is uniformly bounded by the argument in Lemma 3.11. This implies that w is a $C^{\infty}_{\beta(\alpha)}$ Kähler potential and we have

$$\det(\nabla^2\phi_0) = e^{-\alpha(w-P_c\cdot\rho)}.$$

We will then use the continuity method for the following family of Monge–Ampère equations for $t\in[0,\alpha]$,

$$\det(\nabla^2\phi_t) = e^{-t(\phi_t-P_c\cdot\rho)-(\alpha-t)w}. \tag{3-14}$$

Let $\varphi_t = \phi_t-\phi_0$ and $h_\theta \in C^{\infty}_{\beta(\alpha)}(X_P)$ be the unique function satisfying

$$-\sqrt{-1}\partial\bar{\partial}\log\theta^n - \sqrt{-1}\partial\bar{\partial}h_\theta = \alpha\theta + (1-\alpha)[D(\tau)], \quad \int_{X_P}e^{h_\theta}\theta^n = \int_{X_P}\theta^n.$$

Then Equation (3-14) is equivalent to

$$(\sqrt{-1}\partial\bar{\partial}\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{-t\varphi_t}\big(e^{h_{\omega_0}}\omega_0\big)^{t/\alpha}\big(e^{h_\theta}\theta\big)^{\alpha-t/\alpha}, \tag{3-15}$$

where h_{ω_0} is defined for $\theta=\omega_0$. Let

$$S = \{t\in[0,\alpha] \mid \text{(3-14) is solvable for } t \text{ with } \sqrt{-1}\partial\bar{\partial}\phi_t \in C^{\infty}_{\beta(\alpha)}(X_P)\}.$$

Obviously, ϕ_0 solves (3-14) for $t=0$ and so $S\neq\phi$. Notice that

$$\mathrm{Ric}(\omega_t) = t\omega_t + (\alpha-t)\theta + (1-\alpha)[D(\tau)] \geq t\omega_t$$

for $t \in [0, \alpha]$ and $\tau = \frac{\alpha}{1-\alpha} P_c$ if $\alpha \neq 1$. It implies that the first eigenvalue of the Laplace operator $\Delta_t = \mathrm{tr}_{\omega_t}(\sqrt{-1}\partial\bar{\partial})$ is strictly greater than t. By the argument in Proposition 3.5, S is open and it suffices to show that S is closed by proving uniform a priori estimates for $\phi_t - \phi_0$.

Proposition 3.12 *There exists $C > 0$ such that for all $t \in [0, \alpha]$,*

$$\|\phi_t - \phi_0\|_{L^\infty(\mathbb{R}^n)} \leq C. \tag{3-16}$$

Proof We fix some positive $\epsilon_0 \in S$. We let

$$\Phi_t = \alpha^{-1}(\phi_t - P_c \cdot \rho) \quad \text{and} \quad W = \alpha^{-1}(w - P_c \cdot \rho).$$

Then the Equation (3-14) becomes

$$\det \nabla^2 \Phi_t = e^{-(\Phi_t + W)}.$$

Let $W_t = \Phi_t + W$. Immediately, we can see that the moment map with respect to Φ_t is given by

$$F_t \colon \nabla \Phi_t \to P - P_c$$

whose image is the translation of P by $-P_c$. In particular, the barycenter of the new polytope $P - P_c$ coincides with the origin.

Suppose

$$m_t = W_t(\rho_t) = \inf_{\mathbb{R}^n} W_t(\rho)$$

for a unique $\rho_t \in \mathbb{R}^n$ since W_t is asymptotically equivalent to $\log(\sum e^{(p_k - P_c)\cdot\rho})$ where p_k are the vertices of P. We can apply the same argument as Wang and Zhu [45]. First one can show by John's lemma and the maximum principle (see [45, Lemmas 3.1 and 3.2]), that there exists $C > 0$ such that for all $t \in [\epsilon_0, \alpha]$,

$$m_t = W_t(\rho_t) = \inf_{\mathbb{R}^n} W_t(\rho) \leq C.$$

Then by using the fact that the barycenter of $P - P_c$ lies at the origin O, the same argument as in [45, Lemma 3.3] shows that there exists $C > 0$ such that for all $t \in [\epsilon_0, \alpha]$,

$$|\rho_t| \leq C.$$

This then implies that

$$\varphi_t = \alpha^{-1}(\Phi_t - W)$$

is uniformly bounded above for $t \in [\epsilon_0, \alpha]$ by the same argument as in [45, Lemma 3.4].

The uniform lower bound of φ_t can be obtained either by the Harnack inequality

$$-\inf_{X_P}\varphi_t \le C(1+\sup_{X_P}\varphi_t)$$

adapted from the smooth case or directly by the argument in [45, Lemma 3.5]. □

Lemma 3.13 *For any $k \ge 0$, there exists $C_k > 0$ such that for all $t \in [0,\alpha]$,*

$$\|\varphi_t\|_{C^k(X_P)} \le C_k. \tag{3-17}$$

Proof The Laplacian $\Delta_{\beta(\alpha),t}\varphi_t$ is uniformly bounded by Yau's estimates after lifting the calculations to the $\beta(\alpha)$–covering space as in the proof of Proposition 3.5. The C^3–estimates and the Schauder estimates can be applied in the same way. □

Theorem 3.14 *Let X_P be a toric Fano manifold.*

(1) *For any $\beta \in (0, R(X_P))$, there exist a unique smooth toric conical Kähler–Einstein metric ω and a unique effective toric $\mathbb{R}$–divisor $D_\beta \in |-K_X|$ satisfying*

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D_\beta].$$

(2) *For $\beta = R(X_P)$, there exists a unique smooth toric conical Kähler–Einstein metric ω satisfying*

$$\mathrm{Ric}(\omega) = R(X_P)\omega + (1-R(X_P))[D_P] \tag{3-18}$$

for an effective $\mathbb{Q}$–divisor D_P in $c_1(X)$. In particular, if D_j is the toric divisor associated to the face defined by $l_j(x) = v_j \cdot x + \lambda_j = 0$, then

$$D_P = \sum_j \frac{1-\beta_j}{1-R(X)} D_j, \quad \beta_j = \frac{l_j(P_c)}{l_j(0)} R(X_P)$$

and the cone angle of ω along D_j is $2\pi\beta_i$, if $R(X) < 1$.

(3) *For $\beta \in (R(X), 1]$, there does not exist a smooth toric conical Kähler–Einstein metric ω satisfying*

$$\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D],$$

with an effective $\mathbb{R}$–divisor D_β in $[-K_X]$.

Proof (1) and (2) are proved by the uniform estimates from Lemma 3.13. If $\beta > R(X_P)$, there still exists a smooth β–weighted Kähler–Einstein metric satisfying $\mathrm{Ric}(\omega) = \beta\omega + (1-\beta)[D_\beta]$ for some toric divisor D, however, by Lemma 3.8, D is not effective and so (3) is proved. □

Corollary 3.15 *Let X be a Fano toric manifold and $\omega \in c_1(X)$ be a smooth Kähler metric. We define*

$$F_{\omega,\alpha}(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_X \varphi\omega^n - \frac{1}{\alpha}\log\frac{1}{V}\int_X e^{-\alpha\varphi}\omega^n$$

for all $\varphi \in C^\infty(X) \cap \mathrm{PSH}(X,\omega)$. Suppose $R(X) < 1$. Then:

(1) *For $\alpha \in (0, R(X))$, $F_{\omega,\alpha}$ is J–proper.*

(2) *For $\alpha = R(X)$, $F_{\omega,\alpha}$ is bounded below.*

(3) *For $\alpha \in (R(X), 1]$, $\inf_{\varphi\in\mathrm{PSH}(X,\omega)\cap C^\infty(X)} F_{\omega,\alpha}(\varphi) = -\infty$.*

Proof It suffices to prove (2) by Corollary 2.21 . This can be proved by modifying the argument in [4; 2]. By Theorem 3.14, there exists a unique $(S^1)^n$–invariant $\psi \in L^\infty(X_P) \cap \mathrm{PSH}(X_P,\omega)$ satisfying

$$(\omega + \sqrt{-1}\partial\bar\partial\psi) = e^{-\alpha\psi}\mu, \quad \mu = (\Omega_\omega)^\alpha\,(\Omega_D)^{1-\alpha},$$

where Ω_ω is a smooth volume form with $\sqrt{-1}\partial\bar\partial\log\Omega_\omega = -\omega$ and Ω_D is a positive (n,n)–current with $-\sqrt{-1}\partial\bar\partial\log\Omega_D = [D]$ and $D = D(\frac{\alpha}{1-\alpha}P_c)$. For any $\varphi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, let φ_t be the weak geodesic φ_t joining ψ and φ with $\varphi_0 = \psi$ and $\varphi_1 = \varphi$. Then the modified functional

$$f(t) = \mathcal{F}_{\omega,\alpha}(\varphi_t) = J_\omega(\varphi_t) - \frac{1}{V}\int_{X_P}\varphi_t\omega^n - \frac{1}{\alpha}\log\frac{1}{V}\int_{X_P} e^{-\alpha\varphi}\mu$$

is convex on $[0,1]$ and $f'(0) \geq 0$ by applying the same argument as in Theorem 6.2 in [4]. This shows that $\mathcal{F}_{\omega,\alpha}$ is bounded below and since Ω_D is bounded below away from 0, and therefore $F_{\omega,\alpha}$ is bounded from below as well. □

Example 3.16 Let X be $\mathbb{P}^2$ blown-up at one point. Then $R(X) = \frac{6}{7}$ as shown in [34] and X admits a holomorphic $\mathbb{P}^1$ fiber bundle $\pi\colon X \to \mathbb{P}^1$. Let D_∞ be the infinity section of π and H_1 and H_2 be the two toric P^1 fiber of π. Then the divisor D_P in the Equation (3-18) is given by

$$D_P = 2D_\infty + (H_1 + H_2)/2.$$

We remark that for $\beta \in (R(X), 1]$, there exists a smooth toric conical Kähler–Einstein metric in $c_1(X_P)$ satisfying $\mathrm{Ric}(g) = \beta g + (1-\beta)[D_\beta]$ for some toric divisor D_β in $c_1(X_P)$ which can not be effective. However, it is not a conical Kähler metric we are interested in, because the Ricci current of such a Kähler–Einstein metric is not bounded below.

4 The Chern number inequality

4.1 Curvature estimates

In this section, by deriving some curvature estimates for a poly-homogenous conical Kähler metric whose Ricci curvature is bounded, we prove a Chern number inequality for Fano manifolds admitting conical Kähler–Einstein metric.

Let D be the smooth divisor of X. At each point p on D, we can use the holomorphic local coordinates

$$z = (y, \xi) = (z_1, \dots, z_{n-1}, z_n), \quad y = (z_1, \dots, z_{n-1}), \quad \xi = z_n$$

and D is locally defined by $\xi = 0$. We write $\xi = r^{1/\beta} e^{i\theta}$ for $\theta \in [0, 2\pi)$. We use Greek letters $\alpha, \beta, \dots$ as indices for $1, \dots, n$ and letters $i, j, \dots$ for $1, \dots, n-1$.

Let us recall the following result by Jeffres, Mazzeo and Rubinstein [16, Proposition 4.3] on the asymptotic expansion of poly-homogenous conical Kähler–Einstein metrics.

Proposition 4.1 *Suppose ω is a* poly-homogenous *conical Kähler–Einstein metric with conical singularity along a smooth divisor D of angle $2\pi\beta$. Let φ is a local potential of ω, ie $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ in a neighborhood of a conical point (y, ξ), then the asymptotic expansion of φ takes the form*

$$\varphi(r, \theta, y) \sim \sum_{j,k,l \geq 0} a_{jkl}(\theta, z) r^{j+k/\beta} (\log r)^l. \tag{4-1}$$

In particular, if the Ricci curvature of ω is bounded and $\beta \in (\frac{1}{2}, 1)$, φ has the expansion

$$\varphi(r, \theta, y) = a_{00}(y) + (a_{01}(y) \sin\theta + b_{01}(y) \cos\theta) r^{1/\beta} + a_{20}(y) r^2 + O(r^{2+\epsilon}) \tag{4-2}$$

for some $\epsilon(\beta) > 0$.

When the Ricci curvature is bounded and $\beta \in (\frac{1}{2}, 1)$,

$$\varphi = a(y) + b(y)(\xi + \bar{\xi}) + \sqrt{-1} c(y)(\xi - \bar{\xi}) + d(y)|\xi|^{2\beta} + o(|\xi|^{2\beta+\epsilon})$$

for some $\epsilon > 0$. From now on in this section, we will always assume that g is a *regular* conical Kähler metric on X with cone angle $2\pi\beta$ for $\beta \in (\frac{1}{2}, 1)$ along the simple smooth divisor D, since for $\beta \in (0, \frac{1}{2}]$, Proposition 1.8 can be obtained by applying Simon Brendle's curvature asymptotics [5, Section 3].

Let us start with the following lemma, which is a consequence of straightforward calculations.

Lemma 4.2 *Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0,1)$. Let $o(1)$ be the quantity satisfying $\lim_{|\xi|\to 0} o(1) = 0$. Then*

$$g_{z_i\bar{z}_j} \sim \delta_{ij} + o(1), \tag{4-3}$$

$$g_{\xi\bar{\xi}} \sim |\xi|^{-2(1-\beta)} + o(|\xi|^{-2(1-\beta)}), \tag{4-4}$$

$$g_{z_i\bar{\xi}} \sim O(1). \tag{4-5}$$

By taking the inverse, we have the following corollary from Lemma 4.2.

Corollary 4.3 *Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0,1)$. Then*

$$g^{z_i\bar{z}_j} \sim \delta_{ij} + o(1), \tag{4-6}$$

$$g^{\xi\bar{\xi}} \sim |\xi|^{2(1-\beta)} + o(|\xi|^{2(1-\beta)}), \tag{4-7}$$

$$g^{z_i\bar{\xi}} \sim |\xi|^{2(1-\beta)}. \tag{4-8}$$

The following lemma gives estimates for the curvature tensor of g.

Lemma 4.4 *Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0,1)$. If the Ricci curvature of g is bounded, then*

$$R_{z_i\bar{z}_j z_k\bar{z}_l} \sim R_{z_i\bar{z}_j z_k\bar{\xi}} = O(1), \tag{4-9}$$

$$R_{z_i\bar{z}_j\xi\bar{\xi}} = O(|\xi|^{-2(1-\beta)}), \tag{4-10}$$

$$R_{\xi\bar{z}_j\xi\bar{z}_l} = O(|\xi|^{-1}), \tag{4-11}$$

$$R_{\xi\bar{\xi}\xi\bar{z}_l} = O(|\xi|^{-1}), \tag{4-12}$$

$$R_{\xi\bar{\xi}\xi\bar{\xi}} = O(|\xi|^{-\max(1,4(1-\beta))}). \tag{4-13}$$

Proof The estimates (4-9), (4-10) and (4-11) can be shown by straightforward calculation using the curvature formula

$$R_{\alpha\bar{\beta}\gamma\bar{\zeta}} = -g_{\alpha\bar{\beta},\gamma\bar{\zeta}} + g^{\mu\bar{\nu}} g_{\alpha\bar{\nu},\gamma} g_{\mu\bar{\beta},\bar{\zeta}}.$$

The estimates (4-12) and (4-13) follow by combining the boundedness of the Ricci curvature and the estimates (4-10) and (4-11). □

Corollary 4.5 *Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in [\frac{1}{2}, 1)$. If the Ricci curvature of g is* bounded, *then*

$$R^{z_i}_{\xi, z_k \bar{z}_l} \sim R^{\xi}_{\xi, z_k \bar{z}_l} = O(1),$$
$$R^{z_i}_{z_p, z_k \bar{z}_l} \sim R^{\xi}_{z_p, z_k \bar{z}_l} = O(|\xi|^{2(1-\beta)}).$$

From the curvature estimates, we immediately have the following proposition.

Theorem 4.6 *Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0, 1)$. If the Ricci curvature of g is bounded, then we have the following pointwise estimates for $|Rm|^2$:*

$$|Rm(g)|^2_g = O(|\xi|^{-2+4(1-\beta)}).$$

Consequently, the L^2–norm of $Rm(g)$ is bounded, ie there exists $C > 0$ such that

$$\int_X |Rm(g)|^2_g \, dVol(g) \leq C. \tag{4-14}$$

4.2 Chern forms for conical Kähler metrics

Let g be a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along a smooth divisor D. We let θ be the connection form on the tangent bundle TX induced by g, so locally we may write

$$\theta^{\alpha}_{\gamma} = g^{\alpha\bar{\beta}} g_{\gamma\bar{\beta},\eta} dz_{\eta},$$

and $\Omega^{\alpha}_{\gamma} = \bar{\partial}\theta^{\alpha}_{\gamma}$ is the curvature form of θ. Then the total Chern class is defined by

$$\det(tI + \Omega) = \sum_{i=0}^{n} t^{2(n-i)} c_i(\Omega).$$

Let $P_i(\Omega_1, \ldots, \Omega_n)$ be the polarization of $c_i(\Omega) = c_i(X, g)$ for the conical metric g.

Let g_0 be a smooth Kähler metric and θ_0 be the connection induced by g_0 as

$$(\theta_0)^{\alpha}_{\gamma} = (g_0)^{\alpha\bar{\beta}} (g_0)_{\gamma\bar{\beta},\eta} dz_{\eta}.$$

Then $\Omega_0 = \sqrt{-1}\partial\theta_0$ is the curvature form of θ_0.

Let $\theta_t = t\theta + (1-t)\theta_0$ with curvature

$$\Omega_t = \bar{\partial}\theta_t = t\Omega + (1-t)\Omega_0.$$

Then we have

$$c_2(X,\omega)-c_2(X,\omega_0)=2\sqrt{-1}\int_0^1 \overline{\partial} P_2(\theta-\theta_0,\Omega_t)\,dt.$$

We will construct connections on the divisor D from θ and θ_0. Let p be a point in the divisor D. We can choose holomorphic local coordinates

$$z=(z_1,\dots,z_n),\quad \xi=z_n$$

such that D is locally defined by $\xi=0$ as in Section 4.1.

Definition 4.7 We define H and H_0 locally by

$$H_{i\bar{j}}=g_{i\bar{j}},\qquad H^{i\bar{j}}=(H^{-1})_{i\bar{j}},\qquad H_{n\bar{j}}=g_{n\bar{j}}, \tag{4-15}$$

$$(H_0)_{i\bar{j}}=(g_0)_{i\bar{j}},\quad (H_0)^{i\bar{j}}=(H_0^{-1})_{i\bar{j}},\quad (H_0)_{n\bar{j}}=(g_0)_{n\bar{j}}. \tag{4-16}$$

Definition 4.8 For each coordinate system (z,ξ) chosen as above, we define $(1,0)$–forms θ_D and $\theta_{0,D}$ locally by

$$\theta_D=(\theta_i^i)|_D, \tag{4-17}$$

$$\theta_{0,D}=(\theta_0)_i^i|_D+H^{i\bar{j}}H_{n\bar{j}}(\theta_0)_i^n|_D. \tag{4-18}$$

Lemma 4.9 *The $(1,0)$–form*

$$\theta_D=(H^{i\bar{j}}H_{i\bar{j},k}|_D)dz_k=\partial\log\det H|_D \tag{4-19}$$

defines a global smooth Chern connection of the anticanonical line bundle of D. In particular, its curvature form $\sqrt{-1}\partial\theta_D$ is a smooth closed real $(1,1)$–form in $c_1(D)$.

Proof By Corollary 4.3, we have $g^{i\bar{n}}|_D=0$ and hence

$$(\theta_D)_i^i=g^{i\bar{\beta}}g_{i\bar{\beta},k}dz_k=(H^{i\bar{j}}H_{i\bar{j},k}|_D)dz_k=\partial(\log\det H|_D)=\partial\log(\omega|_D)^{n-1}.$$

By Proposition 4.1, the regular part of ω restricted to D is a smooth Kähler form from the expansion in Proposition 4.1 and for different holomorphic local coordinates $z=(z_1,\dots,z_{n-1})$ and $w=(w_1,\dots,w_{n-1})$ on D,

$$\theta_D(z)=\theta_D(w)+\partial\log\left|\det\left(\frac{\partial z_i}{\partial w_j}\right)\right|^2.$$

Therefore θ_D defines a smooth connection on anticanonical bundle of D. □

Lemma 4.10 *The $(1,0)$–form*

$$\begin{aligned}\theta_{0,D} &= \big((\theta_0)^i_i + (H^{i\bar{j}}H_{n\bar{j}})(\theta_0)^n_i\big)|_D \\ &= \big((g_0)^{i\bar{\beta}}(g_0)_{i\bar{\beta},k}|_D dz_k + (H^{i\bar{j}}H_{n\bar{j}})(g_0)^{n\bar{\beta}}(g_0)_{i\bar{\beta},k}|_D dz_k\big)\end{aligned}$$

defines a smooth Chern connection of the anticanonical bundle of D. In particular, $\sqrt{-1}\partial\theta_{0,D}$ is a smooth closed real $(1,1)$ form in $c_1(D)$.

Proof Since g_0 is smooth and the restriction of $H_{i\bar{j}}$, $H^{i\bar{j}}$, $H_{n,\bar{j}}$ to D are all smooth by the asymptotic expansion in Proposition 4.1, $\theta_{0,D}$ is locally smooth. It suffices to show that they patch together give rise to a connection.

To do that we need to show that the transformation of $\theta_{0,D}$ under different coordinate charts satisfies the cocycle condition for the anticanonical bundle of D. Let $(z_1,\dots,z_{n-1})$ and $(w_1,\dots,w_{n-1})$ be two holomorphic local coordinates for some neighborhood of a point p in D. Then they extend to two holomorphic coordinates $(z_1,..,z_{n-1},z_n=\xi)$, $(w_1,\dots,w_{n-1},w_n=\eta)$ in a neighborhood of p in X, where D is locally defined by $\xi=0$ and $\eta=0$. Therefore for $i=1,\dots,n-1$,

$$\frac{\partial\eta}{\partial z_i}\Big|_D = \frac{\partial\xi}{\partial w_i}\Big|_D = 0.$$

By letting

$$A^\alpha_\gamma = \frac{\partial z_\alpha}{\partial w_\gamma}, \quad B^\alpha_\gamma = \frac{\partial w_\alpha}{\partial z_\gamma}, \quad \widetilde{A}^i_k = \frac{\partial z_i}{\partial w_k}, \quad \widetilde{B}^i_k = \frac{\partial w_i}{\partial z_k},$$

we obtain along D,

$$A = \begin{pmatrix}\widetilde{A} & * \\ 0 & \frac{\partial\xi}{\partial\eta}\end{pmatrix}, \quad B = \begin{pmatrix}\widetilde{B} & * \\ 0 & \frac{\partial\eta}{\partial\xi}\end{pmatrix}, \quad A = B^{-1}, \quad \widetilde{A} = \widetilde{B}^{-1}.$$

Straightforward computations show that

$$\theta_{0,D}(z) = \theta_{0,D}(w) + \partial\log|\det\widetilde{A}|^2$$

which completes the proof. □

For any poly-homogenous conical Kähler metric ω with cone angle $2\pi\beta$ along a smooth divisor D, we can define the first and second Chern classes $c_1(X,\omega)$ and $c_2(X,\omega)$. A priori, the intersection numbers among $c_1(X,\omega)$ and $c_2(X,\omega)$ might depend on the choice of ω even if the Ricci curvature of ω is bounded. The following proposition relates the $c_1(X,\omega)$ and $c_2(X,\omega)$ to $c_1(X)$ and $c_2(X)$.

Proposition 4.11 *Let D be a smooth divisor on a Kähler manifold D. Suppose ω is a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along D with bounded Ricci curvature. Then*

$$\int_X c_1(X,\omega)\wedge\omega^{n-1} = (c_1(X)-(1-\beta)[D])\cdot[\omega]^{n-1}, \tag{4-20}$$

$$\int_X c_2(X,\omega)\wedge\omega^{n-2} = (c_2(X)+(1-\beta)(-c_1(X)+[D])\cdot[D])\cdot[\omega]^{n-2}, \tag{4-21}$$

$$\int_X c_1^2(X,\omega)\wedge\omega^{n-2} = (c_1(X)-(1-\beta)[D])^2\cdot[\omega]^{n-2}. \tag{4-22}$$

Proof We break the proof into the following steps.

Step 1 Equations (4-20) and (4-22) follow easily from the following observation. By our assumption, ω is a regular conical Kähler metric (cf Definition 1.3), from which we deduce that $\mathrm{Ric}(\omega)=\eta_0+\sqrt{-1}\partial\bar\partial\psi+(1-\beta)[D]$, where η_0 is a smooth closed real valued $(1,1)$–form and $\psi\in \mathrm{PSH}(X,\theta)\cap L^\infty(X)$ for some smooth Kähler metric θ. In particular, we may assume $\psi\in C^\infty(X\setminus D)$ since ω is smooth outside D. Therefore $\eta=\eta_0+\sqrt{-1}\partial\bar\partial\psi$ is smooth on $X\setminus D$ and $\eta\in c_1(X)-(1-\beta)[D]$ as $\psi\in\mathrm{PSH}(X,\theta)\cap L^\infty(X)$. Therefore, we have

$$c_1(X,\omega)=c_1(X)-(1-\beta)[D]\in H^2(X).$$

Step 2 We first introduce a few notations. Let ω_0 be a smooth Kähler form in the same class of $[\omega]$. Since the curvature tensor can be viewed as the curvature in the tangent bundle, we write $\theta=(\theta^i_j)$, $\theta_0=(\theta^i_j)$ as the Chern connections on the tangent bundle with respect to the Kähler metric ω and ω_0. Their curvature forms are given by Ω and Ω_0 with

$$\Omega=\sqrt{-1}\,\bar\partial\theta,\quad \Omega_0=\sqrt{-1}\,\bar\partial\theta_0.$$

Let s be a defining section of D and h be a smooth hermitian metric on the line bundle associated to $[D]$. We define

$$X_\epsilon=\{p\in X\mid |s|_h^2(p)>\epsilon^2\}.$$

Then locally $\xi=fS$ for some holomorphic function and on ∂X_ϵ, we have

$$\xi=fs=|fs|e^{i\sigma}=\epsilon|f|h^{-1/2}e^{\sqrt{-1}\sigma},\quad \sigma\in[0,2\pi),$$
$$d\xi=\sqrt{-1}\xi d\sigma+\epsilon e^{\sqrt{-1}\sigma}d(|f|h^{-1/2}).$$

Let $\tau=d(|f|h^{-1/2})$. Then τ is a smooth 1–form and on ∂X_ϵ,

$$d\xi=\sqrt{-1}\xi d\sigma+\epsilon\tau|_{\partial X_\epsilon}.$$

Step 3 At each point $p \in \partial X_\epsilon$, we can apply a linear transformation to $(z_1, \dots, z_{n-1})$ such that $g_{i\bar{j}} = \delta_{ij}$ at p and by rescaling ξ so that $g_{n\bar{n}} = |\xi|^{-2(1-\beta)}$ near p. Let

$$H_{i\bar{j}} = g_{i\bar{j}}, \quad H^{i\bar{j}} = (H^{-1})_{i\bar{j}}, \quad H_{n\bar{j}} = g_{n\bar{j}}.$$

then we have

$$\begin{aligned} |\xi|^{-2(1-\beta)} g^{i\bar{n}} &= |\xi|^{-2(1-\beta)} (g_{n\bar{i}} + o(1))(\det(g_{\alpha\beta}))^{-1} \\ &= g_{n\bar{i}} + o(1) = H^{i\bar{j}} H_{n\bar{j}} + o(1) \end{aligned} \tag{4-23}$$

and the connection form θ has the estimates

$$\begin{aligned} \theta^n_n = \theta^\xi_\xi &= \sum g^{n\bar{\beta}} g_{n\bar{\beta},\alpha} dz_\alpha = -(1-\beta+o(1))\xi^{-1} d\xi + \sum_i o(1)\cdot dz_i \\ &= -(1-\beta)\sqrt{-1} d\sigma + \sum_i O(1)\cdot dz_i \quad (\text{since } \xi \sim \epsilon \text{ on } \partial X_\epsilon), \\ \theta^i_n &= \sum g^{i\bar{\beta}} g_{n\bar{\beta},\alpha} dz_\alpha = g^{i\bar{n}} g_{n\bar{n},n} d\xi + o(1)\xi^{-1} d\xi + \sum_i o(1)\cdot dz_i \\ &= g^{i\bar{n}} (|\xi|^{-2(1-\beta)})_\xi d\xi + o(1)\cdot \xi^{-1} d\xi + \sum_i o(1)\cdot dz_i \\ &= -(1-\beta) H^{i\bar{j}} H_{n\bar{j}} \xi^{-1} d\xi + o(1)\cdot \xi^{-1} d\xi + \sum_i o(1)\cdot dz_i \\ &= -\sqrt{-1}(1-\beta) H^{i\bar{j}} H_{n\bar{j}} d\sigma + o(1)\cdot d\sigma + \sum_i O(1)\cdot dz_i, \\ \theta^n_i &= O(1)\cdot d\xi + \sum_i o(1)\cdot dz_i = o(1)\cdot d\sigma + \sum_i o(1)\cdot dz_i, \\ \theta^i_k &= \sum_\alpha O(1)\cdot dz_\alpha = o(1)\cdot d\sigma + \sum_i O(1)\cdot dz_i. \end{aligned}$$

On ∂X_ϵ, by our assumption that $\beta \in (\frac{1}{2}, 1)$ and using (4-23) we deduce

$$\begin{aligned} \Omega^n_n &= \sqrt{-1} \sum_{\alpha,\beta} R^n_{n,\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta \\ &= \sum_\alpha o(1)\cdot d\sigma \wedge dz_\alpha + \sum_\beta o(1)\cdot d\sigma \wedge d\bar{z}_\beta + \sum_{i,j} O(1)\cdot dz_i \wedge d\bar{z}_j, \\ \Omega^i_n &= \big(g^{i\bar{\gamma}} g_{n\bar{\gamma},\alpha}\big)_{\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta \\ &= \sqrt{-1}(g^{i\bar{n}} g_{n\bar{n},n})_{\bar{z}_k} dz_k \wedge d\xi + \sqrt{-1}(g^{i\bar{j}} g_{n\bar{j},n})_{\bar{z}_k} dz_k \wedge d\xi \\ &\quad + \sqrt{-1}(g^{i\bar{j}} g_{n\bar{j},k})_{\bar{\xi}} d\xi \wedge dz_k + \sqrt{-1}(g^{i\bar{j}} g_{n\bar{j},k})_{\bar{z}_l} d\bar{z}_l \wedge dz_k + \sum_{k,l} o(1)\cdot dz_k \wedge d\bar{z}_l \end{aligned}$$

$$= \sqrt{-1}(1-\beta)(H^{i\bar{j}}H_{n\bar{j}})_{\bar{z}_k} dz_k \wedge \xi^{-1} d\xi + \sum_k o(1)\cdot dz_k \wedge d\sigma + \sum_{k,l} O(1)\cdot dz_k \wedge d\bar{z}_l$$

$$= \sqrt{-1}(1-\beta)\bar{\partial}(H^{i\bar{j}}H_{n\bar{j}}) \wedge d\sigma + \sum_k o(1)\cdot dz_k \wedge d\sigma + \sum_{k,l} O(1)\cdot dz_k \wedge d\bar{z}_l,$$

$$\begin{aligned}\Omega_i^n &= \sqrt{-1}R^n_{i,\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta \\ &= \sqrt{-1}R^n_{i,n\bar{l}} d\xi \wedge d\bar{z}_l + \sqrt{-1}R^n_{i,k\bar{n}} dz_k \wedge d\bar{\xi} + \sqrt{-1}R^n_{i,k\bar{l}} dz_k \wedge d\bar{z}_l + o(1)dz_k \wedge d\bar{z}_l \\ &= o(1)dz_k \wedge d\sigma + o(1)dz_k \wedge d\bar{z}_l, \\ \Omega_p^i &= \sqrt{-1}R^i_{p,\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta \\ &= \sqrt{-1}R^i_{p,n\bar{l}} d\xi \wedge d\bar{z}_l + \sqrt{-1}R^i_{p,k\bar{n}} dz_k \wedge d\bar{\xi} + \sqrt{-1}R^i_{p,k\bar{l}} dz_k \wedge d\bar{z}_l + o(1)dz_k \wedge d\bar{z}_l \\ &= o(1)dz_k \wedge d\sigma + O(1)dz_k \wedge d\bar{z}_l.\end{aligned}$$

Step 4 By our assumption, ω is a smooth Kähler metric with cone angle $2\pi\beta$ along $D \in X$ such that it Ricci curvature is bounded on $X \setminus D$. This implies that

$$\Omega_\alpha^\alpha = \mathrm{Ric}(\omega) = \omega_0 - (1-\beta)\sqrt{-1}\partial\bar{\partial}\log|s_D|_h^2, \tag{4-24}$$

where h is a smooth Hermitian metric on the line bundle $\mathcal{O}_X(D)$, ω_0 is a smooth Kähler metric and $s_D \in H^0(X, \mathcal{O}_X(D))$ is a defining section for D. This implies that

$$\int_{X\setminus X_\epsilon} \Omega_\alpha^\alpha \wedge \omega^{n-1} = ([\omega_0] - (1-\beta)[D])\cdot\omega^{n-1} + o(\epsilon),$$

from which we obtain (4-20).

Let $\theta_t = t\theta + (1-t)\theta_0$ be the connection on the tangent bundle TX. The curvature

$$\Omega_t = \sqrt{-1}\bar{\partial}\theta_t = t\Omega + (1-t)\Omega_0.$$

The transgression formula gives

$$c_2(X,\omega) - c_2(X,\omega_0) = 2\sqrt{-1}\int_0^1 \bar{\partial} P_2(\theta-\theta_0, \Omega_t)\, dt$$

and

$$\begin{aligned}2P_2(\theta-\theta_0,\Omega_t) &= (\theta-\theta_0)^n_n \wedge (\Omega_t)^i_i - (\theta-\theta_0)^i_n \wedge (\Omega_t)^n_i + (\theta-\theta_0)^i_i \wedge (\Omega_t)^n_n \\ &\quad - (\theta-\theta_0)^n_i \wedge (\Omega_t)^i_n - (\theta-\theta_0)^i_k \wedge (\Omega_t)^k_i \\ &= -\sqrt{-1}(1-\beta)(t\bar{\partial}\theta + (1-t)\bar{\partial}\theta_0)^i_i \wedge d\sigma \\ &\quad - (1-t)(1-\beta)\sqrt{-1}\big(H^{i\bar{j}}H_{n\bar{j}}\bar{\partial}(\theta_0)^n_i - (\theta-\theta_0)^n_i \wedge \bar{\partial}(H^{i\bar{j}}H_{n\bar{j}})\big) \wedge d\sigma\end{aligned}$$

$$\begin{aligned}&+o(1)\,d\sigma\wedge dz_k\wedge d\bar z_l+O(1)\,dz_i\wedge dz_k\wedge d\bar z_l\\ &=\sqrt{-1}(1-\beta)\big(-t\,\overline{\partial}(\theta_i^i)\wedge d\sigma-(1-t)\overline{\partial}\big((\theta_0)_i^i+H^{i\bar j}H_{n\bar j}(\theta_0)_i^n\big)\wedge d\sigma\big)\\ &\quad+o(1)\,d\sigma\wedge dz_k\wedge d\bar z_l+O(1)\,dz_i\wedge dz_k\wedge d\bar z_l.\end{aligned}$$

Now let η be a smooth Kähler form. Then

$$\begin{aligned}&\int_X(c_2(X,\omega)-c_2(X,\omega_0)\wedge\eta^{n-2}\\ &=2\sqrt{-1}\int_0^1\bigg(\int_X\overline{\partial}P_2(\theta-\theta_0,t\,\Omega+(1-t)\Omega_0)\wedge\eta^{n-2}\bigg)dt\\ &=2\int_0^1\bigg(\int_{\partial X_\epsilon}P_2(\theta-\theta_0,t\,\Omega+(1-t)\Omega_0)\wedge\eta^{n-2}\bigg)dt+o(1)\\ &=(1-\beta)\sqrt{-1}\int_0^1\int_{\partial X_\epsilon}\Big(-t\,\overline{\partial}(\theta_i^i)-(1-t)\overline{\partial}((\theta_0)_i^i+H^{i\bar j}H_{n\bar j}(\theta_0)_i^n)\Big)\wedge d\sigma\wedge\eta^{n-2}dt+o(1)\\ &=(1-\beta)\sqrt{-1}\int_0^1\int_D\Big(-t\,\overline{\partial}\theta_D-(1-t)\overline{\partial}\theta_{0,D}\Big)\wedge(\eta|_D)^{n-2}dt\\ &=(1-\beta)(-c_1(D)/2-c_1(D)/2)\cdot[D]\cdot[\eta]^{n-2}+o(1)\\ &=(1-\beta)(-c_1(X)+[D])\cdot[D]\cdot[\eta]^{n-2}+o(1).\end{aligned}$$

The last three equalities follow from Lemma 4.9, Lemma 4.10 and the adjunction formula. And similarly, we have

$$\begin{aligned}&\int_X(c_1^2(X,\omega)-c_1^2(X,\omega_0)\wedge\eta^{n-2}\\ &\qquad=\int_0^1\bigg(\int_X\overline{\partial}Q(\theta-\theta_0,t\,\Omega+(1-t)\Omega_0)\wedge\eta^{n-2}\bigg)dt\\ &\qquad=2(1-\beta)\int_0^1\int_{\partial X_\epsilon}\big(-t\,\overline{\partial}(\theta_\alpha^\alpha)-(1-t)\overline{\partial}((\theta_0)_\alpha^\alpha)\big)\wedge d\sigma\wedge\eta^{n-2}dt+o(1)\\ \text{(by (4-24))}\quad&\qquad=(1-\beta)\big(-c_1(X)+(1-\beta)[D]-c_1(X)\big)\cdot[D]\cdot[\eta]^{n-2}\\ &\qquad=\big(-2(1-\beta)c_1(X)\cdot[D]+(1-\beta)^2[D]^2\big)\cdot[\eta]^{n-2}\end{aligned}$$

which is equivalent to (4-22).

Step 5 Suppose that $\omega_0\in[\omega]$ is a smooth Kähler form. We want to show that

$$\int_X c_2(X,\omega)\wedge\omega^{n-2}=\int_X c_2(X,\omega)\wedge\omega_0^{n-2},\tag{4-25}$$

$$\int_X c_1^2(X,\omega)\wedge\omega^{n-2}=\int_X c_1^2(X,\omega)\wedge\omega_0^{n-2}.\tag{4-26}$$

Since the proofs are parallel to each other, we will only prove the (4-25). Let φ be defined by $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$.

$$\int_X c_2(X,\omega)\wedge(\omega^{n-2}-\omega_0^{n-2}) = \sum_{i=0}^{n-3}\int_X \sqrt{-1}\partial\bar{\partial}\varphi\wedge c_2(X,\omega)\wedge\omega^i\wedge\omega_0^{n-3-i}.$$

For any $i = 0, 1, \ldots, n-3$,

$$\int_X \sqrt{-1}\partial\bar{\partial}\varphi\wedge c_2(X,\omega)\wedge\omega^i\wedge\omega_0^{n-3-i} = \int_{\partial X_\epsilon} d^c\varphi\wedge c_2(X,\omega)\wedge\omega^i\wedge\omega_0^{n-3-i} + o(1).$$

Note that

$$\begin{aligned} d^c\varphi &= o(1)d\sigma + O(1)dz_k,\\ \omega &= o(1)d\sigma\wedge dz_k + o(1)d\sigma\wedge d\bar{z}_k + O(1)dz_k\wedge d\bar{z}_l \end{aligned}$$

and

$$\begin{aligned} &\Omega_i^i\wedge\Omega_j^j \sim \Omega_j^i\wedge\Omega_i^j \sim \Omega_n^n\wedge\Omega_j^j \sim \Omega_j^n\wedge\Omega_n^j\\ &= o(1)dz_k\wedge d\bar{z}_l\wedge dz_p\wedge d\sigma + o(1)dz_k\wedge d\bar{z}_l\wedge d\bar{z}_q\wedge d\sigma + O(1)dz_k\wedge d\bar{z}_l\wedge dz_p\wedge d\bar{z}_q. \end{aligned}$$

Therefore

$$\int_X \sqrt{-1}\partial\bar{\partial}\varphi\wedge c_2(X,\omega)\wedge\omega^i\wedge\omega_0^{n-3-i} = \int_{\partial X_\epsilon} d^c\varphi\wedge c_2(X,\omega)\wedge\omega^i\wedge\omega_0^{n-3-i} + o(1) = 0$$

after letting ϵ tend to 0.

Step 6 Finally, combining the above estimates, we obtain (4-22) and the proof of the proposition is completed. □

One can apply a similar argument to show that if ω is a poly-homogenous conical Kähler metric ω with cone angle $2\pi\beta$ along a smooth divisor D and if the Ricci curvature of ω is bounded, then the n^{th}–Chern number $c_n(X,\omega)$ is well-defined and does not depend on the choice of ω.

4.3 The Gauss–Bonnet and signature theorems for Kähler surfaces with conical singularities

Definition 4.12 Let X be a Kähler surface and Σ be a smooth holomorphic curve on X. If g is a poly-homogenous conical Kähler metric with cone angle $2\pi\beta$ along Σ, we define the corresponding conical Euler number and signature by

$$\chi(X,g)=\int_{X\setminus\Sigma}\frac{1}{8\pi^2}\left(\frac{S^2}{24}+|W|^2-\frac{|\overset{\circ}{\mathrm{Ric}}|^2}{2}\right)dg,$$
$$\sigma(X,g)=\frac{1}{12\pi^2}\int_{X\setminus\Sigma}(|W^+|^2-|W^-|^2)dg,$$

where S is the scalar curvature of metric g, W is the Weyl tensor for g and $\overset{\circ}{\mathrm{Ric}}$ is the traceless Ricci curvature. In particular, if $\beta=1$, we recover classical characteristic class.

The Gauss–Bonnet and the signature theorems are proved in [1] for smooth compact Riemannian 4–folds with specified conical metrics with cone angle $2\pi\beta$ along a smooth embedded Riemann surface. As an immediate consequence of Proposition 4.11 and Definition 4.12 above, we obtain the following formulas related to the recent result by Atiyah and Lebrun [1] by removing the assumption $\beta\in(0,\frac{1}{3})$ in the Kähler case.

Proposition 4.13 *Let g be a poly-homogenous conical Kähler metric with angle $2\pi\beta$ for $\beta\in(0,1]$ along a holomorphic curve Σ. If the Ricci curvature of g is bounded, we have*

$$\chi(X,g)=\chi(X)-(1-\beta)\chi(D),$$
$$\sigma(X,g)=\sigma(X)-\tfrac{1}{3}(1-\beta^2)[D]^2.$$

Proof To prove the statement, we apply the identities (cf [1])

$$\frac{1}{8\pi^2}\left(\frac{S^2}{24}+|W|^2-\frac{|\overset{\circ}{\mathrm{Ric}}|^2}{2}\right)dg,=c_2(X,g)$$
$$\frac{1}{12\pi^2}(|W^+|^2-|W^-|^2)\,dg=\frac{1}{3}(c_1^2(X,g)-2c_2(X,g))$$

and the statement follows from Proposition 4.11 □

4.4 The Chern number inequality on Fano manifolds

In this section, we will prove Theorem 1.9.

Proposition 4.14 *Let X be an n–dimensional Fano manifold. If $R(X)=1$, then the following Miyaoka–Yau type inequality holds:*

$$c_2(X)\cdot c_1(X)^{n-2}\geq\frac{n}{2(n+1)}c_1(X)^n.$$

Proof We fix a smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$. Such a divisor always exists by Bertini's theorem. Then for any $\beta \in (0,1)$, there exists a poly-homogenous conical Kähler–Einstein metric ω satisfying $\mathrm{Ric}(\omega) = \beta g\omega + (1-\beta)m^{-1}[D]$.

By Chern–Weil theory, if we let

$$\overset{\circ}{R}_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{\mathrm{tr}(R)}{n(n+1)}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}),$$
$$\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{tr}(\mathrm{Ric})}{n}g$$

be the traceless curvature and Ricci curvature tensor, we have

$$\left(\frac{2(n+1)}{n}c_2(X,\omega) - c_1^2(X,\omega)\right)\cdot[\omega]^{n-2} = \frac{1}{n(n-1)}\int_X\left(\frac{n+1}{n}|\overset{\circ}{R}|^2 - \frac{n^2-2}{n^2}|\overset{\circ}{\mathrm{Ric}}|^2\right)\omega^n.$$

We then have

$$\left(\frac{2(n+1)}{n}c_2(X,\omega) - c_1^2(X,\omega)\right)\cdot[\omega]^{n-2} \geq 0.$$

By Proposition 4.11, we have

$$c_1(X,\omega) = \beta c_1(X), \quad c_2(X,\omega) = c_2(X) + (1-\beta)(-c_1(X)+[D])\cdot[D].$$

This implies that

$$\left(\frac{2(n+1)}{n}c_2(X) - \beta^2 c_1^2(X)\right)\cdot(\beta c_1(X))^{n-2} \geq 0.$$

The theorem then follows by letting $\beta \to 1$. □

We also have the following lemma when $R(X) < 1$ with the same argument as in the proof of Proposition 4.14.

Lemma 4.15 *Let $D \in |-K_X|$ be a smooth divisor. If there exists a conical Kähler–Einstein metric g_β for some $\beta \in (0, R(X))$ satisfying*

$$\mathrm{Ric}(g) = \beta g + (1-\beta)[D],$$

then

$$c_2(X)\cdot c_1(X)^{n-2} \geq \frac{n\beta^2}{2(n+1)}c_1(X)^n.$$

Theorem 1.9 is proved by combining Proposition 4.14, Lemma 4.15 and Theorem 1.5.

4.5 The Chern number inequality on minimal manifolds of general type

The following proposition was first claimed in [44], although the analytic part in the proof does not seem to be complete. The first complete proof seems to have been given by Zhang [49] using the Ricci flow. In this section, we apply Proposition 4.11 to complete Tsuji's original approach.

Proposition 4.16 *Let X be a smooth minimal model of general type. Then*

$$\left(\frac{2(n+1)}{n}c_2(X)-c_1^2(X)\right)\cdot(-c_1(X))^{n-2}\geq 0. \tag{4-27}$$

In particular, if the equality holds, the canonical Kähler–Einstein metric is a complex hyperbolic metric on the smooth part of the canonical model of X.

Proof Fix a smooth ample divisor D on X. Since $[K_X]+\epsilon[D]$ is a Kähler class for any $\epsilon>0$, there exists a poly-homogenous conical Kähler–Einstein metric in $[K_X]+\epsilon[D]$ with conical singularity along D satisfying

$$\mathrm{Ric}(\omega)=-\omega+\epsilon[D].$$

By the same argument as in the proof of Proposition 4.14,

$$\left(\frac{2(n+1)}{n}c_2(X,\omega)-c_1^2(X,\omega)\right)\cdot(-c_1(X,\omega))^{n-2}\geq\int_X|\mathring{R}(\omega)|^2\omega^n\geq 0.$$

By standard argument from [14; 50], ω converges as $\epsilon\to 0$ to the unique Kähler–Einstein metric g_{can} on the canonical model X_{can} of X in the C^∞ local topology away from the exceptional locus the pluricanonical system. In particular, ω_{can} is smooth on the smooth part of X_{can}.

By letting ϵ tend to 0, we have

$$\left(\frac{2(n+1)}{n}c_2(X)-c_1^2(X)\right)\cdot(-c_1(X))^{n-2}\geq\int_X|\mathring{R}(\omega_{\mathrm{can}})|^2\omega_{\mathrm{can}}^n\geq 0, \tag{4-28}$$

where X_{can}° is the smooth part of X_{can}. When the equality holds, $\mathring{R}(g_{\mathrm{can}})$ vanishes on X_{can}° and so g_{can} must be a complex hyperbolic metric on X_{can}°. □

Then by the estimate (4-28), we immediately obtain an L^2–bound for the curvature tensor of the canonical Kähler–Einstein metric on the regular part of the canonical model associated to a smooth minimal model of general type.

5 Discussions

In this section, we speculate on the limiting behavior of the conical Kähler–Einstein metrics as β tends to $R(X)$.

Conjecture 5.1 *Let X be a Fano manifold with $R(X) = 1$. Then by Proposition 1.6, there exists a smooth divisor $D \in |-mK_X|$ for some $m \in \mathbb{Z}^+$ such that for any $\epsilon > 0$, there exists a regular conical Kähler–Einstein metric g_ϵ with $\mathrm{Ric}(g_\epsilon) = (1-\epsilon)g_\epsilon + \epsilon m^{-1}[D]$ for all $\epsilon \in (0,1)$. We conjecture that (X, g_ϵ) converges to a $\mathbb{Q}$–Fano variety (X_∞, g_∞) coupled with a canonical Kähler–Einstein metric g_∞ in Gromov–Hausdorff topology.*

The above conjecture is related to the recent result in [40], where the Kähler–Ricci flow is combined with the continuity method to produce a limiting Einstein metric space when $R(X) = 1$. We also make a more general conjecture when $R(X) \neq 1$.

Conjecture 5.2 *Let X be a Fano manifold and D be a smooth divisor in $|-mK_X|$ for some $m \in \mathbb{Z}^+$. We consider the conical Ricci flow defined by*

$$\frac{\partial g}{\partial t} = -\mathrm{Ric}(g) + \beta g + m^{-1}(1-\beta)[D] \tag{5-1}$$

starting with a regular conical Kähler metric $g_0 \in c_1(X)$ with cone angle $2\pi(1-(1-\beta)m^{-1})$ along D. Then for some $m \in \mathbb{Z}^+$ and a generic choice of D, we have:

(1) *If $\beta \in (0, R(X))$, the flow converges to a regular conical Kähler Einstein metric on X with conical singularity along D.*

(2) *If $\beta = R(X)$, the flow converges to a singular Kähler–Einstein metric on a paired $\mathbb{Q}$–Fano variety (X_∞, D_∞) with conical singularities along an effective $\mathbb{Q}$–divisor $D_\infty \in [-K_{X_\infty}]$.*

(3) *If $\beta \in (R(X), 1]$, the flow converges to a singular Kähler–Ricci soliton on a paired $\mathbb{Q}$–Fano variety (X_∞, D_∞) with conical singularities along an effective $\mathbb{Q}$–divisor $D_\infty \in [-K_{X_\infty}]$.*

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